On averaged expected cost control as reliability for 1D ergodic diffusions

S.V. Anulova[5 6], H. Mai[7 8], A.Yu. Veretennikov[9 10]

Mon Nov 27 10:24:42 2017

Abstract

For a Markov model described by a one-dimensional diffusion with ergodic control without discount on the infinite horizon an ergodic Bellman equation is proved for the optimal readiness coefficient; convergence of the iteration improvement algorithm is established.

1 Introduction

According to textbooks in reliability – see, e.g., [7], [19] – coefficient of readiness is one of the main characteristics of reliability of the system. In this paper the model under consideration is presented by an ergodic Markov process described as a one-dimensional diffusion process which is controlled so as to spend more time in a "good domain" on average on the infinite horizon of time. The current readiness of the system is measured by a non-negative function f taking values on the interval [0,1]: one signifies a full readiness, while zero means that the model is in the break down state. Hence, in particular, we do not just split the real line into two parts – where $f = 1$ or $f = 0$ – but allow a soft transition from full readiness ($f = 1$) to a complete failure of the model ($f = 0$). Both coefficients of the diffusion as well as the function f itself may depend on the control. We allow only feedback (Markov) control strategies with values from some compact set. The main result states an ergodic Bellman equation on the optimal readiness characteristic ρ along with some auxiliary function; this ρ may be regarded as the most favourable readiness averaged simultaneously in space and time. Also we state an algorithm of improvement of control which in principle provides a tool to solve the Bellman equation approximately.

Earlier results on ergodic control in continuous time were obtained in [13], [15], [3], et al. The latest works include [1], [2], [18], see also the references therein. In the very first papers and books compact cases with some auxiliary boundary conditions – so as to simplify ergodicity – were studied; convergence of the improvement control algorithms were studied only partially. In the

[5] Institute for Control Sciences, Moscow, Russia; email: anulovas @ ipu.ru
[6] For the first author this research has been supported by the Russian Foundation for Basic Research grant no. 16-08-01285
[7] CREST and ENSAE ParisTech, France; email: hilmar.mai @ gmail.com
[8] The second author thanks the Institut Louis Bachelier for financial support.
[9] University of Leeds, UK, & National Research University Higher School of Economics, & Institute for Information Transmission Problems, Moscow, Russia; email: a.veretennikov @ leeds.ac.uk
[10] The third author is grateful to the financial support by the DFG through the CRC 1283 "Taming uncertainty and profiting from randomness and low regularity in analysis, stochastics and their applications" at Bielefeld University during his stay there in August 2017; also, for this author this study has been funded by the Russian Academic Excellence Project '5-100' and by the Russian Foundation for Basic Research grant no. 17-01-00633_a. All the authors gratefully acknowledge the support and hospitality of the Oberwolfach Research Institute for Mathematics (MFO) during the RiP programme in June 2014 where this study was initiated.

later investigations noncompact spaces are allowed; however, apparently, *ergodic* control in the diffusion coefficient σ of the process was not tackled earlier. About controlled diffusion processes on a finite horizon, or, on infinite horizon with discount (also known as killing) the reader may consult in [3], [10].

Discrete time and space theory was developed simultaneously in the monographs [5], [6, 8], [14], [17] and some others; important journal references can be found therein. Technical difficulties related to control in the diffusion coefficients are not an issue in discrete models. Combination of discrete state spaces and continuous time can be found in [18], et al. Reliability was not an issue in most of the cited works; however, it may be introduced in any Markov model. The paper consists of five sections not counting two lines of the Conclusions: 1 – Introduction, 2 – Setting, 3 – Assumptions and Auxiliaries, 4 – Main result and 5 – Sketch of the Proof.

2 Setting

Given a standard probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ and a one-dimensional $(\mathcal{F}_t)$ Wiener process $B = (B_t)_{t\geq 0}$ on it we consider a one-dimensional SDE with coefficients b, σ and a control parameter α described as follows:

$$\begin{aligned} dX_t^\alpha &= b(\alpha(X_t^\alpha), X_t^\alpha)\, dt + \sigma(\alpha(X_t^\alpha), X_t^\alpha)\, dW_t, \quad t \geq 0, \\ X_0^\alpha &= x \in \mathbb{R}. \end{aligned} \tag{1}$$

Its (weak) solution does exist [11] and under our conditions – 1D, boundedness of all coefficients and uniform non-degeneracy (or ellipticity) of σ^2 – is weakly unique.

