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Key words: reliability theory, mathematical methods, Solov'ev

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A service station offers several services of which one is the required and the others are harmful (or destructive) for each customer. At the time when selected for service customers enter in correct mode of service according to a Bernoulli process with parameter p *which is the probability of being selected in correct mode. Arrival follows Markovian arrival process and service time is Phase type distributed in both undesired and required phases. An exponentially distributed threshold clock starts ticking if a customer enters to incorrect mode and the service is terminated if the clock realizes before the customer is transferred to correct service mode. The rate of loss of customers, rate of customers leaving with correct service starting with incorrect service are computed. Authors examined a queueing model offering* n *distinct services to which arrival is according to a MAP forming a single line. Service time has phase type distribution. A single server serves the customers. The service station provides two types of services - one is desirable and other is unwanted for each customer. If the service starts in an undesirable state then a clock also simultaneously starts ticking. In case this clock realizes before the exact requirement of the server is realized, then that customer leaves the system forever without being eligible for the service that he actually requires. On the other extreme, in case the correct identification of required service occurs before realization of clock, then the customer is served in that state and then leaves the system. In case right at the beginning of service the exact requirement of service is identified, then the customer starts getting that service right at the time when taken for service. Several system performance measures are evaluated. Applications of the model in hospital, telecommunication etc are indicated. Stochastic decomposition of the system state is analyzed. Some particular cases are indicated.*

Keywords: desired/undesired service states, random threshold clock, Markovian arrival process

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Keywords: computational methods, conceptual algorithms, gradual failure, parallel computing, parametric synthesis, projected reliability

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Keywords: the Weibull-Gnedenko distribution, reliability theory, renewal density (intensity), numerical methods, collocation knots, moments generating function

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Keywords: flow with limited aftereffect, thinning of the flow, time-dependent function of thinning, Laplace-Stieltjes transform, inverse Laplace-Stieltjes transform.

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Keywords: Geometric process, Generalized Rayleigh Distribution, Maximum *Likelihood Estimator, Fisher Information Matrix, Asymptotic Confidence Interval, Simulation Study.*

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Keywords: time-dependent analysis, single server queuing system, discouraged arrivals, reneging, Runge-Kutta method, retention.

In anniversary of professor Alexander Solov'ev

Mikhail Yastrebenetsky,
•
Gnedenko-Forum president
53, Chernishevska str., of.2, 61002, Kharkov, Ukraine
E-mail: ma_yastreb@mail.ru

Abstract

Professor Alexander Dmitrievich Solov'ev (1927-2001), professor of Lomonosov Moscow State University, was one of the founders of reliability theory, author the classical books, the nearest colleague of academic Boris Gnedenko. Aim of this short article- to mark 90-th birth anniversary of Alexander Solov'ev.

Key words: reliability theory, mathematical methods, Solov'ev

This issue of our journal is devoted to professor Alexander Dmitrievich Solov'ev (1927-2001), one of the founders of reliability theory. Specialists of reliability from many countries marked Alexander Solov'ev's 90-th birth anniversary by organization of International Conference on "Analytical and Computational Methods in Probability Theory and its Applications", held by Lomonosov Moscow State University and by RUDN university (information support- Gnedenko- Forum). Some participant of this Conference were pupils of prof. Solov'ev, and all participants know Solov'ev works very well. Reports at this conference are the main subject of this issue.

Alexander Solov'ev (below- AS) studied at the Mechanics and Mathematics faculty of Lomonosov Moscow State University and completed his studies in 1951. All of his creative life was connected with this faculty: the postgraduate education, teaching as a teaching assistant, an assistant professor from 1958 and a professor since 1975. His supervisor was professor A. O. Gelfond. AS defended his thesis in 1955; the topic of the dissertation was "The problem of moments for analytic functions". He had the outstanding analytical technique and phenomenal mastery of mathematical analysis.

Then AS changed the area of his interest – it became probabilistic theory by influenced by acad. Boris Gnedenko. AS was one of the first mathematician who began to create the mathematic basis of reliability theory. It was 1965 year when the book Mathematical methods in reliability theory (authors- B.Gnedenko, Yu. Belyaev, A.Solov'ev) was published. AS wrote in this book Chapter 2 "Reliability characteristics", Chapter 5 "Redundancy without restoration", Chapter 6 "Redundancy with restoration". This book was translated into many languages, became a classic for many generations of specialists around the all world. I think that every reliability specialist has this book on his shelf. When I began to read this book for the first time, I thought that all problems in reliability theory had been decided and wasn't new activity in this science. But this opinion changed very quickly- I understood that this book would be a gate to reliability theory and a platform for future works.

AS defended his doctoral dissertation "Queuing systems with fast maintaining" in 1972. In 1979 AS together with B.V. Gnedenko, Yu.K Beliaev, V.A.Kashtanov et al were awarded a State Premium for the work "Elaboration and implementation of the complex of methods for equipment high reliability assurance".

The book Problems of Mathematical Theory of Reliability, where AS was co-author together with B.V. Gnedenko, Yu.K. Beliaev, V.A.Kashtanov at al, was published in 1983. This book evolved the principles what were described in the past book Mathematical methods in reliability theory. AS wrote in new book part 1 "Analitical methods of reliability estimation". This part consisted: Chapter 1 "Reliability of elements", Chapter 2 "Limited theorems", Chapter 3 "Reliability of systems" and included: asymptotic exact double-side estimation creation, convergence to exponential distribution in some classes of random values, limited theorems for regenerative processes and their applications to different tasks of reliability.

AS's activity in reliability problems was very high- it was regular consulting for scientists and engineers in the Reliability Cabinet of Moscow Politechnic Museum, lectures in this Museum, which were published in a set of brochures, participation in seminars on mathematical theory of reliability in Lomonosov Moscow State University. The authority of AS was undisputable, people from different cities arrived to him. I don't know any case when someone didn't receive help from AS.

One of the new directions of AS's activity was history of mathematics (together with his wife Svetlana Petrova who was a professional historian of mathematics).

I'd like to add some personal information about my meetings with AS. I remember very well the day and place of my first acquaintance with him- 1970, October 2, Dilijan resort in the mountain part of Caucasia republic Armenia, rest home for composers. On this day School of queening theory under the leadership of academic Boris Gnedenko began. The School was organized by the Department of Probability Theory of Moscow State Lomonosov University by. Music was heard from open windows of composer cottages, it was wonderful harmony of mountains, mathematics and music. AS was one of the key people at this School.

Many young specialists in reliability and queues theory participated in this School: Alexander Andronov, Illia Gertsbah, Bojan Dimitrov, Victor Kashtanov, Mikhail Fedotkin, Volodymir Rykov, whose names are well- known now. Two weeks of this School was the start of future contacts between us for many years.

I had the good luck to sit in the restaurant during this School at the same table with AS: it wasn't only eating of very tasty national Armenian food, but it was feast of joy. AS was the center of attention- he told interesting stories, jokes; it was a theatre of one actor. We left AS only during the lessons and reports and playing football- our main type of rest.

My contacts with AS continued after Dilijan School. He reviewed my articles in the Journal "Proceedings of the USSR Academy of Sciences. Technical Cybernetics", we discussed different problems and stay in his hospitable home was a big pleasure for me. When I ended my doctoral dissertation "Operation Reliability of Industrial Control Systems" in 1974, academic Gnedenko gave proposers me about the choose of opponents. He proposed AS, prof. Igor Ushakov and as necessary- a member of the Scientific Council at the place of defense (it was Kharkov Polytechnic University).

When I met AS and Ushakov at Kharkov Railway station, it was hard to recognize AS: instead of the long artistic hair he had simple short haircut. AS explained to me that he was afraid of a negative reaction of the conservative provincial scientific council to the bohemian appearance of a Moscow professor. I don't know he could repeat the same action. After defense around of friendly table at my home, AS of course was in the center of attraction to everyone, and especially the women.

I was very glad to see AS in my native city- Kharkov. AS twice reported at my seminars on reliability problems. First his report was devoted to asymptotic methods in reliability theory, second- optimal discipline of renewal systems maintenance. His reports collected full auditorium in Kharkov Technique House and there were a lot of questions and long discussion after his reports.

AS liked to go to Kharkov market, talked with saleswomen, chose the most tasty and fresh. After that we would go to my home and to the horror of my family AS came to the kitchen, dressed an apron and prepared the food himself according his own recipes. It was very tasty!

During one of my visit to AS, he in my presence received the letter from Riga from our colleague Ilia Gertsbakh. He wrote that soon he would leave USSR with his family and migrate to Israel. Immigration from USSR was an unusual decision for that time, connected with a lot of different and difficult troubles. AS read the letter with full understanding. I remembered very well one sentence from Gertsbakh's letter: "You are a good mensch, Alexander Dmitrievich". I think that all my colleagues who knew Alexander Dmitrievich Solov'ev agree with this statement.

Single server queues with several services

[1] A. Krishnamoorthy ^{1,a}, [2] Vladimir Vishnevsky ^{2,b}
[3] Manjunath A. S. ^{3,c}, [4] Dhanya Shajin ^{4,d}

●

^{1} Department of Mathematics Cochin University
of Science and Technology Cochin-682022, India.
^{2}V. A. Trapeznikov Institute of Control Sciences of Russian Academy of Sciences,
65 Profsoyuznaya Street, Moscow 117997, Russia.
^{3}Department of Mathematics Government Victoria
College, Palakkad-678001, India.
^{4}Department of Mathematics Amrita School of Arts and Sciences
Amrita Vishwa Vidyapeetham, Brahmasthanam
Kochi-682024, India.

Abstract

A service station offers several services of which one is the required and the others are harmful (or destructive) for each customer. At the time when selected for service customers enter in correct mode of service according to a Bernoulli process with parameter p which is the probability of being selected in correct mode. Arrival follows Markovian arrival process and service time is Phase type distributed in both undesired and required phases. An exponentially distributed threshold clock starts ticking if a customer enters to incorrect mode and the service is terminated if the clock realizes before the customer is transferred to correct service mode. The rate of loss of customers, rate of customers leaving with correct service starting with incorrect service are computed.

Keywords: desired/undesired service states, random threshold clock, Markovian arrival process

1 Introduction

In the analysis of classic queueing models it is assumed that the server is completely aware of the exact service requirement of a customer (see Gross and Harris [2]). Quite often only one type of service is offered by the system and so conflict does not occur. It is also true that the customer

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[1] Author for correspondence: email: achyuthacusat@gmail.com ^{a} Research is supported by the Department of Science and Technology, Government of India under grant INT/RUS/RSF/P-15 and Kerala State Council for Science, Technology & Environment (No.001/KESS/2013/CSTE).

[2] email: vishn@inbox.ru ^{b} Research supported by the Russian Science Foundation and the Department of Science and Technology (India) via grant No. 16-49-02021 for the joint research project by the V. A. Trapeznikov Institute of control Science and the CMS College Kottayam.

[3] email: manjunadem@gmail.com ^{c} Research is supported by the University Grants Commission, Government of India, under Faculty Development Programme (Grant No. F.FIP/12th Plan/KLMG003TF05) in Department of Mathematics, Cochin University of Science and Technology, Cochin-682022, India.

[4] email: dhanya.shajin@gmail.com ^{d} Research is supported by Kerala State Council for Science, Technology & Environment (No.001/KESS/2013/CSTE).

knows the type of service he needs. Thus there is no conflict on the service provided to the customer. However, there are several real life situations where the server and/customer are (is) not knowledgeable about the exact service requirement. This is especially the case when several types of services are available at a service station. As concrete example we have vehicles for repair at service stations, patients consulting physicians for diagnosis and medication. If the right service required is not identified and instead the diagnosis turned out to be wrong the result could be disastrous. A wrong diagnosis and consequent service provided may sometimes turn out to be even fatal/ may result in the equipment getting service, rendered totally unusable. It is this type of problem that we analyze in this note.

In real life there are several service providing system offering a multitude of service. Neither the server nor the customer may be fully aware of the exact service requirement. This, very often, results in irreperable damage to the customer being served. We study such a system in this paper with each customer requiring a specific service. However, due to wrong diagnosis service does not start necessarily for the required one.

The article is organized as follows. In section 2, the mathematical model is described. This section also provides the steady state analysis and some performance measures. Several cases of the above model are considered in Sections 3 and 4. An illustration is given in Section 5. Numerical illustrative example is described in Section 6. In Section 7 we extend the above results to the case of arbitrarily distributed service time in the undesired and desired stages of service. However, we restrict $n_1 = n_2 = 1$. Further we relabel the undesired service as preliminary and desired as main services to suit certain context of application.

List of notations and abbreviations used:

$CTMC$: Continuous time Markov chain
$LIQBD$: Level independent quasi-birth and death process
MAP: Markovian arrival process
LST: Laplace Stieltjes Transform
e: Column vector of $1'$s with appropriate order

2 Model Description

The assumptions leading to the formulation of the mathematical model are

- An infinite capacity queueing system where a single server is providing both unwanted (incorrect) and required (correct) services.
- Arrival of customers to the system is according to the MAP (Markovian arrival process). In a MAP, the customers arrival is directed by an irreducible $CTMC$ (continuous time Markov chain) $\{\phi_t, t \geq 0\}$ with the state space $\{1,2,\ldots,m\}$. The transition intensities of the Markov chain $\{\phi_t, t \geq 0\}$ which are accompanied by arrival of k customers are described by the matrices $D_k, k = 0,1$. Vector $\boldsymbol{\eta}$ of the stationary distribution of the process $\{\phi_t, t \geq 0\}$ is the unique solution to the system
$$\boldsymbol{\eta}(D_0 + D_1) = \boldsymbol{\eta}D = 0 and \boldsymbol{\eta}e = 1. \qquad (1)$$
Fundamental rate λ of the MAP is given by $\lambda = \boldsymbol{\eta}D_1e$.
- A customer is selected for desired (required) service with probability p or to the incorrect service with probability $q = 1 - p$.
- (β_1, S_1) of order n_1 gives the PH-representation for the duration of the correct service time distribution when the service of a customer starts in correct service mode. Let ${}^{0}_{1}$ be such that $S_1e + S_1^0 = 0$. Let $\mu'_1 = \beta_1(-S_1)^{-1}e$ be the mean of this PH-representation.
- (β_2, S_2) of order n_2 gives the PH-representation for the duration of the incorrect service time distribution when the service of a customer starts in incorrect service mode. The rate

(vector) of loss is given by ${}^{0}_{2}$ and the rate (vector) of getting into correct service mode is given by S_2^0. Note that $S_2 e + \quad S_2^0 + \quad S_2^0 = 0$. Let $\mu'_2 = \beta_2(-S_2)^{-1}e$ be the mean of this PH-representation.

- (β_3, S_3) of order n_3 gives the PH-representation for the duration of the correct service time distribution when the customer has gone through incorrect service initially. Let ${}^{0}_{3}$ be such that $S_3 e + \quad S_3^0 = 0$. Let $\mu'_3 = \beta_3(-S_3)^{-1}e$ be the mean of this PH-representation. [**NOTE**: One can take this to be same as (β_1, S_1) but it looks more meaningful in most applications that the service time after going through incorrect one to be different from the use of directly getting into required service. Just something to keep in mind.]
- The service time of a customer can be modeled as a PH-distribution with representation (β, S) of order $n = n_1 + n_2 + n_3$, where

$$\beta = \left(p\,\beta_1, q\,\beta_{2,0}\right) \tag{2}$$

$$S = \begin{pmatrix} S_1 & 0 & 0 \\ 0 & S_2 & S_2^0\,\beta_3 \\ 0 & 0 & S_3 \end{pmatrix} \tag{3}$$

Let ^{0} be such that $Se + \quad S^0 = 0$ and $\quad S^0$ is given by ${}^{0} = [\,{}^{0}_{1} \quad {}^{0}_{2} \quad {}^{0}_{3} \quad]^T$.

Let $N(t)$ be the number of customers in the system, $N^*(t)$ the mode of service going on whether direct admission to required/ undesired or one that came from undesired service designated by 1,2 and 3 respectively, $S(t)$ the phase of service and $A(t)$ the phase of arrival at time t. With these the process $\{(N(t), N^*(t), S(t), A(t)), t \geq 0\}$ is a continuous time Markov chain with state space $\Omega = \{\underline{0}, \underline{1}, \underline{2}, \ldots\}$, where

$$\underline{0} = \{(0, r); 1 \leq r \leq m\}$$

(in the level zero we need consider only the phase of arrival) and

$$\underline{i} = \{(i, j, k, r); i \geq 1, 1 \leq j \leq 3, 1 \leq k \leq n_j, 1 \leq r \leq m\}.$$

Thus the infinitesimal generator of this $CTMC$ is a $LIQBD$ and is of the form

$$Q = \begin{pmatrix} D_0 & A_{01} & & & \\ A_{10} & A_1 & A_0 & & \\ & A_2 & A_1 & A_0 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}$$

where $A_{01} = \beta \otimes \quad D_1$, $A_{10} = \quad S^0 \otimes \quad I_m$, $A_0 = \quad I_n \otimes \quad D_1$, $A_1 = S \oplus \quad D_0$, $A_2 = \quad S^0 \beta \otimes \quad I_m$.

