Reliability Of A *k*-out-of-*n* System With A Single Server Extending Non-Preemptive Service To External Customers-Part I

A. Krishnamoorthy

•

Dept. of Mathematics, Cochin University of Science & Technology, Kochi-682022
e-mail: *achyuthacusat@gmail.com*

M. K. Sathian

•

Dept. of Mathematics, Panampilly Memorial Govt. College, Chalakudy, Thrissur.
e-mail: *sathianmkk@yahoo.com*

Narayanan C Viswanath

•

Dept. of Mathematics Govt. Engg. College, Thrissur Thrissur 680 009.
e-mail: *narayanan_viswanath@yahoo.com*

Abstract

We study repairable k-out-of-n system with single server who provides service to external customers also. N-policy is employed for the service of main customers. Once started, the repair of failed components is continued until all components become operational. When not repairing main customers, the server attends external customers (if there is any) who arrive according to a Poisson process. Once selected, the external customers receive a service of non-preemptive nature. When at least N main customers accumulate in the system and/or when the server is busy with such customers, external customers are not allowed to join the system. Otherwise, they join an infinite capacity queue of external customers. Life time distribution of components, service time distribution of main and external customers are all assumed to follow independent exponential distributions. Steady state analysis has been carried out and several important system performance measures based on the steady state distribution derived. A numerical study comparing the current model with those in which no external customers are provided service, is carried out. This study suggests that rendering service to external customers helps to utilize the server idle time profitably, without affecting the system reliability.

Keywords: *k-out-of-n system; non-preemptive service.*

1 Introduction

A k-out-of-n system can be defined as an n-component system which works if and only if at least k of its components operational. Application of such systems can be seen in many real-world phenomena. For instance almost all machines, of different complexity, are subjected to failure. One would expect a machine to work, even if some of its components have failed. A hospital providing emergency service is a typical example.We would expect the hospital to run even if some of its doctors/nurses/other staffs are on leave since it is supposed to have these personal in excess of the actual requirement. However, keeping these extra resources could be costly and not even feasible in some cases. A probabilistic study of a real world system such as a k-out-of-n system, often helps to develop an optimal strategy for maintaining high system reliability. Literature on such studies is vast (for example,see Chakravarthy et al.[1]).

Dudin et al.[2], Krishnamoorthy et al.[3, 4, 5] are among the studies on the reliability of a k-out-of-n system, where the server provides service to external customers in addition to repairing failed components of the main system. Such models are suitable for many real world situations. For example, a big telecom company may decide to share its resources like optical cables, mobile towers etc., for additional revenue. In doing so there is the risk that it may lead to dissatisfaction of the companies own customers. Therefore, the company would like to develop an optimal strategy for sharing its resources. Krishnamoorthy et al.[5] studied an N-policy for rendering service to external customers. They gave priority to the main customers through N-policy: the moment N failed components of the main system get accumulated, the ongoing service of an external customer (if there is any) is preempted and service to failed components is started.

In the present study, we consider a variant of the model in [5]. We assume N-policy for starting repair of failed components. However, the priority of the main customers is a bit reduced by assuming that an ongoing service of an external customer is not preempted when the number of failed components reaches N. This can be a serious compromise on the reliability of the k-out-of-n system. As in [5] it is assumed here also that an external customer, not allowed to join the system when the server is busy with service of main customers and/or when there are at least N failed components in the system. The external customer joins a queue of infinite capacity.

This paper is arranged as follows. In section 2 , we define the queuing model; section 3 conducts the steady state analysis, where we have obtained the stability condition explicitly and we also present an efficient method for computing the steady state probability vector. In section 4, we derive some important system performance measures and in section 5 the effect of N-policy and rendering service to external customers on the system reliability is examined. A cost function has also been studied in section 5 .

2 The queueing model

Here we consider a k-out-of-n system with a single server, offering service to external customers also. Commencement of service to failed components of the main system is governed by N-policy. That is at the epoch the system starts with all components operational, the server starts attending one by one the external customers (if there is any).When the number of failed components in the system is $\geq N$, the server in service of external customer (if there is any) is switched on to the service of the main customers after completing the ongoing service of the external customer. We assume that the failure rate of a component is $\frac{\lambda}{i}$, when i components are operational so that the inter-failure time of components of the k-out-of-n system remains exponentially distributed with parameter λ. Arrival of external customers follows a Poisson process with parameter $\bar{\lambda}$. External customers are not allowed to join the system when the server is busy with main customers or when there is $\geq N$ failed components. An external customer, who on arrival finds an idle server is directly taken for service. Service times of main and external customers follow exponential distribution with parameters μ and $\bar{\mu}$ respectively.

2.1 The Markov Chain

Let $X_1(t)$ = number of external customers in the system including the one getting service (if any) at time t,

$X_2(t)$ = number of main customers in the system including the one getting service (if any) at time t,

$$S(t) = \begin{cases} 0, & \text{if the server is idle or is busy with external customers} \\ 1, & \text{if the server is idle or is busy with main customers.} \end{cases}$$

Let $X(t) = (X_1(t), S(t), X_2(t))$ then $X = \{X(t), t \geq 0\}$ is a continuous time Markov chain on the state space

$$S = \{(0,0,j_2)/0 \leq j_2 \leq N-1\} \cup \{(j_1, 0, j_2)/j_1 \geq 1, 0 \leq j_2 \leq n-k+1\}$$
$$\cup \{(j_1, 1, j_2)/j_1 \geq 0, 1 \leq j_2 \leq n-k+1\}.$$

Arranging the states lexicographically and partitioning the state space into levels i, where each level i corresponds to the collection of the states with number of external customers in the system at any time t equal to i, we get an infinitesimal generator of the above chain as

$$Q = \begin{bmatrix} A_{10} & A_{00} & & & & \\ A_{20} & A_1 & A_0 & & & \\ & A_2 & A_1 & A_0 & & \\ & & A_2 & A_1 & A_0 & \\ & & & & \cdots & \\ & & & & & \cdots \end{bmatrix}.$$

In order to describe the entries in the above matrix we introduce some notations below.