Let a non-empty compact set $U \subset \mathbb{R}$ be a range of possible control values. Without any further reminder U being compact is always bounded. Let $b: U \times \mathbb{R} \to \mathbb{R}$, $\sigma: U \times \mathbb{R} \to \mathbb{R}$, $\alpha: \mathbb{R} \to U$ be given Borel functions (some more regularity assumptions will be presented later).

Denote the (extended) generator, which corresponds to the equation (??) with a fixed function $\alpha(\cdot)$ by L^α:

$$L^\alpha(x) = b(\alpha(x), x)\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2(\alpha(x), x)\frac{\partial^2}{\partial x^2}, \quad x \in \mathbb{R}.$$

Given a running cost function $f: U \times \mathbb{R} \to \mathbb{R}$ from a suitable function class we aim to choose an optimal (in some relaxed setting, at least, "nearly-optimal") control strategy $\alpha: \mathbb{R} \to U$ (Markov homogeneous, or, in another language, Markov feedback strategy) such that the corresponding solution X^α maximizes the averaged cost function

$$\rho^\alpha(x) := \liminf_{T\to\infty} \frac{1}{T}\int_0^T \mathbb{E}_x f(\alpha(X_t^\alpha), X_t^\alpha)\, dt. \tag{2}$$

Recall that the function f takes values

$$0 \leq f \leq 1, \tag{3}$$

then this running cost may be regarded as a measure of current readiness of the underlying device. Namely, any value between zero and one we can treat as a measure of availability, while the limit ρ^α if it exists, can be understood as an averaged – with respect to time and "ensemble" – availability (=readiness) of the system. This is especially natural for the set of possible values $\{0; 1\}$ for such a function; however, the whole interval of values $[0,1]$ also makes an evident sense in the context of reliability theory. In the sequel we assume that the assumption (3) is satisfied.

By K we denote the class of strategies $\alpha: \mathbb{R} \to U$ which are Borel measurable. For convenience for every $\alpha \in K$ we define the function $f^\alpha: \mathbb{R} \to \mathbb{R}$, $f^\alpha(x) = f(\alpha(x), x)$, $x \in \mathbb{R}$. Now, instead of(2) we can use the equivalent form,

$$\rho^\alpha(x) = \liminf_{T\to\infty} \frac{1}{T}\int_0^T \mathbb{E}_x f^\alpha(X_t^\alpha)\, dt.$$

The "maximin" cost function – or, in other terms, the ergodic availability or readiness coefficient of the system – is defined by the expression

$$\rho(x) := \sup_{\alpha\in K}\liminf_{T\to\infty} \frac{1}{T}\int_0^T \mathbb{E}_x f^\alpha(X_t^\alpha)\, dt. \tag{4}$$

Suppose that for every $\alpha \in K$ the solution of the equation (??) X^α is an ergodic process, that is, there exists a unique limiting distribution μ^α of $X_t^\alpha,\ t\to\infty$, the same for all initial conditions $X_0 = x\in\mathbb{R}$. Then it is true that for every $x\in\mathbb{R}$,

$$\rho^\alpha(x) \equiv \rho^\alpha := \int f^\alpha(x')\,\mu^\alpha(dx') =: \langle f^\alpha, \mu^\alpha\rangle, \tag{5}$$

and

$$\rho(x) \equiv \rho := \sup_{\alpha\in K}\int f^\alpha(x')\,\mu^\alpha(dx') = \sup_{\alpha\in K}\langle f^\alpha, \mu^\alpha\rangle. \tag{6}$$

Note that under our assumptions ρ does not depend on x. Ergodicity requires special conditions on the characteristics b,σ,α; they will be later specified in the next section. We also define an auxiliary function which depends on x and which looks like a cost function but it is not,

$$v^\alpha(x) := \int_0^\infty \mathbb{E}_x(f^\alpha(X_t^\alpha) - \rho^\alpha)\,dt, \quad \alpha\in K.$$

This integral will converge under the recurrence assumptions below.

Solutions of the equation (??) will be understood as weak ones. Correspondlingly, the ergodic Bellman equation (7) below will be established for weak solutions.

