Reliability Of a *k*-out-of-*n* System With A Single Server Extending Non-Preemptive Service To External Customers-Part II

A. Krishnamoorthy
•
Dept. of Mathematics, Cochin University of Science & Technology, Kochi-682022
achyuthacusat@gmail.com

M. K. Sathian
•
Dept. of Mathematics, Panampilly Memorial Govt. College, Chalakudy, Thrissur.
sathianmkk@yahoo.com

Narayanan C Viswanath
•
Dept. of Mathematics Govt. Engg. College, Thrissur Thrissur 680 009.
narayanan_viswanath@yahoo.com

Abstract

In this paper we study a k-out-of-n system with a single repair facility, which provides service to external customers also. We assume an N-policy for service to failed components(main customers) of the k-out-of-n system starts only on accumulation of N of them. Once started, the repair of external customers is continued until all the components become operational. When not repairing failed components, the server attends external customers(if there is any) who arrive according to a Poisson process. Once selected for service, the external customers receive a service of non-preemptive nature. When there are at least N failed components in the system and/or when the server is busy with failed components, the external customers are not allowed to join the system.Otherwise they join an orbit of infinite capacity. Life time distribution of failed components, service time distribution of main and external customers and the inter retrial time distribution of orbital customers are all assumed to follow independent exponential distributions. Steady state analysis has been carried out and several important system performance measures, based on the steady state distribution, derived. A numerical study comparing the current model with those in which no external customers are considered has been carried out.This study suggests that rendering service to external customers helps to utilize the server idle time profitably, without sacrificing the system reliability.

Keywords: *k-out-of-n system; non-preemptive service.*

1 Introduction

In this paper, we consider a variant of the model studied in Krishnamoorthy et al. [1]. In part I (see Krishnamoorthy et al. [3]) of this paper we studied the reliability of a k-out-of-n system with a single server rendering non-preemptive service to external customers.In this paper we extend it to retrial queue of unsatisfied external customers(orbital customers) with linear retrial rate.In effect we replace the infinite queue of external customers in part I by orbital customers and their retrial. However, the stability condition remains the same in both models.

This paper is arranged as follows. In section 2 , we describe the model and in section 3 , its long run behavior is analyzed. The stability condition is derived explicitly in section 3 and computation of the steady state vector using the Neuts-Rao truncation procedure [2] has been discussed. Some important performance measures are derived in section 4 . The effect of rendering service to external customers and N-policy has been studied numerically in section 5 .

2 The retrial model

Here we consider a variant of the model discussed in section 2 of part I by assuming that an arriving external customer either gets immediate service if it finds the server is idle at that time or joins an orbit of infinite capacity, if the server is busy with external customers with $\leq N-1$ failed components of the k-out-of-n system. As in the model discussed in section 2 of part I, the external customers are not allowed to join the orbit when the server is busy with failed components of the system. An orbital customer retries for service with inter-retrial time following an exponential distribution with parameter θ. All other assumptions and parameters remain the same as in model discussed in section 2 of part I. In this situation the system can be modeled as follows.

Let $X_1(t)$ = the number of external customers in the orbit at time t,

$X_2(t)$ = the number of failed components of the k-out-of-n system, including the one getting service (if any) at time t.

Define

$$S(t) = \begin{cases} 0, & \text{If the server is idle} \\ 1, & \text{If the server is busy with an external customer} \\ 2, & \text{If the server is busy with a main customer} \end{cases}$$

Now, $X(t) = (X_1(t), S(t), X_2(t))$ forms a continuous time Markov chain on the state space

$$S = \{(j_1, 0, j_2)/j_1 \geq 0, 0 \leq j_2 \leq N-1\} \cup \{(j_1, 1, j_2)/j_1 \geq 0, 0 \leq j_2 \leq n-k+1\}$$
$$\cup \{(j_1, 2, j_2)/j_1 \geq 0, 1 \leq j_2 \leq n-k+1\}.$$

Arranging the states lexicographically and partitioning the state space into levels i, where each level i corresponds to the collection of states with number of external customers in the orbit at any time t equal to i, we get an infinitesimal generator of the above chain as

$$Q = \begin{bmatrix} \mathbf{A}_{10} & \mathbf{A}_0 & & & & & \\ \mathbf{A}_{21} & \mathbf{A}_{11} & \mathbf{A}_0 & & & & \\ & \mathbf{A}_{22} & \mathbf{A}_{12} & \mathbf{A}_0 & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \\ & & & & \mathbf{A}_{2p} & \mathbf{A}_{1p} & \mathbf{A}_0 \\ & & & & & \cdot & \cdot & \cdot \\ & & & & & & \cdot & \cdot & \cdot \end{bmatrix}.$$

The entries of Q are described as below: For $i \geq 0$, the transition within level i is represented by the matrix

$$\mathbf{A}_{1i} = \begin{bmatrix} D_{11}^{(i)} & D_{12} & 0 & D_{14} \\ D_{21} & D_{22} & D_{23} & 0 \\ 0 & 0 & D_{33} & D_{34} \\ D_{41} & 0 & 0 & D_{44} \end{bmatrix},$$

where

$$D_{11}^{(i)} = \lambda E_N - \overline{\lambda} I_N - i\theta I_N, D_{12} = \overline{\lambda} I_N,$$

