Flow Thinning With Limited Aftereffect: Differently Distributed Intervals Between the Moments of Customers Entrance

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Abstract

We present some analytical results obtained for probability characteristics of flow thinning with limited aftereffect. The thinning is processed according to a given function which depends on the evolution time and on the number of customers in the thinned flow and the number of lost customers in the original flow. The characteristics are obtained in the form of Laplace-Stieltjes transforms which are defined by the system of recurrence equations using the inverse Laplace-Stieltjes transform.

Keywords: flow with limited aftereffect, thinning of the flow, time-dependent function of thinning, Laplace-Stieltjes transform, inverse Laplace-Stieltjes transform.

1. Introduction

Below we present some results on flow thinning . Renyi, A. [1] in 1956 proved the first theorem on thinning of renewal flow. The customer remains in the thinned flow with constant probability q and is lost with probability 1- q . By changing the time scale, the flow rate remains constant. Let the thinning is performed n times with different probabilities $q_1,\dots,q_n$. Then, provided that $n\to\infty$ and $q_1,\dots,q_n\to 0$, the thinned flow converges to Poisson flow. In his review of "Random threads and theory of recovery" of the book by D. Cox and V. Smith [2],
Yu.K. Belyaev [2] investigated the preservation properties of Poisson flow in the thinning of the original Poisson streams.

Belyaev Yu. K. [3] generalized this fact to an arbitrary stream. In the book Gnedenko B. V. and I.N. Kovalenko I. N. [4], Belyaev's theorem was generalized to the case of non-stationary limit flow. A. D. Solov'ev [1] in 1971 proved that asymptotically the time of the first occurrence of a rare event in a regenerative process with appropriate normalization tends to an exponential random variable with parameter 1. Some other results about thinned flows can be found in [8-12].

For all of these works, the aim was to produce the ultimate results in the infinite thinning under appropriate normalization. Common for the above works was the fact that the thinning was carried out according to rules which were not time-dependent.

The outstanding feature of this paper is that the thinning is performed according to a set of time dependent procedures,

2. Statement of the problem

V. Smith [6] studied the flow of customers with differently distributed intervals between the moments of customer appearance. A. J. Khinchin [7] called these flows by flows with a limited aftereffect. This article considers such flows with thinning. The first customer of this flow enters at the random time having distribution $F_1(x)$. The time interval from the arrival of the first and second customer has a distribution $F_2(x)$. The interval time between the i - 1-st customer to the and i -st customer has a distribution $F_i(x)$, etc.

The thinning goes on as follows. If the customer was received at time t and the number of received customers of the thinned flow up to this point in time is equal i - 1, and the number of lost customersin the original flow is equal j, then the customer joins the customers of the thinned flow with a probability of $P_{i-1+j}(t)$, where the functions $P_{i-1+j}(t)$ are assumed to be known, and the time before admission of the followed customer of flow has distribution function $F_{i+j}(x)$. Otherwise it is lost. It is necessary to find the distribution of the number of received requirements of the thinned stream to an arbitrary point in time , t under the condition that at t = 0 the number of acted customers of thinned flow was equal zero.

3. Problem solution

We introduce the following notation: $\nu(t)-$ the number of received customers of the thinned stream, $\nu_0(t)-$ the number of lost customers from the initial flow with limited aftereffect, $\xi(t)-$ time prior to t of the receipt of the following customer of the flow with limited aftereffect.

First, consider the process $\zeta(t)=(\nu(t), \xi(t))$. This process will not be Markovian random process, since its development after the time t will depend not only on $\nu(t)$ and on $\xi(t)$, but and will also depend on the number of lost customers to the time t of the initial flow with limited aftereffect. This is because lost customers shift the points in time of receiving of customers of the thinned stream on the time axis.

Indeed, consider two consecutive time ξ_0 and ξ_0 $+\xi$. Let both times received customers original flow with limited aftereffect has not joined the thinned stream, the probability of this event equals (1 - $P_0(\xi_0)$)(1 - $P_0(\xi_0+\xi)$). If we consider the process $\zeta(t)=$ $(\nu(t), \xi(t))$, the probability of this event is equal to

(1 - $P_0(\xi_0))^2$, as the shift on the time axis by the amount ξ will not be considered because the value $\xi(t)$ it does not take into account.