The first goal of the paper is to prove that the cost ρ – which is a constant in the ergodic setting – is the component of the pair (V,ρ), which is a unique solution of the *ergodic HJB or Bellman's equation,*

$$\sup_{u\in U}[L^u V(x) + f^u(x) - \rho] = 0, \quad x\in\mathbb{R}, \tag{7}$$

where V will be unique up to an additive constant, while ρ will be unique in the standard sense. The meaning of the function V is that it coincides with v^α for the optimal strategy α if the latter exists, and this function is the main tool for finding an optimal strategy. Note that due to the uni-dimensional setting and the non-degeneracy of σ^2 which will be assumed, the equation (7) is equivalent to the folowing,

$$\sup_{u\in U}\sigma^2(u,x)\left[\frac{1}{2}V''(x) + \frac{b(u,x)}{\sigma^2(u,x)}V'(x) + \frac{f^u(x)}{\sigma^2(u,x)} - \frac{\rho}{\sigma^2(u,x)}\right] = 0, \quad x\in\mathbb{R}. \tag{8}$$

Further, due to the non-degeneracy of σ^2 and in particular because the right hand sides in (7) and (8) are equal to zero, we conclude that they are both equivalent to

$$\sup_{u\in U}\left[\frac{1}{2}V''(x) + \frac{b(u,x)}{\sigma^2(u,x)}V'(x) + \frac{f^u(x)}{\sigma^2(u,x)} - \frac{\rho}{\sigma^2(u,x)}\right] = 0, \quad x\in\mathbb{R}. \tag{9}$$

The second goal is to show that the "RIA" algorithm ("reward improvement algorithm", or, in some papers, "PIA" for "policy improvement algorithm") provides a sequence of convergent approximate costs, $\rho_n\to\rho,\ n\to\infty$. Also let us emphasize that unlike in the finite horizon case, here in the average ergodic control setting, the solution of the HJB equation is *a couple* (V,ρ), where ρ is the desired cost while V is some auxiliary function, which also admits a certain interpretation in terms of control theory.

Note that solutions of the equations (7), (8) and (9) will be studied in Sobolev classes, hence, (second) derivatives will be defined up to almost everywhere with respect to Lebesgue's measure. To keep all strategies Borel, all expressions involving Sobolev derivatives will be uderstood as Borel measurable expressions since for any Lebesgue's function there is a Borel function which coincides with the former almost everywhere. Respectively, all HJB or Poisson equations will be understood in the Sobolev sense with Borel versions of any second order Sobolev derivative. First order derivatives are all continuous due to Sobolev imbedding theorems.

3 Assumptions and auxiliaries

To ensure ergodicity of X^α under any feedback control strategy $\alpha\in K$, we make the following assumptions on the drift and diffusion coefficients.

1. The function b is bounded, C^1 in x, and

$$\lim_{|x|\to\infty}\sup_{u\in U} x\,b(u,x) = -\infty. \tag{10}$$

2. The function σ is bounded, uniformly non-degenerate and C^1 in x.

3. The function f takes values in the interval [0,1].

4. The functions $\sigma(u,x), b(u,x), f(u,x)$ are continuous in (u,x).

5. The set $U \subset \mathbb{R}$ is compact.

Lemma 1 *Let the assumptions (A1) – (A4) be satisfied. Then the function v^α has the following properties:*

1. For any strategy α the function v^α is continuous as well as $(v^\alpha)'$, and there exist $C, m > 0$ such that $\sup_\alpha(|v^\alpha(x)| + |v^\alpha(x)'|) \le C(1+|x|^m)$.

2. $v^\alpha \in W^2_{p,loc}$ for any $p \ge 1$.

3. $v^\alpha \in C^{1,Lip}$ (i.e., $(v^\alpha)'$ is locally Lipschitz).
4. v^α satisfies a Poisson equation in the whole space,

$$L^\alpha v^\alpha(x) + f^\alpha(x) - < f^\alpha, \mu^\alpha > \overset{a.e.}{=} 0, \tag{11}$$

in the Sobolev sense.

5. Solution of this equation is unique up to an additive constant in the class of Sobolev solutions $W^2_{p,loc}$ with a no more than some (any) polynomial growth.

6. $< v^\alpha, \mu^\alpha >= 0$.

Proof. follows from [21] & [16]; see also [9, Lemma 4.13 and Remark 4.3].