$D_{14} = \lambda c_N(N) \otimes r_{n-k+1}(N), D_{21} = \overline{\mu} I_N,$
$D_{22} = D_{11}^{(0)} - \overline{\mu} I_N,$
$D_{23} = \lambda c_N(N) \otimes r_{n-k+2-N}(1),$
$D_{33} = \lambda E_{n-k+2-N} + \lambda c_(n-k+2-N) \otimes r_{(n-k+2-N)}(n-k+2-N) - \overline{\mu} I_{n-k+2-N},$
$D_{34} = [O_{n-k+2-N \times (N-1)} \quad \overline{\mu} I_{(n-k+2-N)}],$
$D_{44} = \lambda E_{n-k+1} + \lambda c_{n-k+1}(n-k+1) \otimes r_{n-k+1}(n-k+1) + \mu E'_{n-k+1},$
$D_{41} = \mu c_{n-k+1}(1) \otimes r_N(1).$

For $i \geq 0$ the transition from level i to $i+1$ is represented by the matrix

$$\mathbf{A}_0 = \begin{bmatrix} 0_{N \times N} & 0 & 0 & 0 \\ 0 & \overline{\lambda} I_N & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

For $i \geq 1$, the transition from level i to $i-1$ is represented by the matrix

$$\mathbf{A}_{2i} = \begin{bmatrix} 0 & i\theta I_N & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

3 Steady state analysis of the retrial model

3.1 Stability condition

For finding the stability condition for the system study, we apply Neuts-Rao truncation [2] by assuming $A_{1i} = A_{1M}$ and $A_{2i} = A_{2M}$ for all $i \geq M$. Then the generator matrix of the truncated system will look like:

$$Q = \begin{bmatrix} \mathbf{A}_{10} & \mathbf{A}_0 & & & & & & \\ \mathbf{A}_{21} & \mathbf{A}_{11} & \mathbf{A}_0 & & & & & \\ & \mathbf{A}_{22} & \mathbf{A}_{12} & \mathbf{A}_0 & & & & \\ & & \cdot & \cdot & \cdot & & & \\ & & & \cdot & \cdot & \cdot & & \\ & & & & \mathbf{A}_{2M} & \mathbf{A}_{1M} & \mathbf{A}_0 & \\ & & & & & \mathbf{A}_{2M} & \mathbf{A}_{1M} & \mathbf{A}_0 \\ & & & & & & \cdot & \cdot \\ & & & & & & \cdot & \cdot \\ & & & & & & & \end{bmatrix}.$$

Define $\mathbf{A}_M = \mathbf{A}_0 + \mathbf{A}_{1M} + \mathbf{A}_{2M}$; then

$$A_M = \begin{bmatrix} D_{11}^{(M)} & D_{12}^{(M)} & 0D_{14} & \\ D_{21} & \widetilde{D}_{22} & D_{23} & 0 \\ 0 & 0 & D_{33} & D_{34} \\ D_{41} & 0 & 0 & D_{44} \end{bmatrix},$$

where $D_{12}^{(M)} = (\overline{\lambda} + M\theta) I_N$, $\widetilde{D}_{22} = \lambda E_N - \mu I_N$.

Let

$\pi_M = (\pi_M(0), \pi_M(1), \tilde{\pi}_M(1), \pi_M(2))$, where
$\pi_M(0) = (\pi_M(0,0), \pi_M(0,1), \dots, \pi_M(0, N-1)),$
$\pi_M(1) = (\pi_M(1,0), \dots, \pi_M(1, N-1)),$
$\tilde{\pi}_M(1) = (\pi_M(1, N), \dots, \pi_M(1, n-k+1)),$
$\pi_M(2) = (\pi_M(2,1), \dots, \pi_M(2, n-k+1)).$

be the steady state vector of the generator matrix $\mathbf{A}_M$. Then the relation $\pi_M \mathbf{A}_M = 0$ gives rise to the following equations:

$$\pi_M(0) D_{11}^{(M)} + \pi_M(1) D_{21} + \pi_M(2) D_{41} = 0, \tag{1}$$

$$\pi_M(0)D_{12}^{(M)} + \pi_M(1)\widetilde{D}_{22} = 0, \tag{2}$$

$$\pi_M(1)D_{23} + \tilde{\pi}_M(1)D_{33} = 0, \tag{3}$$

$$\pi_M(0)D_{14} + \tilde{\pi}_M(1)D_{34} + \pi_M(2)D_{44} = 0. \tag{4}$$

It follows from equation (4) that

$$\pi_M(2) = -\pi_M(0)D_{14}(D_{44})^{-1} - \tilde{\pi}_M(1)D_{34}(D_{44})^{-1}. \tag{5}$$

Substituting for $\pi_M(2)$ in equation (1), we get

$$\pi_M(0)D_{11}^{(M)} + \pi_M(1)D_{21} - \pi_M(0)D_{14}(D_{44})^{-1}D_{41} - \tilde{\pi}_M(1)D_{34}(D_{44})^{-1}D_{41} = 0. \tag{6}$$