[(i)]

1. I_m denotes an identity matrix of order m and I denotes an identity matrix of appropriate order.
2. e_m denotes a $m\times 1$ column matrix of 1s and e denotes a column matrix of 1s of appropriate order.
3. E_m denotes a square matrix of order m defined as

$$E_m(i,j) = \begin{cases} -1 & \text{if } j = i, 1 \leq i \leq m \\ 1 & \text{if } j = i+1, 1 \leq i \leq m-1 \\ 0 & \text{otherwise} \end{cases}$$

4. $E'_m = \text{Transpose }(E_m)$.
5. $r_m(i)$ denotes a $1\times n$ row matrix whose ith entry is 1 and all other entries are zeros.
6. $c_m(i) = \text{Transpose }(r_m(i))$.
7. $\otimes$ denotes Kronecker product of matrices.

The transition within level 0 is represented by the matrix

$$A_{10} = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}, \text{where}$$

$B_1 = \lambda E_N - \overline{\lambda} I_N$.

B_2 is a $N\times(n-k+1)$ matrix whose $(N,N)^{\text{th}}$ entry is λ and all other entries are zeroes.

B_3 is a $N\times(n-k+1)$ matrix whose $(1,1)^{\text{th}}$ entry is μ and all other entries are zeroes.

$B_4 = \lambda E_{n-k+1} + \lambda c_{n-k+1}(n-k+1) \otimes r_{n-k+1}(n-k+1) + \mu E_{n-k+1}'$.

The transition from level 0 to level 1 is represented by the matrix

$$A_{00} = \begin{bmatrix} \overline{\lambda} I_N & O_{N\times(2n-2k+3-N)} \\ O_{(n-k+1)\times N} & O_{(n-k+1)\times(2n-2k+3-N)} \end{bmatrix}.$$

Transition from level 1 to 0 is represented by the matrix

$$A_{20} = \begin{bmatrix} \overline{\mu} I_N & O \\ O & H \\ O_{(n-k+1)\times N} & O \end{bmatrix} \text{where} H = [O_{(n-k+2-N)\times(N-1)} \quad \overline{\mu} I_{(n-k+2-N)}].$$

Transition within level 1 is represented by the matrix

$$A_1 = \begin{bmatrix} H_{11} & H_{12} & 0 \\ 0 & H_{22} & 0 \\ H_{31} & 0 & B_4 \end{bmatrix} \text{where}$$

$$H_{11} = B_1 - \overline{\mu} I_N, H_{12} = \lambda c_N(N) \otimes r_{n-k+2-N}(1),$$

$$H_{22} = \lambda E_{n-k+2-N} + \lambda c_{n-k+2-N}(n-k+2-N) \otimes r_{n-k+2-N}(n-k+2-N) - \overline{\mu} I_{n-k+2-N}.$$

H_{31} is an $(n-k+1)\times N$ matrix whose $(1,1)^{\text{th}}$ entry is μ.

$$A_0 = \begin{bmatrix} \overline{\lambda} I_N & O_{N\times(2n-2k+3-N)} \\ O_{(2n-2k+3-N)\times N} & O_{(2n-2k+3-N)\times(2n-2k+3-N)} \end{bmatrix},$$

$$A_2 = \begin{bmatrix} \overline{\mu} I_N & O & O \\ O & O_{(n-k+2-N)\times(n-k+2-N)} & \widetilde{H} \\ O_{(n-k+1)\times N} & O & O \end{bmatrix},$$

where $\widetilde{H} = [O_{(n-k+2-N)\times(N-1)} \quad \overline{\mu} I_{(n-k+2-N)}]$.

3 Steady state analysis

3.1 Stability condition

Consider the generator matrix $A = A_0 + A_1 + A_2$

$$A = \begin{bmatrix} \lambda E_N & H_{12} & 0 \\ 0 & H_{22} & F_{23} \\ F_{31} & 0 & B_4 \end{bmatrix} \text{with}$$

$$F_{23} = [O_{(n-k+2-N)\times(N-1)} \quad \overline{\mu} I_{n-k+2-N}],$$

$$F_{31} = \mu c_{n-k+1}(1) \otimes r_N(1).$$

Let $\zeta = (\zeta_0, \zeta_1, \zeta_2)$ be the steady state vector of the generator matrix A, where

$\zeta_0 = (\zeta_{(0,0)}, \zeta_{(0,1)}, \dots, \zeta_{(0,N-1)})$,
$\zeta_1 = (\zeta_{(0,N)}, \zeta_{(0,N+1)}, \dots, \zeta_{(0,n-k+1)})$,
$\zeta_2 = (\zeta_{(1,1)}, \zeta_{(1,2)}, \dots, \zeta_{(1,n-k+1)})$.

The Markov chain $\{X(t), t \geq 0\}$ is stable if and only if $\zeta A_0 e < \zeta A_2 e$ (please refer Neuts [6]).

It follows that $\zeta A_0 e = \overline{\lambda}\zeta_0 e$ and $\zeta A_2 e = \overline{\mu}(\zeta_0 e + \zeta_1 e)$. Therefore the stability condition becomes

$$\frac{\overline{\lambda}}{\overline{\mu}} \frac{\zeta_0 e}{(\zeta_0 e + \zeta_1 e)} < 1. \tag{1}$$