Lemma 2 *Let the assumptions (A1) – (A3) hold true. Then,*

• For any $C_1, m_1 > 0$ there exist $C, m > 0$ such that for any strategy $\alpha \in K$ and for any function g growing no faster than $C_1(1+|x|^{m_1})$,

$$\sup_t |\mathbb{E}_x g(X_t^\alpha)| \le C(1+|x|^m). \tag{12}$$

• For any strategy $\alpha \in K$ the function ρ^α is a constant, and there exists $C < \infty$ such that

$$\sup_\alpha |\rho^\alpha| \le C < \infty. \tag{13}$$

• For any $\alpha \in K$, the invariant measure μ^α integrates any polynomial:

$$\int |x|^m \, \mu^\alpha(dx) < \infty.$$

Proof follows from [21] and [16].

4 Main result

Recall that the state space dimension is $D = 1$ and that all SDE solutions with any Markov strategy are weak, unique in distribution, strong Markov and ergodic. All of these follow from [11] and from the assumptions (A1) and (A2) (see [21] about ergodicity).

The "exact RIA" reads as follows. Let us start with some homogeneous Markov strategy α_0, which uniquely determines $\rho_0 = \rho^{\alpha_0} \equiv \langle f^{\alpha_0}, \mu^{\alpha_0} \rangle$ and $v_0 = v^{\alpha_0}$. Next, for any couple (v, ρ) such that $v \in C^2$, or $v \in W^2_{p,loc}$ with any $p > 0$, and for $\rho \in \mathbb{R}$, define

$$F[v,\rho](x) := \sup_{u \in U}[L^u v(x) + f^u(x) - \rho] = \max_{u \in U}[L^u v(x) + f^u(x) - \rho].$$

Recall that unless $v \in C^2$, we consider a Borel version of the expression in the right hand side. Now, by induction given α_n, ρ_n and v_n, the next "improved" strategy α_{n+1} is defined as follows: for any x,

$$L^{\alpha_{n+1}} v_n(x) + f^{\alpha_{n+1}}(x) - \rho_n = F[v_n, \rho_n](x). \tag{14}$$

which is equivalent to

$$L^{\alpha_{n+1}} v_n(x) + f^{\alpha_{n+1}}(x) = \max_u [L^u v_n(x) + f^u(x)] =: G[v_n](x).$$

In the sequel we *assume that a Borel measurable version of such a strategy can be chosen.* In our case existence of such a Borel strategy follows from Stschegolkow's (Shchegolkov's) theorem, see [20], [12, Satz 39], [4, Theorem 1] (the first two references are in German, the last one cites the same result in English), which states that if any section of a (nonempty) Borel set E in the direct product of two complete separable metric spaces is sigma-compact (i.e., equals a countable sum of closed sets) then a Borel selection belonging to this set E exists.

Now, the value ρ_{n+1} is defined as

$$\rho_{n+1} := \langle f^{\alpha_{n+1}}, \mu^{\alpha_{n+1}} \rangle,$$

where, in turn, $\mu^{\alpha_{n+1}}$ is the (unique) invariant measure, which corresponds to the strategy α_{n+1}. Recall that

$$v_n(x) = \int_0^\infty \mathbb{E}_x(f^{\alpha_n}(X_t^{\alpha_n}) - \rho_n)\, dt.$$

Theorem 1 *Let the assumptions (A1) – (A5) be satisfied. Then the Bellman equation (7) holds true for ρ and some auxiliary function $V \in C^2$, solution of this equation is unique for ρ, and for any n, $\rho_{n+1} \geq \rho_n$, the sequence ρ_n is bounded, and there is a limit $\rho_n \uparrow \rho, n \to \infty$.*

5 Sketch of the Proof

Let us show the sketch of the main steps of the proof.

noindent **1**. From (14) and (11) it may be derived that

$$(L^{\alpha_{n+1}} v_n - L^{\alpha_{n+1}} v_{n+1})(x) \overset{a.e.}{\geq} \rho_n - \rho_{n+1}.$$

Further, from Dynkin's formula applied to $(v_n - v_{n+1})(X_t^{\alpha_{n+1}})$ we obtain,

$$\mathbb{E}_x v_n(X_t^{\alpha_{n+1}}) - \mathbb{E}_x v_{n+1}(X_t^{\alpha_{n+1}}) - v_n(x) + v_{n+1}(x) \geq (\rho_n - \rho_{n+1})\, t.$$

Since the left hand side here is bounded for a fixed x, after division of all terms by t and at $t \to \infty$, we obtain,

$$0 \geq \rho_n - \rho_{n+1},$$

as required. Therefore, $\rho_n \leq \rho_{n+1}$, so that $\rho_n \uparrow \tilde{\rho}$ with some $\tilde{\rho}$. Thus, **the RIA does converge**, although so far we do not know whether $\tilde{\rho} = \rho$. Clearly, $\tilde{\rho} \leq \rho$, since ρ is the sup over all Markov strategies, while $\tilde{\rho}$ is the sup over some its countable subset.

