Why We Need Probability Distributions With Periodic Failure Rates In Reliability And Risk

Boyan Dimitrov
•
Kettering University, Flint, Michigan
bdimitro@kettering.edu

Abstract

We discuss situations in real life where probability distributions with periodic failure rates should be considered. This discussion leads us to a new class, called Almost-Lack-of-Memory (ALM) probability distributions. We explain the structure of these distributions, and list some of their important properties. One of the main properties is its periodic failure rate. Throughout this article we notice some areas of possible applications of these distributions, and relate applications to the properties of these distributions. However, periodic variability also is observed. This may be another interesting continuation of this study.

Keywords: ALM class of probability distributions; Periodic failure rate functions; Non-homogeneous Poisson Process; periodic random environment

I. Introduction

A positive r.v. X (life time) is uniquely determined by its *failure rate function* (FRF) $\lambda(t)$. The function $\lambda(t)\Delta t$ is presenting the conditional probability that the object will fail within the nearest time interval $[t, t+\Delta t)$ given that it did not fail before t. For X continuous with c.d.f. $F(x)$ and p.d.f. $f(x)$, it holds

$$\lambda(t) = f(t)/\ 1 - F(t), \text{ for all } t \geq 0, \text{ where } 1 - F(t) \neq 0. \tag{1}$$

The function

$$\Lambda(t) = -\ln[1 - F(t)],\ t \geq 0 \tag{2}$$

is known as *hazard function* (HF) of the object.
The relationship

$$\Lambda(t) = \int_0^t \lambda(x)dx \tag{3}$$

allows to understand that either of the four functions $f(x)$, $F(x)$, $\Lambda(t)$, or $\lambda(t)$ uniquely determines the other three. In demography and survival analysis the FRF $\lambda(t)$ is known as *mortality rate*. In [7] for the needs of age comparison, we proposed to call $\lambda(t)$ risk function (risk to fail, risk to die, risk of something to happen at age t since the aging process, has started).

We proposed to call $\lambda(t)$ with numerous of appropriate names, e.g. *risk function* (risk to fail, risk to die, risk of something to happen at age t since the aging process has started). Another suitable terminology for $\lambda(t)$ is *stress function*. $\Lambda(x)$ is the *accumulated stress* (or *accumulated risk*) during the life up to age x. The FRF $\lambda(t)$ varies over the time. It reflects the impact of the environment and its interaction with the working object. In Risk analysis the FRF usually is related to weariness, fatigue, maintenance to stabilization and improvement, or other internal properties of the working objects. Traditionally, the FRF are considered increasing, decreasing, bathtub, or arc shaped. It is reasonable to consider also periodic FRFs in reliability and in other applications too!

Periodic intensity rate FRF $\lambda(t)$ with period c equal to the periodicity in the environmental changes. The function FRF $\lambda(t)$ should satisfy the equations $\lambda(t + nc)= \lambda(t)$ for any $t \geq 0$, and for any n = 1, 2, . . . An appropriate choice of the function $\lambda(t)$ on the interval [0, c) solves the problem of determination of the distributions from this class, and any related characteristics.

In this work we give several examples where periodic FRF takes place. These examples also pinpoint areas of application for our models. Then we discuss the class of ALM probability distributions (first Introduced in [11]) which best suits (by physical and analytical properties) to model the phenomena described. We also focus some attention on random processes related to the ALM probability distributions and on their unique relationships. Each discussion briefly notices areas of possible applications.

II. Examples where periodic failure rates appear naturally

Here we describe some examples of life time distributions where distributions with periodic FRF can be expected. By the way, periodicity is in front of our eyes: We have daily periodicity in our habits every 24 hours; we have the weekly periodicity in our weekly schedules; we have some kind of monthly periodicity, at least while we pay our bills; we have the quarterly periodicity in some payments or other dues; we have half a year periodicity by season changes, or vacation opportunity; we have yearly periodicity in many means (season changes, insurance or subscription renewals, etc.). Some of these are described in more detail in the following examples.

Example 1: **Periodic Reliability Maintenance with Replacements**. The system requires use of items with strictly limited workload. S.