It follows from the relation $\zeta A = 0$ that

$$\zeta_0 \lambda E_N + \zeta_2 F_{31} = 0, \tag{2}$$

$$\zeta_0 H_{12} + \zeta_1 H_{22} = 0, \tag{3}$$

$$\zeta_1 F_{23} + \zeta_2 B_4 = 0. \tag{4}$$

From (4), it follows that

$$\zeta_2 = -\zeta_1 F_{23} B_4^{-1}. \tag{5}$$

Substituting this in (2) we get

$$\zeta_0 \lambda E_N - \zeta_1 F_{23} B_4^{-1} F_{31} = 0. \tag{6}$$

$$\lambda \zeta_0 e = (-\zeta_1 F_{23} B_4^{-1} F_{31})(-E_N^{-1} e). \tag{7}$$

Notice that the first column of the matrix F_{31} is $-B_4 e$ and all other columns of it are zero columns. This implies that the first column of the matrix $B_4^{-1} F_{31}$ is $-e$ and its all other columns are zero columns. Hence the first column of the matrix $-F_{23} B_4^{-1} F_{31}$ is $\overline{\mu} e$ and all other columns are zero columns. The first entry of the row matrix $-\zeta_1 F_{23} B_4^{-1} F_{31}$ is thus $\overline{\mu}\zeta_1 e$ and its all other entries are zeros. It can be seen that the first entry of the column matrix $-E_N^{-1}e$ is N. These two facts together tell us that $(-\zeta_1 F_{23} B_4^{-1} F_{31})(-E_N^{-1}e)$ is $N\bar{\mu}\zeta_1 e$. Thus, equation (7) becomes

$$\lambda\zeta_0 e = N\bar{\mu}\zeta_1 e.$$

Adding $N\bar{\mu}\zeta_0 e$ on both sides of the above equation, we get

$$(\lambda + N\bar{\mu})\zeta_0 e = N\bar{\mu}(\zeta_0 e + \zeta_1 e),$$

which implies

$$\frac{\zeta_0 e}{(\zeta_0 e+\zeta_1 e)} = \frac{N\bar{\mu}}{(\lambda+N\bar{\mu})}.$$

Hence the stability condition (1) becomes

$$\frac{\bar{\lambda}}{\bar{\mu}}\frac{N\bar{\mu}}{(\lambda+N\bar{\mu})} < 1.$$

3.2 Computation of steady state vector

Let $\pi = (\pi(0), \pi(1), \pi(2), \dots)$ the steady state vector of the Markov chain X, where $\pi(0) = (\pi_{(0,0)}, \pi_{(0,1)})$ with $\pi_{(0,0)} = (\pi_{(0,0,0)}, \pi_{(0,0,1)}, \dots, \pi_{(0,0,N-1)})$ and $\pi_{(0,1)} = (\pi_{(0,1,1)}, \dots, \pi_{(0,1,n-k+1)})$. For $i \geq 1$, $\pi(i) = (\pi_{(i,0)}, \tilde{\pi}_{(i,0)}, \pi_{(i,1)})$ with $\pi_{(i,0)} = (\pi_{(i,0,0)}, \pi_{(i,0,1)}, \dots, \pi_{(i,0,N-1)})$, $\tilde{\pi}_{(i,0)} = (\pi_{(i,0,N)}, \dots, \pi_{(i,0,n-k+1)})$,
$\pi_{(i,1)} = (\pi_{(i,1,1)}, \pi_{(i,1,2)}, \dots, \pi_{(i,1,n-k+1)})$. Now from $\pi Q = 0$, we can write

$$\pi_{(0,0)}B_1 + \pi_{(0,1)}B_3 + \pi_{(1,0)}\bar{\mu}I_N = 0, \tag{8}$$

$$\pi_{(0,0)}B_2 + \pi_{(0,1)}B_4 + \tilde{\pi}_{(1,0)}H = 0, \tag{9}$$

and for $i \geq 1$,

$$\pi_{(i-1,0)}\bar{\lambda}I_N + \pi_{(i,0)}H_{11} + \pi_{(i,1)}H_{31} + \pi_{(i+1,0)}\bar{\mu}I_N = 0, \tag{10}$$

$$\pi_{(i,0)}H_{12} + \tilde{\pi}_{(i,0)}H_{22} = 0, \tag{11}$$

$$\pi_{(i,1)}B_4 + \tilde{\pi}_{(i+1,0)}\tilde{H} = 0. \tag{12}$$

From (11), we get, for $i \geq 1$

$$\tilde{\pi}_{(i,0)} = -\pi_{(i,0)}H_{12}(H_{22}^{-1}). \tag{13}$$

From (12), we get

$$\pi_{(i,1)} = -\tilde{\pi}_{(i+1,0)}\tilde{H}(B_4^{-1}). \tag{14}$$

Substituting (13) in (14), we get

$$\pi_{(i,1)} = \pi_{(i+1,0)}H_{12}(H_{22}^{-1})\tilde{H}(B_4^{-1}). \tag{15}$$

Substituting (15) in (10), we get

$$\pi_{(i-1,0)}\bar{\lambda}I_N + \pi_{(i,0)}H_{11} + \pi_{(i+1,0)}H_{12}(H_{22}^{-1})\tilde{H}(B_4^{-1})H_{31} + \pi_{(i+1,0)}\bar{\mu}I_N = 0. \tag{16}$$

We notice that the first column of the matrix H_{31} is $-B_4 e$ and all other columns of H_{31} are zero columns. Hence the first column of the matrix $(B_4^{-1})H_{31}$ is $-e$ and its all other columns are zero columns. This tells us that the first column of the matrix $\tilde{H}(B_4^{-1})H_{31}$ is $-\bar{\mu}e$ and all other columns are zeros. But $-\bar{\mu}e$ is $H_{22}e$ and hence the first column of the matrix $(H_{22}^{-1})\tilde{H}(B_4^{-1})H_{31}$ is e and all other columns are zeros. This fact leads us to conclude that the first column of the matrix $H_{12}(H_{22}^{-1})\tilde{H}(B_4^{-1})H_{31}$ is $H_{12}e = \lambda c_N(N)$ and all other columns are zeros. In other words

$$H_{12}(H_{22}^{-1})\tilde{H}(B_4^{-1})H_{31} = \lambda c_N(N) \otimes r_N(1).$$