2.1 Stability Condition

Consider $A(= A_0 + A_1 + A_2)$, the generator matrix of the Markov chain corresponding to the phase changes.

$$A = (S + \quad S^0\beta) \oplus D = \begin{pmatrix} (p_1^0 \quad \boldsymbol{\beta}_1 + S_1) \oplus D & q_1^0 \quad \boldsymbol{\beta}_2 \otimes I_m & 0 \\ p_2^0 \quad \boldsymbol{\beta}_1 \otimes I_m & (q_2^0 \quad \boldsymbol{\beta}_2 + S_2) \oplus D & S_2^0\,\boldsymbol{\beta}_3 \otimes I_m \\ p_3^0 \quad \boldsymbol{\beta}_1 \otimes I_m & q_3^0 \quad \boldsymbol{\beta}_2 \otimes I_m & S_3 \oplus D \end{pmatrix} \tag{4}$$

Let $\pi = (\pi_1, \pi_2, \pi_3)$ be the steady-state probability vector of $(S + \quad S^0 \; \beta)$. Then

$$\pi\,(S + \quad S^0 \; \beta) = 0 and\, \pi\, e = 1. \tag{5}$$

From the relation $\pi\,(S + \quad S^0 \; \beta) = 0$ we have

$$\pi_1(p\,{}^{0}_{1}\,\boldsymbol{\beta}_1 + S_1) + \pi_2 p\,{}^{0}_{2}\,\boldsymbol{\beta}_1 + \pi_3 p\,{}^{0}_{3}\,\boldsymbol{\beta}_1 = 0, \tag{6}$$

$$\pi_1 q\,{}^{0}_{1}\,\boldsymbol{\beta}_2 + \pi_2(q\,{}^{0}_{2}\,\boldsymbol{\beta}_2 + S_2) + \pi_3 q\,{}^{0}_{3}\,\boldsymbol{\beta}_2 = 0, \tag{7}$$

$$\pi_2 \quad S_2^0\,\boldsymbol{\beta}_3 + \pi_3 S_3 = 0. \tag{8}$$

Multiplying equation (8) by e on right hand side we get

$$\pi_3\,{}^{0}_{3} = \pi_2 \quad S_2^0\,. \tag{9}$$

Putting this in equation (6) yields

$$\pi_1\,{}^{0}_{1} = -\frac{p}{q}\,\pi_2 S_2 e. \tag{10}$$

Substituting relations (9) and (10) in equation (7) gives

$$\pi_2(\,^0_2\,\boldsymbol{\beta}_2 + \quad S_2^0\,\boldsymbol{\beta}_2 + S_2) = 0. \tag{11}$$

This implies, for an arbitrary constant c

$$\pi_2 = c\boldsymbol{\beta}_2(-S_2)^{-1}. \tag{12}$$

Substituting for π_2 in relation (10) we have

$$\pi_1 = \frac{cp}{q}\boldsymbol{\beta}_1(-S_1)^{-1}. \tag{13}$$

Denote by $\delta = \boldsymbol{\beta}_2(-S_2)^{-1} \quad S_2^0$ the probability that a customer starting with incorrect service leaves the system after getting correct service. Then equation (9) gives

$$\pi_3 = c\delta\boldsymbol{\beta}_3(-S_3)^{-1}. \tag{14}$$

From the normalizing condition $\pi\, e = 1$, the value of c is computed as

$$c = \left[\frac{p}{q}\mu'_1 + \mu'_2 + \delta\mu'_3\right]^{-1}. \tag{15}$$

Now from (1) and (5) we get the steady state probability vector of A as $\hat{\pi} = \pi \otimes \boldsymbol{\eta}$.

Theorem 2.1 *The system is stable if and only if*

$$\lambda < (\pi \otimes \boldsymbol{\eta})(\quad S^0 \quad \beta \otimes \quad I_m\,)e. \tag{16}$$

Proof. The queueing system under study with the $LIQBD$ type generator given in (2) is stable if and only if rate of left drift is less than the rate of right drift (see Neuts [6]), that is,

$$\hat{\pi}\, A_0 e < \hat{\pi}\, A_2 e. \tag{17}$$

The left drift rate is $\hat{\pi}\,(\quad I_n \otimes \quad D_1\,)e$ which when simplified reduces to λ. Now, the right drift rate is $(\pi \otimes \boldsymbol{\eta})(\quad S^0 \quad \beta \otimes \quad I_m\,)e$.

Let $\rho = \frac{\lambda}{(\pi \otimes \boldsymbol{\eta})(\quad S^0 \quad \beta \otimes \quad I_m\,)e}$. Then from (16), we have $\rho < 1$.

2.2 Steady-State probability vector

A brief outline for the computation of the stationary probability vector of the system is as follows. Let $\mathbf{x}$ denote the steady-state probability vector of the generator Q. Then

$$\mathbf{x}Q = 0 and \mathbf{x}e = 1. \tag{18}$$

Assuming that the stability condition (16) holds and partitioning $\mathbf{x}$ as $\mathbf{x} = (\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \dots)$ we obtain

$$\mathbf{x}_n = \mathbf{x}_1 R^{n-1}, n \geq 1 \tag{19}$$

where R is the minimal non negative solution to the matrix quadratic equation $R^2 A_2 + R A_1 + A_0 = 0$. The two boundary equations involving $\mathbf{x}_0$ are

$$\mathbf{x}_0 D_0 + \mathbf{x}_1 A_{10} = 0, \tag{20}$$

$$\mathbf{x}_0 A_{01} + \mathbf{x}_1[A_1 + R A_2] = 0 \tag{21}$$

These together with the normalizing condition in (18) gives

$$\mathbf{x}_1 = \mathbf{x}_0 V where V = -A_{01}[A_1 + R A_2]^{-1} \tag{22}$$

$$\mathbf{x}_0[I + V(I - R)^{-1}]e = 1. \tag{23}$$

To see how the system performs, it is instructive to define $\mathbf{y} = \sum_{i=1}^{\infty} \mathbf{x}_i$. Then $\mathbf{y} = (\mathbf{y}_1 \quad \mathbf{y}_2 \quad \mathbf{y}_3)$ where the $\mathbf{y}_i$, $i = 1,2,3$ indicates mode of service of the customer in service along with other system phases.

2.3 System Performance Measures

1. Probability that system is idle, $P_{idle} = \mathbf{x}_0 e = 1 - \rho$.
2. Rate of loss of customers, $R_{loss} = \mathrm{y}_2 \quad S_2^0 = \lambda q(1 - \delta)$.

3. Probability that a customer is lost, $P_{loss} = q(1-\delta)$.
4. Mean number of customers in the system, $\mu_{NS} = \sum_{i=1}^{\infty} i\mathbf{x}_i\mathrm{e}$.
5. Mean number of customers in the queue is given by $\mu_{NQ} = \sum_{i=2}^{\infty} (i-1)x_i\mathrm{e}$.
6. Probability that the server is serving in required mode, $P_C = \mathrm{y}_1\mathrm{e} + \mathrm{y}_3\mathrm{e} = \rho - \lambda q\mu'_2$.
7. Probability that the server is serving in unwanted mode, $P_I = \mathrm{y}_2\mathrm{e} = \lambda q\mu'_2$.
8. Rate at which customers leave with required service starting in desired service mode, $R_C = \mathrm{y}_1 \ S_1^0 = \lambda p$.
9. Rate at which customers leave with correct service starting with unwanted service, $R_I = \mathrm{y}_3 \ S_3^0 = \lambda q\delta$.
10. Expected waiting time in the system $W_S = \frac{\mu_{NS}}{\lambda}$.
11. We define the system reliability at any time as the probability of customers in service is in desired mode of service $p_{reliability} = (\mathrm{y}_1 + \mathrm{y}_3)\mathrm{e}$.

3 Case of Poisson arrival and phase type service

In this section we consider the system with Poisson arrival process and service times are phase type distributed (see Section 2). Then $\{(N(t), N^*(t), S(t)), t \geq 0\}$ is a continuous time Markov chain with state space $\{0, \underline{1}, \underline{2}, \dots\}$ where

$$\underline{i} = \{(i,j,k), 1 \leq j \leq 3, 1 \leq k \leq n_j\} for i \geq 1.$$

Thus the infinitesimal generator is of the form $Q' = \begin{pmatrix} -\lambda & \lambda\beta & & & \\ S^0 & S-\lambda I & \lambda I & & \\ & S^0\ \beta & S-\lambda I & \lambda I & \\ & & \ddots & \ddots & \ddots \end{pmatrix}$.

Theorem 3.1 *The system is stable if and only if $\rho' < 1$ where*

$$\rho' = \lambda[p\mu_{1'} + q(\mu_{2'} + \delta\mu_{3'})]. \tag{24}$$

Proof. From the relation (17) we have $\lambda < \pi \ \ S^0 \ \ \beta\, e$ where $\pi\ = (\pi_1, \pi_2, \pi_3)$ (with π_i's as given in (12)-(14)) is the steady state probability vector of $S + \ S^0\beta$. The right drift $\pi \ \ S^0\beta\, e = \sum_{i=1}^{3} \ \pi_i \ \ S_i^0$.

Multiplying (6) by e on right hand side we get

$$\sum_{i=1}^{3} \ \pi_i \ \ S_i^0 \ = \frac{c}{q} - \frac{1}{p}\, \pi_i\, S_1 e = \frac{1}{p}\left(\frac{cp}{q}\right) \beta_1\, e \quad (from(13)) = \frac{c}{q}$$

where c is given in (15). Hence the condition for system stability is given by

$$\lambda < \frac{1}{p\mu_1' + q(\mu'_2 + \delta\mu_3')} \tag{25}$$

The generator matrix corresponding to the phase changes is $S + \ S^0\,\beta$ and the stationary probability vector is $\pi\ = (\pi_1, \pi_2, \pi_3)$.

Theorem 3.2 *The steady-state probability vector $x = (x_0, x_1, x_2, \cdots)$ of Q' is given by*

$$x_0 = 1 - \rho', \mathrm{x}_i = (1-\rho')\,\beta\, R^i, i \geq 1, \tag{26}$$

where R is given by

$$R = \lambda \begin{bmatrix} \lambda I - \lambda p\mathrm{e}\,\beta_1 - S_1 & -\lambda q\mathrm{e}\,\beta_2 & 0 \\ -\lambda p\mathrm{e}\,\beta_1 & \lambda I - \lambda q\mathrm{e}\,\beta_2 - S_2 & - \quad S_2^0\ \beta_3 \\ -\lambda p\mathrm{e}\,\beta_1 & -\lambda q\mathrm{e}\,\beta_2 & \lambda I - S_3 \end{bmatrix}^{-1}. \tag{27}$$

Proof. Let x be the steady-state probability vector of Q'. Then $xQ' = 0$ and $xe = 1$.

The steady-state equations are given by

$$-\lambda x_0 + x_1 S^0 = 0, \tag{28}$$

$$\lambda x_0\boldsymbol{\beta} + x_1(S - \lambda I) + x_2 S^0\boldsymbol{\beta} = 0, \tag{29}$$

$$\lambda x_{i-1} + x_i(S - \lambda I) + x_{i+1}S^0\boldsymbol{\beta} = 0, for i \geq 2. \tag{30}$$

From (28) we have

$$x_1 S^0 = \lambda x_0. \tag{31}$$

Multiplying equations (29) and (30) by the column vector e on the right hand side leads to

$$x_{i+1}S^0 = \lambda x_i e for i \geq 1. \tag{32}$$

Since $x_{i+1}S^0\boldsymbol{\beta} = \lambda x_i \mathcal{B} for i \geq 1$ where $\mathcal{B} = e.\boldsymbol{\beta}$, from (29) and (30) we obtain

$$x_1(\lambda I - \lambda\mathcal{B} - S) = \lambda x_0\boldsymbol{\beta} \tag{33}$$

and

$$x_i(\lambda I - \lambda\mathcal{B} - S) = \lambda x_{i-1}, for i \geq 2. \tag{34}$$

Denoting $(\lambda I - \lambda\mathcal{B} - S)$ by $\mathcal{K}$, relation (33) takes the form $x_1 = \lambda x_0\boldsymbol{\beta}\mathcal{K}^{-1}$, provided $\mathcal{K}$ is invertible. We now prove the nonsingularity of $\mathcal{K}$.

Let the vector u be in the left kernal of $\mathcal{K}$. Then

$$\lambda u - uS - \lambda(ue)\boldsymbol{\beta} = 0. \tag{35}$$

Suppose $ue = 0$. Then (35) reduces to $u(\lambda I - S) = 0$. But $(\lambda I - S)$ is nonsingular and hence $u = 0$.

If $ue \neq 0$, normalize u by setting $ue = 1$. Post multiplying (35) by e gives

$$uS^0 = 0. \tag{36}$$

Substituting for ue, (35) reduces to $u = \lambda\boldsymbol{\beta}(\lambda I - S)^{-1}$.

From (36) we have

$$\lambda\boldsymbol{\beta}(\lambda I - S)^{-1}S^0 = 0. \tag{37}$$

In (37) $\boldsymbol{\beta}(\lambda I - S)^{-1}S^0$ is the Laplace-Stieltjes transform at $s = \lambda (> 0)$, of the probability distribution $F(t) = 1 - \boldsymbol{\beta}\exp(St)e$ for $t \geq 0$. Therefore (37) cannot hold and hence $u = 0$. Thus $\mathcal{K}$ is nonsingular.

The irreducibility of the representation $(\boldsymbol{\beta}, S)$ leads to the irreducibility of $\mathcal{K}$, so that the matrix R in (27) is positive.

We have $sp(R) < 1$, if $\rho' < 1$. Therefore the quantity x_0 is given by the normalizing equation

$$x_0 + x_0\boldsymbol{\beta}R(I - R)^{-1}e = 1.$$

Substitution for R leads to

$$x_0 - \lambda x_0\boldsymbol{\beta}(\lambda\mathcal{B} + S)^{-1}e = 1. \tag{38}$$

The inverse of $(\lambda\mathcal{B} + S)$ is calculated as

$$\begin{aligned}(\lambda\mathcal{B} + S)^{-1} &= S^{-1}(I + \lambda\mathcal{B}S^{-1})^{-1} = S^{-1}\textstyle\sum_{n=0}^{\infty}(-1)^n\lambda^n(\mathcal{B}S^{-1})^n \\ &= S^{-1}[I - \lambda[\textstyle\sum_{n=0}^{\infty}(-1)^n\lambda^n(\mathcal{B}S^{-1})^n]\mathcal{B}S^{-1}] = S^{-1}[I - \lambda\textstyle\sum_{n=0}^{\infty}\rho'^n\mathcal{B}S^{-1}] \\ &= S^{-1}[I - \lambda(1 - \rho')^{-1}\mathcal{B}S^{-1}].\end{aligned}$$

From (38) we have

$$\begin{aligned}x_0 - \lambda x_0\boldsymbol{\beta}(\lambda\mathcal{B} + S)^{-1}e &= x_0 - \lambda x_0\boldsymbol{\beta}[S^{-1}(I - \lambda(1 - \rho')^{-1}\mathcal{B}S^{-1})]e \\ &= x_0 - \lambda x_0\boldsymbol{\beta}S^{-1}e + \lambda^2 x_0(1 - \rho)^{-1}\boldsymbol{\beta}S^{-1}\mathcal{B}S^{-1}e \\ &= x_0 + \rho' x_0 + \rho'^2(1 - \rho')x_0 = (1 - \rho')x_0 = 1,\end{aligned}$$

so that $x_0 = (1 - \rho')$.

Letting $\mathrm{y} = \sum_{i=1}^{\infty} x_i$, it is obtained that $\mathrm{y} = \rho'\pi$. In the sequel partition $\mathrm{y} = (\mathrm{y}_1, \mathrm{y}_2, \mathrm{y}_3)$, so that $\mathrm{y}_i = \rho'\pi_i, 1 \leq i \leq 3$.

