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Accelerated Life Testing Design Using Geometric Process for Generalized Rayleigh Distribution with Complete Data

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Abstract

The log-linear function between life and stress which is just a simple re-parameterization of the original parameter of the life distribution is used to obtain the estimates of original parameters in many of the studies concerning Accelerated life testing (ALT). But from the statistical point of view, it is preferable to work with the original parameters instead of developing inferences for the parameters of the log-linear link function. In this study we introduce the geometric process for the analysis of accelerated life testing with Generalized Rayleigh Distribution for constant stress. Assuming that the lifetimes of units under increasing stress levels form a geometric process, the maximum likelihood estimation approach is used for the estimation of parameters. The confidence intervals (CIs) of the model parameters are derived. A Simulation study is also performed to check the statistical properties of estimates of the parameters and the confidence intervals.

Keywords: Geometric process, Generalized Rayleigh Distribution, *Maximum* Likelihood Estimator, Fisher Information Matrix, Asymptotic Confidence Interval, Simulation Study.

1. Introduction

Accelerated life testing is the process of testing a product by subjecting it to conditions (stress, strain, temperatures, voltage, vibration rate, pressure etc.) in excess of its normal service parameters in an effort to uncover faults and potential modes of failure in a short amount of time. By analyzing the product's response to such tests, statisticians can make predictions about the service life and maintenance intervals of a product.

In general, ALT deals with three types of stress patterns: constant stress, step stress and Progressive stress. In the former case, each unit is run at a pre-specified constant stress level which does not vary with time. This means that every item is subjected to only one stress level until the item fails or the test is stopped for other reasons. In use, most products such as semiconductors and microelectronics, capacitors, lamps …etc, run at a constant stress. This type of stress is widely used and preferred because the stress is constant in most applications, it is much easier to apply and quantify constant stress and models for constant stress are available, widely publicized and empirically verified.

There is a lot of literature on constant stress accelerated life testing, for example, Ahmad et al. [1], Islam and Ahmad [2], Ahmad and Islam [3], Ahmad et al.[4] and Ahmad [5] discuss the optimal constant stress accelerated life test designs under periodic inspection and Type-I censoring. Yang [6] proposed an optimal design of 4-level constant stress ALT plans considering different censoring times. Pan et al. [7] proposed a Bivariate constant stress accelerated degradation test model by assuming that the copula parameter is a function of the stress level that can be described by the logistic function. Wilkins and johns [8] considered constant stress accelerated life test based on Weibull distribution with constant shape and a log-linear link between scale the stress factor which is terminated by a Type-II censoring regime at one of the stress levels.

The concept of geometric process in accelerated life testing was first introduced by Lam [9] in the problems of repair replacement. Lam [10] studied the geometric process model for a multistate system and concluded a replacement policy to minimize the long run average cost per unit time. Since then a lot of studies in maintenance problems and system reliability have been shown that a GP model is a good and simple model for analysis of data with a single trend or multiple trends, for example, Lam and Zhang [11], Lam [12] and Zhang [13]. Huang [14] introduced the GP model for the analysis of constant stress ALT with complete and censored exponential samples. Kamal et al. [15] extended the GP model for the analysis of complete Weibull failure data in constant stress ALT. Zhou et al. [16] implement the GP in ALT based on the progressive Type-I hybrid censored Rayleigh failure data. Kamal et al. [17] used the geometric process for the analysis of constant stress accelerated life testing for Pareto Distribution with complete data. S. Saxena [18] introduces the Rayleigh geometric process model for the analysis of accelerated life testing under constant stress. Sadia Anwar et al. [19] presented the mathematical model of accelerated life testing for Marshall-Olkin extended exponential distribution using geometric process and extended her work using type I censored data [20]. Recently Kamal [21] presented an application of the geometric process in accelerated life testing analysis on type-I censored Weibull failure data.

In the present study, the GP model is implemented in the analysis of ALT for the Generalized Rayleigh life distribution under constant stress with complete data. Maximum likelihood (ML) estimates of parameters and their asymptotic confidence intervals (CIs) are obtained. The performance of the estimates is evaluated by a simulation study.