Now equation (16) becomes

$$\pi_{(i-1,0)}\bar{\lambda}I_N + \pi_{(i,0)}H_{11} + \pi_{(i+1,0)}\lambda c_N(N) \otimes r_N(1) + \pi_{(i+1,0)}\bar{\mu}I_N = 0.$$

That is

$$\pi_{(i-1,0)}\bar{\lambda}I_N + \pi_{(i,0)}H_{11} + \pi_{(i+1,0)}(\lambda c_N(N) \otimes r_N(1) + \bar{\mu}I_N) = 0. \tag{17}$$

Now from equation (9), we can write

$$\pi_{(0,1)} = -\pi_{(0,0)}B_2(B_4^{-1}) - \tilde{\pi}_{(1,0)}H(B_4^{-1}). \tag{18}$$

However, from equation (13), we have

$$\tilde{\pi}_{(1,0)} = -\pi_{(1,0)}H_{12}(H_{22}^{-1}). \tag{19}$$

Hence equation (18) becomes

$$\pi_{(0,1)} = -\pi_{(0,0)}B_2(B_4^{-1}) + \pi_{(1,0)}H_{12}(H_{22}^{-1})H(B_4^{-1}). \tag{20}$$

Substituting (20) in (8), we get

$$\pi_{(0,0)}B_1 + (-\pi_{(0,0)}B_2(B_4^{-1}) + \pi_{(1,0)}H_{12}(H_{22}^{-1})H(B_4^{-1}))B_3 + \pi_{(1,0)}\bar{\mu}I_N = 0. \tag{21}$$

Since the first column of the matrix B_3 is $-B_4e$, a similar reasoning as for equation (16) leads us to write:

$$-B_2(B_4^{-1})B_3 = \lambda c_N(N) \otimes r_N(1),$$
$$H_{12}(H_{22}^{-1})H(B_4^{-1})B_3 = \lambda c_N(N) \otimes r_N(1).$$

Hence equation (21) becomes

$$\pi_{(0,0)}(B_1 + \lambda c_N(N) \otimes r_N(1)) + \pi_{(1,0)}(\lambda c_N(N) \otimes r_N(1) + \bar{\mu}I_N) = 0. \tag{22}$$

Equations (17) and (22) shows that the vector $\hat{\pi} = (\pi_{(0,0)}, \pi_{(1,0)}, \pi_{(2,0)}, \dots)$ satisfies the relation $\hat{\pi}\tilde{Q} = 0$, where $\tilde{Q}$ is a generator matrix defined as

$$\tilde{Q} = \begin{bmatrix} \tilde{A}_{10} & \tilde{A}_0 & & & \\ \tilde{A}_2 & \tilde{A}_1 & \tilde{A}_0 & & \\ & \tilde{A}_2 & \tilde{A}_1 & \tilde{A}_0 & \\ & & \cdot & \cdot & \cdot \\ & & & \cdot & \cdot & \cdot \\ & & & & & \end{bmatrix}$$

In the above, $\tilde{A}_{10} = B_1 + \lambda c_N(N) \otimes r_N(1)$, $\tilde{A}_0 = \bar{\lambda}I_N, \tilde{A}_1 = H_{11}$ and $\tilde{A}_2 = \lambda c_N(N) \otimes r_N(1) + \bar{\mu}I_N$. Hence the vector $\hat{\pi}$ is a constant multiple of the steady state vector $\tau = (\tau(0), \tau(1), \dots)$ of the generator matrix $\tilde{Q}$. The vector τ can be obtained by applying the matrix analytic methods (see Neuts [6]) as

$$\tau(i) = \tau(0)R^i, \quad i \geq 0, \tag{23}$$

where the matrix R is the minimal non-negative solution of the matrix quadratic equation:

$$\tilde{A}_0 + R\tilde{A}_1 + R^2\tilde{A}_2 = 0. \tag{24}$$

Equation (23) implies

$$\pi_{(0,0)} = \mathcal{K}\tau(0),$$
$$\pi_{(i,0)} = \pi_{(0,0)}R^i, \quad i \geq 0.$$

Now the vector $\hat{\pi}$ is obtained up to a constant $\mathcal{K}$ as $\hat{\pi} = \mathcal{K}\tau$, the other component vectors $\tilde{\pi}_{(i,0)}$, $i \geq 1$, $\pi_{(i,1)}$, $i \geq 0$ of π can be obtained from the equations (13), (14) and (20), up to the constant $\mathcal{K}$, which is finally obtained from the normalizing condition $\pi e = 1$.

4 Performance measures

4.1 Busy period of the server with the failed components of the main system

Let T_i denote the server busy period with failed components which starts with i failed components and with j external customers in the system. Consider the absorbing Markov chain $Y = \{Y(t), t \geq 0\}$, where $Y(t)$ is the number of failed components of the main system, with the state space $\{0,1,2,\dots,N,N+1,\dots,n-k+1\}$ and having infinitesimal matrix given by

$$\widetilde{H}_{BF} = \begin{bmatrix} 0 & 0 \\ -H_{BF}e & H_{BF} \end{bmatrix},$$

where

$$H_{BF} = \lambda E_{n-k+1} + \lambda c_{n-k+1}(n-k+1) \otimes r_{n-k+1} + \mu E'_{n-k+1}.$$

Notice that $Y(t) = 0$ is an absorbing state. T_i is the time until absorption in the Markov chain $\{Y(t)\}$ assuming that it starts in the state i. The expected value ET_i of T_i is therefore the i^{th} entry of the column matrix $-H_{BF}^{-1}e$ as given by (please see Krishnamoorthy et al. [5]):

$$ET_i = \frac{1}{\mu}\left(i\sum_{j=0}^{n-k+1-i}\left(\frac{\lambda}{\mu}\right)^j + \sum_{j=n-k+2-i}^{n-k}(n-k+1-j)\left(\frac{\lambda}{\mu}\right)^j\right).$$

We notice that once the service of failed components starts, the external customers has no effect on it and hence ET_i is independent of j the number of external customers. Define