- The airplane motor must work 300 hours, and then replaced by a new one;
- In military operations this is very strict that an ammunition can be replaced after reaching certain age;
- Medical prescription require medication intake after expiration of certain fixed time;
- In the food industry the expiration date is strictly enforced. Food supply works as a periodic inventory system.

Assume that a failed item before the expiration date is replaced instantaneously with a working item at the same age as the just failed.

At the times of compulsory maintenance, made periodically any c time units, the operating item is replaced by a completely new one and the process continues under the same rules. The life time of such system has a FRF of period c.

Analogous behavior can be noticed in the inventory systems with periodic refill of the storage rooms, as considered in [16] and [17].

Aerospace engineering uses satellites for various purposes for some fixed periods of time 3-8 years. This example may fit some of their reliability problems too.

In social life and politics the periodic elections form processes of constant periodic nature. Numerous interesting variables with periodic FRFs can be associated with such processes elsewhere.

Example 2: **Service on non-reliable server** (It is borrowed from [9], [11, [16], [18]).
Consider a job, processed on an unreliable server, according to "preemptive – repeat - different" service discipline.

- The server may fail during service of a job. After *each failure it gets an instantaneous repair, and becomes as good as new. The server' life times T_n* are i.i.d. r.v.'s.;
- The job service time has duration Y if not interrupted;
- The interrupted job is continued upon the server's recovery, as new with other independent realization of the r.v. Y;

- • The service will be complete when the new service time Y_n is less than the server's life time T_n;
- • *The r.v.'s* Y_n *and* T_n are mutually independent.

Let X be the total service time of the job, and

$$N = \inf \{n;\ Y_n < T_n,\ n \geq 1\} . \tag{4}$$

Then

$$X = \sum_{n=1}^{N-1} T_n + Y_N \tag{5}$$

is the structure of the total service time X. If assume that the server life times T_n are all equal to one and the same constant c, i.e. if $P(T_n = c)=1$ for n=1,2, . . . , and the required service time Y is a continuous r.v., then X has periodic FRF of period c. The authors of [9] and [16] derived the mathematical expression for the distribution of the total service time.

Example 3: **The time to first event in a Non-Homogeneous Poisson Process (NHPP**) The NHPP $\{N_t : t > 0\}$ is a counting process the random number of events (arrivals) N_t in $[0, t)$, $t \geq 0$ (discussed in [2], [3]). The arrival rate $\lambda(t)$ is a non-negative function with finite integral on any finite interval of integration. The integrated rate function $\Lambda(t)$ of the NHPP, given by (3), represents the expected number of events $E(N_t)$, within $[0, t)$, $t \geq 0$. For any $t \geq 0$ the r.v. N_t has Poisson distribution of parameter $\Lambda(t)$ The arrivals on any non-overlapping intervals are independent r.v.'s. One can always associate a non-negative r.v. X with every NHPP. X is defined by the integrated rate function $\Lambda(t)$ declaring $F_X(x) = 0$ for $x < 0$, and $F_X(x) = 1 - e^{-\Lambda(t)}$, $x \geq 0$, according to relationships (1) - (3). This r.v. represents the waiting time up to the first event since the process starts: $F(x) = 1 - P\{X \geq x\} = 1 - P\{N[0,x) = 0\}$, $x \geq 0$. Reversely, any non-negative r.v. X generates a NHPP $\{N_t : t > 0\}$ whose intensity function coincides with the FRF of X, treated as life time. The times of the events in N_t can be interpreted as the flow of minimal repairs for an operating item whose life time is X (used in [3], [4] and [8]). If the intensity rate $\lambda(t)$ of the NHPP is periodic of period c, so is the FRF of the r.v. X .

Example 4: **Periodicity in Earth' life generates periodic NHPPs.** Periodicity in surrounding environmental conditions may have a considerable impact on the chances of random events to occur, on the random variables and the processes these events generate. As a natural descriptor of the effects of the periodic random environment on the rates of related random events, we suggest to use NHPP with periodic intensity rate $\lambda(t)$ with period c equal to the periodicity in the environmental changes. The function $\lambda(t)$ should satisfy the equations

