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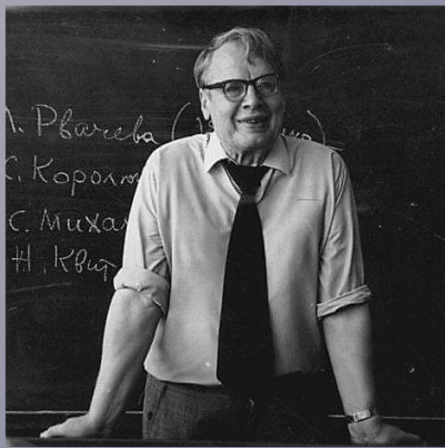

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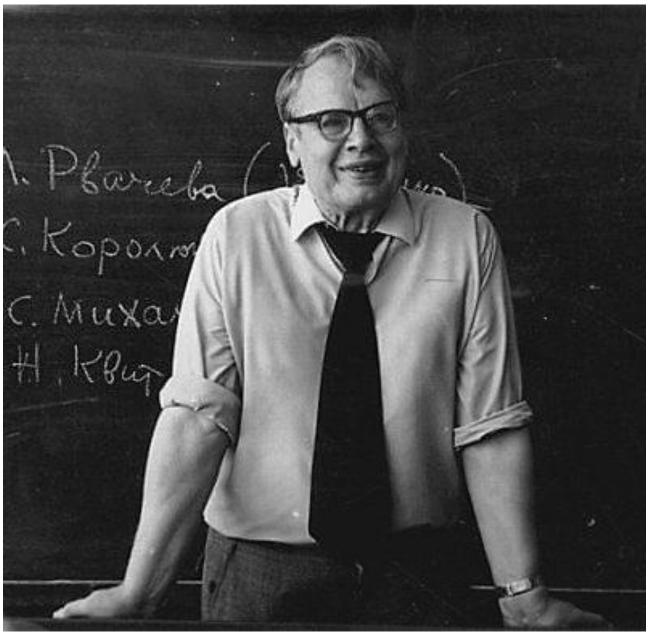

RELIABILITY:

THEORY & APPLICATIONS

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Reliability Of Heterogeneous *(k,r)-out-of-(n,m)* System

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Abstract

We consider a coherent binary system consisting of m components of a-type and n components of b-type. The a-type and b-type components have i.i.d lifetimes with cdf $F_a(t)$ and $F_b(t)$, respectively. The a-type and b-type components are stochastically independent. Our system is UP if at least k a-type components are up and at least r components of b-type are up.We present a simple formula for this system lifetime cumulative distribution function.

Keywords: heterogeneous system; k-out-of-n system; survival signature.

We consider the following generalization of a"standard" *k-out-of-n* system. Our system has two types of components: n components of a-type and m components of b-type. The components of each type have iid lifetimes denoted as $F_a(t)$ and $F_b(t)$, respectively. The a-type and b-type components are stochastically independent. The standard $k-out-of-n$ system is operational (i.e. in state UP) iff at least k of its components are operational, i.e. are up. Our system is defined to be operational if and only if at least k components of a-type **and** at least r of b-type components are up. Formally, our system can be viewed as a series connection of two $k-out-of-n$-type subsystems.

Suppose, without loss of generality, we number the a-type components by numbers $1,2,\ldots,m$ and components of b-type by numbers $m+1,\ldots,m+n$. System state is therefore a binary vector $x=(x_1,\ldots,x_m,\ldots,x_{m+n},$ where $x_i=1$ or $x_i 0$ if component i is *up* or *down*, respectively.

System state is a binary function $\varphi(x)$ which takes values 1 or 0 if the system is UP or *DOWN*, respectively.

If $\varphi(x_1)=1$, then x_1 is called an *UP-vector* or an *UP-set.*If the state vector x is not an *UP-vector*, we call it a *DOWN-vector* or *DOWN-set*

According to the above description of our system, an *UP-set* must have at least k ones on the first m positions of vector x and at least r ones on the last n positions. For example, for $m=5$ and $n=6$, $k=4, r=4$, the vector $x=(0,1,1,1,1;1,1,1,0,1,0)$ is an *UP-vector.*

Denote by $NU(v,w)$ the number of *UP-vectors* which have exactly v ones on the first m positions and w ones on the last n positions. Obviously, $v\geq k, w\geq r$.

The following Lemma follows from the above description:

Lemma

$$NU(v,w) = C_m^v \cdot C_n^w = \frac{m!n!}{v!(m-v)!w!(n-w)!}. \quad (1)$$

The proof is obvious: there are C_m^v ways to locate v ones on the first n positions of the state vector x and C_n^w ways to locate w ones on the last n positions of this vector.#

For sake of brevity, an *UP-vector* with v and w components of a-type and b-type, will be called an (v,w)-*UP-vector*.

Now everything is ready to write the formula for system *UP* probability. Let us take an arbitrary time instant t and denote by $q_a = F_a(t)$ the probability that an a-type component is *down* at time instant t.Smilarly, $q_b = F_b(t)$ is the *down* probability that component of b-type is *down* at time instant t. Denote $p_a = 1 - q_a$ and $p_b = 1 - q_b$.

By independence of all a-type and b-type components and by independence of a-type components from b-type components,the probability that a state vector x is an (v,w)-*UP-vector* equals

$$P(U(v,w)) = p_a^v q_a^{n-v} p_b^w q_b^{m-w}. \quad (2)$$

Now we arrive at

Theorem 1

$$P(\, system\ is\ UP\ at\ time\ t\,) = \sum_{v \geq k, w \geq r} NU(v,w) \cdot P(U(v,w). \quad (3)$$

.

If the system is *UP* at time instant t its lifetime τ_s is greater or equal t, Therefore,

$$P(\tau_s \geq t) = \sum_{v \geq k, w \geq r} C_m^v \cdot C_n^w \cdot [F_a(t)]^v [1 - F_a(t)]^{n-v} [F_b(t)]^w [1 - F_b(t)]^{m-w}.\# \quad (4)$$

Remark 1. The central role for deriving formula (4) is played by the expression for $UN(v,w)$, see (1). Let us note that$NU(v,w)$ depend only on system structure function and they are, therefore, system *structural invariants*. It is quite obvious how to generalize the above derivation for the case when the system has more than two, say $K > 2$ types of components. By definition, this system is *UP* iff it has at least v_i *up* components of each type, $i = 1,2, \dots K$.#

Remark 2. A system consisting of several $k - out - of - n$ subsystems is, to the best of our knowledge, the only lucky case where we can find in a simple form (like in (1)) an explicit formula for the number of system *UP-state* vectors having exactly v_i components of i-th type in *up* state, $i = 1,2, \dots, K$.

Samaniego and Navarro suggested to call the collection of all $NU(v,w)$ values *survival signature*, see [1]. If $ND(u,w)$ is the number of system *DOWN* states with exactly v a-components and w b-components *down*, then it would be natural to call the collection of all $ND(v,w)$ *failure signature*. #

Remark 3. There is a simple relationship between the values of $ND(v,w)$ and $NU(v,w)$:

$$ND(v,w) + NU(n-v, m-w) = \frac{m!n!}{v!(m-v)!w!(n-w)!}. \quad (5)$$

Indeed, let us chose v components of a-type and w components of b-type and let them be *down*. Then we will obtain either a *DOWN* state or an *UP* state vector for the system. But having v, w components *down*, means having the remaining components *up*, which proves (5). From practical point of view, (5) shows that the knowledge of the survival signature provides us the knowledge its dual failure signature.#

Remark 4. Let us return to coherent binary systems consisting of one type iid components. Crucial role in its reliability evaluations play so-called *signature* $f = (f_1, f_2, \dots, f_n)$, see [2]. Let $F(j) = \sum_{k=1}^{j} f_j$, $j = 1,2,\dots,n$ be the so-called cumulative signature or system D-spectrum [3,4]. $F(j)$ is the probability that the system is $DOWN$ if j of its components are *down*, i.e. the probability that system failure appeared after x components have failed, $x = 1,2,\dots,j$. If we know the D-spectrum of the system, we can find the number $ND(r)$-the number of system failure or *DOWN* states with exactly r components *down* and $n-r$ components *up*, by using the following simple formula, see [3,4]:

$$ND(r) = F(r)n!/(r!\,(n-r)!). \tag{6}$$

For systems of real size, having $n > 8-10$ components, there are efficient Monte Carlo algorithms for fast and accurate estimation of $F(j)$, see [4]

In our opinion, in case of coherent systems having two types of independent and identical components, reliability calculations must be based on the knowledge of a two-dimensional analogue of the cumulative D-spectrum. It should be a function $G(k,r)$ expressing the probability that a random permutation of n and m components of both types contains a failure set with k and r *down* components of a- and b-type, respectively.#

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A Minimal Repair Model With Imperfect Fault Detection

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Abstract

In this paper we study a model of a system with imperfect repair, where also detection of failure might fail. For such a system we derive the lifetime distribution function of the system and give bounds for the mean life and the mean residual life function.

Keywords: imperfect repair, redundancy, limiting reliability

I. Introduction

In order to improve the reliability of a system there are mainly two possibilities. The first one is to improve the reliability of the components, the second is to implement redundancy. Mainly this is done by using more than one component to fulfill the same function, see e.g. [1]. Redundancy means that in a technical system there are more possibilities present to ensure a function, than the necessary minimum. In a previous paper the authors have studied a model of a redundant system with imperfect switching to the redundant unit. There, we have studied two cases, one of them with hot standby, the other with cold standby, see [5].

These models with hot standby and cold standby describe just two extreme situations, where the redundant units are either completely unused (pure replacement) or used in parallel under full load (hot standby).

There are many other possible models with different replacement or repair strategies.

In this paper, we will study a model, where in place of switching to a redundant unit, a minimal repair is carried out. In addition, the failure detection mechanism and subsequent minimal repair fail with a probability ꩜. Minimal repair is motivated by the use of a large unit, where only one small component is replaced so that the system itself can be seen as unchanged and the effect of using a new small component is negligible for the entire unit.

In this short paper we will derive expressions for the life time distribution of such a system together with results for conservation of properties of the distribution function as the Increasing (Decreasing) Failure Rate Average.

II. The Model

Assume a unit with minimal repair. The original lifetime distribution of the system is denoted by F(x) and hazard function $\circledcirc(x)$ so that we have

$$F(x) = 1 - \exp(-\circledcirc(x)).$$

Let us further denote the hazard rate, i.e. the derivative of $\circledcirc(x)$, by $\circledcirc(x)$.

Assume further that

$$F(0) = 0$$

and

$$\lim_{x \to \infty} F(x) = 1$$

For a unit undergoing minimal repair at the time of failures, the unit is restored during repair to a functioning state, but with the same unit age as before. The unit is therefore used and re-used after repair and the failure times of the unit follow a Nonhomogeneous Poisson Process with cumulative intensity function $\circledcirc(x)$, see [2]. The distribution of the time of the n-th repair is then defined by

$$F_n(x) = e^{-L(x)} \sum_{i=n}^{\infty} \frac{L(x)^n}{n!} \tag{1}$$

with density

$$f_n(x) = \circledcirc(x)\, e^{-\circledcirc(x)}\, \circledcirc(x)^{n-1}/(n-1)! \tag{2}$$

which gives for n=1

$$f_1(x) = \circledcirc(x)\, e^{-\circledcirc(x)}. \tag{3}$$

Assume further that the repair is successful only with probability $1-\circledcirc$, since $\circledcirc$ is the probability that the repair fails. Then the unit fails at the instant of n-th minimal repair with probability

$$(1-\circledcirc)^{n-1}\circledcirc, \tag{4}$$

i.e. n-1 minimal repairs were successful, the n-th not. Combining (2) and (4) and summing over n one gets the density of the lifetime distribution function of a unit with minimal and imperfect repair

$$g(x) = \sum_{n=1}^{\infty} (1-g)^{n-1} g\, f_n(x) = \sum_{n=1}^{\infty} (1-g)^{n-1} g\, l(x)\, e^{-L(x)}\, L(x)^{n-1}/(n-1)! = \circledcirc\, \circledcirc(x)\, e^{-(1-\circledcirc)\circledcirc(x)}\, e^{-\circledcirc(x)}.$$

$$= \circledcirc\, \circledcirc(x) \exp(-\circledcirc\, \circledcirc(x)). \tag{5}$$

The distribution function reads

$$G(x) = 1 - \exp(-\circledcirc\circledcirc\circledcirc(x)). \tag{6}$$

III. Results for the lifetime of a unit with imperfect and minimal repair

The mean lifetime of the unit is derived by the distribution (6) by

$$m_G = \int_0^{\infty} \exp(-g\, L(x))\, dx. \tag{7}$$

A distribution function F(x) is said to belong to the IFRA class (increasing failure rate average) / DFRA (decreasing failure rate average), if

$$\circledcirc(x)/x$$

is an increasing / decreasing function, see [1]. Then, also G(x) belongs to the IFRA / DFRA class since

$$\circledcirc\circledcirc(x)/\circledcirc x$$

is also increasing / decreasing, if the property holds for $\circledcirc(x)/x$. If now F is IFRA (DFRA), we have

$$\circledcirc(\circledcirc\, x) / (\circledcirc x) \le (\ge)\ \circledcirc(x) / x, \tag{8}$$

since $\circledcirc \le 1$. This inequality gives

$$\circledcirc(\circledcirc\, x) \le (\ge)\ \circledcirc\, \circledcirc(x).$$

From here, it follows

$$\exp(-\circledcirc(\circledcirc\, x)) \ge (\le) \exp(-\circledcirc\, \circledcirc(x)). \tag{9}$$

Integrating from 0 to ∞ yields

$$\int_0^{\infty} \exp(-L(g\, x))\, dx \ge (\le) \int_0^{\infty} \exp(-g\, L(x))\, dx\ . \tag{10}$$

This is equivalent to

$$m_G \le (\ge)\ m_F/\circledcirc,$$

provided F(x) is IFRA (DFRA). Also, bounds on the residual life function can be derived. Writing

$$T_{RL} = \int_x^{\infty}(1\text{-}G(t))\ dt = \int_x^{\infty}\exp(\text{-}g\ L(t))\ dt \le (\ge) \int_x^{\infty}\exp(\text{-}L(g\ t))\ dt = (1/\circledcirc)\ T_{RL,F}(\circledcirc\ x), \quad (11)$$

if F is IFRA (DFRA). Here, $T_{RL,F}$ denotes the residual life function of F. Since a distribution function that is IFRA (DRFRA) is also HNBUE (HNWUE), see [3], the result can be extended

$$T_{RL} \le (\ge)\ (1/\circledcirc)\ m_F \exp(\text{-}\circledcirc x/m_F), \quad (12)$$

Where we have used the HNBUE (HNWUE) property of F, i..e.

$$\int_x^{\infty}(1\text{-}F(t))dt \le (\ge)\ m_F \exp(\text{-}x/m_F).$$

IV. Comparison and Conclusions

In this paper, we have derived results for a model of a unit that undergoes minimal repair with imperfect detection of failures. We have derived the distribution function

$$G(x) = 1 - \exp(\text{-}\circledcirc\circledcirc\circledcirc(x)) = 1\text{-}(1\text{-}F(x))^{\circledcirc}, \quad (13)$$

in closed form. This distribution function can be compared with the lifetimes distribution function of a system with hot standby and imperfect switching (repair), see [5]:

$$G_{hs} = \circledcirc F(x)\ /(1\text{-}(1\text{-}\circledcirc)F(x)) \quad (14)$$

One can now observe that

$$G_{hs}(x) \le G(x).$$

This follows from the fact that

$$(1\text{-}F) \le (1\text{-}F)^{\circledcirc}(1\text{-}(1\text{-}\circledcirc)F).$$

This inequality can be proven using that the function

$$h(x) = \circledcirc x^{\circledcirc}+(1\text{-}\circledcirc)x^{1+\circledcirc}\text{-}x$$

is nonnegative in the interval [0,1]. This property is a result of the following facts

$$h(0) = 0,\ h(1) = 0,$$

$$h'(x) \text{ tends to infinity as } x \to 0,$$

$$h'(1)=0,$$

$$h''(x) \text{ is negative for } x<x_0=\circledcirc/(1+\circledcirc) \text{ and}$$

$$h''(x) \text{ is positive for } x>x_0.$$

$$h''(x_0) = 0 \text{ with } h(x_0) = \circledcirc(3-\circledcirc) >0.$$

Therefore, $h(x)$ must be nonnegative on the interval $[0,x_0]$, since it is convex there. Furthermore, $h(x)$ has an increasing first derivative on $[x_0,1]$, starts at a positive value at x_0 and decreases to zero at 1. There is no inflection point on the interval $(x_0,1]$, which would be necessary for $h(x)$ to take negative values in $(x_0,1]$, since $h'(1)=0$. Therefore, $h(x)$ is also nonnegative on $[x_0,1]$. Hence we have shown that the failure probability of a unit with minimal repair and imperfect failure detection is larger than for a unit with internal hot standby and imperfect failure detection. Now, we can also compare the bounds. If F belongs to the IFRA class, then the same holds for (13) and the following bounds hold are derived:

$$m_G \leq m_F/\circledcirc, \tag{15}$$

$$T_{RL} \leq (1/\circledcirc)\ mF \exp(-\circledcirc x/m_F), \tag{16}$$

These results can be compared with the ones provided in [5] for hot and cold standby systems. We see immediately that the bounds (15) and (16) are the same as for cold standby systems, which is generally the absolutely upper bound for units of types of redundancies. Therefore, we have derived the distribution function, bounds on the mean lifetime and the residual life function for a unit with minimal repair and imperfect failure detection.

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Quantitative Estimation Of The Integrated Parameter Of Safety Ability To Live Of Personnel EPS

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Abstract

Opportunity of a quantitative estimation integrated parameters (IP) safety of ability to live (AL) is one of the basic directions of decrease in a traumatism and destruction of personnel EPS. In a basis of the developed method and algorithm of calculation IP AL there are differences of requirements of Rules AL observable in practice from a level of their execution. Quantitative estimations AL open new, earlier inaccessible opportunities regarding the analysis and the control. For example, to compare AL at enterprises EES, to reveal the directions reducing AL, to estimate AL on various categories of maintenance, tests and repair of the various equipment and devices EES

Keywords: *reliability safety of ability to live, integrated parameters*

I. Introduction

As is known [1] AL concerns to number of the basic properties of reliability. However unlike non-failure operation, maintainability, durability and a storage property, methods of quantitative which estimation of parameters, including, methods of calculation of accuracy and reliability, in many respects are developed, methods of a quantitative estimation of parameters AL are in an initial stage of development. It speaks not only variety of conditions of ability to live (many-sided nature and multidimensionality are characteristic features of a problem of reliability), but, first of all that there is a problem of an estimation of safety not the equipment and devices (objects) EPS, and the personnel, the person.

In practice AL, be provided with strict performance and the control of performance of Rules of a labour safety, the safety precautions, fire-prevention technics at maintenance service and repair (MS&R) objects EPS. However, as well as for noted above the basic properties of reliability, qualitative characteristic AL appears insufficient, first of all, owing to imperfection of existing opportunities of its control. Sad acknowledgement to that is the statistics of a traumatism and destruction of personnel EPS.

In [2] it is marked, that according to the theory of academician A.A.Doronisin in development of any science (and AL is a science about safe interaction of the person and a techno sphere) it is necessary to distinguish two periods: descriptive and exact. Descriptive period provided with accumulation of the information, revealing of influencing factors, "mechanical" classification of the information. The exact period characterized by development of methods of the quantitative characteristic of processes, methods of their modeling. Hence and from the point of view of development of a science, research AL in EPS are in an initial stage of the development, and development of methods and algorithms of quantitative estimation IP AL at operation, tests and repair of objects EPS is one of the basic directions of maintenance AL of personnel EPS.

It is necessary to make a reservation at once, that the official statistics about traumas and destructions of the personnel in EPS, certainly, analyzed and most carefully to exclude similar cases. But, in opinion of authors, it should not and cannot form a basis for calculation of parameters AL. Fortunately, for authentic calculations it is not enough these cases.

II. Method and algorithm of quantitative estimation IP AL

IP AL in a techno sphere average life expectancy of the person [3] is accepted. This parameter is objective enough at an estimation of influence on life expectancy, for example, ecologies of region, but unacceptable for the characteristic of safety at work in EPS.

For quantitative estimation, AL a number of the parameters describing non-failure operation and maintainability of technical objects recommended also. To them concern:
- probability of functioning of objects without undue incident (failures, and accidents) during the set interval of time;
- probability of occurrence even one incident;
- average duration of incident;
- average size of damage.

And though all these incidents, as a rule, occur not on fault of the personnel (but not without its participation), are a source the dangers noted above parameters AL, at presence of corresponding statistical data, can be demanded.

Recommended for practical application IP AL in EPS and a method of its quantitative estimation based on a following axiom: «danger of ability to live arises at infringement of Rules AL. Real AL that above, than above a level of execution of Rules AL». In other words, the reflecting level of execution of Rules AL in EPS recommended IP AL. Generally Rules AL consist of various Rules (in EPS they reflect three directions: a labour safety, the safety precautions, fire-prevention safety), their chapters, sections and positions. We estimate the level of execution of each position of Rules in five-point system. This choice not assignable:
- the five-point system is habitual, since reminds usual system of an estimation of knowledge;
- on the basis of practice of calculations affirms [4], that for the maximal values of the parameters measured in a serial scale, it is expedient to apply estimations within the limits of from 3 up to 6;
- on Strebjes 5 - is an optimum number of intervals when the number of measurements changes in an interval [11÷23] [5] that reflects, as a rule, the maximal number of positions in sections of Rules.

The participant of «ability to live» can execute estimation of a real level of execution of Rules AL, i.e. the executor of concrete work at operation, tests and repair of objects EPS or the Expert on maintenance AL. As the Expert executions of Rules AL responsible for the control, heads of divisions of the enterprise can participate. Advantage of this way (a direct five-ball estimation of a level of execution of Rules AL) is the knowledge the Expert of a real level of execution of positions of Rules AL.

Serious lack is subjectivity of estimations. Well-known, that estimation in this case depends not only on knowledge and on qualification of the Expert. Maintenance of independence of examination, decrease in subjectivity reached by specification of concept corresponding this or that estimation «a level of execution of positions of Rules AL». It reminds tests for examinations, but only reminds. In our case, each position is set by five possible levels of execution of the positions corresponding estimations in five-point system. If besides the made levels of execution to place in the casual order and to limit time of the answer for conformity of each level of execution real, objectivity of the analysis on this computer technology will be practically deprived subjectivity.

Thus, generally some estimation of a level of real execution $L_{i,j,k,s}$ where $i=1,m_g$ is compared with each position of Rules AL; j=1,$m_{h,i}$; k=1,$m_{r,i,j}$; s=1,$m_{c,i,j,k}$; m_g – number of Rules; $m_{h,i}$ - number of chapters in i-th Rule; $m_{r,i,j}$ – number of sections in j -th chapter i-th Rules; $m_{c,i,j,k}$ – number of positions k-th section, j-th chapter i-th Rules.

It would seem to receive IP a level of execution of positions enough to combine these estimations and to divide into the general number of positions. However, it would be a serious mistake, same, as well as an estimation of average progress at school, in classes, etc. [6]. The mistake consists in that, the mathematical theory of measurements does not suppose performance of elementary mathematical operations above parameters with a serial scale of measurements (in this case their addition). And levels of performance of positions of Rules AL just are measured in a serial scale.

Naturally there is a question – whether «much it $5m_{\Sigma}$», where m_{Σ}- - the general number of positions. Probably, one undoubtedly – loading on the Expert will be very big. But this problem is equivalent to a problem of an estimation of reliability and profitability EPS as a whole which as well so, is bulky. Therefore in this bulkiness not anything surprising.

For this reason significant interest estimations IP AL not as a whole, and on each concrete work of personnel EPS (represent analogue: non-failure operation of objects EPS). The Method of calculation IP AL (BJ) generally reduced to following sequence of calculations:

1. For each section of Rules distribution of estimations of a level of execution of positions is calculated. We shall designate number of display L-th estimation, where L=1÷5, through $r_{i,j,k,L}$. It is obvious, that $\sum_{i=1}^{5} r_{i,j,k,L} = m_{c,i,j,k}$
2. Frequency of display L-th estimation under the formula is defined

$$f^*_{i,j,k}(L) = r_{i,j,k,L} / m_{c,i,j,k} \tag{1}$$

where i=1,m_g; j=1,$m_{h,i}$; k=1,$m_{r,i,j}$;

We shall notice, that $r_{i,j,k,L}$ and $f_{i,j,k}(L)$ are measured in a quantitative scale and to them known mathematical operations can be applied. As

$$\left.\begin{aligned} &0 \le f^*_{i,j,k}(L) \le 1 \\ &\sum_{i=1}^{5} f^*_{i,j,k}(L) = 1 \end{aligned}\right\} \tag{2}$$

That sizes $f_{i,j,k}(L)$ can be considered as normalized the random variables unequivocally describing estimations of execution of positions of sections of Rules AL.

3. Discrete values of statistical function of distribution (s.f.d) estimations L=1÷5 under the formula:

$$\left.\begin{aligned} &F^*_{i,j,k}(L) = \sum_{L1=1}^{L} f^*_{i,j,k}(L1) \\ &F^*_{i,j,k}(1) = f^*_{i.j.k}(1) \\ &F^*_{i,j,k}(5) = 1 \\ &i = 1, m_g;\ j = 1, m_{h,i};\ k = 1, m_{r.i.j} \end{aligned}\right\} \tag{3}$$

4. According to order a method [7] estimation IP of a level of execution k-th section j-th chapter i Rules is calculated under the formula;

$$\left.\begin{aligned} &BJ^*_{i,j}(k) = \sqrt[5]{\prod_{L=1}^{5} F^{*\beta_L}_{i,j,k}(L)} \\ &\beta_L = \frac{L}{2^L - 1} \end{aligned}\right\} \tag{4}$$

where i=1,m_g; j=1,$m_{h,i}$; k=1,$m_{r,i,j}$

In table 1 numerical values are resulted β_L for of some values L

Table 1. Settlement values β_L.

L	1	2	3	4	5
β_L	1	2	0,75	0,5	0,31

5. For i Rules estimation IP of execution j-th chapter is calculated under the formula:

$$BJ_i^*(J) = \sqrt{\prod_{k=1}^{m_{c,i,j}} BJ_{i,j}^*(k)} \qquad (5)$$

where $i=1,m_g$; $j=1,m_{h,i}$; $k=1,m_{r,i,j}$

6. Estimation IP of a level of execution i Rules under the formula is calculated

$$BJ^*(I) = \sqrt[m_{h,i}]{\prod_{j=1}^{m_{h,i}} BJ_i^*(J)} \qquad (6)$$

where $i=1,m_g$; $j=1,m_{h,i}$; $k=1,m_{r,i,j}$

7. Estimation IP of a level of execution of Rules AL or parameter AL is calculated

$$BJ^* = \sqrt[m_g]{\prod_{i=1}^{m_g} BJ^*(I)} \qquad (7)$$

The Estimation of importance IP spent on a scale of the importance, which presented in table 2. Here for comparison the known scale of Harrington desirability [4], which with success is put into practice at analysis IP is resulted. Some distinction of a uniform scale (the length of an interval is constant) from scale Harrington formally. It executed for convenience of an estimation of interrelation of discrete argument and function of Harrington desirability looking like:

$$\left.\begin{aligned} d_{i,k} &= \exp[-\exp(-\overline{y_{i,k}})\\ \overrightarrow{y_{i,k}} &= \frac{2\cdot y_{i,k}-(y_{i,\max}-y_{i,\min})}{y_{i,\max}-y_{i,\min}}\\ y_{i,\max} &= \max\left\{y_{i,j}\right\}_{j=1,m_i}\\ y_{i,\min} &= \min\left\{y_{i,j}\right\}_{j=1,m_i} \end{aligned}\right\} \qquad (8)$$

Here $\overline{y_{i,k}}$ - absolute size of argument of Harrington function, having a quantitative scale of measurement

Table 2. A scale of the importance of parameter AL

Categories of the importance	Intervals functions of distribution	Harrington Scale	
		Categories of desirability	Intervals of function of distribution
It is inadmissible	0 – 0,19	Critical (Very badly)	0 – 0,19
Badly	0,0 – 0,39	Dangerous (badly)	0,2 – 0,36
Satisfactory	0,4 – 0,59	Admissible (well)	0,37 – 0,62
Well	0,6 – 0,79	Comprehensible (well)	0,63 – 0,79
It is indicative	0,8 – 1,00	Background (very good))	0,8 – 1,00

Results of calculations under formulas 4÷7 allow:
- to compare with a level of execution of sections, chapters and Rules, to reveal «weak parts » and to plan ways of their elimination;
- to compare with conditions of maintenance AL at different enterprises EES, in shops of power stations, in regional electric networks, etc.