$P_f(N) = \pi_{(0,0,N-1)} + \sum_{j=1}^{\infty} \pi_{(j,0,N)}$ and
$P_f(i) = \sum_{j=1}^{\infty} \pi_{(j,0,i)}$ for $N < i \leq n-k+1$

$P_f(i)$ will then denote the system steady state probability just before starting service to failed components with i number of failed components. The expected length of the busy period of the server with failed components is then given by

$$E_{\hat{H}} = \frac{\sum_{i=N}^{n-k+1} P_f(i) ET_i}{\sum_{i=N}^{n-k+1} P_f(i)}.$$

4.2 Other performance measures

1. Fraction of time the system is down,
$$P_{down} = \sum_{j_1=0}^{\infty} \pi_{(j_1,0,n-k+1)} + \sum_{j_1=0}^{\infty} \pi_{(j_1,1,n-k+1)}.$$

2. System reliability, $P_{rel} = 1 - P_{down}$.
3. Average number of external customers waiting in the queue,
$$N_q = \sum_{j_i=0}^{\infty} j_i \big(\sum_{j_3=0}^{n-k+1} \pi_{(j_1,1,j_3)}\big) + \sum_{j_1=1}^{\infty} (j_1 - 1)\big(\sum_{j_3=0}^{n-k+1} \pi_{(j_1,0,j_3)}\big).$$

4. Average number of failed components of the main system,
$$N_{fail} = \sum_{j_3=0}^{n-k+1} j_3 \big(\sum_{j_1=0}^{\infty} \pi_{(j_1,0,j_3)}\big) + \sum_{j_3=1}^{n-k+1} j_3 \big(\sum_{j_1=0}^{\infty} \pi_{(j_1,1,j_3)}\big).$$

5. Average number of failed components waiting when server is busy with external customers
$$NB_{fail} = \sum_{j_3=0}^{n-k+1} j_3 \big(\sum_{j_1=1}^{\infty} \pi_{(j_1,0,j_3)}\big).$$

6. Expected number of external customers joining the system,
$$\theta_3 = \bar{\lambda}\{\sum_{j_1=1}^{\infty} \big(\sum_{j_3=0}^{N-1} \pi_{(j_1,0,j_3)}\big) + \sum_{j_1=0}^{N-1} \pi_{(0,0,j_3)}\}.$$

7. Expected number of external customers on its arrival gets service directly
$$NEX_{direct} = \sum_{j_3=0}^{N-1} \pi_{(0,0,j_3)}.$$

8. Fraction of time the server is busy with external customers,
$$P_{ext,busy} = \sum_{j_1=1}^{\infty} \big(\sum_{j_3=0}^{n-k+1} \pi_{(j_1,0,j_3)}\big).$$

9. Probability that server is found idle,
$$P_{idle} = \sum_{j_3=0}^{N-1} \pi_{(0,0,j_3)} = N\pi_{(0,0,0)}.$$

10. Probability that the server is found busy,
$$P_{busy} = 1 - \sum_{j_3=0}^{N-1} \pi_{(0,0,j_3)} = 1 - N\pi_{(0,0,0)}.$$

11. Expected loss rate of external customers,
$$\theta_4 = \bar{\lambda}\{\sum_{j_1=0}^{\infty} \big(\sum_{j_3=1}^{n-k+1} \pi_{(j_1,1,j_3)}\big) + \sum_{j_1=1}^{\infty} \big(\sum_{j_3=N}^{n-k+1} \pi_{(j_1,0,j_3)}\big)\}.$$

12. Expected service completion rate of external customers,
$$\theta_5 = \bar{\mu} \sum_{j_1=0}^{\infty} \big(\sum_{j_3=0}^{n-k+1} \pi_{(j_1,0,j_3)}\big).$$

13. Expected number of external customers when server is busy with external customers,
$$\theta_6 = \sum_{j_1=0}^{\infty} j_1 \big(\sum_{j_3=0}^{n-k+1} \pi_{(j_1,0,j_3)}\big).$$

5 Numerical Study of the Performance of the System

5.1 The Effect of N Policy on the Server Busy Probability

The main purpose of introducing N-policy while studying a k-out-of-n system with a single server offering service to external customers, in a non pre-emptive nature, was optimization of the system revenue, by utilizing the server idle time, without compromising the reliability of the system much. Tables 1 and 2 reports the variation in the server busy probability when external customers are allowed and not allowed respectively. A comparison of the two tables suggest that there is an increase in the server busy probability, when external customers are allowed. Table 3, which report the effect of the N-policy level on the fraction of time the server remains busy with external customers, tells that there is an increase in the reported measure with an increase in N. Hence, it can be concluded that the N-policy has helped in improving the attention towards external customers slightly. Now, we want to check whether the introduction of the N-policy has badly affected the system reliability.

5.2 The effect of N policy on system reliability

We study two cases $\lambda < \mu$ and $\lambda > \mu$. We expected a decrease in P_{rel} with an increase in N. This is because as N increases, the server spends more time for external customers, which we thought might cause a decrease in the system reliability. This was verified from Table 4, where we assumed $\lambda < \mu$. However, Table 4 shows very high system reliability over 95 %. The magnitude of decrease in reliability was found lesser when the total number of components n was high. In short Table 4 shows that reliability of the system is not much affected by increasing N-policy level. In Table 5 where it was assumed that the component failure rate λ is greater than their service rate μ, it was again found that P_{rel} decreases with increase in N and that the magnitude of decrease is not high. More importantly, the reliability of the system was found less than 91.5 %. To check whether this was actually due to the introduction of external customers, we compared the system reliability of the current model with that of a k-out-of-n system where no external customers are entertained. Table 6 shows that allowing external customers in the system has only a narrow effect on the system reliability and the decrease in reliability is actually due to the assumption $\lambda > \mu$.