2. The Model and Test Procedure

2.1. The Geometric Process

A geometric process describes a stochastic process $\{X_n, n = 1,2,\ldots\}$, where there exists a real-valued $\lambda > 0$ such that $\{\lambda^{n-1} X_n, n = 1,2,\ldots\}$ forms a renewal process. It can be shown that if $\{X_n, n = 1,2,\ldots\}$ is a GP and the probability density function of X_1 is $f(x)$ with mean μ and variance σ^2 then the probability density function of X_n will be $\lambda^{n-1} f(\lambda^{n-1} x)$ with $E(X_n) = \frac{\mu}{\lambda^{n-1}}$ and $\mathrm{var}(X_n) = \frac{\sigma^2}{\lambda^{2(n-1)}}$. Thus λ, μ and σ^2 are three important parameters of GP.

2.2. The Generalized Rayleigh Distribution

The probability density function (pdf) of a generalized Rayleigh distribution is given by

$$f(x/\alpha,\beta) = \begin{cases} (1 - e^{-\beta x^2})^{\alpha} & x > 0, \alpha > 0, \beta > 0 \\ 0, elsewhere \end{cases}$$

where, $\alpha > 0$ is the shape parameter and $\beta > 0$, is the scale parameter of the distribution. Generalized Rayleigh distribution is a member of the family of Burr distributions which was appeared since 1942. It is known also Burr type X distribution. The cumulative distribution function (cdf) of generalized Rayleigh distribution is

$$F(x/\alpha,\beta)=\begin{cases}\left(1-e^{-\beta x^2}\right)^{\alpha} & ,x>0\,,\alpha>0\,,\beta>0\\ 0\ ,elsewhere\end{cases}$$

The Hazard function of the Generalized Rayleigh distribution takes the following form

$$S(x/\alpha,\beta)=1-\left(1-e^{-\beta x^2}\right)^{\alpha}$$

The failure rate (or hazard rate) for the Generalized Rayleigh distribution is given by

$$h(x/\alpha,\beta)=\frac{2\alpha\beta x e^{x^2}\left(1-e^{-\beta x^2}\right)^{\alpha-1}}{1-\left(1-e^{-\beta x^2}\right)^{\alpha}}$$

The two-parameter Generalized Rayleigh distribution was first proposed by (Raqab and Kundu; 2003) [22] and is denoted by $GR(\alpha,\beta)$. It is observed that the hazard function of a Generalized Rayleigh distribution can be either bathtub type or increasing function, depending on the shape parameter α . For $\alpha\leq\frac{1}{2}$, the hazard function is bathtub type and for $\alpha>\frac{1}{2}$, it has an increasing hazard function. Surles and Padgett (2001) [23] showed that the two-parameter GR distribution can be used quite effectively in modelling strength data and also modelling general lifetime data.

2.3. Assumptions and test procedure

1. Under any constant stress, the time to failure of test unit follows Generalized Rayleigh distribution.

2. The Generalized Rayleigh shape parameter α is constant, i.e. independent of stress.

3. Let the sequence of random variables $X_0, X_1,\ldots X_s$ denote the lifetimes under each stress level, where X_0 denotes lifetime of an item under the design stress. We assume $\{X_k, k=1,2,\ldots,s\}$ is a geometric process with ratio $\lambda>0$.

4. Suppose that an ALT under $z_k, k=1,2,\ldots,s$, arithmetically increasing stress levels is performed. A random sample of $N_i, i=1,2,\ldots,n$, identical items are placed under each stress level and start to operate at the same time. Whenever an item fails, it is removed from the test and its observed failure time x_{ki} is recorded.

5. The scale parameter is a log-linear function of stress that is $\log\beta_i=a+bS_i$, here *a* and *b* are unknown parameters depending on the nature of the product and the test method.

Theorem: *If the stress level in an ALT is increasing with a constant difference then under each stress level the lifetimes of items forms a GP. That is, If* $S_{k+1}-S_k$ *is constant for* $k=1,2,\ldots,s-1$, *then* $\{X_k, k=1,2,\ldots,s\}$ *forms a GP.*

Proof: From assumption (5), we get

$$\log\left(\frac{\beta_{k+1}}{\beta_k}\right)=b\left(S_{k+1}-S_k\right)=b\Delta S$$

This shows that the increased stress levels form an arithmetic sequence with a constant difference ΔS .