It is necessary to note repeatedly, that as IP reliability and profitability (efficiency) of work of power station it is useless at formation of the basic directions of increase of an overall performance of concrete boiler installation of the same power station as the average estimation of a parameter of reliability of power transformers not in a condition to solve questions on classification of a technical condition of power transformers, and IP BJ it is useless at analysis AL of the personnel at performance of concrete version MS&R of objects EPS. This statement in any to a measure does not reduce the importance of an integrated estimation of a level of execution of Rules AL, and only testifies to necessity of automated control AL for set of versions of corresponding operation, a kind and type of test and repair of set of objects EPS. And if, for example, non-failure operation of work of power station is defined by non-failure operation of the equipment and devices AL on power station it is defined AL at performance of separate works.

On fig.1 the integrated block diagram of algorithm of quantitative estimation IP AL is resulted. As initial data of algorithm (fig.1, the block 1) results of an estimation of a level of execution of positions of Rules AL serve.

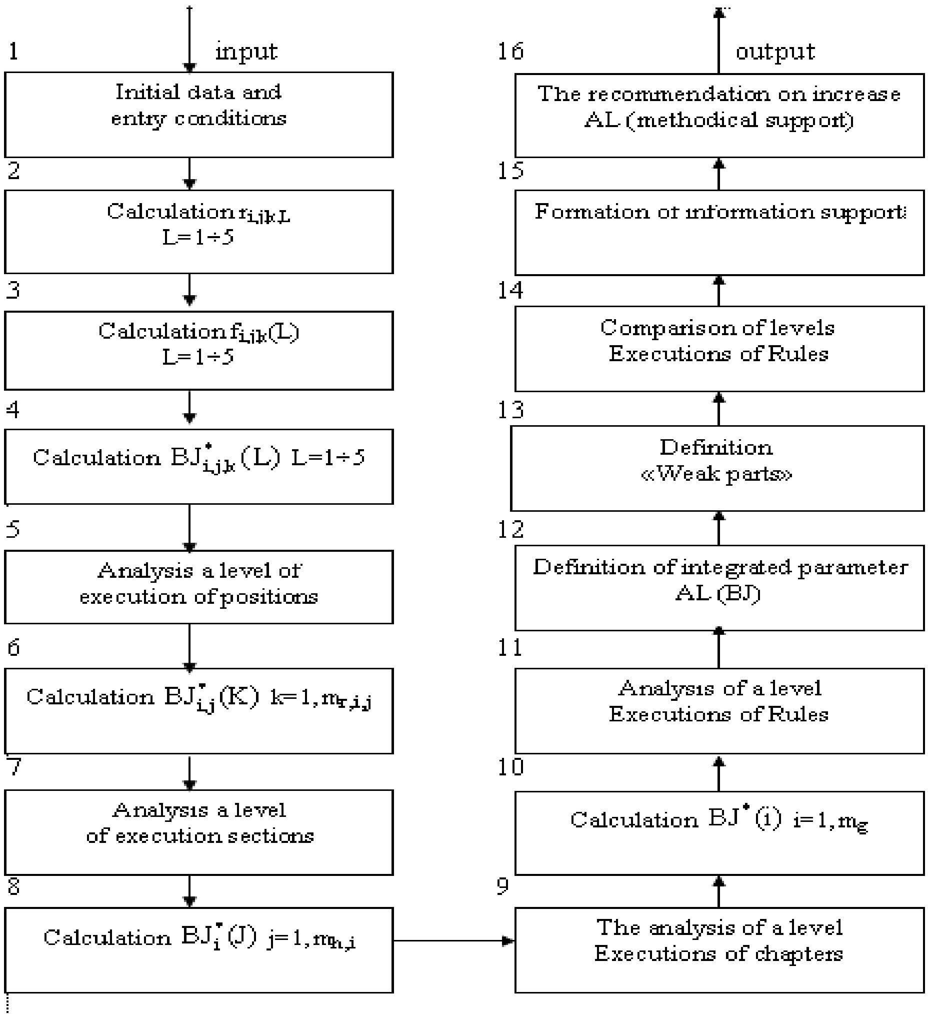

Figure 1 Integrated block diagram of algorithm of estimation IP AL.

The integrated block diagram of the automated system of formation of estimations of a level of execution of positions of Rules AL in dialogue with the Expert is resulted on fig.2. An essence of dialogue – acknowledgement (1) or denying (0) statements about a level of execution of position.

Essential decrease in time of dialogue at acknowledgement of conformity of a level of execution to real execution of positions is reached by automatic deleting from the list of the remained variants of execution of position. The general number of positions of Rules AL is less; the application of a method is more effective. Therefore, the method also recommended for control AL of concrete operative works, test and restoration of deterioration of object EPS.

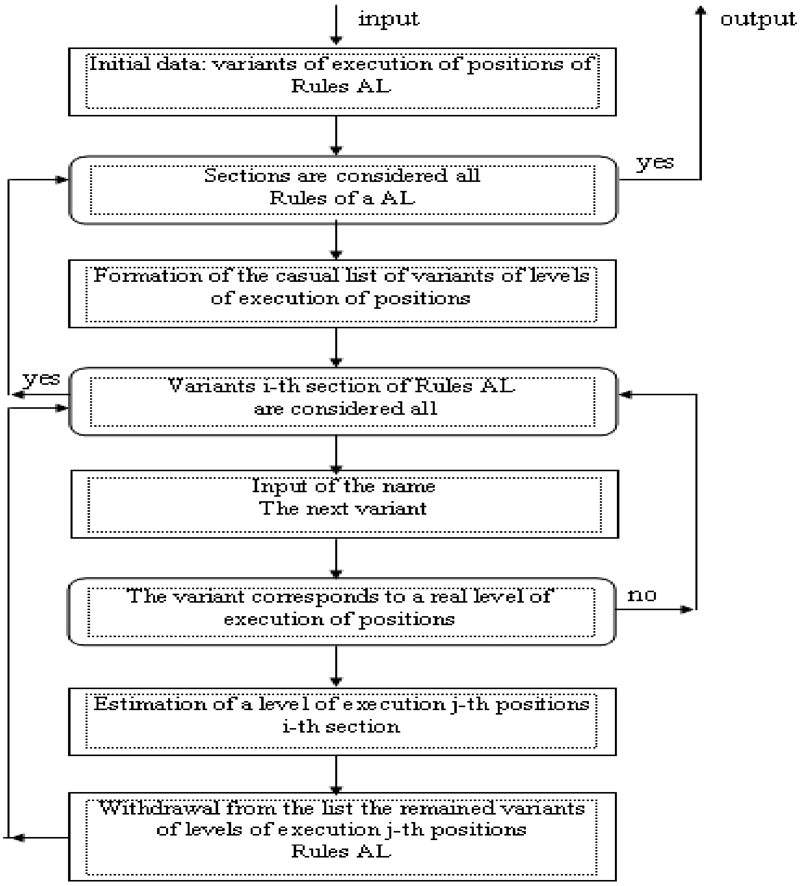

Figure.2. Integrated block diagram of algorithm formation levels execution positions of Rules AL

Conclusion

1. One of the basic directions of decrease in a traumatism and destruction of the personnel on objects EPS, is transition from qualitative to quantitative characteristic AL;
2. For such properties of reliability as non-failure operation, maintainability, durability and storage property questions of a quantitative estimation in many respects solved. Calculation corresponding reliability parameters based mainly on statistical data about refusals and reconstruction deterioration. For AL these questions are in an initial stage of the decision. In many respects it speaks distinction of solved problems;
3. The statistics of a traumatism and destruction of the personnel demands increase of efficiency of the analysis of each case. At the same time it should not form the basis for a quantitative estimation of parameters AL. Parameter AL – average "life expectancy" for characteristic AL of personnel EPS is unacceptable;
4. AL, certainly, depends on damageability of objects EPS. For example, the more service life of objects, their reliability below, and danger of maintenance service above. The likelihood parameters AL describing occurrence of failures and ethnogeny accidents as sources of danger at presence of the corresponding information can be useful.
5. The method and algorithm of a quantitative estimation recommended IP AL developed. In a basis of a method of calculation of this parameter there is an information on levels of execution of positions of Rules AL.;
6. Results of quantitative estimation IP AL allow:
   - to compare AL at various enterprises EES, their divisions;
   - to reveal «weak parts», being a principal cause of discrepancy real AL to shown

requirements;
- to operate AL by liquidation of «weak parts»;
- to calculate objective IP AL on various categories of maintenance, tests and repair of a various kind of objects EPS

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Ability of Logical and Probabilistic Model for Operational Risk Management

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Abstract

In this paper the application of logical and probabilistic (LP) models for operational risk estimation in bank is described, the method for economic capital calculation to cover losses, caused by operational risk, with top and bottom capital limits is offered. The influence of repeated events on credit and operational risks is demonstrated. Significances of such repeated events are higher and actions to their minimization should be top-priority.

Keywords: *bank, management, operational risk, credit risk, structural risk model, logical model, probabilistic model, probability, analysis, business line, economic capital.*

I. Operational risk… What's problem ?

Quality increase, cost reducing, issue of new bank products and services are impossible without upgrading of the risk management framework in a bank. Operational risk plays large role. It leads to financial, human, reputation losses. Operational risk can take place in any bank activity but its realization influences on functioning of whole bank and values of profits. Main feature of operational risk is fundamentality. Operational risk can lead to other risks. We observe in detailed consideration: the reasons of any risk are the human factor, business process defects, technical system failure or external factors. All these reasons are operational risk factors [1]. Moreover, we can suppose the following: the higher operational risk, the higher other bank risks and more losses. Operational risk is the indicator of the bank reliability and reflects the personnel qualification level and possibility of counteraction to unfavorable events.

In comparison with financial risks, operational risk is realized in *events:* power system failure, personnel mistake, flood, earthquake or terrorism actions. Elimination of these events or minimization of consequences requires large resources. Volume of resources should be calculated and resources are reserved beforehand to save a bank from bankruptcy in case of unfavorable events. Operational risk events have accumulative nature, they are similar to "snow ball". In bank's activity some ordinary mistakes and failures occur daily. If personnel will not pay attention to ordinary events, bank administration will finally face consequences, which can be eliminated with large expenses only. Also, another problem is: banks try to hide appeared operational risk events in order to save reputation.

Problem of estimation and identification of operational risk in bank is very complex. Operational risk is caused by different factors and difficult for formalization and modeling. Existing methods of operational risk estimation are used for solution of particular tasks within one business process. Determination of risk value within one business process is not correct, it would be better to determine risk value in interconnection with other bank processes and systems.

Operational risk leads to market and credit risks [2]. Influence and "interaction" of the operational risk with other risks can cause large expenses and reputation loss. Central banks formulate task of development of internal procedures and systems, providing allocation of necessary resources (economic capital), which will equal to business goals of bank, volume of performed operations and risk profile. System for operational risk estimation should be integrated in risk management processes in bank and results should be a part of monitoring process and operational risk profile control [3].

Important task is integration of all risk models in one general model for estimation of "total risk" of bank. This allows calculate integrated risk value (risk index) for bank. Risk index is useful for owners, partners, and Bank's Management Board.

Basel III regulatory framework allows calculates economic capital more precisely [4]. New demands appoint the use of advanced methods only for credit risks while but this event causes banks to develop and implement their internal risk estimation technologies. The Basel Committee on Banking Supervision (BCBS) pays large attention to operational risk. Thus, in Basel III the estimation of economic capital adequacy is equal to coefficient 12,5 (the same coefficient is applied for market risk estimation) rather than 10, as earlier. Basel III cause banks to realize less risky politics, spend money for personal training and implementation of IT systems in order to reduce technological and administrative losses. However, in period of Basel III framework realization the value of operational risk will probably increase.

II. How to formalize operational risk ?

Let consider the logical and probabilistic (LP) model for operational risk estimation [5].

LP-model is constructed with use of the event classification, assigned by BCBS.

In advanced approach every business line is considered separately. In every business line seven kinds of unfavorable operational risk events are considered: internal fraud Z_1; external fraud Z_2; personnel policy and labor safety Z_3; clients, products and business practice Z_4; physical damage of assets Z_5; faults in business and system failures Z_6; execution, delivery and process control Z_7. These are *derivative events*. Every event from $Z_1, \ldots, Z_7$ is caused by concrete elementary events, i.e. *initiating events*. Initiating events are considered as independent casual events. In overall, 98 events were entered. Final derivative event Y is possible losses at business line. The number of initiating events for every business line is equal to 70 and they are the same by description but their probabilities for every business line will be different. Logical variable corresponds to every initiating event. This variable takes values 1 or 0 (events will occur or not) with the certain probability. Initiating events have probabilities of occurrence. These probabilities can be obtained from statistical data accumulated during last period of bank's activity (Basel II Accord recommended three years long period) or by expert way (in case of absence of statistics) [6,7].

Structural, logical and probabilistic risk models are constructed for every business line.

Structural risk model for one line. As example, let consider first business line of bank (Corporate Finance). We construct the structural model and write the logical function of risk for seven kinds of unfavorable events Z_1, Z_2, Z_3, … Z_7 (fig. 1).

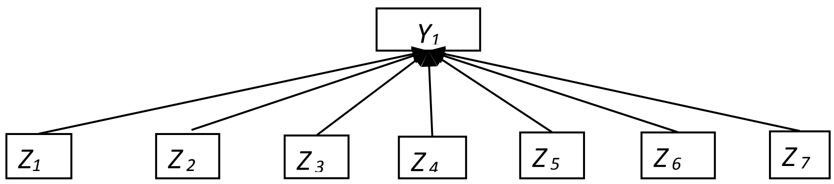

Figure 1. Structural model of operational risk for first business line (Corporate Finance).

Structural model is a *risk scenario*. Scenario is formulated so: event Y_1 (losses at first business-line) will occur if event Z_1 or event Z_2, or Z_3, …, or Z_7 will occur. By other words, Y_1 will occur if, at least, any one event from set $Z_1,\dots, Z_7$, will take place, or any combination of events, or all of them will occur at the same time (probability of such variant is very small but not equal to 0). Let $Z_1,\dots, Z_7$ are logical variables, every Z_j, $j = 1, 2,\dots,7$ is equal 1 (if events took place) or equal to 0 (in opposite case) with some probability.

Logical operational risk model for seven kinds of unfavorable events $Z_1, Z_2,\dots, Z_7$ of operational risk for first business line is written in *disjunctive normal form* by following way:

$$Y_1 = Z_1 \vee Z_2 \vee Z_3 \vee Z_4 \vee Z_5 \vee Z_6 \vee Z_7. \quad (1)$$

In order to obtain *probabilistic model* we have to write equation (1) in *orthogonal disjunctive normal form*. This operation is not simple. Here we are faced with problem of fast increasing of function dimension and we are not stating intermediate mathematical expressions in this paper due to large volume. Methods and procedures of orthogonalization are described in [8]. After orthogonalization procedures we obtain orthogonal logical function

$$Y_1 = Z_1 \vee Z_2 \overline{Z}_1 \vee Z_3 \overline{Z}_1 \overline{Z}_2 \vee Z_4 \overline{Z}_1 \overline{Z}_2 \overline{Z}_3 \vee Z_5 \overline{Z}_1 \overline{Z}_2 \overline{Z}_3 \overline{Z}_4 \vee Z_6 \overline{Z}_1 \overline{Z}_2 \overline{Z}_3 \overline{Z}_4 \overline{Z}_5 \vee$$
$$\vee Z_7 \overline{Z}_1 \overline{Z}_2 \overline{Z}_3 \overline{Z}_4 \overline{Z}_5 \overline{Z}_6$$

where logical variables and signs of logical operations can be substituted with corresponding probabilities and signs of arithmetical operations. In result, we obtain probabilistic operational risk model:

$$\begin{aligned} P\{Y_1=1\} &= P(Z_1)+P(Z_2)(1-P(Z_1)) + P(Z_3)(1-P(Z_1))(1-P(Z_2)) + \\ &+ P(Z_4)(1-P(Z_1))(1-P(Z_2)) (1-P(Z_3)) + P(Z_5)(1-P(Z_1))(1-P(Z_2))(1-P(Z_3))(1-P(Z_4)) + \\ &+ P(Z_6)(1-P(Z_1))(1-P(Z_2))(1-P(Z_3))(1-P(Z_4))(1-P(Z_5)) + \\ &P(Z_7)(1-P(Z_1))(1-P(Z_2))(1-P(Z_3))(1-P(Z_4))(1-P(Z_5))(1-P(Z_6)). \end{aligned} \quad (2)$$

Probabilistic risk model for one business line permits calculate the probability of losses at this business line if probabilities of initiating events are known.

Such models are constructed for eight business lines to calculate probabilities of events $Y_1, \dots, Y_8$.

Let construct probabilistic model for calculation of bank's operational risk. Operational risk of bank is logical sum of probabilities of losses at eight business lines.

Structural operational risk model is represented at fig. 2.

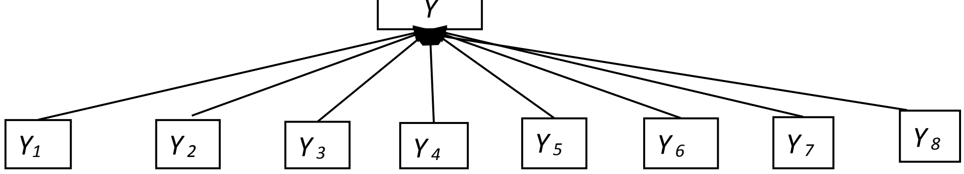

Figure 2. Structural model of bank's operational risk.

Logical model of bank's operational risk in disjunctive normal form is following:

$$Y = Y_1 \vee Y_2 \vee Y_3 \vee Y_4 \vee Y_5 \vee Y_6 \vee Y_7 \vee Y_8, \qquad (3)$$

where:
Y- bank's operational risk,
Y_i – event on i bank's business-line, $i = 1,\dots,8$.

We obtain probabilistic model from logical model by orthogonalization way:

$$P\{Y=1\}=P_1+P_2(1-P_1)+\dots+P_8(1-P_1)(1-P_2)(1-P_3)(1-P_4)(1-P_5)(1-P_6)(1-P_7). \qquad (4)$$

Note, this model can be applied for estimation of bank operational risk by the standardized approach with use of values $P(Y_1)$, $P(Y_2)$, …, $P(Y_8)$ instead of coefficients β in formula of capital resevation. Such modified formula permits determine the volume of the capital for covering losses more precisely because it takes into account functioning features of the concrete bank in comparison with coefficients β, averaged on whole branch [9].

In practice, we don't need use classification of events, offered by Basel II Capital Accord. LP-models can be adopted for business lines and kinds of events in concrete bank. For example, in some Russian banks the additional ninth business line is used. Events, which were not classified on eight standard business lines, are referred to ninth business line. Basel II Capital Accord recommends refer these events to line where the most profit is.

III. How to calculate economic capital volume ?

In general case, for calculation of economic capital we have to calculate probabilities $P_{i,j,k}$ and losses $L_{i,j,k}$ for every initiating event $Z_{i,j,k}$ by statistical data. Here:
$i = 1,2,\dots,8$ – the number of business line;
$j = 1, 2,\dots,7$ – kind of events;
$k =1,2,\dots,N_j$ – initiating events indexes in j-kind of events:
$N_j = 2\div20$ – the number of initiating events of the kind j.
Initiating events probabilities are calculated by formula:

$$P_{i,j,k} = N_{i,j,k} / N, \qquad (5)$$

where: $N_{i,j,k}$ – the number of appearance of losses at business line i due to reason j and initiating event k; N – the number of operations at the business line of the bank in considered time interval.

Estimation of economic capital volume consists of two parts: expected and unexpected losses. Economic capital for expected losses EL is calculated by statistics and can be obtained by summarizing of all losses per a year (true economic capital):

$$EL = \sum_{i=1}^{8}\sum_{j=1}^{7}\sum_{k=1}^{N_j} L_{i,j,k}, \qquad (6)$$

where $L_{i,j,k}$ - summarized losses due to realization (or several realization) event k of kind j at business line i during report period (for example, one year).

Unexpected losses UL^{LP} is suggested to estimate by formula of predictable damage for technical systems [10]:

$$UL^{LP} = P_Y L_{max}, \tag{7}$$

where: P_Y - operational risk of bank is calculated by equation (4),

L_{max} – maximal possible loss at business line, concrete operation (transaction) or in bank as a whole, depending from modeling level.

Risk-manager should decide what losses will be chosen as L_{max}, proceed from the situation. Gross receipt at business line, maximal losses at business line or operation (transaction) can be chosen, or L_{max} can be given on basis of expert evaluation also.

Economic capital volume is calculated by formula:

$$R_{Sub}^{LP} = EL + UL^{LP}. \tag{8}$$

Value R_{sub}^{LP} is bottom limit of economic capital.

The basic indicator approach determines economic capital for operational risk of bank have to be 15% of average gross receipt of bank during three years. For analysis we have to know top limit of possible losses from unfavorable economic situation and unforeseen rare events [11].

Top limit estimation of the reserved capital is performed proceed from the integrated risk of the bank as a whole:

$$R_{Sup}^{LP} = P_Y Q, \tag{9}$$

where: Q – gross receipt of the bank;

P_Y – the probability calculated by probabilistic model (4).

Evaluations by (6), (8) and (9) will be different. Choice of the formula depends on data and expenses of data obtaining. Formula (6) estimates real losses of last years. Formula (8) gives bottom limit of reserved capital under known losses. However, in practice it is difficult to estimate precisely the value of losses due to operational risk event, therefore, we need to know top limit of possible losses. In case of unstable economic and political situation we recommend use formula (9) for calculation of maximal value of economic capital, using the volume of bank's profit which can be lost in case of unfavorable events. Choice of formula depends on situation and this is duty of risk-manager.

IV. Towards to integration

The advantage of LP operational risk model is possibility to unite with other LP risk models. This allows develop a complex model to calculate integrated risk index of a bank.

In activity of any bank there are events that influence on several risks in same time. These events are called "repeated". Losses from repeated events have to be fixed in connection with those risks that were influenced by them. If they influence on several risks then the economic capital is made for every risk. In this case we have too large double economic capital. Of course, if there are several owners of the risk then every owner is responsible. But risk-managers have to pay large attention to these events. We offer to construct the LP risk model with repeated events influencing on several processes simultaneously. The complex LP risk model in bank with logical operations AND or OR, uniting LP operational risk model with LP models of other risks, allows perform quantitative estimation of integrated risk of bank and mark out repeated elements which influence on several risks at once [12].

Operational risk value may be considered as the index of bank's reliability. With small percent (about 5 %) [13] it influences significantly on all bank risks. So, operational risk estimation is important problem and good management of operational risk help to reduce losses from other risks.

Let construct the united LP-model of operational and credit risk with logical connection AND (fig. 3).

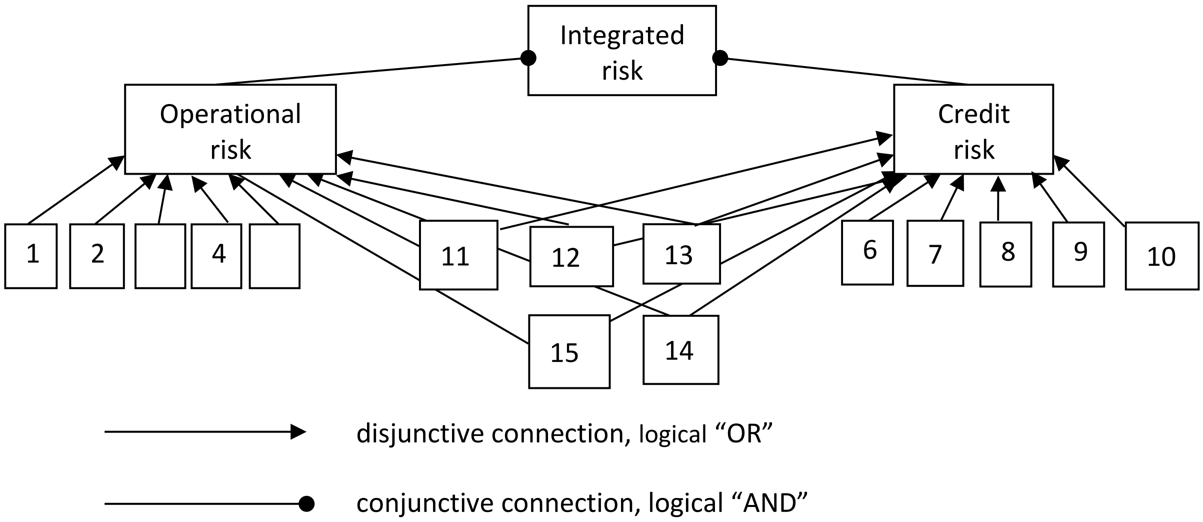

Figure 3. United model of operational and credit risks

where:
1- clients, products, business practice;
2 – operations and process control;
3 – damage of material assets;
4 – organizational violations and system failures;
5 – data transmission process violation;
6 – wrong technique of credit risk estimation;
7 – wrong credit portfolio estimation;
8 – wrong calculation of economic capital volume;
9 – mistake in guarantee estimation;
10 - accident with borrower;
11 - fraud;
12 – economic situation changing;
13 – mistakes in registration of borrower's application;
14 – inaccurate information given to borrower;
15 – mistake in management of bank risks.

Let mark out five events - 1, 2, 3, 4, 5 for operational risk, and events 6, 7, 8, 9, 10 for credit risk. Events 11, 12, 13, 14, 15 are repeated for operational and credit risks. To simplify the calculation, let event 11 is external and internal fraud together.

Under the united structural model (fig.3) we write the logical risk model:

$$Y = Y_{OpR} \wedge Y_{CredR}, \quad \text{or}$$

$$Y = (Y_1 \vee Y_2 \vee Y_3 \vee Y_4 \vee Y_5 \vee Y_{11} \vee Y_{12} \vee Y_{13} \vee Y_{14} \vee Y_{15}) \wedge (Y_6 \vee \vee Y_7 \vee Y_8 \vee Y_9 \vee Y_{10} \vee Y_{11} \vee Y_{12} \vee Y_{13} \vee Y_{14} \vee Y_{15}) \quad (10)$$

Probabilistic functions, written for operational and credit risks separately, will look like function (4). Integrated risk is obtained by multiplication values $P(Y_{OpR}=1)$ and $P(Y_{CredR}=1)$.

Let calculate, for analysis, probabilities of operational and credit risks and integrated risk (table 1) without repeated events 11-15. Further, we enter other events gradually, one by one, make calculations on changing models and fix the changing of integrated risk. Integrated risk will change on some value. This value is *contribution of event* in risk. In table 1 probabilities of initiating events are presented. Probabilities of events were obtained by experts with use of the method of summarized indexes [7, 14], taking into account the percent of operational risk in Russian banks is 5% [13]. Expert estimation were obtained by same method for credit risk also but taking into account the average risk of credit portfolio is 25% and was recognized as satisfactory[1].

Table. 1. Results of calculations on model with repeated events.