4 Poisson arrival with exponential service

In this section we consider customers arrive according to the Poisson process with rate λ and desired (correct) service time follows exponential distribution but the undesired (incorrect) service follows phase type distribution. Let $N(t)$ be the number of customers in the system $N^*(t)$ the type of service and $S(t)$ the phase of service at time t. Then $\{(N(t), N^*(t), S(t)), t \geq 0\}$ is a continuous time Markov chain with state space $\{0, \underline{1}, \underline{2}, \dots\}$ where

$$\underline{i} = \{(i,1,0),(i,3,r+1)\} \cup \{(i,2,j), 1 \le j \le r\} fori \ge 1.$$

Thus the infinitesimal generator is of the form

$$Q = \begin{array}{c|ccccc} & 0 & \underline{1} & \underline{2} & \underline{3} & \cdots\ \cdots \\ \hline 0 & -\lambda & b_0 & & & \\ \underline{1} & c_0 & A_1 & A_0 & & \\ \underline{2} & & A_2 & A_1 & A_0 & \\ \vdots & & & \ddots & \ddots & \ddots \end{array}$$

$$where\ b_0 = \lambda(p, q\boldsymbol{\beta}, 0),\quad c_0 = \begin{pmatrix} \mu \\ \tilde{S}_1^0 \\ \mu \end{pmatrix}, A_0 = \lambda I$$

$$A_1 = \begin{pmatrix} -\lambda - \mu & 0 & 0 \\ 0 & \tilde{S} - \lambda I & \tilde{S}_2^0 \\ 0 & 0 & -\lambda - \mu \end{pmatrix}, A_2 = \begin{pmatrix} \mu p & \mu q\boldsymbol{\beta} & 0 \\ p\ \tilde{S}_1^0 & q\ \tilde{S}_1^0\boldsymbol{\beta} & 0 \\ \mu p & \mu q\boldsymbol{\beta} & 0 \end{pmatrix} with\ \tilde{S}e + \tilde{S}_1^0 + \tilde{S}_2^0 = 0.$$

4.1 Stability condition

Consider $A = A_0 + A_1 + A_2$

$$= \begin{pmatrix} -\mu q & \mu q\boldsymbol{\beta} & 0 \\ p\ \tilde{S}_1^0 & \tilde{S} + q\ \tilde{S}_1^0\boldsymbol{\beta} & \tilde{S}_2^0 \\ \mu p & \mu q\boldsymbol{\beta} & -\mu \end{pmatrix} \tag{39}$$

the generator matrix of the Markov chain corresponding to the phase changes. Let $\Pi = (\pi_0, \hat{\boldsymbol{\pi}}, \pi_{r+1})$ be the steady state probability matrix of A. Solving the relations

$$\Pi A = 0, \quad \Pi e = 1 \tag{40}$$

we obtain

$$-\mu q\ \pi_0 + p\hat{\boldsymbol{\pi}}\ \ \tilde{S}_1^0 + \mu p\ \pi_{r+1} = 0 \tag{41}$$

$$\mu q \pi_0 \boldsymbol{\beta} + \hat{\boldsymbol{\pi}}(\ \tilde{S} + q\ \ \tilde{S}_1^0\ \boldsymbol{\beta}) + \mu q\ \pi_{r+1}\boldsymbol{\beta} = 0 \tag{42}$$

$$\hat{\boldsymbol{\pi}}\ \ \tilde{S}_2^0 - \mu\pi_{r+1} = 0 \tag{43}$$

Equation (43) gives

$$\mu\pi_{r+1} = \hat{\boldsymbol{\pi}}\ \ \tilde{S}_2^0 \tag{44}$$

Putting this in equation(41),

$$\mu q\ \pi_0 = p(\hat{\boldsymbol{\pi}}\ \ \tilde{S}_1^0 + \hat{\boldsymbol{\pi}}\ \ \tilde{S}_2^0) \tag{45}$$

Substituting these in equation(42) and simplifying we get

$$\hat{\boldsymbol{\pi}}(S + q\ \tilde{S}_1^0\ \boldsymbol{\beta}) + p\hat{\boldsymbol{\pi}}\ (\ \tilde{S}_1^0\beta + \ \ \tilde{S}_2^0\ \boldsymbol{\beta}) + q\hat{\boldsymbol{\pi}}\ \ \tilde{S}_2^0\ \boldsymbol{\beta} = 0$$

$$\Rightarrow \hat{\boldsymbol{\pi}}\ (\ \tilde{S} + \ \ \tilde{S}_1^0\ \boldsymbol{\beta} + \ \ \tilde{S}_2^0\ \boldsymbol{\beta}) = 0$$

so that

$$\hat{\boldsymbol{\pi}} = c\ \boldsymbol{\beta}(-\ \tilde{S})^{-1} \tag{46}$$

c being a constant and is computed from the normalizing condition. Let δ be the probability that a customer getting correct service from incorrect services and η the probability of staying back in incorrect services. Then

$$\delta = \boldsymbol{\beta}\ (-\ \tilde{S})^{-1}\ \ \tilde{S}_2^0 \tag{47}$$

and

$$\eta = \left(\boldsymbol{\beta}(-\ \tilde{S})^{-1}e\right)^{-1} \tag{48}$$

Then the probability that a customer leaves the system without getting required service is

$$1 - \delta = \boldsymbol{\beta}\ (-\ \tilde{S})^{-1}\ \ \tilde{S}_1^0 \tag{49}$$

and the mean time a customer stay back in incorrect services is

$$\frac{1}{\eta} = \left(\boldsymbol{\beta}(-\ \tilde{S})^{-1}e\right) \tag{50}$$

The normalizing equation is $\pi_0 + \hat{\boldsymbol{\pi}}\ e + \pi_{r+1} = 1$. Substituting for the components of Π which are now computed as

$$\pi_0 = \frac{pc}{\mu q},\ \hat{\boldsymbol{\pi}}\ e = \frac{c}{\eta},\ \pi_{r+1} = \frac{c\delta}{\mu} \tag{51}$$

we get $\frac{pc}{\mu q} + \frac{c}{\eta} + \frac{c\delta}{\mu} = 1$ which shows

$$c = \frac{\mu q \eta}{p\eta + \mu q + \delta q \eta}. \tag{52}$$

Theorem 4.1 *The system is stable if and only if* $\lambda < \frac{1}{q}\ c$.

Proof. The condition for the stability of the system is $\Pi A_0 e < \Pi A_2 e$. Simplification gives $\Pi A_0 e = \lambda$. Now $A_2 e = \begin{pmatrix} \mu & S_1^0 & \mu \end{pmatrix}^T$. Therefore $\Pi A_2 e = \mu \pi_0 + \hat{\boldsymbol{\pi}}(\ \tilde{S}\,^0_1 +\ \tilde{S}\,^0_2)$ Substituting for $\mu\pi_0$, right hand side becomes $\frac{1}{q}\hat{\boldsymbol{\pi}}(\ \tilde{S}\,^0_1 +\ \tilde{S}\,^0_2)$. Using equation(46) and the fact that $(\ \tilde{S}\ )^{-1}(\ \tilde{S}\,^0_1 + \tilde{S}\,^0_2) = e$, the result follows. Hence the system is stable if $\rho < 1$ where $\rho = \lambda\ \frac{q}{c}$.

4.2 Steady-State probability Vector

Let the steady state probability vector x of Q be $x = (x^*, \mathbf{x}(1), \mathbf{x}(2), \ldots)$ be such that $xQ = 0, xe = 1$. Partitioning gives $\mathbf{x}(i) = (x_0(i), \frown\ x(i), x_{r+1}(i))$. The relation $xQ = 0$ gives the following system of equations.

$$-\lambda x^* + \mathbf{x}(1)c_0 = 0 \tag{53}$$

$$x^* b_0 + \mathbf{x}(1)A_1 + \mathbf{x}(2)A_2 = 0 \tag{54}$$

$$\text{For} i \geq 1, \mathbf{x}(i-1)A_0 + \mathbf{x}(i)A_1 + \mathbf{x}(i+1)A_2 = 0 \tag{55}$$

From the matrix geometric structure we obtain

$$\mathbf{x}(i) = \mathbf{x}(1)R^{i-1}, i \geq 1 \tag{56}$$

where R is the minimal non negative solution to the matrix quadratic equation $R^2 A_2 + RA_1 + A_0 = O$. Equation (53) shows

$$x^* = \frac{1}{\lambda}\mathbf{x}(1)c_0. \tag{57}$$

Equation (54) together with normalizing condition gives

$$x^* b_0 + \mathbf{x}(1)(A_1 + RA_2) = 0 \tag{58}$$

$$\text{subjectto} x^* e + \mathbf{x}(1)(I-R)^{-1}e = 1. \tag{59}$$

Substituting for x^*,

$$\mathbf{x}(1)\left(A_1 + RA_2 + \frac{1}{\lambda}c_0 b_0\right) = 0 \tag{60}$$

$$\text{subjectto}\mathbf{x}(1)\left(\frac{1}{\lambda}c_0 + (I-R)^{-1}e\right) = 1. \tag{61}$$

But $c_0 b_0 = \lambda A_2$ which implies

$$\mathbf{x}(1)(A_1 + RA_2 + A_2) = 0 \tag{62}$$

$$\text{subjectto}\mathbf{x}(1)\left(\frac{1}{\lambda}c_0 + (I-R)^{-1}e\right) = 1. \tag{63}$$

4.2.1 *Computation of* ***R***

R can computed explicitly along the following lines.

We have

$$A_2 = \begin{pmatrix} \mu p & \mu q \boldsymbol{\beta} & 0 \\ p\,\tilde{S}_1^0 & q\,\tilde{S}_1^0 \boldsymbol{\beta} & 0 \\ \mu p & \mu q \boldsymbol{\beta} & 0 \end{pmatrix} = \begin{bmatrix} \mu \\ \tilde{S}_1^0 \\ \mu \end{bmatrix} \begin{bmatrix} p & q\boldsymbol{\beta} & 0 \end{bmatrix} \tag{64}$$

so that

$$A_2 e = \begin{bmatrix} \mu \\ \tilde{S}_1^0 \\ \mu \end{bmatrix} = c_0 \tag{65}$$

Also from the relation $RA_2 e = A_0 e$, we obtain

$$RA_2 e = \lambda e. \tag{66}$$

Now, $R^2 A_2 = R^2 \begin{pmatrix} \mu \\ \tilde{S}_1^0 \\ \mu \end{pmatrix} (p \quad q\boldsymbol{\beta} \quad 0) = R^2 A_2 e (p \quad q\boldsymbol{\beta} \quad 0)$.

Substituting for RA_2 from (66), we get

$$R^2 A_2 = R\lambda e (p \quad q\boldsymbol{\beta} \quad 0) \tag{67}$$

Therefore

$$\lambda Re \ \ (p \quad q\boldsymbol{\beta} \quad 0) + RA_1 + \lambda I = 0. \tag{68}$$

This gives

$$R = \lambda \begin{pmatrix} \mu + \lambda q & -\lambda q\boldsymbol{\beta} & 0 \\ -\lambda pe & \lambda I - \lambda qe\boldsymbol{\beta} - \ \ \tilde{S} & - \ \ \tilde{S}_2^0 \\ -\lambda p & -\lambda q\boldsymbol{\beta} & \lambda + \mu \end{pmatrix}^{-1}. \tag{69}$$

Lemma 4.2 *$x^* = 1 - \rho$ so that $\boldsymbol{x}(1)(I - R)^{-1} e = \rho$.*

Proof. Multiplying by e on the right side of equation (54) and simplifying we get the relation

$$\lambda x^* + \mathbf{x}(1) \begin{pmatrix} -\lambda - \mu \\ \tilde{S} - \lambda I + \ \ \tilde{S}_2^0 \\ -\lambda - \mu \end{pmatrix} + \mathbf{x}(2) \begin{pmatrix} \mu \\ \tilde{S}_1^0 \\ \mu \end{pmatrix} = 0. \tag{70}$$

Equation (53) gives

$$\lambda x^* = \mathbf{x}(1) \begin{pmatrix} \mu \\ \tilde{S}_1^0 \\ \mu \end{pmatrix}. \tag{71}$$

Putting this in (70) the following relation is obtained.

$$\mathbf{x}(2) \begin{pmatrix} \mu \\ \tilde{S}_1^0 \\ \mu \end{pmatrix} = \lambda \mathbf{x}(1) e. \tag{72}$$

Multiplying equation(55) on right side by **e** and recursive use of the relation results in

$$\mathbf{x}(i) \begin{pmatrix} \mu \\ \tilde{S}_1^0 \\ \mu \end{pmatrix} = \lambda \mathbf{x}(i-1) e \quad \text{for } i \geq 3. \tag{73}$$

Adding (71), (72) and (73)

$$\textstyle\sum_{i=1}^{\infty} \mathbf{x}(i) \begin{pmatrix} \mu \\ \tilde{S}_1^0 \\ \mu \end{pmatrix} = \lambda. \tag{74}$$

Adding the system of equations (55) with equation (54) and using the fact that $x^* b_0 = \mathbf{x}(1) A_2$ we get

$$\textstyle\sum_{i=1}^{\infty} \mathbf{x}(i) \ \ A = 0. \tag{75}$$

But the relation (40) says

$$\textstyle\sum_{i=1}^{\infty} \mathbf{x}(i) \ \ = d \ \ \Pi \quad \text{forsomeconstant } c \tag{76}$$

which in turn gives

$$\textstyle\sum_{i=1}^{\infty} \mathbf{x}(i) \ \ = (1 - x^*) \ \ \Pi \tag{77}$$

Multiplying on the right side by $\begin{pmatrix} \mu \\ \tilde{S}_1^0 \\ \mu \end{pmatrix}$ and using the relation in (??)

$$\textstyle\sum_{i=1}^{\infty} \mathbf{x}(i) \ \ \begin{pmatrix} \mu \\ \tilde{S}_1^0 \\ \mu \end{pmatrix} = (1 - x^*) \ \frac{\lambda}{\rho} \tag{78}$$

The result follows from (74) and (78).

4.3 System Performance measures

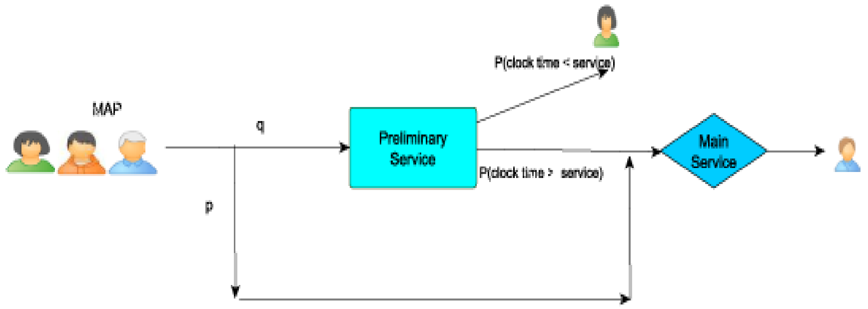
1. Probability that the system is idle = x^*
2. Rate of loss = $\sum_{i=1}^{\infty} \frown\ x(i)\ \tilde{S}\,_1^0 = \lambda q(1-\delta)$
3. Probability of loss = $q(1-\delta)$
4. Mean number of customers in the system = $\sum_{i=1}^{\infty} i\mathbf{x}(i)e = \mathbf{x}(1)(I-R)^{-2}e$
5. Mean number of customers in the queue = $\sum_{i=1}^{\infty} (i-1)\mathbf{x}(i)e = \mathbf{x}(1)(I-R)^{-2}e - \mathbf{x}(1)(I-R)^{-1}e$
6. Probability that the server is busy serving in correct mode

$$\sum_{i=1}^{\infty} \mathbf{x}(i) \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \mathbf{x}(1)(I-R)^{-1} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \rho(\pi_0 + \pi_{r+1}) = \rho - \frac{\lambda q}{\eta}$$

7. Probability that the server is busy serving in correct mode

$$\sum_{i=1}^{\infty} \mathbf{x}(i) \begin{pmatrix} 0 \\ e \\ 0 \end{pmatrix} = \rho \frown \pi e = \frac{\lambda q}{\eta}$$

5 An illustration



In this section we consider a queueing model consisting of two service stations - preliminary service and main service. Customers arrive to this system according to a MAP (Markovian Arrival Process) with representation (D_0, D_1) of order m. A customer, which taken for service is directly selected for main service with probability p or to the preliminary service with probability $q(=1-p)$. A threshold clock starts ticking if a customer enters to preliminary service. When the duration of preliminary service exceeds the threshold clock, the customer move out of the system, else he goes to main service. The threshold clock follows exponential distribution with parameter ζ. Service times of the customers at these stations follow phase type distributions with representation $(\boldsymbol{\alpha}, S_P)$, $(\boldsymbol{\gamma}, S_M)$ and of order a, b respectively. Write $S_P^0 + \zeta e = -S_P e$ and $S_M^0 = -S_M e$ where e is a column vector of 1's of appropriate order. Hence service time of a customer can be modeled as a phase type distribution with representation $(\boldsymbol{\xi}, U)$ of order $a + 2b$ such that $Ue + U^0 = 0$ where

$$\boldsymbol{\xi} = (p\boldsymbol{\gamma} \quad q\boldsymbol{\alpha} \quad 0)$$

$$U = \begin{pmatrix} S_M & 0 & 0 \\ 0 & S_P & S_P^0\boldsymbol{\gamma} \\ 0 & 0 & S_M \end{pmatrix}, U^0 = \begin{pmatrix} S_M^0 \\ \zeta e \\ S_M^0 \end{pmatrix}.$$

Let $N(t), N^*(t), S(t), A(t)$ denote respectively the number of customers in the system, nature of service, phase of service and phase of arrival at time t.

$$N^*(t) = \begin{cases} 1 & mainservice \\ 2 & preliminaryservice \\ 3 & onethatcomefrompreliminaryservice \end{cases}$$

The process $\Omega = \{(N(T), N^*(t), S(t), A(t)), t \geq 0\}$ is a continuous time Markov chain with state space $\{(n, i, j, k); i = 1,3, 1 \leq j \leq b, 1 \leq k \leq m\} \cup \{(n, 2, j, k); 1 \leq j \leq a, 1 \leq k \leq m\}$ for $n \geq 1$. Note that when $N(t) = 0$, the only other component in the state vector is $A(t)$. Thus the infinitesimal generator of Ω is of the form

$$Q^* = \begin{pmatrix} D_0 & A_{01} & & & \\ A_{10} & A_1 & A_0 & & \\ & A_2 & A_1 & A_0 & \\ & & \ddots & \ddots & \ddots \end{pmatrix} \tag{79}$$

where $A_{01} = \boldsymbol{\xi} \otimes D_1, A_{10} = U^0 \otimes I_m, A_0 = I_{a+2b} \otimes D_1, A_1 = U \oplus D_0, A_2 = U^0 \boldsymbol{\xi} \otimes I_m$.

The infinitesimal generator Q^* given by (79) is of the same form as Q of the model described initially. Thus the analysis of the Markov chain with infinitesimal generator Q^* can be done in the same way as for Q.