Now the above equation can be written as

$$\frac{\beta_{k+1}}{\beta_k} = e^{b\Delta S} = \lambda(say) \tag{1}$$

It is clear from (1) that

$$\beta_k = \lambda\beta_{k-1} = \lambda^2\beta_{k-2} = \ldots = \lambda^k\beta$$

The lifetime PDF of an item at the kth stress level is

$$f_{X_k}(x) = 2\alpha\beta_k^2 x e^{-(\beta_k x)^2}\left(1-e^{-(\beta_k x)^2}\right)^{\alpha-1}$$

$$= 2\alpha\left(\lambda^k\beta\right)^2 x e^{-\left(\lambda^k\beta x\right)^2}\left(1-e^{-\left(\lambda^k\beta x\right)^2}\right)^{\alpha-1} \tag{2}$$

This implies that

$$f_{X_k}(x) = \lambda^k f_{X_0}\left(\lambda^k x\right) \tag{3}$$

Now, from the definition of GP and from expression (3) it is clear that, if density function of X_0 is $f_{X_0}(x)$, then the pdf of X_k will be given by $\lambda^k f_{X_0}\left(\lambda^k x\right)$, $k = 1,2,\ldots,s$. Therefore, it is clear that lifetimes under a sequence of arithmetically increasing stress levels form a GP with ratio λ.
Now, the pdf of a lifetime of an item at the k^{th} stress level is

$$f_{x_k}(x/\alpha,\beta,\lambda) = \ = 2\alpha\left(\lambda^k\beta\right)^2 x e^{-\left(\lambda^k\beta x\right)^2}\left(1-e^{-\left(\lambda^k\beta x\right)^2}\right)^{\alpha-1} \tag{4}$$

It is clear from above expression that if lifetimes of items under a sequence of increasing stress level form a geometric process with ratio λ and if the life distribution at design stress level is generalized Rayleigh with characteristic β, then the life distribution at k^{th} stress level will also be generalized Rayleigh with characteristic life $\beta\lambda^k$.

2.4. Maximum likelihood Estimation

The likelihood function for constant stress ALT for complete case generalized Rayleigh distribution failure data using GP for s stress levels is given by:

$$L(\lambda,\alpha,\beta) = \prod_{k=1}^{s}\prod_{i=1}^{n} 2\alpha\left(\lambda^k\beta\right)^2 x_{k_i} e^{-\left(\lambda^k\beta x_{k_i}\right)^2}\left(1-e^{-\left(\lambda^k\beta x_{k_i}\right)^2}\right)^{\alpha-1} \tag{5}$$

The log likelihood of (5) can be written a

$$l(\lambda,\alpha,\beta) = \sum_{k=1}^{s}\sum_{i=1}^{n}\left\{\begin{array}{l}\log 2\alpha + 2k\log\lambda + 2\log\beta + \log x_{k_i} - \left(\lambda^k\beta x_{k_i}\right)^2 + \\ (\alpha-1)\log\left(1-e^{-\left(\lambda^k\beta x_{k_i}\right)^2}\right)\end{array}\right\}$$

Partial derivatives of above equation with respect to λ and β are:

$$\frac{\partial l}{\partial\lambda} = \sum_{k=1}^{s}\sum_{i=1}^{n}\left\{\frac{2k}{\lambda} - 2k\lambda^{2k-1}\left(\beta x_{k_i}\right)^2 + 2k\lambda^{2k-1}(\alpha-1)\left(\beta x_{k_i}\right)^2\frac{A}{D}\right\} \tag{6}$$

$$\frac{\partial l}{\partial \beta} = \sum_{k=1}^{s}\sum_{i=1}^{n} \left\{\frac{2}{\beta} - 2\beta\left(\lambda^k x_{k_i}\right)^2 + 2\beta(\alpha-1)\left(\lambda^k x_{k_i}\right)^2 \frac{A}{D}\right\} \tag{7}$$

Where

$$\text{A}= e^{-\left(\lambda^k \beta x_{k_i}\right)^2} \quad \text{and} \quad \text{D}=\left(1-e^{-\left(\lambda^k \beta x_{k_i}\right)^2}\right)$$

From equations (6) and (7), it is observed that these equations are non-linear. Therefore, the closed forms of MLEs of λ and β do not exist. So, Newton-Raphson method must be used to solve these equations simultaneously to obtain the MLEs of λ and β .