Event	1 variant (without repeated event)	2 variant (1 repeated event)	3 variant (2 repeated events)	4 variant (3 repeated events)	5 variant (4 repeated events)	6 variant (5 repeated event)
Clients, products and business practice	0,00146	0,00146	0,00146	0,00146	0,00146	0,00146
Operation and process control	0,0138	0,0138	0,0138	0,0138	0,0138	0,0138
Damage of material assets	0,0015	0,0015	0,0015	0,0015	0,0015	0,0015
Organizational violations and system failures	0,00041	0,00041	0,00041	0,00041	0,00041	0,00041
Data transmission process violation	0,022	0,022	0,022	0,022	0,022	0,022
Wrong technique of credit risk estimation	0,05677	0,05677	0,05677	0,05677	0,05677	0,05677
Wrong credit portfolio estimation	0,051323	0,051323	0,051323	0,051323	0,051323	0,051323
Wrong calculation of economic capital volume	0,036733	0,036733	0,036733	0,036733	0,036733	0,036733
Mistake in guarantee estimation	0,050401	0,050401	0,050401	0,050401	0,050401	0,050401
Accident with borrower	0,016759	0,016759	0,016759	0,016759	0,016759	0,016759
Fraud		0,0191	0,0191	0,0191	0,0191	0,0191
Economic situation changing			0,0054	0,0054	0,0054	0,0054
Mistakes in registration of borrower's application				0,0041	0,0041	0,0041
Inaccurate information given to borrower					0,00024	0,00024
Mistake in						0,00518

[1] This was obtained by expert way to demonstrate calculations.

Event	1 variant (without repeated event)	2 variant (1 repeated event)	3 variant (2 repeated events)	4 variant (3 repeated events)	5 variant (4 repeated events)	6 variant (5 repeated event)
management of bank risks						
OR	0,0387473	0,057103	0,062195	0,06604	0,066264	0,071101
CR	0,195209	0,210581	0,214844	0,218063	0,218251	0,2223
Integrated risk	0,007563	0,026519	0,031775	0,035745	0,035977	0,04097

Contributions of repeated events are presented in table 2.

Table 2. Contribution of repeated events in changing of risks

Risk	Risk without repeated events, %	Contribution of event 1 in changing of risk	Contribution of event 2 in changing of risk	Contribution of event 3 in changing of risk	Contribution of event 4 in changing of risk	Contribution of event 5 in changing of risk	Difference between final result and primary result
OR	3,87473	1,8087	0,5043	0,3824	0,0223	0,4837	3,23537
CR	1,95209	1,5114	0,4222	0,3202	0,0187	0,4049	2,7091
Integrated risk	0,7563	1,8674	0,5207	0,3948	0,023	0,4993	3,34

Repeated event make different contributions in operational and credit risks. At the same time the operational risk has increased on 3,23 percentage points due to repeated elements, credit risk - on 2,7 percentage points and integrated risk - on 3,34 percentage points.

Significances of initiating events for final event are presented in table 3.

Table 3. Significances of events for final event

Number of the initiating event	Significance of the event
1	+1.81595E-01
2	+1.83867E-01
3	+1.81602E-01
4	+1.81404E-01
5	+1.85409E-01
6	+3.1944E-02
7	+3.17609E-02
8	+3.12798E-02
9	+3.17300E-02
10	+3.06444E-02
11	+9.77704E-01
12	+9.64237E-01
13	+9.62978E-01
14	+9.5926E-01
15	+9.64023E-01

As result we have obtained, repeated events that are initiating events for several risks have larger significance for final event.

Integrated risk can be used as bank risk index. It allows classify bank in corresponding category of quality, reliability and safety. This is useful for investors, creditors, partners and other interesting persons. Also, risk index allows to compare banks with each other.

V. Conclusion

Above-mentioned models are simple but they reflect the sense of the approach, based on tree of events, logics and probability theory. LP method is flexible and allows adopt models for concrete bank without limitation of event classification, suggested by Basel II Capital Accord. For example, in (Karaseva, 2012) there is a model of internal fraud in bank, describing this problem more detailed. Internal fraud model can be included in model (3) to increase the accuracy of operational risk estimation and economic capital calculation.

Advantage of LP-model is possibility to analyze of operational risk (determination of events with largest contribution in losses) and analyze repeated events, which provide contributions in different risks. Taking into account repeated elements gives the more accurate estimation of integrated risk of bank. Integrated risk can be used for management purposes and classification.

Offered approach is simple, transparent, understandable for bank personnel and does not require large expenses of money and resources. However, it can be realized only with a system of effective monitoring of operational risk (events have to be fixed). It is necessary to provide strict event classification in a bank (or use ready event classification from Basel II) in order to every initiating events will be classified to certain kind of certain business line without any doubts.

Main serious problem in realization of logical and probabilistic technique of estimation and analysis of operational risk is providing of good motivation of bank's employers to fix events without fear for their personal mistakes. Every employer has to fix occurred events promptly and pass reports to operational risk manager. The manager classifies events and inputs in database. Database information is used for regular re-training of model (calculation of probabilities of initiating events) in order to take into account change of economic situation and internal processes conditions and keep necessary accuracy of estimation.

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Constructing Tolerance Limits On Order Statistics In Future Samples Coming From Location-Scale Distributions

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Abstract

Although the concept of statistical tolerance limits has been well recognized for long time, surprisingly, it seems that their applications remain still limited. Analytic formulas for the tolerance limits are available in only simple cases. Thus it becomes necessary to use new or innovative approaches which will allow one to construct tolerance limits on future order statistics for many populations. In this paper, a new approach to constructing lower and upper tolerance limits on order statistics in future samples is proposed. Attention is restricted to location-scale distributions under parametric uncertainty. The approach used here emphasizes pivotal quantities relevant for obtaining tolerance factors and is applicable whenever the statistical problem is invariant under a group of transformations that acts transitively on the parameter space. It does not require the construction of any tables and is applicable whether the past data are complete or Type II censored. The proposed approach requires a quantile of the F distribution and is conceptually simple and easy to use. For illustration, the normal and log-normal distributions are considered. The discussion is restricted to one-sided tolerance limits. A practical example is given.

Keywords: *order statistics, F distribution, tolerance limits, location-scale distribution*

1. Introduction

Statistical tolerance limits are an important tool often utilized in areas such as engineering, manufacturing, and quality control for making statistical inference on an unknown population. As opposed to a confidence limit that provides information concerning an unknown population parameter, a tolerance limit provides information on the entire population; to be specific, one-sided tolerance limit is expected to capture a certain proportion or more of the population, with a given confidence level. For example, an upper tolerance limit for a univariate population is such that with a given confidence level, a specified proportion or more of the population will fall below the limit. A lower tolerance limit satisfies similar conditions. It is often desirable to have statistical tolerance limits available for the distributions used to describe time-to-failure data in reliability problems. For example, one might wish to know if at least a certain proportion, say β, of a manufactured product will operate at least T hours. This question can not usually be answered exactly, but it may be possible to determine a lower tolerance limit $L(X_1, \ldots, X_n)$, based on a preliminary random sample $(X_1, \ldots, X_n)$, such that one can say with a certain confidence γ that at least 100β % of the product will operate longer than $L(X_1, \ldots, X_n)$. Then reliability statements can be made based on $L(X_1, \ldots, X_n)$, or, decisions can be reached by comparing $L(X_1, \ldots, X_n)$ to T. Tolerance limits of the type mentioned above are considered in this paper. That is, if $f_\theta(x)$ denotes the density function of the parent population under consideration and if S is any statistic obtained from the preliminary random sample $(X_1, \ldots, X_n)$ of that population, then $L(S)$ is a lower γ probability tolerance limit for proportion β if

$$\Pr\left(\int_{L(S)}^{\infty} f_\theta(x)dx \geq \beta\right) = \gamma, \tag{1}$$

and $U(S)$ is an upper γ probability tolerance limit for proportion β if

$$\Pr\left(\int_{-\infty}^{U(S)} f_\theta(x)dx \geq \beta\right) = \gamma, \tag{2}$$

where θ is the parameter (in general, vector).

The common distributions used in life testing problems are the normal, log-normal, exponential, Weibull, and gamma distributions [1]. Tolerance limits for the normal distribution have been considered in [2], [3], [4], and others.

Tolerance limits enjoy a fairly rich history in the literature and have a very important role in engineering and manufacturing applications. Patel [5] provides a review (which was fairly comprehensive at the time of publication) of tolerance limits for many distributions as well as a discussion of their relation with confidence intervals for percentiles and prediction intervals. Dunsmore [6] and Guenther, Patil, and Uppuluri [7] both discuss 2-parameter exponential tolerance intervals and the estimation procedure in greater detail. Engelhardt and Bain [8] discuss how to modify the formulas when dealing with type II censored data. Guenther [9] and Hahn and Meeker [10] discuss how one-sided tolerance limits can be used to obtain approximate two-sided tolerance intervals by applying Bonferroni's inequality. Tolerance limits on order statistics in future samples coming from a two-parameter exponential distribution have been considered in [11].

In contrast to other statistical limits commonly used for statistical inference, the tolerance limits (especially on order statistics) are used relatively rarely. One reason is that the theoretical concept and computational complexity of the tolerance limits is significantly more difficult than that of the standard confidence and prediction limits. Thus it becomes necessary to use new or innovative approaches which will allow one to construct tolerance limits on future order statistics for many populations.

In this paper, a new approach to constructing lower and upper tolerance limits on order statistics in future samples is proposed. For illustration, the normal and log-normal distributions that are commonly used in reliability and risk theory are considered. Although the concept of statistical tolerance limits has been well recognized for long time, surprisingly, it seems that their applications remain still limited.

2. Mathematical Preliminaries

2.1. Probability Distribution Function of Order Statistic

Theorem 1. If there is a random sample of m ordered observations $Y_1 \le \ldots \le Y_m$ from a known distribution (continuous or discrete) with density function $f_\theta(y)$, distribution function $F_\theta(y)$, then the probability distribution function of the kth order statistic Y_k, $k \in \{1, 2, \ldots, m\}$, is given by

$$P_\theta(Y_k \le y_k) = \int\limits_{\frac{1-F_\theta(y_k)}{F_\theta(y_k)} \frac{2k}{2(m-k+1)}}^{\infty} f_{2(m-k+1),2k}(x)dx, \tag{3}$$

where

$$f_{2(m-k+1),2k}(x) = \frac{1}{\mathrm{B}\left(\frac{2(m-k+1)}{2}, \frac{2k}{2}\right)} \left(\frac{2(m-k+1)}{2k}\right) \left(\frac{2(m-k+1)}{2k} x\right)^{2(m-k+1)/2-1}$$

$$\times \left(1 + \frac{2(m-k+1)}{2k} x\right)^{-[2(m-k+1)+2k]/2}, \quad x > 0, \tag{4}$$

is the probability density function of an F distribution with $2(m-k+1)$ and $2k$ degrees of freedom.

Proof. Suppose an event occurs with probability p per trial. It is well-known that the probability P of its occurring k or more times in m trials is termed a cumulative binomial probability, and is related to the incomplete beta function $I_x(a, b)$ as follows:

$$P \equiv \sum_{j=k}^{m} \binom{m}{j} p^j (1-p)^{m-j} = I_p(k, m-k+1). \tag{5}$$

It follows from (5) that

$$P_\theta\{Y_k \le y_k\} = \sum_{j=k}^{m} \binom{m}{j} [F_\theta(y_k)]^j [1 - F_\theta(y_k)]^{m-j} = I_{F_\theta(y_k)}(k, m-k+1)$$

$$= \frac{1}{\mathrm{B}(k, m-k+1)} \int\limits_0^{F_\theta(y_k)} u^{k-1}(1-u)^{(m-k+1)-1} du = \frac{\left(\frac{2(m-k+1)}{2k}\right)^{2(m-k+1)/2}}{\mathrm{B}\left(\frac{2k}{2}, \frac{2(m-k+1)}{2}\right)} \int\limits_0^{F_\theta(y_k)} u^{\frac{2(m-k+1)+2k}{2}}$$

$$\times\left(\frac{1-u}{u}\frac{2k}{2(m-k+1)}\right)^{2(m-k+1)/2-1}\frac{-2k}{2(m-k+1)}\left(-\frac{du}{u^2}\right)$$

$$=\frac{\left(\frac{2(m-k+1)}{2k}\right)^{2(m-k+1)/2}}{\mathrm{B}\left(\frac{2(m-k+1)}{2},\frac{2k}{2}\right)}\int\limits_{\frac{1-F_\theta(y_k)}{F_\theta(y_k)}\frac{2k}{2(m-k+1)}}^{\infty} x^{2(m-k+1)/2-1}\left(1+\frac{2(m-k+1)}{2k}x\right)^{-[2(m-k+1)+2k]/2}dx, \quad (6)$$

where

$$x=\frac{1-u}{u}\frac{2k}{2(m-k+1)}. \quad (7)$$

This ends the proof.

Corollary 1.1.

$$P_\theta(Y_k>y_k)=1-P_\theta\{Y_k\le y_k\}=\int\limits_0^{\frac{1-F_\theta(y_k)}{F_\theta(y_k)}\frac{2k}{2(m-k+1)}} f_{2(m-k+1),2k}(x)dx. \quad (8)$$

Corollary 1.2. If $y_{k,m;\gamma}$ is the quantile of order γ for the distribution of Y_k, we have from (8) that $y_{k,m;\gamma}$ is the solution of

$$F_\theta(y_{k,m;\gamma})=k\big/[k+(m-k+1)q_{2(m-k+1),2k;1-\gamma}], \quad (9)$$

where $q_{2(m-k+1),2k;1-\gamma}$ is the quantile of order $1-\gamma$ for the F distribution with $2(m-k+1)$ and $2k$ degrees of freedom.

2.2. Normal and Log-Normal Distributions

The normal and log-normal distributions are commonly used to model certain types of data that arise in several fields of engineering as, for example, different types of lifetime data (see, e.g., [12]). The goal of modeling certain types of data is to provide quantitative forecasts of various system performance measures such as service level, expected waiting time, agent's occupancy, schedule efficiency, cost etc. Evaluation of these performance measures is important to making optimal decisions about overall cost, system performance, which has to be within the allowable budget and other performance based constraints.

Particular properties of the log-normal random variable (as the non-negativeness and the skewness) and of the log-normal hazard function (which increases initially and then decreases) make log-normal distribution a suitable fit for some engineering data sets. The log-normal distribution is used to model the lives of units whose failure modes are of a fatigue-stress nature. Since this includes most, if not all, mechanical systems, the log-normal distribution can have widespread application. Consequently, the log-normal distribution is a good companion to the Weibull distribution when attempting to model these types of units. As may be surmised by the name, the log-normal distribution has certain similarities to the normal distribution. A random variable is log-normally distributed if the logarithm of the random variable is normally distributed. Because of this, there are many mathematical similarities between the two distributions. For example, the mathematical reasoning for the construction of the probability plotting scales and the bias of parameter estimators is very similar for these two distributions.

Nevertheless, the log-normal distribution differs from the normal distribution in several ways. A major difference is in its shape: where the normal distribution is symmetrical, a lognormal one is not. Because the values in a lognormal distribution are positive, they create a right skewed curve (Figure 1).

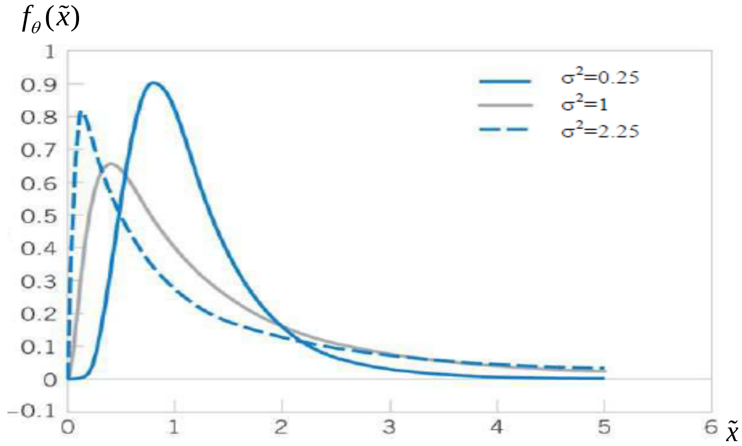

Figure 1. *Log-normal probability density functions with μ=0 for selected values of σ^2.*

The log-normal distribution has played major roles in diverse areas of science. Royston [13] modeled survival time in cancer with an emphasis on prognostic factors using the log-normal distribution. Log-normal distributions gave appropriate description of the overall service times and the service times of administrative, e-mail, miscellaneous and network jobs.

Finally, log-normal distributions are self-replicating under multiplication and division, i.e., products and quotients of log-normal random variables are themselves log-normal distributions (Crow and Shimizu [14]; Aitchison and Brown [15]), a result often exploited in back-of-the-envelope calculations.

A positive random variable $\tilde{X}$ is said to be log-normally distributed with two parameters μ and σ^2 if $X=\ln\tilde{X}$ is normally distributed with mean μ and variance σ^2. The two-parameter log-normal distribution is denoted by $\Lambda(\mu,\sigma^2)$; the corresponding normal distribution is denoted by $N(\mu,\sigma^2)$. The probability density function (pdf) of $\tilde{X}$ having $\Lambda(\mu,\sigma^2)$ is

$$f_\theta(\tilde{x})=\frac{1}{\tilde{x}\sigma\sqrt{2\pi}}\exp\left(-\frac{[\ln\tilde{x}-\mu]^2}{2\sigma^2}\right),\quad \tilde{x}>0,\quad -\infty<\mu<\infty,\quad \sigma>0, \tag{10}$$

where $\theta=(\mu,\sigma^2)$. The cumulative distribution function (cdf)) of $\tilde{X}$ is given by

$$F_\theta(\tilde{x})=\Pr(Z\le\tilde{x})=\Phi\left(\frac{\ln\tilde{x}-\mu}{\sigma}\right). \tag{11}$$

It follows from (10) that

$$X\sim f_\theta(x)=\frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right),\quad -\infty<x<\infty, \tag{12}$$

that is, $X=\ln\tilde{X}\sim N(\mu,\sigma^2)$, where $\theta=(\mu,\sigma^2)$, $-\infty<\mu<\infty$ is the location parameter and $\sigma>0$ is the scale parameter. The cdf of the normal distribution is given by

$$F_\theta(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{y} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx. \tag{13}$$

It is known (Nechval and Vasermanis [16]) that the complete sufficient statistic for the parametric vector θ, based on observations in a random sample (X_1, ..., X_n) of size n from the normal distribution (13) is given by

$$S = \left(\bar{X} = \sum_{i=1}^{n} X_i \,/\, n,\ S_1^2 = \sum_{i=1}^{n} (X_i - \bar{X})^2 \,/\, (n-1) \right). \tag{14}$$

Here the following theorem takes place.

Theorem 2. Let (X_1, ..., X_n) be a preliminary random sample from the normal distribution (13) , where it is assumed that the parametric vector $\theta = (\mu,\sigma^2)$ is unknown. Then the joint probability density function of the pivotal quantities,

$$V_1 = \frac{\sqrt{n}(\bar{X}-\mu)}{\sigma}, \quad V_2 = \frac{(n-1)S_1^2}{\sigma^2}, \tag{15}$$

is given by

$$f(v) = f_1(v_1) f_2(v_2), \tag{16}$$

where

$$V = (V_1, V_2), \tag{17}$$

$$f_1(v_1) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{v_1^2}{2}\right), \quad -\infty < v_1 < \infty, \tag{18}$$

$$f_2(v_2) = \frac{1}{2^{(n-1)/2}\Gamma\big((n-1)/2\big)} v_2^{(n-1)/2-1} \exp(-v_2/2), \quad v_2 \ge 0. \tag{19}$$

Proof. The joint density of X_1, ..., X_n is given by

$$f_\theta(x_1,\ ...,\ x_n) = \prod_{i=1}^{r} f_\theta(x_i) = \prod_{i=1}^{n} (2\pi\sigma^2)^{-1/2} \exp\left(-\frac{1}{2\sigma^2}(x_i-\mu)^2\right)$$

$$= (2\pi\sigma^2)^{-n/2} \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2\right). \tag{20}$$

Using the invariant embedding technique (Nechval et al. [17], [18], [19]), we transform (20) to

$$f_\theta(x_1,\ ...,\ x_n) d\bar{x} ds_1^2 = (2\pi\sigma^2)^{-n/2} \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \bar{x} + \bar{x} - \mu)^2\right) d\bar{x} ds_1^2$$

$$= (2\pi\sigma^2)^{-n/2} \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}[(x_i-\bar{x})^2 + 2(x_i-\bar{x})(\bar{x}-\mu) + (\bar{x}-\mu)^2]\right) d\bar{x} ds_1^2$$

$$= (2\pi\sigma^2)^{-n/2} \exp\left(-\frac{1}{2\sigma^2}\left[\sum_{i=1}^{n}(x_i-\bar{x})^2 + 2(\bar{x}-\mu)\sum_{i=1}^{n}(x_i-\bar{x}) + n(\bar{x}-\mu)^2\right]\right) d\bar{x} ds_1^2$$

$$= (2\pi\sigma^2)^{-n/2} \exp\left(-\frac{1}{2\sigma^2}\left[\sum_{i=1}^{n}(x_i-\bar{x})^2 + n(\bar{x}-\mu)^2\right]\right) d\bar{x} ds_1^2$$

$$= n^{-1/2}(2\pi)^{-1/2} \exp\left(-\frac{n(\bar{x}-\mu)^2}{2\sigma^2}\right) d\left(\frac{\sqrt{n}(\bar{x}-\mu)}{\sigma}\right)$$

$$\times (\pi)^{-(n-1)/2}(n-1)^{-(n-1)/2}(s_1^2)^{-(n-1)/2}\left(\frac{(n-1)s_1^2}{2\sigma^2}\right)^{(n-1)/2-1} \exp\left(-\frac{(n-1)s_1^2}{2\sigma^2}\right) d\left(\frac{(n-1)s_1^2}{2\sigma^2}\right)$$

$$\propto (2\pi)^{-1/2} \exp\left(-\frac{v_1^2}{2}\right) dv_1 \left(\frac{v_2}{2}\right)^{(n-1)/2-1} \exp\left(-\frac{v_2}{2}\right) d\left(\frac{v_2}{2}\right). \tag{21}$$

Normalizing (21), we obtain (16). This ends the proof.

Thus,

$$V_1 \sim N(0,1), \quad V_2 \sim \chi^2_{n-1}, \tag{22}$$

where V_2 is statistically independent of V_1.

Theorem 3. If V_1 is a normally distributed random variable with unit variance and zero mean, and V_2 is a chi-squared distributed random variable with n-1 degrees of freedom that is statistically independent of V_1, then

$$T = \frac{V_1+\Delta}{\sqrt{V_2/(n-1)}} = \frac{V_1+\Delta}{\sqrt{W}} \sim f_{n-1,\Delta}(t), \quad -\infty < t < \infty, \tag{23}$$

is a non-central t-distributed random variable with n-1 degrees of freedom and non-centrality parameter Δ, where

$$W = \frac{V_2}{n-1} = \frac{S_1^2}{\sigma^2} \sim f_{n-1}(w) = \frac{(n-1)^{(n-1)/2}}{2^{(n-1)/2}\Gamma\big((n-1)/2\big)} w^{(n-1)/2-1} \exp(-(n-1)w/2), \quad w \ge 0, \tag{24}$$

$$f_{n-1,\Delta}(t) = \frac{(n-1)^{(n-1)/2}}{\sqrt{\pi}\Gamma\big((n-1)/2\big)2^{n/2}} \frac{\exp\left(-\frac{(n-1)\Delta^2}{2(t^2+n-1)}\right)}{(t^2+n-1)^{n/2}}$$

$$\times \int_0^\infty w_\bullet^{n/2-1} \exp\left(-\frac{1}{2}\left[w_\bullet^{1/2} - \frac{t\Delta}{\sqrt{t^2+n-1}}\right]^2\right) dw_\bullet, \quad -\infty < t < \infty, \tag{25}$$

is the probability density function of T,

$$W_\bullet = W(t^2+n-1), \tag{26}$$

$$F_{n-1,\Delta}(t) = \Pr(T \le t) = \frac{(n-1)^{(n-1)/2}}{2^{(n-1)/2}\Gamma\big((n-1)/2\big)} \int_0^\infty w^{(n-1)/2-1} \exp(-(n-1)w/2)\Phi(t\sqrt{w}-\Delta) dw \tag{27}$$

is the cumulative distribution function of T. $\Phi(x)$ is the standard normal distribution function. Note that the non-centrality parameter Δ may be negative.

Proof. It follows from (23) that

$$\Pr(T \le t \mid W = w) = \Pr\left(\frac{V_1 + \Delta}{\sqrt{w}} \le t \,\middle|\, w \right) = \Pr\left(V_1 \le t\sqrt{w} - \Delta\right) = \Phi(t\sqrt{w} - \Delta). \tag{28}$$

Since it follows from (24) and (28) that

$$\Pr(T \le t) = E\left\{\Pr(T \le t \mid W)\right\} = \int_0^\infty \Phi(t\sqrt{w} - \Delta)\, f_{n-1}(w)dw, \tag{29}$$

we get the cumulative distribution function $F_{n-1,\Delta}(t)$ of the non-central t-distribution given in (27). It is easy to show that the probability density function of T defined in (25) is given by

$$f_{n-1,\Delta}(t) = F'_{n-1,\Delta}(t). \tag{30}$$

This completes the proof.

3. Tolerance Limits on Order Statistic

3.1. Lower Tolerance Limit

Theorem 4. Let $X_1, \ldots, X_n$ be observations from a preliminary sample of size n from a normal distribution defined by the probability density function (12). Then a lower one-sided β-content tolerance limit at a confidence level γ, $L_k \equiv L_k(S)$ (on the kth order statistic Y_k, $k\in\{1, \ldots, m\}$, from a set of m future ordered observations $Y_1 \le \ldots \le Y_m$ also from the distribution (12)), which satisfies

$$\Pr\left\{P_\theta(Y_k > L_k) \ge \beta\right\} = \gamma, \tag{31}$$

is given by

$$L_k = \bar{X} + \eta_L S_1, \tag{32}$$

where

$$\eta_L = -t_{r,\Delta;\gamma} \big/ \sqrt{n}, \tag{33}$$

is the lower tolerance factor, $t_{r,\Delta;\gamma}$ is the quantile of order γ for the non-central t-distribution with $r=n-1$ degrees of freedom and non-centrality parameter $\Delta = -z_{1-\delta_\beta}\sqrt{n}$, $z_{1-\delta_\beta}$ denotes the $1-\delta_\beta$ quantile of the standard normal distribution,

$$\delta_\beta = (m-k+1)q_{2(m-k+1),2k;\beta} \big/ [(m-k+1)q_{2(m-k+1),2k;\beta} + k], \tag{34}$$

$q_{2(m-k+1),2k;\beta}$ is the quantile of order β for the F distribution with $2(m-k+1)$ and $2k$ degrees of freedom.

Proof. It follows from (8), (13) and (31) that

$$\Pr\left\{P_\theta(Y_k > L_k) \ge \beta\right\} = \Pr\left(\int_0^{\frac{1-F_\theta(L_k)}{F_\theta(L_k)}\frac{2k}{2(m-k+1)}} f_{2(m-k+1),2k}(x)dx \ge \beta \right)$$

$$= \Pr\left(\frac{1-F_\theta(L_k)}{F_\theta(L_k)}\frac{2k}{2(m-k+1)} \geq q_{2(m-k+1),2k;\beta}\right) = \Pr\left(F_\theta(L_k) \leq \frac{k}{k+(m-k+1)q_{2(m-k+1),2k;\beta}}\right)$$

$$= \Pr\left(\frac{1}{\sigma\sqrt{2\pi}}\int_{L_k}^{\infty}\exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right)dy \geq \frac{(m-k+1)q_{2(m-k+1),2k;\beta}}{(m-k+1)q_{2(m-k+1),2k;\beta}+k}\right)$$

$$= \Pr\left(\frac{1}{\sqrt{2\pi}}\int_{\frac{L_k-\mu}{\sigma}}^{\infty}\exp\left(-\frac{z^2}{2}\right)dz \geq \delta_\beta\right) = \Pr\left(\frac{1}{\sqrt{2\pi}}\int_{\infty}^{\frac{L_k-\mu}{\sigma}}\exp\left(-\frac{z^2}{2}\right)dz \leq 1-\delta_\beta\right)$$

$$= \Pr\left(\frac{L_k-\mu}{\sigma} \leq z_{1-\delta_\beta}\right) = \Pr\left(\frac{L_k-\bar{X}+\bar{X}-\mu}{\sigma} \leq z_{1-\delta_\beta}\right)$$

$$= \Pr\left(\frac{L_k-\bar{X}}{S_1}\sqrt{n}\frac{S_1}{\sigma}+\frac{\sqrt{n}(\bar{X}-\mu)}{\sigma} \leq z_{1-\delta_\beta}\sqrt{n}\right) = \Pr\left(\eta_L\sqrt{n}\sqrt{W}+V_1 \leq z_{1-\delta_\beta}\sqrt{n}\right)$$

$$= \Pr\left(\frac{V_1-z_{1-\delta_\beta}\sqrt{n}}{\sqrt{W}} \leq -\eta_L\sqrt{n}\right) = \Pr\left(\frac{V_1+\Delta}{\sqrt{W}} \leq -\eta_L\sqrt{n}\right) = \Pr\left(T \leq -\eta_L\sqrt{n}\right) = F_{r,\Delta}(t), \tag{35}$$

where

$$\eta_L = (L_k-\bar{X})\big/S_1, \tag{36}$$

is the lower tolerance factor,

$$\Delta = -z_{1-\delta_\beta}\sqrt{n}, \quad r = n-1, \quad t = -\eta_L\sqrt{n}. \tag{37}$$

It follows from (31), (35) and (37) that the lower tolerance factor η_L should be chosen such that

$$F_{r,\Delta}(t) = F_{r,\Delta}(-\eta_L\sqrt{n}) = F_{r,\Delta}(t_{r,\Delta;\gamma}) = \gamma, \tag{38}$$

where $t_{r,\Delta;\gamma}$ is the quantile of order γ for the non-central *t*-distribution with *r* degrees of freedom and non-centrality parameter Δ. It follows from (38) that

$$\eta_L = -t_{r,\Delta;\gamma}\big/\sqrt{n}. \tag{39}$$

It follows from (36) that $L_k = \bar{X}+\eta_L S_1$. This completes the proof.