The significance of this model is as follows: customer arriving to a single server belong to two categories, though they join the same. Only while taken for service the category will be revealed. Call them category 1 and category 2, respectively. Category 1 are qualified for the main service without undergoing preliminary service. However, category 2 have to be given the preliminary service before admitted to mean service. However, if such customers do not get service in preliminary before realization of the timer (random clock), they get disqualified and so leave the system forever. On the other those among category 2, completing service successfully in preliminary are immediately admitted to main service. On completion of that service such customers leave the system.

Remark 5.1 *In telecommunication it is this type of situation that is often encountered. Packages have to identify the server in idle state; then wait for a while. But in the mean time another message may get through, making the server busy. Then the customer (packet) under consideration has to go through a series of contention windows. These passages could be regarded as unwanted service. In case the process of going through contention windows exceeds a threshold time limit, the message will not get served.*

Remark 5.2 *The problem discussed in Madan [3] and Medhi [5] could be arrived at from our model as follows. Suppose that we reverse the order of preliminary and main service, that is, main service first and preliminary (hereafter we call it optional) service next. Then after completion of main service, the customer asks for an optional service with probability* $1 - q$ *(this optional service time has exponential distribution in Madan [3]). With probability q, the customer leaves the system immediately after main service completion. This model is also the same as a queue with instantaneous feedback after a service and immediate commencement of his service (feedback restricted to one). This feedback policy is referred to as queues with instantaneous feedback as head of the queue.*

6 Numerical illustration

The following numerical illustration is based on the description in Section 2.

We fix parameters $n_1 = 2, n_2 = 3, n_3 = 4, \boldsymbol{\beta}_1 = (0.4 \quad 0.6), \boldsymbol{\beta}_2 = (0.3 \quad 0.5 \quad 0.2), \boldsymbol{\beta}_3 = (0.2 \quad 0.3 \quad 0.3 \quad 0.2)$,

$$S_1 = \begin{bmatrix} * & 6 \\ 8 & * \end{bmatrix}, S_1^0 = \begin{bmatrix} 7 \\ 8 \end{bmatrix} \; with\, S_1 e + S_1^0 = 0,$$

$$S_2 = \begin{bmatrix} * & 5 & 5 \\ 6 & * & 6 \\ 5 & 7 & * \end{bmatrix}, S_2^0 = \begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix}, S_2^0 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} with\ S\_2\ e + S\_2\text{^}0 + S\_2\text{^}0 = 0,$$

$$S_3 = \begin{bmatrix} * & 7 & 8 & 9 \\ 6 & * & 7 & 7 \\ 6 & 6 & * & 6 \\ 8 & 7 & 6 & * \end{bmatrix}, S_3^0 = \begin{bmatrix} 6 \\ 7 \\ 8 \\ 9 \end{bmatrix} \ with\ S_3 e + S_3^0 = 0.$$

For the arrival process, we consider the following two sets of values for D_0 and D_1 as follows. The arrival processes labeled $MNCA$ and $MPCA$ respectively, have negative and positive correlation for two successive inter-arrival times with values -0.48891 and 0.48891. The standard deviation of the inter-arrival times of these two arrival processes are, respectively, 0.2819 and 0.2819.

1. MAP with negative correlation ($MNCA$):

$$D_0 = \begin{pmatrix} -5.0111 & 5.0111 & 0 \\ 0 & -5.0111 & 0 \\ 0 & 0 & -1128.75 \end{pmatrix}, D_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0.05011 & 0 & 4.96099 \\ 1117.4625 & 0 & 11.2875 \end{pmatrix}$$

p	P_{loss}	μ_{NS}	P_C	P_I	R_C	R_I	W_S
.4	0.2136	7.5229	0.5242	0.3921	2	1.9320	1.5046
.5	0.1780	4.9744	0.5483	0.3267	2.5	1.6100	0.9949
.6	0.1424	3.6690	0.5724	0.2614	3	1.2880	0.7338
.7	0.1068	2.8654	0.5965	0.1960	3.5	0.9660	0.5731
.8	0.0712	2.3138	0.6206	0.1307	4	0.6440	0.4628
.9	0.0356	1.9069	0.6447	0.0653	4.5	0.3220	0.3814

Table 1: Effect of p for $MNCA$

2. MAP with positive correlation ($MPCA$):

$$D_0 = \begin{pmatrix} -5.0111 & 5.0111 & 0 \\ 0 & -5.0111 & 0 \\ 0 & 0 & -1128.75 \end{pmatrix}, D_1 = \begin{pmatrix} 0 & 0 & 0 \\ 4.96099 & 0 & 0.05011 \\ 11.2875 & 0 & 1117.4625 \end{pmatrix}$$

p	P_{loss}	μ_{NS}	P_C	P_I	R_C	R_I	W_S
.4	0.2136	546.8179	0.5242	0.3921	2	1.9320	109.3646
.5	0.1780	349.9587	0.5483	0.3267	2.5	1.6100	69.9924
.6	0.1424	250.7699	0.5724	0.2614	3	1.2880	50.1545
.7	0.1068	191.0008	0.5965	0.1960	3.5	0.9660	38.2005
.8	0.0712	151.0402	0.6206	0.1307	4	0.6440	30.2083
.9	0.0356	122.4351	0.6446	0.0653	4.5	0.3220	24.4873

Table 2: Effect of p for $MPCA$

The output in Tables 1 and 2 are on expected lines. Note that P_{loss} decreases with increasing value of p. The value of $P_C(R_C)$ steadily increases with p and values of $P_I(R_I)$ and W_S decrease with increase in value of p, as expected.

The main comparison in Tables 1 and 2 is between values of μ_{NS} in $MNCA$ and $MPCA$. Both decrease with increase in value of p. However, $MNCA$ has much smaller values compared to their $MPCA$ counter parts. This indicates that positive correlation in the arrival process results in accumulation of large number of customers in the system.

7 $M/G/1$ Model

In this section we consider an $M/G/1$ system with two service stations – preliminary service and main service. Customers arrive to this system according to a Poisson process with rate λ. A customer, when taken for service, is directly selected for main service with probability p or to the preliminary service with probability q $(= 1 - p)$. A threshold clock starts ticking if a customer enters to preliminary service. When the duration of preliminary service exceeds the threshold clock, the customer moves out of the system, else he goes to main service. The threshold clock follows exponential distribution with parameter ζ. Here the service times, V_p, V_m of the preliminary and main services are independent having general distributions with distribution function $G_1(.), G_2(.)$, LST $G_1^*(.), G_2^*(.)$ respectively.

The (total) service time V of a unit is

$$V = \begin{cases} V_f & withprobabilityq \bullet P(G_1(.) > \exp(\zeta)) \\ V_p & withprobabilityq \bullet P(G_1(.) < \exp(\zeta)) \\ V_m & withprobabilityp \end{cases}$$

where V_f is the duration of threshold clock realization.

Thus

$$G(t) = P(V \le t) = q\left[\int_0^t \zeta e^{-\zeta u}(1 - G_1(u))du + \int_0^t e^{-\zeta u} G_1(u) dG_2(t-u)\right] + p\int_0^t dG_2(u)$$

and LST $G^*(s)$ of V is given by $G^*(s) = \int_0^\infty e^{-st} dG(t)$.

Remark 7.1 *This modelling closely resembles the protocol IEEE 802.11. This is so because of a message generated has to wait before checking for idle server; if server is busy it has to go through a series of contention windows and then look for idle server. In case this process takes longer duration than the life of message (before its significance is lost), then the message does not serve any purpose. In the opposite case it is transmitted before its expiry time.*

Remark 7.2 *Assume the random clock to be of infinite duration (ie., its rate of realization goes to zero). Now interchange the roles of preliminary and main services (in this case, we call the preliminary service, which is the second one now, as optional service). Invariably main service is given for all customers. Thus the main service is followed by an optional service to which customers, on completion of main service, proceed with probability q. Then our model reduces to Madan [3] with exponentially distributed optional service and to Medhi [5] in the case of arbitrarily distributed optional service time.*

Transient solution

The supplementary variable technique (see Cox [1], Medhi [4]) could be used to get the transient solution. Denote by $h(x) = \frac{dG(x)}{1-G(x)}$, the hazard function of the service time distribution $G(.)$ and the probability density function of V is given by

$$g(x) = h(x)\exp\{-N(x)\}$$

where

$$N(x) = \int_0^x h(u)du \qquad \left(N(0) = 0 and \frac{d}{dx}N(x) = h(x)\right).$$

If V is the total service time, then $h(x)dx = P$(service will be completed in $(x, x + dx)$ given that service time exceeds x) and $E(V) = \int xg(x)dx = -G^{*(1)}(0)$.

The supplementary variable $X(t)$ considered is defined below. Let

$N(t) \quad = systemsizeattimet$
$X(t) \quad = timealreadyspentinserviceuptotofaunitreceivingservice$
$p_n(t) \quad = P(N(t) = n) with p_0(0) = 1$
$p_n(t,x)dx \quad = P(N(t) = n, x \le X(t) < x + dx),\ n \ge 1$

$$p_n(t) = \int_0^\infty p_n(t,x)dx,\ \ Q(t,z) = \sum_{n=0}^\infty p_n(t)z^n,\ \ Q(t,x,z) = \sum_{n=1}^\infty p_n(t,x)z^n$$

Now we have

$$p_0(t+\delta t) = [1-\lambda\delta t + o(\delta t)]p_0(t) + \int_0^\infty p_1(t,x)h(x)dx\delta t.$$

$$As\delta t \to 0,\ \ \frac{\partial}{\partial t}p_0(t) = -\lambda p_0(t) + \int_0^\infty p_1(t,x)h(x)dx. \tag{80}$$

$$For\delta x > 0,\ p_1(t+\delta t, x+\delta x) = [1-\lambda\delta t + o(\delta t)][1-h(x)\delta x + o(\delta x)]p_1(t,x).$$

Subtracting and adding a term $p_1(t, x+\delta x)$ to the LHS, then dividing by $\delta t(\delta x)$ and taking as $\delta t \to 0(\delta x \to 0)$, we get

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)p_1(t,x) = -(\lambda + h(x))p_1(t,x). \tag{81}$$

$$Forn \geq 0,\ \left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)p_n(t,x) = -(\lambda + h(x))p_n(t,x) + \lambda p_{n-1}(t,x). \tag{82}$$

We have the following boundary conditions:

$$p_1(t,0) = \int_0^\infty p_2(t,x)h(x)dx + \lambda p_0(t) \tag{83}$$

and

$$p_n(t,0) = \int_0^\infty p_{n+1}(t,x)h(x)dx, \qquad n \geq 2. \tag{84}$$

Multiplying (82) by $z^n, n = 2,3,\ldots$ and (81) by z, then adding all the terms we get

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)\sum_{n=1}^{\infty} p_n(t,x)z^n = -(\lambda + h(x))\sum_{n=1}^{\infty} p_n(t,x) + \lambda \sum_{n=2}^{\infty} p_{n-1}(t,x) \tag{85}$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)Q(t,x,z) = -(\lambda - \lambda z + h(x))Q(t,x,z). \tag{86}$$

Now multiplying (84) by $z^n, n = 2,3,\ldots$ and (83) by z, then adding the terms we have

$$Q(t,0,z) = \int_0^\infty \left(\sum_{n=1}^{\infty} p_{n+1}(t,x)z^n\right)h(x)dx + \lambda z p_0(t). \tag{87}$$

Now

$$\begin{aligned}
\int_0^\infty \left(\sum_{n=1}^{\infty} p_{n+1}(t,x)z^n\right)h(x)dx &= \int_0^\infty \left(\frac{1}{z}\right)\sum_{n=1}^{\infty} p_{n+1}(t,x)z^{n+1}h(x)dx \\
&= \int_0^\infty \left(\frac{1}{z}\right)\left[\sum_{n=1}^{\infty} p_n(t,x)z^n - p_1(t,x)z\right]h(x)dx \\
&= \left(\frac{1}{z}\right)\int_0^\infty [Q(t,x,z) - p_1(t,x)z]h(x)dx \\
&= \left(\frac{1}{z}\right)\left[\int_0^\infty Q(t,x,z)h(x)dx - z(p'_0(t) + \lambda p_0(t))\right] by (80)
\end{aligned}$$

Thus (87) reduces to

$$\begin{aligned}
Q(t,0,z) &= \left(\frac{1}{z}\right)\left[\int_0^\infty Q(t,x,z)h(x)dx - z(p'_0(t) + \lambda p_0(t))\right] + \lambda z p_0(t) \\
&= \left(\frac{1}{z}\right)\left[\int_0^\infty Q(t,x,z)h(x)dx - z(p'_0(t) + \lambda p_0(t)) + \lambda z^2 p_0(t)\right]
\end{aligned}$$

$$zQ(t,0,z) = \int_0^\infty Q(t,x,z)h(x)dx - zp'_0(t) + \lambda z(z-1)p_0(t). \tag{88}$$

The partial differential equation (86) can be solved using the boundary condition (88) and the normalizing condition $\sum_{n=0}^{\infty} p_n(t) = 1$.

7.1 Steady state distribution

Let

$$lim_{t\to\infty} p_n(t) = p_n,\ n \geq 0$$

and

$$\begin{aligned}
\lim_{t\to\infty} p_n(t,x) &= p_n(x), \qquad x > 0, n \geq 1 \\
&= p_0(x) = 0, \quad x > 0.
\end{aligned}$$

Then $\{p_n, n \geq 0\}$ gives the distribution of the general time system size.

Let

$$\begin{aligned} Q(x,z) &= \sum_{n=1}^{\infty} p_n(x)z^n = \sum_{n=1}^{\infty} \left[\lim_{t\to\infty} p_n(t,x)\right] z^n \\ &= \lim_{t\to\infty}\left[\sum_{n=1}^{\infty} p_n(t,x)z^n\right] = \lim_{t\to\infty} Q(t,x,z) \end{aligned}$$

and

$$Q(z) = \int_0^{\infty} Q(x,z)dx.$$

Then

$$(80) \Rightarrow \lambda p_0 = \int_0^{\infty} p_1(x)h(x)dx \tag{89}$$

$$(81) and (8182) \Rightarrow \frac{\partial}{\partial x} p_n(x) = -(\lambda + h(x))p_n(x) + \lambda p_{n-1}(x),\ n \geq 1 \tag{90}$$

$$(83) \Rightarrow p_1(0) = \int_0^{\infty} p_2(x)h(x)dx + \lambda p_0 \tag{91}$$

$$(84) \Rightarrow p_n(0) = \int_0^{\infty} p_{n+1}(x)h(x)dx,\ n \geq 2. \tag{92}$$

The partial differential equation (86) and the boundary condition (88) reduces to

$$\frac{d}{dx} Q(x,z) = -(\lambda - \lambda z + h(x))Q(x,z) \tag{93}$$

$$zQ(0,z) = \int_0^{\infty} Q(x,z)h(x)dx + \lambda z(z-1)p_0 \tag{94}$$

and

$$p_0 + Q(1) = 1. \tag{95}$$

From relation (93)

$$\begin{aligned} \int \frac{dQ(x,z)}{Q(x,z)} &= \int -\big(\lambda - \lambda z + h(x)\big)dx \\ \log\big(Q(x,z)\big) &= logc\big(-\lambda(1-z)x - N(x)\big) \\ Q(x,z) &= c\exp\big(-\lambda(1-z)x - N(x)\big) \\ Q(0,z) &= c \end{aligned}$$

$$Q(x,z) = Q(0,z)\exp(-\lambda(1-z)x - N(x)) \tag{96}$$

Substituting (96) in (94) we get

$$\begin{aligned} zQ(0,z) &= \int_0^{\infty} Q(0,z)e^{(-\lambda(1-z)x - N(x))}h(x)dx + \lambda z(z-1)p_0 \\ &= Q(0,z)\int_0^{\infty} e^{-\lambda(1-z)x}\big[e^{-N(x)}h(x)\big]dx + \lambda z(z-1)p_0 \\ &= Q(0,z)G^*(\lambda(1-z)) + \lambda z(z-1)p_0. \end{aligned}$$

Thus

$$Q(0,z) = \frac{\lambda z(z-1)p_0}{z - G^*(\lambda - \lambda z)}. \tag{97}$$

Now from (96) we have

$$\begin{aligned} Q(z) &= \int_0^{\infty} Q(x,z)dx \\ &= \int_0^{\infty} Q(0,z)e^{(-\lambda(1-z)x - N(x))}dx \\ &= Q(0,z)\int_0^{\infty} e^{(-\lambda(1-z)x}e^{-N(x)}dx \\ &= \frac{Q(0,z)}{\lambda(1-z)}\left[1 - \int_0^{\infty} e^{-\lambda(1-z)x}\big(e^{-N(x)}h(x)\big)dx\right] \end{aligned}$$

$$Q(z) = \frac{Q(0,z)}{\lambda(1-z)}[1 - G^*(\lambda - \lambda z)] \tag{98}$$

From (97) and (98) we get

$$Q(z) = \frac{z[G^*(\lambda-\lambda z)-1]p_0}{z-G^*(\lambda-\lambda z)} \tag{99}$$

Using L'Hospital rule, we get

$$\begin{aligned} Q(1) &= \lim_{z\to 1} Q(z) \\ &= p_0 \frac{[G^*(\lambda-\lambda z)-1]+z\lambda G^{*(1)}(\lambda-\lambda z)}{1+\lambda G^{*(1)}(\lambda-\lambda z)} \\ &= p_0 \frac{\lambda E(V)}{1-\lambda E(V)} \end{aligned}$$

From (95) we obtain

$$p_0 = 1 - \lambda E(V).$$

Hence

$$Q(z) = \frac{z[G^*(\lambda-\lambda z)-1][1-\lambda E(V)]}{z-G^*(\lambda-\lambda z)}. \tag{100}$$

7.2 Busy period

Let T be the length of a busy period (starting with a customer arrival to an idle server, until the becomes idle again). Define $B(t) = P(T \leq t)$. Then $B(t)$ satisfies the relation

$$B(t) = \int_0^t \sum_{k=0}^{\infty} \frac{(\lambda u)^k}{k!} e^{-\lambda u} B^{*k}(t-u) dG(u) \tag{101}$$

The Laplace Stieltjes Transform (LST) of busy period $B(t)$ be denoted by $B^*(s)$. That is,

$$\begin{aligned} B^*(s) &= \int_0^{\infty} e^{-st} dB(t) \qquad (for Re(s) > 0) \\ &= \int_0^{\infty} e^{-st} \int_0^t \sum_{k=0}^{\infty} \frac{(\lambda u)^k}{k!} e^{-\lambda u} B^{*k}(t-u) dG(u) dt \\ &= \int_0^{\infty} \sum_{k=0}^{\infty} \frac{(\lambda u)^k}{k!} e^{-\lambda u} e^{-su} \int_u^{\infty} e^{-s(t-u)} B^{*k}(t-u) dt dG(u) \\ &= \int_0^{\infty} \sum_{k=0}^{\infty} \frac{(\lambda u)^k}{k!} e^{-\lambda u} e^{-su} (B^*(s))^k dG(u) \\ &= \int_0^{\infty} \sum_{k=0}^{\infty} \frac{(\lambda B^*(s) u)^k}{k!} e^{-(\lambda+s)u} dG(u) \\ &= \int_0^{\infty} e^{-(\lambda+s-\lambda B^*(s))u} dG(u) \end{aligned}$$

Therefore

$$B^*(s) = G^*(\lambda + s - \lambda B^*(s)). \tag{102}$$

From this the mean and higher moments of the number of customers in the system can be computed.