It follows from equation (3) that

$$\tilde{\pi}_M(1) = -\pi_M(1)D_{23}(D_{33}^{-1}). \tag{7}$$

Substituting for $\tilde{\pi}_M(1)$ in equation (6), we get

$$\begin{aligned}\pi_M(0)D_{11}^{(M)} + \pi_M(1)D_{21} - \pi_M(0)D_{14}(D_{44})^{-1}D_{41}\\ +\pi_M(1)D_{23}(D_{33})^{-1}D_{34}(D_{44})^{-1}D_{41} = 0.\end{aligned} \tag{8}$$

We notice that the first column of the matrix D_{41} is $-D_{44}e$ and its all other columns are zero columns. Hence the first column of the matrix $(D_{44})^{-1}D_{41}$ is $-e$ and its all other columns are zero columns. This implies that the first column of the matrix $-D_{14}(D_{44})^{-1}D_{41}$ is $D_{14}e = \lambda c_N(N)$ and its all other columns are zero columns. In other words $-D_{14}(D_{44})^{-1}D_{41} = \lambda c_N(N) \otimes r_N(1)$. Also, the first column of the matrix $D_{34}(D_{44})^{-1}D_{41}$ is $-D_{34}e$ and its all other columns are zero columns. Since $-D_{34}e = D_{33}e$, the first column of the matrix $(D_{33})^{-1}D_{34}(D_{44})^{-1}D_{41}$ is e and its all other columns are zero columns. Hence it follows that $D_{23}(D_{33})^{-1}D_{34}(D_{44})^{-1}D_{41}$ is $D_{23}e = \lambda c_N(N) \otimes r_N(1)$. Thus equation (8) becomes

$$\pi_M(0)(D_{11}^{(M)} + \lambda c_N(N) \otimes r_N(1)) + \pi_M(1)(D_{21} + \lambda c_N(N) \otimes r_N(1)) = 0. \tag{9}$$

Adding equations (2) and (9), we get

$$\pi_M(0)(D_{11}^{(M)} + \lambda c_N(N) \otimes r_N(1) + D_{12}^{(M)}) + \pi_M(1)(\widetilde{D}_{22} + D_{21} + \lambda c_N(N) \otimes r_N(1)) = 0. \tag{10}$$

Since $D_{11}^{(M)} + D_{12}^{(M)} = \widetilde{D}_{22} + D_{21} = \lambda E_N$, equation (10) reduces to

$$(\pi_M(0) + \pi_M(1))(\lambda E_N + \lambda c_N(N) \otimes r_N(1)) = 0. \tag{11}$$

which implies that $\pi_M(0) + \pi_M(1)$ is a constant multiple of the steady state vector $\frac{1}{N}e'_N$ of the generator matrix $\lambda E_N + \lambda c_N(N) \otimes r_N(1)$ and hence,

$$\pi_M(0) + \pi_M(1) = v\frac{1}{N}e'_N. \tag{12}$$

where v is a constant. Equation (2) implies that

$$\pi_M(0) = -\pi_M(1)\widetilde{D}_{22}(D_{12}^{(M)})^{-1}. \tag{13}$$

Since $(D_{12}^{(M)})^{-1} = \frac{1}{(\bar{\lambda}+M\theta)}I_N$, (13) gives

$$\lim_{M\to\infty} \pi_M(0) = 0. \tag{14}$$

and hence

$$\lim_{M\to\infty} \pi_M(1) = v\frac{1}{N}e'_N, \tag{15}$$

and

$$\lim_{M\to\infty} \bar{\lambda}\pi_M(1)e = v\bar{\lambda}. \tag{16}$$

Again from (13),

$$M\theta\pi_M(0)e = -M\theta\pi_M(1)\widetilde{D}_{22}(D_{12}^{(M)})^{-1}e. \tag{17}$$

Since, $\lim_{M\to\infty} M\theta(D_{12}^{(M)})^{-1}e = \lim_{M\to\infty} \frac{M\theta}{(\bar{\lambda}+M\theta)}e_N = e_N$, (17) implies that

$$\lim_{M\to\infty} M\theta\pi_M(0)e = -\lim_{M\to\infty}\pi_M(1)\widetilde{D}_{22}e$$
$$= -\nu\frac{1}{N}e'_N(-\lambda c_N(N) - \bar{\mu}e)$$
$$= \nu(\frac{\lambda}{N} + \bar{\mu}). \qquad (18)$$

The truncated system is stable if and only if

$$\pi_M A_0 e < \pi_M A_{2M} e, \qquad (19)$$
$$\pi_M A_0 e = \bar{\lambda}\pi_M(1)e, \qquad (20)$$
$$\pi_M A_{2M} e = M\theta\pi_M(0)e. \qquad (21)$$

Making use of equations (16), (18), (20) and (21), the stability condition for the truncated system as $M \to \infty$ is given by

$$\nu\bar{\lambda} < \nu(\frac{\lambda}{N} + \bar{\mu}) ,$$

which can be re-arranged as

$$\frac{\bar{\lambda}}{\bar{\mu}}\frac{N\bar{\mu}}{(\lambda + N\bar{\mu})} < 1 .$$

Hence, we conclude that the retrial problem has the same stability condition as the queueing problem, which was obtained in section 3.1 of part I.