Corollary 4.1. It follows from (35) that $\Pr\left(\eta_L\sqrt{n}\sqrt{W}+V_1 \leq z_{1-\delta_\beta}\sqrt{n}\right)$ can be transformed as follows:

$$\Pr\left(\eta_L\sqrt{n}\sqrt{W}+V_1 \leq z_{1-\delta_\beta}\sqrt{n}\right) = \Pr\left(V_1 \leq -\eta_L\sqrt{n}\sqrt{W}+z_{1-\delta_\beta}\sqrt{n}\right)$$

$$= \int_{-\infty}^{-\eta_L\sqrt{n}\sqrt{W}+z_{1-\delta_\beta}\sqrt{n}} f_1(v_1)dv_1 = \Phi(-\eta_L\sqrt{n}\sqrt{W}+z_{1-\delta_\beta}\sqrt{n}) = \Phi(t\sqrt{W}-\Delta), \tag{40}$$

where

$$t=-\eta_L\sqrt{n},\quad \Delta=-z_{1-\delta_\beta}\sqrt{n}. \tag{41}$$

Then it follows from (31) and (40) that t has to be found such that

$$t=\arg\left(E\left\{\Phi(t\sqrt{W}-\Delta)\right\}=\gamma\right)=\arg\left(\int_0^\infty \Phi(t\sqrt{w}-\Delta)\,f_r(w)dw=\gamma\right)$$

$$=\arg\left(\frac{r^{r/2}}{2^{r/2}\Gamma(r/2)}\int_0^\infty w^{r/2-1}\exp(-rw/2)\Phi(t\sqrt{w}-\Delta)\,dw=\gamma\right)=\arg\left(F_{r,\Delta}(t)=\gamma\right)=t_{r,\Delta;\gamma}, \tag{42}$$

where $t_{r,\Delta;\gamma}$ is the quantile of order γ for the non-central t-distribution with r=n–1 degrees of freedom and non-centrality parameter Δ,

$$F_{r,\Delta}(t)=\Pr(T\le t)=\frac{r^{r/2}}{2^{r/2}\Gamma(r/2)}\int_0^\infty w^{r/2-1}\exp(-rw/2)\Phi(t\sqrt{w}-\Delta)dw. \tag{43}$$

is the cumulative distribution function of T,

$$f_{r,\Delta}(t)=F'_{r,\Delta}(t)=\frac{r^{r/2}\exp\left(-r\Delta^2/[2(t^2+r)]\right)}{\sqrt{\pi}\Gamma(r/2)2^{(r+1)/2}(t^2+r)^{(r+1)/2}}$$

$$\times\int_0^\infty w_\bullet^{(r-1)/2}\exp\left(-\frac{1}{2}\left[w_\bullet^{1/2}-\frac{t\Delta}{\sqrt{t^2+r}}\right]^2\right)dw_\bullet,\quad -\infty<t<\infty, \tag{44}$$

is the probability density function of T, where

$$W_\bullet=W(t^2+r). \tag{45}$$

Corollary 4.2. If

$$W_{\bullet\bullet}=W(t^2+r)/2, \tag{46}$$

then

$$f_{r,\Delta}(t)=F'_{r,\Delta}(t)=\frac{r^{r/2}\exp(-\Delta^2/2)}{\sqrt{\pi}\Gamma(r/2)(t^2+r)^{(r+1)/2}}\int_0^\infty w_{\bullet\bullet}^{(r-1)/2}\exp\left(-\left[w_{\bullet\bullet}-\frac{t\Delta\sqrt{2}}{\sqrt{t^2+r}}w_{\bullet\bullet}^{1/2}\right]\right)dw_{\bullet\bullet}$$

$$=\frac{r^{r/2}\exp(-\Delta^2/2)}{\sqrt{\pi}\Gamma(r/2)(t^2+r)^{(r+1)/2}}\sum_{j=0}^\infty\frac{\Gamma((r+j+1)/2)}{j!}\left(\frac{t\Delta\sqrt{2}}{\sqrt{t^2+r}}\right)^j,\quad -\infty<t<\infty. \tag{47}$$

This form of the density function is derived in Rao [20] and appears in Searle [21]. In both Rao and Searle, $\sqrt{\pi}$ is incorrectly omitted from the denominator. It should also be noted that the central t-distribution is just a special case of the non-central t with Δ = 0.

Corollary 4.3. If k=m=1, then

$$\delta_\beta=\beta,\quad \Delta=-z_{1-\beta}\sqrt{n}. \tag{48}$$

Corollary 4.4. Let $\tilde{X}_1\le\ldots\le\tilde{X}_n$ be ordered observations from a preliminary sample of size n from a log-normal distribution defined by the probability density function (10). Then a lower one-sided β-content tolerance limit at confidence level γ, $\tilde{L}_k\equiv\tilde{L}_k(S)$ (on the kth order statistic $\tilde{Y}_k$,

$k \in \{1, \ldots, m\}$, from a set of m future ordered observations $\tilde{Y}_1 \leq \ldots \leq \tilde{Y}_k$ also from the distribution (10)), which satisfies

$$\Pr\left\{ P_\theta(\tilde{Y}_k > \tilde{L}_k) \geq \beta \right\} = \gamma, \tag{49}$$

is given by

$$\tilde{L}_k = \exp(L_k) = \exp(\bar{X} + \eta_L S_1), \tag{50}$$

where

$$X_i = \ln \tilde{X}_i,\ i \in \{1, \ldots, n\},\quad \bar{X} = \sum_{i=1}^{n} X_i / n,\ S_1^2 = \sum_{i=1}^{n} (X_i - \bar{X})^2 / (n-1),$$

$$\Delta = -z_{1-\delta_\beta}\sqrt{n},\quad \delta_\beta = (m-k+1)q_{2(m-k+1),2k;\beta} \Big/ [(m-k+1)q_{2(m-k+1),2k;\beta} + k],$$

$$t_{r,\Delta;\gamma} = \arg\left[F_{r,\Delta}(t) = \gamma\right],\quad r = n-1,\quad \eta_L = -t_{r,\Delta;\gamma} \Big/ \sqrt{n}. \tag{51}$$

3.2. Upper Tolerance Limit

Theorem 5. Let $X_1, \ldots, X_n$ be observations from a preliminary sample of size n from a normal distribution defined by the probability density function (12). Then an upper one-sided β-content tolerance limit at a confidence level γ, $U_k \equiv U_k(S)$ (on the kth order statistic Y_k from a set of m future ordered observations $Y_1 \leq \ldots \leq Y_m$ also from the distribution (12)), which satisfies

$$\Pr\left\{ P_\theta(Y_k \leq U_k) \geq \beta \right\} = \gamma, \tag{52}$$

is given by

$$U_k = \bar{X} + \eta_U S_1, \tag{53}$$

where

$$\eta_U = t_{r,\Delta;1-\gamma} \Big/ \sqrt{n}, \tag{54}$$

is the upper tolerance factor, $t_{r,\Delta;1-\gamma}$ is the quantile of order $1-\gamma$ for the non-central t-distribution with $r=n-1$ degrees of freedom and non-centrality parameter $\Delta = -z_{1-\delta_{1-\beta}}\sqrt{n}$, $z_{1-\delta_{1-\beta}}$ denotes the $1-\delta_{1-\beta}$ quantile of the standard normal distribution,

$$\delta_{1-\beta} = (m-k+1)q_{2(m-k+1),2k;1-\beta} \Big/ [(m-k+1)q_{2(m-k+1),2k;1-\beta} + k], \tag{55}$$

$q_{2(m-k+1),2k;1-\beta}$ is the quantile of order $1-\beta$ for the F distribution with $2(m-k+1)$ and $2k$ degrees of freedom.

Proof. It follows from (3), (13) and (52) that

$$\Pr\left\{ P_\theta(Y_k \leq U_k) \geq \beta \right\} = \Pr\left(\int_{\frac{1-F_\theta(U_k)}{F_\theta(U_k)} \frac{2k}{2(m-k+1)}}^{\infty} f_{2(m-k+1),2k}(x)dx \geq \beta \right)$$

$$= \Pr\left(\int_0^{\frac{1-F_\theta(U_k)}{F_\theta(U_k)}\frac{2k}{2(m-k+1)}} f_{2(m-k+1),2k}(x)dx \le 1-\beta \right) = \Pr\left(\frac{1-F_\theta(U_k)}{F_\theta(U_k)} \frac{2k}{2(m-k+1)} \le q_{2(m-k+1),2k;1-\beta} \right)$$

$$= \Pr\left(F_\theta(U_k) \ge \frac{k}{k+(m-k+1)q_{2(m-k+1),2k;1-\beta}} \right)$$

$$= \Pr\left(\frac{1}{\sigma\sqrt{2\pi}} \int_{U_k}^{\infty} \exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right)dy \le \frac{(m-k+1)q_{2(m-k+1),2k;1-\beta}}{(m-k+1)q_{2(m-k+1),2k;1-\beta}+k} \right)$$

$$= \Pr\left(\frac{1}{\sqrt{2\pi}} \int_{\frac{U_k-\mu}{\sigma}}^{\infty} \exp\left(-\frac{z^2}{2}\right)dz \le \delta_{1-\beta} \right) = \Pr\left(\frac{1}{\sqrt{2\pi}} \int_{\infty}^{\frac{U_k-\mu}{\sigma}} \exp\left(-\frac{z^2}{2}\right)dz \ge 1-\delta_{1-\beta} \right)$$

$$= \Pr\left(\frac{U_k-\mu}{\sigma} \ge z_{1-\delta_{1-\beta}} \right) = \Pr\left(\frac{U_k-\bar{X}+\bar{X}-\mu}{\sigma} \ge z_{1-\delta_{1-\beta}} \right)$$

$$= \Pr\left(\frac{U_k-\bar{X}}{S_1}\sqrt{n}\frac{S_1}{\sigma} + \frac{\sqrt{n}(\bar{X}-\mu)}{\sigma} \ge z_{1-\delta_{1-\beta}}\sqrt{n} \right) = \Pr\left(\eta_U\sqrt{n}\sqrt{W} + V_1 \ge z_{1-\delta_{1-\beta}}\sqrt{n} \right)$$

$$= \Pr\left(\frac{V_1 - z_{1-\delta_{1-\beta}}\sqrt{n}}{\sqrt{W}} \ge -\eta_U\sqrt{n} \right) = \Pr\left(\frac{V_1+\Delta}{\sqrt{W}} \ge -\eta_U\sqrt{n} \right) = \Pr\left(T \ge -\eta_U\sqrt{n} \right) = 1 - F_{r,\Delta}(t), \qquad (56)$$

where

$$\eta_U = (U_k - \bar{X})\big/S_1, \qquad (57)$$

is the upper tolerance factor,

$$\Delta = -z_{1-\delta_{1-\beta}}\sqrt{n}, \quad r = n-1, \quad t = -\eta_U\sqrt{n}. \qquad (58)$$

It follows from (49), (56) and (58) that the upper tolerance factor η_U should be chosen such that

$$F_{r,\Delta}(t) = F_{r,\Delta}(-\eta_U\sqrt{n}) = F_{r,\Delta}(t_{r,\Delta;1-\gamma}) = 1-\gamma, \qquad (59)$$

where $t_{r,\Delta;1-\gamma}$ is the quantile of order $1-\gamma$ for the non-central *t*-distribution with *r* degrees of freedom and non-centrality parameter Δ. It follows from (59) that

$$\eta_U = -t_{r,\Delta;1-\gamma}\big/\sqrt{n}. \qquad (60)$$

It follows from (57) that $U_k = \bar{X} + \eta_U S_1$. This completes the proof.

Corollary 5.1. Let $\tilde{X}_1, \ldots \le \tilde{X}_n$ be observations from a preliminary sample of size *n* from a log-normal distribution defined by the probability density function (10). Then an upper one-sided β-content tolerance limit at confidence level γ, $\tilde{U}_k \equiv \tilde{U}_k(S)$ (on the *k*th order statistic $\tilde{Y}_k$, $k \in \{1, \ldots, m\}$, from a set of *m* future ordered observations $\tilde{Y}_1 \le \ldots \le \tilde{Y}_k$ also from the distribution (10)), which satisfies

$$\Pr\left\{P_\theta(\tilde{Y}_k \le \tilde{U}_k) \ge \beta\right\} = \gamma, \tag{61}$$

is given by

$$\tilde{U}_k = \exp(U_k) = \exp(\bar{X} + \eta_U S_1), \tag{62}$$

where

$$X_i = \ln \tilde{X}_i,\ i \in \{1,\ ...,\ n\},\quad \bar{X} = \sum_{i=1}^{n} X_i / n,\ S_1^2 = \sum_{i=1}^{n} (X_i - \bar{X})^2 / (n-1),$$

$$\Delta = -z_{1-\delta_{1-\beta}} \sqrt{n},\quad \delta_{1-\beta} = (m-k+1)q_{2(m-k+1),2k;1-\beta} \Big/ [(m-k+1)q_{2(m-k+1),2k;1-\beta} + k],$$

$$t_{r,\Delta;1-\gamma} = \arg\left[F_{r,\Delta}(t) = 1-\gamma\right].\quad r = n-1,\quad \eta_U = -t_{r,\Delta;1-\gamma} \Big/ \sqrt{n}. \tag{63}$$

Remark 1. It will be noted that an upper tolerance limit may be obtained from a lower tolerance limit by replacing β by $1-\beta$, γ by $1-\gamma$.

4. Practical Example

A manufacturer of semiconductor lasers has the data on lifetimes (in terms of hours) obtained from testing n=10 semiconductor lasers. These data are given in Table 1.

Table 1. *The data on lifetimes obtained from testing n=10 semiconductor lasers*

Observations (in terms of hours)									
$\tilde{X}_1$	$\tilde{X}_1$	$\tilde{X}_1$	$\tilde{X}_1$	$\tilde{X}_1$	$\tilde{X}_1$	$\tilde{X}_1$	$\tilde{X}_1$	$\tilde{X}_1$	$\tilde{X}_1$
18657	18960	19771	21015	21183	21960	22881	24642	25373	27373

A buyer tells the laser manufacturer that he wants to place two orders for the same type of semiconductor lasers to be shipped to two different destinations. The buyer wants to select a random sample of m=5 semiconductor lasers from each shipment to be tested. An order is accepted only if all of 5 semiconductor lasers in each selected sample meet the warranty lifetime (in terms of hours). What warranty lifetime (in terms of hours) should the manufacturer offer so that all of 5 semiconductor lasers in each selected sample meet the warranty with probability of 0.95?

In order to find this warranty lifetime, the manufacturer wishes to use a random sample of size n=10 given in Table 1 and to calculate the lower one-sided simultaneous tolerance limit $L_{k=1}(S)$ (warranty lifetime) which is expected to capture a certain proportion, say, β=0.95 or more of the population of selected items (m=5), with the given confidence level γ=0.95. This tolerance limit is such that one can say with a certain confidence γ that at least 100β % of the semiconductor lasers in each sample selected by the buyer for testing will operate longer than $L_1(S)$.

Goodness-of-fit testing. It is assumed that the data of Table 1 follow the log-normal probability distribution (10), where the parameters μ and σ are unknown. Thus, for the above example, we have that n =10, m =5, k = 1, β = 0.95, γ = 0.95,

$$S = \left(\bar{X} = \sum_{i=1}^{n} X_i / n = 10,\ S_1^2 = \sum_{i=1}^{n} (X_i - \bar{X})^2 / (n-1) = 0.016302\right). \tag{64}$$

We assess the statistical significance of departures from the model (10) by performing the Anderson–Darling goodness-of-fit test. The Anderson–Darling test statistic value is determined by

$$A^2 = -\left[\sum_{i=1}^{n}(2i-1)\left(\ln F_\theta(x_i) + \ln\left(1 - F_\theta(x_{n+1-i})\right)\right)\right]\Big/ n - n, \tag{65}$$

where $F_\theta(\ )$ is the cumulative distribution function of $X = \ln \tilde{X}$,

$$\theta = (\mu = \bar{x}, \sigma = s_1), \tag{66}$$

n is the number of observations.

The result from (65) needs to be modified for small sampling values. For the normal distribution the modification of A^2 is

$$A^2_{\text{mod}} = A^2(1 + 0.75/n + 2.25/n^2). \tag{67}$$

The A^2_{mod} value must then be compared with critical values, A^2_α, which depend on the significance level α and the distribution type. As an example, for the normal distribution the determined A^2_{mod} value has to be less than the following critical values for acceptance of goodness-of-fit (see Table 2):

Table 2. *Critical values for* A^2_{mod}

α	0.1	0.05	0.025	0.01
A^2_α	0.631	0.752	0.873	1.035

For this example, α=0.05, $A^2_{\alpha=0.05} = 0.752$,

$$A^2 = -\left[\sum_{i=1}^{10}(2i-1)\left(\ln F_\theta(x_i) + \ln\left(1 - F_\theta(x_{n+1-i})\right)\right)\right]\Big/ 10 - 10 = 0.193174, \tag{68}$$

$$A^2_{\text{mod}} = A^2(1 + 0.75/10 + 2.25/10^2) = 0.212 < A^2_{\alpha=0.05} = 0.752. \tag{69}$$

Thus, there is not evidence to rule out the log-normal model (10).

Finding lower tolerance limit (warranty lifetime for semiconductor laser). Now the lower one-sided simultaneous β-content tolerance limit at the confidence level γ, $L_1 \equiv L_1(S)$ (on the order statistic Y_1 from a set of m = 5 future ordered observations $Y_1 \le \ldots \le Y_m$) can be obtained from (50).

Since m=5, k=1, β=0.95, it follows from (51) that:

$$\delta_\beta = (m-k+1)q_{2(m-k+1),2k;\beta}\Big/[(m-k+1)q_{2(m-k+1),2k;\beta} + k] = 0.989796, \tag{70}$$

$$r = n - 1 = 9, \quad \Delta = -z_{1-\delta_\beta}\sqrt{n} = 7.3325, \quad \gamma = 0.95, \tag{71}$$

the quantile of order γ for the non-central t-distribution with r degrees of freedom and non-centrality parameter Δ is given by

$$t_{r,\Delta;\gamma} = \arg\left(F_{r,\Delta}(t) = \gamma\right) = 12.5512, \tag{72}$$

the lower tolerance factor is given by

$$\eta_L = -t_{r,\Delta;\gamma}\Big/\sqrt{n} = -3.969. \tag{73}$$

Now it follows from (50), (64) and (73) that

$$L_{k=1} = \exp\left(\bar{X} + \eta_L S_1\right) = 13270. \tag{74}$$

Statistical inference. Thus, the manufacturer has 95% assurance that at least 100β % of the semiconductor lasers in each sample (m=5) selected by the buyer for testing will operate (in terms of hours) no less than L_1=13270 hours.

5. Conclusion

This paper introduces a methodology to construct the one-sided tolerance limits on order statistics in future samples coming from location-scale distributions under parametric uncertainty. For illustration, the normal and log-normal distributions are considered. These distributions play a vital role in many applied problems of biology, economics, engineering, financial risk management, genetics, hydrology, mechanics, medicine, number theory, statistics, physics, psychology, reliability, etc., and have been extensively studied, both from theoretical and applications point of view, by many researchers, since its inception.

It will be noted that the theoretical concept and computational complexity of the tolerance limits is significantly more difficult than that of the standard confidence and prediction limits. Thus it becomes necessary to use new or innovative approaches which will allow one to construct tolerance limits on future order statistics for many populations. The concept proposed in this paper can be extended to two-sided tolerance limits too.

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On recurrence and availability factor for single–server system with general arrivals

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Abstract

Recurrence and ergodic properties are established for a single–server queueing system with variable intensities of arrivals and service. Convergence to stationarity is also interpreted in terms of reliability theory.

Keywords: *single-server system, arrivals, recurrence*

1 Introduction

In the last decades, queueing systems generalising $M/G/1/\infty$, or $M/G/1$ (cf. [7]) – one of the most important queueing systems – attracted much attention, see [1] – [5], [9]. In this paper a single–server system similar to [10, 11] is considered, in which *intensities* of new arrivals as well as of their service may depend on the "whole state" of the system and the whole state includes the number of customers in the system – waiting and on service – *and* on the elapsed time of the last service, as well as on the elapsed time since the end of the last service. Batch arrivals are not allowed. The news in comparison to [10, 11] is that at any state, even if the system idle (no service), the intensity of new arrivals may depend on the time from the last end of service. The details of the system description will be formalised in the beginning of the next section. By the *m-availability factor* of the system we understand the probability of the idle state if $m = 0$, or probability of m customers in total on the server and in the queue. We do not use notation $G/G/1$ (or $GI/GI/1$) only because some conditions on intensities are assumed, which makes the model slightly less general. The problem addressed in the paper is how to estimate convergence rate of characteristics of the system including the m-availability factors to their stationary values.

The *elapsed* service time is assumed to be known at any moment, but the remaining service times for each customer are not. For definiteness, the discipline of serving is FIFO, although other disciplines may be also considered.

The paper consists of the Section 1 – Introduction, of the setting and main result in the Section 2, of the auxiliary lemmata in the Section 3 and of the short sketch of the proof of the main result in the Section 4.

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2 The setting and main results

2.1 Defining the process

Let us present the class of models under investigation in this paper. Here the state space is a union of subspaces,

$$\mathcal{X} = \{(0, y):\ y \ge 0\} \cup \bigcup_{n=1}^{\infty} \{(n, x, y):\ x, y \ge 0\}.$$

Functions of class $C^1(\mathcal{X})$ are understood as functions with classical continuous derivatives with respect to the variable x. Functions with compact support on $\mathcal{X}$ are understood as functions vanishing outside some domain bounded in this metric: for example, $C_0^1(\mathcal{X})$ stands for the class of functions with compact support and one continuous derivative. There is a generalised Poisson arrival flow with intensity $\lambda(X)$, where $X = (n, x, y)$ $for\ any\ n \ \ge\ 1$, and $X = (0, y)$ $for\ n\ =\ 0$. Slightly abusing notations, it is convenient to write $X = (n, x, y)$ for $n = 0$ as well, assuming that in this case $x = 0$. If $n > 0$, then the server is serving one customer while all others are waiting in a queue. When the last service ends, immediately a new service of the next customer from the queue starts. If $n = 0$ then the server remains idle until the next customer arrival; the intensity of such arrival at state $(0, y) \equiv (0,0, y)$ may be variable depending on the value y, which stands for the elapsed time from the last end of service. Here n denotes the total number of customers in the system, and x stands for the elapsed time of the current service (except for $n = 0$, which was explained earlier), and y is the elapsed from from the last arrival. *Normally*, intensity of arrivals depend on n and y, while intensity of service depends on n and x; however, we allow more general dependence. Denote $n_t = n(X_t)$ – the number of customers corresponding to the state X_t, and $x_t = x(X_t)$, the second component of the process (X_t), and $y_t = y(X_t)$, the third component of the process (X_t) (the third if $n > 0$)). For any $X = (n, x, y)$, intensity of service $h(X) \equiv h(n, x, y)$ is defined; it is also convenient to assume that $h(X) = 0$ for $n(X) = 0$. Both intensities λ and h are understood in the following way, which is a definition: on any nonrandom interval of time $[t, t + \Delta)$, conditional probability given X_t that the current service will *not* be finished and there will be no new arrivals reads,

$$\exp\left(-\int_0^{\Delta} (\lambda + h)(n_t, x_t + s, y_t + s)\, ds\right). \tag{1}$$

In the sequel, λ and h are assumed to be *bounded*. In this case, for $\Delta > 0$ small enough, the expression in (1) may be rewritten as

$$1 - \int_0^{\Delta} (\lambda + h)(n_t, x_t + s, y_t + s)\, ds + O(\Delta^2), \qquad \Delta \to 0, \tag{2}$$

and this what is "usually" replaced by

$$1 - (\lambda(X_t) + h(X_t))\Delta + O(\Delta^2).$$

However, in our situation, the latter replacement may be incorrect because of discontinuities of the functions λ and h. Emphasize that from time t and until the next jump, the evolution of the process X is *deterministic*, which makes the process *piecewise-linear Markov*, see, e.g., [7]. The (conditional given X_t) density of the moment of a new arrival *or* of the end of the current service after t at $x_t + z$, $z \ge 0$ equals,

$$(\lambda(n_t, x_t + z, y_t + z) + h(n_t, x_t + z, y_t + z))\exp\left(-\int_0^{\Delta} (\lambda + h)(n_t, x_t + s, y_t + s))\, ds\right). \tag{3}$$

Further, given X_t, the moments of the next "candidates" for jumps up and down are conditionally independent and have the (conditional – given X_t) density, respectively,

$$\lambda(X_t + z)\exp\left(-\int_0^{z} \lambda(X_t + s)\, ds\right)$$

$$and \tag{4}$$

$$h(X_t + z)\exp\left(-\int_0^{z} h(X_t + s)\, ds\right),\ z \ge 0.$$

(Here $X_t + s := (n_t, x_t + s, y_t + s)$.) Notice that (3) does correspond to conditionally independent densities given in (4).

2.2 Main result

Let

$$\Lambda := \sup_{n,x,y:\, n>0} \lambda(n,x,y) < \infty.$$

For establishing convergence rate to the stationary regime, we assume similarly to [10, 11],

$$\inf_{n>0,y} h(n,x,y) \ge \frac{C_0}{1+x}, \quad x \ge 0. \tag{5}$$

We also assume a new condition related to $\lambda_0(t) = \lambda(0,0,t)$, which was constant in the earlier papers: now it is allowed to be variable and satisfying

$$0 < \inf_{t\ge 0} \lambda_0(t) \le \sup_{t\ge 0} \lambda_0(t) < \infty. \tag{6}$$

Recall that the process has no explosion with probability one due to the boundedness of both intensities, i.e., the trajectory may have only finitely many jumps on any finite interval of time.

Theorem 1 *Let the functions λ and h be Borel measurable and bounded and let the assumptions (5) and (6) be satisfied. Then, under the assumptions above, if C_0 is large enough, then there exists a unique stationary measure μ. Moreover, for any $m > k$, $C > 0$ there exists $\bar{C} > 0$ such that if $C_0 \ge \bar{C}$, then for any $t \ge 0$,*

$$\| \mu_t^{n,x,y} - \mu \|_{TV} \le C\, \frac{(1+n+x+y)^m}{(1+t)^{k+1}}, \tag{7}$$

where $\mu_t^{n,x,y}$ is a marginal distribution of the process $(X_t, t \ge 0)$ with the initial data $X = (n,x,y) \in \mathcal{X}$.