Recall that now we want to show that $v_n \to \tilde{v}$ such that the couple $(\tilde{v}, \tilde{\rho})$ satisfies the HJB equation (7), and that $\tilde{\rho}$ – as well as $\tilde{v}$ in some sense – here is unique.

2. What we want to do is to pass to the limit in the equation

$$L^{\alpha_{n+1}}v_{n+1}(x) + f^{\alpha_{n+1}}(x) - \rho_{n+1} = 0, \quad as \quad n \to \infty,$$

after having showed compactness of the set (v_n) at least in C^1 (and later on, in $C^{1,\beta}$ for any $0 < \beta < 1$). Since

$$\rho_n = L^{\alpha_n}v_n(x) + f^{\alpha_n}(x),$$

we obtain after division by $\sigma^2/2$,

$$v_{n''}(x) = \frac{2\rho_n}{\sigma^2}(x) - \frac{2f}{\sigma^2}(x) - \frac{2v_{n'}}{\sigma^2}(x). \quad (15)$$

Due to the local boundedness and absolute continuity of $v_{n'}$ – see the Lemma 1 – we conclude that the sequence (v_n) is locally (i.e. on any bounded interval) tight in C^1. Hence, there is a subsequence $n' \to \infty$ such that $v_{n'}$ converges in C^1 on any bounded interval to some function $\tilde{v} \in C^1$ (in fact, even $\tilde{v} \in C^{1,Lip} = \{g \in C^1(\mathbb{R}): g' \in Lip\}$ as will be clear in a few lines and follows, e.g., from (15)). Denoting

$$F_1[x, v', \rho] := \max_u[\hat{b}^u v' + \hat{f}^u - \hat{\rho}](x) \equiv \max_u\left[\hat{b}^u v' + \hat{f}^u - \frac{\rho}{a^u}\right](x),$$

where

$$a^u(x) = \frac{1}{2}(\sigma^u(x))^2,\ \hat{b}^u(x) = b^u(x)/a^u(x),\ \hat{f}^u(x) = f^u(x)/a^u(x),\ \hat{\rho}^u(x) = \rho/a^u(x),$$

and by using the bounds as in [10, Chapter 1], *it can be shown* the limiting equation as $n' \to \infty$

$$\widetilde{v}'(x) - \widetilde{v}'(r) + \int_r^x F_1[s, \widetilde{v}'(s), \tilde{\rho}]\, ds = 0, \quad (16)$$

which implies by differentiation that

$$\widetilde{v}''(x) + F_1[x, \widetilde{v}', \tilde{\rho}](x) = 0. \quad (17)$$

This equation is equivalent to (9) and, hence, to (7), as required. In other words, the limiting pair $(\tilde{v}, \tilde{\rho})$ satisfies the HJB equation (7).

3. *Uniqueness for* ρ. Suppose there are two solutions of the (HJB) equation, v^1, ρ^1 and v^2, ρ^2 with a polynomial growth for v^i. Denote $v(x) := v^1(x) - v^2(x)$ and consider two Borel strategies $\alpha_1(x) \in argmax_u(L^u v(x))$ and $\alpha_2(x) \in argmin_u(L^u v(x))$, and denote by X_t^i a (weak) solution of the SDE corresponding to each strategy α_i. (It exists and is weakly unique.) Note that

$$h_2(x) := \max_u(L^u v(x) - \rho^1 + \rho^2) = \max_u(L^u v^1(x) + f^u(x) - \rho^1 - L^u v^1(x) - f^u(x) + \rho^2)$$

$$\geq \max_u(L^u v^1(x) + f^u(x) - \rho^1) - \max_u(L^u v^2(x) + f^u(x) - \rho^2) \overset{a.e.}{=} 0,$$

and similarly,

$$h_1(x) := \min_u(L^u v(x) - \rho^1 + \rho^2) = -\max_u(L^u(-v)(x) - \rho^2 + \rho^1)$$

$$\leq -\left[\max_u(L^u v^2(x) + f^u(x) - \rho^2) - \max_u(L^u v^1(x) + f^u(x) - \rho^1)\right] \overset{a.e.}{=} 0.$$