3. Asymptotic Confidence Interval

Let $I(\lambda,\beta)$ denotes the Fisher Information matrix, then observed Information matrix of $I(\lambda,\beta)$ is given as

$$I(\lambda,\beta) = \begin{bmatrix} \hat{I}_{11} & \hat{I}_{12} \\ \hat{I}_{21} & \hat{I}_{22} \end{bmatrix}$$

Where

$$\hat{I}_{11} = -\left(\frac{\partial^2 l}{\partial \lambda^2}\right)$$

$$= \sum_{k=1}^{s}\sum_{i=1}^{n}\left[\begin{matrix} \frac{2k}{\lambda^2} + 2k(2k-1)Z^2\lambda^{2(k-1)} + (\alpha-1) \\ \times\left\{\frac{4ADZ^4k^2\lambda^{2(k-1)} - 2kAD(2k-1)Z^2\lambda^{2(k-1)} + 4k^2A^2Z^4\lambda^{2(2k-1)}}{D^2}\right\}\end{matrix}\right]$$

$$\hat{I}_{22} = -\left(\frac{\partial^2 l}{\partial \beta^2}\right) = \sum_{k=1}^{s}\sum_{i=1}^{n}\left[\frac{2}{\beta^2} + 2Z^2 + (\alpha-1)\left\{\frac{4AD\beta^2Z^4 - 2ADZ^2 + 4A^2\beta^2Z^4}{D^2}\right\}\right]$$

$$\hat{I}_{12} = -\left(\frac{\partial^2 l}{\partial \lambda \partial \beta}\right) = \hat{I}_{21} = -\left(\frac{\partial^2 l}{\partial \beta \partial \lambda}\right)$$

$$= \sum_{k=1}^{s}\sum_{i=1}^{n}\left[\frac{4\beta kZ^2}{\lambda} + 4\left(\frac{\alpha-1}{\lambda}\right)\left\{\frac{ADk\beta^3Z^4 - ADk\beta Z^2 + Z^4A^2\beta^3k}{D^2}\right\}\right]$$

Where

$$Z = \left(\lambda^k x_{k_i}\right)$$

Now, the variance-covariance matrix can be written as

$$\begin{bmatrix} \text{var}(\hat{\lambda}) \\ \text{var}(\hat{\beta}) \end{bmatrix} = \begin{bmatrix} \hat{I}_{11} \\ \hat{I}_{22} \end{bmatrix}^{-1}$$

The $100(1-\theta)\%$ asymptotic confidence interval for λ and β are then given respectively as

$$\left[\hat{\lambda} \pm z_{1-\frac{\vartheta}{2}} \sqrt{\operatorname{var}(\hat{\lambda})}\right]$$

And

$$\left[\hat{\beta} \pm z_{1-\frac{\vartheta}{2}} \sqrt{\operatorname{var}(\hat{\beta})}\right]$$

4. Simulation Studies:

Simulation of data is the initial task for studying different properties of parameters. It is an attempt to model an assumed condition to study the behaviour of the function.

1. First, to perform the simulation study, first, a random sample is generated from Uniform distribution by using R software.
2. Now, we use inverse cdf method to transform the cdf at kth stress level in terms of u and get the expression of $X_{ki}, k=1,2,\ldots,s;\ i=1,2,\ldots,n.$

$$X_{ki} = -\frac{In\left(1-u\right)^{\frac{1}{2\alpha}}}{\beta\lambda^{k}}, \qquad k=1,2,\ldots,s; \quad i=1,2,\ldots,n.$$

Where X_{ki} is obtained for n=20, 40 and 60.

3. The values of parameters and numbers of the stress levels are chosen to be $\alpha=1, \beta=2.8, \lambda=1.1$ and $s=4\ or\ 6$.
4. By using optim() function, we obtain ML estimates, the mean squared error (MSE), relative absolute bias (RAB), relative error (RE) and lower and upper bound of 95% and 99% confidence intervals for different sample sizes n=20,40 and 60. The results obtained in the above simulation study are summarized in Table1 & 2.

Table 1: Simulation results of Generalized Rayleigh distribution using GP at $\alpha=1, \beta=2.8, \lambda=1.1$ and $s=4$.