Remark 1 *It is plausible that the bound in (7) may be improved so that the right hand side does not depend on y. Moreover, given all other constants, the value C in (7) may be made "computable", with a rather involved but explicit dependence on other constants. Moreover, it is likely that the condition (6) may be replaced by a weaker one,*

$$\frac{C_{0\prime}}{1+t} \le \lambda_0(t) \le \sup_{t\ge 0} \lambda_0(t) < \infty, \tag{8}$$

along with the assumption that C'_0 is large enough. However, all these issues require a bit more accuracy in the calculus and we do not pursue these goals here leaving them until further publications with complete technical details.

3 Lemmata

Recall [6] that the generator of a Markov process $(X_t, t \ge 0)$ is an operator $\mathcal{G}$, such that for a sufficiently large class of functions f

$$\sup_X \lim_{t\to 0} \left\| \frac{E_X f(X_t) - f(X)}{t} - \mathcal{G} f(X) \right\| = 0 \tag{9}$$

in the norm of the state space of the process; the notion of generator does depend on this norm. An operator $\mathcal{G}$ is called a *mild generalised generator* (another name is extended generator) if (9) is replaced by its corollary (10) below called *Dynkin's formula*, or *Dynkin's identity* [6, Ch. 1, 3],

$$E_X f(X_t) - f(X) = E_X \int_0^t \mathcal{G} f(X_s)\, ds, \tag{10}$$

also for a wide enough class of functions f. We will also use the non-homogeneous counterpart of Dynkin's formula,

$$E_X \varphi(t, X_t) - \varphi(0, X) = E_X \int_0^t \left(\frac{\partial}{\partial s} \varphi(s, X_s) + \mathcal{G} \varphi(s, X_s) \right) ds, \tag{11}$$

for appropriate functions of two variables $(\varphi(t, X))$. Both (10) and (11) play a very important role in analysis of Markov models and under our assumptions may be justified similarly to [11]. Here X is a (non-random) initial value of the process. Both formulae (10)–(11) hold true for a large class of functions f, φ with $\mathcal{G}$ given by the standard expression,

$$\mathcal{G}f(X) := \frac{\partial}{\partial x} f(X) 1(n(X) > 0) + \frac{\partial}{\partial y} f(X)$$

$$+\lambda(X)(f(X^+) - f(X)) + h(X)(f(X^-) - f(X)),$$

where for any $X = (n, x, y)$,

$$X^+ := (n+1, x, 0), \quad X^- := ((n-1) \vee 0, 0, y)$$

(here $a \vee b = \max(a, b)$). Under our minimal assumptions on regularity of intensities this may be justified similarly to [11].

Lemma 1 *If the functions λ and h are Borel measurable and bounded, then the formulae (10) and (11) hold true for any $t > 0$ for every $f \in C_b^1(\mathcal{X})$ and $\varphi \in C_b^1([0,\infty)\times\mathcal{X})$, respectively. Moreover, the process $(X_t, t \ge 0)$ is strong Markov with respect to the filtration $(\mathcal{F}_t^X, t \ge 0)$.*

Further, let

$$L_m(X) = (n + 1 + x + y)^m, \quad L_{k,m}(t, X) = (1+t)^k L_m(X). \tag{12}$$

The extensions of Dynkin's formulae for some unbounded functions hold true: we will need them for the Lyapunov functions in (12).

Corollary 1 *Under the assumptions of the Lemma 1,*

$$L_m(X_t) - L_m(X) = \int_0^t \lambda(X_s)\Big[\ \big(L_m(X_s^{(+)}) - L_m(X_s)\big)$$

$$+h(X_s)(L_m(X_s^-) - L_m(X_s)) + \frac{\partial}{\partial x} L_m(X_s) 1(n(X_s) > 0) + \frac{\partial}{\partial y} L_m(X_s)\Big]\, ds + M_t, \tag{13}$$

with some martingale M_t, and also

$$L_{k,m}(t, X_t) - L_{k,m}(0, X) = \int_0^t \big[\lambda(X_s)\big(L_{k,m}(s, X_s^{(+)}) - L_{k,m}(s, X_s)\big)$$

$$+h(X_s)\big(L_{k,m}(s, X_s^-) - L_{k,m}(s, X_s)\big) + \Big(\frac{\partial}{\partial x} 1(n(X_s) > 0) + \frac{\partial}{\partial y} + \frac{\partial}{\partial s}\Big) L_{k,m}(s, X_s)\Big]\, ds + \widetilde{M}_t, \tag{14}$$

with some martingale $\widetilde{M}_t$.

About a martingale approach in queueing models see, for example, [8]. The proof of the Lemma 1 is based on the next three Lemmata. The first of them is a rigorous statement concerning a well-known folklore property that probability of "one event" on a small nonrandom interval of length Δ is of the order $O(\Delta)$ and probability of "two or more events" on the same interval is of the order $O(\Delta^2)$. Of course, this is a common knowledge in queueing theory, yet for discontinuous intensities it has to be, at least, explicitly stated.

Lemma 2 *Under the assumptions of the Theorem 1, for any $t \ge 0$,*

$$P_{X_t}(\,no\ jumps\ on\ (t,\ t+\Delta]\,) = \exp\Big(-\int_0^\Delta (\lambda + h)(X_t + s)\, ds\Big) \quad (= 1 + O(\Delta)), \tag{15}$$

$$P_{X_t}(\,at\ least\ one\ jump\ on\ (t,\ t+\Delta]\,) = O(\Delta), \tag{16}$$

$$P_{X_t}(\,exactly\ one\ jump\ up\ \&\ no\ down\ on\ (t,\ t+\Delta]\,) = \int_0^\Delta \lambda(X_t + s)\, ds + O(\Delta^2), \tag{17}$$

$$P_{X_t}(\,exactly\ one\ jump\ down\ \&\ no\ up\ on\ (t,\ t+\Delta]\,) = \int_0^\Delta h(X_t + s)\, ds + O(\Delta^2), \tag{18}$$

and

$$P_{X_t}(\,at\ least\ two\ jumps\ on\ (t,\ t+\Delta]\,) = O(\Delta^2). \tag{19}$$

In all cases above, $O(\Delta)$ and $O(\Delta^2)$ are uniform with respect to X_t and only depend on the norm $\sup_X(\lambda(X) + h(X))$, that is, there exist $C > 0, \Delta_0 > 0$ such that for any X and any $\Delta < \Delta_0$,

$$\limsup_{\Delta\to 0}\{\Delta^{-1} P_X(\,at\ least\ one\ jumps\ on\ (0,\Delta]\,) +$$
$$\Delta^{-2} P_X(\,at\ least\ two\ jumps\ on\ (0,\ \Delta]\,)$$

$$+\Delta^{-2}\left[P_{X_t}(\,one\ \ jump\ \ up\ \ \&\ \ no\ \ down\ \ on\ \ (t,\ t+\Delta]\,) - \int_0^\Delta \lambda(X_t+s)\,ds\right]$$
$$+\Delta^{-2}\left[P_{X_t}(\,one\ \ jump\ \ down\ \ \&\ \ no\ \ up\ \ on\ \ (t,\ t+\Delta]\,) - \int_0^\Delta h(X_t+s)\,ds\right]\Big\} < C < \infty. \tag{20}$$

The next two Lemmata are needed for the justification that the process with discontinuous intensities is, indeed, strong Markov.

Lemma 3 *Under the assumptions of the Theorem 1, the semigroup* $T_t f(X) = E_X f(X_t)$ *is continuous in* t.

Lemma 4 *Under the assumptions of the Theorem 1 the process* $(X_t, t \ge 0)$ *is Feller, that is,* $T_t f(\cdot) \in C_b(\mathcal{X})$ *for any* $f \in C_b(\mathcal{X})$.

The proofs of all Lemmata may be performed similarly to [11].

4 Sketch of Proof of Theorem 1

The proof of convergence in total variation with rate of convergence repeats the calculus in [10] based on the Lyapunov functions $L_m(X)$ and $L_{k,m}(t,X)$ from (12), and on Dynkin's formulae (10) and (11) due to the Corollary 1. Without big changes, this calculus provides a polynomial moment bound
$$E_X\tau_0^k \le CL_m(X) \le C(n+1+x+y)^m, \tag{21}$$
for certain values of k and for the hitting time
$$\tau_0 := \inf(t \ge 0:\ n_t = 0).$$
Namely, once the process attains the set $\{n = 0\}$, it may be successfully coupled with another (stationary) version of the same process at their joint jump $\{n = 0\} \mapsto \{n = 1\}$. This is because, in particular, immediately after such a jump the state of each process reads as (1,0,0); in other words, this is a regeneration state. The news is only a wider class of intensities, which may be all variable (as well as discontinuous) including λ_0; however, this affects the calculus only a little, once it is established that (10) and (11) hold true, because this calculus involves only time values $t < \tau_0$. (Some change will be in the procedure of coupling, though.) In turn, the inequality (21) provides a bound for the rate of convergence, for the justification of which rate there are various approaches such as versions of coupling as well as renewal theory. Convergence of probabilities in the definition of m-availability factors is a special case of a more general convergence in total variation. We drop further details, which will be specified in a further publication.

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Mean Time To System Failure Assessment Of A Single Unit System Requiring Two Types Of Supporting Device For Operation

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Abstract

This paper studies the mean time to system failure (MTSF) of single unit system operating with the help of two types of external supporting device. Each type of supporting device has two copies I and II. The system is analyzed using differential difference equation to develop the explicit expression for mean time to system failure. Based on assumed numerical values given to system parameters, graphical illustrations are given to highlight important results.

Keywords: *availability, supporting device, probabilistic, single*

I. Introduction

Proper maintenance planning plays a role in achieving high system reliability, availability and production output. It is therefore important to keep the equipments/systems always available and to lay emphasis on system availability at the highest order. In real-life situations we often encounter cases where the systems that cannot work without the help of external supporting devices connect to such systems. These external supporting devices are systems themselves that are bound to fail. Such systems are found in power plants, manufacturing systems, and industrial systems. Large volumes of literature exist on the issue relating to prediction of various systems performance connected to an external supporting device for their operations. Yusuf et al (2014) present mathematical modeling approach to analysis of mean time to system failure of two unit cold standby system with a supporting device. Yusuf et al (2015) performed comparative analysis of MTSF between systems connected to supporting device for operation. Yusuf et al (2014) performed reliability computation of a linear consecutive 2-out-of-3 system in the presence of supporting device.

Existing literatures either ignores the impact of multi-supporting device on system performance. Such works laid emphasis on systems connected to one type of an external supporting device whose failure brings about total breakdown. More sophisticated models of systems connected to multi-external supporting device should be developed to assist in reducing operating costs and the risk of a catastrophic breakdown, to maximize output, system availability, and generated revenue, minimize cost, and assure ongoing quality of the parts being produced. The problem considered in this paper is different from discussed authors above. The purpose of this paper is twofold. The first purpose is to develop the explicit expressions for mean time to system failure. The second is to capture the effect of both failure and repair rates on mean time to system failure based on assumed numerical values given to the system parameters.

The organization of the paper is as follows. Section 2 presents model's description and assumptions. Section 3 presents formulations of the models. Numerical examples are presented and discussed in Section 4. Finally, we make a concluding remark in Section 5.

II. Description of the System

In this paper, a single unit system connected to two types of supporting device is considered. It is assumed that each type of supporting has a copy on standby and the switching is perfect. It is also assumed that the system work with either two copies of type I supporting device or two copies of type II supporting device or one copy of both type I and II. Both unit and supporting devices are assumed to be repairable. Each of the primary supporting devices fails independently of the state of the other and has an exponential failure distribution with parameter λ_1 **and** λ_2 **for type I and II respectively.** Whenever a primary supporting device fails, it is immediately sent to repair with parameter μ_1 **and** μ_2 and the standby supporting device is switch to operation. System failure occur when the unit has failed with parameter λ **and service rate with parameter** with parameter μ **or the failure of all copies of type I and type II.**

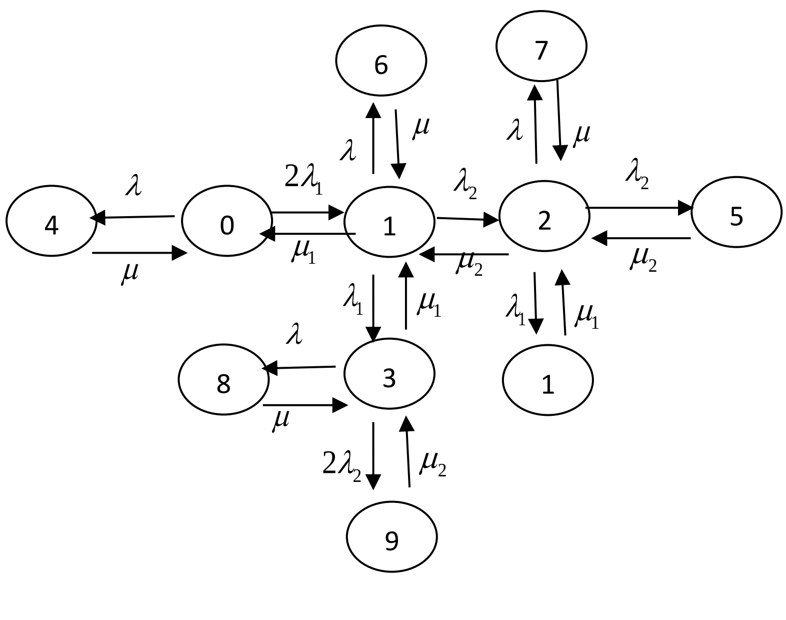

Figure 1: The State transition diagram of System

III. Formulation of the Model

In order to analyze the system availability of the system, we define $P_i(t)$ to be the probability that the system at $t \geq 0$ is in state S_i. Also let $P(t)$ be the row vector of these probabilities at time t. The initial condition for this problem is:

$$P(0) = [p_0(0), p_1(0), p_2(0), \ldots, p_{10}(0)] = [1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]$$

We obtain the following differential difference equations from Figure 1:

$$p_0'(t) = -(\lambda + 2\lambda_1) p_0(t) + \mu_1 p_1(t) + \mu p_4(t)$$
$$p_1'(t) = -(\lambda + \lambda_1 + \lambda_2 + \mu_1) p_1(t) + 2\lambda_1 p_0(t) + \mu_2 p_2(t) + \mu_1 p_3(t) + \mu p_6(t)$$
$$p_2'(t) = -(\lambda + \lambda_1 + \lambda_2 + \mu_2) p_2(t) + \lambda_2 p_1(t) + \mu_2 p_5(t) + \mu_7 p_7(t) + \mu_1 p_{10}(t)$$
$$p_3'(t) = -(\lambda + 2\lambda_2 + \mu_1) p_3(t) + \lambda_1 p_1(t) + \mu p_8(t) + \mu_2 p_9(t)$$
$$p_4'(t) = -\mu p_4(t) + \lambda p_0(t)$$
$$p_5'(t) = -\mu_2 p_5(t) + \lambda_2 p_2(t)$$
$$p_6'(t) = -\mu p_6(t) + \lambda p_1(t)$$
$$p_7'(t) = -\mu p_7(t) + \lambda p_2(t)$$
$$p_8'(t) = -\mu p_8(t) + \lambda p_3(t)$$
$$p_9'(t) = -\mu_2 p_9(t) + 2\lambda_2 p_3(t)$$
$$p_{10}'(t) = -\mu_1 p_{10}(t) + \lambda_1 p_2(t) \quad \textbf{(1)}$$

This can be written in the matrix form as

$$\dot{P} = TP, \quad \textbf{(2)}$$

where

$$T = \begin{pmatrix} -\delta_1 & \mu_1 & 0 & 0 & \mu & 0 & 0 & 0 & 0 & 0 & 0 \\ 2\lambda_1 & -\delta_2 & \mu_2 & \mu_1 & 0 & 0 & \mu & 0 & 0 & 0 & 0 \\ 0 & \lambda_2 & -\delta_3 & 0 & 0 & \mu_2 & 0 & \mu & 0 & 0 & \mu_1 \\ 0 & \lambda_1 & 0 & -\delta_4 & 0 & 0 & 0 & 0 & \mu & \mu_2 & 0 \\ \lambda & 0 & 0 & 0 & -\mu & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 & 0 & -\mu_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 & 0 & -\mu & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 & 0 & 0 & -\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & -\mu & 0 & 0 \\ 0 & 0 & 0 & 2\lambda_2 & 0 & 0 & 0 & 0 & 0 & -\mu_2 & 0 \\ 0 & 0 & \lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu_1 \end{pmatrix}$$

$$\delta_1 = (\lambda + 2\lambda_1),\ \delta_2 = (\lambda + \lambda_1 + \lambda_2 + \mu_1),\ \delta_3 = (\lambda + \lambda_1 + \lambda_2 + \mu_2), \delta_4 = (\lambda_1 + 2\lambda_2 + \mu_1)$$

It is difficult to evaluate the transient solutions, hence following Trivedi (2002), Wang and Kuo (2000), Wang et al. (2006) to develop the explicit for MTSF. The procedures require deleting rows and columns of absorbing states of matrix T and take the transpose to produce a new matrix, say

M . The expected time to reach an absorbing state is obtained from

$$E\left[T_{P(0)\to P(absorbing)}\right] = P(0)\left(-M^{-1}\right)\begin{pmatrix}1\\1\\1\\1\end{pmatrix} \quad \textbf{(3)}$$

where the initial conditions are given by

$P(0) = [p_0(0), p_1(0), p_2(0), p_3(0)] = [1, 0, 0, 0]$ **and**

$$M = \begin{pmatrix} -(2\lambda_1+\lambda) & 2\lambda_1 & 0 & 0 \\ \mu_1 & -(\lambda+\lambda_1+\lambda_2+\mu_1) & \lambda_2 & \lambda_1 \\ 0 & \mu_2 & -(\lambda+\lambda_1+\lambda_2+\mu_2) & 0 \\ 0 & \mu_1 & 0 & -(\lambda_1+2\lambda_2+\mu_1) \end{pmatrix}$$

The procedure above is successful because of the following relations

$$E\left[T_{P(0)\to P(absorbing)}\right] = P(0)\int_0^\infty e^{Mt}dt, \quad \textbf{(4)}$$

$$\text{where} \int_0^\infty e^{Mt}dt = -M^{-1} \quad \textbf{(5)}$$

The explicit expression for is given by MTSF

$$E\left[T_{P(0)\to P(absorbing)}\right] = MTSF = \frac{N}{D} \quad \textbf{(6)}$$

$$N = 2\lambda\lambda_2\mu_2 + \mu_1\mu_2\lambda_1 + 2\mu_1\mu_2\lambda_2 + 2\mu_2\lambda_1\lambda_2 + \lambda_1^3 + 4\mu_1\lambda_1\lambda_2 + \mu_1\lambda_1^2 + 2\mu_1\lambda\lambda_1 + 4\mu_1\lambda\lambda_2 + 4\lambda_1^2\lambda_2 + \mu_2\lambda_1^2 +$$
$$5\lambda_1\lambda_2^2 + \mu_1\mu_2\lambda + \mu_1^2\lambda_2 + 3\mu_1\lambda_2^2 + \mu_2\lambda\lambda_1 + 2\lambda_2^3 + 2\lambda^2\lambda_2 + 6\lambda\lambda_1\lambda_2 + \lambda^2\lambda_1 + \mu_1\lambda^2 + \mu_1^2\lambda_1 + 2\lambda\lambda_1^2 + 4\lambda\lambda_2^2 +$$
$$\mu_1^2\lambda + \mu_1^2\mu_2 + 2\lambda_1(\lambda+\lambda_1+\lambda_2+\mu_2)(2\lambda_2+\lambda_1+\mu_1) + 2\lambda_1\lambda_2(2\lambda_2+\lambda_1+\mu_1) + 2\lambda_1^2(\lambda+\lambda_2+\lambda_1+\mu_2)$$

$$D = 16\lambda\lambda_1^2\lambda_2 + 8\lambda_1^3\lambda_2 + 10\lambda_1^2\lambda_2^2 + 5\lambda\lambda_1^3 + 2\lambda_1^4 + 4\lambda_1\lambda_2^3 + 4\lambda^2\lambda_1^2 + \mu_1\mu_2\lambda^2 + 2\lambda\lambda_2^3 + 2\lambda^3\lambda_2 + \lambda^3\lambda_1 +$$
$$\mu_1\lambda^3 + 4\lambda^2\lambda_2^2 + \mu_1^2\lambda^2 + 2\mu_2\lambda_1^3 + 6\mu_2\lambda\lambda_1\lambda_2 + 4\mu_2\lambda_1^2\lambda_2 + \lambda\lambda_1\lambda_2\lambda_3 + 3\mu_1\mu_2\lambda\lambda_1 + 3\mu_2\lambda\lambda_1^2 + \mu_1^2\mu_2\lambda +$$
$$8\mu_1\lambda\lambda_1\lambda_2 + 2\mu_1\mu_2\lambda\lambda_2 + 3\mu_1\lambda\lambda_1^2 + 2\mu_2\lambda^2\lambda_2 + 4\mu_2\lambda^2\lambda_2 + \mu_2\lambda^2\lambda_1 + 10\lambda^2\lambda_1\lambda_2 + 4\mu_1\lambda^2\lambda_1 + \mu_1^2\lambda\lambda_1 +$$
$$\mu_1^2\lambda\lambda_2 + 3\mu_1\lambda\lambda_2^2 + 2\mu_1\lambda_1^2\lambda_2 + 2\mu_1\lambda_1\lambda_2^2$$

IV. Numerical Examples

Numerical examples are presented to demonstrate the impact of failure and repair rates on **mean time to system failure** based on given values of the parameters. For the purpose of numerical example, the following sets of parameter values are used:

$\mu_1 = 0.3, \mu_2 = 0.5, \mu = 0.5, \lambda_1 = 0.2, \lambda_2 = 0.3, \lambda(0.4, 0.6, 0.8)$ **for Figures 2 – 5.**

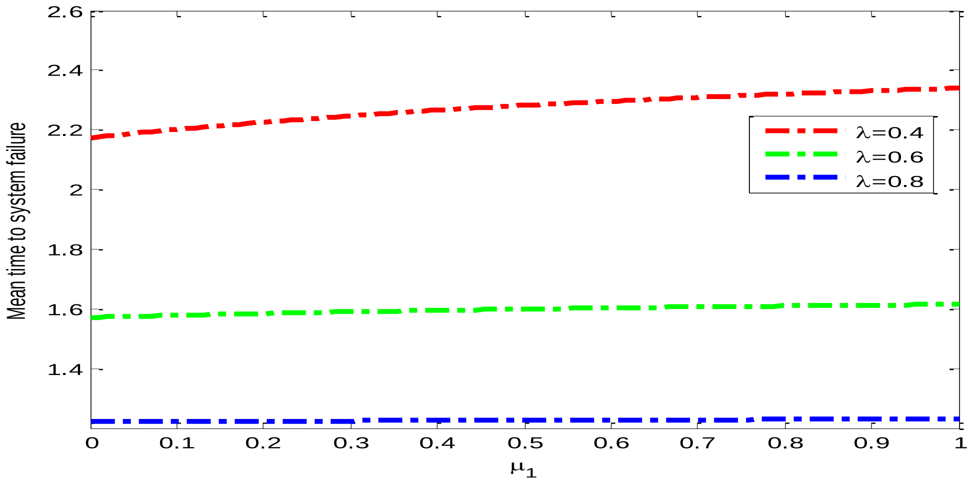

Figure 2: Availability against type I supporting device repair rate μ_1 **for** different values of $\lambda(0.4, 0.6, 0.8)$

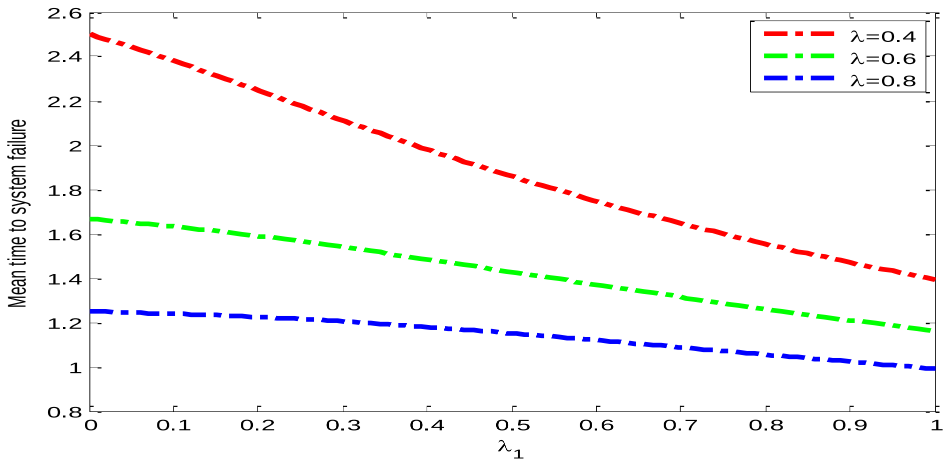

Figure 3: Availability against type I supporting device failure rate λ_1 **for** different values of $\lambda(0.4, 0.6, 0.8)$

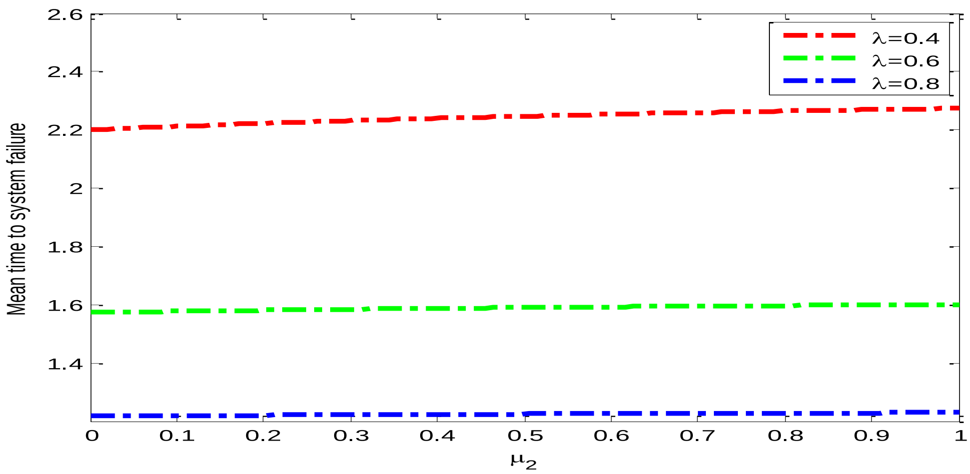

Figure 4: Availability against type II supporting device repair rate μ_2 **for** different values of $\lambda(0.4, 0.6, 0.8)$

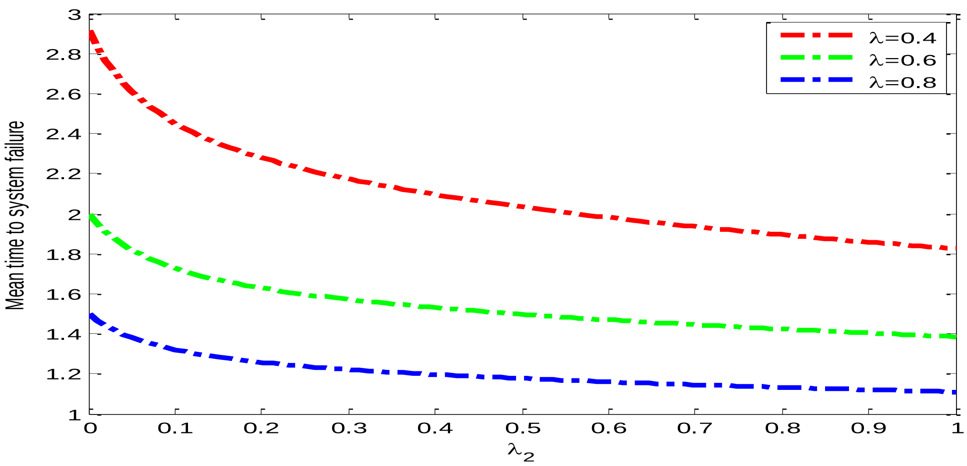

Figure 5: Availability against type II supporting device failure rate λ_2 **for** different values of $\lambda(0.4, 0.6, 0.8)$

IV. Discussion

Numerical results of availability with respect to type k, $k = I, II$ supporting devices repair μ_i and failure rates λ_i, $i = 1, 2$ **for** different values of $\lambda(0.4, 0.6, 0.8)$ **are depicted in Figures 2 - 5 respectively. In Figures 2 and 4, the mean time to system failure** increases as μ_1 and μ_2 for different values of unit failure rate λ. **This sensitivity analysis implies that major maintenance to the unit and supporting devices should be invoked to improve and maximize the mean time to system failure, production output as well as the profit.** On the other hand, Figures 3 and 5 show that the availability decreases as λ_1 and λ_2 increases for different values of unit failure rate λ. **This sensitivity analysis implies that major maintenance should be invoked to the unit and supporting devices to minimize the failure of the system in order to improve and maximize the mean time to system failure, production output as well as the profit.**

V. Conclusion

This paper studied a single system connected to two types of supporting device type I and II for its operation. Explicit expression for the **mean time to system failure** was derived. The numerical simulations presented in Figures 2 – 5 provide a description of the effect of failure rate and repair rate on **mean time to system failure** for different values of unit failure rate λ . On the basis of the numerical results obtained for particular cases, it is suggested that the system mean time to system failure can be improved significantly by:

(i) Adding more cold standby units.
(ii) Increasing the repair rate.
(iii) Reducing the failure rate of the system by hot or cold duplication method.
(iv) Exchange the system when old with new one before failure.