We have, $L^{\alpha_2}v(x) = h_2(x) - \rho^2 + \rho^1$, and $L^{\alpha_1}v(x) = h_1(x) - \rho^2 + \rho^1$. Further, Dynkin's formula is applicable. So,

$$\mathbb{E}_x v(X_t^1) - v(x) = \mathbb{E}_x \int_0^t L^{\alpha_1}v(X_s^1)\, ds$$

$$= \mathbb{E}_x \int_0^t h_1(X_s^1)\, ds + (\rho^1 - \rho^2)\, t \overset{(h_1 \leq 0)}{\leq} (\rho^1 - \rho^2)\, t.$$

The last inequality here is due to the $h_1 \overset{a.e.}{\leq} 0$ along with Krylov's bounds [10]. Here the left hand side is bounded (x fixed) due to the Lemma 2, so, we obtain,

$$\rho_1 - \rho_2 \geq 0.$$

Absolutely similarly we show that also

$$\rho^1 - \rho^2 \overset{(h_2 \geq 0)}{\leq} 0.$$

Thus, eventually,

$$\rho_1 = \rho_2.$$

4. *Proof of the equality* $\rho = \tilde{\rho}$. We have seen that for any initial (α_0, ρ_0), the sequence ρ_n

converges monotonically to $\tilde{\rho}$, which is a component of solution of the Bellman equation (7), as shown earlier in the step 2, and this component $\tilde{\rho}$ is unique as was just shown in the step 3. Hence, given some (any) $\varepsilon > 0$, take any initial strategy α_0 such that

$$\rho_0 = \rho^{\alpha_0} > \rho - \varepsilon.$$

Then, clearly, the corresponding limit $\tilde{\rho}$ will satisfy the same inequality,

$$\tilde{\rho} = \lim_n \rho_n > \rho + \varepsilon.$$

Due to uniqueness of $\tilde{\rho}$ as a component of solution of the equation (7), and since $\varepsilon > 0$ is arbitrary, and because it is already established that $\tilde{\rho} \le \rho$, we now conclude that

$$\tilde{\rho} = \rho.$$

The sketch of the proof of the Theorem 1 is thus completed.

6 Discussion

Thus, we have an approach which in principle allows to evaluate the ergodic readiness coefficient in certain diffusion Markov models.

7 Addendum: Borel measurability

In the presentation of RIA we have assumed existence of a Borel measurable version of such a strategy to be chosen which maximizes some function ofr a fixed x. In our case existence of such a Borel strategy follows from Stschegolkow's (Shchegolkov's) theorem, see [20], [12, Satz 39], [4, Theorem 1] (the first two references are in German, the last one cites the same result in English), which states that if any section of a (nonempty) Borel set E in the direct product of two complete separable metric spaces is sigma-compact (i.e., equals a countable sum of closed sets) then a Borel selection belonging to this set E exists. In our case $E = \{(u,x): F[u,x] = \phi(x) := \max_{v\in U} F[v,x], x \in \mathbb{R}\}$. This set is nonempty and closed and, hence, Borel. Indeed, if $E \ni (u_n, x_n) \to (u,x), n \to \infty$, then $F[u_n, x_n] \to F[u,x]$ due to continuity of F. Also, due to continuity of F, $\phi(x_n) \to \phi(x)$. Since each u_n is a point of $argmax\ (F[\cdot, x_n])$ where $F[u_n, x_n] = \phi(x_n)$ we have, $F[u,x] = \lim_{n\to\infty} F[u_n, x_n] = \lim_{n\to\infty} \phi(x_n) = \phi(x)$, we find that $(u,x) \in E$, i.e., E is closed. Further, any section E_x of E is also closed itself again due to continuity of F, as if $(u_n, x) \in E$ and $u_n \to u$, then $F[u_n, x] \to F[u,x]$, i.e., actually, $F[u_n, x] = F[u,x]$. Thus, Stschegolkow's theorem is applicable.

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