Let us now consider the process $\zeta(t)=(\nu(t), \nu_0(t), \xi(t))$. This process already takes into account the fact that the lost customers shift points in time of receipt of customers of the thinned stream on the time axis. Therefore, the process

$\zeta(t)=(\nu(t), \nu_0(t), \xi(t))$ will already be a Markov random process, its development after the time t will depend on $\nu(t)$, $\nu_0(t)$ and $\xi(t)$, i.e. will not depend on its states before time t. We introduce the notation

$\varphi_{i,j}(t,x)$ = $P(\nu(t)=i, \nu_0(t)=j, \xi(t)<x)$, $\varphi_{i,j}(t)$ = $\varphi_{i,j}(t,\infty)$, $i=0,1,2,\ldots$, $j=0,1,2,\ldots$.

The problem of finding the distribution of the number of received customers of the thinned flow $\varphi_{i,j}(t)$ to a fixed point in time t is placed.

First we find the distribution of the number of received requirements of the thinned stream together with an additional variable x to a fixed point t, i.e. $\varphi_{i,j}(t,x)$. At the initial time of number customers is zero.

For desired quantities $\varphi_{i,j}(t,x)$ we derive the corresponding system of differential equations. We have the following system of difference equations

$$\varphi_{0,0}(t+\Delta t, x-\Delta t) = \varphi_{0,0}(t,x) - \varphi_{0,0}(t,\Delta t),$$

$$\ldots$$

$$\varphi_{0,j}(t+\Delta t, x-\Delta t) = \varphi_{0,j}(t,x) - \varphi_{0,j}(t,\Delta t) + \varphi_{0,j-1}(t,\Delta t)(1-P_{0,j-1}(t))F_j(x),\ j>0,$$

$$\ldots$$

$$\varphi_{i,0}(t+\Delta t, x-\Delta t) = \varphi_{i,0}(t,x) - \varphi_{i,0}(t,\Delta t) + \varphi_{i-1,0}(t,\Delta t)P_{i-1,0}(t)F_i(x), i>0,$$

$$\varphi_{i,1}(t+\Delta t, x-\Delta t) = \varphi_{i,1}(t,x) - \varphi_{i,1}(t,\Delta t) + \varphi_{i-1,1}(t,\Delta t)P_{i-1,1}(t)F_{i+1}(x) +$$

$$\varphi_{i,0}(t,\Delta t)(1-P_{i,0}(t))F_{i+1}(x), i>0,$$

$$\ldots$$

$$\varphi_{i,j}(t+\Delta t, x-\Delta t) = \varphi_{i,j}(t,x) - \varphi_{i,j}(t,\Delta t) + \varphi_{i-1,j}(t,\Delta t)\ P_{i-1,j}(t)\ F_{i+j}(x) +$$

$$\varphi_{i,j-1}(t,\Delta t)(1-P_{i,j-1}(t))F_{i+j}(x),\ i>0. \qquad (1)$$

This yields the following system of differential equations for $\varphi_{i,j}(t,x)$:

$$\frac{\partial}{\partial t}\varphi_{0,0}(t,x) - \frac{\partial}{\partial x}\varphi_{0,0}(t,x) = -\frac{\partial}{\partial x}\varphi_{0,0}(t,0),$$

$$\ldots$$

$$\frac{\partial}{\partial t}\varphi_{0,j}(t,x) - \frac{\partial}{\partial x}\varphi_{0,j}(t,x) = -\frac{\partial}{\partial x}\varphi_{0,j}(t,0) + \frac{\partial}{\partial x}\varphi_{0,j-1}(t,0)(1-P_{0,j-1}(t))F_j(x),\ j>0,$$

