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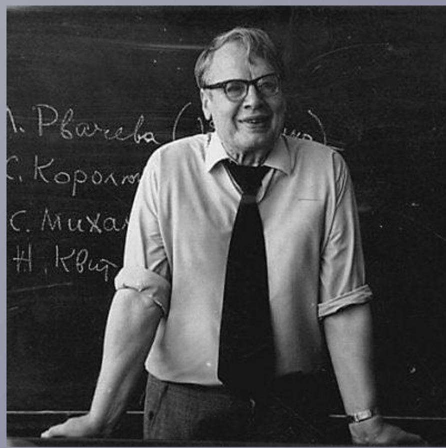

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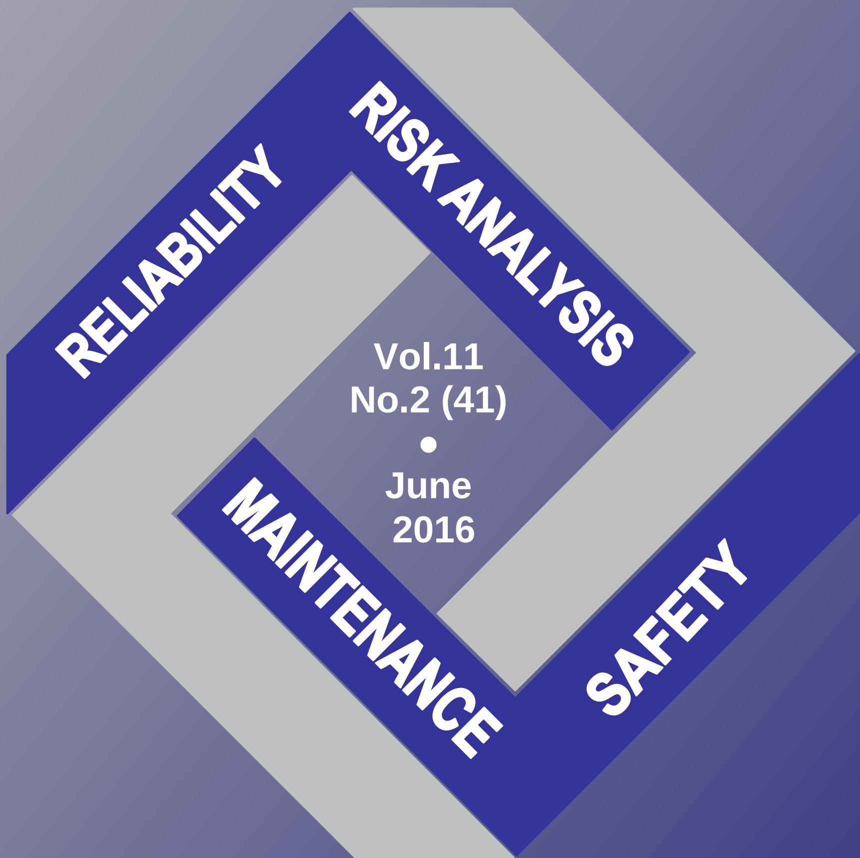

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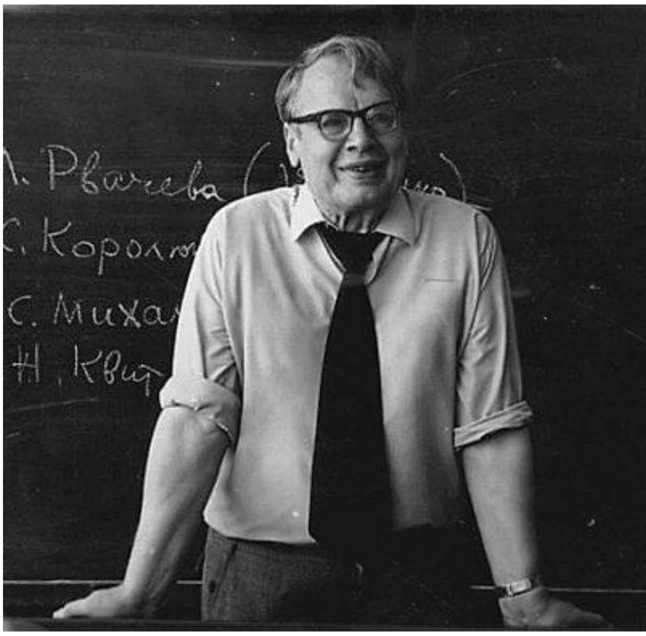

RELIABILITY:

THEORY & APPLICATIONS

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Analysis of statistical data about refusals, maintenance service and repair of objects of electric systems, is an indispensable condition of a quantitative estimation of reliability of their work. The analysis assumes classification of data on a sort of attributes and their versions. Thus, the basic question there is a revealing significant version of attributes. The importance of an attribute established based on statistical criteria of check of hypotheses. The criteria based on boundary fiducially estimation of intervals are offered.

This paper deals with the equal marginal location-scale Generalized Absolutely Continuous Multivariate Exponential model. The distributional properties and applications of the location-scale model arising out of the k-parameter Generalized Absolutely Continuous Multivariate Exponential distribution are studied. Standby, parallel, series and relay systems of order k with location-scale Generalized Absolutely Continuous Multivariate Exponential failuretimes are discussed and their performance measures are obtained. The optimal estimators of the meantime before failure times are also derived.

In this paper we study the reliability of a k-out-of-n system, with a single technician, who also renders service to external customers besides repairing the failed components in the system. For optimizing the revenue from external service without compromising the system reliability, we introduce the N-policy in which the repair of the internal customers (failed components) starts only on accumulation of N failed components. The service to external customers is of preemptive nature in the sense that their service can be interrupted on accumulation of N failed components. It is assumed that an external customer, who finds the server busy with an external customer at the epoch of its arrival joins a queue of infinite capacity; whereas an external customer who finds the server busy with an internal customer leaves the system forever. The failure times of the components of the k-out-of-n system follow an exponential distribution; the arrival of external customers is according to a Poisson process and the service times of the internal and external customers follow non-identical phase-type distributions. Using matrix-analytic methods, we discuss the system stability and steady state distribution. A special case of the model where the underlying distributions are all exponential has been considered, to obtain an expression for the stability condition and a product form solution for the steady state have been obtained for this case. Also several system performance measures have been obtained explicitly. Analysis of a cost function indicates that N-policy does help to optimize the system revenue maintaining high system reliability.

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INFORMATION TECHNOLOGY IN THE HOLY LAND

Sergey Grodzensky
•
Professor of Moscow Institute of Radiotechnics and Electronics.
Doctor of Sciences. Member of the Academy of Quality Control
e-mail: grodzensky44@mail.ru

Abstract

The Second International Symposium on Applied stochastic models in reliability theory, the science of life and management of processes that determined the abbreviation SMRLO'16 (The Second International Symposium on Stochastic Models in Reliability Engineering, Life Science and Operations Management) has been held in the city Be'er Sheva (Israel).

Keywords: symposium, SMRLO, reliability theory

The Second International Symposium on Applied stochastic models in reliability theory, the science of life and management of processes that determined the abbreviation SMRLO'16 (The Second International Symposium on Stochastic Models in Reliability Engineering, Life Science and Operations Management) has been held in the city Be'er Sheva (Israel). The City Be'er Sheva is located in the Negev desert, in a place where according to the Bible, the patriarch Abraham dug a well to water the flock of sheep and is also formed an alliance with Abimelech, king of Gerar, bringing with it a sacrifice of seven sheep, which gave the name to the settlement (Be'er Sheva in Hebrew, "Well of seven"). If we consider the age of the city from the first mention in the Torah, it would be about 3700 years.

Due to the UN resolution on the partition of Palestine, Be'er Sheva was departed to Arab state, but after the Arab-Israeli war of 1947-1949 Be'er Sheva became a part of Israel. The policy of dispersal of the population of the country led to the rapid growth of the city, especially in the 1990-ies, when tens of thousands of immigrants from the former USSR settled here. Now as the result the Russian-speaking population of 200-strong Be'er Sheva almost makes up the majority, and the local chess club created by immigrants from the Soviet Union became the largest in the country and regularly holds international tournaments of high level.

Modern Be'er Sheva is first of all, the city of science, the venue of the Symposium was the Engineering College. S. Shamoon. The history of this University is the following: Businessman Sami Shamoon has been involved in the so-called "Orange deal" — a famous agreement reached in the last days of N.S. Khrushchev in power, of the attend sale by the government of the USSR to Israel most of the property of the Russian spiritual mission in Jerusalem, for which Israel paid the oranges. In his later years S. Shamoon decided to immortalize himself by investing capital in the organization of the College of Engineering in Be'er Sheva, opened in 1996 and now it is bearing his name.

The forum on the "promised land" was attended by 125 experts from 24 countries. Most were from Israel and the former Soviet republics, but someone made a long-distance flight from Chile, Taiwan, and Professor Esa Sahib arrived from Iraq. I asked Sahib how he managed to come from a country which disclaims Israel's right to exist. It turned out that the presence of the second (Sweden) citizen ship helped him to get to the symposium. S. Esa explores both local dependencies in reliability and in the political situation of his country on the basis of probabilistic and statistical concepts. Perhaps, when a scientist from Iraq completes the study, we will be able to talk about the usefulness of its findings for other countries.

At the same time the plenary session and three sections corresponding to the declared directions of the Symposium (applied theory of reliability, life science and process control) took place.

All submitted papers were distributed to three concurrent sections, corresponding to the declared directions of the Symposium (applied theory of reliability, life science and process control). The first plenary session was opened by M. A. Yastrebenetsky (Ukraine) and A.V. Bochkov (Russia), dedicated to the memory of the outstanding scientist Igor Alekseevich Ushakov (1935-2015). Professor I. A. Ushakov is known in the scientific world due to the fundamental results obtained in the theory of reliability, operations research, discrete optimization and related disciplines. In addition, he founded «Forum Gnedenko" on the internet, which became an informal association of professionals in the field of reliability. Also Igor Alexeyevich was a historian, poet and perhaps most importantly, a very faithful friend, in what the author had the opportunity to make sure not once during more than 40 years of acquaintance with him.

One of the organizers of the Symposium — Anatoly Lisnianski, an employee of the company "The Israel Electric Corp."(the main supplier of electricity in Israel and the Palestinian territories) in the overview "Application of Extended Universal Generating Function Technique to Dynamic Reliability Analysis of a Multi-state System" have shown the effectiveness of the new approach to assessing the reliability of complex multi-state systems (e.g. system with different levels of performance and availability). When using the classical Markov method in building model of Multi-state system it requires to solve a system of large (sometimes very large) number of differential equations. The task is greatly simplified if we apply Lz-transform method, based on the approach called as the universal generating function technique, the idea of which was first expressed by I. A. Ushakov in the mid 80-ies of the last century.

This method allowed the hosts of the Symposium under the guidance of Professor I. Frenkel (SCE - Shamoon College of Engineering, Be'er Sheva, Israel) to solve the problem of "Availability and Unloading Capacity Assessment of Multi-state Material Handling System, Operate in a Stochastic Material Handling Demands". The system of loading and unloading operations is presented in the form of a Markov model with 96 different states, corresponding to possible levels of process performance. In the end, the model described by a system of 288 (!) differential equations. The complex problem was solved by using Lz-transform.

The message of international group of authors Alex Karagrigoriou (University of the Aegean, Greece) Andreas Makrides (University of Cyprus, Cyprus) Vlad Barbu (Université de Rouen, LMRS, France) "On Semi-Markov Modelling and Inference for Multi-State Systems" is interesting in the fact that the moments of failures of the studied system aredistributed according to the Weibull distribution, which is a fairly realistic model.

In the report of L. Epstein (School of Business, Universidad de los Andes, Chile) "Optimal Times to Adjust the Mix of Owned and Borrowed Items" the solution to the task of inventory management is given. A situation where the service provider owns the equipment, which he rents out is simulated. If its own reserves are insufficient to meet demands, the supplier may rent additional equipment from a third party. The model allows to determine the optimal timing and the corresponding amount of equipment that the service provider needs to acquire in order to replenish reserves. The approach considers a finite number of users or clients, as well as forecasted demand in the future, using past experience.

Ososkov G. A. (Joint Institute for Nuclear Research, Russia) in the message "Combined

Approach to Reliability of Great Software Complexes for Distributed Computing with Big Data in Contemporary Physical Experiments" presented the results of the development of modeling tools for grid-cloud services for systems of storing and processing large amounts of data in physics experiments. Each of these services is a complex merging of powerful computers interconnected with the software and network components. To calculate the reliability of the software a new simulation method is used, that takes into account the parameters and interconnection of components, as well as the quality of functioning of some real system by combining simulation programs with monitoring system of real grid cloud service through a special database.

In the report of Vladimir Skliarov (National Scientific Centre "Institute of Metrology", Kharkov, Ukraine) the issues of assessment of ageing and degradation of the equipment, the operation life of which reaches up to 30 years and more are considered. Based on the laws of irreversible thermodynamics the possible mechanisms of aging of equipmentare analyzed. For example, the evaluation of aging and degradation of sealed enclosure system of nuclear power plant localizing safety system is shown.

The purpose of the work of Yefim Mikhlin and Ofer Sahama (Technion - Israel Institute of Technology, Haifa, Israel) "Validation of Updated Sequential Test for Standard IEC 61123" — is to validate the method of test planning based on the Wald criterion. The standard summarizes the experience of researchers of various countries on optimal truncation and an assessment of the actual risks.

The speech of the journal "Methods of quality management" editorial Board member L. Papić aroused nostalgic memories. Once Ljubisa Papic was known in Yugoslavia, a chess composer (a composer of chess problems and studies), and even published a journal in which the author of these lines — Moscow champion in chess composition, 1969 — published his works. It was so long ago! … Now Professor L. Papic (Serbia) presented a report on the current topic "Human Factor in Mining Machines Maintenance Operations". Sad experience shows that one of the causes of the problems in the mining industries, often leading to tragedies, is "the human factor", which is manifested in different forms (negligence, omission, incorrect decision when identifying hazardous conditions, poor planning and inappropriate behavior in unpredictable circumstances, etc.). To reduce the probability of occurrence of specified event L. Papic proposes to combine a top-down approach" (Causes-effect diagram and the "Five "why?") and bottom-up (event tree analysis). Method of Causes-effect diagram and Five why?" help to identify human error, i.e. the sources of problems, and analysis of "tree event" is used as a secondary method of preventing (or reducing) damage.

At the final plenary meeting, the author made a report, co-authored with A. N. Chesalin and Ya. S. Grodzenskiy (MIREA, Moscow)"About the Effectiveness of the Statistical Sequential Analysis in the Reliability Trials", which shows the results of the comparison of the effectiveness of the classical Wald criterion, the most well-known ways of modifying it and proposed a new method for reducing the average length of the test procedures, while ensuring the required accuracy.

For the Symposium participants a trip to the ancient fortress of Masada was organized— the symbol of steadfastness of the people of Israel during the First Jewish war against the Roman Empire (66-71 years). The road went winding along the gentle slope of a hill in the Negev desert, here we passed the sign "sea Level" and we all continued to descend until he reached the shores of the Dead sea — the lowest land area on Earth, located at 422 m below sea level. No one volunteered to swim but some of the professors yet ventured to wet hands in the most salty natural water reserve of the planet.

The impressions of visiting Israel remained bright enough. I will mention only a few episodes. On the tour I turned to the young man in a long coat who I had not met at the Symposium, with the duty question "Where do you come from?". Instead of answering, he threw his coat open, showing a gun hanging on the side. It was a "security guard", responsible for the safety of our excursion. In those days there was the "knife intifada". The words "Israel is now going through hard times" have become common because there was not a single peaceful day during almost 70 years existence of the country.

In response to my question: "How do you live in Israel?", Palestinian tour bus driver replied with confidence of propagandist of soviet times: "If you have an israeli passport, it does not matter what is your nationality and religion. It is only necessary to work for the good of the country", but after a pause he added — in real life, of course, all is not so simple." It was clear, that "all is not so simple". In classrooms in College I watched Muslim students chattering with their teacher of pronounced semitic appearance. But I noticed that at the entrance to the territory of University a girl in hijab put out the contents of the bagin front of the security guard, passed through the metal detector, with a straight face waited patiently until the guard made sure that she was not trying to get in with something forbidden. I was standing behind her, ready to turn out the contents of my pockets, he saw my business card "Russia", grinned and allowed to pass almost without inspection.

During a sightseeing tour of Jerusalem after praying with Orthodox Jews at the Wailing Wall, having blessed candles in the Holy fire in the Church of the Holy Sepulchre, I wanted to bow to the Muslim Holy sacred place. But I failed — the entrance for non-Muslims is restricted. The Symposium provided an opportunity to visit the memorial Yad Vashem — the Holocaust Museum, or as it is called in Israel "Museum of the Holocaust and Heroism". Without going into details of my innumerous impressions I'd like to say that moving from room to room, the understanding why Victory Day on May,9 is recognized in Israel as one of the major holidays grew stronger…

Usually, speaking about the "economic miracle" I mean Japan, sometimes remembering the countries of South-East Asia too. But isn't it amazing the very existence of Israel — the state that emerged in the desert, which has become advanced in the development of science and high technology. Many years ago the organizers of the Tokyo Olympics 1964 chose its motto: "the World is one — the Olympics!". Listening to the speeches of representatives of different countries at the Symposium in Be'er Sheva, I felt like exclaiming several times "the World is one – the information technologies!". And, I admit, I have a great desire to participate once again in some scientific or chess forum in the Holy land.

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ABOUT TRIGONOMETRIC DISTRIBUTIONS TO DESCRIBE THE FAILURE OF TECHNICAL DEVICES

Volodarsky V. A.

•

PhD, senior researcher, associate Professor, Institute
of Railway Transport, Krasnoyarsk, Russia
E-mail: volodarsky.vladislav@yandex.ru

Abstract

There has been proposed and investigated the non-traditional trigonometric distribution to describe the gradual failure of technical devices.

1. The starting statements

At present the calculations of indicators of the technical devices (TD) reliability are carried out, as a rule, with the assumption of the constant failure rate for their component elements. It corresponds to the case when the component elements are subjected only to sudden failures because of external influences. The gradual failures of the component elements connected with internal processes of wear and aging at the same time aren't considered. This doesn't correspond to reality. For example, in [1, 2] there were described the degradation processes which cause wear and aging of the component elements of railway systems power supply, automatic equipment, telemechanics and communication. These processes lead to the gradual failure of the elements and they are described in the theory of reliability by the class of distributions having the increasing failure rate function that is so-called VFI – distributions [3]. We will call the elements with gradual failures as the growing old type elements.

By the collection and processing the data about the failures of these component elements which were restored while performing there were obtained only the indices of the constant values of ω failure stream parameter [1] or the time between failures as $T = 1/\omega$ [2]. However it doesn't mean that failure rate of component elements of the growing old type is a constant value.

According to the definition the failure stream parameter is the relation of the number of the failed products during the time interval of n(dt) to the number of the tested products during this interval of dt provided that the failed products are replaced be the serviceable ones (new or repaired), that is $\omega(t) = n(dt)/Ndt$ where N – the number of tested products which remains to be constant. From the theory of reliability it is known that the failure stream parameter at any kind of distribution under operation strives for the stationary value equal to $\omega = 1/T$. It is also evident when collecting statistical data on TU failures under real conditions of operation.

According to the definition the failure rate is the relation of the number of the failed products during the time interval of n(dt) to the average of the products Ncp which have regularly worked during this time interval of dt, that is $\lambda(t) = n(dt)/Ncp\ dt$. By doing so the Ncp decreases because of failures of products with each interval, and $\lambda(t)$ elements of the growing old type increases.

The laws of distribution for the time between failures of component elements of the growing old type being, as a rule, unknown, the problem of the TU reliability indicators calculation has to be solved under uncertain conditions.

The purpose of the article is to offer and to investigate nonconventional trigonometrical distributions for the description of gradual failures of technical devices.

When it is possible to estimate only the value of the time between failures, for example, from expression of $T = 1/\omega$, one could suggest two following methods of the approximate description of the technical devices reliability indicators.

2. Distribution of a cosine

First, as the failure stream parameter at t = T approaches the stationary value equal to 1/T, it is offered to approximate the ω (t) dependence by the piecewise and linear function of the type [4]:

$$\text{at } t < T \quad \omega(t) = t/T2; \quad \text{at } t \geq T \quad \omega(t) = 1/T. \tag{1}$$

Other indicators of the reliability are defined with the use of transformation by Laplace. We will find the density of distribution of f(t) from the equation connecting it in an operator form with the failure stream parameter of $f(s) = \omega(s) / (1+\omega(s))$ as

$$f(t) = (1/T) \sin(t/T). \tag{2}$$

Then the probability of no-failure operation of P (t) and the failure rate of $\lambda(t)$ are defined from the equations:

$$P(t) = 1 - \int_0^t f(t)dt = \cos(t/T); \tag{3}$$

$$\lambda(t) = f(t) / P(t) = (1/T)\ \mathrm{tg}\ (t/T). \tag{4}$$

The argument of t/T in formulas for definition of the reliability indicators is measured in radians. We will call the received distribution as the distribution of the cosine which definition range lies in the range of $0 < t/T < \pi/2$.

The not own integral from the distribution density within the distribution definition range according to [5] has to be equal to one unit. We check

$$\int_0^{\pi T/2} (1/\mathrm{T})\, sin(t/T)\, dt = 1.$$

The distribution variation coefficient is defined from expression

$$V = \mu_2^{0.5}/\mu_1\,, \tag{5}$$

where μ_1- the first initial moment;

μ_2 - second central moment.

$$\mu_1 = \int_0^{\pi T/2} tf(t)dt = \int_0^{\pi T/2} (t/\mathrm{T}) \sin(t/T)\, dt = T;$$

$$\mu_2 = \int_0^{\pi T/2} (t - \mu_1)^2 f(t)dt = \int_0^{\pi T/2} (t - \mathrm{T})^2 (t/\mathrm{T}) \sin(t/T)\, dt = (\pi - 3)\, T^2\,.$$

Having substituted the values of μ_1 and μ_2 into the expression (5), we will receive $V = 0{,}376$.

As the failure rate of this distribution according to (4) is a monotonously increasing function of time, and the value of the variation coefficient is less than one unit, it belongs to the class of the VFI - distributions and can be used for the description of the TU gradual failures. In [4] article there were defined the asymmetry and excess coefficients and it has been noted that the distribution of the cosine can be presented at the Pearson's areas by a point with coordinates of p2 = 0,18 and β = 2,23. It has been shown what according to [3] the cosine function is also the distribution with the increasing average failure rate, the distribution like "the new thing is better than the used one" and the distribution like "the new thing is on average better than the used one".

With use of the equations of (1), (2), (3) and (4) there have been defined the dependences of the reliability indicators on the relative time of t/T operation of the offered distribution. The results of the calculations are shown in table 1.

Table 1

t/T	0	0,2	0,4	0,6	0,8	1,0	1,2	1,4	π/2
Tf(t)	0	0,20	0,39	0,56	0,72	0,84	0,93	0,98	1,0
Tλ(t)	0	0,20	0,42	0,68	1,03	1,56	2,57	5,80	∞
Tω(t)	0	0,20	0,40	0,60	0,80	1,0	1,0	1,0	1,0
P(t)	1	0,98	0,92	0,83	0,70	0,54	0,36	0,17	0

3. Distribution of cosine square

Secondly, it is offered to approximate the dependence of the technical devices failures probability density of f(t) depending on the operation time by t function of the sine of the type

$$f(t) = (1/\mathrm{T})\, sin(2t/T) \tag{6}$$

with the range of definition $0 < t < \pi T/2$.

The not own integral from the distribution density within the distribution definition range according to [5] has to be equal to one unit. We check

$$\int_0^{\pi T/2} (1/\mathrm{T})\, sin(2t/T)\, dt = 1.$$

Considering that $\sin(2t/T) = 2\sin(t/T)$ cos(t/T), we will present the expression (6) in the form

$$f(t) = (2/T) sin(t/T) cos(t/T). \tag{6a}$$

The probability of non-failure operation of P (t) considering (6) is defined from the expression

$$P(t) = 1 - \int_0^t f(t)dt = (1 + \cos(2t/T))/2\ . \tag{7}$$

Considering that, we will $\cos(2t/T) = cos^2(t/T) - sin^2(t/T)$,
present the expression (7) in the form

$$P(t) = cos^2(t/T). \tag{7a}$$

We will call the received distribution as the distribution of the cosine square.

The failure rate of λ(t) taking into account of (6a) and (7a) is defined as

$$\lambda(t) = f(t)/P(t) = (2t/T)tg(t/T). \tag{8}$$

The failure stream parameter of ω(t) is defined by Laplace's transformation of the type $\omega(s) = f(s)/(1 - f(s))$ taking into account of (6) as

$$\omega(t) = (\sqrt{2}/T)sin(\sqrt{2}t/T) \,. \tag{9}$$

The variation coefficient is defined with the use of expression (5).

$$\mu_1 = \int_0^{\pi T/2} tf(t)dt = \int_0^{\pi T/2} (t/\mathrm{T}) \sin(2t/T) \, dt = \pi T/4 \, ;$$

$$\mu_2 = \int_0^{\pi T/2} (t - \mu_1)^2 f(t)dt \int_0^{\pi T/2} \left(t - \frac{\pi T}{4}\right)^2 \sin(2t/T) \, dt = (\frac{\pi^2}{16} - 0.5) \, T^2.$$

Having substituted the values of μ_1 and μ_2 into the expression (5), we will receive $V = 0{,}435$.

With use of the equations of (6), (7a), (8) and (9) there have been defined the dependences of the reliability indicators on the relative time of t/T operation of the offered distribution. The result of the calculation is reduced in table 2 and presented in figures 1 and 2.

Table 2

t/T	0	0,2	0,4	0,6	π/4	1,0	1,2	1,4	π/2
Tf(t)	0	0,39	0,72	0,93	1,0	0,91	0,68	0,33	0
Tλ(t)	0	0,40	0,84	1,36	2,0	3,12	5,14	11,6	∞
Tω(t)	0	0,39	0,76	1,06	1,26	1,40	1,39	1,30	1,10
P(t)	1,0	0,96	0,85	0,68	0,50	0,29	0,13	0,03	0

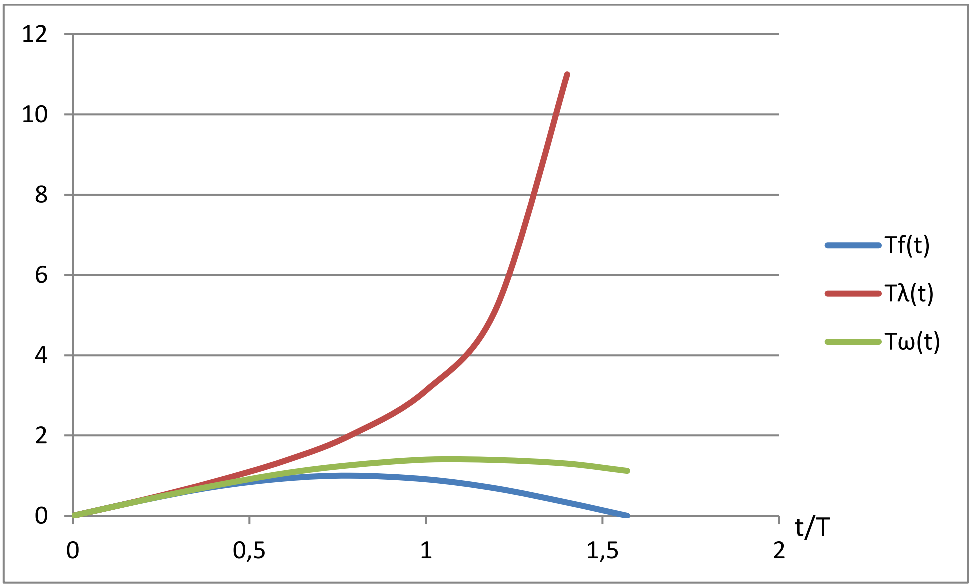

Fig. 1

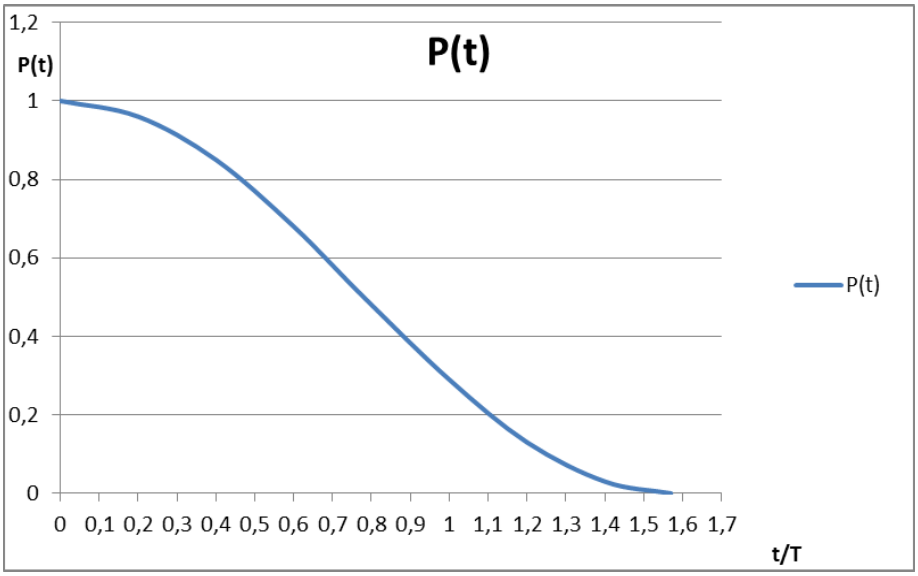

Fig. 2

As it is evident from the formula (8) and the figure 1 the failure rate increases monotonously under operation and considering that the value of variation coefficient is less than one unit, the offered distribution belongs to the class of the VFI - distributions and can be used for the description of the gradual failures of the TU elements. From figure 1 it is also evident that the failure stream parameter strives for the value equal to 1/T. It confirms the famous statement of the theory of reliability that at any distribution the failure stream parameter under operation strives for the established value, the return value of the time between failures.

