Time-Dependent Analysis of a Single-Server Queuing Model with Discouraged Arrivals and Retention of Reneging Customers

Rakesh Kumar∗ & Sapana Sharma
•
Department of Mathematics, Shri Mata Vaishno Devi University,
Katra Jammu and Kashmir, India-182320
Email: rakesh stat kuk@yahoo.co.in,
sapanasharma736@gmail.com
∗ Corresponding Author

Abstract

In this paper, a finite capacity Markovian single-server queuing system with discouraged arrivals, reneging, and retention of reneging customers is studied. The time-dependent probabilities of the queuing system are ob- tained by using a computational technique based on the 4th order Runge- Kutta method. With the help of the time-dependent probabilities, we develop some important measures of performance of the system, such as expected system size, expected reneging rate, and expected retention rate. The time-dependent behavior of the system size probabilities and the ex- pected system size is also studied. Further, the variations in the expected system size, the expected reneging rate, and the expected retention rate with respect to the probability of retaining a reneging customer are also studied. Finally, the effect of discouragement in the same model is ana- lyzed.

Keywords: time-dependent analysis, single server queuing system, discouraged arrivals, reneging, Runge-Kutta method, retention

Introduction

Queuing systems are used in the design and analysis of computer-communication networks, production systems, surface and air traffic systems, service systems etc. The study of queueing systems help to manage waiting lines and to construct an optimal system for balancing customer waiting time with the idle time of the server Gnedenko and Kovalenko (1989). The enormous literature in queuing theory is available where the customers always wait in the queue until their service is completed. But in many practical situations customers become impatient and leave the systems without getting service. Therefore, queuing systems with customers' impatience have attracted a lot of attention. The study of customers' impatience in queueing theory is started in the early 1950's. Haight (1959) studies a single-server queue in steady-state with a Poisson input and exponential holding time, for various reneging distributions. Ancker and Gafarian [(1963a), (1963b)] analyze an $M/M/1/N$ queuing system with balking and reneging. In addition, the effect of reneging on an $M/M/1/N$ queue is investigated in the works of Abou El-Ata (1991), Zhang et al. (2006), Al Seddy et al. (2009), and Wang and Chang (2002). Kovalenko (1961) discusses some queuing systems with restrictions.

Queuing systems with discouraged arrivals are widely studied due to their significant role in managing daily queueing situations. In many practical situations, the service facility possesses defense mechanisms against long waiting lines. For instance, the congestion control mechanism prevents the formation of long queues in computer and communication systems by controlling the transmission rates of packets based on the queue length(of packets) at source or destination. Moreover, a long waiting line may force the servers to increase their rate of service as well as discourage prospective customers which results in balking. Hence, one should study queueing systems by taking into consideration the state-dependent nature of the system. In state-dependent queues the arrival and service rates depend on the number of customers in the system. The discouragement affects the arrival rate of the queueing system. Customers arrive in a Poisson fashion with rate that depends on the number of customers present in the system at that time i.e. $\frac{\lambda}{n+1}$. Morse (1958) considers discouragement in which the arrival rate falls according to a negative exponential law. Natvig (1974), Van Doorn (1981), Sharma and Maheswar (1993), and Parthasarathy and Selvaraju (2001) have also studied the discouraged arrivals queuing systems. Ammar et. al (2012) derive the transient solution of an M/M/1/N queuing model with discouraged arrivals and reneging by employing matrix method. Abdul Rasheed and Manoharan (2016) study a Markovian queueing system with discouraged arrivals and self-regulatory servers. They discuss the steady-state behavior of the system. Rykov (2001) considers a multi-server controllable queuing systems with heterogeneous servers. He studies several monotonicity properties of optimal policies for this system. Koba and Kovalenko (2002) study retrial queuing systems which are used in the analysis of aircraft landing process. Efrosinin and Rykov (2008) study a multi-server heterogeneous queuing system and obtain its steady-state solution. They derive the waiting and sojourn time distributions. They also study the optimal control of the queuing system. Rykov (2013) generalizes the slow server problem to include additional cost structure. He finds that the optimal policy for the problem has a monotone property. Sani et al. (2017) perform the reliability analysis of a system subjected to deterioration before failure. They use system state transition probabilities to derive the Markov models of the system.