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Reliability Of A *k*-out-of-*n* System With A Single Server Extending Non-Preemptive Service To External Customers- Part I

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Abstract

We study repairable k-out-of-n system with single server who provides service to external customers also. N-policy is employed for the service of main customers. Once started, the repair of failed components is continued until all components become operational. When not repairing main customers, the server attends external customers (if there is any) who arrive according to a Poisson process. Once selected, the external customers receive a service of non-preemptive nature. When at least N main customers accumulate in the system and/or when the server is busy with such customers, external customers are not allowed to join the system. Otherwise, they join an infinite capacity queue of external customers. Life time distribution of components, service time distribution of main and external customers are all assumed to follow independent exponential distributions. Steady state analysis has been carried out and several important system performance measures based on the steady state distribution derived. A numerical study comparing the current model with those in which no external customers are provided service, is carried out. This study suggests that rendering service to external customers helps to utilize the server idle time profitably, without affecting the system reliability.

Keywords: *k-out-of-n system; non-preemptive service.*

1 Introduction

A k-out-of-n system can be defined as an n-component system which works if and only if at least k of its components operational. Application of such systems can be seen in many real-world phenomena. For instance almost all machines, of different complexity, are subjected to failure. One would expect a machine to work, even if some of its components have failed. A hospital providing emergency service is a typical example.We would expect the hospital to run even if some of its doctors/nurses/other staffs are on leave since it is supposed to have these personal in excess of the actual requirement. However, keeping these extra resources could be costly and not even feasible in some cases. A probabilistic study of a real world system such as a k-out-of-n system, often helps to develop an optimal strategy for maintaining high system reliability. Literature on such studies is vast (for example,see Chakravarthy et al.[1]).

Dudin et al.[2], Krishnamoorthy et al.[3, 4, 5] are among the studies on the reliability of a k-out-of-n system, where the server provides service to external customers in addition to repairing failed components of the main system. Such models are suitable for many real world situations. For example, a big telecom company may decide to share its resources like optical cables, mobile towers etc., for additional revenue. In doing so there is the risk that it may lead to dissatisfaction of the companies own customers. Therefore, the company would like to develop an optimal strategy for sharing its resources. Krishnamoorthy et al.[5] studied an N-policy for rendering service to external customers. They gave priority to the main customers through N-policy: the moment N failed components of the main system get accumulated, the ongoing service of an external customer (if there is any) is preempted and service to failed components is started.

In the present study, we consider a variant of the model in [5]. We assume N-policy for starting repair of failed components. However, the priority of the main customers is a bit reduced by assuming that an ongoing service of an external customer is not preempted when the number of failed components reaches N. This can be a serious compromise on the reliability of the k-out-of-n system. As in [5] it is assumed here also that an external customer, not allowed to join the system when the server is busy with service of main customers and/or when there are at least N failed components in the system. The external customer joins a queue of infinite capacity.

This paper is arranged as follows. In section 2 , we define the queuing model; section 3 conducts the steady state analysis, where we have obtained the stability condition explicitly and we also present an efficient method for computing the steady state probability vector. In section 4, we derive some important system performance measures and in section 5 the effect of N-policy and rendering service to external customers on the system reliability is examined. A cost function has also been studied in section 5 .

2 The queueing model

Here we consider a k-out-of-n system with a single server, offering service to external customers also. Commencement of service to failed components of the main system is governed by N-policy. That is at the epoch the system starts with all components operational, the server starts attending one by one the external customers (if there is any).When the number of failed components in the system is $\geq N$, the server in service of external customer (if there is any) is switched on to the service of the main customers after completing the ongoing service of the external customer. We assume that the failure rate of a component is $\frac{\lambda}{i}$, when i components are operational so that the inter-failure time of components of the k-out-of-n system remains exponentially distributed with parameter λ. Arrival of external customers follows a Poisson process with parameter $\bar{\lambda}$. External customers are not allowed to join the system when the server is busy with main customers or when there is $\geq N$ failed components. An external customer, who on arrival finds an idle server is directly taken for service. Service times of main and external customers follow exponential distribution with parameters μ and $\bar{\mu}$ respectively.

2.1 The Markov Chain

Let $X_1(t)$ = number of external customers in the system including the one getting service (if any) at time t,

$X_2(t)$ = number of main customers in the system including the one getting service (if any) at time t,

$$S(t) = \begin{cases} 0, & \text{if the server is idle or is busy with external customers} \\ 1, & \text{if the server is idle or is busy with main customers.} \end{cases}$$

Let $X(t) = (X_1(t), S(t), X_2(t))$ then $X = \{X(t), t \geq 0\}$ is a continuous time Markov chain on the state space

$$S = \{(0,0,j_2)/0 \leq j_2 \leq N-1\} \cup \{(j_1, 0, j_2)/j_1 \geq 1, 0 \leq j_2 \leq n-k+1\}$$
$$\cup \{(j_1, 1, j_2)/j_1 \geq 0, 1 \leq j_2 \leq n-k+1\}.$$

Arranging the states lexicographically and partitioning the state space into levels i, where each level i corresponds to the collection of the states with number of external customers in the system at any time t equal to i, we get an infinitesimal generator of the above chain as

$$Q = \begin{bmatrix} A_{10} & A_{00} & & & & \\ A_{20} & A_1 & A_0 & & & \\ & A_2 & A_1 & A_0 & & \\ & & A_2 & A_1 & A_0 & \\ & & & & \cdots & \\ & & & & & \cdots \end{bmatrix}.$$

In order to describe the entries in the above matrix we introduce some notations below.

[(i)]

1. I_m denotes an identity matrix of order m and I denotes an identity matrix of appropriate order.
2. e_m denotes a $m \times 1$ column matrix of 1s and e denotes a column matrix of 1s of appropriate order.
3. E_m denotes a square matrix of order m defined as

$$E_m(i,j) = \begin{cases} -1 & \text{if } j = i, 1 \leq i \leq m \\ 1 & \text{if } j = i+1, 1 \leq i \leq m-1 \\ 0 & \text{otherwise} \end{cases}$$

4. $E'_m = \text{Transpose}\ (E_m)$.
5. $r_m(i)$ denotes a $1 \times n$ row matrix whose ith entry is 1 and all other entries are zeros.
6. $c_m(i) = \text{Transpose}\ (r_m(i))$.
7. $\otimes$ denotes Kronecker product of matrices.

The transition within level 0 is represented by the matrix

$$A_{10} = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}, \text{where}$$

$B_1 = \lambda E_N - \bar{\lambda} I_N$.

B_2 is a $N \times (n-k+1)$ matrix whose $(N,N)^{\text{th}}$ entry is λ and all other entries are zeroes.

B_3 is a $N \times (n-k+1)$ matrix whose $(1,1)^{\text{th}}$ entry is μ and all other entries are zeroes.

$B_4 = \lambda E_{n-k+1} + \lambda c_{n-k+1}(n-k+1) \otimes r_{n-k+1}(n-k+1) + \mu E_{n-k+1}'$.

The transition from level 0 to level 1 is represented by the matrix

$$A_{00} = \begin{bmatrix} \bar{\lambda} I_N & O_{N \times (2n-2k+3-N)} \\ O_{(n-k+1) \times N} & O_{(n-k+1) \times (2n-2k+3-N)} \end{bmatrix}.$$

Transition from level 1 to 0 is represented by the matrix

$$A_{20} = \begin{bmatrix} \bar{\mu} I_N & O \\ O & H \\ O_{(n-k+1) \times N} & O \end{bmatrix} \text{where} H = [O_{(n-k+2-N) \times (N-1)} \quad \bar{\mu} I_{(n-k+2-N)}].$$

Transition within level 1 is represented by the matrix

$$A_1 = \begin{bmatrix} H_{11} & H_{12} & 0 \\ 0 & H_{22} & 0 \\ H_{31} & 0 & B_4 \end{bmatrix} \text{where}$$

$$H_{11} = B_1 - \bar{\mu} I_N, H_{12} = \lambda c_N(N) \otimes r_{n-k+2-N}(1),$$

$$H_{22} = \lambda E_{n-k+2-N} + \lambda c_{n-k+2-N}(n-k+2-N) \otimes r_{n-k+2-N}(n-k+2-N) - \bar{\mu} I_{n-k+2-N}.$$

H_{31} is an $(n-k+1)\times N$ matrix whose $(1,1)^{\text{th}}$ entry is μ.

$$A_0 = \begin{bmatrix} \bar{\lambda} I_N & O_{N\times(2n-2k+3-N)} \\ O_{(2n-2k+3-N)\times N} & O_{(2n-2k+3-N)\times(2n-2k+3-N)} \end{bmatrix},$$

$$A_2 = \begin{bmatrix} \bar{\mu} I_N & O & O \\ O & O_{(n-k+2-N)\times(n-k+2-N)} & \widetilde{H} \\ O_{(n-k+1)\times N} & O & O \end{bmatrix},$$

where $\widetilde{H} = [O_{(n-k+2-N)\times(N-1)} \quad \bar{\mu} I_{(n-k+2-N)}]$.

3 Steady state analysis

3.1 Stability condition

Consider the generator matrix $A = A_0 + A_1 + A_2$

$$A = \begin{bmatrix} \lambda E_N & H_{12} & 0 \\ 0 & H_{22} & F_{23} \\ F_{31} & 0 & B_4 \end{bmatrix} \text{with}$$

$$F_{23} = [O_{(n-k+2-N)\times(N-1)} \quad \bar{\mu} I_{n-k+2-N}],$$

$$F_{31} = \mu c_{n-k+1}(1) \otimes r_N(1).$$

Let $\zeta = (\zeta_0, \zeta_1, \zeta_2)$ be the steady state vector of the generator matrix A, where

$$\zeta_0 = (\zeta_{(0,0)}, \zeta_{(0,1)}, \dots, \zeta_{(0,N-1)}),$$
$$\zeta_1 = (\zeta_{(0,N)}, \zeta_{(0,N+1)}, \dots, \zeta_{(0,n-k+1)}),$$
$$\zeta_2 = (\zeta_{(1,1)}, \zeta_{(1,2)}, \dots, \zeta_{(1,n-k+1)}).$$

The Markov chain $\{X(t), t \geq 0\}$ is stable if and only if $\zeta A_0 e < \zeta A_2 e$ (please refer Neuts [6]).

It follows that $\zeta A_0 e = \bar{\lambda} \zeta_0 e$ and $\zeta A_2 e = \bar{\mu}(\zeta_0 e + \zeta_1 e)$. Therefore the stability condition becomes

$$\frac{\bar{\lambda}}{\bar{\mu}} \frac{\zeta_0 e}{(\zeta_0 e + \zeta_1 e)} < 1. \tag{1}$$

It follows from the relation $\zeta A = 0$ that

$$\zeta_0 \lambda E_N + \zeta_2 F_{31} = 0, \tag{2}$$

$$\zeta_0 H_{12} + \zeta_1 H_{22} = 0, \tag{3}$$

$$\zeta_1 F_{23} + \zeta_2 B_4 = 0. \tag{4}$$

From (4), it follows that

$$\zeta_2 = -\zeta_1 F_{23} B_4^{-1}. \tag{5}$$

Substituting this in (2) we get

$$\zeta_0 \lambda E_N - \zeta_1 F_{23} B_4^{-1} F_{31} = 0. \tag{6}$$

$$\lambda \zeta_0 e = (-\zeta_1 F_{23} B_4^{-1} F_{31})(-E_N^{-1} e). \tag{7}$$

Notice that the first column of the matrix F_{31} is $-B_4 e$ and all other columns of it are zero columns. This implies that the first column of the matrix $B_4^{-1} F_{31}$ is $-e$ and its all other columns are zero columns. Hence the first column of the matrix $-F_{23} B_4^{-1} F_{31}$ is $\bar{\mu} e$ and all other columns are zero columns. The first entry of the row matrix $-\zeta_1 F_{23} B_4^{-1} F_{31}$ is thus $\bar{\mu} \zeta_1 e$ and its all other entries are zeros. It can be seen that the first entry of the column matrix $-E_N^{-1}e$ is N. These two facts together tell us that $(-\zeta_1 F_{23} B_4^{-1} F_{31})(-E_N^{-1}e)$ is $N\bar{\mu}\zeta_1 e$. Thus, equation (7) becomes

$$\lambda\zeta_0 e = N\bar{\mu}\zeta_1 e.$$

Adding $N\bar{\mu}\zeta_0 e$ on both sides of the above equation, we get

$$(\lambda + N\bar{\mu})\zeta_0 e = N\bar{\mu}(\zeta_0 e + \zeta_1 e),$$

which implies

$$\frac{\zeta_0 e}{(\zeta_0 e+\zeta_1 e)} = \frac{N\bar{\mu}}{(\lambda+N\bar{\mu})}.$$

Hence the stability condition (1) becomes

$$\frac{\bar{\lambda}}{\bar{\mu}}\frac{N\bar{\mu}}{(\lambda+N\bar{\mu})} < 1.$$

3.2 Computation of steady state vector

Let $\pi = (\pi(0), \pi(1), \pi(2), \dots)$ the steady state vector of the Markov chain X, where $\pi(0) = (\pi_{(0,0)}, \pi_{(0,1)})$ with $\pi_{(0,0)} = (\pi_{(0,0,0)}, \pi_{(0,0,1)}, \dots, \pi_{(0,0,N-1)})$ and $\pi_{(0,1)} = (\pi_{(0,1,1)}, \dots, \pi_{(0,1,n-k+1)})$. For $i \geq 1$, $\pi(i) = (\pi_{(i,0)}, \tilde{\pi}_{(i,0)}, \pi_{(i,1)})$ with $\pi_{(i,0)} = (\pi_{(i,0,0)}, \pi_{(i,0,1)}, \dots, \pi_{(i,0,N-1)})$, $\tilde{\pi}_{(i,0)} = (\pi_{(i,0,N)}, \dots, \pi_{(i,0,n-k+1)})$, $\pi_{(i,1)} = (\pi_{(i,1,1)}, \pi_{(i,1,2)}, \dots, \pi_{(i,1,n-k+1)})$. Now from $\pi Q = 0$, we can write

$$\pi_{(0,0)}B_1 + \pi_{(0,1)}B_3 + \pi_{(1,0)}\bar{\mu}I_N = 0, \tag{8}$$

$$\pi_{(0,0)}B_2 + \pi_{(0,1)}B_4 + \tilde{\pi}_{(1,0)}H = 0, \tag{9}$$

and for $i \geq 1$,

$$\pi_{(i-1,0)}\bar{\lambda}I_N + \pi_{(i,0)}H_{11} + \pi_{(i,1)}H_{31} + \pi_{(i+1,0)}\bar{\mu}I_N = 0, \tag{10}$$

$$\pi_{(i,0)}H_{12} + \tilde{\pi}_{(i,0)}H_{22} = 0, \tag{11}$$

$$\pi_{(i,1)}B_4 + \tilde{\pi}_{(i+1,0)}\tilde{H} = 0. \tag{12}$$

From (11), we get, for $i \geq 1$

$$\tilde{\pi}_{(i,0)} = -\pi_{(i,0)}H_{12}(H_{22}^{-1}). \tag{13}$$

From (12), we get

$$\pi_{(i,1)} = -\tilde{\pi}_{(i+1,0)}\tilde{H}(B_4^{-1}). \tag{14}$$

Substituting (13) in (14), we get

$$\pi_{(i,1)} = \pi_{(i+1,0)}H_{12}(H_{22}^{-1})\tilde{H}(B_4^{-1}). \tag{15}$$

Substituting (15) in (10), we get

$$\pi_{(i-1,0)}\bar{\lambda}I_N + \pi_{(i,0)}H_{11} + \pi_{(i+1,0)}H_{12}(H_{22}^{-1})\tilde{H}(B_4^{-1})H_{31} + \pi_{(i+1,0)}\bar{\mu}I_N = 0. \tag{16}$$

We notice that the first column of the matrix H_{31} is $-B_4 e$ and all other columns of H_{31} are zero columns. Hence the first column of the matrix $(B_4^{-1})H_{31}$ is $-e$ and its all other columns are zero columns. This tells us that the first column of the matrix $\tilde{H}(B_4^{-1})H_{31}$ is $-\bar{\mu}e$ and all other columns are zeros. But $-\bar{\mu}e$ is $H_{22}e$ and hence the first column of the matrix $(H_{22}^{-1})\tilde{H}(B_4^{-1})H_{31}$ is e and all other columns are zeros. This fact leads us to conclude that the first column of the matrix $H_{12}(H_{22}^{-1})\tilde{H}(B_4^{-1})H_{31}$ is $H_{12}e = \lambda c_N(N)$ and all other columns are zeros. In other words

$$H_{12}(H_{22}^{-1})\tilde{H}(B_4^{-1})H_{31} = \lambda c_N(N) \otimes r_N(1).$$

Now equation (16) becomes

$$\pi_{(i-1,0)}\bar{\lambda}I_N + \pi_{(i,0)}H_{11} + \pi_{(i+1,0)}\lambda c_N(N) \otimes r_N(1) + \pi_{(i+1,0)}\bar{\mu}I_N = 0.$$

That is

$$\pi_{(i-1,0)}\bar{\lambda}I_N + \pi_{(i,0)}H_{11} + \pi_{(i+1,0)}(\lambda c_N(N) \otimes r_N(1) + \bar{\mu}I_N) = 0. \tag{17}$$

Now from equation (9), we can write

$$\pi_{(0,1)} = -\pi_{(0,0)}B_2(B_4^{-1}) - \tilde{\pi}_{(1,0)}H(B_4^{-1}). \tag{18}$$

However, from equation (13), we have

$$\tilde{\pi}_{(1,0)} = -\pi_{(1,0)}H_{12}(H_{22}^{-1}). \tag{19}$$

Hence equation (18) becomes

$$\pi_{(0,1)} = -\pi_{(0,0)} B_2 (B_4^{-1}) + \pi_{(1,0)} H_{12} (H_{22}^{-1}) H (B_4^{-1}). \tag{20}$$

Substituting (20) in (8), we get

$$\pi_{(0,0)} B_1 + (-\pi_{(0,0)} B_2 (B_4^{-1}) + \pi_{(1,0)} H_{12} (H_{22}^{-1}) H (B_4^{-1})) B_3 + \pi_{(1,0)} \bar{\mu} I_N = 0. \tag{21}$$

Since the first column of the matrix B_3 is $-B_4 e$, a similar reasoning as for equation (16) leads us to write:

$$-B_2 (B_4^{-1}) B_3 = \lambda c_N(N) \otimes r_N(1),$$
$$H_{12} (H_{22}^{-1}) H (B_4^{-1}) B_3 = \lambda c_N(N) \otimes r_N(1).$$

Hence equation (21) becomes

$$\pi_{(0,0)} (B_1 + \lambda c_N(N) \otimes r_N(1)) + \pi_{(1,0)} (\lambda c_N(N) \otimes r_N(1) + \bar{\mu} I_N) = 0. \tag{22}$$

Equations (17) and (22) shows that the vector $\hat{\pi} = (\pi_{(0,0)}, \pi_{(1,0)}, \pi_{(2,0)}, \dots)$ satisfies the relation $\hat{\pi}\tilde{Q} = 0$, where $\tilde{Q}$ is a generator matrix defined as

$$\tilde{Q} = \begin{bmatrix} \tilde{A}_{10} & \tilde{A}_0 & & & \\ \tilde{A}_2 & \tilde{A}_1 & \tilde{A}_0 & & \\ & \tilde{A}_2 & \tilde{A}_1 & \tilde{A}_0 & \\ & & \cdot & \cdot & \cdot \\ & & & \cdot & \cdot & \cdot \\ & & & & & \end{bmatrix}$$

In the above, $\tilde{A}_{10} = B_1 + \lambda c_N(N) \otimes r_N(1)$, $\tilde{A}_0 = \bar{\lambda} I_N, \tilde{A}_1 = H_{11}$ and $\tilde{A}_2 = \lambda c_N(N) \otimes r_N(1) + \bar{\mu} I_N$. Hence the vector $\hat{\pi}$ is a constant multiple of the steady state vector $\tau = (\tau(0), \tau(1), \dots)$ of the generator matrix $\tilde{Q}$. The vector τ can be obtained by applying the matrix analytic methods (see Neuts [6]) as

$$\tau(i) = \tau(0) R^i, \quad i \geq 0, \tag{23}$$

where the matrix R is the minimal non-negative solution of the matrix quadratic equation:

$$\tilde{A}_0 + R\tilde{A}_1 + R^2 \tilde{A}_2 = 0. \tag{24}$$

Equation (23) implies

$$\pi_{(0,0)} = \mathcal{K}\tau(0),$$
$$\pi_{(i,0)} = \pi_{(0,0)} R^i, \quad i \geq 0.$$

Now the vector $\hat{\pi}$ is obtained up to a constant $\mathcal{K}$ as $\hat{\pi} = \mathcal{K}\tau$, the other component vectors $\tilde{\pi}_{(i,0)}$, $i \geq 1$, $\pi_{(i,1)}$, $i \geq 0$ of π can be obtained from the equations (13), (14) and (20), up to the constant $\mathcal{K}$, which is finally obtained from the normalizing condition $\pi e = 1$.

4 Performance measures

4.1 Busy period of the server with the failed components of the main system

Let T_i denote the server busy period with failed components which starts with i failed components and with j external customers in the system. Consider the absorbing Markov chain $Y = \{Y(t), t \geq 0\}$, where $Y(t)$ is the number of failed components of the main system, with the state space $\{0,1,2, \dots, N, N+1, \dots, n-k+1\}$ and having infinitesimal matrix given by

$$\tilde{H}_{BF} = \begin{bmatrix} 0 & 0 \\ -H_{BF} e & H_{BF} \end{bmatrix},$$

where

$$H_{BF} = \lambda E_{n-k+1} + \lambda c_{n-k+1}(n-k+1) \otimes r_{n-k+1} + \mu E'_{n-k+1}.$$

Notice that $Y(t) = 0$ is an absorbing state. T_i is the time until absorption in the Markov chain $\{Y(t)\}$ assuming that it starts in the state i. The expected value ET_i of T_i is therefore the i^{th} entry of the column matrix $-H_{BF}^{-1} e$ as given by (please see Krishnamoorthy et al. [5]):

$$ET_i = \frac{1}{\mu}\left(i \sum_{j=0}^{n-k+1-i} \left(\frac{\lambda}{\mu}\right)^j + \sum_{j=n-k+2-i}^{n-k} (n-k+1-j) \left(\frac{\lambda}{\mu}\right)^j \right).$$

We notice that once the service of failed components starts, the external customers has no effect on it and hence ET_i is independent of j the number of external customers. Define

$P_f(N) = \pi_{(0,0,N-1)} + \sum_{j=1}^{\infty} \pi_{(j,0,N)}$ and
$P_f(i) = \sum_{j=1}^{\infty} \pi_{(j,0,i)}$ for $N < i \leq n-k+1$

$P_f(i)$ will then denote the system steady state probability just before starting service to failed components with i number of failed components. The expected length of the busy period of the server with failed components is then given by

$$E_{\hat{H}} = \frac{\sum_{i=N}^{n-k+1} P_f(i) ET_i}{\sum_{i=N}^{n-k+1} P_f(i)}.$$

4.2 Other performance measures

1. Fraction of time the system is down,
$$P_{down} = \sum_{j_1=0}^{\infty} \pi_{(j_1,0,n-k+1)} + \sum_{j_1=0}^{\infty} \pi_{(j_1,1,n-k+1)}.$$

2. System reliability, $P_{rel} = 1 - P_{down}$.
3. Average number of external customers waiting in the queue,
$$N_q = \sum_{j_i=0}^{\infty} j_i \left(\sum_{j_3=0}^{n-k+1} \pi_{(j_1,1,j_3)}\right) + \sum_{j_1=1}^{\infty} (j_1 - 1)\left(\sum_{j_3=0}^{n-k+1} \pi_{(j_1,0,j_3)}\right).$$

4. Average number of failed components of the main system,
$$N_{fail} = \sum_{j_3=0}^{n-k+1} j_3 \left(\sum_{j_1=0}^{\infty} \pi_{(j_1,0,j_3)}\right) + \sum_{j_3=1}^{n-k+1} j_3 \left(\sum_{j_1=0}^{\infty} \pi_{(j_1,1,j_3)}\right).$$

5. Average number of failed components waiting when server is busy with external customers
$$NB_{fail} = \sum_{j_3=0}^{n-k+1} j_3 \left(\sum_{j_1=1}^{\infty} \pi_{(j_1,0,j_3)}\right).$$

6. Expected number of external customers joining the system,
$$\theta_3 = \bar{\lambda}\left\{\sum_{j_1=1}^{\infty} \left(\sum_{j_3=0}^{N-1} \pi_{(j_1,0,j_3)}\right) + \sum_{j_1=0}^{N-1} \pi_{(0,0,j_3)}\right\}.$$

7. Expected number of external customers on its arrival gets service directly
$$NEX_{direct} = \sum_{j_3=0}^{N-1} \pi_{(0,0,j_3)}.$$

8. Fraction of time the server is busy with external customers,
$$P_{ext,busy} = \sum_{j_1=1}^{\infty} \left(\sum_{j_3=0}^{n-k+1} \pi_{(j_1,0,j_3)}\right).$$

9. Probability that server is found idle,
$$P_{idle} = \sum_{j_3=0}^{N-1} \pi_{(0,0,j_3)} = N\pi_{(0,0,0)}.$$

10. Probability that the server is found busy,
$$P_{busy} = 1 - \sum_{j_3=0}^{N-1} \pi_{(0,0,j_3)} = 1 - N\pi_{(0,0,0)}.$$

11. Expected loss rate of external customers,
$$\theta_4 = \bar{\lambda}\left\{\sum_{j_1=0}^{\infty} \left(\sum_{j_3=1}^{n-k+1} \pi_{(j_1,1,j_3)}\right) + \sum_{j_1=1}^{\infty} \left(\sum_{j_3=N}^{n-k+1} \pi_{(j_1,0,j_3)}\right)\right\}.$$

12. Expected service completion rate of external customers,
$$\theta_5 = \bar{\mu} \sum_{j_1=0}^{\infty} \left(\sum_{j_3=0}^{n-k+1} \pi_{(j_1,0,j_3)}\right).$$

13. Expected number of external customers when server is busy with external customers,
$$\theta_6 = \sum_{j_1=0}^{\infty} j_1 \left(\sum_{j_3=0}^{n-k+1} \pi_{(j_1,0,j_3)}\right).$$

5 Numerical Study of the Performance of the System

5.1 The Effect of N Policy on the Server Busy Probability

The main purpose of introducing N-policy while studying a k-out-of-n system with a single server offering service to external customers, in a non pre-emptive nature, was optimization of the system revenue, by utilizing the server idle time, without compromising the reliability of the system much. Tables 1 and 2 reports the variation in the server busy probability when external customers are allowed and not allowed respectively. A comparison of the two tables suggest that there is an increase in the server busy probability, when external customers are allowed. Table 3, which report the effect of the N-policy level on the fraction of time the server remains busy with external customers, tells that there is an increase in the reported measure with an increase in N. Hence, it can be concluded that the N-policy has helped in improving the attention towards external customers slightly. Now, we want to check whether the introduction of the N-policy has badly affected the system reliability.