As it is evident figure 2 the probability of TU non-failure operation under operation decreases, and at value of $t = \pi T/2$ it approaches to zero. And the $P(t)$ curve at first is convex up, and then – down.

For comparison in figures 3 – 5 there are presented the curves of the distribution density, failure rate and failure stream parameter of distributions of the cosine constructed with use of these tables 1 and 2 (are designated in the blue color 1) and the cosine square (are designated in the red color 2). As it is evident from formulas of (4), (8) and figure 4 the failure rate at distribution of the cosine square increases under operation twice as faster than at distribution of the cosine.

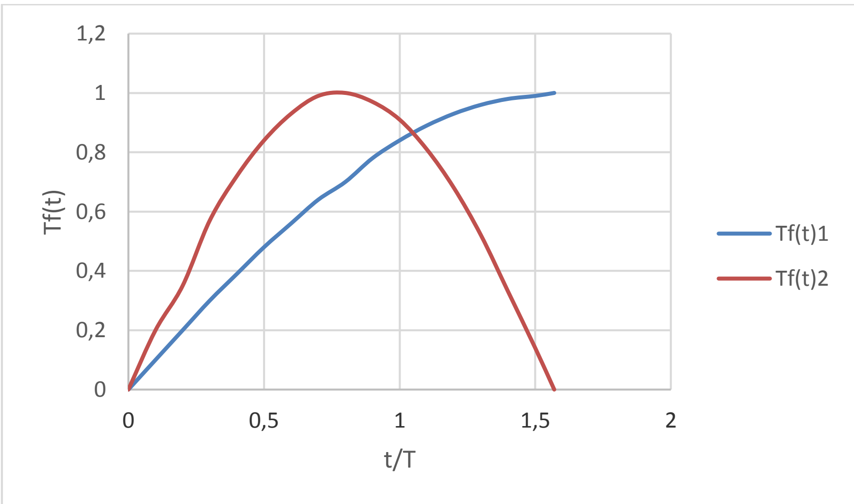

Fig. 3

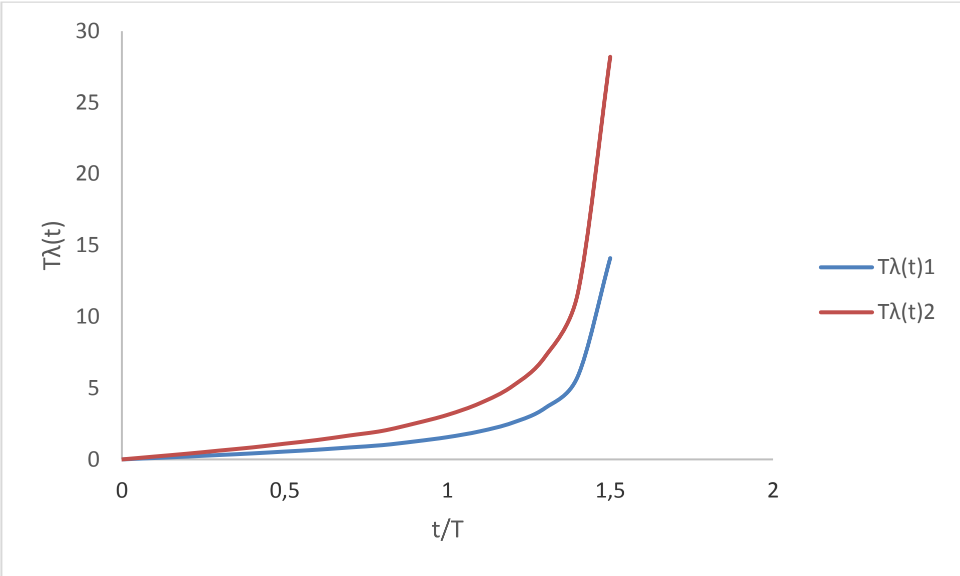

Fig. 4

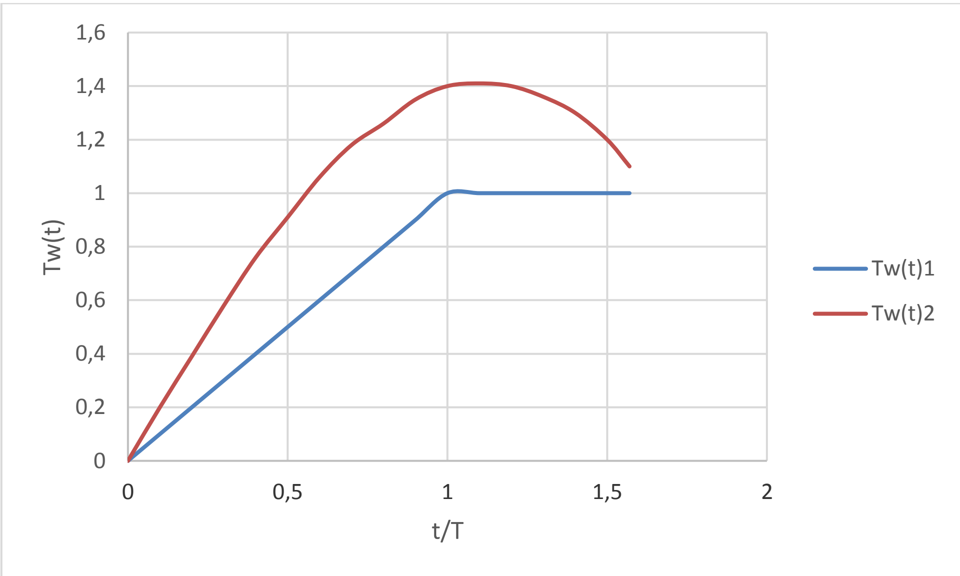

Fig. 5

Conclusion

In the case when only the time between failures is defined, and it is known that elements of technical devices are subjected to wear and aging, for the description of their gradual failures under the conditions of uncertainty it is expedient to use the offered distributions of a cosine and a cosine a square. It is evident from the received results the reliability indicators at these distributions are expressed by the elementary functions that can simplify carrying out calculations of the indicators of systems reliability at different connection of the component elements.

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METHOD AND ALGORITHM OF FUZZY CONTROL OF REACTIVE CAPACITY AND VOLTAGE PROVIDING REGIME RELIABILITY OF ELECTRIC NETWORKS

Guliyev H.B.

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Azerbaijan Scientific-Research
and Design-Prospecting Power Engineering Institute
AZ1012, Ave. H.Zardabi- 94
E-mail: huseyngulu@mail.ru

Abstract

The structure and algorithm of the voltage and reactive power control system for distribution networks with on site power sources containing fuzzy logic controller (FLC) is presented. The controlling parameters are: the transformers voltage ratio and capacities of the reactive power sources in distribution networks. The placement of reactive power sources, their values and also transformers regulator's positions are determined using traditional methods of optimization for selected networks. The structure of reactive power sources and transformers voltage ratio control system containing the fuzzy logic controller is presented in this paper. The problem of optimal correction of transformers voltage ratio and power sources at time of their deviation from the preset values to minimize losses in studied network and maintaining of nodes voltages on the necessary level is considered. The algorithm of membership function formation for input variables of FLC to control / correct capacitors value is shown. Modeling results for real electrical circuit, reactive capacity correction in nodes and transformers impact on losses and voltage profile in studied network are presented.

Key words: voltage, reactive power, fuzzy logic, power losses, optimal placement of static capacitor, controller, membership function, electric network.

I. Introduction

For mode profitability conditions and voltage quality support in distributive electric networks the adjustable batteries of static capacitors and voltage control units for transformers under loading regulation are used. Among voltage and reactive power regulating devices the automatic excitation regulators for local sources (synchronous generators, diesel or gas-turbine units) in distributed generation networks also are used.

The choice of a placement position and static condensers batteries rate planned for installation is the optimizing problem which essence consists of total active power losses minimization. Now methods of nonlinear optimization [1-5] and also heuristic methods are applied to the decision of the given problem [6].

With help of [1-6] methods for planned schemes and predicted modes the batteries of static condensers optimum rates assumed for installation in network knots are defined. In real operation conditions the loadings consumption capacity in a network continuously changes, that leads to a deviation of a current production schedule from planned on the set period (days, weeks etc.). Actual values of reactive power in network knots will differ from optimum chosen values for their covering of condenser units capacities. The losses levels and knots voltages will change according to current mode changes in a network. Such current losses values and voltages will differ from their corresponding values in optimum modes.

The difference between current knot's reactive power value and optimum chosen capacity of the condenser battery is possible to compensate operatively by change of minimum share of capacity correction pre-setted in knot in a direction of losses reduction in a network.

The power factor correction condensers module usually consists of several separate elements or groups of elements, everyone with own contactor or switch. Reactive power covering demand and a power factor are continuously estimated and the condenser modules connected and disconnected necessarily for optimum level achievement.

The algorithm of indistinct logic realized in block of current mode condition estimation in distributed generation (DG) network is developed for definition of optimum number of modules in each knot. In the same block the necessity of planned parameters values updating - rates of condensers capacities installed in controllable knots and transformer's voltage ratio is checked.

The problem solution on definition of necessity of connected condenser's capacity rate correction and a choice of transformer's voltage ratio is spent by developed indistinct system's algorithm in which as inputs the knots voltages and power losses indicators are defined. Thus necessity of condenser correction for this or that knot will be defined by an importance indicator of condenser's capacity variability. Necessity of correction of condenser's capacity for this or that knot is defined in case of large value of this indicator.

For practical correction of condenser's capacity value in knots of installation the indistinct logic regulator is used.

II. Structure of Reactive Power and Voltage Control System in Distributed Generation Networks

In dispatching management modern practice the operative modes correction in power system electric networks has a great value at control solutions acceptance at a network mode deviation on Q and U from their values received on the base of optimum modes calculation. Thus for a choice of correcting actions for reactive power and voltage (RCV) management the criterion of a minimum of losses is used at performance of preservation conditions of the standard deviation of knots voltages [2,3,7-9]. RCV control in distribution networks basically is carried out by means of batteries of static condensers (BSC), and also generating sources, placing in a network for a local loads covering and transformers regulated under loading. The choice of adjustable static condensers batteries number and generating sources, their placing in network is an optimizing problem. The decision of given problem for electric network normal scheme defines the optimum number of regulating devices.

At an operational control in process of scheme and mode change current optimum values of voltage in knots $U_{i,max}$ and total losses values in network $\Delta P_{i,min}$. are defined. In accordance with calculated new values $U_{i,max}$, $\Delta P_{i,min}$ the setting of numbers of individual condensers $K_{конд,i}$ for knots in which their installation is accepted, and transformers regulating devices positions $K_{t,i.}$ are defined.

Such optimizing calculations can be executed in frame of the program complex for power system condition estimation. Algorithms used by these programs are known [10-15] and basically consist in periodic optimizing calculations carrying out according to a current scheme condition and a system mode. On the basis of current optimizing calculations results comparison - knots voltages values and total losses in a network, with the optimum values established for base normal modes, necessity of condensers batteries capacities $(\mathrm{C}_{\mathrm{ki}})$ correction, transformers voltage ratio for remote adjustable transformers, voltages of generators placed in distributed generation system is defined. Depending on a deviation value of current optimum values of the voltage in controllable knots and network losses value from their corresponding values in nominal base mode the operating influences sizes for condenser batteries modules established in controllable knots switching on and positions of transformers regulating device are defined.

Following the above-stated distributed network reactive power control mode it is possible to present the general control scheme in form of the following block structure of the static condensers batteries, position of transformers switching and synchronous generators voltages co-ordinate control.

The general management concept for the purpose of optimum mode support in an electric network with the distributed generation consists in a choice of static condensers capacity from among the set condensers in knots, and also in transformer voltage ratio definition installed in connection point of DG network with a power system and its switching to position providing a minimum of power losses in a network. Necessity of correcting actions on condensers and the transformer arises at deviations of network current mode losses from their (planned) values calculated for network optimum modes.

In considered statement correcting control influence on change of condensers batteries modules in network knots and transformers voltage ratio accepted in form of linear dependence on a deviation of current conditions (changes of active and reactive power of knots loadings) [16]:

$$\Delta Y = f(k, \Delta d) \tag{1}$$

where $\Delta Y = \overline{Y} - Y$, $\Delta d = \overline{d} - d$

$\overline{Y}$, $\bar{d}$ – planned values of adjustable and initial data

$$\overline{Y} = \begin{Vmatrix} C_{ki} \\ \cdot \\ K_{ti} \end{Vmatrix}, \qquad d = \begin{Vmatrix} P_1 + jQ_1 \\ \cdot \\ P_i + jQ_i \end{Vmatrix}, \qquad i = 1...n$$

$\Delta \mathrm{Y}$ – operating influences on change of condenser capacity rate on $\Delta \mathrm{C}_{\mathrm{k,i}}$ and change of adjustable transformer's voltage ratio $\Delta \mathrm{K}_{\mathrm{t,i}}$; Δd – initial data changes of knot loadings $\Delta \mathrm{P}_{\mathrm{i}} + \mathrm{j}\Delta \mathrm{Q}_{\mathrm{i}}$.

Condensers and transformers control equation adjusting parameters are defined from optimization conditions:

$$\min_{k,\ \Delta \mathrm{P}_i,\ \Delta \mathrm{Q}_i} \mathrm{M}\, \Delta \mathrm{P}(\overline{\mathrm{Y}} + f(k, \Delta d), x, d) \tag{2}$$

where x - dependent parameters:

$$\mathrm{x} = \begin{Vmatrix} U_1 \\ \cdot \\ \cdot \\ U_n \end{Vmatrix}, \quad i = 1 \dots n$$ - knots voltages vector.

III. Correction of DG Network Mode Parameters by a Fuzzy Logic Method

The probabilistic and indistinct-defined character of scheme and network mode parameters variability (knots power and voltage) and also the electric systems modes (ESM) models nonlinearity, its parametrical uncertainty and unpredictability complicates application of the known determined methods for active and reactive power flows control in RG network. For the problem solution in choice of correcting control in [12, 14, 17-20] the algorithms – as solving rules generated on the base of linear dependences in form of (1) are used. Besides, for correcting values for $\mathrm{C}_{k,i}$ and $\mathrm{K}_{t,j}$ an estimation of the determined active power losses equivalent is defined. But even in this case the problem becomes complicated when operating vector $"\overline{\mathrm{Y}}"$. dimension increases.

In frame of indistinct system the correcting actions choice on sizes of knots capacities and transformers voltage ratio is formalized on base of linguistic rules defined by membership functions. The purpose of reactive power flow mode correction adds up to the "max-min" problem solution [9,10].

For a problem of correcting action definition on change of installed in knots condensers rate a resultant membership function of an admissibility of condenser rate $\mu_{S_C}(\mathrm{i})$ correction in i mode and at *k* accepted rules:

$$\mu_{S_C}(\mathrm{i}) = \max \mathrm{k}\,[\min[\mu_P(\mathrm{i}),\ \mu_U(\mathrm{i})]] \tag{3}$$

where $\mu_{\mathrm{P}}(\mathrm{i}),\ \mu_U(\mathrm{i})$ membership functions of power losses and voltage indicators.

From the determined optimizing problem solution with taking into account the forecast of initial data:

$$\overline{d} = \left|\overline{\pi}, \overline{P}_{H,i}, \overline{Q}_{H,i}\right|$$

The planned targets for capacities rates in knots and values ratio of adjustable transformers are defined as:

$$\overline{Y} = \left|C_{k,1}, C_{k,2}, \cdots, C_{k,n}, K_{t,1}, K_{t,2}, \cdots, K_{t,m}\right|$$

where, $\overline{\pi}$ - active power losses; $\overline{P}_{H,i}$, $\overline{Q}_{H,i}$ - predictably values of active both reactive power in the *i*- th loading knot.

In frame of the is indistinct-defined statement the problem solution of an estimation of a share of correcting action on condenser batteries rate change in knots and position of the transformers regulating devices, can be realized in the form of following stages:

1. To define total active power losses for DG system base structure (are carried out on the base of flow distribution calculation programs). The program complex ETAP which provides steady stage calculations, and also calculations of Q sources optimum placement in a network is used in

this research.

2. By change of a reactive power compensation share in each knot to carry out the flow distribution calculations and define the total active power losses in each case $\Delta C_{k,i.}$.

3. To calculate losses decreasing indicators as:

$$\Pi_{\Delta P}(i) = \frac{(\Delta P(i) - \Delta P_{\min})}{(\Delta P_{\max} - \Delta P_{\min})} \tag{4}$$

Where i=2, 3, … n - number of knots in which batteries of condensers are placed.

By indicator value (4) the capacity correction suitability for knot "i" is defined. If this indicator is highest for any i th knot the capacity correction in this knot is most comprehensible.

4. The membership functions for power losses indicators $\mu(L_{\Delta P})$ and voltages in each knot $\mu_U(\mathrm{i})$ are accepted as model (3) inputs.

5. Indistinct model's (3) target parameter - a resultant membership function $\mu_{S_c}(\mathrm{i})$ defines an acceptability of capacity correction in the given knot.

$$\tilde{Y} = \tilde{U} \circ \tilde{\Pi}_{\Delta P} \circ R(U, L_{\Delta P}, Y) \tag{5}$$

Where, «$\circ$» -the "мax-мin" composition's symbol; R - the indistinct relation.

6. Dephasification of an indistinct control output signal for $C_{k,i}$ condensers batteries capacity and transformers voltage ratio $K_{t,i}$ correction:

$$Y = F^{-1}[\tilde{Y}] \tag{6}$$

where,

$$\tilde{Y} = \max \{\min[\mu_P(\mathrm{i}),\ \mu_U(\mathrm{i})]\}$$

F –the phasing symbol.

According to the offered algorithm for network knot definition in which it would be preferable the battery of static condensers capacity correction, in the indistinct logic regulator the knots voltages and losses index (IL) $\Pi_{\Delta P}(\mathrm{i})$ calculated on (4) are accepted as input parameters. The higher limiting value for $\Pi_{\Delta P}(\mathrm{i})$ for i knot is considered as the priority knot in which it is necessary to carry out the correction established in knot where the condensers battery was connected.

Indistinct variable of knots voltages, losses indexes $\Pi_{\Delta P}(\mathrm{i})$, and also an indicator of network knot preference in which the condensers battery will be corrected, are described in terms of indistinct definitions: Critical Low, Low, Low-Medium, Medium, High-Medium and High.

Subsets fuzzy logic A_{1i} of an indicator of loss of capacity on terms to linguistic variables it is resulted below:

$A_{11} = CL$ (Critical Low) $\underline{\underline{\Delta}}(p, \mu_{11}(p))$

$A_{12} = L$ (Low) $\underline{\underline{\Delta}}(p, \mu_{12}(p))$

$A_{13} = LM$ (Low-Medium) $\underline{\underline{\Delta}}(p, \mu_{13}(p))$

$A_{14} = M$ (Medium) $\underline{\underline{\Delta}}(p, \mu_{14}(p))$

$A_{15} = HM$ (High-Medium) $\underline{\underline{\Delta}}(p, \mu_{15}(p))$

$A_{16} = H$ (High) $\underline{\underline{\Delta}}(p, \mu_{16}(p))$

Defined A_1 a universum of a subset of fuzzy-ligic sets the generalised kind it is possible to write a below-mentioned variant

$$\underline{\underline{\Delta}}(p, \mu(p)) = \sum_{p \in A_1} \mu_{1i}(p_i))/p_i \ , \ \forall p_i \in A_1$$

Subsets fuzzy logic A_{2j} of an indicator of voltage knots on terms to linguistic variables it is resulted analogycaly below:

$A_{21} = CL$ (Critical Low) $\underline{\underline{\Delta}}(V, \mu_{21}(V))$
$A_{22} = L$ (Low) $\underline{\underline{\Delta}}(V, \mu_{22}(V))$
$A_{23} = LM$ (Low-Medium) $\underline{\underline{\Delta}}(V, \mu_{23}(V))$
$A_{24} = M$ (Medium) $\underline{\underline{\Delta}}(V, \mu_{24}(V))$
$A_{25} = HM$ (High-Medium) $\underline{\underline{\Delta}}(V, \mu_{25}(V))$
$A_{26} = H$ (High) $\underline{\underline{\Delta}}(V, \mu_{26}(V))$

But, defined A_2 a universum of a subset of fuzzy-ligic sets the generalised kind it is possible to write a below-mentioned variant

$$\underline{\underline{\Delta}}(V, \mu_{2j}(V)) = \sum_{V \in A_2} \mu_{2j}(V_j)/V_j \ , \quad \forall V_j \in A_2$$

In Tables 1 and 2 the membership functions for the above-stated indistinct linguistic variables are presented.

Table 1

The membership functions for losses and voltage indicators

Description of variables	Critical Low	Low	Low-Medium	Medium	High-Medium	High
Indicators of capacity losses	< 0,15	0 – 0,25	0,12 – 0,5	0,32 – 0,75	0,5 – 1,0	>0,75
Voltages	< 0,92	0,9 – 0,94	0,91 – 0,96	0,95 – 1,0	0,98 – 1,05	1,02 – 1,1

Table 2

The membership functions of an indicator of correction preference (ICP) for condensers battery capacity in network knots

Variable	Critical Low	Low	Low-Medium	Medium	High-Medium	High
ICP(*i*)	<0,15	0 – 0,25	0,12 – 0,5	0,32 – 0,75	0,5 – 1,0	≥0,75

For of network knot definition with the revealed preference of connected condensers battery's capacity correction it is necessary to calculate the losses and voltage indicators for each knot, and then to present each of them as they own membership functions. Using the values of knot's voltages and losses indicators $L_{\Delta P}(\mathrm{i})$ the rules in form of the indistinct logic conclusions set matrix are formulated and generalized in Table. 3: CL-Critical Low; L- Low; LM- Lw - Medium; M-Medium; HM- High- Medium; H- High;

Table 3

Matrix of solutions for knot definition in which the condensers battery capacity correction is preferable

Parameters		**Voltage in knots**					
		CL	L	LM	M	HM	M
$L_{\Delta P}$ (i)	CL	L	L	L	L	L	L
	L	L	L	L	L	LM	LM
	LM	L	L	L	LM	LM	M
	M	L	L	L	LM	M	HM
	HM	L	L	LM	M	HM	H
	H	L	LM	LM	M	HM	H

IV. The Results of Modeling

The application of indistinct regulator algorithm is reviewed on an example of one of IEEE 30 BUS electric network. Investigated network contains 30 knots. With use of ETAP program complex for the given network depending on knots loading the optimum points (network knots) for condensers batteries placing and treir capacity rates are defined. The knots voltages, power factors, quantity and capacity of placed batteries, and also the total expenses necessary for condensers installation and operation are defined for three various loading modes. Calculations results are presented in Table 4-6.

Table 4

Calculation results of condensers batteries optimum distribution at 70 %loading

Knot name	U_{calc} в %	$\cos\varphi$	Information about BSC			Total cost (thousand $)
			кVAR/sect.	No of sect.	Total cap кVAR	
Bus1	100,0	0,858	1000	3	3000	122,4
Bus2	99,2	1,0	1000	3	3000	122,4
Bus3	99,2	0,997	1000	1	1000	41,6
Bus4	97,8	0,644	1000	3	3000	122,4
Bus5	97,6	0,999	1000	2	2000	82,0
Bus6	97,0	0,78	1000	3	3000	122,4
Bus7	97,3	1,0	1000	3	3000	122,4
Bus8	96,7	1,0	1000	3	3000	122,4
Bus9	96,9	0,998	1000	3	3000	122,4
Bus10	97,8	0,494	1000	3	3000	122,4
Bus11	96,5	1,0	1000	1	1000	41,6
Bus12	97,8	0,994	1000	3	3000	122,4
Bus13	96,3	0,997	1000	1	1000	41,6
Bus14	97,4	0,998	1000	3	3000	122,4
Bus15	96,1	0,968	1000	1	1000	41,6
Bus16	98,7	0,934	1000	2	2000	82,0
Bus17	101,7	0,90	1000	3	3000	122,4
Bus18	99,5	1,0	1000	2	2000	82,0
Bus19	98,1	0,992	1000	2	2000	82,0
In total:	-	-	-	45	45000	1840,8

Table 5

Calculation results of condensers batteries optimum distribution at 85 %loading

Knot name	U_{calc} в %	$\cos\varphi$	Information about BSC			Total cost (thousand $)
			кVAR/sections	No of sections	Total capacity кVAR	
Bus1	104,6	0,993	1000	3	3000	122,4
Bus2	104,3	0,941	1000	7	7000	284,0
Bus3	104,2	0,889				
Bus4	104,1	0,958	1000	4	4000	162,8
Bus5	103,5	0,999	1000	1	1000	41,6
Bus6	103,4	0,992	1000	6	6000	243,6
Bus7	103,7	1,0	1000	1	1000	41,6
Bus8	103,8	0,971	1000	9	9000	364,8
Bus9	103,9	1,0	1000	2	2000	82,0
Bus10	103,4	0,968	1000	1	1000	41,6
Bus11	103,2	0,999	1000	5	5000	203,2
Bus12	103,9	1,0	1000	1	1000	41,6
Bus13	101,9	0,914	1000	1	1000	41,6
Bus14	104,3	0,926	1000	5	5000	203,2
Bus15	102,5	0,999	1000	2	2000	82,0
Bus16	105,1	1,0	1000	11	11000	445,6
Bus17	105,5	0,91	1000	3	3000	122,4
Bus18	104,3	1,0	1000	2	2000	82,0
Bus19	104.7	0,935	1000	3	3000	122,4
In total:	-	-	-	67	67000	2728,4

Table 6

Calculation results of condensers batteries optimum distribution at 100% loading

Knot name	U_{calc} в %	$\cos\varphi$	Information about BSC			Total cost (thousand $)
			кVAR/sections	No of sections	Total capacity кVAR	
Bus1	104,7	0,904	1000	6	6000	243,6
Bus2	104,0	1,0	1000	3	3000	122,4
Bus3	104,1	0,958	1000	1	1000	41,6
Bus4	104,0	0,989	1000	6	6000	243,6
Bus5	103,8	0,984	1000	2	2000	82,0
Bus6	103,8	0,92	1000	10	10000	405,2
Bus7	103,9	1,0	1000	1	1000	41,6
Bus8	103,4	0,933	1000	6	6000	243,6
Bus9	103,6	1,0	1000	2	2000	82,0
Bus10	103,4	0,966	1000	1	1000	41,6
Bus11	103,5	0,948	1000	13	13000	526,4
Bus12	104,6	1,0	1000	1	1000	41,6
Bus13	102,0	0,836				
Bus14	105,8	0,925	1000	6	6000	243,6
Bus15	104,0	0,999	1000	3	3000	122,4
Bus16	106,2	1,0	1000	14	14000	566,8

Knot name	U_{calc} в %	cos φ	Information about BSC			Total cost (thousand $)
			кVAR/sections	No of sections	Total capacity кVAR	
Bus17	105,2	0,831				
Bus18	104,7	1,0	1000	14	14000	566,8
Bus19	104.4	0,958	1000	13	13000	526,4
In total:	-	-	-	102	102000	4141,2

As evident from Table 4, at 70 % of network loading on 19 knots the 45 sections of condensers batteries should run, at loading of 85 % on 18 knots – the 67 should run and at last, at 100 % to network loading on 17 knots – the 102 sections should run. Thus total capacities of sections of the condenser accordingly make 45,0 MVar, 67,0 MVar and 102,0 MVar, and total expenses 1840,8; 2728,4 and 4141,2 thousand US dollars. I.e. at loading reduction the optimum capacity of sections running concerning to initial mode has decreased for 34,0 %, and for the third mode on 56,0 %.