Sample	Estimate	Mean	SE	√MSE	LCL	UCL
20	β	3.078	0.319	0.095	2.452 2.254	3.703 3.901
	λ	1.107	0.103	0.099	0.797 0.732	1.202 1.267
40	β	3.039	0.256	0.061	2.536 2.377	3.542 3.701
	λ	1.081	0.103	0.100	0.797 0.732	1.202 1.267
60	β	3.003	0.241	0.054	2.529 2.380	3.477 3.627
	λ	1.072	0.103	0.099	0.797 0.733	1.202 1.267

Table 2: Simulation results of Generalized Rayleigh distribution using GP at $\alpha = 1, \beta = 2.8, \lambda = 1.1$ and $s = 4$.

Sample	Estimate	Mean	SE	√MSE	LCL	UCL
20	β	3.078	0.218	0.078	2.491 2.311	3.628 3.807
	λ	0.977	0.112	0.103	0.797 0.732	1.202 1.267
40	β	3.039	0.209	0.074	2.520 2.348	3.609 3.781
	λ	0.981	0.103	0.101	0.797 0.732	1.202 1.267
60	β	2.953	0.192	0.044	2.551 2.416	3.407 3.543
	λ	0.992	0.020	0.100	0.784 0.732	1.202 1.267

5. Conclusions

In this study, the geometric process is introduced for the analysis of accelerated life testing under constant stress when the life data are from a generalized Rayleigh distribution. It is a better choice for life testing because of its simplicity in nature. The Mean, SE and RMSE of the parameters are obtained. Based on the asymptotic normality, the 95% and 99% confidence intervals of the parameters are also obtained.

The results show in Table 1 and Table 2 that the estimated values of β and λ are very close to true (or initial) value with very small SE and RMSE. As sample size increases, the value of SE and RMSE decreases and the confidence interval become narrower. For the Table 2, the maximum likelihood estimators have good statistical properties than the Table 1 for all sample sizes.

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Time-Dependent Analysis of a Single-Server Queuing Model with Discouraged Arrivals and Retention of Reneging Customers

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Abstract

In this paper, a finite capacity Markovian single-server queuing system with discouraged arrivals, reneging, and retention of reneging customers is studied. The time-dependent probabilities of the queuing system are ob- tained by using a computational technique based on the 4th order Runge- Kutta method. With the help of the time-dependent probabilities, we develop some important measures of performance of the system, such as expected system size, expected reneging rate, and expected retention rate. The time-dependent behavior of the system size probabilities and the ex- pected system size is also studied. Further, the variations in the expected system size, the expected reneging rate, and the expected retention rate with respect to the probability of retaining a reneging customer are also studied. Finally, the effect of discouragement in the same model is ana- lyzed.

Keywords: time-dependent analysis, single server queuing system, discouraged arrivals, reneging, Runge-Kutta method, retention

Introduction

Queuing systems are used in the design and analysis of computer-communication networks, production systems, surface and air traffic systems, service systems etc. The study of queueing systems help to manage waiting lines and to construct an optimal system for balancing customer waiting time with the idle time of the server Gnedenko and Kovalenko (1989). The enormous literature in queuing theory is available where the customers always wait in the queue until their service is completed. But in many practical situations customers become impatient and leave the systems without getting service. Therefore, queuing systems with customers' impatience have attracted a lot of attention. The study of customers' impatience in queueing theory is started in the early 1950's. Haight (1959) studies a single-server queue in steady-state with a Poisson input and exponential holding time, for various reneging distributions. Ancker and Gafarian [(1963a), (1963b)] analyze an $M/M/1/N$ queuing system with balking and reneging. In addition, the effect of reneging on an $M/M/1/N$ queue is investigated in the works of Abou El-Ata (1991), Zhang et al. (2006), Al Seddy et al. (2009), and Wang and Chang (2002). Kovalenko (1961) discusses some queuing systems with restrictions.