Conclusion:

We examined a queueing model offering n distinct services to which arrival is according to a MAP forming a single line. Service time has phase type distribution. A single server serves the customers. The service station provides two types of services - one is desirable and other is unwanted for each customer. If the service starts in an undesirable state then a clock also simultaneously starts ticking. In case this clock realizes before the exact requirement of the server is realized, then that customer leaves the system forever without being eligible for the service that he actually requires. On the other extreme, in case the correct identification of required service occurs before realization of clock, then the customer is served in that state and then leaves the system. In case right at the beginning of service the exact requirement of service is identified, then the customer starts getting that service right at the time when taken for service. Several system performance measures are evaluated. Applications of the model in hospital, telecommunication etc are indicated. Stochastic decomposition of the system state is analyzed. Some particular cases are indicated.

In a future work we propose to extend the model to multi server case.

References

[1] *Cox, D. R.* (1955): The analysis of non-Markovian stochastic processes by the inclusion of supplementary variables. Proc. Cambridge Phil. Soc. 51, 433-441.

[2] *Gross, D. and Harris, C. M.* (1988): Fundamentals of Queueing Theory, John Wiley and Sons, New York.

[3] *Madan, K. C.* (2000): An M/G/1 queue with second optional service. Queueing systems 34, 37-46.

[4] *Medhi, J.* (1994): Stochastic Processes. 2^{nd} ed. Wiley, New York and Wiley Eastern, New Delhi.

[5] *Medhi, J.* (2002): A Single Server Poisson Input Queue with a Second Optional Channel. Queueing systems 42, 239-242.

[6] *Neuts, M.F.* (1981): Matrix-Geometric Solutions in Stochastic Models, Johns Hopkins University Press, Baltimore

On averaged expected cost control as reliability for 1D ergodic diffusions

S.V. Anulova[5] [6], H. Mai[7] [8], A.Yu. Veretennikov[9] [10]

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Abstract

For a Markov model described by a one-dimensional diffusion with ergodic control without discount on the infinite horizon an ergodic Bellman equation is proved for the optimal readiness coefficient; convergence of the iteration improvement algorithm is established.

1 Introduction

According to textbooks in reliability – see, e.g., [7], [19] – coefficient of readiness is one of the main characteristics of reliability of the system. In this paper the model under consideration is presented by an ergodic Markov process described as a one-dimensional diffusion process which is controlled so as to spend more time in a "good domain" on average on the infinite horizon of time. The current readiness of the system is measured by a non-negative function f taking values on the interval [0,1]: one signifies a full readiness, while zero means that the model is in the break down state. Hence, in particular, we do not just split the real line into two parts – where $f = 1$ or $f = 0$ – but allow a soft transition from full readiness ($f = 1$) to a complete failure of the model ($f = 0$). Both coefficients of the diffusion as well as the function f itself may depend on the control. We allow only feedback (Markov) control strategies with values from some compact set. The main result states an ergodic Bellman equation on the optimal readiness characteristic ρ along with some auxiliary function; this ρ may be regarded as the most favourable readiness averaged simultaneously in space and time. Also we state an algorithm of improvement of control which in principle provides a tool to solve the Bellman equation approximately.

Earlier results on ergodic control in continuous time were obtained in [13], [15], [3], et al. The latest works include [1], [2], [18], see also the references therein. In the very first papers and books compact cases with some auxiliary boundary conditions – so as to simplify ergodicity – were studied; convergence of the improvement control algorithms were studied only partially. In the

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[5] Institute for Control Sciences, Moscow, Russia; email: anulovas @ ipu.ru

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[7] CREST and ENSAE ParisTech, France; email: hilmar.mai @ gmail.com

[8] The second author thanks the Institut Louis Bachelier for financial support.

[9] University of Leeds, UK, & National Research University Higher School of Economics, & Institute for Information Transmission Problems, Moscow, Russia; email: a.veretennikov @ leeds.ac.uk

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later investigations noncompact spaces are allowed; however, apparently, *ergodic* control in the diffusion coefficient σ of the process was not tackled earlier. About controlled diffusion processes on a finite horizon, or, on infinite horizon with discount (also known as killing) the reader may consult in [3], [10].

Discrete time and space theory was developed simultaneously in the monographs [5], [6, 8], [14], [17] and some others; important journal references can be found therein. Technical difficulties related to control in the diffusion coefficients are not an issue in discrete models. Combination of discrete state spaces and continuous time can be found in [18], et al. Reliability was not an issue in most of the cited works; however, it may be introduced in any Markov model. The paper consists of five sections not counting two lines of the Conclusions: 1 – Introduction, 2 – Setting, 3 – Assumptions and Auxiliaries, 4 – Main result and 5 – Sketch of the Proof.

2 Setting

Given a standard probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ and a one-dimensional $(\mathcal{F}_t)$ Wiener process $B = (B_t)_{t\geq 0}$ on it we consider a one-dimensional SDE with coefficients b, σ and a control parameter α described as follows:

$$dX_t^\alpha = b(\alpha(X_t^\alpha), X_t^\alpha)\, dt + \sigma(\alpha(X_t^\alpha), X_t^\alpha)\, dW_t, \quad t \geq 0,$$
$$X_0^\alpha = x \in \mathbb{R}. \tag{1}$$

Its (weak) solution does exist [11] and under our conditions – 1D, boundedness of all coefficients and uniform non-degeneracy (or ellipticity) of σ^2 – is weakly unique.

Let a non-empty compact set $U \subset \mathbb{R}$ be a range of possible control values. Without any further reminder U being compact is always bounded. Let $b: U \times \mathbb{R} \to \mathbb{R}$, $\sigma: U \times \mathbb{R} \to \mathbb{R}$, $\alpha: \mathbb{R} \to U$ be given Borel functions (some more regularity assumptions will be presented later).

Denote the (extended) generator, which corresponds to the equation (??) with a fixed function $\alpha(\cdot)$ by L^α:

$$L^\alpha(x) = b(\alpha(x), x)\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2(\alpha(x), x)\frac{\partial^2}{\partial x^2}, \quad x \in \mathbb{R}.$$

Given a running cost function $f: U \times \mathbb{R} \to \mathbb{R}$ from a suitable function class we aim to choose an optimal (in some relaxed setting, at least, "nearly-optimal") control strategy $\alpha: \mathbb{R} \to U$ (Markov homogeneous, or, in another language, Markov feedback strategy) such that the corresponding solution X^α maximizes the averaged cost function

$$\rho^\alpha(x) := \liminf_{T\to\infty} \frac{1}{T}\int_0^T \mathbb{E}_x f(\alpha(X_t^\alpha), X_t^\alpha)\, dt. \tag{2}$$

Recall that the function f takes values

$$0 \leq f \leq 1, \tag{3}$$

then this running cost may be regarded as a measure of current readiness of the underlying device. Namely, any value between zero and one we can treat as a measure of availability, while the limit ρ^α if it exists, can be understood as an averaged – with respect to time and "ensemble" – availability (=readiness) of the system. This is especially natural for the set of possible values {0; 1} for such a function; however, the whole interval of values [0,1] also makes an evident sense in the context of reliability theory. In the sequel we assume that the assumption (3) is satisfied.

By K we denote the class of strategies $\alpha: \mathbb{R} \to U$ which are Borel measurable. For convenience for every $\alpha \in K$ we define the function $f^\alpha: \mathbb{R} \to \mathbb{R}$, $f^\alpha(x) = f(\alpha(x), x)$, $x \in \mathbb{R}$. Now, instead of(2) we can use the equivalent form,

$$\rho^\alpha(x) = \liminf_{T\to\infty} \frac{1}{T}\int_0^T \mathbb{E}_x f^\alpha(X_t^\alpha)\, dt.$$

The "maximin" cost function – or, in other terms, the ergodic availability or readiness coefficient of the system – is defined by the expression

$$\rho(x) := \sup_{\alpha\in K}\liminf_{T\to\infty} \frac{1}{T}\int_0^T \mathbb{E}_x f^\alpha(X_t^\alpha)\, dt. \tag{4}$$

Suppose that for every $\alpha \in K$ the solution of the equation (??) X^α is an ergodic process, that is, there exists a unique limiting distribution μ^α of $X_t^\alpha,\ t \to \infty$, the same for all initial conditions $X_0 = x \in \mathbb{R}$. Then it is true that for every $x \in \mathbb{R}$,

$$\rho^\alpha(x) \equiv \rho^\alpha := \int f^\alpha(x')\,\mu^\alpha(dx') =: \langle f^\alpha, \mu^\alpha\rangle, \tag{5}$$

and

$$\rho(x) \equiv \rho := \sup_{\alpha\in K} \int f^\alpha(x')\,\mu^\alpha(dx') = \sup_{\alpha\in K}\langle f^\alpha, \mu^\alpha\rangle. \tag{6}$$

Note that under our assumptions ρ does not depend on x. Ergodicity requires special conditions on the characteristics b, σ, α; they will be later specified in the next section. We also define an auxiliary function which depends on x and which looks like a cost function but it is not,

$$v^\alpha(x) := \int_0^\infty \mathbb{E}_x(f^\alpha(X_t^\alpha) - \rho^\alpha)\,dt, \quad \alpha \in K.$$

This integral will converge under the recurrence assumptions below.

Solutions of the equation (??) will be understood as weak ones. Correspondlingly, the ergodic Bellman equation (7) below will be established for weak solutions.

The first goal of the paper is to prove that the cost ρ – which is a constant in the ergodic setting – is the component of the pair (V, ρ), which is a unique solution of the *ergodic HJB or Bellman's equation,*

$$\sup_{u\in U}[L^u V(x) + f^u(x) - \rho] = 0, \quad x \in \mathbb{R}, \tag{7}$$

where V will be unique up to an additive constant, while ρ will be unique in the standard sense. The meaning of the function V is that it coincides with v^α for the optimal strategy α if the latter exists, and this function is the main tool for finding an optimal strategy. Note that due to the uni-dimensional setting and the non-degeneracy of σ^2 which will be assumed, the equation (7) is equivalent to the folowing,

$$\sup_{u\in U}\sigma^2(u,x)\left[\frac{1}{2}V''(x) + \frac{b(u,x)}{\sigma^2(u,x)}V'(x) + \frac{f^u(x)}{\sigma^2(u,x)} - \frac{\rho}{\sigma^2(u,x)}\right] = 0, \quad x \in \mathbb{R}. \tag{8}$$

Further, due to the non-degeneracy of σ^2 and in particular because the right hand sides in (7) and (8) are equal to zero, we conclude that they are both equivalent to

$$\sup_{u\in U}\left[\frac{1}{2}V''(x) + \frac{b(u,x)}{\sigma^2(u,x)}V'(x) + \frac{f^u(x)}{\sigma^2(u,x)} - \frac{\rho}{\sigma^2(u,x)}\right] = 0, \quad x \in \mathbb{R}. \tag{9}$$

The second goal is to show that the "RIA" algorithm ("reward improvement algorithm", or, in some papers, "PIA" for "policy improvement algorithm") provides a sequence of convergent approximate costs, $\rho_n \to \rho,\ n \to \infty$. Also let us emphasize that unlike in the finite horizon case, here in the average ergodic control setting, the solution of the HJB equation is *a couple* (V, ρ), where ρ is the desired cost while V is some auxiliary function, which also admits a certain interpretation in terms of control theory.

Note that solutions of the equations (7), (8) and (9) will be studied in Sobolev classes, hence, (second) derivatives will be defined up to almost everywhere with respect to Lebesgue's measure. To keep all strategies Borel, all expressions involving Sobolev derivatives will be uderstood as Borel measurable expressions since for any Lebesgue's function there is a Borel function which coincides with the former almost everywhere. Respectively, all HJB or Poisson equations will be understood in the Sobolev sense with Borel versions of any second order Sobolev derivative. First order derivatives are all continuous due to Sobolev imbedding theorems.

3 Assumptions and auxiliaries

To ensure ergodicity of X^α under any feedback control strategy $\alpha \in K$, we make the following assumptions on the drift and diffusion coefficients.

1. The function b is bounded, C^1 in x, and

$$\lim_{|x|\to\infty}\sup_{u\in U} x\, b(u,x) = -\infty. \tag{10}$$

2. The function σ is bounded, uniformly non-degenerate and C^1 in x.

3. The function f takes values in the interval [0,1].

4. The functions $\sigma(u,x), b(u,x), f(u,x)$ are continuous in (u,x).

5. The set $U \subset \mathbb{R}$ is compact.

Lemma 1 *Let the assumptions (A1) – (A4) be satisfied. Then the function* v^α *has the following properties:*

1. For any strategy α the function v^α is continuous as well as $(v^\alpha)'$, and there exist $C, m > 0$ such that $\sup_\alpha(|v^\alpha(x)| + |v^\alpha(x)'|) \leq C(1 + |x|^m)$.

2. $v^\alpha \in W^2_{p,loc}$ for any $p \geq 1$.

3. $v^\alpha \in C^{1,Lip}$ (i.e., $(v^\alpha)'$ is locally Lipschitz).
4. v^α satisfies a Poisson equation in the whole space,

$$L^\alpha v^\alpha(x) + f^\alpha(x) - < f^\alpha, \mu^\alpha > \overset{a.e.}{=} 0, \tag{11}$$

in the Sobolev sense.

5. Solution of this equation is unique up to an additive constant in the class of Sobolev solutions $W^2_{p,loc}$ with a no more than some (any) polynomial growth.

6. $< v^\alpha, \mu^\alpha >= 0$.

Proof. follows from [21] & [16]; see also [9, Lemma 4.13 and Remark 4.3].

Lemma 2 *Let the assumptions (A1) – (A3) hold true. Then,*

- For any $C_1, m_1 > 0$ there exist $C, m > 0$ such that for any strategy $\alpha \in K$ and for any function g growing no faster than $C_1(1 + |x|^{m_1})$,

$$\sup_t |\mathbb{E}_x g(X^\alpha_t)| \leq C(1 + |x|^m). \tag{12}$$

- For any strategy $\alpha \in K$ the function ρ^α is a constant, and there exists $C < \infty$ such that

$$\sup_\alpha |\rho^\alpha| \leq C < \infty. \tag{13}$$

- For any $\alpha \in K$, the invariant measure μ^α integrates any polynomial:

$$\int |x|^m \, \mu^\alpha(dx) < \infty.$$

Proof follows from [21] and [16].

4 Main result

Recall that the state space dimension is $D = 1$ and that all SDE solutions with any Markov strategy are weak, unique in distribution, strong Markov and ergodic. All of these follow from [11] and from the assumptions (A1) and (A2) (see [21] about ergodicity).