On Fig. 1 the profiles of voltage levels for bus 10 кV consumers of network district are shown at various modes. Apparently, in some knots the bus 10 кV voltage has decreased on 5 % average. It has been defined, that voltage reduction on consumer buses up to permissible level is connected not with condenser batteries placing, but with discrepancy of distributive network lines leghth.

The above calculations results analysis shows that depending on electric network modes for increasing of electric power distribution efficiency, the periodical correction, i.e. optimum condensers batteries capacity control in knots is necessary.

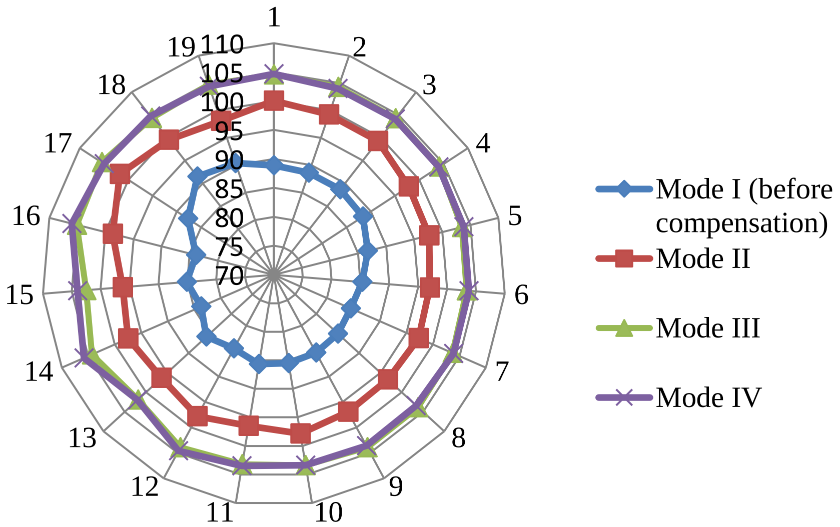

Fig. 1. Voltage profiles in 10 кV knots

V. Conclusion

1. For optimum electric network mode correction the model of reactive power and voltages indistinct control is developed allowing improving the knots voltage values and reducing power losses.
2. An algorithm realizing the regulator indistinct logic principle is developed for condensers batteries capacity operative correction in knots by criterion of a network's mode optimality.
3. On the base of researches provided on an example of 30-knots IEEE network scheme, are established that the operative condensers batteries capacity correction on the by means of the

indistinct logic regulator allows to keep optimum conditions for DG mode at current loading deviations on network buses.

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DANIELS' EPSILON-SEQUENCE AND MODELLING OF RELIABILITY OF UNIDIRECTIONAL FIBROUS COMPOSITE

Yu. Paramonov
•
Institute of Aeronautic, Riga Technical University,
Riga, LV 1019, Latvia
yuri.paramonov@gmail.com

Abstract

This is a new version of the Daniels' sequence (DS) for analyzing the relation of the static strength of unidirectional fiber composite (UFC) component with a fatigue life, a static and a residual tensile strength of UFC itself. The DS allows to explain the existence of fatigue strength, the residual strength and the dependence of static strength on the rate of a tensile test. It explains the structure of these processes of the UFC failure, but to get a numerical result some new hypotheses are needed. New version of the DS allows to simplify a problem of the regression analysis of test data and the prediction of the UFC parameter changes in case of changes of the parameters of its component.

Keywords: Daniels' sequence, strength, fatigue life, composite

1. Introduction

The concept of the DS, with reference to the description of the process of fatigue failure, was first introduced in [1]. Its successful application to describing the relation between the strength of LI and the fatigue life of UFCs is discussed in [2,3]. The more general definition, which can be applied to testing of the UFC specimens both in the fatigue and the tensile strength was considered in [4]. In these papers it was assumed that specimen failure takes place if DS item become more than some critical level which does not depend on the process of loading. But in paper [5] the definition of the Daniels' residual strength was introdused and decreasing of its value was considered as the reason of the failure: the failure of the UFC takes place if the residual strength become lower than the applied load.

Lately, composite materials are widely used in various engineering areas, in particular, aviation. Therefore, the strength and fatigue life of these materials are now a very urgent problem. The first scientific publication on this topic appears to be the Peirce's work [6], which gives an approximation formula for the average strength of the bundle of LIs forming the foundation of the unidirectional fibrous composite. The correctness of the normal approximation of the strength distribution law of the LI parallel system was proved by Daniels [7,8]. His result was refined by Smith [9] already with reference to the series-parallel system (SPS), which was earlier proposed for consideration in [10]. A lot of papers are devoted to the reliability of composite (detail review, for example, is given in [11]) but here we take into account only of the papers connected with the DS.

First, we consider the modified definition of the DS and some other functions, connected with the DS. We present an unified approach to the description of relation between the static strength of LIs (fibers or bundles) and the fatigue life, the fatigue strength and the ultimate static strength of UFC specimens made with the use of such bundles. The definition of the Daniels' epsilon-sequence (DeS) is introduced. The presentation is accompanied by a comparison of calculation results with experimental data.

The modelling of the composite behavior under static and fatigue loading can be made for different purpose. There are different types of structure and component of material, there are different types of definition of composites failure, there are different requirements to the details and precision of description of studied phenomenon. So we consider different versions of the models based on the DS definition.

2. Structure of UFC Specimens for Tension and Fatigue Tests

We consider a composite specimen for the static strength or the fatigue life tests as a series-parallel system: series system, every link of which is a parallel system, or more specifically, a bundle of n_C longitudinal items (fibers or bundles) immersed into a composite matrix (CM). We make an assumption that the CM is a composition of the matrix itself and all the layers with stacking different from the longitudinal one. Here we make an additional assumption also that only LIs carry the longitudinal load but the matrix only redistributes the loads after the failure of some LIs. The case when the failure of the matrix is considered as failure of the specimen also is studied in [12]. It is supposed that the composite is divided into n_L "links" of the same length, l_1. The total length of the composite specimen is equal to $L = n_L\, l_1$. In random K_L links, $1 \le K_L \le n_L$, some defects already exist or can appear, while, in ($n_L - K_L$) links the defects cannot appear. In general case the defected LIs have a different strength distribution function. In some special cases we assume that the strength and fatigue life of the defected LIs are equal to zero. In these cases in flawless links, there are n_C LIs without defects, but in defected links (the links in which there are defected LIs) there are only $N_C^+ = n_C - K_C$ LIs, where K_C ($1 \le K_C \le n_C - 1$) is the random number of missing LIs. The equality $K_C = n_C$ means the failure of both the link and the whole SPS.

Here we assume additionally that the failure of the system can occur only due to failures of links with defects. Inside this link there is some weak micro volume (WMV) the failure of which is a failure of link and the failure of the tested specimen also. In fact we consider a composite as a series WMVs. This paper is devoted really to the fatigue life and tensile strength of one WMV. The relation of the cumulative distribution functions (c.d.f.-s) of these random values r.v.s with the c.d.f.s of the same r.v.s of the whole composite is considered in [13].

3. Basic Daniel's sequence

So we consider the UFC as SPS and we study the reliability of this system. It was usually assumed in the classical theory of reliability that the failure of an element does not influence the operation of the other elements. In this paper, we consider the situation when the failure of an element increases the load on the still workable elements. To describe the loading process model, we use of the Daniels random sequence concept the deterministic version of which was introduced, as it was told already, in [1].

Let in WMV there is n_C LIs and $X_{(1)}, X_{(2)}, ..., X_{(n_C)}$ be the ordered values of the random strengths of them. Assuming the independence of $X_1, X_2, ..., X_{n_C}$, Daniels showed in [7,8] that the random variable

$$R_D = \max(X_{(k)}(n_C - k + 1)/n_C : 1 \le k \le n_C) \tag{1}$$

has an asymptotically normal distribution with the average and standard deviation

$$\mu_D = \max x(1 - F_X(x)) = x^*(1 - F_X(x^*)), \quad \sigma_D = (\mu_D x^* F_X(x^*)/n_C)^{1/2}, \tag{2}$$

which are determined by the c.d.f. $F_X(x)$ of corresponding positive random variable X.

Let us mention three salient features of Daniels' model:

(1) a continuous increase in the external load is assumed, but its rate is not taken into account;

(2) it is assumed that destructions of one by one fibers with a successively growing strength are accumulated;

(3) The strength of a fiber bundle corresponds to the the instant when the load becomes higher than the load-carrying ability of the specimen tested. There after, the destruction, i.e., rupture, of the bundle is presumed.

By "unwrapping" this model in time for specific sample $x_{1:n_C} = (x_1, ..., x_{n_C})$ which is a realization of the random vector $X_{1:n_C} = (X_{(1)}, ... X_{(n_C)})$ and assuming that the process of loading is described by sequence $s^+_{0:\infty} = \{s^+_0, s^+_1, s^+_2, ...\}$, we obtain a sequence of local stresses $s_{0:\infty} = \{s_0, s_1, s_2, ...\}$ (in the link where the damage develops) described by the equation

$$s_{i+1} = s_i^+ / (1 - k(s_i)/n_c), \quad i = 0, 1, 2, ..., \tag{3}$$

where $k(s_i)$ is the number of LIs, the strength of which is lower or equal to s_i, $k(s_i)/n_C$ can be considered as an estimation of the c.d.f. $F_{X,n_C}(s_i)$, obtained by the use of the sample $x_{1:n_C}$.

This sequence can be used for describing both: 1) the fatigue test, if $s_i^+ = s^+$, $i = 1, 2, 3, ...$, (here we consider the fatigue test only for describing the fatigue curve and s^+ is some constant; for example, s^+ is the maximum stress of the cyclic load; in general case $s^+_{0:\infty} = \{s^+_0, s^+_1, s^+_2, ...\}$ can be defined by specific program) and 2) the tensile static test, if items of sequence $s^+_{0:\infty}$ increase up to infinity. The sequence of local stresses is called Daniels' sequence for a constant external load (DS_CL) for the loading process of the first case and Daniels' sequence for an increasing external load (DS_EL) in the loading process of the second case. In what follows, the concept of DS_CL will be used to describe fatigue tests, while the DS_EL is a tension tests. The realization of these random processes is described by the pair $(s^+_{0:\infty}, x_{1:n_C})$.

For specific pair $(s^+_{0:\infty}, x_{1:n_C})$ let us define the following functions:

the K-Daniels' function

$$r_{D,K}(k) = x_{(K)}(n_C - k + 1)/n_C, \tag{4}$$

the S-Daniels' function

$$r_{D,s}(s) = s(n_C - k(s) + 1)/n_C, \tag{5}$$

the Daniels' residual function

$$r_{D\max}(x) = \max_{s > x} r_{D,s}(s). \tag{6}$$

The transition from s_i to s_{i+1} we call a step of DS. It corresponds to the destruction of all LIs with the strength in the interval $(s_i, s_{i+1}]$. So the value

$$n_D = 1 + \max(i : r_{D,S}(s_i) > s^+, \ s_i \in \{s^+_0, s_1, s_2 ...\}) \tag{7}$$

we call the *DS-fatigue life* . It is *a number of DS steps up to failure: DS residual strength become lower than external load or some specific value.*

And let us define the following values:

the *DS-ultimate tensile strength* by equation

$$r_{D,u} = a^+_{n_{D.S}} \tag{8}$$

where $n_{D,S} = 1 + \max(i : r_{D,S}(s_i) > s_i^+,\ s_i \in \{s_0^+, s_1^+, s_2^+ ...\})$;

the DS-fatigue strength

$$s_D = \max(s^+ : s^+ \in S^+_{N_D=\infty}) \tag{9}$$

where $S^+_{n_D=\infty}$ is a set of s^+ for which there is solution of equation

$$x = s^+ / (1 - k(x) / n_c)$$

In this case there is such i^*, that $s_{i^*+1} = s_{i^*}$. Then $R_{D,n_C}(s_{i^*+1}) = R_{D,n_C}(s_{i^*})$. The changes of processes of s_i and $R_{D,n_C}(s_i)$ will be stopped, some LIs never will be destroyed and n_D will be equal to infinity.

In following we denote random DS by $S_{0:\infty} = \{s_0, S_1, S_2, ...\}$ and use the symbol R instead of r in notations $R_{D,K}(k)$, $R_{D,S}(s)$ … for random versions of corresponding functions if instead of sample $x_{1:n_C}$ we consider random vector $X_{1:n_C}$. For example, random fatigue life is defined by

$$N_D = 1 + \max(i : R_{D,S}(S_i) > s^+,\ s_i \in \{s_0^+, S_1, S_2 ...\}) \tag{7a}$$

The random processes DS_CL and DS_EL and corresponding functions and values are completely determined by the triple $(s^+_{0:\infty}, F_X(x), n_C)$.

4. Modeling of fatigue tests

4.1. Daniels' epsilon-sequence

In [12] it is shown that basic DS is very similar to a S-type changes of some physical parameter of composite during fatigue loading. The use of DS-approuch allows to explain the existence of fatigue strength. So it explains the structure of a fatigue phenomenon but numerical results are very pure: the values of N_D are very small, the values of S_D are too large. So there is necessity to make some additional „patches"-assumptions in order to make the considered mathematics useful for numerical description of the fatigue phenomenon.

1) The accumulation of the damages takes a place only in local WMV in which there is some stress concentration which is different in the fatigue and in the tension tests. In simplest case it can be assumed that real **local** external stress is equal to $k_c s^+$ in the DS_CL and $k_T s_i^+$, $i = 1, 2, ...$, in the DS_EL.

2) The c.d.f. of the local tensile strength does not coincide with the c.d.f. of the strength of LI in a static strength test: the "size" of WMV and adjacent LIs have a specific influence on the local strength. In the following the specific c.d.f. $F_{X_L}(x)$ will be used in the numerical calculation. In the simplest case it can be assumed that $F_{X_L}(x) = F_X(k_F x)$, where k_F is some constant or (in general case) some random variable, or some another hypothesis can be used also.

3) The failure of the LIs does not take place in only one cross section (a plain) of a tested composite specimen. Here we suppose that only some part, $r_F(s)$, of the failure of the LIs is in the same cross section. This part changes in time growing up to unit. In the following numerical example the "mean" value of this function, $r_F(s)$, will be used.

4) During one step of DS, when the stress s_i grows up to s_{i+1} for the failure all LEs with the

strength $s_i < X_L \le s_{i+1}$ some energy is needed the value of which may be is not enough to cause the failure of all these LEs in one fatigue cycle. In simplest case it can be assumed that only part, $\varepsilon_D(s^+)$, of these LEs will be destroyed in one DS-step.

So here for modeling the fatigue test we use the equation

$$s_{i+1} = (k_C s^+ / (1 - rk(s_i)/n_c) - s^+)\varepsilon_D(s^+) + s_i), \quad i = 0,1,2,... \tag{10}$$

instead of equation (3). The sequence (10) we call the Daniels' epsilon-sequence (DeS). The function $\varepsilon_D(s^+)$ will be discussed some later.

The modified Daniels' residual function

$$r_{D\max}(x) = \max_{s>x} r_{D,s}(s)/k_T \quad . \tag{6a}$$

From the condition of existence of infinite life : $s_{i+1} = s_i$,- now we get the following definition of DeS fatigue strength $s_D = (1/k_C)\max x(1 - rk(x)/n_C)$.

5) Two types of connections between the number of steps of the DeS and number of fatigue cycles was considered in our previous works [1, 12, 13].

If we are interested only to know the mean fatigue curve then the deterministic version of SeD can be used for corresponding regression analysis.

When the number of steps, *i*, is small, the DeS items, $s_0, s_1, s_2,...$, and the value of N_D are easily calculated using recurrent formula. For the large *i*, it is possible to assume that *i* is the continuous variable. The value ($s_{i+1} - s_i$) can be considered as derivative, ds/dn, and for $n_C = \infty$ for the initial value s^+ it is possible to obtain an approximate connection of the number of steps needed to reach the critical value, s^*. For $\varepsilon_D = 1$ it can be found, that approximately:

$$N(s,s^*) = \{\arctan((s^* - s_h)/(a_2/a_0)^{1/2}) - \arctan((s_0 - s_h)/(a_2/a_0)^{1/2}))\}/(a_2 a_0)^{1/2},$$

where a_0 and a_2 are the values of the function $h(x) = s/(1-F(x)) - x$ and its second derivative at $x = s_h$, where s_h is the value of the argument x at which the first derivative is equal to zero (for small x the function $h(x)$ decreases and then goes up; therefore, its expansion into a Taylor's series is used at the minimum point, while disregarding the members with an order of more than 2.). Approximately $N_D(s) = N(s,\infty)$. In [1], some version of this approach was successfully used for the regression analysis of the fatigue data set provided in [14].

The other way it is the use the theory of absorbing Markov chains (MCh) and semi-Markov process (sMP) theory.

The DeS can be limited by of conditions of the two types: 1) some item of DeS become more than some critical value s^* which defines the value of N_D, $N_D < \infty$; 2) there is such i^* that $s_{i^*+1} = s_{i^*}$, $N_D = \infty$.The items of the limited DeS can be considered as the states of some absorbing MCh in which there are two types of corresponding absorbing states. This approach (with $\varepsilon_D = 1$, critical value s^* do not depend on s^+) was successfully used for regression analysis of the data of the fatigue tests of carbon-fibers composites for describing of the fatigue curve. The **c.d.f. of random fatigue strength** was obtained also [3,4]. But for analysis of the data described in [11] the use of the theory of semi-Markov process (sMP) with rewords (as some generalization of MCh) appears to be more appropriate. Details of this analysis with the use of the definition of DeSs are given in next section.

4.2. Absorbing sMP

Every fatigue test of several specimens for description of fatigue curve can be releted with the set of possible limited DeSs : $\{(s_0, s_1,...)_j,\ j = 1,2,..\}$, where $(s_0, s_1,...)$ is a realization of a random process $(s_0, S_1, S_2,...)_j$ for specific load level s^+_{0j}. The items of limited DeS, $s_0, s_1,...$, we consider as the states of an absorbing sMP. The second type of the sMP must be studied in case when in a program fatigue loading there are load levels corresponding to $N_D = \infty$. Here we consider only loading with the probability of the event $N_D < \infty$ which is very close to the unit.

Different types of matrix of transition probabilities for specific set of states $\{s_0, s_1, ..., s_{N_D}\}$ is considered in [12]. Here we consider the simplest case. We suppose that that only transition to the same and to the next state is possible and these probabilities are functions of a sample $(x_1, x_2, ..., x_{n_C})$ which is a realization of the random vector $(X_1, X_2, ..., X_{n_C})$. We define

$$p_{1,2} = 1 - k(s^+)/n_C,\ \ p_{1,1} = 1 - p_{1,2},$$

$$p_{i,i+1} = ((k(s_{i+1}) - k(s_i))/n_C)/(1 - k(s_i)/n_C),\ p_{i,i} = 1 - p_{i,i+1},\ \ i = 0,1,...,n_D - 1,\ \ p_{n_D,n_D} = p_{n_D,n_D}. \quad (11)$$

All the other probabilities are equal to 0.

As it was told already, the transition (s_i, s_{i+1}) corresponds to the failure of some part of the LIs with the strength in interval $(s_i, s_{i+1}]$. We assume that time of the transition is proportional to $(k(s_{i+1})/n_c - k(s_i)/n_c)/h(s_i)$, where $h(s_i)$ is proportional to the an area of a hysteresis loop of one fatigue cycle. Let us approximate a strain-stress curve of a composite by an equation

$$\varepsilon(s) = s/E + (s/a)^b$$

where E is the elastic modulus, a, b are some constants, $a >> s$, $b > 1$.

It can be assumed, that the area of hysteresis is proportional to the area between the straight line $\varepsilon_1(s) = \varepsilon(s_{\max})s/s_{\max}$ where $s_{\max}$ is the maximal value of fatigue cycle and the curve $\varepsilon(s)$ that is proportional to the integral $\int_0^{s_{\max}} (\varepsilon_1(s) - \varepsilon(s))ds$ which is proportional to the value $(s_{\max})^{b+1}$. So it can be assumed that the time of transition has the exponential distribution with the parameter

$$\lambda_i = (s^+/R_U)^m n_C/(k(s_{i+1}) - k(s_i))$$

where R_U, m are some model parameters.

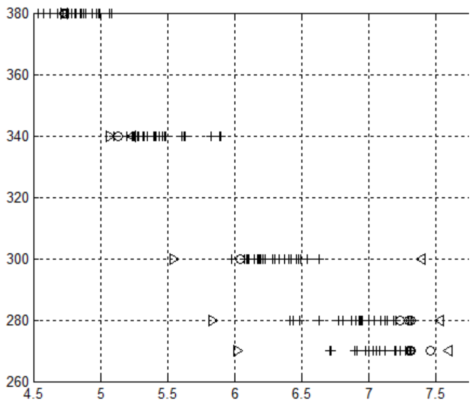
The random time to absorption is equal to the sum of corresponding random variables.

5. Numerical examples

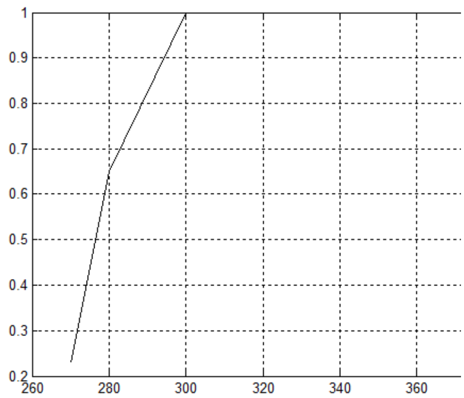
Here we consider the examples of processing of some date in order to show the use of DeS for modeling fatigue strength, residual strength, c.d.f. of static strength of UFC.

Example 1. The estimation of c.d.f of the fatigue strength. The dataset employed in this analysis was kindly given to the authors by W.Q. Meeker, who already studied them [15] and provides the following description of the data: "the data come from 125 specimens analysed in four-point out-of-plane bending tests of carbon eight-harness-satin/epoxy laminate. Both fiber fracture and final specimen fracture occurred simultaneously. Thus, fatigue life is defined as the number of cycles until specimen fracture. The dataset includes 10 right-censored observations (referred to as "runouts" in the fatigue literature)".

Processing of the data was made inaccordance with the definition (10) but not all parameters was used. In Fig.1a the fatigue life test data (+) (in logarithm scale) and result of Monte Carlo calculations (mean (o) and extreme values (►,◄)) and in Fig. 1b the c.d.f. of fatigue strength are shown. The following model parameters were used : $\theta_0 = 6.44,\ \ \theta_1 = 0.2$ for lognormal distribution of fatigue life of LIs, $n_C = 100,\ \ k_C = 1.58,\ \ \varepsilon = 0.3,\ \ r = 1$.



a



b

Fig. 1. Fatigue curve (a) and c.d.f. of fatigue strength (b).

It can be noted that in accordance with the model calculation the random fatigue strength is in the interval 270-300 MPa but "The dataset includes 10 right-censored observations (referred to as "runouts" in the fatigue literature)". More exactly: there are 2 runouts for the load level 280 MPa and there are 8 runouts for the load level 270 MPa.

Example 2. The estimation of the residual strength. Within the framework of the European research project „OPTIMAT Blades" a series of tests was performed on a glass/epoxy material used in manufacturing of wind turbine rotor blades. Here we consider the processing of the result of the fatigue test the description of which is given in [11]:

1) „To defermine the material response under cycling loading, 15 coupons were tested under sinusoidal loading (at a stress ratio equal to 0.1)…. at stress levels… SL1=48.50 MPa, SL2=63.60 MPa and SL3=78.31 MPa" .

2) „ A total of 74 residual strength tests were performed … The three stress levels chosen … for the same ones used for the S-N curve determination exept the high stress level wich was truncated to $5*10^3$ cycles. 25,25 and 24 specimens at each stress level were tested. These in turn, where divided into three groups of approximately 8 coupons, each one cycled for a specific life fraction; 20%, 50% and 80% of the nominal life time".

The considered fatigue models were applied to experimental data set available in [16,17,18].

In Fig.2 the result of calculation of the DeSs (Δ), the residual strength (∇) and fatigue load level(*) (a1, b1 and c1) as the functions of DeS step number, the same and the test of the residual strength (o) as a functions of fatigue cycle number (a2,b2 and c2) are shown. The values of the DeS functions are limited by 200 MPa in order to show all functions in the same scale. In Fig. 3 the fatigue lives are shown: test fatigue lives (+), calculated mean (*) and the extreem values (►,◄) corresponding to two standard deviation from mean value. The test residual strength for preliminary load levels 48.5 MPa, 63.6 MPa and 78.3 MPa for 20%, 50% and 80% of the corresponding nominal lifetime is shown also.

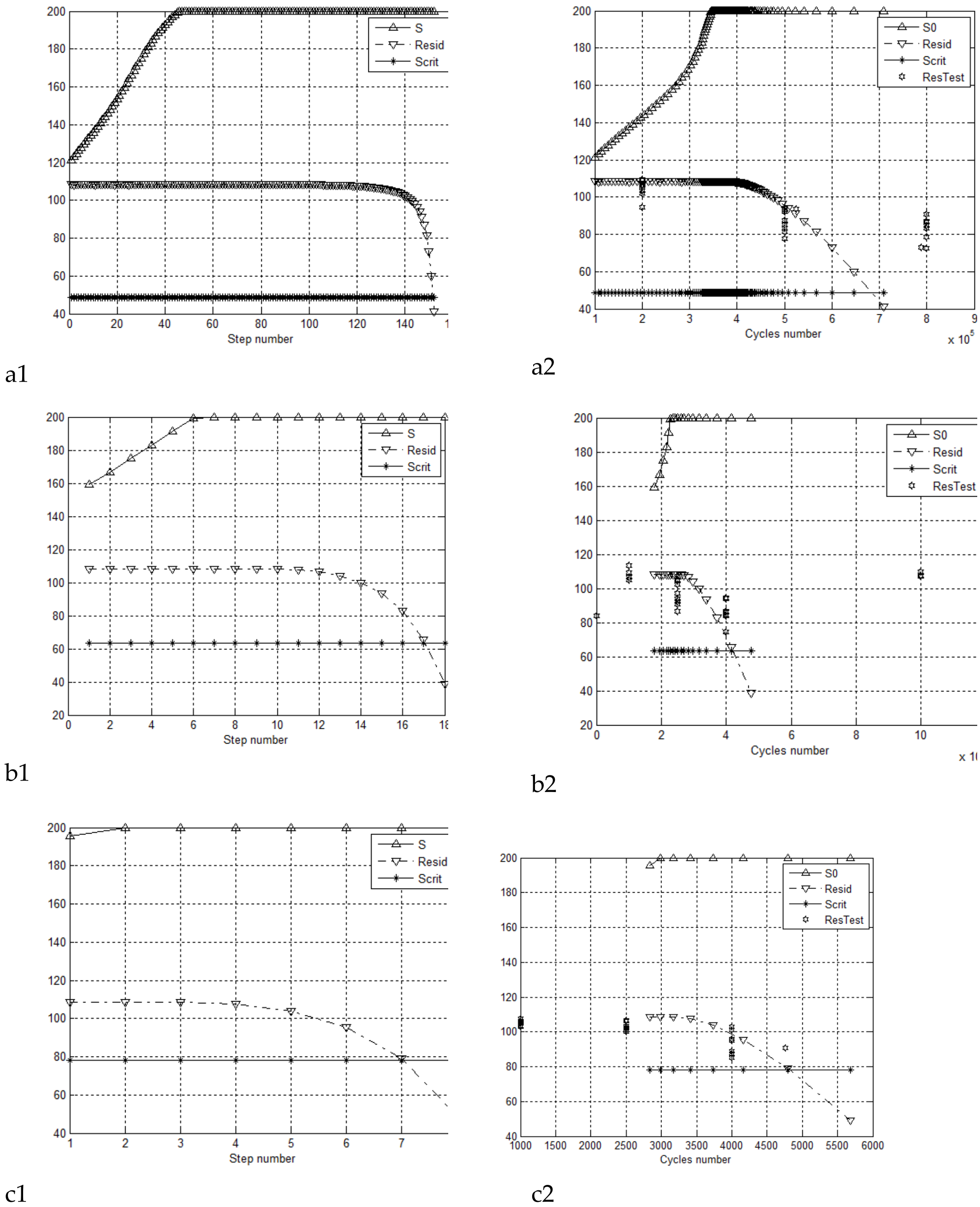


Fig. 2. DeSs (Δ), Daniels' resid. functions (∇), test resid. strength(o) and fatigue load (*).