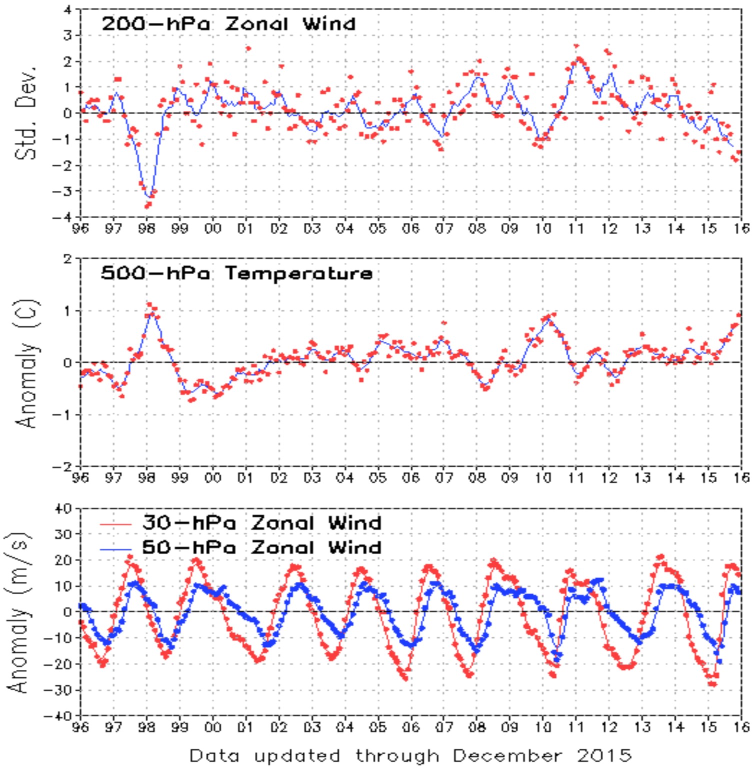
$$\lambda(t + nc) = \lambda(t) \text{ for any } t \geq 0, \text{ and for any } n = 1, 2, \ldots . \tag{6}$$

An appropriate choice of the function $\lambda(t)$ on the interval $[0, c)$ solves the problem of determination of the NHPP and any related characteristics.

Car accidents: The winter intensity rates due to wet, snowy or foggy weather are higher compare to what is in dry summer months; A bathtub-shape for the claims rate can be repeatedly expected every year in the northern countries. Natural periodicity of 1 year is reasonable, but periodic behavior of several years also seems admissible.

Hurricane activities: More than hundred years of records show that "the hurricane season starts by the end of May, has a season of high activity from second half of August to the first half of October and vanishes by the second half of December". The hurricane activity at coastal U.S. is modeled with a bell-shaped intensity function with a mode about the second half of the year. Natural periodicity seems to be 1 year; More realistic however, is the 6 year periodic alternation "El Ni˜no - La Ni˜na" seasons. In the South Hemisphere the shape of hurricane's intensity function should be bathtub (in a yearly interval starting in January) Sinusoidal up/down shifts with depression in the yearly behavior on a 3 year segment are reported and clearly visible on the site

http://www.cpc.ncep.noaa.gov/products/CDB/Tropics/figt3.gif
Go there, and you'll see the picture



We do believe that you clearly see periodicity and agree to our suggestions. But, *this variability is observed in the standard deviation too. This is a deep challenge to see what class of life time distributions it may call.*

Forest fires: Their intensity is low during wet seasons and high during dry seasons of the year. Usually, wet are Fall, Winter and Spring, and dry is the Summer. Respectively, a symmetric bell-shaped intensity function for the counts of the forest fires could be an appropriate model.

Flooding related events: Their intensity obviously goes high with snow melting when Winter ends and Spring comes. Somewhere the flooding may be related to hurricanes, or Summer thunderstorms. The respective behavior of $\lambda(t)$ may depend on the local specifics. Generally, $\lambda(t)$ is expected to have a bell shape with maximum shifted to the first third of the period. For some areas $\lambda(t)$ may be a multi-modal shaped flooding intensity.

House and car sales: Here everything can be expected. Most likely, there are two picks of the sale' intensity function, in the Spring and in the Fall. In such cases a mixture of two one pick functions, or a polynomial function of third degree may fit an appropriate model for the sale' intensity function $\lambda(t)$.