5.2 The effect of N policy on system reliability

We study two cases $\lambda < \mu$ and $\lambda > \mu$. We expected a decrease in P_{rel} with an increase in N. This is because as N increases, the server spends more time for external customers, which we thought might cause a decrease in the system reliability. This was verified from Table 4, where we assumed $\lambda < \mu$. However, Table 4 shows very high system reliability over 95 %. The magnitude of decrease in reliability was found lesser when the total number of components n was high. In short Table 4 shows that reliability of the system is not much affected by increasing N-policy level. In Table 5 where it was assumed that the component failure rate λ is greater than their service rate μ, it was again found that P_{rel} decreases with increase in N and that the magnitude of decrease is not high. More importantly, the reliability of the system was found less than 91.5 %. To check whether this was actually due to the introduction of external customers, we compared the system reliability of the current model with that of a k-out-of-n system where no external customers are entertained. Table 6 shows that allowing external customers in the system has only a narrow effect on the system reliability and the decrease in reliability is actually due to the assumption $\lambda > \mu$.

5.3 Analysis of a Cost function

Table 1 shows that as N increases, even though the server busy probability increases first, it decreases as N crosses some value. Note that the overall server busy probability is the sum of the server busy probability with external customers and the server busy probability with main customers. Table 3 shows that the fraction of time server remaining busy with external customers is ever increasing with N. Now as N increases, there is a decrease in the server busy probability with main customers. Hence, the above said behavior of the overall server busy probability can be concluded to be due to the conflicting nature of the two entities constituting it. This behavior of the server busy probability lead us to construct a cost function in the hope of finding an optimal value for the N-policy level defined as follows:

$$\begin{aligned} &Expected\ cost\ per\ unit\ time \\ &= C_1 \cdot P_{\text{down}} + C_2 \cdot N_q + C_4 \cdot \theta_4 + C_5 \cdot N_{fail} + \frac{C_3}{E_{\widehat{H}}} + C_6 \cdot P_{idle} \end{aligned}$$

In the above, C_1 denote the cost per unit time incurred if the system is down, C_2 denote the holding cost per unit time per external customer in the queue, C_3 denote the cost incurred for starting failed components service, C_4 denote the cost due to loss of 1 external customer, C_5 denote the holding cost per unit time of one failed component, C_6 denote the cost per unit time if the server is idle. The values of the cost function presented in Table 7, for various failure rates of the components, shows an optimal value for N in each case.

Table 1: Variation in the server busy probability when external customers are allowed $k = 20, \lambda = 4, \bar{\lambda} = 3.2, \mu = 5.5, \bar{\mu} = 8$

N	n=45	n=50	n=60	n=65
1	0.823494	0.823522	0.823529	0.823529
3	0.829935	0.829973	0.829983	0.831354
5	0.832187	0.832243	0.832256	0.832891
7	0.833255	0.833338	0.833358	0.833717
9	0.833839	0.833968	0.834	0.83423
11	0.834162	0.834367	0.834417	0.834577
13	0.834295	0.834627	0.834708	0.834827
15	0.834239	0.834789	0.834923	0.835093
17	0.833936	0.834861	0.835085	0.835224
19	0.833252	0.834829	0.835211	0.835329
21	0.831922	0.834652	0.835306	0.835413
23	0.829445	0.834239	0.835375	0.83548
25	0.824871	0.833426	0.835412	0.83553

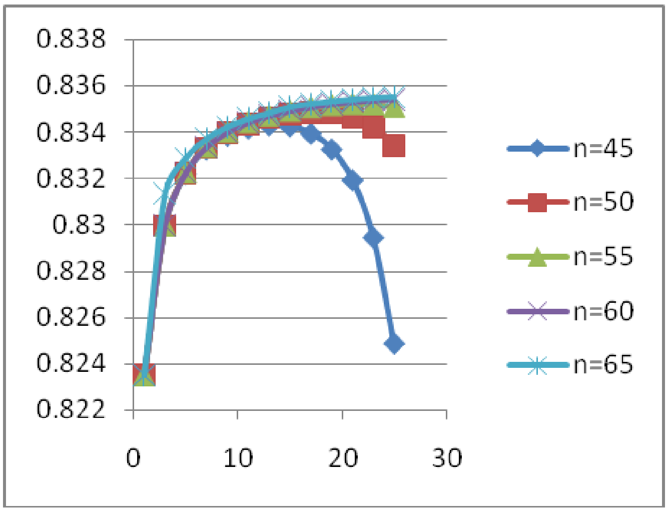


Table 2: Variation in the server busy probability when external customers are not allowed $k = 20, \lambda = 4, \mu = 5.5$

N	n=45	n=50	n=60	n=65
1	0.72722	0.72726	0.72727	0.72727
3	0.7272	0.72726	0.72727	0.72727
5	0.72717	0.72725	0.72727	0.72727
7	0.72711	0.72724	0.72727	0.72727
9	0.72703	0.72722	0.72727	0.72727
11	0.72688	0.72719	0.72727	0.72727
13	0.72663	0.72714	0.72727	0.72727
15	0.72622	0.72706	0.72726	0.72727
17	0.7255	0.72691	0.72726	0.72727
19	0.72425	0.72666	0.72725	0.72727
21	0.72206	0.72623	0.72723	0.72726
23	0.71814	0.72546	0.7272	0.72726

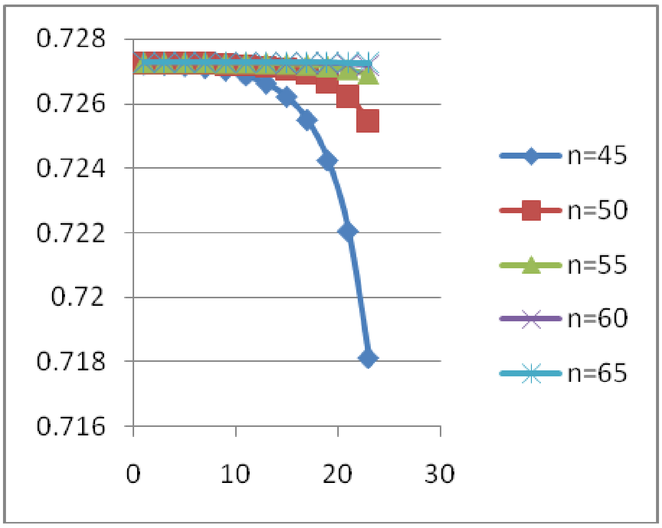


Table 3: Effect of the N-policy level on the fraction of time server is busy with external customers with $k = 20, \lambda = 4, \bar{\lambda} = 3.2, \mu = 5.5, \bar{\mu} = 8$

N	n=40	n=45	n=50	n=55	n=60
1	0.096351	0.096276	0.096261	0.096257	0.096257
2	0.100557	0.100464	0.100445	0.100441	0.10044
3	0.102853	0.10274	0.102717	0.102712	0.102711
4	0.104255	0.104117	0.104089	0.104083	0.104082
5	0.105198	0.105028	0.104993	0.104986	0.104985
6	0.105882	0.105672	0.105629	0.105621	0.105619
7	0.106413	0.106153	0.1061	0.106089	0.106087
8	0.106853	0.106528	0.106462	0.106449	0.106446
9	0.107241	0.106832	0.106749	0.106733	0.106729
10	0.107605	0.107088	0.106984	0.106963	0.106958
11	0.107968	0.107313	0.10718	0.107153	0.107148
12	0.108354	0.107517	0.107348	0.107314	0.107307
13	0.108786	0.107711	0.107495	0.107451	0.107442
14	0.109291	0.107904	0.107626	0.10757	0.107559
15	0.109905	0.108106	0.107747	0.107675	0.10766
17	0.111651	0.108581	0.107976	0.107854	0.107829
19	0.114606	0.109249	0.108092	0.108008	0.107966
21		0.110301	0.108216	0.108153	0.10808
23		0.112079	0.10851	0.108308	0.108182
25		0.115216	0.108928	0.1085	0.108281
27			0.110699	0.108771	0.108387
29			0.112652	0.109196	0.108516
31			0.116153	0.10991	0.108697
33				0.111158	0.108978
35				0.113399	0.109446

Table 4: Variation in the system reliability with increase in N ($\lambda < \mu$ case) $k = 20, \lambda = 4, \bar{\lambda} = 3.2, \mu = 5.5, \bar{\mu} = 8$

N	n=40	n=45	n=50	n=55	n=60	n=65
1	0.99963	0.99993	0.99998	1	1	1
3	0.99948	0.99989	0.99998	1	1	1
5	0.99924	0.99985	0.99997	0.99999	1	1
7	0.99885	0.99977	0.99995	0.99999	1	1
9	0.9982	0.99964	0.99993	0.99998	1	1
11	0.99712	0.99942	0.99988	0.99998	1	1
13	0.9953	0.99905	0.99981	0.99996	0.99999	1
15	0.99217	0.99843	0.99968	0.99994	0.99999	1
17	0.98668	0.99736	0.99947	0.99989	0.99998	1
19	0.97689	0.9955	0.99909	0.99982	0.99996	0.99999
21	0.95915	0.99223	0.99844	0.99968	0.99994	0.99999
23		0.98638	0.9973	0.99945	0.99989	0.99998
25		0.97578	0.99528	0.99905	0.99981	0.99996
27			0.99165	0.99833	0.99966	0.99993
29			0.98509	0.99705	0.9994	0.99988
31			0.97315	0.99475	0.99894	0.99979

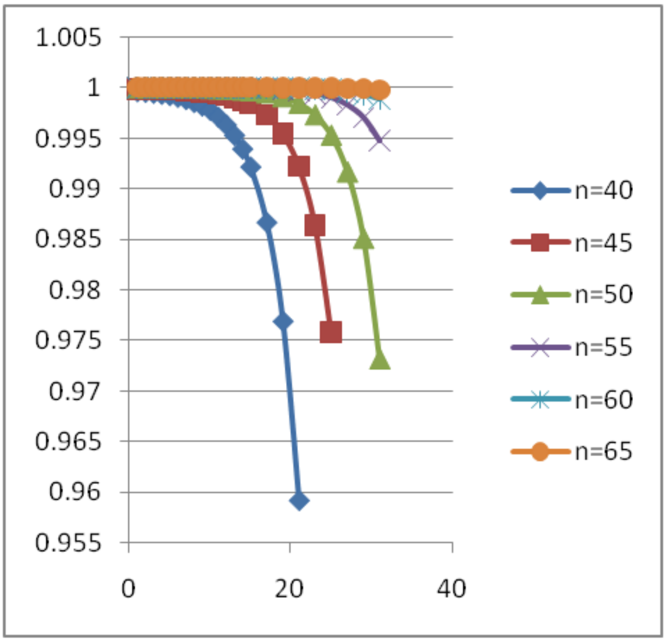


Table 5: Variation in the system reliability with increase in N ($\lambda > \mu$ case) $\lambda = 6, \mu = 5.5, \bar{\lambda} = 3.2, \bar{\mu} = 8$

N	n=40	n=50	n=55	n=60
1	0.90191	0.91106	0.91312	0.91441
2	0.90118	0.91081	0.91297	0.91431
3	0.90041	0.91055	0.91281	0.91421
4	0.89961	0.91028	0.91264	0.91411
5	0.89876	0.91	0.91247	0.914
6	0.89758	0.90971	0.91229	0.91389
7	0.89696	0.90941	0.91211	0.91377
8	0.896	0.9091	0.91192	0.91366
9	0.895	0.90878	0.91173	0.91354
10	0.89396	0.90845	0.91153	0.91341
11	0.89287	0.90812	0.91133	0.91329
12	0.89174	0.90777	0.91112	0.91316
13	0.89055	0.90741	0.9109	0.91303
14	0.88932	0.90705	0.91068	0.91289
15	0.88804	0.90667	0.91046	0.91275
16	0.8867	0.90628	0.91	0.91261
17	0.88531	0.90589	0.90951	0.91247
18	0.88386	0.90548	0.90901	0.91232
19	0.88235	0.90507	0.90848	0.91217
21	0.88079	0.90464	0.90794	0.91186
23	0.87916	0.90421	0.90738	0.91155
25		0.90331	0.90679	0.91122
27		0.90237	0.9062	0.91088
29		0.90139	0.90558	0.91053
31		0.90036	0.90494	0.91018
33		0.8993	0.90462	0.90981
35				0.90944
37				0.90905
39				0.90866
41				0.90827

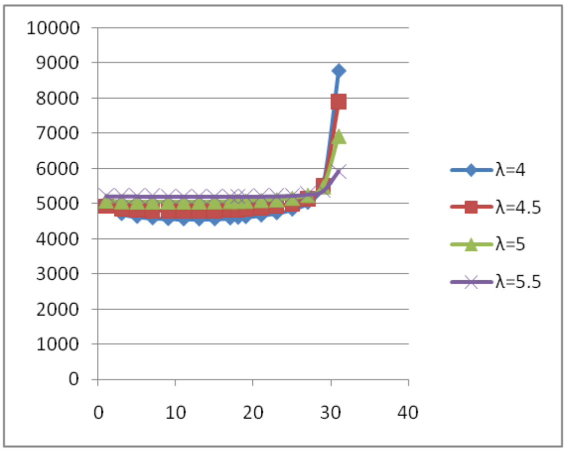
Table 6: Variation in the system reliability with increase in N (case when no external customers are allowed) $k = 20, \lambda = 6, \mu = 5.5$

N	n=40	n=45	n=50	n=55	n=60	n=65
1	0.902225	0.907874	0.911180	0.913196	0.914453	0.915247
3	0.900740	0.907001	0.910662	0.912877	0.914252	0.915120
5	0.899093	0.906080	0.910108	0.912537	0.914040	0.914985
7	0.897301	0.905082	0.909519	0.912176	0.913815	0.914843
9	0.895355	0.904014	0.908894	0.911796	0.913578	0.914693
11	0.893242	0.902873	0.908232	0.911395	0.913329	0.914537
13	0.890948	0.901655	0.907531	0.910974	0.913069	0.914373
15	0.888461	0.900358	0.906793	0.910533	0.912797	0.914202
17	0.885763	0.898979	0.906016	0.910071	0.912514	0.914025

19	0.882837	0.897514	0.905200	0.909589	0.912219	0.913841
21	0.879662	0.895960	0.904345	0.909087	0.911913	0.913651
23		0.894313	0.903450	0.908566	0.911597	0.913454
25		0.892570	0.902514	0.908025	0.911271	0.913252
27			0.901539	0.907465	0.910934	0.913044
29			0.900523	0.906886	0.910588	0.912831
31			0.899465	0.906289	0.910233	0.912613
33				0.905674	0.909868	0.912390
35				0.905041	0.909495	0.912162
37					0.909114	0.911930
39					0.908724	0.911693
41					0.908327	0.911453
43						0.911209
45						0.910961

Table 7: Analysis of a cost function for finding optimal N value , $n = 50, k = 20, \mu = 5.5, \bar{\lambda} = 3.2, \bar{\mu} = 8, C_1 = 2000, C_2 = 20, C_3 = 800, C_4 = 1000, C_5 = 10, C_6 = 200$

N	$\lambda = 4$	$\lambda = 4.5$	$\lambda = 5$	$\lambda = 5.5$
1	4925.877	4937.695	5079.029	5226.181
3	4710.059	4856.852	5057.425	5221.212
5	4630.354	4825.835	5050.332	5218.775
7	4591.702	4812.151	5048.243	5216.965
9	4571.3	4806.745	5048.411	5215.313
11	4561.086	4806.248	5049.849	5213.713
13	4558.217	4809.556	5052.345	5212.268
15	4563.915	4817.604	5056.578	5211.373
17	4588.216	4835.444	5064.896	5211.922
18	4605.19	4846.938	5070.21	5212.65
19	4624.185	4859.68	5076.196	5213.701
21	4670.646	4890.628	5091.4	5217.34
23	4735.585	4934.206	5114.597	5224.719
25	4837.829	5004.721	5155.522	5240.069
27	5032.125	5144.138	5241.815	5274.736
29	5546.901	5525.659	5482.957	5371.341
31	8780.95	7911.995	6932.789	5918.758



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Reliability Of a *k*-out-of-*n* System With A Single Server Extending Non-Preemptive Service To External Customers- Part II

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Abstract

In this paper we study a k-out-of-n system with a single repair facility, which provides service to external customers also. We assume an N-policy for service to failed components(main customers) of the k-out-of-n system starts only on accumulation of N of them. Once started, the repair of external customers is continued until all the components become operational. When not repairing failed components, the server attends external customers(if there is any) who arrive according to a Poisson process. Once selected for service, the external customers receive a service of non-preemptive nature. When there are at least N failed components in the system and/or when the server is busy with failed components, the external customers are not allowed to join the system.Otherwise they join an orbit of infinite capacity. Life time distribution of failed components, service time distribution of main and external customers and the inter retrial time distribution of orbital customers are all assumed to follow independent exponential distributions. Steady state analysis has been carried out and several important system performance measures, based on the steady state distribution, derived. A numerical study comparing the current model with those in which no external customers are considered has been carried out.This study suggests that rendering service to external customers helps to utilize the server idle time profitably, without sacrificing the system reliability.

Keywords: *k-out-of-n system; non-preemptive service.*

1 Introduction

In this paper, we consider a variant of the model studied in Krishnamoorthy et al. [1]. In part I (see Krishnamoorthy et al. [3]) of this paper we studied the reliability of a k-out-of-n system with a single server rendering non-preemptive service to external customers.In this paper we extend it to retrial queue of unsatisfied external customers(orbital customers) with linear retrial rate.In effect we replace the infinite queue of external customers in part I by orbital customers and their retrial. However, the stability condition remains the same in both models.

This paper is arranged as follows. In section 2 , we describe the model and in section 3 , its long run behavior is analyzed. The stability condition is derived explicitly in section 3 and computation of the steady state vector using the Neuts-Rao truncation procedure [2] has been discussed. Some important performance measures are derived in section 4 . The effect of rendering service to external customers and N-policy has been studied numerically in section 5 .

2 The retrial model

Here we consider a variant of the model discussed in section 2 of part I by assuming that an arriving external customer either gets immediate service if it finds the server is idle at that time or joins an orbit of infinite capacity, if the server is busy with external customers with $\leq N-1$ failed components of the k-out-of-n system. As in the model discussed in section 2 of part I, the external customers are not allowed to join the orbit when the server is busy with failed components of the system. An orbital customer retries for service with inter-retrial time following an exponential distribution with parameter θ. All other assumptions and parameters remain the same as in model discussed in section 2 of part I. In this situation the system can be modeled as follows.

Let $X_1(t)=$ the number of external customers in the orbit at time t,

$X_2(t)=$ the number of failed components of the k-out-of-n system, including the one getting service (if any) at time t.

Define

$$S(t)=\begin{cases}0, & \text{If the server is idle}\\ 1, & \text{If the server is busy with an external customer}\\ 2, & \text{If the server is busy with a main customer}\end{cases}$$

Now, $X(t)=(X_1(t),S(t),X_2(t))$ forms a continuous time Markov chain on the state space

$$S=\{(j_1,0,j_2)/j_1\geq 0, 0\leq j_2\leq N-1\}\cup\{(j_1,1,j_2)/j_1\geq 0, 0\leq j_2\leq n-k+1\}$$
$$\cup\{(j_1,2,j_2)/j_1\geq 0, 1\leq j_2\leq n-k+1\}.$$

Arranging the states lexicographically and partitioning the state space into levels i, where each level i corresponds to the collection of states with number of external customers in the orbit at any time t equal to i, we get an infinitesimal generator of the above chain as

$$Q=\begin{bmatrix}\mathbf{A}_{10} & \mathbf{A}_0 & & & & & \\ \mathbf{A}_{21} & \mathbf{A}_{11} & \mathbf{A}_0 & & & & \\ & \mathbf{A}_{22} & \mathbf{A}_{12} & \mathbf{A}_0 & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \\ & & & & \mathbf{A}_{2p} & \mathbf{A}_{1p} & \mathbf{A}_0 \\ & & & & & \cdot & \cdot & \cdot \\ & & & & & & \cdot & \cdot & \cdot\end{bmatrix}.$$

The entries of Q are described as below: For $i\geq 0$, the transition within level i is represented by the matrix

$$\mathbf{A}_{1i}=\begin{bmatrix}D_{11}^{(i)} & D_{12} & 0 & D_{14}\\ D_{21} & D_{22} & D_{23} & 0\\ 0 & 0 & D_{33} & D_{34}\\ D_{41} & 0 & 0 & D_{44}\end{bmatrix},$$

where

$$D_{11}^{(i)}=\lambda E_N-\overline{\lambda}I_N-i\theta I_N, D_{12}=\overline{\lambda}I_N,$$

$D_{14} = \lambda c_N(N) \otimes r_{n-k+1}(N), D_{21} = \overline{\mu} I_N,$
$D_{22} = D_{11}^{(0)} - \overline{\mu} I_N,$
$D_{23} = \lambda c_N(N) \otimes r_{n-k+2-N}(1),$
$D_{33} = \lambda E_{n-k+2-N} + \lambda c_(n-k+2-N) \otimes r_{(n-k+2-N)}(n-k+2-N) - \overline{\mu} I_{n-k+2-N},$
$D_{34} = [O_{n-k+2-N \times (N-1)} \quad \overline{\mu} I_{(n-k+2-N)}],$
$D_{44} = \lambda E_{n-k+1} + \lambda c_{n-k+1}(n-k+1) \otimes r_{n-k+1}(n-k+1) + \mu E'_{n-k+1},$
$D_{41} = \mu c_{n-k+1}(1) \otimes r_N(1).$

For $i \geq 0$ the transition from level i to $i+1$ is represented by the matrix

$$\mathbf{A}_0 = \begin{bmatrix} 0_{N \times N} & 0 & 0 & 0 \\ 0 & \overline{\lambda} I_N & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

For $i \geq 1$, the transition from level i to $i-1$ is represented by the matrix

$$\mathbf{A}_{2i} = \begin{bmatrix} 0 & i\theta I_N & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

3 Steady state analysis of the retrial model

3.1 Stability condition

For finding the stability condition for the system study, we apply Neuts-Rao truncation [2] by assuming $A_{1i} = A_{1M}$ and $A_{2i} = A_{2M}$ for all $i \geq M$. Then the generator matrix of the truncated system will look like:

$$Q = \begin{bmatrix} \mathbf{A}_{10} & \mathbf{A}_0 & & & & & \\ \mathbf{A}_{21} & \mathbf{A}_{11} & \mathbf{A}_0 & & & & \\ & \mathbf{A}_{22} & \mathbf{A}_{12} & \mathbf{A}_0 & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \\ & & & & \mathbf{A}_{2M} & \mathbf{A}_{1M} & \mathbf{A}_0 \\ & & & & & \mathbf{A}_{2M} & \mathbf{A}_{1M} & \mathbf{A}_0 \\ & & & & & & \cdot & \cdot \\ & & & & & & \cdot & \cdot \end{bmatrix}.$$

Define $\mathbf{A}_M = \mathbf{A}_0 + \mathbf{A}_{1M} + \mathbf{A}_{2M}$; then

$$A_M = \begin{bmatrix} D_{11}^{(M)} & D_{12}^{(M)} & 0 D_{14} & \\ D_{21} & \widetilde{D}_{22} & D_{23} & 0 \\ 0 & 0 & D_{33} & D_{34} \\ D_{41} & 0 & 0 & D_{44} \end{bmatrix},$$

where $D_{12}^{(M)} = (\overline{\lambda} + M\theta) I_N$, $\widetilde{D}_{22} = \lambda E_N - \mu I_N$.

Let

$\pi_M = (\pi_M(0), \pi_M(1), \tilde{\pi}_M(1), \pi_M(2))$, where
$\pi_M(0) = (\pi_M(0,0), \pi_M(0,1), \dots, \pi_M(0, N-1)),$
$\pi_M(1) = (\pi_M(1,0), \dots, \pi_M(1, N-1)),$
$\tilde{\pi}_M(1) = (\pi_M(1,N), \dots, \pi_M(1, n-k+1)),$
$\pi_M(2) = (\pi_M(2,1), \dots, \pi_M(2, n-k+1)).$

be the steady state vector of the generator matrix $\mathbf{A}_M$. Then the relation $\pi_M \mathbf{A}_M = 0$ gives rise to the following equations:

$$\pi_M(0) D_{11}^{(M)} + \pi_M(1) D_{21} + \pi_M(2) D_{41} = 0, \tag{1}$$

$$\pi_M(0)D_{12}^{(M)} + \pi_M(1)\widetilde{D}_{22} = 0, \tag{2}$$

$$\pi_M(1)D_{23} + \tilde{\pi}_M(1)D_{33} = 0, \tag{3}$$

$$\pi_M(0)D_{14} + \tilde{\pi}_M(1)D_{34} + \pi_M(2)D_{44} = 0. \tag{4}$$

It follows from equation (4) that

$$\pi_M(2) = -\pi_M(0)D_{14}(D_{44})^{-1} - \tilde{\pi}_M(1)D_{34}(D_{44})^{-1}. \tag{5}$$

Substituting for $\pi_M(2)$ in equation (1), we get

$$\pi_M(0)D_{11}^{(M)} + \pi_M(1)D_{21} - \pi_M(0)D_{14}(D_{44})^{-1}D_{41} - \tilde{\pi}_M(1)D_{34}(D_{44})^{-1}D_{41} = 0. \tag{6}$$

It follows from equation (3) that

$$\tilde{\pi}_M(1) = -\pi_M(1)D_{23}(D_{33}^{-1}). \tag{7}$$

Substituting for $\tilde{\pi}_M(1)$ in equation (6), we get

$$\begin{aligned}&\pi_M(0)D_{11}^{(M)} + \pi_M(1)D_{21} - \pi_M(0)D_{14}(D_{44})^{-1}D_{41}\\ &+\pi_M(1)D_{23}(D_{33})^{-1}D_{34}(D_{44})^{-1}D_{41} = 0.\end{aligned} \tag{8}$$

We notice that the first column of the matrix D_{41} is $-D_{44}e$ and its all other columns are zero columns. Hence the first column of the matrix $(D_{44})^{-1}D_{41}$ is $-e$ and its all other columns are zero columns. This implies that the first column of the matrix $-D_{14}(D_{44})^{-1}D_{41}$ is $D_{14}e = \lambda c_N(N)$ and its all other columns are zero columns. In other words $-D_{14}(D_{44})^{-1}D_{41} = \lambda c_N(N) \otimes r_N(1)$. Also, the first column of the matrix $D_{34}(D_{44})^{-1}D_{41}$ is $-D_{34}e$ and its all other columns are zero columns. Since $-D_{34}e = D_{33}e$, the first column of the matrix $(D_{33})^{-1}D_{34}(D_{44})^{-1}D_{41}$ is e and its all other columns are zero columns. Hence it follows that $D_{23}(D_{33})^{-1}D_{34}(D_{44})^{-1}D_{41}$ is $D_{23}e = \lambda c_N(N) \otimes r_N(1)$. Thus equation (8) becomes

$$\pi_M(0)(D_{11}^{(M)} + \lambda c_N(N) \otimes r_N(1)) + \pi_M(1)(D_{21} + \lambda c_N(N) \otimes r_N(1)) = 0. \tag{9}$$

Adding equations (2) and (9), we get

$$\pi_M(0)(D_{11}^{(M)} + \lambda c_N(N) \otimes r_N(1) + D_{12}^{(M)}) + \pi_M(1)(\widetilde{D}_{22} + D_{21} + \lambda c_N(N) \otimes r_N(1)) = 0. \tag{10}$$

Since $D_{11}^{(M)} + D_{12}^{(M)} = \widetilde{D}_{22} + D_{21} = \lambda E_N$, equation (10) reduces to

$$(\pi_M(0) + \pi_M(1))(\lambda E_N + \lambda c_N(N) \otimes r_N(1)) = 0. \tag{11}$$

which implies that $\pi_M(0) + \pi_M(1)$ is a constant multiple of the steady state vector $\frac{1}{N}e'_N$ of the generator matrix $\lambda E_N + \lambda c_N(N) \otimes r_N(1)$ and hence,

$$\pi_M(0) + \pi_M(1) = v\frac{1}{N}e'_N. \tag{12}$$

where v is a constant. Equation (2) implies that

$$\pi_M(0) = -\pi_M(1)\widetilde{D}_{22}(D_{12}^{(M)})^{-1}. \tag{13}$$

Since $(D_{12}^{(M)})^{-1} = \frac{1}{(\bar{\lambda}+M\theta)}I_N$, (13) gives

$$\lim_{M\to\infty}\pi_M(0) = 0. \tag{14}$$

and hence

$$\lim_{M\to\infty}\pi_M(1) = v\frac{1}{N}e'_N, \tag{15}$$

and

$$\lim_{M\to\infty}\bar{\lambda}\pi_M(1)e = v\bar{\lambda}. \tag{16}$$

Again from (13),

$$M\theta\pi_M(0)e = -M\theta\pi_M(1)\widetilde{D}_{22}(D_{12}^{(M)})^{-1}e. \tag{17}$$

Since, $\lim_{M\to\infty}M\theta(D_{12}^{(M)})^{-1}e = \lim_{M\to\infty}\frac{M\theta}{(\bar{\lambda}+M\theta)}e_N = e_N$, (17) implies that

$$\lim_{M\to\infty} M\theta\pi_M(0)e = -\lim_{M\to\infty}\pi_M(1)\widetilde{D}_{22}e$$
$$= -\nu\frac{1}{N}e'_N(-\lambda c_N(N) - \bar{\mu}e)$$
$$= \nu(\frac{\lambda}{N} + \bar{\mu}). \quad (18)$$

The truncated system is stable if and only if

$$\pi_M A_0 e < \pi_M A_{2M} e, \quad (19)$$
$$\pi_M A_0 e = \bar{\lambda}\pi_M(1)e, \quad (20)$$
$$\pi_M A_{2M} e = M\theta\pi_M(0)e. \quad (21)$$

Making use of equations (16), (18), (20) and (21), the stability condition for the truncated system as $M \to \infty$ is given by

$$\nu\bar{\lambda} < \nu(\frac{\lambda}{N} + \bar{\mu}) ,$$

which can be re-arranged as

$$\frac{\bar{\lambda}}{\bar{\mu}}\frac{N\bar{\mu}}{(\lambda + N\bar{\mu})} < 1 .$$

Hence, we conclude that the retrial problem has the same stability condition as the queueing problem, which was obtained in section 3.1 of part I.