$$\frac{\partial}{\partial t}\varphi_{i,0}(t,x) - \frac{\partial}{\partial x}\varphi_{i,0}(t,x) = -\frac{\partial}{\partial x}\varphi_{i,0}(t,0) + \frac{\partial}{\partial x}\varphi_{i-1,0}(t,0)\ P_{i-1,0}(t)\ F_i(x),\ i>0,$$

$$\frac{\partial}{\partial t}\varphi_{i,1}(t,x) - \frac{\partial}{\partial x}\varphi_{i,1}(t,x) = -\frac{\partial}{\partial x}\varphi_{i,1}(t,0) + \frac{\partial}{\partial x}\varphi_{i-1,1}(t,0)\ P_{i-1,1}(t)\ F_{i+1}(x)$$

$$+\frac{\partial}{\partial x}\varphi_{i,0}(t,0)(1-P_{i,0}(t))F_{i+1}(x),\ i>0,$$

$$\ldots$$

$$\frac{\partial}{\partial t}\varphi_{i,j}(t,x) - \frac{\partial}{\partial x}\varphi_{i,j}(t,x) = -\frac{\partial}{\partial x}\varphi_{i,j}(t,0) + \frac{\partial}{\partial x}\varphi_{i-1,j}(t,0)\ P_{i-1,j}(t)\ F_{i+j}(x)$$

$$+\frac{\partial}{\partial x}\varphi_{i,j-1}(t,0)(1-P_{i,j-1}(t))F_{i+j}(x),\ i>0,\ j>1. \qquad (2)$$

We introduce the notation:

$$\tilde{\varphi}_0^{(0)}(s) = \int_0^\infty e^{-sx}dF_1(x) = \tilde{\varphi}_1(s),,\ \tilde{\varphi}_i(s) = \int_0^\infty e^{-sx}dF_i(x),\ i>0,$$

$$\tilde{\varphi}_{i,j}(u,s) = \int_0^\infty\int_0^\infty e^{-sx-ut}d_x\varphi_{i,j}(t,x)dt,\ \tilde{\varphi}_{i,j}(u) = \int_0^\infty e^{-ut}\varphi_{i,j}(t)dt,\ \frac{\partial}{\partial x}\tilde{\varphi}_{i,j}(u,0) = \int_0^\infty e^{-ut}\frac{\partial}{\partial x}$$

$$\varphi_{i,j}(t,0)dt,\ \frac{\partial}{\partial x}\tilde{\tilde{\varphi}}_{i,j}(u,0) = \int_0^\infty e^{-ut}\frac{\partial}{\partial x}\varphi_{i,j}(t,0)\ P_{i,j}(t)dt,\ i=0,1,2,\ldots,\ j=0,1,2,\ldots.$$

Then we have the following theorem:

Theorem 1. *For the Laplace-Stieltjes* $\tilde{\varphi}_{i,j}(u,s)$ *of function* $\varphi_{i,j}(t,x)$ *fair following formulas*

$$\tilde{\varphi}_{0,0}(u,s) = (u-s)^{-1}(\tilde{\varphi}_1(s) - \tilde{\varphi}_1(u)), \quad (3)$$

$$\ldots$$

$$\tilde{\varphi}_{i,j}(u,s) = (u-s)^{-1}\Big(-\frac{\partial}{\partial x}\tilde{\varphi}_{i,j}(u,0) + \frac{\partial}{\partial x}\tilde{\tilde{\varphi}}_{i-1,j}(u,0)\,\tilde{\varphi}_{i+j}(s)$$
$$+\frac{\partial}{\partial x}\tilde{\varphi}_{i,j-1}(u,0)\,\tilde{\varphi}_{i+j}(s) - \frac{\partial}{\partial x}\tilde{\tilde{\varphi}}_{i,j-1}(u,0)\,\tilde{\varphi}_{i+j}(s), \; i>0,\, j>1.$$

where $\frac{\partial}{\partial x}\tilde{\varphi}_{i,j}(u,0)$ *are determined sequentially from the following recurrent equations*

