Heuristic Principles of Phase Merging in Reliability Analysis

Volodymyr S. Koroliuk, Dmitri Koroliouk
•
Institute of Mathematics,
Institute of Telecommunications & Global Information Space
National Academy of Sciences of Ukraine
mailto:vskorol@yahoo.comvskorol@yahoo.com, dimitri.koroliouk@ukr.net

Abstract

B.V. Gnedenko was the founder of reliability analysis for stochastic systems. His works [1]-[2] have inspirited, in reliability theory, the development of analytical methods of phase state merging principles for Markov and semi-Markov processes.

Keywords: duplicated system, phase merging principles, potential matrix.

1 Introduction

The fundamental works of Boris V. Gnedenko in the reliability analysis for stochastic systems [1]-[3] laid the foundation in many areas of specific research.

In particular, there where developed the methods of phase space merging in reliability theory for Markov and semi-Markov processes with the corresponding heuristic approach [4, 5]. A surprising property of such heuristic principles is that any results obtained with their use can be justified rigorously by mean of the phase merging algorithms [6]. The stationary phase merging techniques represent a particular cluster analysis, based on asymptotic properties of semi-Markov systems and is useful for simplification of reliability analysis, as shown for a duplicated renewal system.

2 The duplicated renewal system

B.V. Gnedenko in [1, 2] has studied the reliability problem for the stochastic systems with two identical working devices and one repairing facility.

The description of such a duplicated renewal system is determined by the working times α_k, $k = 1{,}2$ of devices with an arbitrary distribution function $F_k(t) = P(\alpha_k \le t)$, and by the repairing times β_k, $k = 1{,}2$ with the distribution function $G_k(t) = P(\beta_k \le t)$.

The working times of the system up to the first failure $\tau_k, k = 1{,}2$ are dependent on the type of the initial working components.

The Laplace transform functions of the working times of the system, that are

$$\varphi_k(s) := Ee^{-s\tau_k} = \int_0^\infty e^{-st} d\Phi_k(t) \ , \ = 1{,}2$$

may be obtained by using the stochastic relations (see [1, 2] and also [5])

$$\begin{aligned} \tau_1 &\doteq \alpha_1 + I(\alpha_1 \ge \beta_2)\tau_2, \\ \tau_2 &\doteq \alpha_2 + I(\alpha_2 \ge \beta_1)\tau_1. \end{aligned} \tag{1}$$

The equality $\doteq$ means that the left and the right parts are identically distributed.

The stochastic relations (1) mean that during the working times α_k, $k = 1,2$, the failure of the system can occur under the condition $\alpha_k < \beta_{k'}$, $k = 1,2$, $k' = 2,1$ with probabilities

$$q_k = \int_0^\infty \overline{G}_{k'}(t) dF_k(t) \ , \quad k = 1,2 \ , \quad k' = 2,1.$$

So, the relations (1) imply the following system of algebraic equations:

$$Q(s)\varphi(s) = \psi(s), \tag{2}$$

where $\varphi(s) = (\varphi_1(s), \varphi_2(s))$, $\psi(s) = (\psi_1(s), \psi_2(s))$,

$$\psi_1(s) := \int_0^\infty e^{-st}\overline{G}_1(t)dF_2(t) \ , \quad \psi_2(s) := \int_0^\infty e^{-st}\overline{G}_2(t)dF_1(t). \tag{3}$$

The matrix Q is defined as follows:

$$Q(s) := \begin{bmatrix} 1 & -g_1(s) \\ -g_2(s) & 1 \end{bmatrix}, \tag{4}$$

where

$$g_1(s) := \int_0^\infty e^{-st} G_2(t) dF_1(t), g_2(s) := \int_0^\infty e^{-st} G_1(t) dF_2(t). \tag{5}$$

3 The duplicated renewal system in the series scheme

In order to simplify the duplicated renewal system, described by the linear algebraic equations (2)-(5), let's introduce the series scheme with a small series parameter $\varepsilon \to 0$ ($\varepsilon > 0$), under the following asymptotical conditions

$$C1: 1cm\psi_k^\varepsilon(s) = \varepsilon \int_0^\infty e^{-\varepsilon st}\overline{G}_{k'}(t)dF_k(t) = \varepsilon q_k + o(\varepsilon) \ , \quad q_k := P\{\beta_{k'} > \alpha_k\} \ , \quad k = 1,2 \ ; \quad k' = 2,1.$$

$$C2: 1cm1 - f_k^\varepsilon(s) = \varepsilon s \int_0^\infty e^{-\varepsilon st}\overline{F}_k(t)dt = \varepsilon s a_k + o(\varepsilon) \ , \quad a_k := E\alpha_k \ , \quad k = 1,2.$$

The asymptotical conditions C1 – C2 mean that the probabilities of falure $q_k^\varepsilon := P\{\beta_{k'}^\varepsilon > \alpha_k^\varepsilon\}$ tend to zero together with the mean values $a_k^\varepsilon = E\alpha_k^\varepsilon$ such that the ratio $q_k^\varepsilon \lambda_k^\varepsilon = q_k^\varepsilon / a_k^\varepsilon$ tend to finite values $\Lambda_k = q_k \lambda_k$, $k = 1,2$.