3.2 Computation of Steady State Vector

We find the steady state vector of $\{X(t), t \geq 0\}$, by approximating it with the steady state vector of the truncated system. Let $\pi = (\pi_0, \pi_1, \pi_2, \dots)$ where each $\pi_i = (\pi_i(0,0), \pi_i(0,1), \dots, \pi_i(0, N-1), \pi_i(1,1), \dots, \pi_i(1, n-k+1), \pi_i(2,0), \ \pi_i(2,1), \dots, \pi_i(2, n-k+1))$ be the steady state vector of the Markov chain $\{X(t), t \geq 0\}$.

Suppose $A_{1i} = A_{1M}$ and $A_{2i} = A_{2M}$ for all $i \geq M$. Let $\pi_{M+r} = \pi_{M-1}R^{r+1}, r \geq 0$, then from $\pi Q = 0$ we get

$$\pi_{M-1}A_0 + \pi_M A_{1M} + \pi_{M+1}A_{2M} = 0,$$
$$\pi_{M-1}A_0 + \pi_{M-1}RA_{1M} + \pi_{M-1}R^2 A_{2M} = 0,$$
$$\pi_{M-1}(A_0 + RA_{1M} + R^2 A_{2M}) = 0.$$

Choose R such that $A_0 + RA_{1M} + R^2 A_{2M} = 0$. We call this R as R_M. Also we have

$$\pi_{M-2}A_0 + \pi_{M-1}A_{1M-1} + \pi_M A_{2M} = 0,$$
$$\pi_{M-2}A_0 + \pi_{M-1}(A_{1M-1} + R_M A_{2M}) = 0,$$
$$\pi_{M-1} = -\pi_{M-2}A_0(A_{1M-1} + R_M A_{2M})^{-1}$$
$$= \pi_{M-2}R_{M-1} .$$

where

$$R_{M-1} = -A_0(A_{1M-1} + R_M A_{2M}) .$$

Next,

$$\pi_{M-3}A_0 + \pi_{M-2}A_{1M-2} + \pi_{M-1}A_{2M-1} = 0,$$
$$\pi_{M-3}A_0 + \pi_{M-2}(A_{1M-2} + \pi_{M-1}A_{2M-1}) = 0,$$
$$\pi_{M-2} = -\pi_{M-3}A_0(A_{1M-2} + R_{M-1}(A_{2M-1})^{-1}$$
$$= \pi_{M-3}R_{M-2}.$$

Where

$$R_{M-2} = -A_0(A_{1M-2} + R_{M-1}A_{2M-1})^{-1}.$$

and so on.

Finally

$$\pi_0 A_{10} + \pi_1 A_{21} = 0$$

becomes

$$\pi_0(A_{10} + R_1 A_{21}) = 0.$$

For finding π, first we take π_0 as the steady state vector of $A_{10} + R_1 A_{21}$.Then π_i for $i \geq 1$ can be found using the recursive formula, $\pi_i = \pi_{i-1}R_i$ for $1 \leq i \leq M$.

Now the steady state probability distribution of the truncated system is obtained by dividing each π_i with the normalizing constant

$$[\pi_0 + \pi_1 + \cdots]e = [\pi_0 + \pi_1 + \cdots + \pi_{N-2} + \pi_{M-1}(I - R_M)^{-1}]e.$$

3.3 Computation of the matrix R_M

Consider the matrix quadratic equation

$$A_0 + R_M A_{1M} + R_M^2 A_{2M} = 0, \tag{22}$$

which implies

$$R_M = -A_0(A_{1M} + R_M A_{2M})^{-1}. \tag{23}$$

The structure of the A_0 matrix implies that the matrix R_M has the form:

$$R_M = \begin{bmatrix} 0 & 0 & 0 & 0 \\ R_{M1} & R_{M2} & R_{M3} & R_{M4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{24}$$

In other words, the non-zero rows of the R_M matrix are those, where the A_0 matrix has at least one nonzero entry. Now,

$$R_M^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ R_{M2}R_{M1} & R_{M2}^2 & R_{M2}R_{M3} & R_{M2}R_{M4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{25}$$

Equation (22) gives rise to the following equations:

$$R_{M1}D_{11}^{(M)} + R_{M2}D_{21} + R_{M4}D_{41} = 0, \tag{26}$$

$$R_{M2}R_{M1}M\theta I_N + R_{M1}D_{12} + R_{M2}D_{22} + \bar{\lambda} I_N = 0, \tag{27}$$

$$R_{M2}D_{23} + R_{M3}D_{33} = 0, \tag{28}$$

$$R_{M1}D_{14} + R_{M3}D_{34} + R_{M4}D_{44} = 0. \tag{29}$$

From equation (28), we can write

$$R_{M3} = -R_{M2}D_{23}(D_{23})^{-1}. \tag{30}$$

From equation(29), we can write

$$R_{M4} = -R_{M1}D_{14}(D_{44})^{-1} - R_{M3}D_{34}(D_{44})^{-1}. \tag{31}$$

Substituting for R_{M3} from (30) in equation (31), we get

$$R_{M4} = -R_{M1}D_{14}(D_{44})^{-1} + R_{M2}D_{23}(D_{33})^{-1}D_{34}(D_{44})^{-1}. \tag{32}$$