$$\frac{\partial}{\partial x}\tilde{\varphi}_{i,j}(u,0) = \tilde{\varphi}_{i+j}(u)\Big(\frac{\partial}{\partial x}\tilde{\tilde{\varphi}}_{i-1,j}(u,0) + \frac{\partial}{\partial x}\tilde{\varphi}_{i,j-1}(u,0) - \frac{\partial}{\partial x}\tilde{\tilde{\varphi}}_{i,j-1}(u,0)\Big) =$$
$$\tilde{\varphi}_{i+j}(u)\Big(\int_0^\infty e^{-ut}\frac{\partial}{\partial x}\varphi_{i-1,j}(t,0)\,P_{i-1,j}(t)dt + \int_0^\infty e^{-ut}\frac{\partial}{\partial x}\varphi_{i,j-1}(t,0)\,(1 - P_{i,j-1},(t))dt\Big), \quad (4)$$

Consistent application of recurrent equations (4) given in proof of this theorem ".

Proof. Applying to (2) transform of the Laplace-Stieltjes obtained

$$\tilde{\varphi}_{0,0}(u,s)(u-s) = -\frac{\partial}{\partial x}\tilde{\varphi}_{0,0}(u,0) + \tilde{\varphi}_1(s),$$

$$\ldots$$

$$\tilde{\varphi}_{0,j}(u,s)\,(u-s) = -\frac{\partial}{\partial x}\tilde{\varphi}_{0,j}(u,0) + \frac{\partial}{\partial x}\tilde{\varphi}_{0,j-1}(u,0)\,\tilde{\varphi}_j(s) - \frac{\partial}{\partial x}\tilde{\tilde{\varphi}}_{0,j-1}(u,0)\,\tilde{\varphi}_j(s), \; j>0,$$

$$\tilde{\varphi}_{i,0}(u,s)\,(u-s) = -\frac{\partial}{\partial x}\tilde{\varphi}_{i,0}(u,0) + \frac{\partial}{\partial x}\tilde{\tilde{\varphi}}_{i-1,0}(u,0)\,\tilde{\varphi}_i(s), \; i>0,$$

$$\tilde{\varphi}_{i,1}(u,s)\,(u-s) = -\frac{\partial}{\partial x}\tilde{\varphi}_{i,1}(u,0) + \frac{\partial}{\partial x}\tilde{\tilde{\varphi}}_{i-1,1}(u,0)\,\tilde{\varphi}_{i+1}(s)$$
$$+\frac{\partial}{\partial x}\tilde{\varphi}_{i,0}(u,0)\,\tilde{\varphi}_{i+1}(s) - \frac{\partial}{\partial x}\tilde{\tilde{\varphi}}_{i,0}(u,0)\,\tilde{\varphi}_{i+1}(s), \; i>0,$$

$$\ldots$$

$$\tilde{\varphi}_{i,j}(u,s)\,(u-s) = -\frac{\partial}{\partial x}\tilde{\varphi}_{i,j}(u,0) + \frac{\partial}{\partial x}\tilde{\tilde{\varphi}}_{i-1,j}(u,0)\,\tilde{\varphi}_{i+j}(s)$$
$$+\frac{\partial}{\partial x}\tilde{\varphi}_{i,j-1}(u,0)\,\tilde{\varphi}_{i+j}(s) - \frac{\partial}{\partial x}\tilde{\tilde{\varphi}}_{i,j-1}(u,0)\,\tilde{\varphi}_{i+j}(s), \; i>0,\, j>1. \quad (5)$$