Queuing systems with customers' impatience have negative impact on the performance of the system, because it leads to the loss of potential customers. Kumar and Sharma (2012a) take this practically valid aspect into account and study an $M/M/1/N$ queuing system with reneging and retention of reneging customers. Kumar (2013) obtains the transient solution of an $M/M/c$ queue with balking, reneging and retention of reneging customers. Kumar and Sharma (2014) obtain the steady-state solution of a Markovian single server queueing system with discouraged arrivals and retention of reneging customers by using iterative method.

The steady-state results do not reveal the actual functioning of the system. Moreover, stationary results are mainly used within the system design process and it cannot give insight into the transient behavior of the system. That is why, we extend the work of Kumar and Sharma (2014) in the sense that the time-dependent analysis of the model is performed. The time-dependent numerical behavior is studied by using a numerical technique Runge-Kutta method.

1 Queuing Model Description

In this section, we describe the queueing model. The model is based on following assumptions:

1. We consider a single-server queueing system in which the customers arrive in a Poisson fashion with rate that depends on the number of customers present in the system at that time i.e. $\frac{\lambda}{n+1}$.
2. There is single server and the service time distribution is negative exponential with parameter μ.
3. Arriving customers form a single waiting line based on the order of their arrivals and are served according to the first-come, first-served (FCFS) discipline.
4. The capacity of the system is finite (say N).
5. A queue gets developed when the number of customers exceeds the number of

servers, that is, when $n > 1$. After joining the queue each customer will wait for a certain length of time T (say) for his service to begin. If it has not begun by then he may get renege with probability p and may remain in the queue for his service with probability $q\ (= 1 - p)$ if certain customer retention strategy is used. This time T is a random variable which follows negative exponential distribution with parameter ξ. The reneging rate is given by

$$\xi_n = \begin{cases} 0, & 0 < n \leq 1 \\ (n-1)\xi, & n \geq 2 \end{cases}$$

2 Mathematical Model

Let $\{X(t), t \geq 0\}$ be the number of customers present in the system at time t. Let $P_n(t) = P\{X(t) = n\}, n = 0,1, \ldots$ be the probability that there are n customers in the system at time t. We assume that there is no customer in the system at $t = 0$.

The differential-difference equations of the model are:

$$\frac{dP_0(t)}{dt} = -\lambda P_0(t) + \mu P_1(t), \tag{1}$$

$$\frac{dP_n(t)}{dt} = -\left[\left(\frac{\lambda}{n+1}\right) + \mu + (n-1)\xi p\right] P_n(t) + \left(\frac{\lambda}{n}\right) P_{n-1}(t) + (\mu + n\xi p)P_{n+1}(t), 1 \leq n < N \tag{2}$$

$$\frac{dP_N(t)}{dt} = \left(\frac{\lambda}{N}\right) P_{N-1}(t) - (\mu + (N-1)\xi p)P_N(t), \tag{3}$$

3 Transient analysis of the model

In this section, we perform the time-dependent analysis of a finite capacity single-server Markovian Queuing model with discouraged arrivals and retention of reneging customers using Runge-Kutta method of fourth order (R-K 4). The "ode45" function of MATLAB software is used to find the time-dependent numerical results corresponding to the differential-difference equations of the model.

We study the following performance measures in transient state:

1. Expected System Size ($L_s(t)$)

$$L_s(t) = \sum_{n=0}^{N} nP_n(t)$$

2. Average Reneging Rate ($R_r(t)$)

$$R_r(t) = \sum_{n=1}^{N} (n-1)\xi p P_n(t)$$

3.Average Retention Rate ($R_r(t)$)

$$R_R(t) = \sum_{n=1}^{N} (n-1)\xi q P_n(t)$$

Now, we perform the time-dependent numerical analysis of the model with the help of a numerical example. We take $N = 10$, $\lambda = 2, \mu = 3, \xi = 0.1$, and $p = 0.4$. The results are presented in the form of Figures 1-5. Following are the main observations:

In Figure 1, the probabilities of number of customers in the system at different time points are plotted. We observe that the probability values $P_1(t)$, $P_2(t)$, ..., $P_{10}(t)$ increase gradually until they reach stable values except the probability curve $P_0(t)$ which decreases rapidly in the beginning and then attains steady-state with the passage of time.