Substituting for R_{M4} from (32) in equation (26), we get

$$\begin{aligned} R_{M1}D_{11}^{(M)} \quad &+ R_{M2}D_{21} - R_{M1}D_{14}(D_{44})^{-1}D_{41} \\ &+ R_{M2}D_{23}(D_{33})^{-1}D_{34}(D_{44})^{-1}D_{41} = 0. \end{aligned} \tag{33}$$

Using the same reasoning, that lead us to equation (9), equation (33) becomes

$$R_{M1}(D_{11}^{(M)} + \lambda c_N(N) \otimes r_N(1)) + R_{M2}(D_{21} + \lambda c_n(N) \otimes r_N(1)) = 0. \tag{34}$$

From (34), it follows that

$$R_{M1} = -R_{M2}(D_{21}\lambda c_N(N) \otimes r_N(1))(D_{11}^{(M)} + \lambda c_N(N) \otimes r_N(1))^{-1}. \tag{35}$$

Substituting for R_{M1} in (27), we get

$$\begin{aligned} &-R_{M2}^2(D_{21} + \lambda c_N(N) \otimes r_N(1))(D_{11}^{(M)} + \lambda c_N(N) \otimes r_N(1))^{-1}M\theta I_N \\ &-R_{M2}(D_{21} + \lambda c_N(N) \otimes r_N(1))(D_{11}^{(M)} + \lambda c_N(N) \otimes r_N(1))^{-1}D_{12} \\ &+R_{M2}D_{22} + \bar{\lambda} I_N = 0. \end{aligned}$$

That is

$$\begin{aligned} &R_{M2}^2\big(-(D_{21} + \lambda c_N(N) \otimes r_N(1))(D_{11}^{(M)} + \lambda c_N(N) \otimes r_N(1))^{-1}M\theta I_N\big) \\ &\quad + R_{M2}\big(-(D_{21} + \lambda c_N(N) \otimes r_N(1))(D_{11}^{(M)} + \lambda c_N(N) \otimes r_N(1))^{-1}D_{12} + D_{22}\big) \\ &\quad + \bar{\lambda} I_N = 0. \end{aligned} \tag{36}$$

We notice that $-(D_{11}^{(M)} + \lambda c_N(N) \otimes r_N(1))e = (D_{12} + M\theta I_N)e$. and therefore

$$\begin{aligned} &-(D_{21} + \lambda c_N(N) \otimes r_N(1))(D_{11}^{(M)} + \lambda c_N(N) \otimes r_N(1))^{-1}(D_{12} + M\theta I_N)e \\ &= (D_{21}\lambda c_N(N) \otimes r_N(1))e. \end{aligned} \tag{37}$$

Also,

$$D_{22}e + (D_{21} + \lambda c_N(N) \otimes r_N(1))e + \bar{\lambda} e = 0.$$

and hence

$$\begin{aligned}&\left(-(D_{21} + \lambda c_N(N) \otimes r_N(1))(D_{11}^{(M)} + \lambda c_N(N) \otimes r_N(1))^{-1} M\theta I_N\right)e + \\ &\left(-(D_{21} + \lambda c_N(N) \otimes r_N(1))(D_{11}^{(M)} + \lambda c_N(N) \otimes r_N(1))^{-1} D_{12} + D_{22}\right)e \\ &+\bar{\lambda} e = 0.\end{aligned} \tag{38}$$

Equation (38) shows that the matrix R_{M2} is the minimal non-negative solution of the matrix quadratic equation (36). Once obtaining R_{M2}, the matrices R_{M1}, R_{M3}, and R_{M4} can be found using equations (35), (30) and (31) respectively. Hence the matrix R_M can be found. From the form of the matrix $D_{11}^{(M)}$, we notice that,

$$\begin{aligned}&-\left(D_{11}^{(M)} + \lambda c_N(N) \otimes r_N(1)\right) \\ &\quad = M\theta I_N - \left(\lambda E_N - \bar{\lambda} I_N + \lambda c_N(N) \otimes r_N(1)\right) \\ &\quad = M\theta \left(I_N - \frac{1}{M\theta}(\lambda E_N - \bar{\lambda} I_N + \lambda c_N(N) \otimes r_N(1))\right).\end{aligned}$$

and hence

$$\begin{aligned}&-\left(D_{11}^{(M)} + \lambda c_N(N) \otimes r_N(1)\right)^{-1} \\ &\quad = \frac{1}{M\theta}\left(I_N - \frac{1}{M\theta}(\lambda E_N - \bar{\lambda} I_N + \lambda c_N(N) \otimes r_N(1))\right)^{-1} \\ &\quad = \frac{1}{M\theta}\left(I_N + \frac{1}{M\theta}(\lambda E_N - \bar{\lambda} I_N + \lambda c_N(N) \otimes r_N(1)) + \cdots\right).\end{aligned}$$

Therefore

$$\lim_{M\to\infty}\left(-(D_{11}^{(M)} + \lambda c_N(N) \otimes r_N(1))^{-1} M\theta I_N\right) = I_N.$$

and

$$\lim_{M\to\infty}\left(-(D_{11}^{(M)} + \lambda c_N(N) \otimes r_N(1))^{-1} D_{12}\right) = 0.$$

Hence as $M \to \infty$ equation (36) becomes

$$R_{M2}^2(D_{21} + \lambda c_N(N) \otimes r_N(1)) + R_{M2} D_{22} + \bar{\lambda} I_N = 0. \tag{39}$$

We identify $D_{21} + \lambda c_N(N) \otimes r_N(1)$ as $\tilde{A}_2$, D_{22} as $\tilde{A}_1$ and $\bar{\lambda} I_N$ as $\tilde{A}_0$, which were defined in section 3.2 of part I. Hence equation (39) is the same as equation (24) of section 3.2 of part I. That is the matrix R_M tends to the matrix R, the minimal non-negative solution of (24) of section 3.2 of part I, as $M \to \infty$. This fact can be utilized in determining the truncation level M.