Assuming in (5) $u = s$, get

$$\frac{\partial}{\partial x}\tilde{\varphi}_{0,0}(u,0) = \tilde{\varphi}_1(u),$$

$$\frac{\partial}{\partial x}\tilde{\varphi}_{0,j}(u,0) = \frac{\partial}{\partial x}\tilde{\varphi}_{0,j-1}(u,0)\,\tilde{\varphi}_j(u) - \frac{\partial}{\partial x}\tilde{\tilde{\varphi}}_{0,j-1}(u,0)\,\tilde{\varphi}_j(u), \; j>0,$$

$$\ldots$$

$$\frac{\partial}{\partial x}\tilde{\varphi}_{i,0}(u,0) = \frac{\partial}{\partial x}\tilde{\tilde{\varphi}}_{i-1,0}(u,0)\,\tilde{\varphi}_i(u), \; i>0,$$

$$\frac{\partial}{\partial x}\tilde{\varphi}_{i,1}(u,0) = \frac{\partial}{\partial x}\tilde{\tilde{\varphi}}_{i-1,1}(u,0)\,\tilde{\varphi}_{i+1}(u) + \frac{\partial}{\partial x}\tilde{\varphi}_{i,0}(u,0)\,\tilde{\varphi}_{i+1}(u) - \frac{\partial}{\partial x}\tilde{\tilde{\varphi}}_{i,0}(u,0)\,\tilde{\varphi}_{i+1}(u), \; i>0,$$

$$\frac{\partial}{\partial x}\tilde{\varphi}_{i,j}(u,0) = \frac{\partial}{\partial x}\tilde{\tilde{\varphi}}_{i-1,j}(u,0)\,\tilde{\varphi}_{i+j}(u) + \frac{\partial}{\partial x}\tilde{\varphi}_{i,j-1}(u,0)\,\tilde{\varphi}_{i+j}(u) - \frac{\partial}{\partial x}\tilde{\tilde{\varphi}}_{i,j-1}(u,0)\,\tilde{\varphi}_{i+j}(u),$$
$$i>0,\, j>1. \quad (6)$$

Inversing the first equation (6) we obtain the following expression

$$\frac{\partial}{\partial x}\varphi_{0,0}(t,0) = F_1(t). \tag{7}$$

From the second equation (6) have at $j = 1$

$$\frac{\partial}{\partial x}\tilde{\varphi}_{0,1}(u,0) = \tilde{\varphi}_1(u)\,\tilde{\varphi}_1(u) - \frac{\partial}{\partial x}\tilde{\tilde{\varphi}}_{0,0}(u,0)\,\tilde{\varphi}_1(u) = \tilde{\varphi}_1(u)\,(\tilde{\varphi}_1(u) - \int_0^\infty e^{-ut}\,F_1(t)\,P_{0,0}(t)dt). \tag{8}$$

Inversing (8), we can find the unknown function $\frac{\partial}{\partial x}\varphi_{0,1}(t,0)$.

Then from the second equation (6) have at $j = 2$

$$\frac{\partial}{\partial x}\tilde{\varphi}_{0,2}(u,0) = \tilde{\varphi}_2(u)\,(\frac{\partial}{\partial x}\tilde{\varphi}_{0,1}(u,0) - \int_0^\infty e^{-ut}\,\frac{\partial}{\partial x}\varphi_{0,1}(t,0)\,P_{0,1}(t)dt). \tag{9}$$

Inversing (9), we can find the unknown function $\frac{\partial}{\partial x}\varphi_{0,2}(t,0)$.

Then from the second equation (6) with arbitrary *j*>0,we have the following recursive sequence completely determines $\frac{\partial}{\partial x}\varphi_{0,j}(t,0)$, namely, from the following expression

$$\frac{\partial}{\partial x}\tilde{\varphi}_{0,j}(u,0) = \tilde{\varphi}_j(u)\,(\frac{\partial}{\partial x}\tilde{\varphi}_{0,j-1}(u,0) - \int_0^\infty e^{-ut}\,\frac{\partial}{\partial x}\varphi_{0,j-1}(t,0)\,P_{0,j-1}(t)dt) \tag{10}$$

by his conversion it is possible to find the unknown function $\frac{\partial}{\partial x}\varphi_{0,j}(t,0)$.