The examples may assume that the intensity at the yearend smoothly matches the value of this intensity at the beginning of the next year. There are possible situations with picks either at the beginning of the year (e.g. intensity of new loan contracts, purchase intensity of certain consumer's goods), or picks located near the end of the year (e.g. donations, spending funds related to taxes, and similar reasons in financial models). Respective $\lambda(t)$.can be decreasing in t, increasing, any, and no need to match the starting values with the values at the end of a cycle, and is a periodic function. The waiting time up to the occurrence of the first event in either of these processes is having periodic FRF.

Example 5: **Extended in time Bernoulli Trials (BTs) generate periodic NHPP** Numerous processes with established periodicity may be related to certain events (”successes”) in an artificially created sequence of extended in time BTs.

The traditional sequence of Bernoulli trials is based on the following conditions: – The consecutive trials are independent;

– Each trial has only two possible outcomes conditionally called “success” (S) and ”failure” (F)

– The probability P(S)=1−α does not depend on the current trial (so does the probability for failure P(F) = α), and is the same for each trial.

Let us add three more components (requirements to the sequence of BTs as in [9]) due to the facts that: Each trial essentially takes some considerable time to be conducted. As soon as success is observed it can be immediately recognized and recorded.

Hence:

– There is a need of considerable time $c > 0$ to complete each one of these BT;

– If success occurs within a trial, the time from the start of this trial until the success occurs, is a r.v. Y. Its distribution has support on the interval [0, c).

– The fact that a particular trial fails can be confirmed only when it is completed.

We list some of the potential contenders to the BTs constructions.

– Periodic alternations between cold and warm eras on Earth (of yet to be established periodicity);

– The 24 hours rotation of the Earth around its axis;

– The 11 years periodic cycles in solar activity;

– Other periodic configurations between planets or other spatial objects;

– The 24 hours and 50 minutes in length cycle in the tidal events (tides are cyclic rises and falls of seawater);

– The mentioned yearly changing climatic and meteorological conditions. Numerous pollution characteristics (chemical concentrations, emissions) can be related to this periodic process;

Equidistant check-points (measured in time, in miles, or in other workload units) in the maintenance procedures, associated with the normal work of a production system, a technical item, or a social system, are good sources for consideration of embedded extended ”in time” BTs;

Risk-associated events, such as house fires, diseases, fertility, mutation, bankruptcy, consumption rates, currency exchange rates, investment’ revenue, etc. may be involved in extended in time BTs of appropriate periodicity (time to perform a trial);

Human bio-rhythmic cycles like breathing cycles, heartbeats, eating cycles, use of prescription drugs, and others;

The waiting time until the occurrence of the first success in extended in time BTs has a periodic FRF (Khalil and Dimitrov [9]).

III. The role of origin in the time count

In the definition of periodic NHPP the origin is fixed as the beginning of a period, e.g. 1st of January each year. Imagine, the time origin t_0 is moving within the year, and can be located at any other day on the calendar. For instance

– April 15th as the last day to submit tax documents in the United States

. – July 1st as start of the new Academic year for most American

– A 12-month fiscal year, begins on October 1 and ends on September 30 next year,

In the sequel it will be important to have the same fixed start of any next period of time of duration c, as shown in [8] and [12]. The flows of events related to such determination of the time origin may change the analytical expression that fits best. Such variability creates some challenging statistical problems, namely to find the best position of the time origin that might produce the best goodness of fit to given observations.

The nature of the waiting time up to the occurrence of the first event of the process is not changed no matter what is the time origin. The properties of the r.v.'s related with periodic NHPPs, remain stable despite of some variability in the explicit mathematical forms in their description.

IV. Presentations of the distributions with periodic failure ratesons with periodic failure rates: equivalent analytical presentations

In this section we give a summary of results found by various authors during last 20 years on establishing properties and analytical forms of the distributions with periodic failure rates (see [6], [7], [8], [10], [15]).