4 System Performance Measures

The following system performance measures were calculated numerically.

1. Fraction of time the system is down,
$$P_{down} = \sum_{j_1=0}^{\infty} \left(\pi_{j_1}(1, n-k+1) + \pi_{j_1}(2, n-k+1)\right).$$

2. System reliability, $P_{rel} = 1 - P_{down}$
$$= 1 - \sum_{j_1=0}^{\infty} \left(\pi_{j_1}(1, n-k+1) + \pi_{j_1}(2, n-k+1)\right).$$

3. Average number of external customers in the orbit,
$$N_{orbit} = \sum_{j_1=0}^{\infty} j_1\left(\sum_{j_3=1}^{n-k+1} \pi_{j_1}(1, j_3)\right) + \sum_{j_1=0}^{\infty} j_1\left(\sum_{j_3=0}^{n-k+1} \pi_{j_1}(2, j_3)\right).$$

4. Average number of failed components in the system,
$$N_{fail} = \sum_{j_3=0}^{n-k+1} j_3\left(\sum_{j_1=0}^{\infty} \pi_{j_1}(0, j_3)\right) + \sum_{j_3=1}^{n-k+1} \left(\sum_{j_1=0}^{\infty} \pi_{j_1}(2, j_3)\right).$$

5. Average number of failed components waiting when server is busy with external customers
$$N_{failextb} = \sum_{j_3=0}^{n-k+1} j_3\left(\sum_{j_1=1}^{\infty} \pi_{j_1}(0, j_3)\right).$$

6. Expected rate at which external customers joining the system
$$E_{extrate} = \bar{\lambda}\left\{\sum_{j_1=1}^{\infty} \left(\sum_{j_3=0}^{n-k+1} \pi_{j_1}(0, j_3)\right) + \sum_{j_3=0}^{N-1} \pi_0(0, j_3)\right\}.$$

7. Expected number of external customers on its arrival gets service directly,
$$E_{extdirect} = \sum_{j_3=0}^{N-1} \pi_0(0, j_3).$$

8. Fraction of time server is busy with external customers,
$$P_{extbusy} = \sum_{j_1=1}^{\infty} \left(\sum_{j_3=0}^{n-k+1} \pi_{j_1}(0, j_3)\right).$$

9. Probability that the server is found idle,
$$P_{idle} = \sum_{j_3=0}^{N-1} \pi_0(0, j_3) = N\pi_0(0,0).$$

10. Probability that the server is found busy,
$$P_{busy} = 1 - \sum_{j_3=0}^{N-1} \pi_0(0, j_3) = 1 - N\pi_0(0,0).$$

11. Expected loss rate of external customers
$$\theta_4 = \bar{\lambda}\left\{\sum_{j_1=0}^{\infty} \left(\sum_{j_3=1}^{n-k+1} \pi_{j_1}(1, j_3)\right) + \sum_{j_1=1}^{\infty} \left(\sum_{j_3=N}^{n-k+1} \pi_{j_1}(0, j_3)\right)\right\}.$$

12. Expected service completion rate of external customers,
$$\theta_5 = \bar{\mu} \sum_{j_1=0}^{\infty} \left(\sum_{j_3=0}^{n-k+1} \pi_{j_1}(0, j_3)\right).$$

13. Expected number of external customers when server is busy with external customers
$$\theta_6 = \sum_{j_1=0}^{\infty} j_1\left(\sum_{j_3=0}^{n-k+1} \pi_{j_1}(0, j_3)\right).$$

14. Expected successful retrial rate
$$\theta_7 = \theta \cdot \sum_{j_1=1} \left(\sum_{j_3=0}^{N-1} \pi_{J_1}(0, j_3)\right).$$

5 Numerical study of the performance of the system

5.1 The effect of N policy on the server busy probability

A comparison of Table 1 of part I, which report the behaviour of server busy probability with variation in the N-policy level, with that of part II shows that the models described in section 2 of part I and its variant where external customers are sent to the orbit, which was described in section 2 of part II have similar behaviour as far as the server busy probability is considered. Comparison of Table 3 of part I, which report the variation in the fraction of time the server remains busy with external customers with increase in N, with table 2 of part II also points to similar behaviour for both models. Table 4 of part I and table 3 of part II indicate that the two models have similar reliability.