Queuing systems with discouraged arrivals are widely studied due to their significant role in managing daily queueing situations. In many practical situations, the service facility possesses defense mechanisms against long waiting lines. For instance, the congestion control mechanism prevents the formation of long queues in computer and communication systems by controlling the transmission rates of packets based on the queue length(of packets) at source or destination. Moreover, a long waiting line may force the servers to increase their rate of service as well as discourage prospective customers which results in balking. Hence, one should study queueing systems by taking into consideration the state-dependent nature of the system. In state-dependent queues the arrival and service rates depend on the number of customers in the system. The discouragement affects the arrival rate of the queueing system. Customers arrive in a Poisson fashion with rate that depends on the number of customers present in the system at that time i.e. $\frac{\lambda}{n+1}$. Morse (1958) considers discouragement in which the arrival rate falls according to a negative exponential law. Natvig (1974), Van Doorn (1981), Sharma and Maheswar (1993), and Parthasarathy and Selvaraju (2001) have also studied the discouraged arrivals queuing systems. Ammar et. al (2012) derive the transient solution of an M/M/1/N queuing model with discouraged arrivals and reneging by employing matrix method. Abdul Rasheed and Manoharan (2016) study a Markovian queueing system with discouraged arrivals and self-regulatory servers. They discuss the steady-state behavior of the system. Rykov (2001) considers a multi-server controllable queuing systems with heterogeneous servers. He studies several monotonicity properties of optimal policies for this system. Koba and Kovalenko (2002) study retrial queuing systems which are used in the analysis of aircraft landing process. Efrosinin and Rykov (2008) study a multi-server heterogeneous queuing system and obtain its steady-state solution. They derive the waiting and sojourn time distributions. They also study the optimal control of the queuing system. Rykov (2013) generalizes the slow server problem to include additional cost structure. He finds that the optimal policy for the problem has a monotone property. Sani et al. (2017) perform the reliability analysis of a system subjected to deterioration before failure. They use system state transition probabilities to derive the Markov models of the system.

Queuing systems with customers' impatience have negative impact on the performance of the system, because it leads to the loss of potential customers. Kumar and Sharma (2012a) take this practically valid aspect into account and study an $M/M/1/N$ queuing system with reneging and retention of reneging customers. Kumar (2013) obtains the transient solution of an $M/M/c$ queue with balking, reneging and retention of reneging customers. Kumar and Sharma (2014) obtain the steady-state solution of a Markovian single server queueing system with discouraged arrivals and retention of reneging customers by using iterative method.

The steady-state results do not reveal the actual functioning of the system. Moreover, stationary results are mainly used within the system design process and it cannot give insight into the transient behavior of the system. That is why, we extend the work of Kumar and Sharma (2014) in the sense that the time-dependent analysis of the model is performed. The time-dependent numerical behavior is studied by using a numerical technique Runge-Kutta method.

1 Queuing Model Description

In this section, we describe the queueing model. The model is based on following assumptions:

1. We consider a single-server queueing system in which the customers arrive in a Poisson fashion with rate that depends on the number of customers present in the system at that time i.e. $\frac{\lambda}{n+1}$.
2. There is single server and the service time distribution is negative exponential with parameter μ.
3. Arriving customers form a single waiting line based on the order of their arrivals and are served according to the first-come, first-served (FCFS) discipline.
4. The capacity of the system is finite (say N).
5. A queue gets developed when the number of customers exceeds the number of

servers, that is, when $n > 1$. After joining the queue each customer will wait for a certain length of time T (say) for his service to begin. If it has not begun by then he may get renege with probability p and may remain in the queue for his service with probability q $(= 1 - p)$ if certain customer retention strategy is used. This time T is a random variable which follows negative exponential distribution with parameter ξ. The reneging rate is given by

$$\xi_n = \begin{cases} 0, & 0 < n \leq 1 \\ (n-1)\xi, & n \geq 2 \end{cases}$$

2 Mathematical Model

Let $\{X(t), t \geq 0\}$ be the number of customers present in the system at time t. Let $P_n(t) = P\{X(t) = n\}, n = 0,1, \ldots$ be the probability that there are n customers in the system at time t. We assume that there is no customer in the system at $t = 0$.

The differential-difference equations of the model are:

$$\frac{dP_0(t)}{dt} = -\lambda P_0(t) + \mu P_1(t), \quad (1)$$

$$\frac{dP_n(t)}{dt} = -\left[\left(\frac{\lambda}{n+1}\right) + \mu + (n-1)\xi p\right] P_n(t) + \left(\frac{\lambda}{n}\right) P_{n-1}(t) + (\mu + n\xi p)P_{n+1}(t), 1 \leq n < N \quad (2)$$

$$\frac{dP_N(t)}{dt} = \left(\frac{\lambda}{N}\right) P_{N-1}(t) - (\mu + (N-1)\xi p)P_N(t), \quad (3)$$

3 Transient analysis of the model

In this section, we perform the time-dependent analysis of a finite capacity single-server Markovian Queuing model with discouraged arrivals and retention of reneging customers using Runge-Kutta method of fourth order (R-K 4). The "ode45" function of MATLAB software is used to find the time-dependent numerical results corresponding to the differential-difference equations of the model.