Theorem 1: **(A)** The cdf of the waiting time X up to the first event in a periodic NHPP has the form

$$F_X(t) = 1 - \alpha^{\left[\frac{x}{c}\right]}\left(1-(1-\alpha)F_Y(t-\left[\frac{t}{c}\right]c)\right), \quad t \geq 0, \tag{7}$$

where $\alpha \in [0, 1]$, and $F_Y(y)$ is a cdf with support on the interval $[0, c)$. These are determined by the equations

– $\alpha = e^{-\int_0^c \lambda_X(u)du}$;

– The r.v. Y is defined either by its cdf

$$F_Y(y) = \frac{1}{1-e^{-\int_0^c \lambda_X(u)du}}\left[1-e^{-\int_0^y \lambda_X(u)du}\right], \text{ for } y \in [0, c].$$

Its pdf is

$$f_Y(y) = \frac{\lambda_X(y)}{1-e^{-\int_0^c \lambda_X(u)du}} \; e^{-\int_0^y \lambda_X(u)du}, \text{ with } y \in [0, c).$$

It is said that a r.v. X with a cdf of the form (7) belongs to the class of ALM(α, c, $F_Y(y)$) distributions. α, c , and $F_Y(y)$ are parameters of this class of probability distributions;

(B) In the continuous case the pdf $f_X(t)$ exists and has the form

$$f_X(t) = \alpha^{\left[\frac{x}{c}\right]}(1-\alpha)\, f_Y(t-\left[\frac{t}{c}\right]c), \quad t \geq 0; \tag{8}$$

(C) If a r.v. X has the distribution given by (7), then it satisfies the conditional probability property

$$P\{X - nc \geq t \mid X \geq nc\} = P\{X \geq t\} \tag{9}$$

for all $t \geq 0$, and for arbitrary integer $n = 0, 1, 2, \ldots$.

Namely, this property is called "Almost-Lack-of-Memory (ALM) property**"**, first in [13].

(D) Let a non-negative r.v. **X** have the lack of memory at a point c, $c > 0$, i.e. let

$$P\{X - c \geq t \mid X \geq c\} = P\{X \geq t\} \tag{10}$$

holds for all values of $t \geq 0$ and for the given value of the constant c. Then it holds:

(Di) X possesses the LM(nc) property with the value of c, for arbitrary integer $n = 1, 2, \ldots$;

(Dii) The cdf $F_X(t)$ is given by equation (7), where $F_Y(y)$ is a cdf with support on the interval $[0, c)$ if and only if (10) holds;

(E) A r.v. X has the LM property at some moment $c > 0$ if and only if X belongs to the class of ALM(α, c, $F_Y(y)$) distributions;

(F) A r.v. X belongs to the class of ALM(α, c, $F_Y(y)$) distrib*utions if and only if it can be represented as a sum* $X = Y + cZ$ *of two independent r.v.'s* Y *and* Z, where

– Y is located on the interval $[0, c)$ with probability 1, and

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(G) The Laplace-Stieltjes transform of the life time X is given by the formula

$$\varphi_X(s) = \varphi_Y(s)\,\frac{1-\alpha}{1-\alpha e^{-sc}}\ ;$$

(H) The Hazard Distribution Function (HDF), defined as a two argument function by the relation

$$\Lambda_X(x, t) = P\{X - x < t \mid X \ge x\} = \frac{F_X(x+t) - F_X(x)}{1 - F_X(x)} \tag{11}$$

is periodic with respect to first argument x, i.e. it satisfies

$$\Lambda_X(x + c, t) = \Lambda_X(x, t),\ x \ge 0, \text{ for any } t \ge 0\ ;$$

(I) If a life time r.v. X has periodic HDF of period $c > 0$ then its cdf $F_X(t)$ has the form (7), where $\alpha \in [0, 1]$, and $F_Y(y)$ is a cdf with support on the interval [0, c). Moreover, it is fulfilled:

(Ia) The FRF of X has the form

$$\lambda_X(t) = \frac{(1-\alpha)\, f_Y(t - \left[\frac{t}{c}\right]c)}{1-(1-\alpha)F_Y(t - \left[\frac{t}{c}\right]c)}\ ,\quad t \ge 0;$$

(Ib) The FRF $\lambda_X(t)$ is periodic function of period c;

(J) If a r.v. $X \sim ALM(\alpha, c, F_Y(y))$ distribution, then

$$P\{X - (nc + y) < t \mid X \ge nc + y\} = P\{X - y < t \mid X \ge y\} \tag{12}$$

holds for all $t \ge 0$, for arbitrary $y \in [0, c)$, and for arbitrary integer $n = 1, 2, \ldots$. This property explains that the ALM property is irrelevant in respect of the location of the origin within the interval $[0, c)$;

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$$P\{X \ge M\} > 0\ .$$

Precise expressions for cases of discrete distributions and discrete times are also feasible. Some of these are shown in [11] and [12].