Then the matrix of system (2) in the series scheme has the following asymptotic representation:

$$Q^\varepsilon(s) = Q_0 + \varepsilon Q_1(s) + o(\varepsilon), \tag{6}$$

where

$$Q_0 = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \ , \quad Q_1 = \begin{bmatrix} 0 & q_1 + s a_1 \\ q_2 + s a_2 & 0 \end{bmatrix}. \tag{7}$$

The singularity of the matrix Q_0 ($\det Q_0 = 0$) means that the phase merging algorithm [5] may be applied to solve the singularly perturbed (truncated!) equation

$$[Q_0 + \varepsilon Q_1(s)]\varphi^\varepsilon(s) = \psi^\varepsilon(s). \tag{8}$$

According to the phase merging principles (see [4, 5, 6]), the average (limit) result takes place in the following form:

$$\varphi_1^0(s) = \varphi_2^0(s) = q/(q + sa) \ , \quad q = (q_1 + q_2)/2 \ , \quad a = (a_1 + a_2)/2. \tag{9}$$

The times-to-failure limits of the duplicated renewal systems, under the asymptotical assumptions C1-C2, have identical exponential distribution

$$\lim_{\varepsilon \to 0} P\{\tau_k^\varepsilon > t\} = e^{-\Lambda t} \ , \quad \Lambda = q/a. \tag{10}$$

Remark 1. Let us introduce the mean intensity of the working time $\lambda := 1/a$. Then the intensity of the failure time is $\Lambda = q\lambda$. So the formula (10) represents the failure time of the duplicated system with the failure probability q and with intensity λ.

4 Heuristic principles of the phase merging

The phase merging algorithms described in [5, 6] may be formulated as some *heuristic phase merging principles* in the reliability analysis of redundant renewal stochastic systems with N elements (see [5], Ch.3).

1) The lack of memory. The common working time of a system till the instant of failure τ is determined by exponential distribution:

$$P(\tau > t) = e^{-\Lambda t} t \geq 0. \tag{11}$$

2) The superposition of failures. The intensity of the system failure is determined by the sum of intensities of system failures in every renewal state:

$$\Lambda = \sum_{k=1}^{N} \Lambda_k \ , \quad \tau = \min_{1 \leq n \leq N} \tau_n. \tag{12}$$

According to the Principle 2), the failure of the system can occur in every renewal state as was explained in Section 3 for duplicated systems.

3) The independence of the elements failures. The system failures for every element are determined by the failure rule as follows:

$$1/E\tau_k = \Lambda_k = q_k \lambda_k, \tag{13}$$

where q_k is the probability of failure for k-th state and λ_k is the stationary intensity of working time for k-th state.

The heuristic principles action can be illustrated by analysis of *the duplicated renewal system.* Namely, two working devices are described by independent working-repearing processes with given distribution functions of the working times α_k and the repairing times β_k

$$F_k(t) = P(\alpha_k \leq t) \ , \quad G_k(t) = P(\beta_k \leq t) \ , = 1{,}2. \tag{14}$$

Such a classical example of the system is usually called "two lifts system" [9, 10].

The heuristic principles of the phase merging technique are based on use of the limit renewal theorem [7] for the stationary residual time α^* expressed as:

$$P(\alpha^* \leq t) = \lambda \int_0^t \overline{F}(s) ds \ , \quad \lambda = 1/E\alpha.$$

According to heuristic principles, the described above the failure intensity of two lift system is the following:

$$\Lambda = q_1\lambda_1 + q_2\lambda_2 \ , \quad \lambda_k = 1/E\alpha_k \ , \quad k = 1,2. \tag{15}$$

The failure probabilities $q_k, k = 1,2$ are determined as follows:

$$q_1 = P(\alpha_2^* > \beta_1) \ , \quad q_2 = P(\alpha_1^* > \beta_2). \tag{16}$$

Here the stationary remaining working times α_k^*, $k = 1,2$, have the following distribution functions:

$$P(\alpha_k^* \le t) = \lambda_k \int_0^t \overline{F}_k(s)ds \ , \quad k = 1,2. \tag{17}$$

Under the natural *assumption of the repairing relative brevity*:

$$E\beta_k \ll E\alpha_k \ , \quad k = 1,2, \tag{18}$$

the intensity of the system failure for the duplicated renewal system may be estimated as follows:

$$\Lambda \simeq [E[\alpha_2 \wedge \beta_1] + E[\alpha_1 \wedge \beta_2]]/E\alpha_1 E\alpha_2. \tag{19}$$

The phase merging algorithms in [5] are the basis to verify the heuristic phase merging principles.