Figure 2 shows the effect of the probability of retaining a reneging customer on the expected system size in transient state. One can observe that as the probability of retaining a reneging customer increases, the expected system size also increases. This establishes the role of probability of retention associated with any customer retention strategy.

In Figure 3, the change in average reneging rate with the change in probability of retaining a reneging customer is shown. One can observe that there is a proportional decrease in average reneging rate with the increase in probability of retention, q.

The variation in average retention rate with probability of retention is shown in Figure 4. We can see that there is a proportional increase in $R_R(t)$ with increase in q, which justifies the functioning of the model.

In figure 5, the impact of discouraged arrivals on the performance of the system is shown. We compare two single server finite capacity Markovian queuing systems having retention of reneging customers with and without discouraged arrivals. One can see from Figure 5 that the expected system size is always lower in case of discouraged arrivals as compare to the queuing model without discouragement.

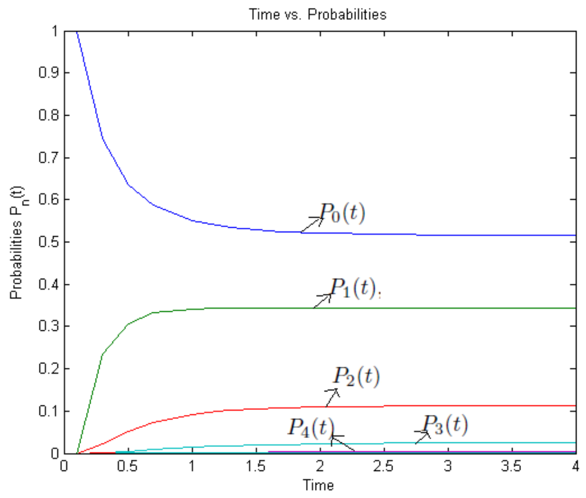

Figure 1: The probability values for different time points are plotted for the case $N = 10$, $\lambda = 2, \mu = 3, \xi = 0.1$, and $p = 0.4$

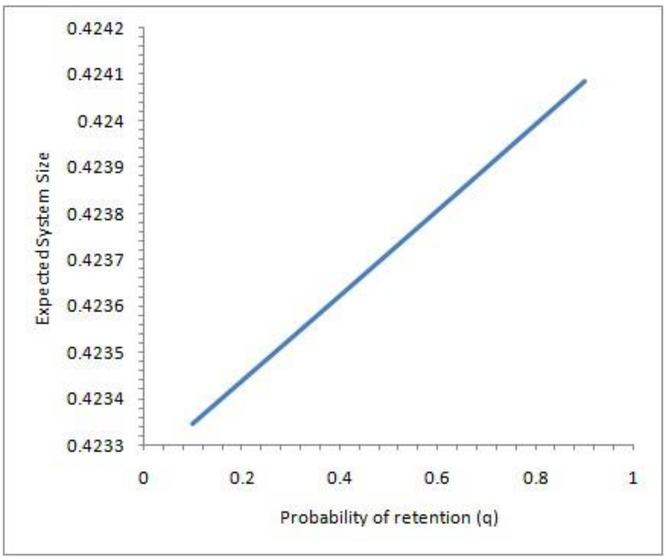

Figure 2: The expected system sizes versus probability of retention (q) are plotted for the case $N = 10$, $\lambda = 2, \mu = 3, \xi = 0.1, t = 0.5$, and $q = 0.1, 0.2, \ldots, 0.9$