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(Ib) The FRF $\lambda_X(t)$ is periodic function of period c;

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holds for all $t \ge 0$, for arbitrary $y \in [0, c)$, and for arbitrary integer $n = 1, 2, \ldots$. This property explains that the ALM property is irrelevant in respect of the location of the origin within the interval $[0, c)$;

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$$P\{X \ge M\} > 0 .$$

Precise expressions for cases of discrete distributions and discrete times are also feasible. Some of these are shown in [11] and [12].

VI. Properties of periodic NHPP

The only process which can be uniquely associated to the r.v.' s with periodic failure rates are the NHPPs with periodic intensity functions and with same interval of periodicity [2], [3], [5], [8]. They possess some interesting and important properties which can be used in applications. These can also be treated as an indivisible part of the properties of the life times. It holds

Theorem 2: An integer valued random process $\{N_t\}$ with periodic intensity rate $\lambda(t)$ of period $c > 0$ is a NHPP if and only if the following two properties hold:

(A) **(i)** For some constant c and for arbitrary $t \ge 0$ it is true that

$$P\{N[c,c+t) = m\} = P\{N[0,t) = m\} \text{ for any integer } m = 0, 1, 2, \ldots;$$

(ii) The r.v.'s $N[c,c+t)$ and $N[0,c)$ are independent for any $t \ge 0$;

(B) A counting process $\{N_t,\ t \ge 0\}$ is generated by random environment with periodic behavior of period $c > 0$

$$N_t =^d M_1 + M_2 + \ldots + M_{[t/c]} + N_{t - [t/c]c}, \qquad (13)$$

where $\{M_n\}_{n=0}^{\infty}$ are i.i.d. Poisson r.v.'s of parameter $\Lambda(c) = \int_0^c \lambda(x)dx$, independent of the component $N_{t - [t/c]c}$, and $\lambda(x) \ge 0$ is some periodic function of period $c > 0$, presenting the intensity of occurrence of events.

The notation "$=^d$" means equality in distribution;

(C) A NHPP $\{N_t,\ t \ge 0\}$ of intensity rate $\lambda(t)$ is periodic of period $c > 0$ if and only if the associated random variable X belongs to the class of $\mathrm{ALM}(\alpha, c, F_Y(y))$ distributions with parameters defined as in Theorem 1 (A);

(D) If each of the K mutually independent point processes $N_t^{(1)}, \ldots, N_t^{(K)}$ is a periodic NHPP with integrated intensity functions $\Lambda_1(t), \ldots, \Lambda_K(t)$, and all of the K processes have a common period c^*, then their superposition is a periodic NHPP with integrated intensity function $\Lambda(t) = \Lambda_1(t) + \ldots + \Lambda_K(t)$, and its period is c^*.

The random sum

$$Z_t = \sum_{n=0}^{N_t} \xi_n \text{ , with } \xi_0 = 0$$

is called Process of accumulation;

(E) If the sequence of sum components $\{\xi_n,\ n \ge 1\}$ consists of i.i.d. random variables, the aggregate claim process $\{Z_t\, ;\ t \ge 0\}$ driven by a periodic NHPP can be decomposed into the form

$$Z_t =^d Z^{(1)} + Z^{(2)} + \ldots + Z^{([t/c])} + Z_{t-[t/c]c} . \qquad (14)$$

The variables $\{Z^{(n)},\ n \ge 0\}$ are i.i.d. r.v.'s distributed as the compound Poisson sum

$$Z^{(1)} = \sum_{n=0}^{M_1} \xi_n ,$$

and M_1 is a Poisson r.v. as defined in part (A). The last term $Z_{t-[t/c]c}$ in (14) is also a compound Poisson term, independent of the other components $Z^{(n)}$, and with parameter $\Lambda(t)$, for $t \le$ c. Some applications of these results in modeling of processes in reliability and environmental studies can be seen in [2],[3] and [14].