The "exact RIA" reads as follows. Let us start with some homogeneous Markov strategy α_0, which uniquely determines $\rho_0 = \rho^{\alpha_0} \equiv \langle f^{\alpha_0}, \mu^{\alpha_0}\rangle$ and $v_0 = v^{\alpha_0}$. Next, for any couple (v, ρ) such that $v \in C^2$, or $v \in W^2_{p,loc}$ with any $p > 0$, and for $\rho \in \mathbb{R}$, define

$$F[v, \rho](x) := \sup_{u\in U}[L^u v(x) + f^u(x) - \rho] = \max_{u\in U}[L^u v(x) + f^u(x) - \rho].$$

Recall that unless $v \in C^2$, we consider a Borel version of the expression in the right hand side. Now, by induction given α_n, ρ_n and v_n, the next "improved" strategy α_{n+1} is defined as follows: for any x,

$$L^{\alpha_{n+1}} v_n(x) + f^{\alpha_{n+1}}(x) - \rho_n = F[v_n, \rho_n](x). \tag{14}$$

which is equivalent to

$$L^{\alpha_{n+1}} v_n(x) + f^{\alpha_{n+1}}(x) = \max_u [L^u v_n(x) + f^u(x)] =: G[v_n](x).$$

In the sequel we *assume that a Borel measurable version of such a strategy can be chosen.* In our case existence of such a Borel strategy follows from Stschegolkow's (Shchegolkov's) theorem, see [20], [12, Satz 39], [4, Theorem 1] (the first two references are in German, the last one cites the same result in English), which states that if any section of a (nonempty) Borel set E in the direct product of two complete separable metric spaces is sigma-compact (i.e., equals a countable sum of closed sets) then a Borel selection belonging to this set E exists.

Now, the value ρ_{n+1} is defined as

$$\rho_{n+1} := \langle f^{\alpha_{n+1}}, \mu^{\alpha_{n+1}}\rangle,$$

where, in turn, $\mu^{\alpha_{n+1}}$ is the (unique) invariant measure, which corresponds to the strategy α_{n+1}. Recall that

$$v_n(x) = \int_0^\infty \mathbb{E}_x (f^{\alpha_n}(X_t^{\alpha_n}) - \rho_n)\, dt.$$

Theorem 1 *Let the assumptions (A1) – (A5) be satisfied. Then the Bellman equation (7) holds true for ρ and some auxiliary function $V \in C^2$, solution of this equation is unique for ρ, and for any n, $\rho_{n+1} \geq \rho_n$, the sequence ρ_n is bounded, and there is a limit $\rho_n \uparrow \rho, n \to \infty$.*

5 Sketch of the Proof

Let us show the sketch of the main steps of the proof.

noindent **1**. From (14) and (11) it may be derived that

$$(L^{\alpha_{n+1}} v_n - L^{\alpha_{n+1}} v_{n+1})(x) \overset{a.e.}{\geq} \rho_n - \rho_{n+1}.$$

Further, from Dynkin's formula applied to $(v_n - v_{n+1})(X_t^{\alpha_{n+1}})$ we obtain,

$$\mathbb{E}_x v_n(X_t^{\alpha_{n+1}}) - \mathbb{E}_x v_{n+1}(X_t^{\alpha_{n+1}}) - v_n(x) + v_{n+1}(x) \geq (\rho_n - \rho_{n+1})\, t.$$

Since the left hand side here is bounded for a fixed x, after division of all terms by t and at $t \to \infty$, we obtain,

$$0 \geq \rho_n - \rho_{n+1},$$

as required. Therefore, $\rho_n \leq \rho_{n+1}$, so that $\rho_n \uparrow \tilde{\rho}$ with some $\tilde{\rho}$. Thus, **the RIA does converge**, although so far we do not know whether $\tilde{\rho} = \rho$. Clearly, $\tilde{\rho} \leq \rho$, since ρ is the sup over all Markov strategies, while $\tilde{\rho}$ is the sup over some its countable subset.

Recall that now we want to show that $v_n \to \tilde{v}$ such that the couple $(\tilde{v}, \tilde{\rho})$ satisfies the HJB equation (7), and that $\tilde{\rho}$ – as well as $\tilde{v}$ in some sense – here is unique.

2. What we want to do is to pass to the limit in the equation

$$L^{\alpha_{n+1}}v_{n+1}(x) + f^{\alpha_{n+1}}(x) - \rho_{n+1} = 0, \quad as \quad n \to \infty,$$

after having showed compactness of the set (v_n) at least in C^1 (and later on, in $C^{1,\beta}$ for any $0 < \beta < 1$). Since

$$\rho_n = L^{\alpha_n}v_n(x) + f^{\alpha_n}(x),$$

we obtain after division by $\sigma^2/2$,

$$v_{n''}(x) = \frac{2\rho_n}{\sigma^2}(x) - \frac{2f}{\sigma^2}(x) - \frac{2v_{n'}}{\sigma^2}(x). \tag{15}$$

Due to the local boundedness and absolute continuity of $v_{n'}$ – see the Lemma 1 – we conclude that the sequence (v_n) is locally (i.e. on any bounded interval) tight in C^1. Hence, there is a subsequence $n' \to \infty$ such that $v_{n'}$ converges in C^1 on any bounded interval to some function $\tilde{v} \in C^1$ (in fact, even $\tilde{v} \in C^{1,Lip} = \{g \in C^1(\mathbb{R}) \colon g' \in \ Lip\ \}$ as will be clear in a few lines and follows, e.g., from (15)). Denoting

$$F_1[x, v', \rho] := \max_u[\hat{b}^u v' + \hat{f}^u - \hat{\rho}](x) \equiv \max_u \left[\hat{b}^u v' + \hat{f}^u - \frac{\rho}{a^u}\right](x),$$

where

$$a^u(x) = \frac{1}{2}(\sigma^u(x))^2,\ \hat{b}^u(x) = b^u(x)/a^u(x),\ \hat{f}^u(x) = f^u(x)/a^u(x),\ \hat{\rho}^u(x) =$$
$\rho/a^u(x)$,

and by using the bounds as in [10, Chapter 1], *it can be shown* the limiting equation as $n' \to \infty$

$$\widetilde{v}'(x) - \widetilde{v}'(r) + \int_r^x F_1[s, \widetilde{v}'(s), \tilde{\rho}]\, ds = 0, \tag{16}$$

which implies by differentiation that

$$\widetilde{v}''(x) + F_1[x, \widetilde{v}', \tilde{\rho}](x) = 0. \tag{17}$$

This equation is equivalent to (9) and, hence, to (7), as required. In other words, the limiting pair $(\tilde{v}, \tilde{\rho})$ satisfies the HJB equation (7).

3. *Uniqueness for* ρ. Suppose there are two solutions of the (HJB) equation, v^1, ρ^1 and v^2, ρ^2 with a polynomial growth for v^i. Denote $v(x) := v^1(x) - v^2(x)$ and consider two Borel strategies $\alpha_1(x) \in argmax_u(L^u v(x))$ and $\alpha_2(x) \in argmin_u(L^u v(x))$, and denote by X_t^i a (weak) solution of the SDE corresponding to each strategy α_i. (It exists and is weakly unique.) Note that

$$h_2(x) := \max_u(L^u v(x) - \rho^1 + \rho^2) = \max_u(L^u v^1(x) + f^u(x) - \rho^1 - L^u v^1(x) - f^u(x) + \rho^2)$$

$$\geq \max_u(L^u v^1(x) + f^u(x) - \rho^1) - \max_u(L^u v^2(x) + f^u(x) - \rho^2) \overset{a.e.}{=} 0,$$

and similarly,

$$h_1(x) := \min_u(L^u v(x) - \rho^1 + \rho^2) = -\max_u(L^u(-v)(x) - \rho^2 + \rho^1)$$

$$\leq -\left[\max_u(L^u v^2(x) + f^u(x) - \rho^2) - \max_u(L^u v^1(x) + f^u(x) - \rho^1)\right] \overset{a.e.}{=} 0.$$

We have, $L^{\alpha_2}v(x) = h_2(x) - \rho^2 + \rho^1$, and $L^{\alpha_1}v(x) = h_1(x) - \rho^2 + \rho^1$. Further, Dynkin's formula is applicable. So,

$$\mathbb{E}_x v(X_t^1) - v(x) = \mathbb{E}_x \int_0^t L^{\alpha_1}v(X_s^1)\, ds$$

$$= \mathbb{E}_x \int_0^t h_1(X_s^1)\, ds + (\rho^1 - \rho^2)\, t \overset{(h_1 \leq 0)}{\leq} (\rho^1 - \rho^2)\, t.$$

The last inequality here is due to the $h_1 \overset{a.e.}{\leq} 0$ along with Krylov's bounds [10]. Here the left hand side is bounded (x fixed) due to the Lemma 2, so, we obtain,

$$\rho_1 - \rho_2 \geq 0.$$

Absolutely similarly we show that also

$$\rho^1 - \rho^2 \overset{(h_2 \geq 0)}{\leq} 0.$$

Thus, eventually,

$$\rho_1 = \rho_2.$$

4. *Proof of the equality* $\rho = \tilde{\rho}$. We have seen that for any initial (α_0, ρ_0), the sequence ρ_n

converges monotonically to $\tilde{\rho}$, which is a component of solution of the Bellman equation (7), as shown earlier in the step 2, and this component $\tilde{\rho}$ is unique as was just shown in the step 3. Hence, given some (any) $\varepsilon > 0$, take any initial strategy α_0 such that

$$\rho_0 = \rho^{\alpha_0} > \rho - \varepsilon.$$

Then, clearly, the corresponding limit $\tilde{\rho}$ will satisfy the same inequality,

$$\tilde{\rho} = \lim_n \rho_n > \rho + \varepsilon.$$

Due to uniqueness of $\tilde{\rho}$ as a component of solution of the equation (7), and since $\varepsilon > 0$ is arbitrary, and because it is already established that $\tilde{\rho} \le \rho$, we now conclude that

$$\tilde{\rho} = \rho.$$

The sketch of the proof of the Theorem 1 is thus completed.

6 Discussion

Thus, we have an approach which in principle allows to evaluate the ergodic readiness coefficient in certain diffusion Markov models.

7 Addendum: Borel measurability

In the presentation of RIA we have assumed existence of a Borel measurable version of such a strategy to be chosen which maximizes some function ofr a fixed x. In our case existence of such a Borel strategy follows from Stschegolkow's (Shchegolkov's) theorem, see [20], [12, Satz 39], [4, Theorem 1] (the first two references are in German, the last one cites the same result in English), which states that if any section of a (nonempty) Borel set E in the direct product of two complete separable metric spaces is sigma-compact (i.e., equals a countable sum of closed sets) then a Borel selection belonging to this set E exists. In our case $E = \{(u,x): F[u,x] = \phi(x) := \max_{v \in U} F[v,x], x \in \mathbb{R}\}$. This set is nonempty and closed and, hence, Borel. Indeed, if $E \ni (u_n, x_n) \to (u,x), n \to \infty$, then $F[u_n, x_n] \to F[u,x]$ due to continuity of F. Also, due to continuity of F, $\phi(x_n) \to \phi(x)$. Since each u_n is a point of $argmax\ (F[\cdot, x_n])$ where $F[u_n, x_n] = \phi(x_n)$ we have, $F[u,x] = \lim_{n\to\infty} F[u_n, x_n] = \lim_{n\to\infty} \phi(x_n) = \phi(x)$, we find that $(u,x) \in E$, i.e., E is closed. Further, any section E_x of E is also closed itself again due to continuity of F, as if $(u_n, x) \in E$ and $u_n \to u$, then $F[u_n, x] \to F[u,x]$, i.e., actually, $F[u_n, x] = F[u,x]$. Thus, Stschegolkow's theorem is applicable.

References

[1] A. Arapostathis. On the policy iteration algorithm for nondegenerate controlled diffusions under the ergodic criterion in book. In: *Optimization, control, and applications of stochastic systems*, Systems Control Found. Appl., 1–12. Birkhäuser/Springer, New York, 2012.

[2] A. Arapostathis, V. S. Borkar, M. K. Ghosh. *Ergodic control of diffusion processes.* Encyclopedia of Mathematics and its Applications 143. Cambridge: Cambridge University Press, 2012.

[3] V. S. Borkar. *Optimal control of diffusion processes.* Harlow: Longman Scientific & Technical; New York: John Wiley & Sons, 1989.

[4] L.D. Brown, R. Purves. Measurable selections of extrema. *Ann. Stat.*, 1: 902–912, 1973.

[5] E.B. Dynkin and A.A. Yushkevich. *Upravlyaemye markovskie protsessy i ikh prilozheniya*, Moskva: "Nauka", 1975 (in Russian).

[6] R. A. Howard. *Dynamic programming and Markov processes.* New York-London: John Wiley &; Sons, Inc. and the Technology Press of the Massachusetts Institute of Technology, 1960.

[7] B. V. Gnedenko, Yu. K. Belyaev, A. D. Solovyev. *Mathematical Methods in Reliability*

Theory. Academic Press, New York, 1969.

[8] R. A. Howard. *Dynamic probabilistic systems. Vol. II: Semi-Markov and decision processes. Reprint of the 1971 original ed.* Mineola, NY: Dover Publications, 577-1108, 2007.

[9] R. Khasminskii. *Stochastic stability of differential equations. With contributions by G.N. Milstein and M.B. Nevelson. 2nd completely revised and enlarged ed.* Berlin: Springer, 2012.

[10] N.V. Krylov. *Controlled diffusion processes, 2nd ed.* Berlin, et al., Springer, 2009.

[11] N. V. Krylov, On the selection of a Markov process from a system of processes and the construction of quasi-diffusion processes. *Math. USSR Izv.* 7: 691–709, 1973.

[12] A.A. Ljapunow, E.A. Stschegolkow, and W.J. Arsenin. *Arbeiten zur deskriptiven Mengenlehre.* Mathematische Forschungsberichte. 1. Berlin: VEB Deutscher Verlag der Wissenschaften, 1955.

[13] P. Mandl. *Analytical treatment of one-dimensional Markov processes.* Berlin-Heidelberg-New York: Springer, 1968.

[14] H. Mine and S. Osaki. *Markovian decision processes.* New York: American Elsevier Publishing Company, Inc., 1970.

[15] R. Morton. On the optimal control of stationary diffusion processes with inaccessible boundaries and no discounting. *J. Appl. Probab.*, 8: 551–560, 1971.

[16] È. Pardoux and A.Yu. Veretennikov. On Poisson equation and diffusion approximation. II. *Ann. Probab.*, 31(3): 1166–1192, 2003.

[17] M. L. Puterman. *Markov Decision Processes: Discrete Stochastic Dynamic Programming.* Wiley Series in Probability and Statistics. John Wiley & Sons, Inc., Hoboken, 2005.

[18] V. V. Rykov, Controllable Queueing Systems: From the Very Beginning up to Nowadays, *RT&A,* 2(45), vol. 12, 39-61, 2017.

[19] A. D. Solovyev. *Basics of mathematical reliability theory, vol. 1.* Moscow: Znanie, 1975 (in Russian).

[20] E.A. Shchegol'kov. Über die Uniformisierung gewisser B-Mengen. *Dokl. Akad. Nauk SSSR, n. Ser.*, 59: 1065–1068, 1948.

[21] A.Yu. Veretennikov. On polynomial mixing for SDEs with a gradient-type drift. *Theory Probab. Appl.*, 45(1): 160–164, 2000

Reliability Design in Gradual Failures: A Functional-Parametric Approach

O. V. Abramov, B. N. Dimitrov
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Institute for Automation and Control Processes,
Far Eastern Branch of the Russian Academy of Sciences, Vladivostok, Russia
E-mail: abramov@iacp.dvo,ru
Department of Mathematics, Kettering University, Flint, MI 48504
Email: bdimitro@kettering.edu

Abstract

This conceptual paper discusses the main provisions of the functional-parametrical (FP) approach in reliability studies. The FP itself is a detailed frame algorithm suggested for use in design of new and unique technical items. The article also presents possibilities and perspectives of using FP approach in problems for "building in" reliability in analogy to these for technical devices and systems. It is pointed out that for solving problems of analysis and ensuring of desired reliability it is appropriate to use parallel and distributed processing techniques. We discuss the idea of constructing efficient parallel algorithms for multivariate statistical analysis necessary to calculate estimates of the probability for failure-free operation with different nominal values of internal parameters. More use of parallel algorithms, including in continuous media via discretization are discussed

Keywords: computational methods, conceptual algorithms, gradual failure, parallel computing, parametric synthesis, projected reliability, scanning method.

I. Introduction

In modern reliability analysis there are several methodological trends, the dominant position of which is the probabilistic and statistical trend. The methodology of this trend is based on empirically established fact for statistical stability failure rates. This enables the use the analytic probability theory and some elements of queuing studies.

Calculation of reliability within a probabilistic-statistical approach is based on the construction of block diagrams for the processes running in the studied system (reliability model). When constructing a model for system reliability each of its elements is usually allowed to have only two possible states - full functionality or complete failure. Consequently, the system can be also in only two states - full functionality or complete failure. Any possibility of partial functioning of the system or of its components is usually excluded. Thus, in estimation of reliability the real system is replaced by a logical (Boolean) model. Its various modifications, such as models in the form of a fault tree and even as a Markov model with finite space of states, does not change the fundamental nature of the reliability model. The main element of the design characteristics of the reliability of such models is the failure rate. Methods of this trend are quite simple, convenient for engineering calculations, and do not require (in majority of the application) the use of modern computer equipment, since the solution of the main tasks in this direction can be obtained in closed form.

At the same time, the two-staged model used in probabilistic and statistical approach does not reflect the reliability indices with performance of multiple functions by the item, i.e. model cannot be functional and realistic. The process of developing technical objects with desired characteristics is associated mostly with current research. It is based on functional models (physical, mathematical, or combined). Functional reliability model are supposed to reflect the relationships between the specified functions (output parameters) of the system and the parameters of its elements. There are tolerance types of interactions between the given system functions and operational factors; between the functions of the elements and the physical-chemical processes (that cause changes in their parameters during the operation process); between different types of a prior information about the processes of changes in element parameters and in the system as a whole.