For these calculations it was assumed that there are some LIs with some defects (W) and there are some LIs without defects (U). The c.d.f of the loacal strength is defined by equation

$$F_{X_L}(x) = (1 - p_U) F_{X_{LW}}(x) + p_U F_{X_{LU}}(x),$$

where

$F_{X_{LW}}(x) = 1 - \exp(-\exp((\ln(x) - \theta_{0W})/\theta_{1W}))$, $F_{X_{LU}}(x) = 1 - \exp(-\exp((\ln(x) - \theta_{0U})/\theta_{1U}))$.

The following parameters was used for calculation: $n_C = 30$, $\theta_{0U} = \log R_U = 5{,}704$ (where $R_U = 300$), $\theta_{1U} = 0{,}175$, $\theta_{0W} = \theta_{0U} - \log k_F = 4{,}962$, $k_F = 2{,}1$, $\theta_{1W} = 0.175$, $k_C = 2{,}5$, $k_T = 1{,}1$, $p_U = 0.6$, $r_F = 0{,}93$, $m = 10$. $\varepsilon = 0.1$.

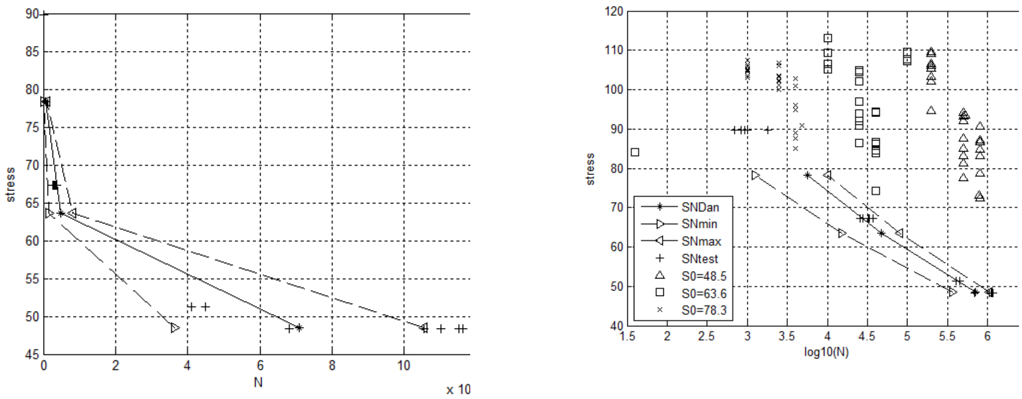


Fig.3. The test result (+), calculated mean (*) and the extreem values (►,◄) corresponding to two standard deviation from mean value. The test residual strength for preliminary load levels 48.5 MPa, 63.6 MPa and 78.3 MPa for 20%, 50% and 80% of the corresponding nominal life time.

Example 3. The modelling of tensile tests. In [14], the results of tests for the tensile strength of bundles of carbon fibers and of 14 specimens made with the use of these bundles are presented.

The data already have been studied in [4] but here for similar analysis we use the residual Daniels' function. The results of verification of the normal, lognormal and Weibull distribution laws of their strength are presented in [4] also. Usually, the verification of such hypotheses is performed visually according to the results of fitting of experimental data on the corresponding probability paper. But in [4] such a verification was carried out by using numerical ρ -criteria [12] with calculation of their efficiency. It allowes us to estimate the degree of confidence to the final conclusion objectively. As follows from results of the verification there are grounds to prefer the hypothesis about the lognormal distribution of the strength of CFRP bundles with c.d.f. $F_X(x,\theta_0,\theta_1)=\Phi((\log(x)-\theta_0)/\theta_1)$, $\theta_{0X}=6.44$, $\theta_{1X}=0.1816$.

Using this c.d.f. and Monte Carlo method for $n_C=100$ the calculation 10 realizations of the functions $r_{D,S}(s)$ and $r_{D,K}(k)$ was made. These realizations are shown in Fig.4.

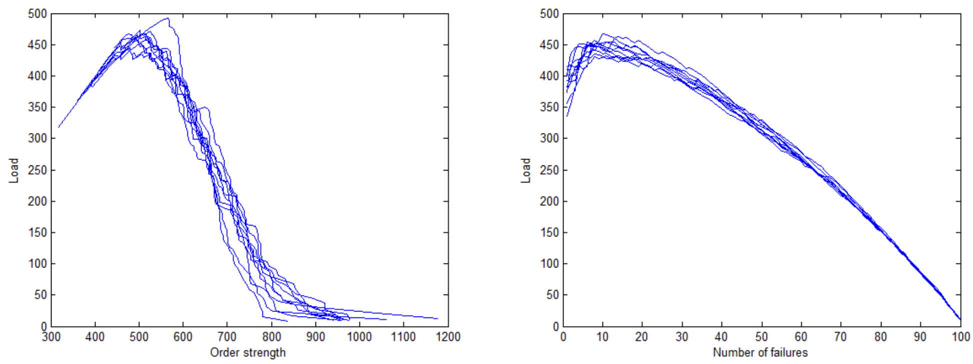


Fig. 4. Modelling of 10 realizations of the functions $R_{D,s}(s)$ and $R_{D,k}(k)$.

To get the hystogram of the value of the random variable, $K_{\max}$, corresponding to the maximum of the function $R_{D,k}(k)$, Fig. 5, the calculations of 100 realizations of this function was made.

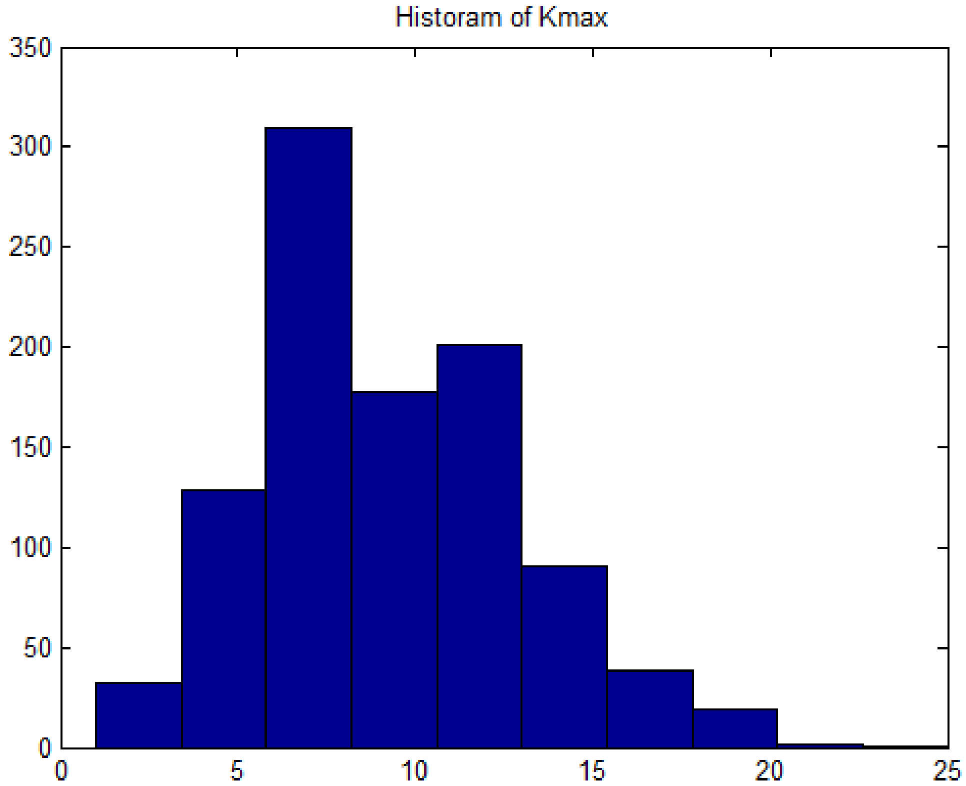


Fig.5. Hystogramm of $K_{\max}$.

It is important to notice that the value of $K_{\max}$ corresponding to the maximum of the function $R_{D,k}(k)$, is equal to small part of n_C. Mean value and standard deviation of $K_{\max}$ are equal to 9.3 and 3.8 correspondingly . After failure of this small part of LIs the „domino effect" and failure of WMV and whole composite take part.

For modelling the DS it was assumed that local strength X_L in framework of composite structure is equal to X / K_F where $\log(K_F)$ is a random variable with normal distribution with the mean value and standard deviation of which is equal to $\log(1.25) = 0.223$ and 0.07 correspondingly.

Result of modeling of the DS (-)

$s_{i+1} = k_C s_i^+ / (1 - k(s_i) / n_C),\ \ i = 0,1,2,...$

for $n_C = 100$, the local stress concentration coefficient $k_T = 1.5$ is shown in Fig. 6. The residual strength (--) and process of loading (-.-) are shown in the same figure also. The sequence of growth in the nominal load is defined by

$s_i^+ = s_0^+ + (i-1)k_s E(X_L)$, $i = 1,2,...$

where $E(X_L)$ is the expected value of X_L, $k_S = 0.01$ is the conditional relative rate of external loading (the details of the choice of k_S is discussed in [4]).

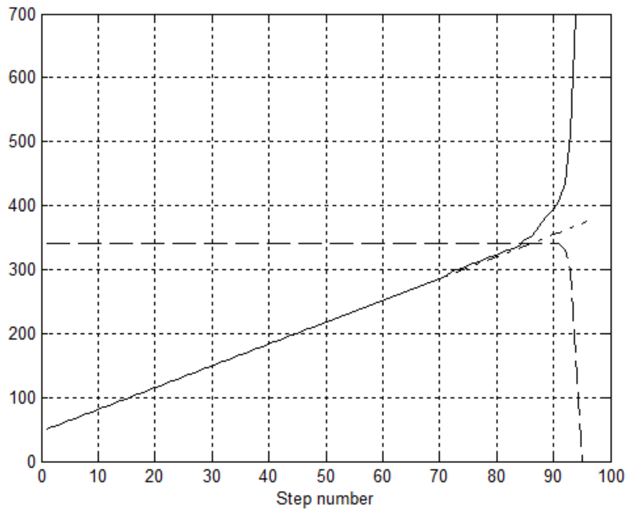


Fig. 6. DS (-) , the residual strength (--) and process of loading (-.-) for tensile strength test.

We should note that the local stress s_i is proportional to the local relative strain ε . Therefore, the relation between s_i^+ and s_i must be similar to the stress-strain curve (see Fig. 7).

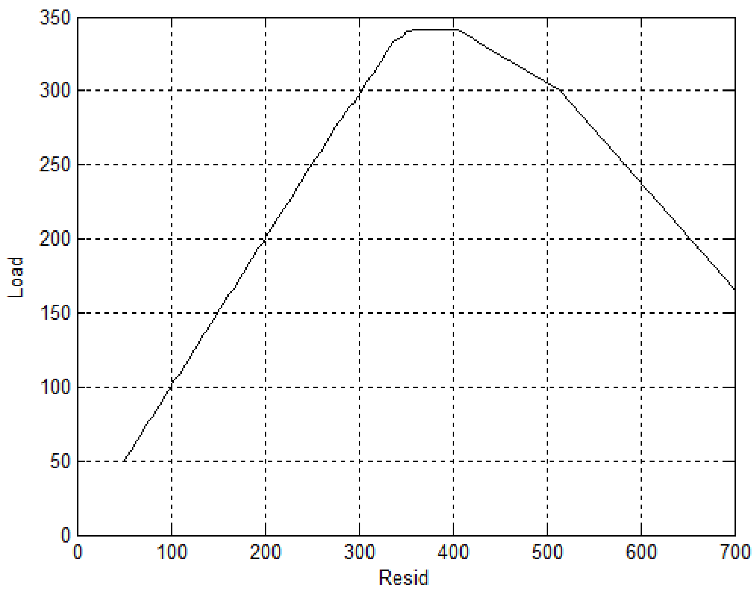


Fig. 7. The curve"the load versus residual strength" is similar to stress-strain curve in static strenth test

The repeated calculations by the use of the Monte Carlo method of specimen strength, S_D^+, enable us to find the average values of order statistics in a sample of the same size as in the test and to compare the predictions with real data. In [14], the test data of 14 specimens made with the use of mentioned carbon-fiber bundles are presented. The mean values (-) calculated according to the studied model and test order statistics (-.-) versus test order statistics are shown in Fig. 8. The problem on approximation of experimental data by the model calculations can be regarded as the problem of nonlinear regression. We see that the estimates of the model parameters used in corresponding calculations give a reasonably good fitting of test data.

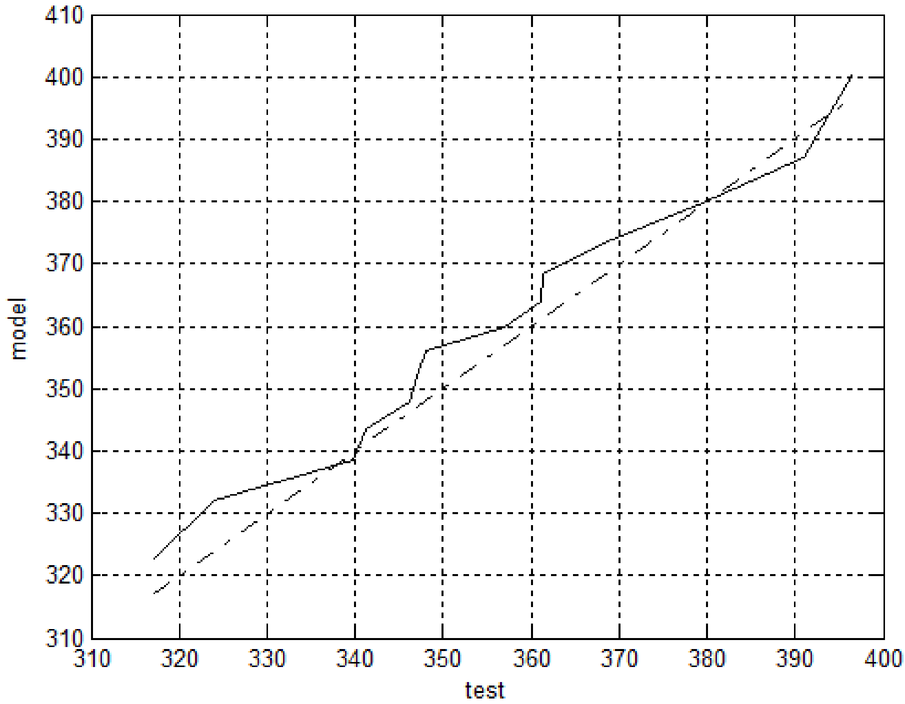


Fig. 8. The calculated mean values (-) and test order statistics (-.-) versus test order statistics.

Conclusions

The analysis of the examples of using of the developed DS approach to the description and modelling of the fatigue and the static strength tests of UFC shows that the corresponding models allows:

- to explain some specific features of these tests : 1) the existence of the fatigue strength and the possibility to have the infinite fatigue life , 2) the existence of a long period of a constancy of the residual strength;

- interpretation of the parameters of the studied models as parameters of local static strength (this is the main difference of these models from the others models).

The difference of the parameters of the c.d.f. of the local strength of LIs in framework of UFC and ones of isolated single LIs do not allows to make prediction of the parameters of UFC using parameters of LI. But these models can be used for regression analysis of test data. The search for the parameters of nonlinear regression for the above-described models is a difficult task, but we think that, in due course, the structure of models suggested will be of interest not only for graduation theses of students, but also for engineering applications, in particular, for predicting the variations in the parameters of the strength and the fatigue life of UFCs upon changes in the parameters of their components. The mathematics of the DeS allows to make the modeling and the prediction for any loading sequence, $s^+_{0:\infty} = \{s^+_0, s^+_1, s^+_2, ...\}$, in **any program of the fatigue and the tensile tests** but an **accuracy** of this prediction needs to be studied.

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CRITERIA THE ESTIMATION EXPEDIENCY OF CLASSIFICATION INFORMATION ON RELIABILITY OF THE EQUIPMENT AND DEVICES EPS

Farhadzadeh E.M., Muradaliyev A.Z., Abdullayeva S.A.

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Azerbaijan Scientific-Research and Design-Prospecting Institute of Energetic
AZ1012, Ave. H.Zardabi-94, e-mail:fem1939@rambler.ru

Abstract

Analysis of statistical data about refusals, maintenance service and repair of objects of electric systems, is an indispensable condition of a quantitative estimation of reliability of their work. The analysis assumes classification of data on a sort of attributes and their versions. Thus, the basic question there is a revealing significant version of attributes. The importance of an attribute established based on statistical criteria of check of hypotheses. The criteria based on boundary fiducially estimation of intervals are offered.

INTRODUCTION

Necessity and methodology of the analysis of reliability of work of the equipment and devices (objects) of electro power systems (EPS) according to about their deterioration, refusals, tests and restoration of a technical condition are regulated by corresponding Normative materials. Prominent feature of these data is dependence on the big number of attributes and their versions. This dependence carries data to a class multivariate. Multivariate character of data not considered at a quantitative estimation of parameters of reliability (PR) objects EPS.

1. Classification parameters of reliability of objects EPS

The variety of properties of reliability and describing these properties PR causes necessity of grouping PR on a way of calculation. As an example in table 1 formulas of calculation of the some PR power units (PU) TPS are resulted. Following designations are accepted: $\mathbf{n}_{б}$ – number of power units; $\mathbf{m}_{от,i}$ – number of switching-off i-th PU; $\mathbf{m}_{от,\Sigma}$ - number of switching-off of all PU; $\tau_{p,i,}$ - j- th realization of duration of the worker (p) conditions i-th PU; $\mathbf{m}_{ав,i}$ – number of emergency switching-off i- th PU; $\mathbf{m}_{ав,\Sigma}$ – number of emergency switching-off of all PU; $\tau_{ав,i,j}$ - j-th duration of emergency switching-off i-th PU; $\mathbf{m}_{н.п.,i}$ – number of unsuccessful start-up i-th PU; $\mathbf{m}_{п\ i}$ – number of start-up i- th PU; $\tau_{нп,\ i,j}$ - duration j- th idle time after unsuccessful start-up i-th PU; $\mathbf{m}_{x.p.,i}$ – number of switching-off in a cold reserve i-th PU; $\mathbf{m}_{x.p.ав,i}$ – number of translations PU from a condition of cold reserve in an emergency condition; $\tau_{п.р.,i}$ – duration of idle time in scheduled repair i-th PU.

Let's distinguish PR, calculated as:

- an average arithmetic random variables. According to tabl.1 is $T_{р}^{*}$ and $T_{ав}^{*}$;

- relative size of total duration of a condition. As an example parameters $P^*_{ав}$, $P^*_{р}$, $P^*_{х.р.}$ serve (see tabl.1.)
- probability of occurrence of event. In tabl.1 it- $Q^*_{ав}$, $Q^*_{п}$, and $Q^*_{х.р.}$
- specific number of events. In table 1 it is specific number of emergency switching-off PU ($h^*_{ав}$) and its versions (automatic switching-off PU, emergency switching-off PU by the personnel, manually, switching-off PU under the emergency application).
- function noted above individual parameters. In table 1 the factor of readiness $K^*_{г}$ and factor of technical use $K^*_{т.и.}$ are resulted.

Table 1. Sample of parameters of reliability of power units 300 MWt TES

Condition	Parameters of reliability	Formula of calculation
	Individual parameters	
Finding PU in work	Relative size of total duration of work PU within year, % Average duration of continuous work, cl	$P^*_p = \sum_{i=1}^{n_б}\sum_{j=1}^{m_{om,i}} \tau_{p,i,j} / n_б \cdot 8760$ $T^*_p = \sum_{i=1}^{n_б}\sum_{j=1}^{m_{om,i}} \tau_{p.i,j} / \sum_{i=1}^{n_б} m_{om,i}$
Emergency остановов PU	Specific number of emergency switching-off (on 1000 cl. works) Specific number of switching-off under the emergency application; Specific number of automatic switching-off at system failures Relative number of emergency switching-off	$h^*_{ав} = 10^3 \sum_{i=1}^{n_б} m_{ав,i} / \sum_{i=1}^{n_б}\sum_{j=1}^{m_{OT}} \tau_{p,i,j}$ $h^*_{а.з.} = 10^3 \sum_{i=1}^{n_б} m_{аз,i} / \sum_{i=1}^{n_б}\sum_{j=1}^{m_{OT}} \tau_{p,i,j}$ $h^*_{а.о} = 10^3 \sum_{i=1}^{n_б} m_{а.о.,i} / \sum_{i=1}^{n_б}\sum_{j=1}^{m_{OT}} \tau_{p,i,j}$ $Q^*_{ав} = \sum_{i=1}^{n_б} m_{ав.,i} / \sum_{i=1}^{n_б} m_{om,i}$
Finding in emergency repair PU	Average idle time in emergency repair, cl. Relative size of total duration of idle time in emergency repair, cl.	$T^*_{ав} = \sum_{i=1}^{n_б}\sum_{j=1}^{m_{ав,i}} \tau_{ав.i,j} / \sum_{i=1}^{n_б} m_{ав,i}$ $P^*_{ав} = \sum_{i=1}^{n_б}\sum_{j=1}^{m_{ав,i}} \tau_{ав.i,j} / n_б \cdot 8760$
Preservation of equipment PU	Probability of refusal PU at start-up Probability of translation in a condition emergency repair from a condition of a cold reserve	$Q^*_{НП} = \sum_{i=1}^{n_б} m_{н.п.,i} / \sum_{i=1}^{n_б} m_{п,i}$ $Q^*_{рез,о.р.} = \sum_{i=1}^{n_б} m_{рез,о.р.} / \sum_{i=1}^{n_б} m_{рез,i}$
Cold reserve	Relative size of total duration of a finding in a cold reserve	$P^*_{рез} = \sum_{i=1}^{n_б}\sum_{j=1}^{m_{рез,i}} \tau_{рез.i,j} / n_б \cdot 8760$
Scheduled repair	Average duration of scheduled repair	$T^*_{п.р} = \sum_{i=1}^{n_б}\sum_{j=1}^{m_{п.р,i}} \tau_{п.р.i,j} / \sum_{i=1}^{n_б} m_{п.р.,i}$
Complex parameters		
Factor of readiness		$K^*_{Г} = P^*_p / (P^*_p + P^*_{ав})$
Factor technical uses		$K^*_{Т.И}. = (P^*_p + P^*_{ав} + P^*_{рез})$

2. Criteria of an estimation of character of a divergence $M^*_{\Sigma,Э}(\tau_{ав})$ and $M^*_{V,Э}(\tau_{ав})$.

Following criteria recommended:

Criterion K1. It is based on comparison of probability of empirical value of absolute size of a relative deviation $M^*_{V,Э}(\tau_{ав})$ from $M^*_{\Sigma,Э}(\tau_{ав})$ with critical (c) value of this probability equal α_k. Calculations spent in following sequence:

- the absolute size of a relative deviation $M^*_{V,Э}(\tau_{ав})$ from $M^*_{\Sigma,Э}(\tau_{ав})$ under the formula defined:

$$\delta M^*_{V,Э}(\tau_{ав}) = \frac{[(M^*_{V,Э}(\tau_{ав}) - M^*_{\Sigma,Э}(\tau_{ав})]}{M^*_{\Sigma.Э}(\tau_{ав})} \tag{1}$$

- tabulated data of [1] discrete values of statistical function fiducially distributions (s.f.f.d.) $F^*[\delta M^{**}_V(\tau_{ав})]$ the required probability is defined $\alpha_{v=1-}F^*[\delta M^{**}_V(\tau_{ав})]$. Here $\delta M^{**}_V(\tau_{ав})$ - counted on the basis of imitating modeling (**) possible values of absolute size of a relative deviation $M^*_{V,Э}(\tau_{ав})$ from $M^*_{\Sigma,Э}(\tau_{ав})$.

Note. *Фидуциальные distributions for the first time have been entered into R.E.Fisherom's consideration in 30th years of the last century. These distributions characterize likelihood distributions of possible values of estimations of parameters of distribution of random variables. It specified that it is necessary to trust only statistical conclusions which basis is empirical data. A.N.Kolmogorov in 1942-year note, that fiducially probabilities and intervals are the most suitable at small number of supervision [2]. Transition from confidential intervals to fiducially is expedient at multivariate character of statistical data. Application to multivariate data of the mathematical device of functions of distribution of random variables of casual sample of general set can lead to erroneous recommendations. As it has noted been above the statistical data describing reliability of objects EPS concern to a class multivariate, and samples of these data on set version of an attribute (VA) – are not casual.*

- if α_v does not exceed α_k classification of set of statistical data on j- th versions i- th an attribute is inexpedient. In engineering calculations α_k is accepted usually equal 0,05 or 0,1.

Lack of this criterion is necessity of the reference to the tables setting discrete values fiducially distribution $F^*[\delta M_V(\tau_{ав})]$ allowing for set n_v and $\delta M^*_{V,Э}(\tau)$ roughly to define probability $F^*[\delta M^*_{V.Э}(\tau_{ав})]$.

Example 1. Let some data set is known about $\tau_{ав}$ PU TPS capacity 300 MWt $M^*_{V,Э}(\tau_{ав}) = 72{,}4c.$. Sample of realizations is received $\tau_{ав}$ for one of PU and settlement size $M^*_{V,Э}(\tau_{ав}) = 51c.$ at n_v=2. It is required to establish character of a divergence $M^*_{\Sigma,Э}(\tau_{ав})$ and $M^*_{V,Э}(\tau_{ав})$. For the decision of this problem:

- it is calculated $\delta M^*_{V,Э}(\tau_{ав}) = \dfrac{[|51-72{,}4|]}{72{,}4} = 0{,}296$

- on tabl.1 [1] for n_v=2 we find probability $F^*[\delta M^{**}_V(\tau_{ав})] \cong 0{,}5$

- as α_v=0,5. and $\alpha_к$=0,05<<α_v, classification of a data set about $\tau_{ав}$ it is inexpedient.

Criterion K2. It is based on comparison $\delta M^*_{V,Э}(\tau_{ав})$ and $\delta M^*_{V,K}(\tau_{ав})$. According to the [1] size $\delta M^{**}_{V,K}(\tau_{ав})$ corresponding $\alpha_к$=1-$F^*[\delta M^{**}_{V,K}(\tau_{ав})]$, it is calculated under the formula

$$\delta M^{**}_{V,K}(\tau_{ав}) = A\big/\sqrt{n_V} \tag{2}$$

0,994 critical values $\delta M^{**}_{V,K}(\tau_{ав})$ approximating with accuracy for set α_k and n_v.

Here $\delta M^{**}_{V}(\tau_{ав})$ - realization of possible absolute values of a relative deviation $M^{**}_{V}(\tau_{ав})$ and $M^{*}_{\Sigma,Э}(\tau_{ав})$; $M^{**}_{V}(\tau_{ав})$ - realizations of the possible values $M^{*}_{V}(\tau_{ав})$ modeled on s.f.d. $F^{*}_{\Sigma}(\tau_{ав})$ for a preset value n_v; A – constant factor.

The size defined α_k. For $\alpha_к$=0,05 factor A=1,13, and for $\alpha_к$=0,1 – it is equal 0,95. Thus, classification of a data set $\tau_{ав}$ it is expedient, if $\delta M^{*}_{V,Э}(\tau_{ав})$ exceed $\delta M^{**}_{V,K}(\tau_{ав})$.

Example 2. We shall take advantage of data of an example 1. $M^{*}_{\Sigma,Э}(\tau_{ав}) = 72{,}4$c. and $M^{*}_{V,Э}(\tau_{ав}) = 51$c.; n_v=2. Define critical value $\delta M^{**}_{V,K}(\tau_{ав}) = 1{,}13/\sqrt{2} = 0{,}80$. Thus, as $\delta M^{*}_{V,Э}(\tau_{ав}) << \delta M^{**}_{V,K}(\tau_{ав})$ classification of a data set with a significance value 0,05 is inexpedient.

The considered example shows, that criterion K2 is simple enough. It is necessary to execute only two elementary calculations.

Criterion K3. It is based on comparison of experimental estimations $M^{*}_{V,Э}(\tau_{ав})$ and $M^{*}_{\Sigma,Э}(\tau_{ав})$ with boundary values fiducially intervals

$$\left[\underline{M^{**}_{V,(1-\alpha_L)}}(\tau_{ав});\quad \overline{M^{**}_{V,\alpha_L}(\tau_{ав})}\right],$$

Calculated on s.f.f.d. $F^{*}[\delta M^{**}_{V,K}(\tau_{ав})]$.