5.3 Analysis of a Cost function

Table 1 shows that as N increases, even though the server busy probability increases first, it decreases as N crosses some value. Note that the overall server busy probability is the sum of the server busy probability with external customers and the server busy probability with main customers. Table 3 shows that the fraction of time server remaining busy with external customers is ever increasing with N. Now as N increases, there is a decrease in the server busy probability with main customers. Hence, the above said behavior of the overall server busy probability can be concluded to be due to the conflicting nature of the two entities constituting it. This behavior of the server busy probability lead us to construct a cost function in the hope of finding an optimal value for the N-policy level defined as follows:

$$\begin{aligned}&\mathit{Expected\ cost\ per\ unit\ time}\\&= C_1 \cdot P_{\text{down}} + C_2 \cdot N_q + C_4 \cdot \theta_4 + C_5 \cdot N_{fail} + \frac{C_3}{E_{\hat{H}}} + C_6 \cdot P_{idle}\end{aligned}$$

In the above, C_1 denote the cost per unit time incurred if the system is down, C_2 denote the holding cost per unit time per external customer in the queue, C_3 denote the cost incurred for starting failed components service, C_4 denote the cost due to loss of 1 external customer, C_5 denote the holding cost per unit time of one failed component, C_6 denote the cost per unit time if the server is idle. The values of the cost function presented in Table 7, for various failure rates of the components, shows an optimal value for N in each case.

Table 1: Variation in the server busy probability when external customers are allowed $k = 20, \lambda = 4, \bar{\lambda} = 3.2, \mu = 5.5, \bar{\mu} = 8$

N	n=45	n=50	n=60	n=65
1	0.823494	0.823522	0.823529	0.823529
3	0.829935	0.829973	0.829983	0.831354
5	0.832187	0.832243	0.832256	0.832891
7	0.833255	0.833338	0.833358	0.833717
9	0.833839	0.833968	0.834	0.83423
11	0.834162	0.834367	0.834417	0.834577
13	0.834295	0.834627	0.834708	0.834827
15	0.834239	0.834789	0.834923	0.835093
17	0.833936	0.834861	0.835085	0.835224
19	0.833252	0.834829	0.835211	0.835329
21	0.831922	0.834652	0.835306	0.835413
23	0.829445	0.834239	0.835375	0.83548
25	0.824871	0.833426	0.835412	0.83553

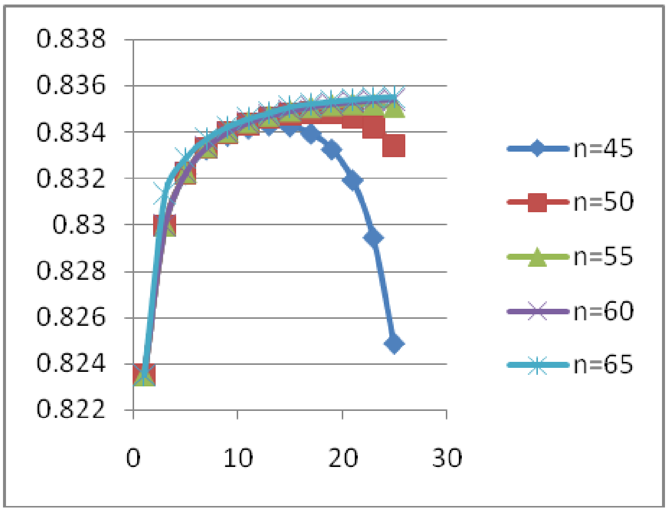


Table 2: Variation in the server busy probability when external customers are not allowed $k = 20, \lambda = 4, \mu = 5.5$

N	n=45	n=50	n=60	n=65
1	0.72722	0.72726	0.72727	0.72727
3	0.7272	0.72726	0.72727	0.72727
5	0.72717	0.72725	0.72727	0.72727
7	0.72711	0.72724	0.72727	0.72727
9	0.72703	0.72722	0.72727	0.72727
11	0.72688	0.72719	0.72727	0.72727
13	0.72663	0.72714	0.72727	0.72727
15	0.72622	0.72706	0.72726	0.72727
17	0.7255	0.72691	0.72726	0.72727
19	0.72425	0.72666	0.72725	0.72727
21	0.72206	0.72623	0.72723	0.72726
23	0.71814	0.72546	0.7272	0.72726

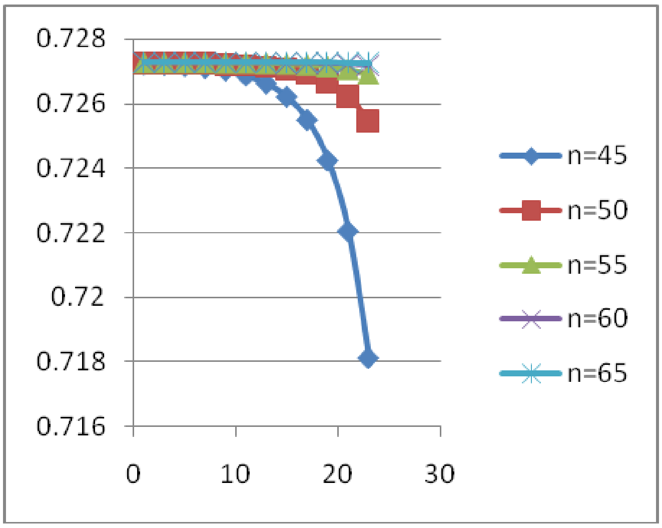


Table 3: Effect of the N-policy level on the fraction of time server is busy with external customers with $k = 20, \lambda = 4, \bar{\lambda} = 3.2, \mu = 5.5, \bar{\mu} = 8$