Note also, that using of probabilistic and statistical approach does not give trustful results in solving the reliability for unique objects and systems for critical applications where failures are not massive and do not present statistically regular phenomenon.

The study of the reliability problems for technical systems from the viewpoint of the random walk theory in phase space is the most common and promising. Reliability models of this type have been proposed originally (Gnedenko, Beliaev, Soloviev, 1969). It is allowed to find a deep connection between reliability modeling and the general theory of random functions, and allows formulating a detailed methodological approach which we call *functional-parametric* (FP) approach.

The possibilities and feasibilities of establishing a FP-approach not only arise from the deficiencies in the classical approach to solve reliability problems. The FP approach is pre-determined by the successful development of methods of mathematical modeling of complex systems, automation, the mathematical description of the processes during their operation, as well as research methods of reliability of gradual failures (Becker and Jensen 1977). Nowadays we also may rely on much better computational equipment and digital technology which make even the wildest dreams to get true.

II. The main idea of the functional- parametric approach

The functional-parametric approach naturally follows from the generally accepted conventional definition of reliability as *a property of an object to keep the values of its parameters in prescribed limits, which characterize object' ability to perform the required functions in specified modes.* In accordance with this definition any mathematical model for determination of the reliability should reflect possible relation between the reliability rates and the functions of operating items in changing conditions and time. We grab this main idea of the FP-approach to describe in detail how the problems of system's reliability can be resolved by following the next basic principles:

1. The process of operation of the system and its technical conditions at any time is determined by some finite set of variables - parameters of the process and the system;

2. The accumulation of various impacts on the system leads to the evolution of its indicators (changes in the parameter values with the time). Therefore, to keep track on the possibility of switching to a different qualitative state is an important action;

3. Failures are results of deviations of parameter' values from their original prescription to current. The forms of identification of a failure consist in the departure of a parameter outside the allowed tolerance range (acceptability region);

4. When the process of parameters change is admissible (observable, predictable, or controllable), then there is (in principle) a possibility to prevent failures;

5. In FP-approach the problems of evaluating and assessment of desired reliability during the system design, its manufacture and operation are interrelated. They can all be represented as a kind of management tasks to model, simulate and analyze appropriate stochastic processes. Respective decision should be based on the results of the forecasting intervals of possible parameter's changes (technical condition) and overall assessment of the reliability of the studied objects.

The forecasting and control methods need to keep into account the specifics of modelled random processes and descriptions in drifts of parameter changes. These may belong to the class of non-stationary and locally controlled processes. Sometimes characteristics of such control may have the form of a pulse or shock correction.

Thus, when solving a problem related to reliability of technical objects based on the FP-approach it could be necessary to take into account several things: (a) Possible deviations true parameters from the calculated values; (b) To foresee the consequences of these deviations and (c) To develop a set of measures that provide sufficient information about required characteristics of the object in terms of these deviations. It is understandable that in the framework of the FP-approach in reliability evaluation is a natural extension of conventional engineering approach at the design stage. Performance assessment in the framework of the FP-approach is based on creation and optimal use of reserves of admissible variations of the system parameters. In addition, the possible control of the most important parameters, prediction of parameter changes for prevention their exit out of the permissible limits, availability of parameter' adjustments, or replacements of worn out components should be included in considerations. The reliability evaluation problem can be considered as an extended form of the optimal parametric synthesis problem (Abramov 2006).

III. Reliability models creation for technical systems and the FP-approach

The reliability and quality of technical devices and systems is built within numerous activities performed on stages of development, production (manufacturing), testing and operation. The stage of development is of a particular importance, since at this moment the principles of quality assurance and future reliability are planned. A significant part of the activities aimed at the implementation of these principles, subsequently implemented, in particular in the production and exploitation. However, their successful implementation is built in the items the most during development stage. It is essential to find out, to what extent there are taken into account the processes expected to occur in subsequent phases.

The basis in the development phase is the technical design and specification, which decides required parameter values (requirements needed for operations in the field), Numerical data describing the ranges of possible variability in environmental parameters (conditions of operation), a qualitative description of the restrictions, ergonomic requirements and conditions that are not directly measurable – each detail may affect the development phase run.

One main part of the technical specifications is the inclusion of the requirements about outcome parameters from the object of development (technical requirements). Certain relationships between output parameters and technical requirements will be called serviceability specifications. These will be later given to the item users as instructions in operation. It is needed in the design process and it is necessary to find solutions that are acceptable. First of all, from the point of view of the designer, it is important, to ensure the achievement of the conditions of functionality. Together with that, the designer should build in the pre-determined quality indicators. This is a set of quality indicators determines the ability of an object to perform its functions and therefore, characterizes the generalized states of the object.

The concept of quality should include a description of the object properties that determine the success of the problem solution in certain conditions. Such properties can be efficiency, productivity, accuracy, stability, reliability, survivability, safety, ergonomic solutions, etc. Indicators (criteria) of quality are necessary for qualitative estimation of object features. They can be used also as quantitative measures for the degree of conformity of the object to its intended purpose. The review article (B. Dimitrov, 1998) may serve in great extend in this phase.

In the traditional sense, *the problem of parametric synthesis* is reduced to the selection of parameter values (for a given structure) under which serviceability specifications are performed. Consideration of possible parameters deviations from the calculated values and development of measures that ensure the operation of the object in the presence of such deviations are transferred to the subsequent stages of the design. Sometimes they appear in the stages of the production and operation. This approach to the problem of parametric tolerance synthesis is most common in practice.

The *stage of parametric synthesis* involves two procedures. The first one is the choice of initial values of the internal parameters, which is usually carried out on the basis of simple calculations, or is based on the experience and intuition of the engineer. The second one consists in appropriate correction of the initial values of the parameters. Synthesis vector of internal parameters found at this stage suggests only that the "tentative" project is operational.

Deviations of the parameters from the calculated nominal values can lead to loss of efficiency, so the next step of parametric synthesis is to set optimal values of internal parameters. For example, those which provide greatest perform ability margins or maximum probabilities of the specified performance fulfillment deserve special attention.

Selection *of the optimal parameter values* does not always allow creation of an object with the required consumer properties, i.e. to provide the functions of a given quality. Thus, defining the nominal parameter values, which guarantee maximum probability for non-failure work of the object within a certain period of time $P_{max}(T)$. Comparing the obtained values with desired ones, $P_d(T)$, the developer cannot consider the design process completed, if $P_{max}(T) < P_d(T)$. In this case, it is necessary to look for ways of further design improvement. The required reliability can be achieved by adjusting (tuning) some parameters. Thus, to ensure the required quality of operation for the item, it is necessary to select and implement some strategies for control of its parameters.

Next step is to determine some *parametric synthesis in a narrow sense*, as evaluation of the nominal values of the parameters of the item, and in wide sense as a result of certain strategy for parameters control. The main content of the methodology of parametric synthesis in wide sense are hidden in the answers of the following two related questions: which control parameters to choose, if it is necessary to keep them under control and which values should take these controlled parameters.

Parameters deviations are formed under the influence of factors in the manufacture, storage, service (use), and may have random character. Therefore, the internal parameters have to be considered as some random functions depending on time. Consequently, serviceability conditions can be met only with a certain probability like the following

$$P(T) = P\{\boldsymbol{X}(t) \in D_x,\ \forall\ \ t \in [0, T]\},$$

Here $\boldsymbol{X}(t)$ is a random process of the internal parameters changes; D_x is the set of acceptable changes for internal parameters (region of acceptability); T is the projected operation time. Selected nominal values of the parameters ratings (nominal ones) $\mathbf{x}_r=(x_{1r}\ ,..., \ x_{nr})$ can be considered as components of the vector of average according to distribution of the random process $\boldsymbol{X(t)}$ at time t =0, i.e. $\mathbf{x}_r^{(1)} = M\ [\boldsymbol{X}(0)]$.

If probability for *correct fulfillment of the performance specifications* with selected in the previous step nominal values of the internal parameters within given time $P(T,\ \mathbf{x}_r^{(1)})$ is below the desired $P_d\ (T)$, it will be necessary to make a switch to parametric synthesis as in the second level. Here we mean a choice of the nominal values for the parameters with respect of their patterns in the conditions of manufacturing and operational variations.

Parametric synthesis in the second level is to select such nominal values of internal parameters, which provide maximum efficiency (the maximum probability for non-failure operations during a given time, or maximum operation time between failures, etc.).

Thus, the parametric synthesis at the second level is an *optimization problem* with stochastic criteria. The result of its solution is the values of the internal parameters

$$\mathbf{x}_r^{(2)} = arg\ max\ P\{\boldsymbol{X}(t) \in D_x, \text{ for any } t \in [0, T]\},\ \mathbf{x}_r \in D_c.$$

Here $D_c \subseteq D_x$ is the space of acceptable internal controlled parameters.

When the probability of failure does not meet the requirements with the values of parameters selected in parametric synthesis on second level, then proceed to parametric synthesis of the next level.

Increase of the probability of performance specifications can be achieved if some of the internal object and process parameters are tuned (adjusted). Synthesis of tuned objects will be called *parametric synthesis in the third level.* The task of selecting the set of parameters at which most appropriate to carry out adjusting (of controlled parameters), and to change their reasonable ranges becomes of independent importance in the process of synthesis. After its solution arises the problem of choosing the optimal values of tuning parameters (as shown in Abramov 2016).

The one-time adjustment problem can be solved in the process of parametric synthesis in the third level, but does not guarantee the quality of synthesized object.

Next level of parametric synthesis is the synthesis of items with multiple adjustable parameters. We call it parametric synthesis of the fourth level. It aim is to give an answer to all the above questions: what parameters, when and how should be changed to ensure that specified requirements for the object performance quality are achieved.

In the *parametric synthesis of the fourth level* the optimization of parameters is carried out to prevent possible loss of efficiency, and has the character of preventive corrections. It is necessary to choose for tuned items a set of tuning options (the control parameters), which will allow to determine the appropriate moments of preventive corrections (settings) of parameters, and to give recommendations for choosing optimal values of the tuned parameters. For non-tuned objects it is necessary to assign the timescales of preventive measures, to give recommendations for finding components needed to be replaced, and to determine the parameters of the replacement components.

At this level of synthesis it is necessary to distinguish between item parameters which are not controlled during the operation, and item parameters with control. Moments of change and optimal values of the controlled parameters for the first group are defined by some prior information about the processes operational changes in parameters. The previous experience with a set of similar items (of a group nature), can be determined on the basis of prior data and as results of monitoring of particular item parameters. The just obtained recommendations are valid only for a particular item, since strategies of parameters control are strictly individual. In each case necessity of identifying the specific ways of control measurements will arise.

We give now a *general frame algorithm formulation* for solving reliability optimization) problems, formulated as problems of parametric synthesis.

1. *The problem of optimal selection of nominal parameter values.* With determined characteristics $\boldsymbol{X}(t)$, defined D_x and T, find such nonrandom initial nominal values $E_1, E_2, \dots, E_n$, for which

$$P\ \{(X_1(t)+E_1,\ X_2(t)+E_2, \dots, X_n(t)+E_n) \in D_x,\ \forall\ t \in [0, T]\}, = \max P.$$

2. *The problem of optimal adjustment.* With determined characteristics of random process under varying non-adjustable parameters, $X_1(t), \dots X_k(t)$ and adjustable parameters, $X_{k+1}(t), \dots X_n(t)$ find such values $E_{k+1}, \dots, E_n$, for which

$$P\ \{(X_1(t),\ \dots\ X_k(t),\ X_{k+1}(t) + E_{k+1}\ ,\ \dots\ ,\ X_n(t)+E_n) \in D_x,\ \forall\ \ t\in\ [0,\ T]\},\ = \max P$$

3. The *problem* of prevention *maintenance:*

a) *Heuristic prevention maintenance.* With determined characteristics from the random priory selected process $\mathbf{X}^{P_r}(t)$, given the tolerance box D_x and the desired time T find a non-random function $\boldsymbol{E}(t)$, for which

$$P\{(X_1^{P_r}(t)+E_1(t),\dots,X_n^{P_r}(t)+E_n(t))\in D_x,\forall t\in[0,T]\}=\max P\ ,$$

In each case check if $C(t) \le C_0$, where $C(t)=\int_0^t C(E(x))dx$ is the costs related to the parameters correction (needed maintenance), where C_0 is an acceptable level of maintenance costs;

b) *Posterior (individual) preventions maintenance.* With the known characteristics of the posterior stochastic process $\mathbf{X}^{Ps}(t)$ derived from prior data with respect of the control results, define D_x and T, find a function $\boldsymbol{E}(t)$, for which

$$P\{(X_1^{P_s}(t)+E_1(t),\dots,X_n^{P_s}(t)+E_n(t))\in D_x,\forall t\in[0,T]=\max P$$

In each case check if $C_1\ (t)\ \le C_0$, where $C_1\ (t)$ is the costs related to the control and correction of parameters.

Analyzing the given algorithm formulation, it is important to verify if there are fundamental commonalities, and if problems at stages 1 and 2 are special cases of the problem in stage 3. Inherently, they all belong to the class of control problems for stochastic processes. Their solution should be based on the results of the forecasting of parameters drift process (the technical condition) and reliability of an optimized item.

Mathematical and computational complexity of the methods of optimal synthesis of technical systems is taking into account the laws of random variations of their parameters and proposed reliability requirements. The difficulty of obtaining the necessary initial information about parametric perturbations raises certain pessimism regarding the practical usefulness of FP-approach. However, in recent years an active development of sufficiently radical ways for reducing the complexity in solving complex computational problems is observed. It is based on the idea of parallel processes of searching final results. Currently it is gained some experience in creation of algorithms and software tools for calculation and parametric optimization in reliability design of technical devices and systems, based on the use of technology in parallel and distributed computing.

To overcome the complexities associated with the deficiency or absence of information about the patterns in stochastic processes for the parameters changes of the studied systems. Possible solutions are offered, based on using the ideas of robustness and minimax approaches. In other words, the necessary level of reliability is ensured either by the creation of systems that are insensitive to variations of their parameters (robustness), or as a result of giving them a required stamina. These approaches take into account the most unfavorable variations of the system parameters.

IV. The technology of parallel computing in parametric synthesis problems

The stochastic nature of the optimality criterions, the dimensionality of the search space, the need to solve global optimization problems forced researchers to seek ways of creating efficient numerical methods for solving problems of parametric synthesis (PS). One of the most radical solutions to the problems of high computational complexity is the *parallelization of the solutions search process*.

There are some versions of PS strategies using the technology of parallel computing. The base of the first strategy is the idea of parallel methods for calculation of the objective functions using optimization techniques.

Creation and implementation of a parallel analogue of the method *of statistical simulations* (Monte Carlo) does not cause essential difficulties. The use of parallel computation in this method is quite logical, as far as the idea of parallelism – the repetition of a certain process model with varying sets of data is inherent in the very structure of the method. It is intuitively clear that the use of *k* independent processors and the distribution between them as independent trials reduces the complexity of statistical simulation about almost *k* times. The cost of the final summation and averaging of results is almost negligible (see Abramov 2010).

Further, reducing the time of solving the problem of PS can be achieved by parallelization of the algorithm of search of extreme values of the objective functions.

The simplest method for global optimization, having the property of potential parallelism, is *the scanning* (full enumeration) *method*. The essence of the method lies in the fact that the search area is divided into unit cells, in each of them (by a particular algorithm) is chosen the point: in the center of the cell, or on the edges or the vertices. In each cell there is a consequent view for the values of the objective function and finding among them the extreme values is just a simple task. The accuracy of the method, naturally, is determined by how tightly the points in the search area are selected.

The main advantage of the scanning method is that using it with fairly dense points of location, it always guarantees to find the global optimum. However, this method requires significant amount of computation, which can be reduced by parallelization of the computational algorithm.

In the tasks of the PS sample, the set of nominal parameters is in most cases finite. This is due to the values of most parameters of radio components (resistors, capacitors, inductors, operational amplifiers, etc.) are regulated by technical specifications and standards. On the basis of experience and intuition, the developer can usually set the necessary options as used in active elements. Therefore, the nominal values of their parameters are known. In cases where it is possible to select nominal values of parameters in a continuous range, the researcher can use *discretization procedure*. Thus, in most cases it can be assumed that the sample set of nominal parameters for solving PS problem is discrete.

In the simplest case, the search of the solutions is reduced to exhaustive enumeration on the full set of possible nominal values of the internal parameters $D_r^{in} = \{x_r^{in} \mid x_r \in D_x\}$, in each point $\mathbf{x}_r^{in}$ at which is needed to find the value of the objective function. Taking into account the cyclical nature of the calculation procedure for objective functions, it is easy to apply data parallelism.

Let the solution process can be performed using *k* processors (slaves). The set D_r^{in} is partitioned into non-overlapping subset $D_r^{in} = \bigcup_{j=1}^{k} \{D_{r\,j}^{in}\}$, for which to the *j*-th processor is assigned a subset of original data. Thus, each *j*-th processor calculates the objective function for all elements of the set $D_{r\,j}^{in}$ and finds the best vector of values of parameters for the assigned subareas. The results are transmitted to the master processor: which selects the optimal vector from the values throughout the area D_r^{in}. Such partitioning of the entire search set on non-overlapping subsets constitutes the essence of blocks of the scheduling parallel distributed process.