We study the following performance measures in transient state:

1. Expected System Size ($L_s(t)$)

$$L_s(t) = \sum_{n=0}^{N} nP_n(t)$$

2. Average Reneging Rate ($R_r(t)$)

$$R_r(t) = \sum_{n=1}^{N} (n-1)\xi p P_n(t)$$

3.Average Retention Rate ($R_r(t)$)

$$R_R(t) = \sum_{n=1}^{N} (n-1)\xi q P_n(t)$$

Now, we perform the time-dependent numerical analysis of the model with the help of a numerical example. We take $N = 10$, $\lambda = 2, \mu = 3, \xi = 0.1$, and $p = 0.4$. The results are presented in the form of Figures 1-5. Following are the main observations:

In Figure 1, the probabilities of number of customers in the system at different time points are plotted. We observe that the probability values $P_1(t)$, $P_2(t)$, ..., $P_{10}(t)$ increase gradually until they reach stable values except the probability curve $P_0(t)$ which decreases rapidly in the beginning and then attains steady-state with the passage of time.

Figure 2 shows the effect of the probability of retaining a reneging customer on the expected system size in transient state. One can observe that as the probability of retaining a reneging customer increases, the expected system size also increases. This establishes the role of probability of retention associated with any customer retention strategy.

In Figure 3, the change in average reneging rate with the change in probability of retaining a reneging customer is shown. One can observe that there is a proportional decrease in average reneging rate with the increase in probability of retention, q.

The variation in average retention rate with probability of retention is shown in Figure 4. We can see that there is a proportional increase in $R_R(t)$ with increase in q, which justifies the functioning of the model.

In figure 5, the impact of discouraged arrivals on the performance of the system is shown. We compare two single server finite capacity Markovian queuing systems having retention of reneging customers with and without discouraged arrivals. One can see from Figure 5 that the expected system size is always lower in case of discouraged arrivals as compare to the queuing model without discouragement.

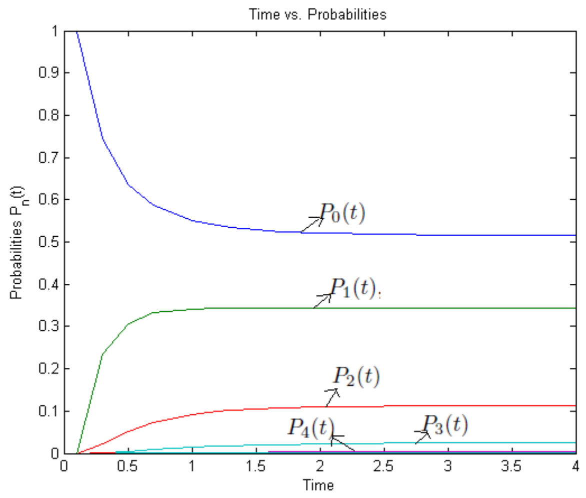

Figure 1: The probability values for different time points are plotted for the case $N = 10$, $\lambda = 2, \mu = 3, \xi = 0.1$, and $p = 0.4$

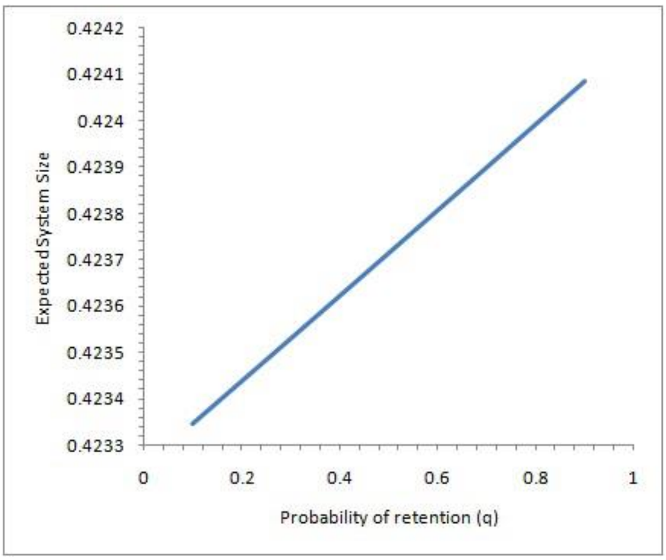

Figure 2: The expected system sizes versus probability of retention (q) are plotted for the case $N = 10$, $\lambda = 2, \mu = 3, \xi = 0.1, t = 0.5$, and $q = 0.1, 0.2, \ldots, 0.9$