5 The duplicated renewal system without failure

The duplicated renewal system without failure ($\beta_k = 0, k = 1,2$) is described by a superposition of two renewal processes given by sums

$$S_n^{\pm} = \sum_{r=1}^{n} \alpha_r^{\pm}, \tag{20}$$

of jointly independent and identically distributed (by $r \ge 1$) random variables $\alpha_r^{\pm}$, $r \ge 1$. For simplicity, we denote the working times α_1 and α_2 as α^+ and α^-, correspondingly.

The duplicated renewal system without failure and with working times α_k^+, α_k^-, $k \ge 0$, means that the working device substitution is accompanied by its instantaneous repairing.

The phase merging principles provide the base model of the duplicated renewal system without failure as a Markov chain $\hat{x}_n, n \ge 0$ on the phase space $E = \{+,-\}$, is given by the sojourn times

$$\hat{\theta}_n^{\pm} = \alpha_n^{\pm} \wedge \alpha_n^{\mp *} \ , \quad n \ge 1. \tag{21}$$

The transition probabilities of the Markov chain $\hat{x}_n, n \ge 0$ with the sojourn times (21) are calculated as follows:

$$q_{\pm} = P\{\hat{x}_{n+1} = \mp | \hat{x}_n = \pm\} = P(\alpha_n^{\pm} > \alpha_n^{\mp *}),$$

that is

$$q_{\pm} = q\lambda_{\mp} \ , \quad q = \int_0^{\infty} \overline{F}_+(t)\overline{F}_-(t)dt \ , \quad \lambda_{\pm} = 1/E\alpha^{\pm}. \tag{22}$$

Its generating matrix has the following form:

$$Q = P - I = \begin{bmatrix} -q_+ & q_+ \\ q_- & -q_- \end{bmatrix}. \tag{23}$$

The stationary distribution of the Markov chain with the generating matrix (23) is given by

$$\Pi = \begin{bmatrix} \rho_+ & \rho_- \\ \rho_+ & \rho_- \end{bmatrix} \ , \quad \rho_\pm = \lambda_\pm/\lambda \ , \ \lambda = \lambda_+ + \lambda_-. \tag{24}$$

Introduce the orthogonal matrix

$$\overline{\Pi} := \Pi - I = \begin{bmatrix} -\rho_- & \rho_- \\ \rho_+ & -\rho_+ \end{bmatrix}. \tag{25}$$

It is easy to note that the generating matrix (23) has the following representation:

$$Q = \lambda q \overline{\Pi}. \tag{26}$$

Now let us define the potential matrix R_0 as a solution of the following equation:

$$QR_0 = R_0 Q = \overline{\Pi} \ , \quad R_0 \Pi = \emptyset. \tag{27}$$

It is easy to verify that

$$R_0 = -(\lambda q)^{-1} \overline{\Pi}. \tag{28}$$

Now, using the Markov chain, given by the generating matrix (26) and the potential matrix (28), we can analyze the asymptotic properties of the reward functional, defined on the duplicated renewal system with failure.

The limit working time of the system with failure gives us the following approximation estimate:

$$\begin{aligned} E\zeta_\pm &\simeq [\lambda_+ c_+ + \lambda_- c_-]/q, \\ c_\pm &:= E\gamma^\pm. \end{aligned} \tag{29}$$

The real valued random variables $\gamma_n^\pm := \gamma_n(\pm)$ is given by the distribution functions

$$\Gamma_\pm(u) = P(\gamma^\pm \le u) \ , \ \in R. \tag{30}$$

The heuristic principles of the phase merging formulated in Section IV, are based on limit theorems for semi-Markov processes with absorbing state.

References

[1] Gnedenko B. V. (1964). On spare duplication. *Engrg. Cybernet.*, 4:3–12.

[2] Gnedenko B.V. (1964). On duplication with renewal. *Engrg. Cybernet.*, 5:111–118.

[3] Gnedenko B.V., Belyaev Yu.K., Solovyev A.D. Mathematical Methods of Reliability Theory, New York, Academic Press, 1969.

[4] Korolyuk V.S., Turbin A.F. Markov renewal processes in problems of reliability of systems, Kyiv, Naukova Dumka, 1982 (in Russian).

[5] Korolyuk V.S., Korolyuk V.V. Stochastic models of systems, Kluwer, 1999.

[6] Korolyuk V.S., Limnios N. Stochastic systems in merging phase space, WSP, 2005.

[7] Feller W. An Introduction to Probability Teory and its Applications, Vol.1-2, Wiley, 1966.

[8] Gnedenko B., Ushakov I. Probabilistic Reliability Engineering, Section 6.2, New York, Wiley, 1995.

[9] Szász D. (1977). A problem of two lifts. *Ann. Probab.*, 5,4:550-559. *Engrg. Cybernet.*, 5:111–118.

[10] Zubkov, A.M. (1975). On the rate of convergence of a renewal density. *Math. USSR-Sb.*, 27-1:131-142.