3.2 Computation of Steady State Vector

We find the steady state vector of $\{X(t), t \geq 0\}$, by approximating it with the steady state vector of the truncated system. Let $\pi = (\pi_0, \pi_1, \pi_2, \dots)$ where each $\pi_i = (\pi_i(0,0), \pi_i(0,1), \dots, \pi_i(0, N-1), \pi_i(1,1), \dots, \pi_i(1, n-k+1), \pi_i(2,0), \pi_i(2,1), \dots, \pi_i(2, n-k+1))$ be the steady state vector of the Markov chain $\{X(t), t \geq 0\}$.

Suppose $A_{1i} = A_{1M}$ and $A_{2i} = A_{2M}$ for all $i \geq M$. Let $\pi_{M+r} = \pi_{M-1}R^{r+1}, r \geq 0$, then from $\pi Q = 0$ we get

$$\pi_{M-1}A_0 + \pi_M A_{1M} + \pi_{M+1}A_{2M} = 0,$$
$$\pi_{M-1}A_0 + \pi_{M-1}RA_{1M} + \pi_{M-1}R^2 A_{2M} = 0,$$
$$\pi_{M-1}(A_0 + RA_{1M} + R^2 A_{2M}) = 0.$$

Choose R such that $A_0 + RA_{1M} + R^2 A_{2M} = 0$. We call this R as R_M. Also we have

$$\pi_{M-2}A_0 + \pi_{M-1}A_{1M-1} + \pi_M A_{2M} = 0,$$
$$\pi_{M-2}A_0 + \pi_{M-1}(A_{1M-1} + R_M A_{2M}) = 0,$$
$$\pi_{M-1} = -\pi_{M-2}A_0(A_{1M-1} + R_M A_{2M})^{-1}$$
$$= \pi_{M-2}R_{M-1} .$$

where

$$R_{M-1} = -A_0(A_{1M-1} + R_M A_{2M}) .$$

Next,

$$\pi_{M-3}A_0 + \pi_{M-2}A_{1M-2} + \pi_{M-1}A_{2M-1} = 0,$$
$$\pi_{M-3}A_0 + \pi_{M-2}(A_{1M-2} + \pi_{M-1}A_{2M-1}) = 0,$$
$$\pi_{M-2} = -\pi_{M-3}A_0(A_{1M-2} + R_{M-1}(A_{2M-1})^{-1}$$
$$= \pi_{M-3}R_{M-2}.$$

Where

$$R_{M-2} = -A_0(A_{1M-2} + R_{M-1}A_{2M-1})^{-1}.$$

and so on.

Finally

$$\pi_0 A_{10} + \pi_1 A_{21} = 0$$

becomes

$$\pi_0(A_{10} + R_1 A_{21}) = 0.$$

For finding π, first we take π_0 as the steady state vector of $A_{10} + R_1 A_{21}$.Then π_i for $i \geq 1$ can be found using the recursive formula, $\pi_i = \pi_{i-1}R_i$ for $1 \leq i \leq M$.

Now the steady state probability distribution of the truncated system is obtained by dividing each π_i with the normalizing constant

$$[\pi_0 + \pi_1 + \cdots]e = [\pi_0 + \pi_1 + \cdots + \pi_{N-2} + \pi_{M-1}(I - R_M)^{-1}]e.$$

3.3 Computation of the matrix R_M

Consider the matrix quadratic equation

$$A_0 + R_M A_{1M} + R_M^2 A_{2M} = 0, \tag{22}$$

which implies

$$R_M = -A_0 (A_{1M} + R_M A_{2M})^{-1}. \tag{23}$$

The structure of the A_0 matrix implies that the matrix R_M has the form:

$$R_M = \begin{bmatrix} 0 & 0 & 0 & 0 \\ R_{M1} & R_{M2} & R_{M3} & R_{M4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{24}$$

In other words, the non-zero rows of the R_M matrix are those, where the A_0 matrix has at least one nonzero entry. Now,

$$R_M^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ R_{M2}R_{M1} & R_{M2}^2 & R_{M2}R_{M3} & R_{M2}R_{M4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{25}$$

Equation (22) gives rise to the following equations:

$$R_{M1} D_{11}^{(M)} + R_{M2} D_{21} + R_{M4} D_{41} = 0, \tag{26}$$

$$R_{M2} R_{M1} M\theta I_N + R_{M1} D_{12} + R_{M2} D_{22} + \bar{\lambda} I_N = 0, \tag{27}$$

$$R_{M2} D_{23} + R_{M3} D_{33} = 0, \tag{28}$$

$$R_{M1} D_{14} + R_{M3} D_{34} + R_{M4} D_{44} = 0. \tag{29}$$

From equation (28), we can write

$$R_{M3} = -R_{M2} D_{23} (D_{23})^{-1}. \tag{30}$$

From equation(29), we can write

$$R_{M4} = -R_{M1} D_{14} (D_{44})^{-1} - R_{M3} D_{34} (D_{44})^{-1}. \tag{31}$$

Substituting for R_{M3} from (30) in equation (31), we get

$$R_{M4} = -R_{M1} D_{14} (D_{44})^{-1} + R_{M2} D_{23} (D_{33})^{-1} D_{34} (D_{44})^{-1}. \tag{32}$$

Substituting for R_{M4} from (32) in equation (26), we get

$$\begin{aligned} R_{M1} D_{11}^{(M)} \quad & + R_{M2} D_{21} - R_{M1} D_{14} (D_{44})^{-1} D_{41} \\ & + R_{M2} D_{23} (D_{33})^{-1} D_{34} (D_{44})^{-1} D_{41} = 0. \end{aligned} \tag{33}$$

Using the same reasoning, that lead us to equation (9), equation (33) becomes

$$R_{M1} (D_{11}^{(M)} + \lambda c_N(N) \otimes r_N(1)) + R_{M2} (D_{21} + \lambda c_n(N) \otimes r_N(1)) = 0. \tag{34}$$

From (34), it follows that

$$R_{M1} = -R_{M2} (D_{21} \lambda c_N(N) \otimes r_N(1)) (D_{11}^{(M)} + \lambda c_N(N) \otimes r_N(1))^{-1}. \tag{35}$$

Substituting for R_{M1} in (27), we get

$$\begin{aligned} & -R_{M2}^2 (D_{21} + \lambda c_N(N) \otimes r_N(1)) (D_{11}^{(M)} + \lambda c_N(N) \otimes r_N(1))^{-1} M\theta I_N \\ & -R_{M2} (D_{21} + \lambda c_N(N) \otimes r_N(1)) (D_{11}^{(M)} + \lambda c_N(N) \otimes r_N(1))^{-1} D_{12} \\ & + R_{M2} D_{22} + \bar{\lambda} I_N = 0. \end{aligned}$$

That is

$$\begin{aligned} & R_{M2}^2 \big(-(D_{21} + \lambda c_N(N) \otimes r_N(1)) (D_{11}^{(M)} + \lambda c_N(N) \otimes r_N(1))^{-1} M\theta I_N \big) \\ & \quad + R_{M2} \big(-(D_{21} + \lambda c_N(N) \otimes r_N(1)) (D_{11}^{(M)} + \lambda c_N(N) \otimes r_N(1))^{-1} D_{12} + D_{22} \big) \\ & \qquad + \bar{\lambda} I_N = 0. \end{aligned} \tag{36}$$

We notice that $-(D_{11}^{(M)} + \lambda c_N(N) \otimes r_N(1)) e = (D_{12} + M\theta I_N) e$. and therefore

$$\begin{aligned} & -(D_{21} + \lambda c_N(N) \otimes r_N(1)) (D_{11}^{(M)} + \lambda c_N(N) \otimes r_N(1))^{-1} (D_{12} + M\theta I_N) e \\ & = (D_{21} \lambda c_N(N) \otimes r_N(1)) e. \end{aligned} \tag{37}$$

Also,

$$D_{22}e + (D_{21} + \lambda c_N(N) \otimes r_N(1))e + \bar{\lambda}e = 0.$$

and hence

$$\begin{aligned}&\left(-(D_{21} + \lambda c_N(N) \otimes r_N(1))(D_{11}^{(M)} + \lambda c_N(N) \otimes r_N(1))^{-1} M\theta I_N\right)e + \\ &\left(-(D_{21} + \lambda c_N(N) \otimes r_N(1))(D_{11}^{(M)} + \lambda c_N(N) \otimes r_N(1))^{-1} D_{12} + D_{22}\right)e \\ &+\bar{\lambda}e = 0.\end{aligned} \tag{38}$$

Equation (38) shows that the matrix R_{M2} is the minimal non-negative solution of the matrix quadratic equation (36). Once obtaining R_{M2}, the matrices R_{M1}, R_{M3}, and R_{M4} can be found using equations (35), (30) and (31) respectively. Hence the matrix R_M can be found. From the form of the matrix $D_{11}^{(M)}$, we notice that,

$$\begin{aligned}&-\left(D_{11}^{(M)} + \lambda c_N(N) \otimes r_N(1)\right) \\ &\quad = M\theta I_N - \left(\lambda E_N - \bar{\lambda} I_N + \lambda c_N(N) \otimes r_N(1)\right) \\ &\quad = M\theta\left(I_N - \frac{1}{M\theta}(\lambda E_N - \bar{\lambda} I_N + \lambda c_N(N) \otimes r_N(1))\right).\end{aligned}$$

and hence

$$\begin{aligned}&-\left(D_{11}^{(M)} + \lambda c_N(N) \otimes r_N(1)\right)^{-1} \\ &\quad = \frac{1}{M\theta}\left(I_N - \frac{1}{M\theta}(\lambda E_N - \bar{\lambda} I_N + \lambda c_N(N) \otimes r_N(1))\right)^{-1} \\ &\quad = \frac{1}{M\theta}\left(I_N + \frac{1}{M\theta}(\lambda E_N - \bar{\lambda} I_N + \lambda c_N(N) \otimes r_N(1)) + \cdots\right).\end{aligned}$$

Therefore

$$\lim_{M\to\infty}\left(-(D_{11}^{(M)} + \lambda c_N(N) \otimes r_N(1))^{-1} M\theta I_N\right) = I_N.$$

and

$$\lim_{M\to\infty}\left(-(D_{11}^{(M)} + \lambda c_N(N) \otimes r_N(1))^{-1} D_{12}\right) = 0.$$

Hence as $M \to \infty$ equation (36) becomes

$$R_{M2}^2(D_{21} + \lambda c_N(N) \otimes r_N(1)) + R_{M2}D_{22} + \bar{\lambda} I_N = 0. \tag{39}$$

We identify $D_{21} + \lambda c_N(N) \otimes r_N(1)$ as $\tilde{A}_2$, D_{22} as $\tilde{A}_1$ and $\bar{\lambda} I_N$ as $\tilde{A}_0$, which were defined in section 3.2 of part I. Hence equation (39) is the same as equation (24) of section 3.2 of part I. That is the matrix R_M tends to the matrix R, the minimal non-negative solution of (24) of section 3.2 of part I, as $M \to \infty$. This fact can be utilized in determining the truncation level M.

4 System Performance Measures

The following system performance measures were calculated numerically.

1. Fraction of time the system is down,
$$P_{down} = \sum_{j_1=0}^{\infty}\left(\pi_{j_1}(1, n-k+1) + \pi_{j_1}(2, n-k+1)\right).$$

2. System reliability, $P_{rel} = 1 - P_{down}$
$$= 1 - \sum_{j_1=0}^{\infty}\left(\pi_{j_1}(1, n-k+1) + \pi_{j_1}(2, n-k+1)\right).$$

3. Average number of external customers in the orbit,
$$N_{orbit} = \sum_{j_1=0}^{\infty} j_1\left(\sum_{j_3=1}^{n-k+1}\pi_{j_1}(1, j_3)\right) + \sum_{j_1=0}^{\infty} j_1\left(\sum_{j_3=0}^{n-k+1}\pi_{j_1}(2, j_3)\right).$$

4. Average number of failed components in the system,
$$N_{fail} = \sum_{j_3=0}^{n-k+1} j_3\left(\sum_{j_1=0}^{\infty}\pi_{j_1}(0, j_3)\right) + \sum_{j_3=1}^{n-k+1}\left(\sum_{j_1=0}^{\infty}\pi_{j_1}(2, j_3)\right).$$

5. Average number of failed components waiting when server is busy with external customers
$$N_{failextb} = \sum_{j_3=0}^{n-k+1} j_3\left(\sum_{j_1=1}^{\infty}\pi_{j_1}(0, j_3)\right).$$

6. Expected rate at which external customers joining the system
$$E_{extrate} = \bar{\lambda}\left\{\sum_{j_1=1}^{\infty}\left(\sum_{j_3=0}^{n-k+1}\pi_{j_1}(0, j_3)\right) + \sum_{j_3=0}^{N-1}\pi_0(0, j_3)\right\}.$$

7. Expected number of external customers on its arrival gets service directly,
$$E_{extdirect} = \sum_{j_3=0}^{N-1} \pi_0(0, j_3).$$

8. Fraction of time server is busy with external customers,
$$P_{extbusy} = \sum_{j_1=1}^{\infty} \left(\sum_{j_3=0}^{n-k+1} \pi_{j_1}(0, j_3)\right).$$

9. Probability that the server is found idle,
$$P_{idle} = \sum_{j_3=0}^{N-1} \pi_0(0, j_3) = N\pi_0(0,0).$$

10. Probability that the server is found busy,
$$P_{busy} = 1 - \sum_{j_3=0}^{N-1} \pi_0(0, j_3) = 1 - N\pi_0(0,0).$$

11. Expected loss rate of external customers
$$\theta_4 = \bar{\lambda}\left\{\sum_{j_1=0}^{\infty} \left(\sum_{j_3=1}^{n-k+1} \pi_{j_1}(1, j_3)\right) + \sum_{j_1=1}^{\infty} \left(\sum_{j_3=N}^{n-k+1} \pi_{j_1}(0, j_3)\right)\right\}.$$

12. Expected service completion rate of external customers,
$$\theta_5 = \bar{\mu} \sum_{j_1=0}^{\infty} \left(\sum_{j_3=0}^{n-k+1} \pi_{j_1}(0, j_3)\right).$$

13. Expected number of external customers when server is busy with external customers
$$\theta_6 = \sum_{j_1=0}^{\infty} j_1\left(\sum_{j_3=0}^{n-k+1} \pi_{j_1}(0, j_3)\right).$$

14. Expected successful retrial rate
$$\theta_7 = \theta \cdot \sum_{j_1=1} \left(\sum_{j_3=0}^{N-1} \pi_{J_1}(0, j_3)\right).$$

5 Numerical study of the performance of the system

5.1 The effect of N policy on the server busy probability

A comparison of Table 1 of part I, which report the behaviour of server busy probability with variation in the N-policy level, with that of part II shows that the models described in section 2 of part I and its variant where external customers are sent to the orbit, which was described in section 2 of part II have similar behaviour as far as the server busy probability is considered. Comparison of Table 3 of part I, which report the variation in the fraction of time the server remains busy with external customers with increase in N, with table 2 of part II also points to similar behaviour for both models. Table 4 of part I and table 3 of part II indicate that the two models have similar reliability.

5.2 Cost Analysis

As in the case of the queueing model discussed in section 2 of part I, we analyzed a cost function for the retrial model for finding an optimal value for the N-policy level. For defining the cost function, let C_1 be the cost per unit time incurred if the system is down, C_2 be the holding cost per unit time per external customer in the orbit, C_3 is the cost incurred for starting failed components service after accumulation of N of them, C_4 be the cost due to loss of 1 external customer, C_5 be the holding cost per unit time of one failed component, C_6 be the cost per unit time if the server is idle. We define the cost function as:

$$Expected\ cost\ per\ unit\ time = C_1 \cdot P_{down} + C_2 \cdot N_{orbit} + C_4 \cdot \theta_4 + C_5 \cdot N_{fail} + \frac{C_3}{E_{\bar{H}}} + C_6 \cdot Pidle.$$

where $E_{\bar{H}}$ is found exactly in the same lines as in section 4.1 of part I.

Our numerical study, as presented in Table 4, show that an optimal value for N can be found for different parameter choices and also that this optimal value happens to be a much smaller value like $N = 6$. This shows the care needed in selecting the N-policy level.

6 Conclusion

We analyzed a k-out-of-n system where the server renders service to external customers also. In the case of a system where a minimum number of working components is necessary for its operation, the service to external customer should be carefully managed so that it does not affect the system reliability much. Krishnamoorthy et al. [1] managed to do that by introducing an N-policy , in which the ongoing service of an external customer is preempted at the moment when N failed components have accumulated for repair. Differing from Krishnamoorthy et al.[1], here we considered a non-preemptive service for external customers thereby making their service more attractive. We analyzed two models: one in which the external customers joins a queue and another in which they moving to an orbit of infinite capacity. Our numerical study showed that rendering non-preemptive service to external customers has not affected the system reliability much, thereby re-asserted that the same could be an effective idea for utilizing the server idle time and there by earning more profit to the system. Analysis of a cost function has helped us in finding an optimal value for the N-policy level.

Table 1: Variation in the server busy probability when external customers are allowed $k = 20, \lambda = 4, \bar{\lambda} = 3.2, \mu = 5.5, \bar{\mu} = 8, \theta = 5$

N	n=45	n=50	n=55	n=60
1	0.82349	0.82352	0.82353	0.82353
3	0.82995	0.82999	0.83	0.83
5	0.83222	0.83228	0.83229	0.83229
7	0.83328	0.83336	0.83338	0.83338
9	0.83385	0.83398	0.83401	0.83401
11	0.83417	0.83437	0.83442	0.83442
13	0.8343	0.83463	0.8347	0.83471
15	0.83424	0.83479	0.8349	0.83493
17	0.83394	0.83486	0.83505	0.83509
19	0.83325	0.83483	0.83515	0.83521
21	0.83192	0.83465	0.8352	0.83531
23	0.82945	0.83424	0.83518	0.83538

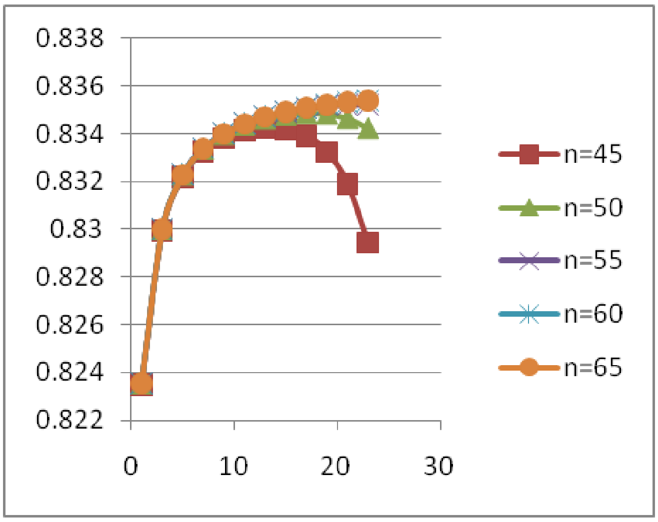

Table 2: Effect of the N-policy level on the fraction of time server is busy with external customers
$k = 20, \lambda = 4, \bar{\lambda} = 3.2, \mu = 3.2, \bar{\mu} = 8, \theta = 5$

N	n=40	n=45	n=50	n=55	n=60
1	0.09635	0.09628	0.09626	0.09626	0.09626
3	0.10287	0.10276	0.10273	0.10273	0.10273
5	0.10523	0.10506	0.10503	0.10502	0.10502
7	0.10644	0.10618	0.10612	0.10611	0.10611
9	0.10725	0.10685	0.10676	0.10675	0.10674
11	0.10798	0.10732	0.10719	0.10716	0.10716
13	0.10879	0.10772	0.1075	0.10746	0.10745
15	0.10991	0.10811	0.10775	0.10768	0.10766
17	0.11461	0.10858	0.10798	0.10786	0.10783
19	0.11983	0.10925	0.10822	0.10801	0.10797
21		0.1103	0.10851	0.10815	0.10808
23		0.11208	0.10893	0.10831	0.10818
25		0.11522	0.10959	0.1085	0.10828
27			0.1107	0.10877	0.10839
29			0.11265	0.1092	0.10852
31			0.11615	0.10991	0.1087
33				0.11116	0.10898
35				0.1134	0.10945
37					0.11026
39					0.11172
41					0.11435

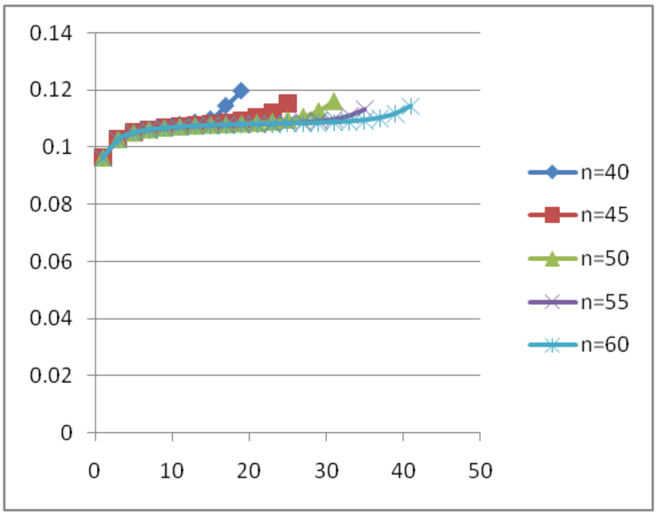

Table 3: Variation in the system reliability with increase in N $k = 20, \lambda = 4, \bar{\lambda} = 3.2, \mu = 5.5, \bar{\mu} = 8,$ $\theta = 5$

N	n=40	n=45	n=50	n=55	n=60
1	0.99963	0.99993	0.99998	1	1
3	0.99948	0.99989	0.99998	1	1
5	0.99924	0.99985	0.99997	0.99999	1
7	0.99885	0.99977	0.99995	0.99999	1
9	0.9982	0.99964	0.99993	0.99998	1
11	0.99712	0.99942	0.99988	0.99998	1
13	0.9953	0.99905	0.99981	0.99996	0.99999
15	0.99217	0.99843	0.99968	0.99994	0.99999
17	0.9769	0.99736	0.99947	0.99989	0.99998
19		0.9955	0.99909	0.99982	0.99996
21		0.99223	0.99844	0.99968	0.99994
23		0.98638	0.9973	0.99945	0.99989
25		0.97578	0.99528	0.99905	0.99981
27			0.99165	0.99833	0.99966
29			0.98509	0.99705	0.9994
31			0.97315	0.99475	0.99894
33				0.99058	0.99812
35				0.98297	0.99663
37					0.99393
39					0.989

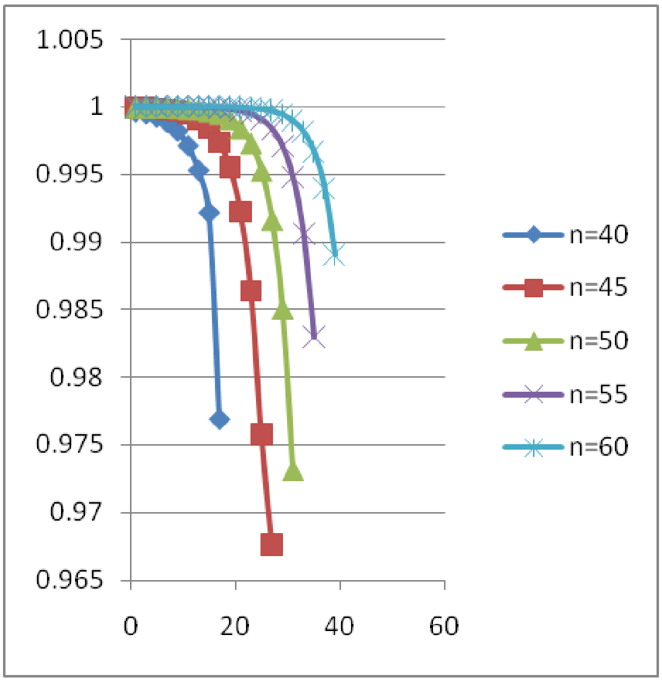

Table 4: Analysis of a cost function $n = 50, \bar{\lambda} = 3.2, \mu = 5.5, \bar{\mu} = 8, C_1 = 2000, C_2 = 1000, C_3 = 800, C_4 = 1000, C_5 = 10, C_6 = 200, \theta = 5$

N	$\lambda = 4$	$\lambda = 4.5$	$\lambda = 5$
1	6235.23047	6440.20947	6671.65918
2	6137.3877	6343.84668	6576.75928
3	6109.98389	6317.7207	6551.88965
4	6102.75391	6311.82178	6547.30566
5	6102.27734	6312.30322	6548.71436
6	6104.71094	6315.28613	6552.17676
7	6108.70947	6319.521	6556.51709
8	6113.67188	6324.50439	6561.33057
9	6119.2749	6329.98047	6566.44873
10	6125.32666	6335.80176	6571.76465
11	6131.69824	6341.87891	6577.22021
12	6138.31006	6348.14307	6582.78711
13	6145.10449	6354.55762	6588.43018
14	6152.04492	6361.09961	6594.13086
15	6159.104	6367.74854	6599.88428
17	6173.53564	6381.33594	6611.51611
19	6188.38672	6395.33936	6623.31689
21	6203.78809	6409.88037	6635.37354
23	6220.13477	6417.44531	6647.98535
25	6238.73828	6443.09375	6662.8042
27	6266.49854	6471.54688	6690.0752
29	6356.05566	6571.71631	6799.88672
31	7073.24658	7340.11523	7618.78223

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