N	n=40	n=45	n=50	n=55	n=60
1	0.096351	0.096276	0.096261	0.096257	0.096257
2	0.100557	0.100464	0.100445	0.100441	0.10044
3	0.102853	0.10274	0.102717	0.102712	0.102711
4	0.104255	0.104117	0.104089	0.104083	0.104082
5	0.105198	0.105028	0.104993	0.104986	0.104985
6	0.105882	0.105672	0.105629	0.105621	0.105619
7	0.106413	0.106153	0.1061	0.106089	0.106087
8	0.106853	0.106528	0.106462	0.106449	0.106446
9	0.107241	0.106832	0.106749	0.106733	0.106729
10	0.107605	0.107088	0.106984	0.106963	0.106958
11	0.107968	0.107313	0.10718	0.107153	0.107148
12	0.108354	0.107517	0.107348	0.107314	0.107307
13	0.108786	0.107711	0.107495	0.107451	0.107442
14	0.109291	0.107904	0.107626	0.10757	0.107559
15	0.109905	0.108106	0.107747	0.107675	0.10766
17	0.111651	0.108581	0.107976	0.107854	0.107829
19	0.114606	0.109249	0.108092	0.108008	0.107966
21		0.110301	0.108216	0.108153	0.10808
23		0.112079	0.10851	0.108308	0.108182
25		0.115216	0.108928	0.1085	0.108281
27			0.110699	0.108771	0.108387
29			0.112652	0.109196	0.108516
31			0.116153	0.10991	0.108697
33				0.111158	0.108978
35				0.113399	0.109446

Table 4: Variation in the system reliability with increase in N ($\lambda < \mu$ case) $k = 20, \lambda = 4, \bar{\lambda} = 3.2, \mu = 5.5, \bar{\mu} = 8$

N	n=40	n=45	n=50	n=55	n=60	n=65
1	0.99963	0.99993	0.99998	1	1	1
3	0.99948	0.99989	0.99998	1	1	1
5	0.99924	0.99985	0.99997	0.99999	1	1
7	0.99885	0.99977	0.99995	0.99999	1	1
9	0.9982	0.99964	0.99993	0.99998	1	1
11	0.99712	0.99942	0.99988	0.99998	1	1
13	0.9953	0.99905	0.99981	0.99996	0.99999	1
15	0.99217	0.99843	0.99968	0.99994	0.99999	1
17	0.98668	0.99736	0.99947	0.99989	0.99998	1
19	0.97689	0.9955	0.99909	0.99982	0.99996	0.99999
21	0.95915	0.99223	0.99844	0.99968	0.99994	0.99999
23		0.98638	0.9973	0.99945	0.99989	0.99998
25		0.97578	0.99528	0.99905	0.99981	0.99996
27			0.99165	0.99833	0.99966	0.99993
29			0.98509	0.99705	0.9994	0.99988
31			0.97315	0.99475	0.99894	0.99979

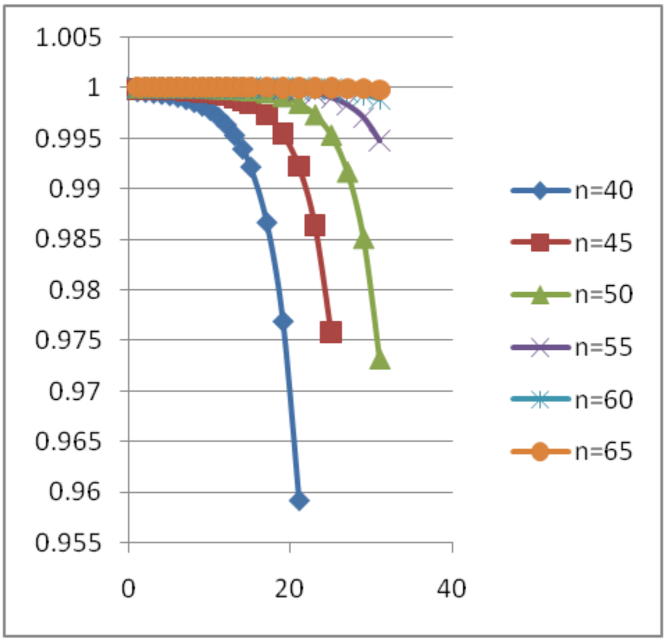


Table 5: Variation in the system reliability with increase in N ($\lambda > \mu$ case) $\lambda = 6, \mu = 5.5, \bar{\lambda} = 3.2, \bar{\mu} = 8$

N	n=40	n=50	n=55	n=60
1	0.90191	0.91106	0.91312	0.91441
2	0.90118	0.91081	0.91297	0.91431
3	0.90041	0.91055	0.91281	0.91421
4	0.89961	0.91028	0.91264	0.91411
5	0.89876	0.91	0.91247	0.914
6	0.89758	0.90971	0.91229	0.91389
7	0.89696	0.90941	0.91211	0.91377
8	0.896	0.9091	0.91192	0.91366
9	0.895	0.90878	0.91173	0.91354
10	0.89396	0.90845	0.91153	0.91341
11	0.89287	0.90812	0.91133	0.91329
12	0.89174	0.90777	0.91112	0.91316
13	0.89055	0.90741	0.9109	0.91303
14	0.88932	0.90705	0.91068	0.91289
15	0.88804	0.90667	0.91046	0.91275
16	0.8867	0.90628	0.91	0.91261
17	0.88531	0.90589	0.90951	0.91247
18	0.88386	0.90548	0.90901	0.91232
19	0.88235	0.90507	0.90848	0.91217
21	0.88079	0.90464	0.90794	0.91186
23	0.87916	0.90421	0.90738	0.91155
25		0.90331	0.90679	0.91122
27		0.90237	0.9062	0.91088
29		0.90139	0.90558	0.91053
31		0.90036	0.90494	0.91018
33		0.8993	0.90462	0.90981
35				0.90944
37				0.90905
39				0.90866
41				0.90827