5.2 Cost Analysis

As in the case of the queueing model discussed in section 2 of part I, we analyzed a cost function for the retrial model for finding an optimal value for the N-policy level. For defining the cost function, let C_1 be the cost per unit time incurred if the system is down, C_2 be the holding cost per unit time per external customer in the orbit, C_3 is the cost incurred for starting failed components service after accumulation of N of them, C_4 be the cost due to loss of 1 external customer, C_5 be the holding cost per unit time of one failed component, C_6 be the cost per unit time if the server is idle. We define the cost function as:

$$Expected\ cost\ per\ unit\ time$$
$$= C_1 \cdot P_{down} + C_2 \cdot N_{orbit} + C_4 \cdot \theta_4 + C_5 \cdot N_{fail} + \frac{C_3}{E_{\bar{H}}} + C_6 \cdot Pidle.$$

where $E_{\bar{H}}$ is found exactly in the same lines as in section 4.1 of part I.

Our numerical study, as presented in Table 4, show that an optimal value for N can be found for different parameter choices and also that this optimal value happens to be a much smaller value like $N = 6$. This shows the care needed in selecting the N-policy level.

6 Conclusion

We analyzed a k-out-of-n system where the server renders service to external customers also. In the case of a system where a minimum number of working components is necessary for its operation, the service to external customer should be carefully managed so that it does not affect the system reliability much. Krishnamoorthy et al. [1] managed to do that by introducing an N-policy , in which the ongoing service of an external customer is preempted at the moment when N failed components have accumulated for repair. Differing from Krishnamoorthy et al.[1], here we considered a non-preemptive service for external customers thereby making their service more attractive. We analyzed two models: one in which the external customers joins a queue and another in which they moving to an orbit of infinite capacity. Our numerical study showed that rendering non-preemptive service to external customers has not affected the system reliability much, thereby re-asserted that the same could be an effective idea for utilizing the server idle time and there by earning more profit to the system. Analysis of a cost function has helped us in finding an optimal value for the N-policy level.

Table 1: Variation in the server busy probability when external customers are allowed $k = 20, \lambda = 4, \bar{\lambda} = 3.2, \mu = 5.5, \bar{\mu} = 8, \theta = 5$

N	n=45	n=50	n=55	n=60
1	0.82349	0.82352	0.82353	0.82353
3	0.82995	0.82999	0.83	0.83
5	0.83222	0.83228	0.83229	0.83229
7	0.83328	0.83336	0.83338	0.83338
9	0.83385	0.83398	0.83401	0.83401
11	0.83417	0.83437	0.83442	0.83442
13	0.8343	0.83463	0.8347	0.83471
15	0.83424	0.83479	0.8349	0.83493
17	0.83394	0.83486	0.83505	0.83509
19	0.83325	0.83483	0.83515	0.83521
21	0.83192	0.83465	0.8352	0.83531
23	0.82945	0.83424	0.83518	0.83538

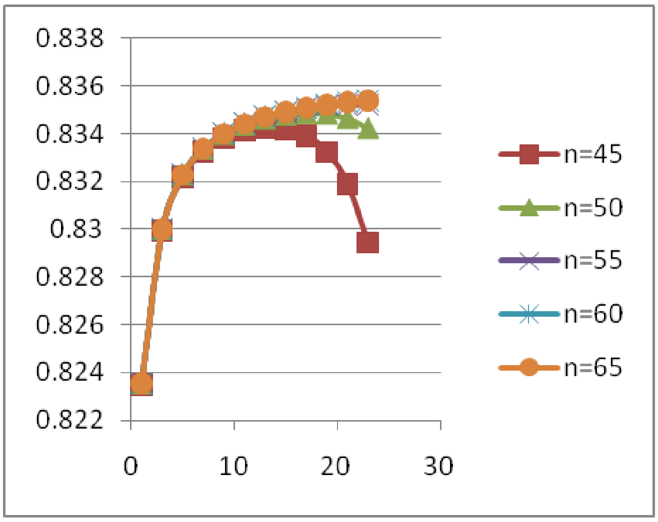

Table 2: Effect of the N-policy level on the fraction of time server is busy with external customers
$k = 20, \lambda = 4, \bar{\lambda} = 3.2, \mu = 3.2, \bar{\mu} = 8, \theta = 5$

N	n=40	n=45	n=50	n=55	n=60
1	0.09635	0.09628	0.09626	0.09626	0.09626
3	0.10287	0.10276	0.10273	0.10273	0.10273
5	0.10523	0.10506	0.10503	0.10502	0.10502
7	0.10644	0.10618	0.10612	0.10611	0.10611
9	0.10725	0.10685	0.10676	0.10675	0.10674
11	0.10798	0.10732	0.10719	0.10716	0.10716
13	0.10879	0.10772	0.1075	0.10746	0.10745
15	0.10991	0.10811	0.10775	0.10768	0.10766
17	0.11461	0.10858	0.10798	0.10786	0.10783
19	0.11983	0.10925	0.10822	0.10801	0.10797
21		0.1103	0.10851	0.10815	0.10808
23		0.11208	0.10893	0.10831	0.10818
25		0.11522	0.10959	0.1085	0.10828
27			0.1107	0.10877	0.10839
29			0.11265	0.1092	0.10852
31			0.11615	0.10991	0.1087
33				0.11116	0.10898
35				0.1134	0.10945
37					0.11026
39					0.11172
41					0.11435