Comparison carried out in following sequence:

If $$M^{*}_{V,Э}(\tau_{ав}) < M^{*}_{\Sigma,Э}(\tau_{ав}) \tag{3}$$

and $$M^{*}_{V,Э}(\tau_{ав}) < \underline{M^{**}_{V,(1-\alpha_k)}(\tau_{ав})} \tag{4}$$

where $$\underline{M^{**}_{V,(1-\alpha_k)}(\tau_{ав})} = M^{*}_{\Sigma,Э}(\tau)\cdot\left(1 - A\big/\sqrt{n_v}\right) \tag{5}$$

That classification is expedient

If $$M^{*}_{V,Э}(\tau_{ав}) > M^{*}_{\Sigma,Э}(\tau_{ав}) \tag{6}$$

and $$M^{*}_{V,Э}(\tau_{ав}) > \overline{M^{**}_{V,\alpha_k}(\tau_{ав})} \tag{7}$$

where $$\overline{M^{**}_{V,\alpha_k}(\tau_{ав})} = M^{*}_{\Sigma,Э}(\tau)\cdot\left(1 + A\big/\sqrt{n_v}\right) \tag{8}$$

That classification is expedient.

If the condition (3) is carried out,

and $$\underline{M^{**}_{V,(1-\alpha_k)}(\tau_{ав})} < M^{*}_{V,Э}(\tau_{ав}) \tag{9}$$

and $$M^{*}_{\Sigma,Э}(\tau_{ав}) > \overline{M^{**}_{V,\beta_k}(\tau_{ав})} \tag{10}$$

where $$\overline{M^{**}_{V,\beta_k}(\tau_{ав})} = M^{*}_{V,Э}(\tau)\cdot\left(1 + A\big/\sqrt{n_v}\right) \tag{11}$$

That classification is expedient

If the condition (6) is carried out,

and $$\overline{M^{**}_{V,\alpha_l}(\tau_{ав})} > M^{*}_{V,Э}(\tau_{ав}) \tag{12}$$

and $$M^{*}_{\Sigma,Э}(\tau_{ав}) < \underline{M^{**}_{V,(1-\beta_k)}(\tau_{ав})} \tag{13}$$

where $$\underline{M^{**}_{V,(1-\beta_k)}(\tau_{ав})} = M^{*}_{V,Э}(\tau)\cdot\left(1 - A\big/\sqrt{n_v}\right) \tag{14}$$

That classification is expedient

In all other cases, classification is inexpedient.

Example 3. We shall address again to initial data of an example 1. As $M^{*}_{\Sigma,Э}(\tau_{ав}) > M^{*}_{V,Э}(\tau_{ав})$, we shall checkup conformity (4). For what it is calculated $\underline{M^{**}_{V,(1-\alpha_k)}(\tau_{ав})}$.

$$\underline{M^{**}_{V,0,95}(\tau_{ав})} = 72,4 \cdot \left(1 - 1,13/\sqrt{2}\right) = 14,4\,с.$$

It is necessary to note, that for any αk=βk the size of constant factor A can be calculated under the formula [1]

$$A=18,5a2 - 7,33a + 1,48 \qquad (15)$$

As $M^{*}_{V,Э}(\tau_{ав}) > \underline{M^{**}_{V,0,95}(\tau_{ав})}$, we shall check up a condition (10). For what we shall calculate $\overline{M^{**}_{V,\beta_k}(\tau_{ав})}$. At αk=βk=0,5

$$\overline{M^{**}_{V,\beta_k}(\tau_{ав})} = 51 \cdot \left(1 + 1,13/\sqrt{2}\right) = 91,9\,с.$$

But as $\overline{M^{**}_{V,0б5}(\tau_{ав})} > M^{*}_{\Sigma,Э}(\tau_{ав})$ classification is inexpedient.

Resulted calculations show, that, despite of a seeming bulkiness of criterion, calculation are small and simple. The opportunity concerns to advantages of this criterion to consider a mistake of the second sort. In the illustrative purposes in figure 1 the scheme of algorithm of application of criterion K3 is resulted the block.

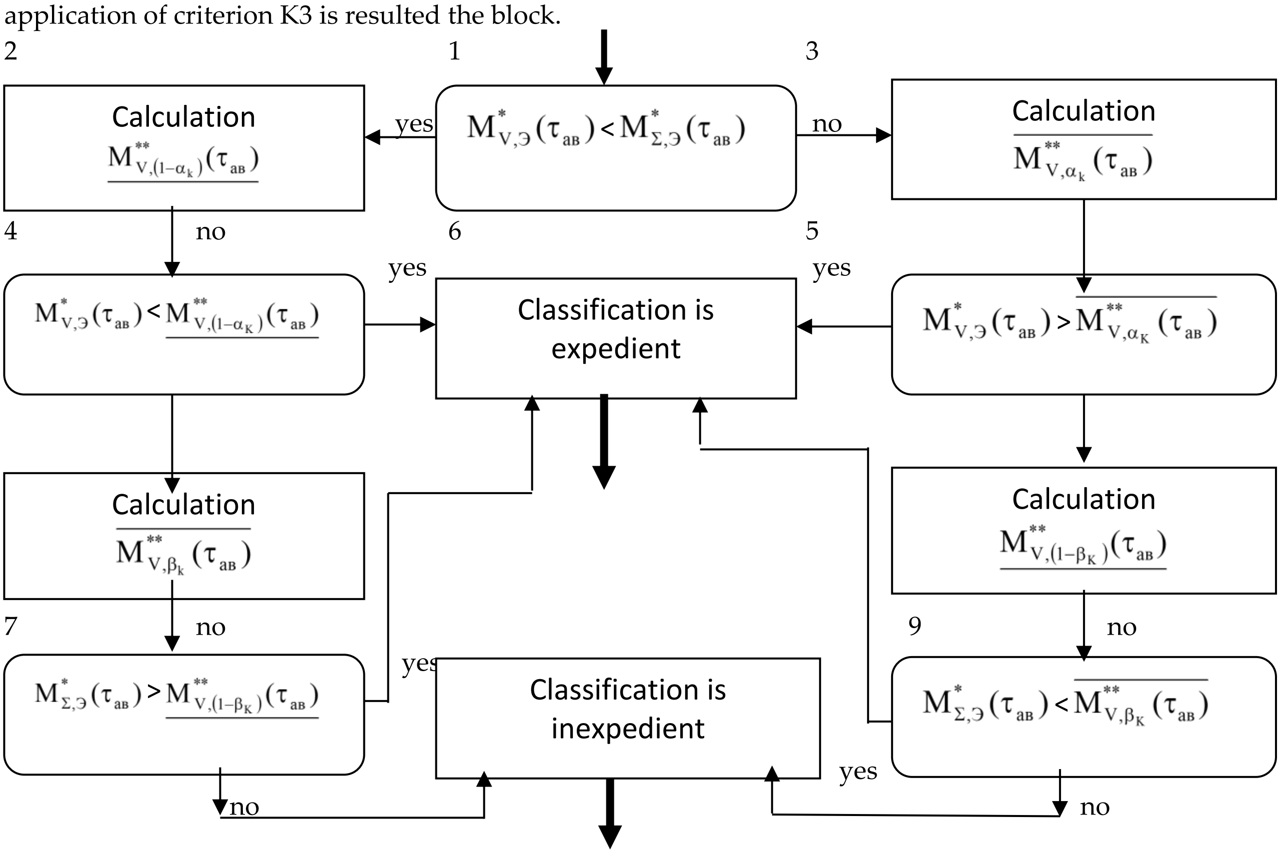

Fig.1. The block the scheme of algorithm of application of criterion K3.

The analysis of the considered criteria allows establishing, that at the manual account VA it is necessary to consider as the basic criterion of check of expediency of classification of statistical data of some criterion K3. Criteria K2 and K1 should be considered as auxiliary mistakes for exception of the manual account.

Conclusion

1. It is necessary to distinguish five versions of ways of calculation of parameters of reliability of objects EPS on multivariate statistical data. These are the parameters calculated as average arithmetic random variables, parameters of relative duration of idle time, the parameters describing probability т frequency of occurrence of casual event, complex parameters.

2. Criteria check of hypotheses about the importance of versions of attributes for the parameters of reliability calculated as average arithmetic random variables recommended. As random variables modeled realizations of parameters of reliability considered, and as critical values boundary values fiducially intervals with the established factor of the importance entered.

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RELIABILITY PERFORMANCE MEASURES OF SYSTEMS WITH LOCATION-SCALE GENERALIZED ABSOLUTELY CONTINUOUS MULTIVARIATE EXPONENTIAL FAILURETIME DISTRIBUTION

S. Amala Revathy, B. Chandrasekar

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Loyola College, Chennai 600034, India
e-mail: sarevathy@gmail.com
bchandrasekar2003@yahoo.co.in,

Abstract

This paper deals with the equal marginal location-scale Generalized Absolutely Continuous Multivariate Exponential model. The distributional properties and applications of the location-scale model arising out of the k-parameter Generalized Absolutely Continuous Multivariate Exponential distribution are studied. Standby, parallel, series and relay systems of order k with location-scale Generalized Absolutely Continuous Multivariate Exponential failuretimes are discussed and their performance measures are obtained. The optimal estimators of the meantime before failure times are also derived.

Keywords: Equivariant estimation, location-scale, multivariate exponential, performance measures

1. Introduction

Though, there is an extensive literature on the reliability aspects of systems with independent failure times, not much work seems to have been carried out on systems with dependent component failure times. Rau (1970) discusses reliability analysis of systems with independent components. Chandrasekar and Paul Rajamanickam (1996), Paul Rajamanickam and Chandrasekar (1997, 1998a, 1998b), Paul Rajamanickam (1999) discuss repairable systems with dependent structures mainly assuming Marshall - Olkin type of joint distributions for the system component failure and repair times. Recently Chandrasekar and Sajesh (2013) and Chandrasekar and Amala Revathy (2016) discussed reliability applications of location-scale equal marginal absolutely continuous bivariate and multivariate exponential distributions respectively.

By considering location-scale Generalized Absolutely Continuous Multivariate Exponential (GACMVE) failuretime distribution, for k unit systems, we derive the reliability performance measures and obtain optimal estimators. In Section 2, we propose the probability density function for the location- scale GACMVE model. In Section 3, we derive some important distributional results required for further discussion. In Section 4, we consider a k unit standby system and obtain the mean time before failure (MTBF) and the reliability function of the system. Further the minimum risk equivariant estimator (MREE) and the uniformly minimum variance unbiased estimator (UMVUE) of the MTBF are derived. Similar results for parallel, series and relay systems are presented in Sections 5, 6 and 7 respectively.

2. Generalized Absolutely Continuous Multivariate Exponential location scale model

The joint pdf of GACMVE is

$$f(x_1, x_2 \ldots, x_k) = \frac{1}{k!} \prod_{l=0}^{k-1} \sum_{i=l}^{k-1} \sum_{j=0}^{i} \binom{i}{j} \lambda_{j+1} \exp\left[-\lambda_1 \sum_{i=1}^{k} x_i - \lambda_2 \sum_{i=1}^{k} \sum_{i<j=1}^{k} \left(x_i \vee x_j \right) - \ldots - \lambda_k \left(x_1 \vee x_2 \vee \ldots \vee x_k \right) \right]$$

$$x_i \geq 0 \ \forall i; \ \lambda_1 > 0, \ \lambda_i \geq 0, \ i = 2, 3 \ldots k \quad \ldots (2.1)$$

Here $x_1 \vee x_2 \vee \ldots \vee x_k = \max\{x_1, x_2, \ldots x_k\}$.

Let X be a random variable (vector) with the distribution function $F_{\xi,\tau}(.)$, $\xi \in R, \tau > 0$.

Let $\{F_{\xi,\tau}; \xi \in R, \tau > 0\}$ be a location-scale family, so that $F_{\xi,\tau}(x) = F\left(\frac{x-\xi}{\tau}\right)$ for some distribution function F.

The location-scale GACMVE has the pdf

$$f_{\xi,\tau}(x_1, x_2 \ldots, x_k) = \frac{1}{\tau^k k!} \prod_{l=0}^{k-1} \sum_{i=l}^{k-1} \sum_{j=0}^{i} \binom{i}{j} \lambda_{j+1}$$

$$\exp\left\{ -\frac{1}{\tau} \left[\lambda_1 \sum_{i=1}^{k} x_i + \lambda_2 \sum_{i=1}^{k} \sum_{i<j=1}^{k} \left(x_i \vee x_j \right) \ldots + \lambda_k \left(x_1 \vee x_2 \vee \ldots \vee x_k \right) - \sum_{p=1}^{k} \binom{k}{p} \lambda_p \xi \right] \right\}$$

$$x_i > \xi \ \forall i, \ \xi \in R, \ \tau > 0, \ \lambda_1 > 0, \ \lambda_2 \geq 0 \quad \ldots (2.2)$$

For fixed $(\lambda_1, \lambda_2, \ldots, \lambda_k)$, the distribution of $\left(\frac{X_1-\xi}{\tau}, \frac{X_2-\xi}{\tau}, \ldots, \frac{X_k-\xi}{\tau}\right)$ does not depend on $(\xi, \tau)'$. Therefore the above family is a location – scale family with the location – scale parameter $(\xi, \tau)'$. Let us refer to the distribution as location-scale GACMVE. When τ = 1, the resulting family is the location GACMVE family. When ξ = 0, the resulting family is the scale GACMVE family. Since we are interested in the location-scale parameter, it is assumed that the parameters $\lambda_1, \lambda_2, \ldots, \lambda_k$ are known.

3. Distributional properties

Theorem 3.1

Let $\left(X_1, X_2, ... X_k\right) \sim$ GACMVE distribution given in (2.1), and $Y_1, Y_2 \ldots, Y_k$ denote the order statistics based on $X_1, X_2 \ldots, X_k$. Define $W_1 = Y_1$, $W_2 = Y_2 - Y_1$, …… $W_k = Y_k - Y_{k-1}$. Then $B_0 W_1$, $B_1 W_2$,..., $B_{k-1} W_k$ are independent and identical standard exponential random variables, where $B_l = \sum_{i=l}^{k-1} A_i$, $l = 0,1,2,....,k-1$ and $A_i = \sum_{j=0}^{i} \binom{i}{j} \lambda_{j+1}$; $\quad i = 0,1,2,...,k-1.$

Proof

The joint pdf of $\left(X_1, X_2 \ldots, X_k\right)$ is

$$f(x_1, x_2 \ldots, x_k) = \frac{1}{k!} \prod_{l=0}^{k-1} \sum_{i=l}^{k-1} \sum_{j=0}^{i} \binom{i}{j} \lambda_{j+1}$$

$$\exp\left[-\lambda_1 \sum_{i=1}^{k} x_i - \lambda_2 \sum_{i=1}^{k} \sum_{i<j=1}^{k} \left(x_i \vee x_j\right) - .... - \lambda_k \left(x_1 \vee x_2 \vee .... \vee x_k\right)\right]$$

$$x_i \geq 0 \, \forall i; \;\; \lambda_1 > 0, \lambda_i \geq 0, i = 2,3....k$$

The pdf of $\left(Y_1, Y_2 \ldots, Y_k\right)$ is

$$g(y_1, y_2 \ldots, y_k) = \left(\prod_{l=0}^{k-1} B_l\right) \exp\left[-\lambda_1 \sum_{i=1}^{k} y_i - \lambda_2 \left(y_2 + 2y_3 + 3y_4 + ...\overline{k-1}y_k\right)\right.$$

$$\left. - \lambda_3 \left(y_3 + \binom{3}{2} y_4 + \binom{4}{2} y_5 + ... \binom{k-1}{2} y_k\right) ..... - \lambda_k y_k\right]$$

$$y_1 < y_2 < .....y_k; \; \lambda_1 > 0, \lambda_i \geq 0, i = 2,3,...k.$$

Consider the pdf of $\left(Y_1, Y_2 \ldots, Y_k\right)$

$$g\left(y_1, y_2, ...,, y_k\right) = B_0 \, B_1 ... B_{k-1} \exp\left\{-\lambda_1 y_1 - \left(\lambda_1 + \lambda_2\right) y_2\right\} \exp\left\{-\left(\lambda_1 + \binom{2}{1} \lambda_2 + \lambda_3\right) y_3\right\}$$

$$\exp\left\{-\left(\lambda_1 + \binom{3}{1} \lambda_2 + \binom{3}{2} \lambda_3 + \lambda_4\right) y_4\right\} .... \exp\left\{-\left(\lambda_1 + \binom{k-1}{1} \lambda_2 + ... + \lambda_k\right) y_k\right\}.$$

$$= B_0 \, B_1 ... B_{k-1} \exp\left\{-A_0 \, y_1 - A_1 \, y_2 ... - A_{k-1} y_k\right\}$$

In order to find the distribution of $\left(W_1, W_2 \ldots, W_n\right)$, consider the transformation

$w_i = y_i - y_{i-1}$, $i = 1,2,...k$, with $y_0 \equiv 0$.

Then $y_j = w_1 + w_2 + ... + w_j$, j = 1,2,3,…,k.

Note that the Jacobian of the transformation is 1.

The joint pdf of $\left(W_1, W_2 \ldots, W_k\right)$ is

$$h(w_1, w_2, \ldots w_k) = B_0\, B_1 \ldots B_{k-1} \exp\left\{-A_0\, w_1 - A_1\left(w_1 + w_2\right) - \ldots - A_{k-1}\sum_{i=1}^{k} w_i\right\}$$

$$= B_0\, B_1 \ldots B_{k-1} \exp\left\{-B_0\, w_1 - B_1\, w_2 - \ldots - B_{k-1}\, w_k\right\}$$

Hence B_0W_1, B_1W_2 ….. $B_{k-1}W_k$ are independent and identical E(0,1) random variables.

Sufficient statistic

Let $X_p = \left(X_{1p}, X_{2p} \ldots, X_{kp}\right)'$; p = 1,2,….n be a random sample of size n from (2.2).

The joint pdf of $\left(X_{1p}, X_{2p} \ldots, X_{kp}\right)$; j = 1,2,….n is

$$p(x;\xi,\tau) = \left\{\frac{1}{\tau^k k!}\prod_{l=0}^{k-1}\sum_{i=l}^{k-1}\sum_{j=0}^{i}\binom{i}{j}\lambda_{j+1}\right\}^n$$

$$\exp\left\{-\frac{1}{\tau}\sum_{p=1}^{n}\left[\lambda_1\sum_{i=1}^{k}x_{ip} + \lambda_2\sum_{i=1}^{k}\sum_{i<j=1}^{k}\left(x_{ip} \vee x_{jp}\right) \ldots + \lambda_k\left(x_{1p} \vee x_{2p} \vee \ldots \vee x_{kp}\right) - \sum_{m=1}^{k}\binom{k}{m}\lambda_m \xi\right]\right\}$$

$$\min_{1 \le p \le n}\left(x_{1p} \wedge x_{2p} \wedge \ldots \wedge x_{kp}\right) > \xi$$

Let $U_p = \left(X_{1p} \wedge X_{2p} \wedge \ldots \wedge X_{kp}\right)$; *and* $U_{(1)} = \min_{1 \le p \le n} U_p$.

$$p(x;\xi,\tau) = \left\{\frac{1}{\tau^k k!}\prod_{l=0}^{k-1}\sum_{i=l}^{k-1}\sum_{j=0}^{i}\binom{i}{j}\lambda_{j+1}\right\}^n$$

$$\exp\left\{-\frac{1}{\tau}\sum_{p=1}^{n}\left[\lambda_1\sum_{i=1}^{k}x_{ip} + \lambda_2\sum_{i=1}^{k}\sum_{i<j=1}^{k}\left(x_{ip} \vee x_{jp}\right) + \ldots + \lambda_k\left(x_{1p} \vee x_{2p} \vee \ldots \vee x_{kp}\right) - \sum_{m=1}^{k}\binom{k}{m}\lambda_m U_{(1)} + \sum_{m=1}^{k}\binom{k}{m}\lambda_m\left(U_{(1)} - \xi\right)\right]\right\}$$

$$= g_{\xi,\tau}\left(t_1, t_2\right) h\left(\underline{x}\right),$$

where , $T_1^* = U_{(1)}$ and

$$T_2^* = \sum_{p=1}^{n}\left\{\lambda_1\sum_{i=1}^{k}x_{ip} + \lambda_2\sum_{i=1}^{k}\sum_{i<j=1}^{k}\left(x_{ip} \vee x_{jp}\right) \ldots + \lambda_k\left(x_{1p} \vee x_{2p} \vee \ldots \vee x_{kp}\right)\right\}.$$

By factorization theorem, $T^* = \left(T_1^*, T_2^*\right)$ is a sufficient statistic.

Theorem 3.2

(i) $T_1^* \sim E\left[\xi, \dfrac{\tau}{n B_0}\right]$

(ii) $T_2^* \sim G\left(nk-1, \tau\right)$ and

(iii) T_1^* and T_2^* are independent.

Proof

(i) Let $X_p = \left(X_{1p}, X_{2p} \ldots, X_{kp}\right)'$; p = 1,2,….n be a random sample of size n from (2.2).

The joint pdf of $\left(X_{1p}, X_{2p} \ldots, X_{kp}\right)$; j = 1,2,….n is

$$p(x;\xi,\tau) = \left\{\frac{1}{\tau^k k!}\prod_{l=0}^{k-1}\sum_{i=l}^{k-1}\sum_{j=0}^{i}\binom{i}{j}\lambda_{j+1}\right\}^n$$

$$\exp\left\{-\frac{1}{\tau}\sum_{p=1}^{n}\left[\lambda_1\sum_{i=1}^{k}x_{ip}+\lambda_2\sum_{i=1}^{k}\sum_{i<j=1}^{k}\left(x_{ip}\vee x_{jp}\right).....+\lambda_k\left(x_{1p}\vee x_{2p}\vee....\vee x_{kp}\right)-\sum_{m=1}^{k}\binom{k}{m}\lambda_m\xi\right]\right\}$$

$$min\left(x_{1p}\wedge x_{2p}\wedge\ldots\wedge x_{kp}\right)>\xi$$

Let $U_p = \left(X_{1p}\wedge X_{2p}\wedge\ldots\wedge X_{kp}\right)$; *and* $U_{(1)} = min\ U_p$.

Then $U_{(1)} > \xi$ and $\dfrac{n B_0}{\tau}\left(U_{(1)}-\xi\right) \sim E(0,1)$

Therefore $U_{(1)} \sim E\left(\xi, \dfrac{\tau}{n B_0}\right)$

(ii) Let $Y_{1j}, Y_{2j,} \ldots, Y_{kj}$ denote the order statistics based on $\left(X_{1j}, X_{2j} \ldots, X_{kj}\right)$, j = 1,2,….n. Note that $Y_{1j} = U_j$, j = 1,2,….n,. Define $W_{rj} = Y_{rj} - Y_{(r-1)j}$, r = 1,2,3…..k; j = 1,2,3,….n.
$Y_{0j} = 0$ for all j.
Consider

$$T_2^* = \sum_{p=1}^{n}\left\{\lambda_1\sum_{i=1}^{k}X_{ip}+\lambda_2\sum_{i=1}^{k}\sum_{i<j=1}^{k}\left(X_{ip}\vee X_{jp}\right).....+\lambda_k\left(X_{1p}\vee X_{2p}\vee....\vee X_{kp}\right)\right\}$$

$$= \sum_{p=1}^{n}\left\{\lambda_1\sum_{i=1}^{k}Y_{ip}+\lambda_2\sum_{i=2}^{k}(i-1)Y_{im}+\lambda_3\sum_{i=3}^{k}(i-2)Y_{im}+....+\lambda_k Y_{km}-\sum_{m=1}^{k}\binom{k}{m}\lambda_m U_{(1)}\right\}$$

$$=\sum_{p=1}^{n}\left\{\lambda_1\left(kW_{1p}+(k-1)W_{2p}+.....+W_{kp}\right)\right.$$

$$+\lambda_2\left(W_{1p}\sum_{i=1}^{k-1}i+W_{2p}\sum_{i=1}^{k-1}i+W_{3p}\sum_{i=1}^{k-1}i.....+(k-1)W_{kp}\right)$$

$$+\lambda_3\left(W_{1p}\sum_{i=1}^{k-2}i+W_{2p}\sum_{i=1}^{k-2}i+W_{3p}\sum_{i=1}^{k-2}i.+W_{4p}\sum_{i=2}^{k-2}i+W_{5p}\sum_{i=3}^{k-2}i.+....+(k-2)W_{kp}\right)+......$$

$$\left.+\lambda_k\sum_{i=1}^{k}W_{im}-\sum_{m=1}^{k}\lambda_m\binom{k}{m}U_{(1)}\right\}$$

$$=\sum_{p=1}^{n}\left\{W_{1p}\left(\lambda_1 k+\lambda_2\sum_{i=1}^{k-1}i+\lambda_3\sum_{i=1}^{k-2}i+\lambda_4\sum_{i=1}^{k-3}i+....+\lambda_k\right)+\right.$$

$$W_{2p}\left(\lambda_1(k-1)+\lambda_2\sum_{i=1}^{k-1}i+\lambda_3\sum_{i=1}^{k-2}i+\lambda_4\sum_{i=1}^{k-3}i+....+\lambda_k\right)+$$

$$W_{3p}\left(\lambda_1(k-2)+\lambda_2\sum_{i=1}^{k-1}i+\lambda_3\sum_{i=1}^{k-2}i+\lambda_4\sum_{i=1}^{k-3}i+....+\lambda_k\right)+......+$$

$$\left.W_{kp}\left(\lambda_1 k+\lambda_2(k-1)+\lambda_3(k-2)+\lambda_4(k-3)+....+\lambda_k\right)+\sum_{m=1}^{k}\lambda_m\binom{k}{m}U_{(1)}\right\}$$

$$=\sum_{p=1}^{n}\left\{W_{1p}\sum_{i=0}^{k-1}\sum_{j=0}^{i}\binom{i}{j}\lambda_{j+1}+W_{2p}\sum_{i=1}^{k-1}\sum_{j=0}^{i}\binom{i}{j}\lambda_{j+1}+......W_{kp}\sum_{i=k-1}^{k-1}\sum_{j=0}^{i}\binom{i}{j}\lambda_{j+1}+\sum_{m=1}^{k}\lambda_m\binom{k}{m}U_{(1)}\right\}$$

$$=\sum_{p=1}^{n}\left\{\sum_{i=0}^{k-1}\sum_{j=0}^{i}\binom{i}{j}\lambda_{j+1}\left(W_{1m}-U_{(1)}\right)+W_{2p}\sum_{i=1}^{k-1}\sum_{j=0}^{i}\binom{i}{j}\lambda_{j+1}+......+W_{kp}\sum_{i=k-1}^{k-1}\sum_{j=0}^{i}\binom{i}{j}\lambda_{j+1}\right\}$$

$$=\sum_{p=1}^{n}\left\{\sum_{i=0}^{k-1}\sum_{j=0}^{i}\binom{i}{j}\lambda_{j+1}\left(U_{(m)}-U_{(1)}\right)+W_{2p}\sum_{i=1}^{k-1}\sum_{j=0}^{i}\binom{i}{j}\lambda_{j+1}+......+W_{kp}\sum_{i=k-1}^{k-1}\sum_{j=0}^{i}\binom{i}{j}\lambda_{j+1}\right\}$$

Since $U_{(1)},U_{(2)}....U_{(n)}$ are order statistics from $E\left(\xi,\frac{\tau}{B_0}\right)$, it follows that the first term on the right hand side follows $G(n-1,\tau)$.

By Theorem 3.1, each of the other (k-1) terms on the right hand side follows $G(n,\tau)$. Since $W_{1j},W_{2j,},\ldots,W_{kj}$ are independent for each j, the k random variables on the right hand side are independent.

Hence $T_2^*\sim G(nk-1,\tau)$.

(iii) For fixed τ, the joint distribution of $\left(X_{1j},X_{2j}\ldots,X_{kj}\right)$, j = 1,2,….n, belongs to a location family with the location parameter ξ. The statistic T_2^* is ancillary and T_1^* is complete sufficient. Hence T_1^* and T_2^* are independent (Basu, 1955).