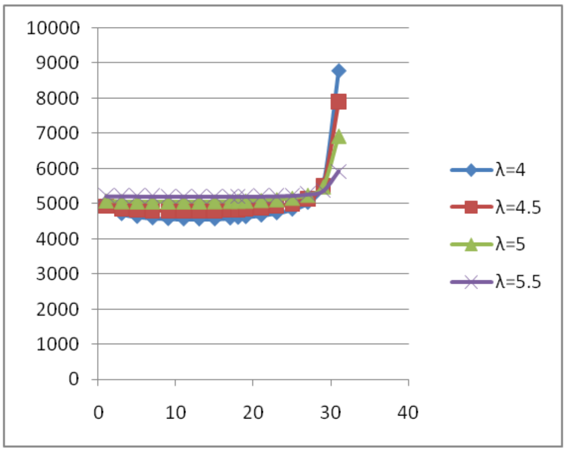
Table 6: Variation in the system reliability with increase in N (case when no external customers are allowed) $k = 20, \lambda = 6, \mu = 5.5$

N	n=40	n=45	n=50	n=55	n=60	n=65
1	0.902225	0.907874	0.911180	0.913196	0.914453	0.915247
3	0.900740	0.907001	0.910662	0.912877	0.914252	0.915120
5	0.899093	0.906080	0.910108	0.912537	0.914040	0.914985
7	0.897301	0.905082	0.909519	0.912176	0.913815	0.914843
9	0.895355	0.904014	0.908894	0.911796	0.913578	0.914693
11	0.893242	0.902873	0.908232	0.911395	0.913329	0.914537
13	0.890948	0.901655	0.907531	0.910974	0.913069	0.914373
15	0.888461	0.900358	0.906793	0.910533	0.912797	0.914202
17	0.885763	0.898979	0.906016	0.910071	0.912514	0.914025

19	0.882837	0.897514	0.905200	0.909589	0.912219	0.913841
21	0.879662	0.895960	0.904345	0.909087	0.911913	0.913651
23		0.894313	0.903450	0.908566	0.911597	0.913454
25		0.892570	0.902514	0.908025	0.911271	0.913252
27			0.901539	0.907465	0.910934	0.913044
29			0.900523	0.906886	0.910588	0.912831
31			0.899465	0.906289	0.910233	0.912613
33				0.905674	0.909868	0.912390
35				0.905041	0.909495	0.912162
37					0.909114	0.911930
39					0.908724	0.911693
41					0.908327	0.911453
43						0.911209
45						0.910961

Table 7: Analysis of a cost function for finding optimal N value , $n = 50, k = 20, \mu = 5.5, \bar{\lambda} = 3.2, \bar{\mu} = 8, C_1 = 2000, C_2 = 20, C_3 = 800, C_4 = 1000, C_5 = 10, C_6 = 200$

N	$\lambda = 4$	$\lambda = 4.5$	$\lambda = 5$	$\lambda = 5.5$
1	4925.877	4937.695	5079.029	5226.181
3	4710.059	4856.852	5057.425	5221.212
5	4630.354	4825.835	5050.332	5218.775
7	4591.702	4812.151	5048.243	5216.965
9	4571.3	4806.745	5048.411	5215.313
11	4561.086	4806.248	5049.849	5213.713
13	4558.217	4809.556	5052.345	5212.268
15	4563.915	4817.604	5056.578	5211.373
17	4588.216	4835.444	5064.896	5211.922
18	4605.19	4846.938	5070.21	5212.65
19	4624.185	4859.68	5076.196	5213.701
21	4670.646	4890.628	5091.4	5217.34
23	4735.585	4934.206	5114.597	5224.719
25	4837.829	5004.721	5155.522	5240.069
27	5032.125	5144.138	5241.815	5274.736
29	5546.901	5525.659	5482.957	5371.341
31	8780.95	7911.995	6932.789	5918.758



Acknowlegment: Krishnamoorthy acknowledges financial support from Kerala State Council for Science, Technology and Environment, Grant

No.001/KESS/2013/CSTE, under the Emeritus Scientist scheme.

Authors would like to thank Prof.V.Rykov for some suggestion which improved the presentation of the paper.

References

[1] S. R. Chakravarthy, A. Krishnamoorthy and P. V. Ushakumari (2001); A k-out-of-n reliability system with an unreliable server and Phase type repairs and services: The (N,T) policy; Journal of Applied Mathematics and Stochastic Analysis; 14(4): 361-380.

[2] A. N. Dudin, A. Krishnamoorthy and C. N. Viswanath (2004); Reliability of a k-out-of-n-system through Retrial Queues; Trans. of XXIV Int. Seminar on Stability Problems for Stochastic Models, Jurmala, Latvia, September, 10–17, 2004.

Also appeared in the Journal of Mathematical Sciences,Vol.191,No.4,June 2013,506-517.

[3] A Krishnamoorthy, C. N. Viswanath and T. G. Deepak (2004); Reliability of a k-out-of-n-system with Repair by a Service Station Attending a Queue with Postponed Work; Advanced Reliability Modelling: Proceedings of the 2004 Asian International Workshop (AIWARM 2004), Hiroshima, Japan, 26-27 August,2004, pp.293-300. Also published in Journal of Applied Mathematics & Computing (JAMC), PP.389-405, Volume 25, Numbers 1-2/September 2007, Springer

[4] A. Krishnamoorthy, C. N. Viswanath and T. G. Deepak; Maximizing of Reliability of a k-out-of-n-system with Repair by a facility attending external customers in a Retrial Queue; Proceedings of V International Workshop on Retrial Queues, Ed. Bong Dae Choi, September, 2004, TMRC, Korea University, Seoul, pp.31-38. Also published in International Journal of Reliability, Quality and Safety Engineering (IJRQSE), Volume 14, No. 4, 2007, pp. 379-398

[5] A. Krishnamoorthy, M. K. Sathian and C. N. Viswanath; Reliability of a k-out-of n system with repair by a single server extending service external customers with pre-emption; Electronic Journal "Reliability:Theory and Applications" (Gnedenko forum,volume 11, June 2016,pp.61-93)

[6] M.F. Neuts; Matrix Geometric solutions in stochastic processes-An Algorithmic Approach,The John Hopkins University Press, 1981.

[7] M.F. Neuts and B.M. Rao; Numerical investigation of a multiserver retrial model; Queuing Systems; Volume 7, Number 2, 169-189, June 1990.