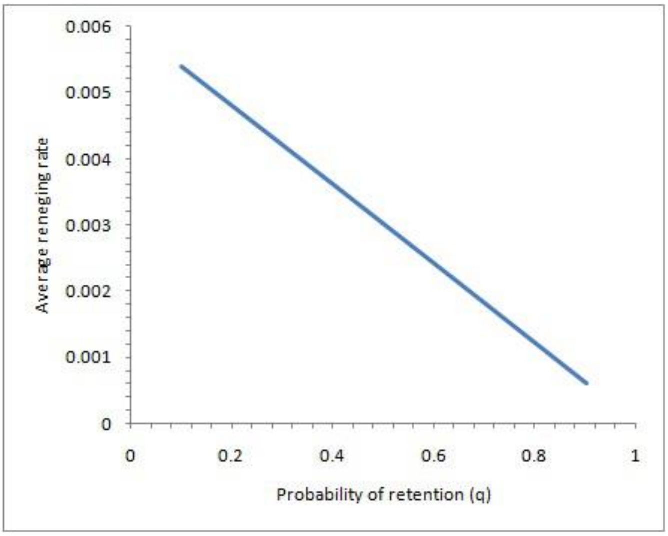

Figure 3: Variation of average reneging rate with the variation in probability of retention for the case $N = 10$, $\lambda = 2, \mu = 3, \xi = 0.1, t = 0.5$, and $q = 0.1, 0.2, \ldots, 0.9$

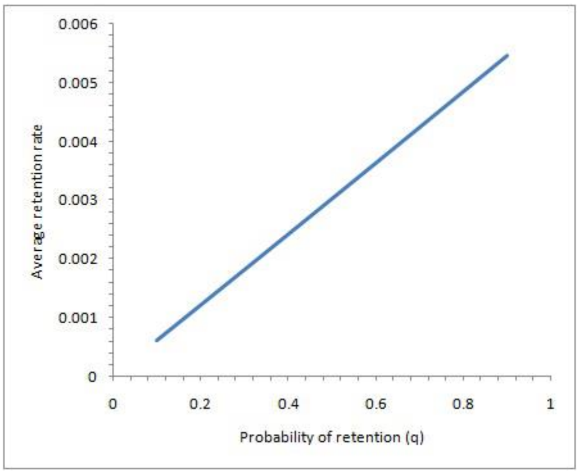

Figure 4: Variation of average retention rate with the variation in probability of retention for the case $N = 10$, $\lambda = 2, \mu = 3, \xi = 0.1, t = 0.5$, and $q = 0.1, 0.2, \ldots, 0.9$

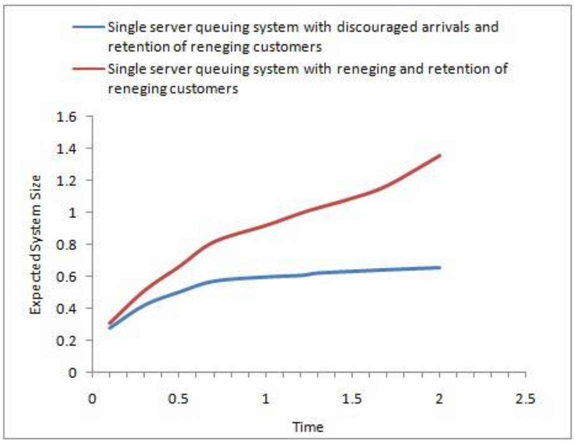

Figure 5: The impact of discouragement on expected system size

4 Conclusions

The time-dependent analysis of a single-server queuing system with discouraged arrivals, reneging and retention of reneging customers is performed by using Runge Kutta method. The numerical results are computed with the help of MATLAB software. The effect of probability of retaining a reneging customer on various performance measures is studied. We also study the impact of discouraged arrivals on the system performance.

Acknowledgements

One of the authors (Dr. Rakesh Kumar) would like to thank the UGC, New Delhi, India for financial support given to him for this research work under the Major Research Project vide letter no. F.-43-434/2014(SR).

References

[1] Abdul Rasheed, K.V., Manoharan, M. (2016). Markovian queueing system with discouraged arrivals and self-regulatory servers. *Advances in Operations Research*, -Vol. 2016, Article ID 2456135, -P. 11 pages, doi:10.1155/2016/2456135.