The following theorem will help us in obtaining the reliability performance measures of standby and parallel syste

Theorem 3.3

Let $\left(T_1, T_2, \ldots, T_k\right)$ follow $GACMVE\left(\lambda_1, \lambda_2, \ldots \lambda_k ; \xi, \tau\right)$ with pdf given in equation (2.2). Then

(i) $\sum_{i=1}^{k} T_i - k\xi \underset{=}{d} V_1 + V_2 + \ldots + V_k$, where V_1, V_2,…. V_k are independent and

$$V_l \sim E\left(0, \frac{\left\{k-\left(l-1\right)\tau\right\}}{B_{l-1}}\right), \text{ for all } l = 1, 2,.., k. \qquad \ldots(3.1)$$

(ii) $\left(T_1 \vee T_2 \vee \ldots T_k\right) - k\ \xi \underset{=}{d} V_1^* + V_2^* + \ldots.. V_k^*$, where V^*_1, V^*_2,…. V^*_k are independent and

$$V_j^* \sim E\left(0, \frac{\tau \sum_{i=l}^{k-1} \binom{i}{k-1}}{\sum_{i=l}^{k-1} \binom{i}{k-1} \lambda_k}\right), \text{ for all } j = 1, 2,.., k. \qquad \ldots.(3.2)$$

Proof:

The MGF of $\left(\sum_{i=1}^{k} T_i, \sum_{i<j=1}^{k} \sum_{j=1}^{k} T_i \vee T_j, \ldots, T_1 \vee T_2 \vee \ldots T_k\right)$ at $\left(U_1, U_2, \ldots, U_k\right)$ is

$$M(u_1, u_2 \ldots, u_k) = \int_{\xi}^{\infty}\int_{\xi}^{\infty} \ldots \int_{\xi}^{\infty} \frac{1}{\tau^k k!} \prod_{l=0}^{k-1} \sum_{i=l}^{k-1} \sum_{j=0}^{i} \binom{i}{j} \lambda_{j+1}$$

$$\exp\left\{ u_1 \sum_{i=1}^{k} t_i + u_2 \sum_{i=1}^{k} \sum_{i<j=1}^{k} \left(t_i \vee t_j\right) + \ldots.. + u_k \left(t_1 \vee t_2 \vee \ldots. \vee t_k\right) \right\}$$

$$\exp\left\{ -\frac{1}{\tau}\left[\lambda_1 \sum_{i=1}^{k} t_i + \lambda_2 \sum_{i=1}^{k} \sum_{i<j=1}^{k} \left(t_i \vee t_j\right) + \ldots.. + \lambda_k \left(t_1 \vee t_2 \vee \ldots. \vee t_k\right) \right] - \sum_{p=1}^{k} \binom{k}{p} \lambda_p \xi \right\}$$

$$dt_1\, dt_2 \ldots\ldots dt_k$$

$$M(u_1,u_2\ldots,u_k)=\int_{\xi}^{\infty}\int_{\xi}^{\infty}\cdots\int_{\xi}^{\infty}\ \frac{1}{\tau^k k!}\prod_{l=0}^{k-1}\sum_{i=l}^{k-1}\sum_{j=0}^{i}\binom{i}{j}\lambda_{j+1}\exp\left\{-\frac{1}{\tau}\left[(\lambda_1-\tau u_1)\sum_{i=1}^{k}t_i+\right.\right.$$

$$(\lambda_2-\tau u_2)\sum_{i=1}^{k}\sum_{i<j=1}^{k}\left(t_i\vee t_j\right)+\ldots..+(\lambda_k-\tau u_k)(t_1\vee t_2\vee\ldots.\vee t_k)$$

$$\left.\left.-\sum_{p=1}^{k}\binom{k}{p}\lambda_p\xi\right]\right\}\ dt_1\,dt_2\ldots\ldots.dt_k$$

$$=\prod_{l=0}^{k-1}\frac{\exp\left(-\sum_{p=1}^{k}\binom{k}{p}u_p\,\xi\right)}{\left(1-\dfrac{\tau\sum_{i=l}^{k-1}\sum_{j=0}^{i}\binom{i}{j}u_{j+1}}{B_l}\right)}$$

(i) $M\left(u_1,0,\ldots,0\right)=\prod_{l=0}^{k-1}\dfrac{\exp\left(-k\,u_1\,\xi\right)}{\left(1-\dfrac{(k-l)\tau\,u_1}{B_l}\right)}$

$$\therefore\ \sum_{i=1}^{k}T_i-k\,\xi\,\underset{=}{d}\sum_{i=1}^{k}V_i\ ,$$

where Vi's are independent and

$$V_l\sim E\left[0,\frac{(k-l)\tau}{B_l}\right],\qquad l=1,2,\ldots.k.$$

(ii) $M\left(0,0,\ldots,u_k\right)=\prod_{l=0}^{k-1}\dfrac{\exp\left(-\,u_k\,\xi\right)}{\left(1-\dfrac{\tau\sum_{i=l}^{k-1}\binom{i}{k-1}u_k}{B_l}\right)}$

$$\therefore\left(T_1\vee T_2\vee\ldots T_k\right)-k\,\xi\,\underset{=}{d}\ V_1^*+V_2^*+\ldots..V_k^*\ ,$$

where Vi*'s are independent and $V_l^*\sim E\left[0,\dfrac{\tau\sum_{i=l}^{k-1}\binom{i}{k-1}}{B_l}\right],\quad i=0,1,\ldots.k-1\,.$

The following Lemma helps us in finding the reliability function.

Lemma 3.1

Let $M(u)=\left[\prod_{j=1}^{k}\left(1-\alpha_j u\right)\right]^{-1}$; $u<\frac{1}{\alpha_j}\ \forall\ j$.

Then $M(u)=\sum_{j=1}^{k}\frac{w_j}{\left(1-\alpha_j u\right)}$, where $w_j=\frac{\alpha_j^{\,k-1}}{\prod_{\substack{r=1\\ \neq j}}^{k}\left(\alpha_j-\alpha_r\right)}$, and $\sum_{j=1}^{k}w_j=1$.

Proof

$$M(u)=\frac{1}{(1-\alpha_1 u)(1-\alpha_2 u)......(1-\alpha_k u)}$$

Resolving into partial fractions,

$$M(u)=\frac{w_1}{(1-\alpha_1 u)}+\frac{w_2}{(1-\alpha_2 u)}+......+\frac{w_k}{(1-\alpha_k u)} =$$

$$\frac{w_1(1-\alpha_2 u).....(1-\alpha_k u)+w_2(1-\alpha_1 u)(1-\alpha_3 u)....(1-\alpha_k u)+....+w_k(1-\alpha_1 u)....(1-\alpha_{k-1} u)}{\prod_{j=1}^{k}\left(1-\alpha_j u\right)}$$

$$=\frac{w_1\prod_{j=2}^{k}\left(1-\alpha_j u\right)+w_2\prod_{\substack{j=1\\ \neq 2}}^{k}\left(1-\alpha_j u\right)+....+w_k\prod_{j=1}^{k-1}\left(1-\alpha_j u\right)}{\prod_{j=1}^{k}\left(1-\alpha_j u\right)}$$

Thus, for j = 1,2,3,…,k, we get $w_j=\frac{\alpha_j^{\,k-1}}{\prod_{\substack{r=1\\ \neq j}}^{k}\left(\alpha_j-\alpha_r\right)}$.

Corollary 3.1

The survival function corresponding to M(u) is

$$\overline{G}(u)=\sum_{j=1}^{k}w_j\exp\left(-\frac{1}{\alpha_j}u\right),\ u>0.$$

4 Standby system

Consider a k unit standby system with component failure times $T_1,T_2,\ldots,T_k$ having location-scale GACMVE distribution.

Then the system failure time is $T=\sum_{i=1}^{k}T_i$.

The MTBF of the system is

MTBF = E (T)

$$= E\left(V_1 + V_2 + \ldots. + V_k\right) + k\xi$$

$$= \sum_{l=1}^{k} \frac{\{k-l\}\tau}{B_{l-1}} + k\,\xi \text{, in view of (3.1).}$$

Following the arguments of Chandrasekar and Amala Revathy (2016), the MREE of $\eta = \alpha\,\xi + \beta\,\tau$, $\alpha, \beta \in R$, is given by

$$\delta^{*} = \alpha\,\delta_{01} + \frac{1}{kn}\left[\beta - \frac{\alpha}{n\,B_0}\right]\delta_{02}$$

Define,

$$\delta_{01} = \min_{1\le p\le n}\left\{X_{1p} \wedge X_{2p} \wedge \ldots \wedge X_{kp}\right\} \text{ and}$$

$$\delta_{02} = \sum_{p=1}^{n}\left\{\lambda_1 \sum_{i=1}^{k} X_{ip} + \lambda_2 \sum_{i=1}^{k}\sum_{i<j=1}^{k}\left(X_{ip} \vee X_{jp}\right) + \ldots.. + \lambda_k\left(X_{1p} \vee X_{2p} \vee \ldots. \vee X_{kp}\right)\right\}$$

By taking α = k and $\beta = \sum_{l=1}^{k} \frac{(k-1)}{B_{l-1}}$, the MREE of the MTBF is given by

$$k\,\delta_{01} + \frac{1}{kn}\left[\sum_{l=1}^{k} \frac{(k-1)}{B_{l-1}} + \frac{\alpha}{n\,B_0}\right]\delta_{02}$$

Reliability function of the standby system is

R(t) = P(T > t)

$$= P\left(\sum_{i=1}^{k} T_i - k\,\xi > t\right),\ t > 0$$

$$= P\left(\sum_{i=1}^{k} V_i > t\right),\ t > 0$$

$$= \sum_{l=1}^{k} \beta_l \exp\left(-\frac{1}{\alpha_l} t\right)$$

in view of Lemma 3.1.

Here $\alpha_l = \frac{(k-l)\tau}{B_{l-1}} \quad \forall\, l = 1, 2, \ldots, k$, and $\beta_l = \frac{\alpha_l^{k-1}}{\prod_{\substack{r=1\\ \neq l}}^{k}(\alpha_l - \alpha_r)}$, and $\sum_{j=1}^{k} \beta_j = 1$.

Therefore,

$$R(t) = \sum_{l=1}^{k} \frac{\left(\frac{(k-l)\tau}{B_{l-1}}\right)^{k-1}}{\prod_{\substack{r=1\\ \neq l}}^{k}\left(\frac{(k-l)\tau}{B_{l-1}} - \frac{(k-r)\tau}{B_{r-1}}\ _r\right)} \exp\left(-\frac{1}{\frac{(k-l)\tau}{B_{l-1}}} t\right)$$

5 Parallel system

Consider a k unit parallel system with component failure times $T_1, T_2, \ldots, T_k$ having the GACMVE distribution. Then the system failure time is $T = \underset{1\le i\le k}{Max}\, T_i$.

MTBF = E (T)

$$= E\left(V_1^* + V_2^* + \ldots + V_k^*\right) + k\xi$$

$$= \tau \sum_{l=0}^{k-1}\left[\frac{\sum_{i=l}^{k-1}\binom{i}{k-1}}{B_l}\right] + k\,\xi, \text{ in view of (3.2)}$$

When η = α ξ +β τ, α, β ϵ R, the MREE of η is given by $\delta^* = \alpha\,\delta_{01} + \frac{1}{kn}\left[\beta - \frac{\alpha}{n\,B_0}\right]\delta_{02}$

By taking α = k and $\beta = \sum_{l=0}^{k-1}\left[\frac{\sum_{i=l}^{k-1}\binom{i}{k-1}}{B_l}\right]$ the MREE of the MTBF is given by

$$\delta^* = k\,\delta_{01} + \frac{1}{kn}\left[\sum_{l=0}^{k-1}\left[\frac{\sum_{i=l}^{k-1}\binom{i}{k-1}}{B_l}\right] - \frac{k}{n\,B_0}\right]\delta_{02}$$

Reliability function

R (t) = P(T > t)

$$= P\left(\sum_{i=1}^{k} V_i^* > t\right), \; t > 0$$

$$= \sum_{l=1}^{k} w_l \,\exp\left(-\frac{1}{\alpha_l} t\right)$$

Here $\alpha_l = \frac{\sum_{i=l}^{k-1}\binom{i}{k-1}}{B_{l-1}}$ $\forall\, l = 0,1,,\ldots,k-1$, and $w_l = \frac{\alpha_l^{k-1}}{\prod_{\substack{r=1\\ \neq l}}^{k}(\alpha_1 - \alpha_r)}\ \forall\, l = 0,1,,\ldots,k-1$

6 Series system

Consider a k unit series system with component failure times $T_1, T_2, \ldots, T_k$ having the GACMVE distribution.

Then the system failure time is $T = \underset{1\le i\le k}{Min}\, T_i$.

From Theorem 3.2, $\underset{1\le l\le k}{Min}\, T_l \sim E\left[\xi, \frac{\tau}{B_0}\right]$

Thus, MTBF = $\frac{\tau}{B_0} + \xi$

When η = α ξ +β τ, α, β ϵ R, the MREE of η is given by $\delta^* = \alpha\,\delta_{01} + \frac{1}{kn}\left[\beta - \frac{\alpha}{n\,B_0}\right]\delta_{02}$

By taking α = 1 and $\beta = \frac{1}{B_l}$, the MREE of the MTBF is given by $\delta^* = \delta_{01} + \frac{1}{kn}\left[\frac{1}{B_l} - \frac{1}{n\,B_0}\right]\delta_{02}$

Reliability function

R(t) = P(T > t)

$$= \exp\left[\frac{B_0}{\tau}(t-\xi)\right],\ t>\xi$$

7 Relay system

Consider a k unit relay system with component failure times $T_1, T_2, \ldots, T_k$ having the GACMVE distribution. A relay system of order k operates if the first component and anyone of the remaining (k-1) components operate. Therefore, the failure time of the system is $T = T_1 \wedge \left(T_2 \vee T_3 \vee \ldots \vee T_k\right)$.

The reliability function of the system is

$$R(t) = P(T>t)$$
$$= \sum_{r=2}^{k}(-1)^r \binom{k-1}{r-1}\overline{F}_r\,(t,t,\ldots,t,0,\ldots,0),$$

using distributive law and routine arguments.

Here $\overline{F}_r\,(t,t,\ldots,t,0,\ldots,0)$ represents $P\left(X_1>t, X_2>t,\ldots,X_r>t, X_{r+1}>0,\ldots,X_k>0\right)$.

Let us discuss in detail the case when k = 3.

Here

$$R(t) = P(T>t)$$
$$= \sum_{r=2}^{3}(-1)^r \binom{k-1}{r-1}\overline{F}_r\,(t,t,0)$$
$$= 2\,\exp\left\{-\frac{(2\lambda_1+\lambda_2)}{\tau}(t-\xi)\right\} - \exp\left\{-\frac{(3\lambda_1+3\lambda_2+\lambda_3)}{\tau}(t-\xi)\right\}.$$

The MTBF is given by

$$\text{MTBF} = \frac{2\tau}{(2\lambda_1+\lambda_2)} - \frac{\tau}{(3\lambda_1+3\lambda_2+\lambda_3)} + \xi$$
$$= \left[\frac{(4\lambda_1+5\lambda_2+2\lambda_3)}{(2\lambda_1+\lambda_2)(3\lambda_1+3\lambda_2+\lambda_3)}\right]\tau + \xi$$

When η = α ξ +β τ, α, β ϵ R, the MREE of η is given by $\delta^* = \alpha\,\delta_{01} + \dfrac{1}{kn}\left[\beta - \dfrac{\alpha}{n\,B_0}\right]\delta_{02}$.

By taking α = 1 and $\beta = \left[\dfrac{(4\lambda_1 + 5\lambda_2 + 2\lambda_3)}{(2\lambda_1 + \lambda_2)(3\lambda_1 + 3\lambda_2 + \lambda_3)}\right]$, in the above equation, we get the MREE of the MTBF.

Therefore, MREE of the MTBF is

$$\delta^* = \delta_{01} + \frac{1}{kn}\left[\left(\frac{(4\lambda_1 + 5\lambda_2 + 2\lambda_3)}{(2\lambda_1 + \lambda_2)(3\lambda_1 + 3\lambda_2 + \lambda_3)}\right) - \frac{1}{n\,B_0}\right]\delta_{02}$$

Remark 7.1

From Theorem 3.2, we can obtain the UMVUE's of ξ and τ, and hence obtain the UMVUE of α ξ +β τ:

$$\delta^{**} = \alpha\,\delta_{01} + \frac{1}{kn-1}\left[\beta - \frac{\alpha}{n\,B_0}\right]\delta_{02}$$

Hence one can obtain the UMVUE's of the MTBF in each of the four systems discussed in this chapter.

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RELIABILITY OF A k-OUT-OF-n SYSTEM WITH REPAIR BY A SINGLE SERVER EXTENDING SERVICE TO EXTERNAL CUSTOMERS WITH PRE-EMPTION

A Krishnamoorthy *

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Dept. of Mathematics
Cochin University of Science & Technology, Kochi-682022.
achyuthacusat@gmail.com

M. K. Sathian

Dept. of Mathematics
Panampilly Memorial Govt. College, Chalakudy, Thrissur
sathianmkk@yahoo.com

Viswanath C Narayanan

•

Dept. of Mathematics
Govt. Engg. College, Thrissur-680009
narayanan viswanath@yahoo.com.

Abstract

In this paper we study the reliability of a k-out-of-n system, with a single technician, who also renders service to external customers besides repairing the failed components in the system. For optimizing the revenue from external service without compromising the system reliability, we introduce the N-policy in which the repair of the internal customers (failed components) starts only on accumulation of N failed components. The service to external customers is of preemptive nature in the sense that their service can be interrupted on accumulation of N failed components. It is assumed that an external customer, who finds the server busy with an external customer at the epoch of its arrival joins a queue of infinite capacity; whereas an external customer who finds the server busy with an internal customer leaves the system forever. The failure times of the components of the k-out-of-n system follow an exponential distribution; the arrival of external customers is according to a Poisson process and the service times of the internal and external customers follow non-identical phase-type distributions. Using matrix-analytic methods, we discuss the system stability and steady state distribution. A special case of the model where the underlying distributions are all exponential has been considered, to obtain an expression for the stability condition and a product form solution for the steady state have been obtained for this case. Also several system performance measures have been obtained explicitly. Analysis of a cost function indicates that N-policy does help to optimize the system revenue maintaining high system reliability.

1. **Introduction**

A k-out-of-n system can be defined as an n-component system, which works if and only if at least k of the n components are operational. The literature on k-out-of-n systems is vast (see Chakravarthy *et al.* [1] and the references therein). In a highly competitive world organizations pay high attention on giving service to external customers in addition to their internal customers. One main intention behind this is the additional income collected through external service. In addition, it may be expected that the expertise of the server be improved by attending jobs that are more diverse. The main drawback of providing service to external customers is that this decreases the attention on the internal customers. Also there is a chance of the service facility getting overloaded with too much of work. Hence keeping a proper balance between the internal and external services is much needed and at the same time much harder a task. In this context studying the reliability of a k-out-of-n system where the server attends external customers also could be of great value. There had been a few studies by Dudin *et al.* [2], Krishnamoorthy *et al.* [3, 4] in this area. In [2], the external customers are sent to an orbit from where they can try to access the idle server. Once selected for service, an external customer is assumed to get a non-preemptive service. Through numerical illustrations they show that providing service to external customers in this fashion is economical to the system in comparison with the decrease in the reliability caused due to external service. In [3] it is assumed that the external customers, finding the service station busy on arrival, are directed to a pool of infinite capacity. They also assume that if the size of the buffer of internal customers is less than L, a pooled customer is selected for service with probability p. In [4], a finite pool and an orbit of infinite capacity accommodate the external customers in such a manner that external customers join the orbit with some probability and from there try to enter the pool. The external customers are selected for service from the pool. The internal customers (failed components) are served based on an N-policy In addition they assume that the on-going service of an external customer is not pre-empted on accumulation of N-failed components. Under this assumption, numerical illustrations on [3, 4] indicate a decrease in the server idle probability, and an increase in the overall system revenue as in [2].

In the present paper we study a k-out-of-n system, where the sever also offers service to external customers for additional income. For optimizing revenue by way of providing external service, maintaining a high system reliability, we introduce an N-policy in which the service of the failed components starts on accumulation of N failed components at the beginning of each cycle (a cycle starts with the server being switched over to service of the failed components of the system on accumulation of N components until all of them, and the subsequent failed components get repaired. In other words the moment all failed components of the k-out of-n system are repaired, the server switches over to serve external customers; the service to external customers continue until the next epoch at which N failed components of the system again get accumulated). The service to the external customers is of preemptive nature in the sense that their service is interrupted on accumulation of N failed components. The external customers join a queue of infinite capacity on finding a busy server, provided the customer in service is an external arrival. The current study differs from that in [4] in that the pool (waiting space) of external customers is of infinite capacity and here there is no orbit of retrying customers. Also in contrast to [4], in the present work the service of external customers is assumed to be preemptive in nature. Under these stronger assumptions we obtain an explicit steady state distribution of the underlying Markov chain has been obtained.

This paper is arranged as follows: In section 2, we perform the Stochastic Modeling of the above problem and in section 3, we perform the steady state analysis of the underlying Markov chain after finding a necessary and sufficient condition for the stability of the system. Section 4, discusses a special case of the model discussed in Section 2, where the service time distributions are assumed to follow exponential distribution. In section 5 we conduct a numerical study of the model discussed in

Section 4 and compares it with a model in which no external customers are allowed. Section 6 concludes the discussion.

2. Modeling and Analysis

In this paper we study the reliability of a k-out-of-n system with repair by a single repair facility which also provides service to external customers. The system consists of two parts.

(1) A main queue consisting of customers (failed components of the k-out-of-n system) and

(2) A queue of external customers.

A k-out-of-n system is in the up state (working state) as long as at least k components are in operational state. Otherwise the system is in the down state.

The arrival process.

Arrival of main customers have inter-occurrence time exponentially distributed with parameter λ_i when the number of operational components of the k-out-of-n system is i. By taking $\lambda_i = \frac{\lambda}{i}$ we notice that the failure rate is a constant λ. Arrival of external customers have inter-occurrence time exponentially distributed with parameter $\overline{\lambda}$. Arrival of external customers is temporarily halted while serving the main customers (the failed components of the k-out -of-n system).

The service process.

Commencement of service to the failed components of the main system is governed by the N-policy, that is at the epoch the system starts with all components operational, the server starts attending one by one the customers from the queue of external customers (if there is any waiting). At the epoch when the accumulated number of failed components of the main system reaches N, the external customer in service will get pre-empted and the server is switched on to the service of main customers. Service times of main customers and external customers follow phase-type distributions with representations (α, S) and (β, T) of orders m_1 and m_0 respectively.

Objective.

To maximize the reliability of a k-out-of-n system with repair by a single server, who provides service to external customers also, based on N-policy.

The Markov Chain.

Let $X_1(t)$ denotes at time t number of external customers in the system including the one getting service (if any) ,

$X_2(t)$ denotes the server status at time t defined as;

$$X_2(t) = \begin{cases} 0, & \text{if the server is idle or serving an external customer} \\ 1, & \text{if the server is busy with a failed component.} \end{cases}$$

$X_3(t)$ denotes number of main customers in the system at time t including the one getting service (if any). $X_4(t)$ denotes the phase of the service process.

Let $X(t) = (X_1(t), X_2(t), X_3(t), X_4(t))$ then $\{X(t),\ t \geq 0\}$ is a continuous time Markov chain on the state space whose levels are designated

$$l(0) = \{(0,0,j_1)/0 \leq j_1 \leq N-1\} \cup \{(0,1,j_1,j_2)/1 \leq j_1 \leq n-k+1, 1 \leq j_2 \leq m_1\},$$

$$l(i) = l(i,0) \cup l(i,1),$$

$$l(i,0) = \{(i,0,j_1,j_2)/0 \leq j_1 \leq N-1, 1 \leq j_2 \leq m_0\}$$

$$l(i,1) = \{(i,1,j_1,j_2)/1 \leq j_1 \leq n-k+1, 1 \leq j_2 \leq m_1\}.$$

In the sequel,

(i) I_n denotes the identity matrix of order n;
(ii) I denotes an identity matrix of appropriate size;
(iii) e_n denotes a $n \times 1$ column matrix of 1's
(iv) e denotes a column matrix of 1's of appropriate order;
(v) E_n denotes a square matrix of order n defined as

$$E_n(i,j) = \begin{cases} -1, & \text{if } i = j; 1 \leq i \leq n \\ 1, & \text{if } j = i+1; 1 \leq i \leq n-1 \\ 0, & \text{otherwise} \end{cases}$$

(vi) E'_n = Transpose of E_n

(vii) $r_n(i)$ denotes a $1 \times n$ row matrix whose i^{th} entry is 1 and all other entries are zeros

(viii) $C_n(i)$ = Transpose of $r_n(i)$

(ix) $\otimes$ denotes Kronecker product of matrices

(x) $S^0 = -Se$, $T^0 = -Te$.

The infinitesimal generator matrix of $\{X(t)\}$ is given by

$$Q = \begin{bmatrix} \widetilde{A}_1 & \widetilde{A}_0 & & & & \\ \widetilde{A}_2 & A_1 & A_0 & & & \\ & A_2 & A_1 & A_0 & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \cdot \\ & & & & \cdot & \cdot \end{bmatrix}, \text{ where } \widetilde{A}_1 = \begin{bmatrix} \widetilde{A}_{00} & \widetilde{A}_{01} \\ \widetilde{A}_{10} & \widetilde{A}_{11} \end{bmatrix}$$

$$\widetilde{A}_{00} = \lambda E_N - \overline{\lambda} I_N, \widetilde{A}_{01} = [C_N(N) \otimes r_{n-k+1}(N)] \otimes \lambda\alpha, \widetilde{A}_{10} = [C_{n-k+1}(1) \otimes r_N(1)] \otimes S^0,$$

$$\widetilde{A}_{11} = I_{n-k+1} \otimes S + (E'_{n-k+1} + I_{n-k+1}) \otimes \left(S^0\alpha\right)$$
$$+ [E_{n-k+1} + C_{n-k+1}(n-k+1) \otimes r_{n-k+1}(n-k+1)] \otimes \lambda I_{m_1};$$

$$A_1 = \begin{bmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{bmatrix};$$

$$A_{00} = E_N \otimes \lambda I_{m_0} + I_N \otimes \left(T - \overline{\lambda} I_{m_0}\right), A_{01} = [C_N(N) \otimes r_{n-k+1}(N)] \otimes (\lambda e_{m_0}\alpha);$$

$$A_{10} = [C_{n-k+1}(1) \otimes r_N(1)] \otimes (S^0\beta), \quad A_{11} = \widetilde{A}_{11};$$

$$\widetilde{A}_0 = \begin{bmatrix} I_N \otimes (\overline{\lambda}\beta) & 0 \\ 0 & 0 \end{bmatrix}, \widetilde{A}_2 = \begin{bmatrix} I_N \otimes T^0 & 0 \\ 0 & 0 \end{bmatrix}, A_0 = \begin{bmatrix} I_N \otimes (\overline{\lambda} I_{m_0}) & 0 \\ 0 & 0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} I_N \otimes (T^0\beta) & 0 \\ 0 & 0 \end{bmatrix}$$

3. **Steady State Analysis**

3.1. **Stability condition.**

Let $A = A_0 + A_1 + A_2$ and π be the steady state vector of A. That is π satisfies the equations

$$\boldsymbol{\pi} A = 0 \quad \text{and} \tag{3.1}$$

$$\boldsymbol{\pi}\, \boldsymbol{e} = 1. \tag{3.2}$$

Partitioning $\boldsymbol{\pi}$ as $\boldsymbol{\pi} = (\boldsymbol{\pi_0}, \boldsymbol{\pi_1})$, equation (3.1) gives

$$\boldsymbol{\pi_0}\left[E_N \otimes \lambda I_{m_0} + I_N \otimes (T + T^0\beta)\right] + \boldsymbol{\pi_1} A_{10} = 0 \tag{3.3}$$

$$\boldsymbol{\pi_0} A_{01} + \boldsymbol{\pi_1} A_{11} = 0. \tag{3.4}$$

From equation (3.4), $\boldsymbol{\pi_1} = -\boldsymbol{\pi_0} A_{01} A_{11}^{-1}$.
Substituting in equation (3.3), we get

$$\boldsymbol{\pi_0}\left[E_N \otimes \lambda I_{m_0} + I_N \otimes (T + T^0\beta)\right] - \boldsymbol{\pi_0} A_{01} A_{11}^{-1} A_{10} = 0 \tag{3.5}$$

We notice that $A_{10} = (-A_{11}e)(r_N(1) \otimes \beta)$ and therefore $-A_{11}^{-1}A_{10} = e(r_N(1) \otimes \beta)$

$$-A_{01}A_{11}^{-1}A_{10} = (C_N(N) \otimes \lambda e_{m_0})\,(r_N(1) \otimes \beta)\,. \tag{3.6}$$

Thus equation (3.5) reduce to

$$\boldsymbol{\pi_0}\left[E_N \otimes \lambda I_{m_0} + (C_N(N) \otimes r_N(1)) \otimes (\lambda e_{m_0}\beta) + I_N \otimes (T + T^0\beta)\right] = 0. \tag{3.7}$$

Further partitioning $\boldsymbol{\pi_0} = (\pi_{0,0}, \pi_{0,1}, \ldots, \pi_{0,N-1})$, equation (3.7) give rise to the following set equations

$$\pi_{0,0}\left(T + T^0\beta - \lambda I_{m_0}\right) + \pi_{0,N-1}\lambda e_{m_0}\beta = 0 \tag{3.8}$$

$$\pi_{0,i}\lambda I_{m_0} + \pi_{0,i+1}\left(T + T^0\beta - \lambda I_{m_0}\right) = 0, 0 \le i \le N-1. \tag{3.9}$$

Postmultiply both sides of equation (3.8) and (3.9) by the column vector e, we get

$$\pi_{0,0}\left(T + T^0\beta - \lambda I_{m_0} + \lambda e_{m_0}\beta\right) = 0 \tag{3.10}$$

$$\pi_{0,i}\boldsymbol{e} = \pi_{0,i+1}\boldsymbol{e}, 0 \le i \le N-1. \tag{3.11}$$

And this gives

$$\pi_{0,0} = a\eta \tag{3.12}$$

where η is the steady state vector of the generator matrix $T + T^0\beta - \lambda I_{m_0} + \lambda e_{m_0}\beta$ and 'a' is a constant.