From the third equation of (6) obtained by $i = 1$

$$\frac{\partial}{\partial x}\tilde{\varphi}_{1,0}(u,0) = \frac{\partial}{\partial x}\tilde{\tilde{\varphi}}_{0,0}(u,0)\,\tilde{\varphi}_1(u) = \tilde{\varphi}_1(u)\int_0^\infty e^{-ut}\,\frac{\partial}{\partial x}\varphi_{0,0}(t,0)\,P_{0,0}(t)dt =$$

$$\tilde{\varphi}_1(u)\int_0^\infty e^{-ut}\,F_1(t)\,P_{0,0}(t)dt. \tag{11}$$

Inversing (11), we can find the unknown function $\frac{\partial}{\partial x}\varphi_{1,0}(t,0)$.

Further, from the third equation of (6) at $i > 1$ have

$$\frac{\partial}{\partial x}\tilde{\varphi}_{i,0}(u,0) = \frac{\partial}{\partial x}\tilde{\tilde{\varphi}}_{i-1,0}(u,0)\,\tilde{\varphi}_i(u) = \tilde{\varphi}_i(u)\int_0^\infty e^{-ut}\,\frac{\partial}{\partial x}\varphi_{i-1,0}(t,0)\,P_{i-1,0}(t)dt,\ i > 1. \tag{12}$$

Reversing (12), we can find the unknown function $\frac{\partial}{\partial x}\varphi_{i,0}(t,0)$, since (12) together with (11) is a recurrence formula for finding $\frac{\partial}{\partial x}\varphi_{i,0}(t,0)$ at $i > 1$.

Let us consider the fourth equation of (6) at $i > 1,\ j = 1$. It can be converted to the form

$$\frac{\partial}{\partial x}\tilde{\varphi}_{i,1}(u,0) = \tilde{\varphi}_{i+1}(u)\,(\frac{\partial}{\partial x}\tilde{\tilde{\varphi}}_{i-1,1}(u,0) + \frac{\partial}{\partial x}\tilde{\varphi}_{i,0}(u,0) - \frac{\partial}{\partial x}\tilde{\tilde{\varphi}}_{i,0}(u,0)) = \tilde{\varphi}_{i+1}(u) \times$$

$$(\int_0^\infty e^{-ut}\,\frac{\partial}{\partial x}\varphi_{i-1,1}(t,0)\,P_{i-1,1}(t)dt + \tilde{\varphi}_i(u)\int_0^\infty e^{-ut}\,\frac{\partial}{\partial x}\varphi_{i-1,0}(t,0)\,P_{i-1,0}(t)dt -$$

$$\tilde{\varphi}_i(u)\int_0^\infty e^{-ut}\,\frac{\partial}{\partial x}\varphi_{i,0}(t,0)\,P_{i,0}(t)dt),\ i > 0,\ j = 1. \tag{13}$$

Reversing (13), we can find the unknown function $\frac{\partial}{\partial x}\varphi_{i,1}(t,0)$, since (13) together with (12) is a recurrence formula for finding $\frac{\partial}{\partial x}\varphi_{i,1}(t,0)$ at $i>1,\ j=1$.

Let us consider the last fifth of equation (6) at $i>1,\ j>1$. It can be transform to the form

$$\frac{\partial}{\partial x}\tilde{\varphi}_{i,j}(u,0) = \tilde{\varphi}_{i+j}(u)\,(\frac{\partial}{\partial x}\tilde{\tilde{\varphi}}_{i-1,j}(u,0)+\frac{\partial}{\partial x}\tilde{\varphi}_{i,j-1}(u,0) - \frac{\partial}{\partial x}\tilde{\tilde{\varphi}}_{i,j-1}(u,0)) =$$

$$\tilde{\varphi}_{i+j}(u)\,(\int_0^\infty e^{-ut}\frac{\partial}{\partial x}\varphi_{i-1,j}(t,0)\,P_{i-1,j}(t)dt+\int_0^\infty e^{-ut}\frac{\partial}{\partial x}\varphi_{i,j-1}(t,0)\,(1 - P_{i,j-1}(t))dt. \quad (14)$$

Equation (14) is a recurrence relation expressed $\frac{\partial}{\partial x}\varphi_{i,j}(u,0)$ through $\frac{\partial}{\partial x}\varphi_{i-1,j}(t,0)$ and $\frac{\partial}{\partial x}\varphi_{i,j-1}(t,0)$. The beginning of this recurrence relation initiated by formulas (12) together with (11) and formulas (13) together with (12).