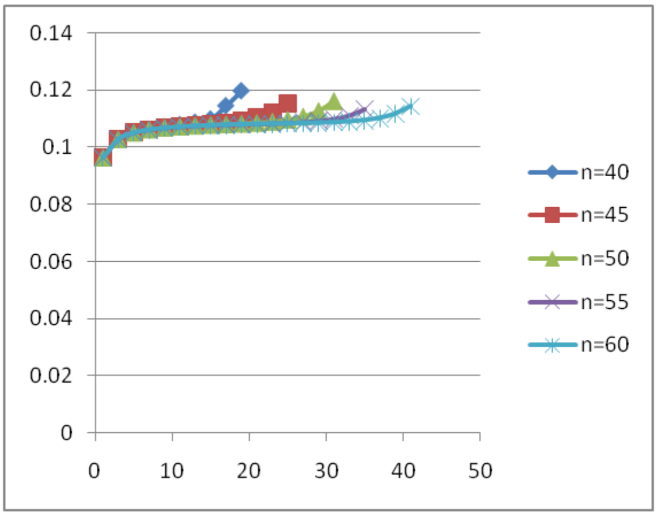

Table 3: Variation in the system reliability with increase in N $k = 20, \lambda = 4, \bar{\lambda} = 3.2, \mu = 5.5, \bar{\mu} = 8,$ $\theta = 5$

N	n=40	n=45	n=50	n=55	n=60
1	0.99963	0.99993	0.99998	1	1
3	0.99948	0.99989	0.99998	1	1
5	0.99924	0.99985	0.99997	0.99999	1
7	0.99885	0.99977	0.99995	0.99999	1
9	0.9982	0.99964	0.99993	0.99998	1
11	0.99712	0.99942	0.99988	0.99998	1
13	0.9953	0.99905	0.99981	0.99996	0.99999
15	0.99217	0.99843	0.99968	0.99994	0.99999
17	0.9769	0.99736	0.99947	0.99989	0.99998
19		0.9955	0.99909	0.99982	0.99996
21		0.99223	0.99844	0.99968	0.99994
23		0.98638	0.9973	0.99945	0.99989
25		0.97578	0.99528	0.99905	0.99981
27			0.99165	0.99833	0.99966
29			0.98509	0.99705	0.9994
31			0.97315	0.99475	0.99894
33				0.99058	0.99812
35				0.98297	0.99663
37					0.99393
39					0.989

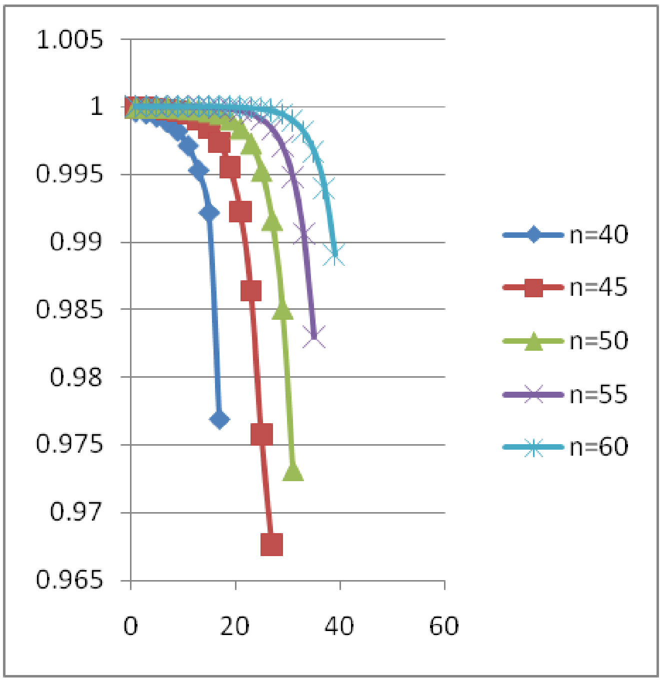

Table 4: Analysis of a cost function $n = 50, \bar{\lambda} = 3.2, \mu = 5.5, \bar{\mu} = 8, C_1 = 2000, C_2 = 1000, C_3 = 800, C_4 = 1000, C_5 = 10, C_6 = 200, \theta = 5$

N	$\lambda = 4$	$\lambda = 4.5$	$\lambda = 5$
1	6235.23047	6440.20947	6671.65918
2	6137.3877	6343.84668	6576.75928
3	6109.98389	6317.7207	6551.88965
4	6102.75391	6311.82178	6547.30566
5	6102.27734	6312.30322	6548.71436
6	6104.71094	6315.28613	6552.17676
7	6108.70947	6319.521	6556.51709
8	6113.67188	6324.50439	6561.33057
9	6119.2749	6329.98047	6566.44873
10	6125.32666	6335.80176	6571.76465
11	6131.69824	6341.87891	6577.22021
12	6138.31006	6348.14307	6582.78711
13	6145.10449	6354.55762	6588.43018
14	6152.04492	6361.09961	6594.13086
15	6159.104	6367.74854	6599.88428
17	6173.53564	6381.33594	6611.51611
19	6188.38672	6395.33936	6623.31689
21	6203.78809	6409.88037	6635.37354
23	6220.13477	6417.44531	6647.98535
25	6238.73828	6443.09375	6662.8042
27	6266.49854	6471.54688	6690.0752
29	6356.05566	6571.71631	6799.88672
31	7073.24658	7340.11523	7618.78223

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