[2] Abou-El-Ata, M.O. and Hariri, A.M.A. (1992). The M/M/c/N queue with balking and reneging. *Computers and Operations Research*, 19:713-716.

[3] Al-Seedy, R.O., El-Sherbiny, A.A., El-Shehawy, S.A, Ammar, S.I. (2009). Transient solution of the M/M/c queue with balking and reneging. *Computers and Mathematics with Applications*, 57:1280-1285.

[4] Ammar, S.I., El-Sherbiny, A.A., Al-Seedy, R.O. (2012). A matrix approach for the transient solution of an M/M/1/N queue with discouraged arrivals and reneging, *International Journal of Computer Mathematics*, 89:482-491.

[5] Ancker. Jr., C.J., Gafarian A.V. (1963a). Some queuing problems with balking and reneging I. *Operations Research*, 11:88-100

[6] Ancker. Jr., C.J., Gafarian A.V. (1963b). Some queuing problems with balking and reneging II. *Operations Research*, 11:928-937.

[7] Efrosinin, D., Rykov, V. (2008). On performance characteristics for queueing systems with heterogeneous servers. *Automation and Remote Control*, 69:61-75.

[8] Gnedenko, B.V., Kovalenko, I. N. (1989). Introduction to Queuing Theory. *Birkhäuser, 2nd Edition*.

[9] Haight, F. A. (1959). Queuing with reneging. *Metrika*, 2:186-197.

[10] Koba, E. V., Kovalenko, I. N. (2002). Three Retrial Queueing Systems Representing Some Special Features of Aircraft Landing. *Journal of Automation and Information Sciences*, 34:DOI · 10.1615/JAutomatInfScien.v34.i4.10, 4 pages.

[11] Kovalenko, I. N. (1961). Some Queuing Problems with Restrictions. *Theory of Probability and Its Applications*, 6:204-208.

[12] Kumar, R. and Sharma, S. K. (2012a). An M/M/1/N queuing system with retention of reneged customers. *Pakistan Journal of Statistic and Operation Research*, 8:859-866.

[13] Kumar, R. and Sharma, S. K. (2014). A single-server Markovian queuing system with discouraged arrivals and retention of reneged customers. *Yugoslav Journal of Operations Research*, 24:119-216.

[14] Kumar, R. (2013). Economic analysis of an M/M/c/N queuing model with balking, reneging and retention of reneged customers. *Opsearch*, 50:383-403.

[15] Morse, P.M. Queues, inventories and maintenance. Wiley, New York, 1958.

[16] Natvig, B. (1974). On the transient state probabilities for a queueing model where potential customers are discouraged by queue length. *Journal of Applied Probability*, 11:345-354.

[17] Parthasarathy, P.R. and Selvaraju, N. (2001). Transient analysis of a queue where potential customers are discouraged by queue length. *Mathematical Problems in Engineering*, 7:433-454.

[18] Rykov, V. (2001). Monotone Control of Queueing Systems with Heterogeneous Servers. *Queueing Systems*, 37391-403.

[19] Rykov, V. (2013). On a Slow Server Problem. *Stochastic Orders in Reliability and Risk*, 351-361.

[20] Sani, B., Gatawa, R. I. and Yusuf, I. (2017). Reliability Assessment of Deteriorating System. *Reliability: Theory and Applications*, 12:20-29.

[21] Sharma, O.P., Maheswar, M.V.R. (1993). Transient behavior of a simple queue with discouraged arrivals. *Optimization*, 27:283-291.

[22] Van Doorn, E. A. (1981). The transient state probabilities for a queueing model where potential customers are discouraged by queue length. *Journal of Applied Probability,* 18:499-506.

[23] Wang, K.H., Chang, Y.C. (2002). Cost analysis of a finite M/M/R queueing system with balking, reneging and server breakdowns. *Mathematical Methods of Operations Research* 56:169-180.

[24] Zhang, Y., Yue, D., Yue, W. (2006). Optimal performance analysis of an M/M/1/N queue system with balking, reneging and server vacation. *International Journal of Pure and Applied Mathematics*, 28:101-115.