Thus, the expression for $\frac{\partial}{\partial x}\varphi_{i,j}(u,0)$ it is possible to obtain by the above method. Substituting these expressions into the formula (5), we can obtain expressions for the desired quantities $\tilde{\varphi}_{i,j}(u,s)$.

Corollary 1. "*For the Laplace-Stieltjes* $\tilde{\varphi}_{i,j}(u)$ *function* $\varphi_{i,j}(t)$ *fair following formulas*

$$\tilde{\varphi}_{0,0}(u)= u^{-1}(1 - \tilde{\varphi}_1(u))), \quad (15)$$

$$\ldots$$

$$\tilde{\varphi}_{i,j}(u) = (u-s)^{-1}\,(- \frac{\partial}{\partial x}\tilde{\varphi}_{i,j}(u,0) + \frac{\partial}{\partial x}\tilde{\tilde{\varphi}}_{i-1,j}(u,0)\,\tilde{\varphi}_{i+j}(s)$$

$$+\frac{\partial}{\partial x}\tilde{\varphi}_{i,j-1}(u,0)\,\tilde{\varphi}_{i+j}(s) - \frac{\partial}{\partial x}\tilde{\tilde{\varphi}}_{i,j-1}(u,0)\,\tilde{\varphi}_{i+j}(s),\ i>0,\ j>1,$$

where $\frac{\partial}{\partial x}\tilde{\varphi}_{i,j}(u,0)$ *consistently determined from recurrent equations* (4)."

4. Special case

We now turn to the consideration of the problem of thinning of the flow, when the probability of thinning of this thread $P_{i,j}(t)$ not time-dependent, and depend only on the received number of customers thinned flow and the number of lost customers of the initial flow, i.e. have the form $P_{i,j}$. This gives the following results. Function $\frac{\partial}{\partial x}\tilde{\tilde{\varphi}}_{i,j}(u,0)$ takes the form

$$\frac{\partial}{\partial x}\tilde{\tilde{\varphi}}_{i,j}(u,0)= P_{i,j}\int_0^\infty e^{-ut}\frac{\partial}{\partial x}\varphi_{i,j}(t,0)\,dt = P_{i,j}\frac{\partial}{\partial x}\tilde{\varphi}_{i,j}(u,0),\ i=0,1,2,\ldots,\ j=0,1,2,\ldots.$$

Theorem 1 transforms into theorem 2, which has the following form.

Theorem 2. "*For the Laplace-Stieltjes* $\tilde{\varphi}_{i,j}(u,s)$ *function* $\varphi_{i,j}(t,x)$ fair following formulas

$$\tilde{\varphi}_{0,0}(u,s)=(u-s)^{-1}(\tilde{\varphi}_1(s) - \tilde{\varphi}_1(u)), \quad (16)$$

$$\ldots$$

$$\tilde{\varphi}_{i,j}(u,s) = (u-s)^{-1}\,(- \frac{\partial}{\partial x}\tilde{\varphi}_{i,j}(u,0) + \tilde{\varphi}_{i+j}(s)\,(P_{i-1,j}\frac{\partial}{\partial x}\tilde{\varphi}_{i-1,j}(u,0)$$

$$+(1-P_{i,j-1})\frac{\partial}{\partial x}\tilde{\varphi}_{i,j-1}(u,0))),\ i>0,\ j>1.$$

where $\frac{\partial}{\partial x}\tilde{\varphi}_{i,j}(u,0)$ are determined sequentially from the following recurrent formulas

$$\frac{\partial}{\partial x}\tilde{\varphi}_{i,j}(u,0)=\tilde{\varphi}_{i+j}(u)\,(P_{i-1,j}\frac{\partial}{\partial x}\tilde{\varphi}_{i-1,j}(u,0)+(1-P_{i,j-1})\frac{\partial}{\partial x}\tilde{\varphi}_{i,j-1}(u,0)),\tag{17}$$