Now equation (3.9) gives

$$\pi_{0,i} = (-1)^i a\lambda^i \eta \left(T + T^0\beta - \lambda I_{m_0}\right)^{-i}, 0 \leq i \leq N-1. \tag{3.13}$$

Equation (3.13) determines the vector π_0 up to the multiplicative constant.

It follows from equations (3.11) and (3.13) that

$$\begin{aligned}\boldsymbol{\pi} A_0 \boldsymbol{e} &= \bar{\lambda}\boldsymbol{\pi_0}\,\boldsymbol{e}\\ &= \bar{\lambda}\,aN\\ \boldsymbol{\pi} A_2 \boldsymbol{e} &= \sum_{i=0}^{N-1}\pi_{0,i}T^0\\ &= a\sum_{i=0}^{N-1}(-1)^i\lambda^i\eta\left(T + T^0\beta - \lambda I_{m_0}\right)^{-i}T^0.\end{aligned}$$

Here $\boldsymbol{\pi} A_0\boldsymbol{e} < \boldsymbol{\pi} A_2 \boldsymbol{e}$ becomes

$$N\bar{\lambda} < \sum_{i=0}^{N-1}(-1)^i\lambda^i\eta\left(T + T^0\beta - \lambda I_{m_0}\right)^{-i}T^0.$$

This leads to the following theorem for the stability of the system.

Theorem 3.1. *The Markov chain $\{X(t)\}$ is stable if and only if*

$$N\bar{\lambda} < \sum_{i=0}^{N-1}(-1)^i\lambda^i\eta\left(T + T^0\beta - \lambda I_{m_0}\right)^{-i}T^0.$$

3.2. **Steady State Vector.**

The steady state vector $\boldsymbol{x}$ is partitioned as $\boldsymbol{x} = (x_0, x_1, x_2, \ldots)$ satisfies the equations

$$\begin{aligned}x_0\widetilde{A}_1 + x_1\widetilde{A}_2 &= 0\\ x_0\widetilde{A}_0 + x_1A_1 + x_2A_2 &= 0\\ x_iA_0 + x_{i+1}A_1 + x_{i+2}A_2 &= 0, i \geq 1.\end{aligned}$$

Matrix theoretic approach (See Neuts [5]*) gives*

$$x_i = x_1 R^{i-1},\ i \geq 1 \tag{3.14}$$

where R is the minimal non negative solution of the matrix quadratic equation

$$R^2 A_2 + RA_1 + A_0 = 0. \tag{3.15}$$

It then follows that

$$x_1 = -x_0 \widetilde{A}_0 (A_1 + RA_2)^{-1} \tag{3.16}$$

and that x_0 satisfies the system of equations

$$x_0 \left(\widetilde{A}_1 - \widetilde{A}_0 \left(A_1 + RA_2\right)^{-1} \widetilde{A}_2 \right) = 0. \tag{3.17}$$

From the structure of the matrix A_0, it follows that the R matrix has the form

$$R = \begin{bmatrix} R_1 & R_2 \\ 0 & 0 \end{bmatrix} \tag{3.18}$$

where R_1 is a square matrix of order Nm_0 and R_2 is a matrix of order $Nm_0 \times (n-k+1)m_1$.

$$R^2 = \begin{bmatrix} R_1^2 & R_1 R_2 \\ 0 & 0 \end{bmatrix}$$

Equation (3.15) *then reduces to the following equations*

$$R_1^2 \left(I_N \otimes T^0 \beta \right) + R_1 A_{00} + R_2 A_{10} + I_N \otimes \bar{\lambda} I_{m_0} = 0 \tag{3.19}$$

$$R_1 A_{01} + R_2 A_{11} = 0 \tag{3.20}$$

$$\textit{Equation } (3.20) \textit{ gives} \quad R_2 = -R_1 A_{01} A_{11}^{-1} \tag{3.21}$$

which when substituted in Equation (3.19) *gives*

$$R_1^2 \left(I_N \otimes T^0 \beta \right) + R_1 A_{00} - R_1 A_{01} A_{11}^{-1} A_{10} + \bar{\lambda} I_{Nm_0} = 0$$

$$\text{i.e., } R_1^2 \left(I_N \otimes T^0 \beta \right) + R_1 \left(A_{00} - A_{01} A_{11}^{-1} A_{10} \right) + \bar{\lambda} I_{Nm_0} = 0.$$

Using equation (3.6), *the above equation can be rewritten as*

$$R_1^2\left(I_N \otimes T^0\beta\right) + R_1\left[A_{00} + (C_N(N) \otimes r_N(1)) \otimes (\lambda e_{m_0}\beta)\right] + \bar{\lambda} I_{Nm_0} = 0. \tag{3.22}$$

Solving equation (3.22), *we get* R_1 *and hence the steady state vector of* $\{X(t)\}$. *For Solving equation* (3.22) *we use Logarithmic reduction algorithm (refer Latouche and Ramaswami* [6]*).*

4. A Special Case

We now concentrate on a special case of the problem discussed in Section 2 where the service time distributions of main and external customers follow exponential distributions with parameters μ and $\bar{\mu}$ respectively. As expected, this resulted in arriving at explicit expression for the stability condition, steady state distribution and several performance measures.

4.1. The Markov Chain Model.

With $X_1(t)$, $X_2(t)$ and $X_3(t)$ having same definition as in section 2, $\widetilde{X}(t) = (X_1(t), X_2(t), X_3(t))$ is a continuous time Markov chain on the state space

$$\{(j_1, 0, j_2) | j_1 \geq 0; 0 \leq j_2 \leq N-1\} \cup \{(j_1, 1, j_2) | j_1 \geq 0; 0 \leq j_2 \leq n-k+1\}.$$

Arranging the states lexicographically and then partitioning the state space into levels i, where each level i corresponds to the collection of states with number of external customers in the system including the one getting service (if any) at time t as i. We get the infinitesimal generator of the above chain as

$$\widetilde{Q} = \begin{bmatrix} F_{10} & F_0 & & & \\ F_2 & F_1 & F_0 & & \\ & F_2 & F_1 & F_0 & \\ & & \cdots & \cdots & \cdots \\ & & & \cdots & \cdots & \cdots \end{bmatrix} \tag{4.1}$$

The entries of the matrix are described below.

The transition from level i to level $i+1$ is represented by the matrix

$$F_0 = \begin{bmatrix} \bar{\lambda} I_N & 0_{N \times n-k+1} \\ 0_{(n-k+1)\times N} & 0_{(n-k+1)\times(n-k+1)} \end{bmatrix}.$$

The transition from level i to level $i-1$ is represented by the matrix

$$F_2 = \begin{bmatrix} \bar{\mu} I_N & 0_{N \times n-k+1} \\ 0_{(n-k+1)\times N} & 0_{(n-k+1)\times(n-k+1)} \end{bmatrix}.$$

The transition within level 0 to level 0 is represented by the matrix

$$F_{10} = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}$$

where $\quad B_1 = \lambda E_N - \bar{\lambda} I_N;$

B_2 is a $N \times (n-k+1)$ matrix whose $(N, N)^{\text{th}}$ entry is λ and all other entries are zeroes. B_3 is a $(n-k+1) \times N$ matrix whose $(1, 1)^{\text{th}}$ entry is μ and all other entries are zeroes.

$$B_4 = \lambda E_{n-k+1} + \mu E'_{n-k+1} + \lambda C_{n-k+1}(n-k+1) \otimes r_{n-k+1}(n-k+1).$$

The transitions within level i, $i \geq 1$, is represented by matrix

$$F_1 = \begin{bmatrix} D_1 & B_2 \\ B_3 & B_4 \end{bmatrix}$$

where $D_1 = \lambda E_N - (\bar{\lambda} + \bar{\mu}) I_N$.

4.2. **Steady State Analysis.** First we derive the condition for stability of the system.

4.2.1. Stability condition.

Consider the generator matrix

$$F = F_0 + F_1 + F_2 = \begin{bmatrix} H_1 & H_2 \\ H_3 & B_4 \end{bmatrix}, \quad \text{where} \quad H_1 = \lambda E_N.$$

H_2 is a $N \times (n-k+1)$ matrix whose $(N,N)^{\text{th}}$ entry is λ and all other entries are zeroes.
H_3 is a $(n-k+1) \times N$ matrix whose $(1,1)^{\text{th}}$ entry is μ and all other entries are zeroes.

The stationary probability vector $\widetilde{\Pi} = (\widetilde{\pi}_{(0,0)}, \widetilde{\pi}_{(0,1)}, \cdots, \widetilde{\pi}_{(0,N-1)}, \widetilde{\pi}_{(1,1)}, \cdots, \widetilde{\pi}_{(1,N)} \cdots, \widetilde{\pi}_{(1,n-k+1)})$ of the generator matrix A satisfies the equations $\widetilde{\Pi}F = 0$ and $\widetilde{\Pi}\boldsymbol{e} = 1$.
$\widetilde{\Pi}F = 0$ gives the following equations

$$\widetilde{\pi}_{(0,i)} = \widetilde{\pi}_{(0,0)}, 1 \leq i \leq N-1 \quad \text{and}$$

$$\widetilde{\pi}_{(1,i)} = \begin{cases} \alpha_i \widetilde{\pi}_{(0,0)}, & \text{where } \alpha_i = \sum\limits_{j=1}^{i} (\lambda/\mu)^j, i = 1, 2, \ldots N \\ \beta_i \widetilde{\pi}_{(0,0)}, & \text{where } \beta_i = \sum\limits_{j=1-N+1}^{i} (\lambda/\mu)^j, i = N+1, \ldots n-k+1 \end{cases}$$

The normalizing condition $\widetilde{\Pi}\boldsymbol{e} = 1$ gives $\widetilde{\pi}_{(0,0)} = \frac{1}{\varphi - \psi}$, where

$$\varphi = N + \frac{\left(\mu^{N-2} - \lambda^{N-2}\right)\lambda}{(\mu - \lambda)\mu^N} \left\{ N + \frac{\lambda\left(\mu^{n-k+1-N} - \lambda^{n-k+1-N}\right)}{\mu^{n-k+1-N}(\mu - \lambda)} \right\} \quad \text{and}$$

$$\psi = \frac{(\mu - \lambda)\left(\mu^{N-1} - (N-1)\lambda^N\right) + \lambda\mu\left(\mu^{N-2} - \lambda^{N-2}\right)}{\mu^{N-1}(\mu - \lambda)}$$

Thus we arrive at the following

Theorem 4.1. *The process $\{\widetilde{X}(t), t \geq 0\}$ is positive recurrent if and only if $\overline{\lambda} < \overline{\mu}$.*

Proof. It is well known (see Neuts [5]) that the Markov chain with infinitesimal generator $\widetilde{Q}$ is stable if and only if $\widetilde{\pi}F_0e < \widetilde{\pi}F_2e$, that is if and only if the left drift rate exceeds that to the right.
We have $\widetilde{\pi}F_0e = N\overline{\lambda}\widetilde{\pi}_{(0,0)}$ and $\widetilde{\pi}F_2e = N\overline{\mu}\widetilde{\pi}_{(0,0)}$. Thus $\{\widetilde{X}(t), t \geq 0\}$ is positive recurrent if and only if $\overline{\lambda} < \overline{\mu}$. □

4.2.2. Steady State Distribution.

Here using the steady state vector $\widetilde{\Pi}$ of the generator matrix F, we proceed construct the steady state vector $\widetilde{X} = (\widetilde{X}(0), \widetilde{X}(1), \widetilde{X}(2), \ldots)$ of the Markov chain $\{\widetilde{X}(t), t \geq 0\}$ by defining, $\widetilde{X}(i) = \eta \left(\frac{\overline{\lambda}}{\overline{\mu}}\right)^i \widetilde{\Pi}$, for $i \geq 0$, where η is a positive constant to be found out.

First we will prove that $\widetilde{X}$ satisfies the equation $\widetilde{X}\widetilde{Q} = 0$. For this, notice that we can decompose the infinitesimal generator matrix $\widetilde{Q}$ as $\widetilde{Q} = \widetilde{Q}_1 + \widetilde{Q}_2$, where

$$\widetilde{Q}_1 = \begin{bmatrix} F & & & & \\ & F & & & \\ & & F & & \\ & & & \cdots & \\ & & & & \cdots \end{bmatrix} \quad \text{and}$$

$$\widetilde{Q}_2 = \begin{bmatrix} -F_0 & F_0 & & & & \\ F_2 & \overline{F_1} & F_0 & & & \\ & F_2 & \overline{F_1} & F_0 & & \\ & & \cdots & \cdots & \cdots & \\ & & & \cdots & \cdots & \cdots \end{bmatrix},$$

where each entry is a square matrix of order $N + n - k + 1$ listed as:

$$\overline{F}_1 = \begin{bmatrix} -(\overline{\lambda} + \overline{\mu})I_N & 0_{N \times n-k+1} \\ 0_{(n-k+1)\times N} & 0_{(n-k+1)\times(n-k+1)} \end{bmatrix}.$$

Since $\widetilde{\Pi}F = 0$ and $\widetilde{X}(i) = \eta \left(\frac{\overline{\lambda}}{\overline{\mu}}\right)^i \widetilde{\Pi}$, we have

$$\widetilde{X}\widetilde{Q}_1 = 0. \tag{4.2}$$

Now,

$$\widetilde{X}\widetilde{Q}_2 = \left[\widetilde{X}(0)(-F_0) + \widetilde{X}(1)F_2, \widetilde{X}(0)F_0 + \widetilde{X}(1)\overline{F_1} + \widetilde{X}(2)F_2, \widetilde{X}(1)F_0 + \widetilde{X}(2)\overline{F_1} + \widetilde{X}(3)F_2, \cdots\right].$$

Notice that $(-F_0) + \frac{\overline{\lambda}}{\overline{\mu}}F_2 = 0$ and

$$F_0 + \left(\frac{\overline{\lambda}}{\overline{\mu}}\right)\overline{F_1} + \left(\frac{\overline{\lambda}}{\overline{\mu}}\right)^2 F_2 = F_0 + \left(\frac{\overline{\lambda}}{\overline{\mu}}\right)\left(\overline{F_1} + \frac{\overline{\lambda}}{\overline{\mu}}F_2\right)$$

$$= F_0 - \frac{\overline{\lambda}}{\overline{\mu}} F_2$$
$$= 0,$$

which leads us to $\widetilde{X}(0)(-F_0) + \widetilde{X}(1)F_2 = 0$ and

$$\widetilde{X}(i)F_0 + \widetilde{X}(i+1)\overline{F_1} + \widetilde{X}(i+1)F_2 = \left(\frac{\overline{\lambda}}{\overline{\mu}}\right)^i \widetilde{X}(0)\left[F_0 + \frac{\overline{\lambda}}{\overline{\mu}}F_1 + \left(\frac{\overline{\lambda}}{\overline{\mu}}\right)^2 F_2\right]$$
$$= 0,\ i = 0, 1, 2, 3, \ldots.$$

Hence
$$\widetilde{X}Q_2 = 0. \tag{4.3}$$
From (4.2) and (4.3), we have $\widetilde{X}\widetilde{Q}_1 + \widetilde{X}\widetilde{Q}_2 = 0$, which implies that $\widetilde{X}\widetilde{Q} = 0$.
Finally, $\widetilde{X}\boldsymbol{e} = 1$ gives the unknown constant $\eta = \frac{(\overline{\mu}-\overline{\lambda})}{\overline{\mu}}$.

Hence, $\widetilde{X} = (\widetilde{X}(0), \widetilde{X}(1), \widetilde{X}(2)\cdots)$, where $\widetilde{X}(i) = \left(\frac{\overline{\mu}-\overline{\lambda}}{\overline{\mu}}\right)\left(\frac{\overline{\lambda}}{\overline{\mu}}\right)^i \widetilde{\Pi}$ is the steady state vector for the matrix $\widetilde{Q}$ and we have the following theorem;

Theorem 4.2. *Let* $\widetilde{\Pi} = (\widetilde{\pi}_{(0,0)}, \widetilde{\pi}_{(0,1)}, \cdots, \widetilde{\pi}_{(0,N-1)}, \widetilde{\pi}_{(1,1)}, \cdots, \widetilde{\pi}_{(1,N)}, \cdots \widetilde{\pi}_{(1,n-k+1)})$ *be the steady state vector for the matrix F, where*

$$\widetilde{\pi}_{(0,i)} = \widetilde{\pi}_{(0,0)}, 1 \le i \le N-1 \quad \textit{and}$$
$$\widetilde{\pi}_{(1,i)} = \begin{cases} \alpha_i \widetilde{\pi}_{(0,0)}, & \textit{with } \alpha_i = \sum_{j=1}^{i} (\lambda/\mu)^j, i = 1, 2, \ldots N \\ \beta_i \widetilde{\pi}_{(0,0)}, & \textit{for } \beta_i = \sum_{j=1-N+1}^{i} (\lambda/\mu)^j, i = N+1, \ldots n-k+1 \end{cases}.$$

Further $\widetilde{\pi}_{(0,0)} = \frac{1}{\varphi-\psi}$, *where*

$$\varphi = N + \frac{\left(\mu^{N-2} - \lambda^{N-2}\right)\lambda}{(\mu-\lambda)\mu^N}\left\{N + \frac{\lambda\left(\mu^{n-k+1-N} - \lambda^{n-k+1-N}\right)}{\mu^{n-k+1-N}(\mu-\lambda)}\right\} \quad \textit{and}$$
$$\psi = \frac{(\mu-\lambda)\left(\mu^{N-1} - (N-1)\lambda^N\right) + \lambda\mu\left(\mu^{N-2} - \lambda^{N-2}\right)}{\mu^{N-1}(\mu-\lambda)}.$$

Then $\widetilde{X} = (\widetilde{X}(0), \widetilde{X}(1), \widetilde{X}(2)\cdots)$, *where* $\widetilde{X}(i) = \left(1 - \frac{\overline{\lambda}}{\overline{\mu}}\right)\left(\frac{\overline{\lambda}}{\overline{\mu}}\right)^i \widetilde{\Pi}$ *is the steady state probability vector for the Markov chain* $\{\widetilde{X}(t), t \ge 0\}$.

4.3. **Performance Measures.**

Here we derive certain important performance measures of the system under study.

4.3.1. Busy period of the server with the failed components of the main system.

The busy period of the server with failed components starts the instant when N failed components accumulate and it ends when no failed components are left in the system. Let $T_N(i)$, for $i \geq 0$, denote the server busy period with failed components, which starts with i external customers in the system. Note that, the number of external customers does not affect the busy period of the server with the failed components. Hence, $T_N(i) = T_N$, for $i \geq 0$. For analyzing the time T_N, we consider the Markov chain $\{Y(t)\}$ with state space $\{0, 1, 2, \ldots, N, N+1, \ldots, n-k+1\}$ and infinitesimal generator given by:

$$B_N = \begin{bmatrix} 0 & 0 \\ -\widehat{B}_N e & \widehat{B}_N \end{bmatrix}, \quad \text{where}$$

$$\widehat{B}_N = \lambda E_{n-k+1} + \mu E'_{n-k+1}.$$

Note that $Y(t)$ denotes the number of failed components of the main system and $Y(t) = 0$ is considered as an absorbing state; so that the busy period T_N is the time until absorption in the Markov chain $\{Y(t)\}$, assuming that it starts at the state N. Hence, the busy period T_N has a phase type distribution with representation $(\omega, \widehat{B}_N)$, where the probability vector $\omega = (0, \ldots, 0, 1, 0, \ldots, 0)$, with 1 appearing in the N^{th} position. The expected value of T_N is therefore given by $ET_N = -\omega(B_N^{-1})\boldsymbol{e}$ where $\boldsymbol{e}$ is a column vector with $n-k+1$ elements all equal to 1. Now for finding ET_N, let us partition the column vector $(\widehat{B}_N^{-1})\boldsymbol{e}$ as $(t_1, t_2, \ldots, t_{n-k+1})^T$. Then the identity $\widehat{B}_N(\widehat{B}_N^{-1})\boldsymbol{e} = \boldsymbol{e}$ leads us to the following equations:

$$-(\lambda + \mu)t_1 + \lambda t_2 = 1$$

$$\mu t_{i-1} - (\lambda + \mu)t_i + \lambda t_{i+1} = 1, \quad \text{for } 2 \leq i \leq n-k$$

$$\mu t_{n-k} - \mu t_{n-k+1} = 1.$$

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The above equations give

$$t_i - t_{i+1} = \frac{1}{\mu}\sum^{n-k-i} j = 0(\lambda/\mu)^j, 1 \le i \le n-k$$

$$t_{n-k} - t_{n-k+1} = \frac{1}{\mu} \quad \text{and} \quad -\mu t_1 = \sum_{j=0}^{n-k}(\lambda/\mu)^j.$$

Hence

$$ET_N = -t_N = \frac{1}{\mu}\left(N\sum_{j=0}^{n-k-N+1}(\lambda/\mu)^j + \sum_{j=n-k-N+2}^{n-k}(n-k+1-j)(\lambda/\mu)^j\right). \tag{4.4}$$

The expected value of the busy period of the server with failed components, which starts with an arbitrary number of external customers is given by

$$E_B = ET_N\sum_{j_1=0}^{\infty}\widetilde{x}(j_1, 0, N-1)$$

$$= \frac{1}{(\varphi-\psi)}\frac{1}{\mu}\left(N\sum_{j=0}^{n-k-N+1}(\lambda/\mu)^j + \sum_{j=n-k-N+2}^{n-k}(n-k+1-j)(\lambda/\mu)^j\right). \tag{4.5}$$

We sum up the above results in

Theorem 4.3. *The busy period of the server with the repair of the components of the k-out-of-n system has phase type distribution with representation* $(\omega, \widehat{B}_N)$*. The expected length of the busy period is given by* (4.5).

4.3.2. Expected number of pre-emptions of an external customer who is taken for service.

Consider the Markov process $X_p(t) = (N_p(t), J(t))$, where $N_p(t)$ is the number of pre-emptions occurred upto time t (measured from the time he is taken for service) of a particular external customer who is taken for service and $J(t)$ is the number of failed components of the main system. Then $X_p(t)$ has the state space

$$\left\{(j_1, j_2)\big/ j_1 = 0, 1, 2, \ldots, 0 \le j_2 \le N-1\right\} \cup \{\Delta\}$$

where Δ is an absorbing state which denotes the service completion of the external customer. The infinitesimal generator of this process is

$$Q = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & \cdots & \cdots \\ \widetilde{T}^0 & \widetilde{T} & \widehat{A}_0 & 0 & \cdots & \cdots & \cdots \\ \widetilde{T}^0 & 0 & \widetilde{T} & \widehat{A}_0 & \cdots & \cdots & \cdots \\ \widetilde{T}^0 & 0 & 0 & \widetilde{T} & \widehat{A}_0 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}, \text{ where } \quad \widetilde{T}^0 = \overline{\mu} e_N$$

$$\widetilde{T} = \lambda E_N - \overline{\mu} I_N$$

and $\widehat{A}_0$ is an $N \times N$ matrix whose $(N, 1)^{\text{th}}$ entry is λ.

If p_{k_i} is the probability for k pre-emptions of an external customer who starts service with i failed components, then $p_{0_i} = \left(-\widetilde{T}^{-1}\widetilde{T}^0\right)_i = 1 - \left(\frac{\lambda}{\lambda+\overline{\mu}}\right)^{N-i}$, $0 \le i \le N-1$ and for $k \ge 1$,

$$\begin{aligned} p_{k_i} &= \left(\left(-\widetilde{T}^{-1}\widehat{A}^0\right)\left(-\widetilde{T}^{-1}\widetilde{T}^0\right)\right)_i \\ &= \left(\frac{\lambda}{\lambda+\overline{\mu}}\right)^{N-i}\left(\frac{\lambda}{\lambda+\overline{\mu}}\right)^{N(k-1)}\left(1-\left(\frac{\lambda}{\lambda+\overline{\mu}}\right)^N\right) \\ &= \left(\frac{\lambda}{\lambda+\overline{\mu}}\right)^{Nk-i}\left(1-\left(\frac{\lambda}{\lambda+\overline{\mu}}\right)^N\right). \end{aligned}$$

Expected number of pre-emptions of an external customer, starting service with i failed components

$$= \sum_{k=0}^{\infty} k p_{k_i} = \left(1-\left(\frac{\lambda}{\lambda+\overline{\mu}}\right)^N\right)^{-1}\left(\frac{\lambda}{\lambda+\overline{\mu}}\right)^{N-i}.$$

4.3.3. Expected waiting time of an external customer.

For computing the expected waiting time of an external customer who joins as the r^{th} customer in the queue of external customers, we consider the Markov process $X_w(t) = (J_1(t), S(t), J_2(t))$, where $J_1(t)$ is the rank of the external customer, $S(t) = 0$ if the server is busy with external customers and $S(t) = 1$ if the server is busy with a main customer. $J_2(t)$ is the number of main customers in the system. The rank

$J_1(t)$ of an external customer is assumed to be 'l' if it finds $l-1$ external customers ahead of it. The rank of an external customer may decrease by 1 if an external customer ahead of it leaves the system after completing the service. Now consider the Markoov process $X_w(t)$ for a tagged external customer who finds $l-1$ external customers ahead of it while joining the system. The state space for this process is given by $\{*\} \cup \{\{1,2,\ldots,l\} \times (\{0\} \times \{0,1,\ldots,N-1\} \cup \{1\} \times \{1,2,\ldots,n-k+1\})\}$, where $*$ is an absorbing state, which denotes the service completion of the tagged customer. The infinitesimal generator Q_w of this process is $Q_w = \begin{bmatrix} 0 & 0 \\ W_l^0 & W_l \end{bmatrix}$, where

$$W_l = \begin{bmatrix} w_{11} & & & & \\ w_{22} & w_{12} & & & \\ & w_{23} & w_{13} & & \\ & & & \ddots & \\ & & & w_{2l} & w_{1l} \end{bmatrix}$$

with
$$w_{1i} = F_1 + F_0; 1 \le i \le l$$
$$w_{2i} = F_2; 1 \le i \le l$$
$$w_l^0 = C_l(1) \otimes (F_2 e)$$

The waiting time of the tagged customer is the time until absorption in the Markov process $X_w(t)$. Let $E_W^{(i)}(l)$ denote the expected waiting time of a tagged customer who joins the system with rank l, who finds 'i' failed components. Defining the row vector $\widetilde{\theta}_i$ as $\widetilde{\theta}_i = r_l(l) \otimes r_{N+n-k+1}(i+1)$, $0 \le i \le N-1$. Then $E_W^{(i)}(l) = -\widetilde{\theta}_i W_l^{-1} \boldsymbol{e}$, $0 \le i \le N-1$. Let $E_W(l)$ be the $N \times 1$ column matrix whose $(i,1)^{\text{th}}$ entry is $E_W^{(i-1)}(l)$. Taking the probability that an external customer see i external customers, j failed components and server busy with external customers on its arrival as $\left(1-\frac{\bar{\lambda}}{\bar{\mu}}\right)\left(\frac{\bar{\lambda}}{\bar{\mu}}\right)^i \frac{1}{\varphi-\psi}$, the expected waiting time of an arbitrary external customers is given by

$$\sum_{i=0}^{\infty} \left(1-\frac{\bar{\lambda}}{\bar{\mu}}\right)\left(\frac{\bar{\lambda}}{\bar{\mu}}\right)^i \frac{1}{\varphi-\psi} \sum_{j=0}^{N-1} E_W^{(j)}(i+1).$$

4.4. Other Performance measures.

(1) Fraction of time the system is down is given by,

$$P_{down} = \sum_{j_1=0}^{\infty} x(j_1, 1, n-k+1) = \frac{\lambda^{n-k+2-N}\left(\mu^N - \lambda^N\right)}{\mu^{n-k+1}(\mu-\lambda)(\varphi-\psi)}.$$

(2) System reliability defined as the probability that at least k components are operational

$$P_{rel} = 1 - P_{down} = 1 - \frac{\lambda^{n-k+2-N}(\mu^N - \lambda^N)}{\mu^{n-k+1}(\mu-\lambda)(\varphi-\psi)}.$$