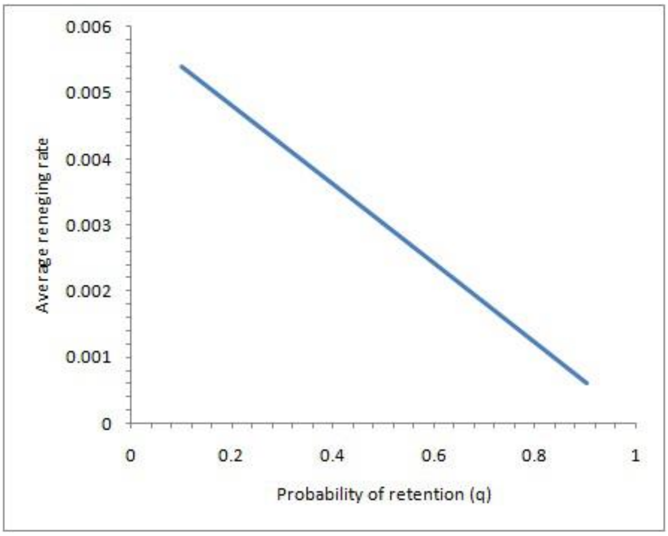

Figure 3: Variation of average reneging rate with the variation in probability of retention for the case $N = 10$, $\lambda = 2, \mu = 3, \xi = 0.1, t = 0.5$, and $q = 0.1, 0.2, \ldots, 0.9$

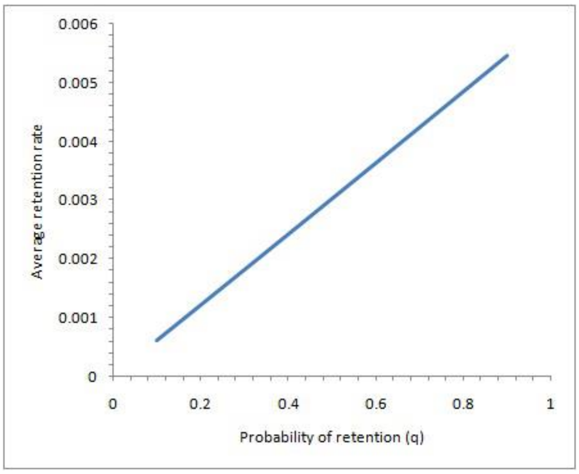

Figure 4: Variation of average retention rate with the variation in probability of retention for the case $N = 10$, $\lambda = 2, \mu = 3, \xi = 0.1, t = 0.5$, and $q = 0.1, 0.2, \ldots, 0.9$

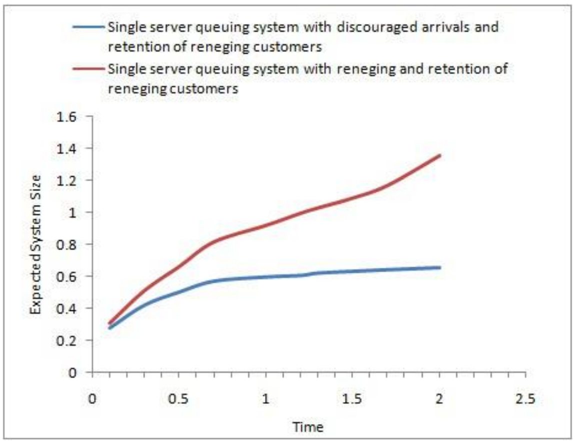

Figure 5: The impact of discouragement on expected system size

4 Conclusions

The time-dependent analysis of a single-server queuing system with discouraged arrivals, reneging and retention of reneging customers is performed by using Runge Kutta method. The numerical results are computed with the help of MATLAB software. The effect of probability of retaining a reneging customer on various performance measures is studied. We also study the impact of discouraged arrivals on the system performance.

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