VII. Conclusions

It is natural to consider life times with periodic FRF in reliability, risk, insurance, environmental modeling, financial mathematics, political theory, climate changes and various other applications. The reason is that the periodicity in surrounding environmental conditions has a considerable impact on generated random events, variables and processes.

An easy to understand description of these effects is suggested in the present article, and simple analytical relationships between its mathematical models are presented.

NHPP's with periodic failure rates provide good models for counting processes in any periodic random environment.

The periodic nature of the intensity $\lambda(t)$ in modeling risk processes could become a powerful tool in the study of cost models and financial problems related to various maintenance policies. Statistical challenges for best estimation of this function fir varying origin and fixed length of periodicity are raised.

References

[1] H.W. Block, Y. Li, T.H. Savits (2003). Initial and final behavior of failure rate functions in for mixtures and systems. *Journal of Applied Probability*, **40,** 721–740.

[2] S. Chukova, B. Dimitrov, J. Garrido (1993). Renewal processes generated by distributions with periodic failure rates, *Actuarial Research Clearing House*, 83–95.

[3] S. Chukova, B. Dimitrov, J. Garrido (1993). Renewal and non-homogeneous Poisson processes generated by distributions with periodic failure rate. *Statistics and Probability Letters.* **17,** 19–25.

[4] S. Chukova, B. Dimitrov, Z. Khalil (1993). A characterization of probability distributions similar to the exponential. *The Canadian Jour. of Statistics*, **21**, 269–276.

[5] E.Cinlar (1975). *Introduction to Stochastic Processes*. Prentice-Hall, New Jersey.

[6] L. R. Cui, M. Xie (2001). Availability analysis of periodically inspected systems with random walk model. *Journal of Applied Probability,* **38**, 860–871.

[7] B. Dimitrov (2019), Ages in reliability and bio systems, interpretations, control and applications. In: Rykov, V., Balakrishnan, N. (eds.*) Mathematical and Statistical Models and Methods in Reliability*, pp. 2005–2010. Birkhoiser.

[8] B. Dimitrov, V. Rykov, Z. Krougly (2004). Periodic Poisson processes and almost-lack-of-memorydistributions. *Automation and Remote Control*, **65**, 1597–1610.

[9] B. Dimitrov, Z. Khalil (1992). A class of new probability distributions for modeling environmental evolution with periodic behavior. *Environmetrics*, **3**, 447–464.

[10] B. Dimitrov, S. Chukova, D. Green (1997). Probability distributions in periodic random environment and their applications. *SIAM J. Appl. Math.* **57**, 501–517.

[11] B. Dimitrov (1996). Periodic random environment generates compound counting processes with amazing properties. applications. In: Laxon, J. (ed.) *Proceedings of the 7-th Annual GMI Industry Symposium: Technology & The Quality Revolution GMI*, pp. 148–163.

[12] B. Dimitrov (2011). Distributions with periodic failure rats in reliability. In: Zhao, L.C..X. (ed.) *Proceedings of The 7-th International Conference on "Mathematical Methods in Reliability": Theory. Methds. Applications,* Beijing Institute of Technology Press, pp. 721–726.

[13] B. Dimitrov, S. Chukova (1992). On the distributions having the almost lack-of-memory property, J. Appl. Prob. **v. 429**, No. 3, pp. 691 – 698.

[14] B. Dimitrov, S. Chukova, M. A. El-Saidi (1999). Modeling uncertainty in periodic random environment: Applications to environmental studies. *Environmetrics*, **10**, 467–485.

[15] B. Dimitrov, D, Green P. Stanchev (2011). Distributions with periodic failure rats and the relevation transformation. In: Zhao, L.C..X. (ed.) *Proceedings of The 7-th International Conference*

on "Mathematical Methods in Reliability": Theory. Methds. Applications, Beijing Institute of Technology Press, pp. 397–401.

[16] G. D. Lin (1994). A note on distributions having the almost-lack-of-memory property. *J. Appl. Prob*. **31**, 854–856.

[17] E. A. Silver, D. F. Pyke, R. Peterson (1998). *Inventory Management and Production Planning and Scheduling*. 3rd Edition. Wiley, New York.

[18] D. Yue, F. Tu (2005). On the completion time and the interruption time of a job processed on an unreliable machine. *Journal of Systems Science and Complexity* **18**, 205–209.