and $\frac{\partial}{\partial x}\tilde{\varphi}_{0,0}(u,0)=\tilde{\varphi}_1(u)$, $\frac{\partial}{\partial x}\tilde{\varphi}_{0,1}(u,0)=\tilde{\varphi}_1^2(u)(1-P_{0,0})$,

$$\frac{\partial}{\partial x}\tilde{\varphi}_{0,j}(u,0)=\tilde{\varphi}_j(u)(1-P_{0,j-1})\frac{\partial}{\partial x}\tilde{\varphi}_{0,j-1}(u,0)=\tilde{\varphi}_1(u)\prod_{l=1}^{j}\tilde{\varphi}_l(u)(1-P_{0,l-1}),\ j>1,$$

$$\frac{\partial}{\partial x}\tilde{\varphi}_{1,0}(u,0)=\tilde{\varphi}_1^2(u)\,P_{0,0},$$

$$\frac{\partial}{\partial x}\tilde{\varphi}_{i,0}(u,0)=\tilde{\varphi}_i(u)\,P_{i-1,0}\frac{\partial}{\partial x}\tilde{\varphi}_{i-1,0}(u,0)=\tilde{\varphi}_1(u)\prod_{l=1}^{i}\tilde{\varphi}_l(u)\,P_{l-1,0},\ i>1."$$

Here are a few of the subsequent formulas. Have

$$\frac{\partial}{\partial x}\tilde{\varphi}_{1,1}(u,0)=\tilde{\varphi}_2(u)\,(P_{0,1}\frac{\partial}{\partial x}\tilde{\varphi}_{0,1}(u,0)+(1-P_{1,0})\frac{\partial}{\partial x}\tilde{\varphi}_{1,0}(u,0))=$$

$$\tilde{\varphi}_2(u)\,\tilde{\varphi}_1^2(u)(P_{0,1}(1-P_{0,0})+(1-P_{1,0})\,P_{0,0}),$$

$$\frac{\partial}{\partial x}\tilde{\varphi}_{2,1}(u,0)=\tilde{\varphi}_3(u)(P_{1,1}\frac{\partial}{\partial x}\tilde{\varphi}_{1,1}(u,0)+(1-P_{2,0})\frac{\partial}{\partial x}\tilde{\varphi}_{2,0}(u,0))=$$

$$\tilde{\varphi}_3(u)(P_{1,1}(\tilde{\varphi}_2(u)\,\tilde{\varphi}_1^2(u)(P_{0,1}(1-P_{0,0})+(1-P_{1,0})\,P_{0,0}))+$$

$$(1-P_{2,0})\,\tilde{\varphi}_1(u)\prod_{l=1}^{2}\tilde{\varphi}_l(u)\,P_{l-1,0}),$$

$$\frac{\partial}{\partial x}\tilde{\varphi}_{1,2}(u,0)=\tilde{\varphi}_3(u)(P_{0,2}\frac{\partial}{\partial x}\tilde{\varphi}_{0,2}(u,0)+(1-P_{1,1})\frac{\partial}{\partial x}\tilde{\varphi}_{1,1}(u,0))=$$

$$\tilde{\varphi}_3(u)(P_{0,2}\,\tilde{\varphi}_1(u)\prod_{l=1}^{2}\tilde{\varphi}_l(u)(1-P_{0,l-1})+$$

$$(1-P_{1,1})\,\tilde{\varphi}_2(u)\,\tilde{\varphi}_1^2(u)(P_{0,1}(1-P_{0,0})+(1-P_{1,0})\,P_{0,0})).$$

Thus, in the article, obtained some analytical results for probability characteristics of a thinning of the flow with different-distributed intervals between the moments of customers entrance (flow with limited aftereffect). The thinning is processed according to a given function which depends on evolution time and on the number customers of the thinned flow and the number of lost customers in theoriginal flow. The characteristics are obtained in the form of Laplace-Stieltjes transforms which are defined by the system of recurrence equations with using inversion of Laplace-Stieltjes transforms.

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