System Reliability for Shock and Lottery Models

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Abstract

In this note we consider how system signatures (D-spectra) can be used in computing system reliability for "shock" and "lottery" models of system reliability.

Key words: shock model, lottery model, signatures, D-spectra.

Suppose you have a coherent binary system with n binary components subject to failure. To make this note more visual, imagine that the system is a network and the components subject to failure are the edges. So, any edge can be in two states, up and $down$, i.e. operational or not, respectively. The network can be in two states UP and $DOWN$. For example, the network is UP if two nodes of the network, S and T, are connected, and $DOWN$, otherwise. Let the components be numbered as $1,2,\dots,n$. Let us consider two situations which seem quite different. The first we will call "The shock model".

1. The shock model

Suppose there is an external source of "shocks" which act on our system in the following way. A shock chooses randomly one component of our system and hits (erases) it as a result of which this component goes from up to $down$. The next shock chooses randomly one of the remaining (non hit, up) components and hits it. This process continues until the system goes $DOWN$. This model has been considered in literature many times, see for example [1] and references there.

Suppose we check system state after each shock. Initially, before the shock process starts, the system is UP. Sooner or later the shocks will cause the system to go $DOWN$. Let us register the ordinal number of the shock which turns the system from UP to $DOWN$.

If it happens on the first hit, this number is one, if on the second - this number will be 2, and so on. By the definition of the shock process, all random sequences of component numbers hit by shocks are equally probable, and each particular sequence has probability $1/n!$ So, we can speak about random events $\{A_k\}$ and their probability $\{f_k\}$

$$A_k = (\ \mathit{system\ went\ DOWN\ on\ the\ k-th\ shock}\ ), f_k = P(A_k).$$

Obviously the collection of numbers $f = (f_1, f_2, \dots, f_n)$ is a discrete density and $\sum_1^n f_k = 1$. F. Samaniego [4,5] called the collection f *signature* . M.Lomonosov [5] suggested the name -"ID" (*internal distribution*).

Let us look now into $F(x) = f_1 + f_2 + \dots + f_x = \sum_{k=1}^{x} f_k$ which is called *cumulative signature* or *D-spectrum* [6].

The probabilistic meaning of $F(x)$ is the following. Suppose we know that the system is $DOWN$. Given this fact, the probability that the system has suffered k shocks equals $F(k)$. If the shocks process starts at $t = 0$ and shocks come with interval 1 hour, then $F(k)$ will be the CDF of system lifetime in hours. Or in other words: in the shock model scheme, $F(k)$ is the probability that the system failed on the first or the second,..., or the k-th shock.

In the shock model, finding $F(k)$ is the central issue of the resilience study of the system, see [6] where we describe Monte Carlo algorithms for obtaining an unbiased estimates for $F(x)$. These algorithms are based on simulating the process of sequential destruction of system components to locate the position of the $UP -> DOWN$ transition -from this process comes the prefix "D"-destruction.

Our personal impression that reliability engineers don't like too much the signature issue. For them the shock process looks as something artificial and not relevant to the main problem which is finding system reliability. Let us describe this problem.

2. The lottery model

Suppose the system consists of independent components and each component is up with probability p and $down$ with probability $q = 1 - p$. We can think also that these probabilities are related to a particular instant t, i.e. the component is up at t with probability $p = p(t)$. If the components have i.i.d. lifetimes, then $p(t)$ is the probability that component lifetime $\tau \geq t$.The central problem is finding system reliability, i.e. $P(\; system \; is \; UP)$, or $P(DOWN) = 1 - P(UP)$.

We will call this situation "the lottery" model. Assume that for each system component we carry out an independent lottery. In this lottery, the component is declared to be in state up with probability p and $down$with probability $q = 1 - p$. After the lottery ends, the system will be either in UP or in $DOWN$, and we are interested in finding $P(UP)$.

This is a solid reliability problem and its solution is an important practical issue. From the first sight, this problem has nothing in common with the above artificial shock model. How the reliability engineer would solve his problem? Most probably, by using the following formula

$$P(DOWN) = \sum_{k=1}^{n} C(k) q^k (1-p)^{(n-k)}, \quad (1)$$

where $C(k)$ is the number of *failure sets* having exactly k components $down$ and the remaining $(n - k)$ components up. The real issue is finding the $C(k)$'s.