(3) Average number of external units waiting in the queue is given by,

$$N_q = \sum_{j_1=0}^{\infty} j_1 \sum_{j_3=1}^{n-k+1} X_{(j_1,1,j_3)} + \sum_{j_1=2}^{\infty}(j_1-1)\sum_{j_3=1}^{N-1} X_{(j_1,0,j_3)}$$

$$= \bar{\lambda}\left[\frac{1}{\bar{\mu}-\bar{\lambda}} - \frac{N}{\bar{\mu}(\varphi-\psi)}\right]$$

(4) Average number of failed components of the main system,

$$N_{fail} = \sum_{j_3=0}^{N-1} J_3\left(\sum_{j_1=0}^{\infty} X_{(j_1,0,j_3)}\right) + \sum_{j_3=0}^{n-k+1} j_3\left(\sum_{j_1=0}^{\infty} X_{(j_1,1,j_3)}\right)$$

$$= \frac{1}{(\varphi-\psi)}\left\{\frac{N(N-1)}{2} + \sum_{i=1}^{N-1} i\left(\sum_{j=1}^{i}(\lambda/\mu)^j\right) + \frac{\lambda(\mu^N-\lambda^N)}{\mu^N(\mu-\lambda)}\left(\sum_{i=N}^{n-k+1} i(\lambda/\mu)^{i-N}\right)\right\}$$

(5) Average number of failed components waiting when the server is busy with external customers

$$= \sum_{j_3=0}^{N-1} j_3\left(\sum_{j_1=1}^{\infty} x_{(j_1,0,j_3)}\right)$$

$$= \frac{N(N-1)\bar{\lambda}}{2\bar{\mu}(\varphi-\psi)}$$

(6) Expected number of external customers joining the system,

$$\theta_3 = \bar{\lambda} \sum_{j_1=0}^{\infty} \left(\sum_{j_3=0}^{N-1} x_{(j_1,0,j_3)} \right) = N \frac{\bar{\lambda}}{(\varphi - \psi)}.$$

(7) Expected number of external customers, on arrival, getting service directly

$$= \bar{\mu} \sum_{j_3=0}^{N-1} x_{(0,0,j_3)} = N \frac{(\bar{\mu} - \bar{\lambda})}{(\varphi - \psi)}.$$

(8) Fraction of time the server is busy with external customers,

$$P_{ex.busy} = \sum_{j_1=1}^{\infty} \left(\sum_{j_3=0}^{N-1} x_{(j_1,0,j_3)} \right) = \frac{N \cdot \bar{\lambda}}{\bar{\mu}(\varphi - \psi)}.$$

(9) Probability that the server is found idle,

$$P_{idle} = \sum_{j_3=0}^{N-1} x_{(0,0,j_3)} = N \frac{(\bar{\mu} - \bar{\lambda})}{\bar{\mu}(\varphi - \psi)}.$$

(10) Probability that the server is found busy,

$$P_{busy} = 1 - P_{idle} = 1 - N \frac{(\bar{\mu} - \bar{\lambda})}{\bar{\mu}(\varphi - \psi)}.$$

(11) Expected loss rate of external customers,

$$\theta_4 = \bar{\lambda} \sum_{j_1=0}^{\infty} \left(\sum_{j_3=1}^{n-k+1} x_{(j_1,1,j_3)} \right) = \bar{\lambda} \left(1 - \frac{N}{(\varphi - \psi)} \right).$$

(12) Expected service completion rate of external customers,

$$\theta_5 = \bar{\mu} \sum_{j_1=0}^{\infty} \sum_{j_3=0}^{N-1} x_{(j_1,0,j_3)} = \frac{N\bar{\mu}}{(\varphi - \psi)}.$$

(13) Expected number of external customers in the system when the server is busy with external customers

$$\theta_6 = \sum_{j_1=0}^{\infty} j_1 \left(\sum_{j_3=0}^{N-1} x_{(j_1,0,j_3)} \right) = \frac{N\overline{\lambda}}{(\overline{\mu} - \overline{\lambda})(\varphi - \psi)}.$$

4.5. **Another Special case.** Next we consider second special case of the problem discussed in section 4.1, where we take $N = 1$; that is the case where no special policy has been applied for providing service to external customers. Notice that in this case, at most importance is given to the failed components and an external customer can get service only when there are no failed components in the system. Further, an ongoing external customer's service may be pre-empted if a component of the system fails during the service of the former. Since in this case, knowing the number of external as well as the failed components is enough for determining the server status, the Markov chain becomes $\widehat{X}(t) = (X_1(t), X_3(t))$, with state space $\widehat{S} = \{(j_1, j_2) | j_1 \geq 0, 0 \leq j_2 \leq n - k + 1\}$ and infinitesimal generator

$$\widehat{Q} = \begin{bmatrix} \widetilde{A}_{10} & \widetilde{A}_0 & & & \\ \widetilde{A}_2 & \widetilde{A}_1 & \widetilde{A}_0 & & \\ & \widetilde{A}_2 & \widetilde{A}_1 & \widetilde{A}_0 & \\ & & \cdots & \cdots & \cdots \\ & & & \cdots & \cdots & \cdots \end{bmatrix}, \text{where}$$

$$\widetilde{A}_{10} = \lambda E_{n-k+2} + \lambda C_{n-k+2}(n-k+2) \otimes r_{n-k+2}(n-k+2)$$
$$+ \mu E'_{n-k+2} + (\mu - \overline{\lambda}) C_{n-k+2}(1) \otimes r_{n-k+2}(1);$$

$\widetilde{A}_0$ is a $(n-k+2) \times (n-k+2)$ matrix whose $(1,1)$ entry is $\overline{\lambda}$ and all other entries are zeroes;

$\widetilde{A}_2$ is a $(n-k+2) \times (n-k+2)$ matrix whose $(1,1)$ entry is $\overline{\mu}$ and all other entries are zeroes;

$$\widetilde{A}_1 = \widetilde{A}_{10} - \overline{\mu} C_{n-k+2}(1) \otimes r_{n-k+2}(1).$$

Let $\widetilde{A} = \widetilde{A}_0 + \widetilde{A}_1 + \widetilde{A}_2$; then

$$\widetilde{A} = \lambda E_{n-k+2} + \lambda C_{n-k+2}(n-k+2) \otimes r_{n-k+2}(n-k+2) + \mu E_{n-k+2} + \mu C_{n-k+2}(1) \otimes r_{n-k+2}(1)$$

The stationary probability vector $\widehat{\Pi} = (\widehat{\pi}_{(0,0)}, \widehat{\pi}_{(0,1)}, \ldots \widehat{\pi}_{(0,N-1)}, \widehat{\pi}_{(1,1)}, \ldots \widehat{\pi}_{(1,N)}, \ldots \widehat{\pi}_{(1,n-k+1)})$ of the generator matrix $\widetilde{A}$ is given by $\widehat{\pi}_{(1,i)} = \left(\frac{\lambda}{\mu}\right)^i \widehat{\pi}_{(0,0)}$, $i = 1, 2, \ldots n-k+1$, where

$$\widehat{\pi}_{(0,0)} = \frac{\mu^{n-k+1}(\mu - \lambda)}{(\mu^{n-k+2} - \lambda^{n-k+2})}.$$

Here again, from the condition $\widetilde{\pi}\widetilde{A}_0 e < \widetilde{\pi}\widetilde{A}_2 e$, it can be easily verified that the necessary and sufficient condition for the stability of the Markov chain $\widehat{X}(t)$ is $\overline{\lambda} < \overline{\mu}$.
Applying the same technique as in section 4.2.2, we can easily prove that the vector $\widehat{X} = \left(\widehat{X}(0), \widehat{X}(1), \widehat{X}(2), \ldots\right)$, with $\widehat{X}(i) = \left(1 - \frac{\overline{\lambda}}{\overline{\mu}}\right)\left(\frac{\overline{\lambda}}{\overline{\mu}}\right)^i \widehat{\Pi}$, is the steady state probability vector for the matrix $\widehat{Q}$.

Performance Measures for the case $N = 1$

(1) Fraction of time the system is down,

$$P_{down} = \sum_{j_1=0}^{\infty} x(j_1, 1, n-k+1) = \frac{\lambda^{n-k+1}(\mu - \lambda)}{(\mu^{n-k+2} - \lambda^{n-k+2})}.$$

(2) System reliability,

$$P_{rel} = 1 - P_{down} = 1 - \sum_{j_1=0}^{\infty} x(j_1, 1, n-k+1) = \frac{\mu\left(\mu^{n-k+1} - \lambda^{n-k+1}\right)}{(\mu^{n-k+2} - \lambda^{n-k+2})}.$$

(3) Average number of customers waiting in the queue,

$$N_q = \sum_{j_1=2}^{\infty} X_{(j_1,0,1)} + \sum_{j_1=0}^{\infty} j_1 \left(\sum_{j_3=1}^{n-k+1} x(j_1, 1, j_3)\right)$$

$$= \frac{\overline{\mu}}{(\overline{\mu} - \overline{\lambda})} \frac{\mu^{n-k+1}(\mu - \lambda)}{(\mu^{n-k+2} - \lambda^{n-k+2})} \left[\left(\frac{\overline{\lambda}}{\overline{\mu}}\right)^2 + \frac{\lambda\left(\mu^{n-k+1} - \lambda^{n-k+1}\right)}{\mu^{n-k+1}(\mu - \lambda)}\right]$$

(4) Average number of failed components,

$$N_{fail} = \sum_{j_3=1}^{n-k+1} J_3 \left(\sum_{j_1=0}^{\infty} X_{(j_1,1,j_3)} \right) = \frac{\lambda \mu^{n-k+2}}{(\mu - \lambda)\left(\mu^{n-k+2} - \lambda^{n-k+2}\right)}.$$

(5) Expected number of external customers joining the system in unit time,

$$\theta_3 = \bar{\lambda} \sum_{j_1=0}^{\infty} x_{(j_1,0,0)} = \frac{\bar{\lambda}\mu^{n-k+1}(\mu - \lambda)}{\left(\mu^{n-k+2} - \lambda^{n-k+2}\right)}.$$

(6) Expected number of external customers, on arrival, getting service directly

$$= \bar{\mu} x_{(0,0,0)}$$

$$= \frac{(\bar{\mu} - \bar{\lambda})\mu^{n-k+1}(\mu - \lambda)}{\left(\mu^{n-k+2} - \lambda^{n-k+2}\right)}.$$

(7) Fraction of time the server is busy with external customers,

$$P_{ex.busy} = \sum_{j_1=0}^{\infty} x_{(j_1,0,0)}$$

$$= \frac{\bar{\lambda}\mu^{n-k+1}(\mu - \lambda)}{\left(\mu^{n-k+2} - \lambda^{n-k+2}\right)}.$$

(8) Probability that the server is idle,

$$P_{idle} = x_{(0,0,j_3)} = \frac{(\bar{\mu} - \bar{\lambda})}{\bar{\mu}} \frac{\mu^{n-k+1}(\mu - \lambda)}{\left(\mu^{n-k+2} - \lambda^{n-k+2}\right)}.$$

(9) Probability that the server is found busy,

$$P_{busy} = 1 - P_{idle} = 1 - \frac{(\bar{\mu} - \bar{\lambda})}{\bar{\mu}} \frac{\mu^{n-k+1}(\mu - \lambda)}{\left(\mu^{n-k+2} - \lambda^{n-k+2}\right)}.$$

(10) Expected loss rate of external customers,

$$\theta_4 = \bar{\lambda} \sum_{j_1=0}^{\infty} \left(\sum_{j_3=1}^{n-k+1} x_{(j_1,1,j_3)} \right) = \bar{\lambda} \frac{\mu\left(\mu^{n-k+1} - \lambda^{n-k+1}\right)}{\left(\mu^{n-k+2} - \lambda^{n-k+2}\right)}.$$

(11) Expected service completion rate of external customers,

$$\theta_5 = \bar{\mu} \sum_{j_1=0}^{\infty} x_{(j_1,0,0)} = \bar{\mu} \frac{\mu^{n-k+1}(\mu - \lambda)}{\left(\mu^{n-k+2} - \lambda^{n-k+2}\right)}.$$

(12) Expected number of external customers in the system when the server is busy with external customers

$$\theta_6 = \sum_{j_1=0}^{\infty} j_1 x_{(j_1,0,0)} = \overline{\lambda}\frac{\mu^{n-k+1}(\mu - \lambda)}{(\mu^{n-k+2} - \lambda^{n-k+2})}.$$

5. Numerical illustrations

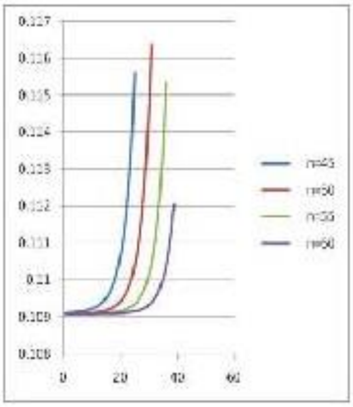
Here, we perform a numerical study on the effect of the N-policy on the system performance. Unless otherwise stated, the parameter values for the numerical study are the following: $\overline{\lambda} = 3.2$, $\mu = 5.5$, $\overline{\mu} = 8$.

5.1. Effect of the N-policy on the probability that server is busy with external customers.

While studying a k-out-of-n system, where the server provides service to external customers also, the main purpose of N-policy is to provide improved attention to external customers for optimizing the system revenue. According to the N-policy considered here, the moment the number of failed components of the main system reaches N, the external customer's service ('if there is any') is pre-empted to attend the failed components. Hence, an increase in the value of N will extend the time during which external customers can get service and so it is expected that the probability that the server is busy with external customers increases with an increase in the value of N. The column wise increase in Table 1 supports this intuition. The high service rate for the external customers, as compared to their arrival rate can be considered as the reason for the slow increase in the above probability. The row wise decrease in Table 1 points to the decrease in the probability that the server is busy with external customers with an increase in the total number of components in the system. We have the following reasoning for this behavior: With an increase in the total number of components n in the system, there can be more number of failed components in the system for a fixed N, which leads to an increase in the probability that the server is attending failed components, resulting in a decrease in the probability $P_{ex.busy}$. A closer scrutiny of Table 1 shows that, by increasing the policy level N with an increase in the number of components n, the same value for the fraction $P_{ex.busy}$ can be achieved as that when

TABLE 1. Dependence of the probability $P_{ex\,busy}$ on the N-policy level

N	$n = 45$	$n = 50$	$n = 55$	$n = 60$
1	0.10910	0.10909	0.10909	0.10909
2	0.10910	0.10910	0.10909	0.10909
3	0.10912	0.10910	0.10909	0.10909
6	0.10914	0.10910	0.10909	0.10909
7	0.10915	0.10910	0.10909	0.10909
10	0.10922	0.10912	0.10910	0.10909
11	0.10925	0.10912	0.10910	0.10909
12	0.10929	0.10913	0.10910	0.10910
15	0.10952	0.10918	0.10911	0.10910
18	0.11002	0.10928	0.10913	0.10910
21	0.11118	0.10952	0.10918	0.10911
22	0.11185	0.10965	0.10921	0.10912
23	0.11275	0.10982	0.10925	0.10913
24	0.11397	0.11006	0.10929	0.10914
25	0.11562	0.11037	0.10935	0.10915
26		0.11078	0.10944	0.10917
27		0.11134	0.10955	0.10919
28		0.11209	0.10970	0.10922
29		0.11310	0.10989	0.10926
30		0.11448	0.11016	0.10932
31		0.11638	0.11051	0.10939



n has a lesser value. For example, when $n = 45$ and $N = 7$, $P_{ex.busy} = 0.10915$ and $P_{ex.busy} = 0.10909$, when $n = 60$ with the same N. Now with $n = 60$ and when N is increased to 25, we see that $P_{ex.busy} = 0.10915$. This suggests that, when n increases, the N-policy level can be adjusted in favor of the external customers, which was our objective while introducing the N policy. However, when N increases, it is probable that the server spends more time for failed components, once he starts attending them, which leads to a loss of the external customers who finds the server busy with internal customers. In Table 1, one can see that the probability $P_{ex.busy}$ has a lesser value when $n = 60$, $N = 30$ than in the case when $n = 45$, $N = 15$, which points to the loss of external customers. Another challenge here is that, while increasing the N-policy level, the system reliability is not affected significantly.

5.2. **Effect of the N-policy on the system reliability.**

In the previous section, we discussed how N-policy helps in longer duration of attention to external customers and the challenge there is the possibility of a decrease in the system reliability. Here we discuss how the N-policy level affects the system reliability P_{rel}. We study two cases with $\frac{\lambda}{\mu} < 1$ and $\frac{\lambda}{\mu} > 1$ respectively, results of which are given in Table 2(a) and (b) respectively. While studying the impact of the N-policy on the system reliability, a decrease in P_{rel} is expected with an increase in value of N. Hence, the purpose of the Tables 2(a) and (b) is to show the magnitude of this impact. Table 2(a) shows that when $\frac{\lambda}{\mu} < 1$, $n = 45$ and when N increased from 3 to 25, there is a decrease in reliability of magnitude equal to 0.02. As the total number of components n increases, the magnitude of decrease in reliability reduces. This is because, when n increases, k being fixed, $n - k + 1$ increases; as a result, once the server starts attending the failed components on accumulation of N of them, he spends more time for the failed components, which maintains a high system reliability even when N increases. In Table 1 we have seen that as n increases, the probability $P_{ex.busy}$ decreases and that increasing the N-policy level can remedy this to some extent; Table 2(a) shows that the reliability of the system is not much affected by increasing the N-policy level. However, the magnitude of drop in the system reliability increases with the increase in N-policy level. Table 2(b) studies the system reliability when the failure rate of the components λ is larger than their repair rate μ. As expected, there is a drop in the system reliability compared to the case $\lambda < \mu$. Other behaviour of the system reliability are similar to that in Table 2(a).

TABLE 2. **(a)**: Dependence of the system reliability on the N-policy level in the $\lambda < \mu$ case $\lambda = 4$

N	$n = 45$	$n = 50$	$n = 55$	$n = 60$	$n = 65$
1	0.999930799	0.999985933	0.999997139	0.999999404	0.999999881
3	0.999901652	0.999979973	0.999995947	0.999999166	0.999999821
5	0.999855518	0.999970615	0.999994040	0.999998808	0.999999762
9	0.999660194	0.999930918	0.999985933	0.999997139	0.999999404
13	0.999121249	0.999821544	0.999963701	0.999992609	0.999998510
17	0.997560024	0.999506116	0.999899626	0.999979556	0.999995828
21	0.992828071	0.998562694	0.999708474	0.999940693	0.999987960
25	0.977587163	0.995647013	0.999122441	0.999821782	0.999963760
26		0.994222760	0.998838782	0.999764323	0.999952078
29		0.986251056	0.997281969	0.999450147	0.999888241
31		0.974976659	0.995165646	0.999026358	0.999802291
34			0.984254420	0.996900022	0.999531090
35			0.978649259	0.995844364	0.999373376
38				0.989870846	0.998496175
39				0.986294508	0.979825020
40				0.981382251	0.972903130
41					0.996356070
45					0.987866700
46					0.983495116

TABLE 2. **(b)**: Dependence of the system reliability on the N-policy level in the $\lambda > \mu$ case $\lambda = 6$

N	$n = 45$	$n = 50$	$n = 55$	$n = 60$	$n = 65$
1	0.907874525	0.911180377	0.913196325	0.914452970	0.915247083
3	0.907009840	0.910661936	0.912876606	0.914252222	0.915119767
5	0.906079888	0.910108566	0.912536800	0.914039671	0.914985061
9	0.904014528	0.908894181	0.911796451	0.913578153	0.914693415
11	0.902873158	0.908231616	0.911395609	0.913329482	0.914536774
13	0.901655436	0.907531500	0.910974264	0.913069129	0.914373279
17	0.898979187	0.906016290	0.910070777	0.912513614	0.914025128
21	0.895960152	0.904344857	0.909087002	0.911913455	0.913650930
25	0.892570674	0.902514517	0.908024848	0.911270797	0.913252294
26		0.902032018	0.907747209	0.911103785	0.913149118
29		0.900522947	0.906886399	0.910588324	0.912831187
31		0.89946568	0.906289339	0.910232842	0.912612915
34			0.905359924	0.909682870	0.912276387
35			0.905041218	0.909495234	0.912169460
38				0.908919990	0.911812007
39				0.908724248	0.911693335
40				0.908526540	0.911573648
41				0.908326924	0.911453009
45					0.910961330
46					0.910836279

5.3. **Cost analysis.**

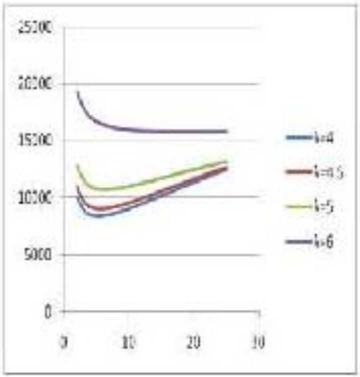
In sections 1.5.1 and 1.5.2, we have seen that by increasing N, we can provide uninterrupted service over a long duration to more external customers and without compromising the system reliability significantly. However, the magnitude of decrease in the system reliability increases with N. Hence, it is worth finding whether there exists an optimal value for the N-policy level. For this, we construct the following cost function. Let C_1 be the cost per unit time incurred if the system is down; C_2, the holding cost per unit time per external customer in the queue; C_3 is the cost incurred towards set up (instantaneous) of the server to serve main customers; C_4 be the cost due to loss of an external customer, C_5, be the holding cost per unit time of one failed component and C_6 be the cost per unit idle time.

Expected Cost per unit time $= C_1 \cdot P_{down} + C_2 \cdot N_q + C_4 \cdot \theta_4 + C_5 \cdot N_{fail} + \left(\frac{C_3}{E_B}\right) + C_6 \cdot P_{idle}.$

Table 3 studies the variation of cost function as N varies. We study the cost function for different failure rates of the components. In all the 4 cases studied, for the various costs assumed, we get a concave nature for the cost curve, which gives an optimal value for N. Table 3 shows that when $\lambda < \mu$, the optimal values for N are 5,6 and 6 when λ equal to 4, 4.5 and 5 respectively; whereas when $\lambda = 6 > 5.5 = \mu$, we get a much higher optimal value 18 for N. This is as expected, since when λ is greater than μ, there will be a heavier traffic of failed components so that the server has to spend more time attending the failed components. Hence, the policy level N needs to be increased to a much higher value than in the $\lambda < \mu$ situation, for the system to earn maximum profit. Also note that the optimal value of the cost function is much higher in the $\lambda > \mu$ case, when compared to the opposite situation.

TABLE 3. Variation in the cost function $n = 50$, $k = 20$, $C_1 = 2000$, $C_2 = 1000$, $C_3 = 1600$, $C_4 = 1000$, $C_5 = 500$, $C_6 = 100$

N	$\lambda = 4$	$\lambda = 4.5$	$\lambda = 5$	$\lambda = 6$
2	10139.47	10923.82	12783.28	19330.75
3	8910.199	9663.817	11496.69	17827.49
4	8489.626	9199.57	10981.16	17095.61
5	**8370.844**	9038.382	10764.71	16671.58
6	8396.2	**9024.268**	**10694.5**	16401.28
7	8500.631	9092.232	10706.09	16218.57
8	8652.372	9210.307	10767.47	16090.46
12	9474.447	9919.942	11245.17	15847.97
14	9939.594	10337.5	11542.35	15805.31
16	10416.93	10769	11849.9	15786.79
17	10657.6	10986.45	12003.95	15783.35
18	10898.62	11203.53	12156.71	**15782.6**
19	11139.36	11419.23	12307.2	15783.87
20	11379.19	11632.57	12454.51	15786.61
23	12085.88	12248.03	12868.57	15799.16
24	12313.97	12441.01	12994.22	15803.64
25	12536.01	12625.49	13111.97	15807.79

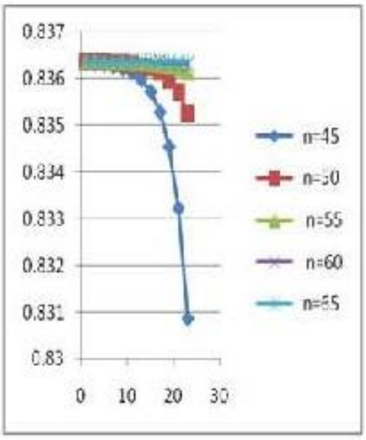


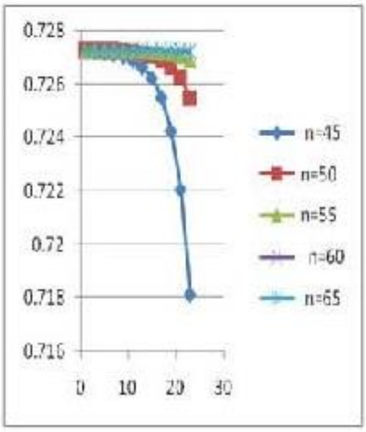
5.4. Comparison with a k-out-n system where no external customers are serviced.

Here we compare the model discussed above with another model where no external customers are allowed but N-policy is maintained. Notice that because of the assumption of the preemption of service of an external customer on accumulation of N failed components, the two systems will have the same reliability. The nature of the steady state distribution obtained in Theorem 4.2 further substantiates this claim. Hence, it can be concluded that the external customers when allowed as in this study, utilizes the server idle time without affecting the performance of the k-out-of-n system. In Table 4, we present the results of the numerical study conducted for comparing the increase in the server busy probability, when external customers are allowed. In that Table, case 1 refers to the model discussed above and case 2 stands for k-out-of-n system where no external customers are allowed. Table 4 shows that when external customers are allowed, there is an increase, of magnitude 0.11, in the server busy probability.

TABLE 4. Variation in the server busy probability

Case 1 $\lambda = 4$						Case 2 $\lambda = 4$					
N	$n = 45$	$n = 50$	$n = 55$	$n = 60$	$n = 65$	N	$n = 45$	$n = 50$	$n = 55$	$n = 60$	$n = 65$
1	0.83633	0.83635	0.83636	0.83636	0.83636	1	0.72722	0.72726	0.72727	0.72727	0.72727
3	0.83632	0.83635	0.83636	0.83636	0.83636	3	0.7272	0.72726	0.72727	0.72727	0.72727
5	0.8363	0.83635	0.83636	0.83636	0.83636	5	0.72717	0.72725	0.72727	0.72727	0.72727
7	0.83626	0.83634	0.83635	0.83636	0.83636	7	0.72711	0.72724	0.72727	0.72727	0.72727
9	0.83621	0.83633	0.83635	0.83635	0.83635	9	0.72703	0.72722	0.72726	0.72727	0.72727
11	0.83612	0.83631	0.83634	0.83635	0.83635	11	0.72688	0.72719	0.72726	0.72727	0.72727
13	0.83597	0.83627	0.83634	0.83635	0.83635	13	0.72663	0.72714	0.72725	0.72727	0.72727
15	0.83572	0.83622	0.83632	0.83635	0.83635	15	0.72622	0.72706	0.72723	0.72726	0.72727
17	0.83528	0.83613	0.83631	0.83634	0.83635	17	0.7255	0.72691	0.7272	0.72726	0.72727
19	0.83453	0.83598	0.83627	0.83633	0.83634	19	0.72425	0.72666	0.72715	0.72725	0.72727
21	0.83322	0.83572	0.83622	0.83632	0.83634	21	0.72206	0.72623	0.72706	0.72723	0.72726
23	0.83086	0.83526	0.83613	0.8363	0.83634	23	0.71814	0.72546	0.72691	0.7272	0.72726





6. Conclusion

Rendering service to external customers could be an effective idea for utilizing the server idle time and thereby earning more revenue to the system. However, in the case of a system, where a minimum number of working components is necessary for its operation, the external service should be carefully managed so that it does not affect the system reliability considerably. In the present paper, we have adopted N-Policy for managing the external service. Precisely, we assume that the server starts attending failed system components only on the accumulation of N of them. During this idle period, he/she renders service to external customers (if there is any). This scenario has been modeled using a continuous time Markov chain. Further, we make the reasonable assumption that the external service is pre-empted on accumulation of N failed components and also that the external arrivals which finds the server busy with failed components of the main system, are blocked from entering the system. These assumptions lead us to a product form solution the system in steady state. For this purpose, we employed a novel matrix decomposition approach. Though we have an explicit expression for the system reliability, due to the complex involvement of the parameters in the same, we studied the effect of the N-policy on the system reliability, numerically. This study reveals that by introducing the N-policy, we can maximize the system revenue, by rendering service to external customers, maintaining high system reliability. In future, we plan to study the effect of pre-emption on the system reliability as well as on the product form nature of the system steady state under the N-policy. Another extension is to allow external customers to join the system even when the server is busy with internal customers.

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