But it turns out that the solution of the shock model provides easily the solution of the lottery model and vice versa. It turns out that there is a simple formula connecting $F(k)$ and $C(k)$:

$$C(k) = F(k) \frac{n!}{k!(n-k)!} \quad (2)$$

The proof of (2) can be carried out by purely combinatorial arguments or analytically. We will present both proofs in the

Appendix Important is the following fact: $F(k)$ and $C(k)$ do not depend on p or q. They are what we call a *combinatorial invariant*, depending only on system structure and not depending on probabilistic properties of its components.

Let us consider an

Example

Figure 1: (S-a) -edge 1, (a-T)-edge 2, (a-T)-edge 3. UP is S-T connection

The figure shows a network with three edges, which is UP if S is connected to T. In shock model, the first shock "kills" the system if it hits component 1. So, $f_1 = 1/3$. If the system survives the first shock, then the second shock always kills the system. So, $f_2 = 1 - f_1 = 2/3$. Then $f_3 = 0$. So, $F(1) = 1/3$, $F(2) = 1, F(3) = 1$. By (1), $C(1) = (1/3) \cdot 3!/2! = 1$. Indeed, there is only one failure set with one component down: {1}. $C(2) = 1 \cdot 3!/2! = 3$. The failure sets with two components down are: $\{1,2\}, \{1,3\}, \{2,3\}$. There is only one failure set with three components down- {1,2,3}. So, system is $DOWN$ with probability

$$P(DOWN) = qp^2 + 3p\dot{q}^2 + q^3.$$

The traditional reliability analysis would be the following. Denote the $downcomponent$ by zero and the up component by one. The list of all $2^3 = 8$ system states is the following:

$$000,001,010,011,100,101,110,111.$$

The numbers 000,001,010,100,011 correspond to the $DOWN$ state. There is exactly one state with three zeroes, one state with only one zero on the first position (shown bold), and three states with two zeroes, which are failure states with two $down$ components. This is exactly the above result obtained in the shock model.#

APPENDIX

a. Combinatorial proof of (2) ([2], page 114-115).

Consider random permutation of component numbers $\pi = i_1, i_2, \ldots, i_n$. Declare the first x of its members as system component's numbers which are *down* and all the rest -as being up.

If this permutation now determines system *DOWN* state, call it the $(x; D)$-type permutation. Denote by $N(x)$ the total number of of $(x; D)$ permutations. Obviously, the probability to have an $(x; D)$ permutation is $N(x)/n!$. On the other hand, this probability equals $f_1 + f_2 + \ldots + f_x$, which follows from the definition of the destruction process. Suppose that the permutation π has the property that the system failure was observed at the instant of k-th failure, $1 \leq k \leq x$. Declare for this permutation that all components whose numbers appear on the next $x - k$ positions as being $down$, and all the other components -as being up. In this way we will reconstruct all permutations of $(x; D)$-type. Note also that any permutation which in the destruction process produces $DOWN$ state **after** the x-th step is not of $(x; D)$-type. Therefore,

$$N(x) = (s_1 + s_2 + \ldots s + x) \cdot n!$$

Now note that we define system $DOWN$ state with exactly x components being $down$, the order of their appearance is not relevant. All permutations obtained by permuting x *down* components between themselves, and $(n - x)$ remaining also between themselves, determine, in fact, the **same** system failure state. Therefore,

$$C(x) = \frac{N(x)}{x!(n-x)!} \#$$

b. Analytic proof of (2)

Take the well known Samaniego formula for system lifetime probability [4,5]:

$$P(\tau_s \leq t) = \sum_{k=1}^{n} f_k F_{(k:n)}(t),$$

where $F_{(k:n)}$ is the CDF of the k-th order statistics from the sample of n i.i.d random variables with CDF $F(t)$. Substitute the explicit formulas for the order statistics and change the order of summation. You will obtain the expression

$$P(\tau_s \leq t) = \sum_{k=1}^{n} (f_1 + \ldots + f_k) q^k (1-q)^{(n-k)} n!/(k!\,(n-k)!),.$$

where $q = F(t)$. But the right-hand side of this expression is system $DOWN$ probability expressed via its failure sets:

$$P(DOWN) = \sum_{k=1}^{n} C(k) q^k (1-q)^{(n-k)}.\ \#$$

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