For symmetric computing cluster consisting of *k* equal computing nodes, the total number of selected points is divided into equal amounts for each of the subordinated processes. In the case of an asymmetric cluster it is necessary to perform a preliminary procedure for the complexity assessment. As a typical procedure in the optimization method, it acts as a single simulation of the item's operation, verification of performance specifications, and calculates the criterion of optimality. In this case the computational load is divided between the components of the complex in proportion to their productivity.

Upon completion of the program scheduling of parallel computation process, each computing component of the complex receives original data from the borders of the subset $D_{r\,j}^{in}$. On the end of computation the main processor receives the results from subordinates and generates the final result of discrete optimization on the entire set D_r^{in}.

Another possible strategy of the PS is based on the design of the region of admissible values of the internal parameters (region of acceptability) D_x. The attractiveness of this strategy is related to the possibility of decomposition of the general task of the PS into two subtasks. The first one consists in the construction, analysis and approximation of the area D_x. This is a task of highest computational complexity, as it is connected with the necessity of multiple calculations of the values for output parameters of the system (see Abramov, Katueva and Nazarov 2006). The second subtask involves the calculation of the objective function and finding optimal values of the nominal parameters, using information about the field of D_x. Obtaining the solution in this case is not related to the need to referee to the model of the investigated system, which significantly reduces the complexity of the parametric synthesis.

Thus, the strategy of PS in this case will consist of two stages, the first concerned with design of the regions of permissible values of parameters (variations region of acceptability) D_x. Parallel algorithms of the region of acceptability design are given in Abramov, Nazarov 2015.

The second stage is focused on the search of optimal solutions. With the known region of acceptability, the complexity of calculating the values of the objective function and the search for extreme is significantly reduced. In this case it is not necessary to compute values of the system's output parameters.

In addition, a significant reduction in computational costs and efforts can be achieved by the use of area D_x for parallel analogues of search optimization models. Thus, when using the PS strategy, based on region of acceptability design, the solution of this problem is carried out in two phases, the first of which can be considered as a pre-requisite (construction of area D_x), and the second – the optimization one.

V. Other areas of use the functional-parametrical approach

Using of the FP-approach appears to be promising in solving the problems of ensuring, assessing and maintaining reliable and safe operation of complex technical systems for critical important applications.

A, The problem of preventing failures and reducing the techno-genic risks is of particular relevance to the technical objects, which failures are associated with large financial losses, or with catastrophic consequences. Most of these complex systems are produced in small numbers, operated in variable conditions and realizing extreme technologies. Such systems are commonly called *unique.*

In the study of the techno genic risks, where a risk event is considered as a loss of functionality (failure) of a technical object, the correct design is a key issue. The characteristics of the objects are the operating time (uptime) or the times to failure, and the probability characteristics which are determined by methods of mathematical statistics and reliability theory. Unfortunately, in the study of complex systems to get fairly representative statistics of failures is not always possible. This is because such systems are manufactured in small numbers when needed by specific instances, and are operated in varying conditions. Their failures are rare events.

Moreover, the problem is to prevent failures rather than to accumulate its statistic.

B. In solving the problem for management of technogenic risks on the basis of the FP-approach, the constructors and designers should be able to estimate the current technical state of the system in order to predict changes in it (the time of transition to the critical state). Moreover, they have to determine the respective total and operating costs related to the monitoring of the states, carrying out preventive measures and the damages upon occurrence of a risk event.

C. It is essential to note that risk management related to the solution of the problem of individual planning of the operation is an important factor. The basis of individual approach to the problems of control of the operations is to predict changes in the parameters of technical states, based on the results of monitoring. Predicting of the technical state based on observations of each particular item can be carried out only if there are known a sufficient amount of prior characteristics of the processes occurring in similar items (or on models of a random process in different parameter variations). As theoretical basis of technical state predicting it may serve the classical methods of optimal estimation and extrapolation. The result of such predictions is the point estimates of the controlled parameter at some future point in time. The problem of risk management is associated with finding the moment of the first outage of the parameters outside the regions of acceptability. The use of this approach is allowable, if the probabilistic characteristics of the measured errors and the accepted model of parameters drift are known exactly.

The main difficulties in solving the problem of forecasting the state of synthesis of the management strategy of techno-genic risks are, that the predicting has to be carried out for individual items. They are produced in small volumes and sources of information are based on a small sample sets from control tests. In the presence of interference between control errors statistical properties of estimates are not known. In these circumstances, classical methods of engineering statistics and theory of random processes lose their attractive properties. Their use in predicting leads to significant errors and low accuracy in prediction. Therefore, it is necessary to expand the initial information base by conducting a comprehensive object surveys and following monitoring of its states. Development of new methods for predicting, supplementing the already existing knowledge is imminent. Some approaches to solving the problem of individual prediction of the reliability issues in complex technical engineering systems and operations planning in conditions of deficiency and incomplete initial information were presented here. This approach allows us to obtain suitable results in these circumstances, and get sufficiently reliable results, as reviewed in Abramov and Nazarov 2016.

D. The basic ideas of FP-approach are useful in solving problems for ensuring the survivability and safety of technical objects.

Survivability is usually described as the ability of a system to maintain its basic functions (albeit with some loss of quality of performance) in adverse effects of environmental factors, beyond the designed operating conditions. It is worthy to notice that adverse impacts may cause abnormal changes of external parameters, which can lead to unacceptable deviations of the internal and output parameters. It seems obvious that basic ideas of FP-approach can be used in the solution of the ensuring the survivability. In this case the description of possible anomalous effects and parametric changes is necessary. In such way the levels of allowable degradation of the system (change of quality indicators of functioning) can be specified. Ensuring survivability in general is also based on creating a certain (limit-exceeding) redundancy of the alleged anomalous deviations in the parameters and the optimal strategy for using this reserve can be established. In this case, some of reserve components should be used only when an abnormal situation occurs.

E. Understand safety is a key property of dynamical systems, which allow maintaining a safe condition at any stage of the life cycle. The safety theory is the science about foreseeing dangerous conditions (catastrophic, emergency, danger, etc.), threatening the destruction of systems or environment, and measures to prevent them. The safety theory cannot proceed from multiple phenomena, with dangerous consequences: from system destruction, sufficient to create a single emergency. It is important also that the basis for the safety management (prevention of dangerous situations) is the need for monitoring, assessing and forecasting of systems studied and used. Thus, an objective assessment of the safety of a system can be made, observing the process of changing its states. For this purpose it is necessary to build a region of the safe state (similar to the region of acceptability in the reliability described above). By highlighting all modes leading to breach of safety (e.g. crash in the system) such monitoring can be achieved. In order to prevent a breach in the safety it is needed to create a safety reserve, to predict the changes (decrease) in such reserve and to take decisions on timely reserve refreshments or termination of the system operation.

VI. Conclusions

In this paper we formulate a conceptual algorithmic approach which combines traditional theoretical and practical steps in assessing reliability of technical systems subject to gradual, time depending failures. This approach is called functional-parametric (FP) since it uses possible parameter variability in tolerances type of restrictions. Under FP approach reliability assessment if based on the creation and optimal use of reserves (margin) of admissible variations of the system and process parameters. Monitoring of the governing parameters, prediction of changes in the parameters and prevention of exits out of admissible range are essential components of the FP approach. Correction in parameter values, adjustments or replacements of worn out components are part of the system control. Reliability support successfully can be used in several enhanced forms of parametric synthesis tasks. The FP approach naturally arises with the numerous of technical research tools and algorithms in reliability and risk studies authors discussed earlier,

It is outlined that the basic ideas of the FP approach and the tools of its practical implementations can be applied in solving a wide range of other problems in risks, theory, safety, survivability and many other fields.

References

[1] Abramov O., Katueva Y. and Nazarov D. (2006). The Definition of Acceptability Region for Parametric Synthesis Problem. *Proceedings of the 6th Asian Control Conference (ASCC'2006)*: pp. 780-786.

[2] Abramov O. (2006). Reliability-Directed Computer-Aided Design System. *Reliability: Theory & Application*. No. 1. Pp. 35–40.

[3] Abramov O. (2010). Parallel Algorithms for Computing and Optimizing Reliability with Respect to Gradual Failures. *Automation and Remote Control.* Vol.71. N7. Pp. 1394-1402.

[4] Abramov O., Nazarov D. (2015). Region of Acceptability using Reliability-oriented Design. *Resend Development on Reliability, Maintenance and Safety/ WIT Transaction on Engineering Sciences.* Vol.108. Pp. 376-387.

[5] Abramov O. (2016). Choosing Optimal Values of Tuning Parameters for Technical Devises and Systems. *Automation and Remote Control.* Vol.77. N4. Pp. 594-603.

[6] Abramov O., Nazarov D. (2016). Condition-based maintenance by minimax criteria. *Applied Mathematics in Engineering and Reliability: Proceedings of the 1st International Conference on Applied Mathematics in Engineering and Reliability*, pp.91-94.

[7] Becker P., Jensen F. (1977). *Design of Systems and Circuits for Maximum Reliability or Maximum Production Yield*. New York: McGraw Buck Company.

[8] Dimitrov, B. (1998) Quality Evaluation Methods - A Review, *Economics, Quality, Control* (*EQC Journal and Newsletter for Quality and Reliability*), Vol. 13, No. 2, pp. 117 - 128.

[9] Gnedenko, B.V., Beliaev Y.K. and Soloviev, A.D. (1969) *Mathematical Methods of Reliability* New York, Wiley, 1969.

Probabilistic Dispersion Models In Large Water Areas. Statistics And Stochastic Computational Algorythms

O. Sorokovikova, D. Dzama, D. Asfandiyarov, D. Blagodatskikh

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The Nuclear Safety Institute (IBRAE)
olga_sorokov@mail.ru, dzama.dv@mail.ru,
dasfandiyarov@ibrae.ac.ru, blagodat@ibrae.ac.ru

Abstract

A Monte Carlo method for calculation of dispersion in large water areas with complex coastlines is presented. Having utilized a multi-year database of sea currents and of distributions of the mixed layer depth, a model of statistical ranking of water areas based on contamination levels is proposed. Techniques for parallelization of the Monte-Carlo method and statistical postprocessing of results have been realized. The method and the techniques have been validated against releases into a particular large water area.

Keywords: Monte Carlo method, ocean dispersion model, pseudorandom number generator, parallel calculations

I. Introduction

Now a significant number of objects related to the Cold War nuclear legacy are present in Russian coastal water areas [1-2]. Such objects are concentrated in the Arctic and Far-East regions. The inventory of these objects comprises coastal radioactive waste repositories (Andreeva Bay, Gremikha Bay, Sayda Bay in the Arctic region and Razboynik Bay in the Far-East region) as well as about 18 000 radiation hazardous objects resting on the seafloor of seas of the Arctic ocean including containers with solid radioactive waste and sunken nuclear submarines.

Therefore, the Arctic and the Far-East regions are considered to pose potential hazards. This is mainly due to nuclear submarines which are kept afloat near coastal storage facilities. Though, the probability of occurrence of pertinent emergency situations is very low a radiation safety analysis of possible consequences of emergency situations for the environment was conducted. It was demonstrated that under unfavorable conditions high concentrations of contamination can be maintained in plume [3].

According to the International Atomic Energy Agency (IAEA) recommendations state-of-the-art software packages are to be used in optimization of monitoring strategies near radiation hazardous objects through discerning dominant contamination pathways in the environment [4].

In the Nuclear Safety Institute, a software package allowing calculation of probabilistic characteristics of possible levels of contamination and tracing of key water areas for establishing monitoring sites has been developing. The exceedance of concentration in those water areas may be considered as a trigger for extending of the routine monitoring strategy.

The necessity of a probabilistic approach to modelling of consequences of radioactive releases into coastal waters is justified by the seasonal variability of ocean currents as well as uncertainty in the source parameters.

Ensemble approaches based on multy-year meteorological data have been widely used for modelling of dispersion of contaminants in the atmosphere and allow conservative estimates to be obtained for various localizations of the source. Finnish software package SILAM [5] can be considered as an example of such an approach.

As regards ocean currents, similar approaches are still at early stage of development due to the fact that for large water areas pertinent databases have been compiled relatively recently. Four-dimensional data in the ocean can only be obtained through reanalysis of the coupled ocean - atmosphere circulation based on data assimilation. Furthermore, information on ocean currents is sparse even in the vicinity of coastlines. Yet, in recent years the situation has changed for good due to ever increasing amount of corresponding data. Sophisticated ocean models allowing free access to reanalysis data have been developing by scientific groups in various countries.

Therefore, in order to tackle such problems, a real-time mathematical model and input data for a long period of observations with a high space- and time- resolution are needed. Furthermore, a probabilistic analysis is a computationally demanding task that cannot be performed without introducing an efficient parallelization method. Thus, an appropriate database of ocean currents for a particular water area, efficient usage of computing resources and tools for postprocessing are at the core of practical implementation of any probabilistic approach for estimation of contamination in large water areas.

The probabilistic model presented in this article is intended for performing analysis of ocean currents and for determining various contamination pathways in large water areas in the Arctic and Far-East regions of Russia. As a test case a hypothetical release into Razboynik Bay in the Far-East region has been simulated. The choice of that particular water area is not arbitrary due to the presence of one of the largest coastal radioactive waste repositories.

II. Mathematical formulation of the model

I. General Assumptions

In this article a quasi three-dimensional ocean dispersion model for calculation of surface contamination is presented. It is assumed that in the upper layer of the ocean contaminants drift with the current. Contamination is regarded either as a solution or fine-disperse aerosol affected by gravitational deposition.

The main assumption of the model is that a homogeneous distribution in the mixed layer (above the seasonal thermocline) is present. Thus, it is supposed that turbulent mixing is negligible below the seasonal thermocline whereas above it intense vertical turbulent mixing occurs due to wind forcing. Vertical currents into deeper layers of the ocean are attributed to large-scale horizontal divergence of surface currents.

The mathematical model employs the transport equation for Lagrangian particles and one which determines the increase in particles dimension. In addition the model utilizes a parametrization of the turbulent diffusion coefficient, takes into account large-scale downwelling and interaction of particles with a coastline.

II. Governing equations

The governing equation of the model is the convection-diffusion transport equation in the spherical coordinates:

$$\frac{\partial HC}{\partial t}+\frac{\partial HCU}{R\cos\theta\partial\phi}+\frac{\partial HCV}{R\partial\theta}=\frac{\partial}{R^2\cos\theta\partial\phi}\left(\mu\frac{\partial HC}{\cos\theta\partial\phi}\right)+\frac{\partial}{R^2\partial\theta}\left(\mu\frac{\partial HC}{\partial\theta}\right)+Q(t) \qquad (1)$$

where the following notation is used: C - concentration (Bq/l), ϕ and θ - longitude and latitude, R - Earth radius (m), H - the mixed layer depth (m), U and V - the zonal and meridional components of wind velocity (m/s), μ - the turbulent diffusion coefficint (m^2/s), $Q(t)$ - the time - dependent source rate (Bq·m^2/s). H, C, U, V, μ depend on ϕ, θ and t. The source can also occupy a finite volume.

As a computational algorithm of the proposed model, a Monte Carlo method utilizing Lagrangian particles is used. At any moment contamination is regarded as a set of Lagrangian particles. Each particle is characterized by the longitude and the latitude of the center of the particle, the horizontal dimension and the activity of the particle. The distribution of concentration within each particle is described by the two-dimensional normal distribution with the dispersion equal to the horizontal dimension of the particle. The vertical dimension of the particle depends on the mixed layer depth. The distribution of concentration in the Eulerian coordinates is calculated as the total contribution from the particles.

The equation (1) is solved applying the Lagrangian coordinates. Hence, the probability density function of Lagrangian particles is a solution for the equation if the motion of the particles is governed by the Ito stochastic differential equation:

$$d\vec{X} = \vec{U}dt + B \cdot d\vec{W} \tag{2}$$

where $d\vec{X}$ is the change in the position a particle over the time interval dt, $\vec{U}$ - the velocity of the particle, B - diagonal matrix with non-zero elements determining the intensity of turbulence, $\vec{W}$ - the Wiener process.

The advection and diffusion members of the equation (2) are numerically discretized as follows:

$$\begin{aligned}
\phi^{n+\frac{1}{2}} &= \phi^n + \frac{U\tau}{R\cos\theta^n} \\
\theta^{n+\frac{1}{2}} &= \theta^n + \frac{V\tau}{R} \\
\phi^{n+1} &= \phi^{n+\frac{1}{2}} + \frac{\xi_1\sqrt{2\mu\tau\beta}}{R\cos\theta^n} \\
\theta^{n+1} &= \theta^{n+\frac{1}{2}} + \frac{\xi_2\sqrt{2\mu\tau\beta}}{R}
\end{aligned} \tag{3}$$

where τ - the time integration step; β is a variable model parameter ranging from 0 to 1, σ - the horizontal dimension of the particle. When $\beta=1$ a classical Lagrangian stochastic model is realized, i.e. turbulent diffusion affects only the displacement of particles, whereas when $\beta=0$ turbulent diffusion has no influence on the positions of the centers of particles. In this model the coefficient β is equal to 0,9.

The increase of the horizontal dimension of particles is given by the following equation:

$$\left(\sigma^2\right)^{n+1} = \left(\sigma^2\right)^n + 2\left(1-\beta\right)\mu\tau \tag{4}$$

As parametrization schemes of the turbulent diffusion coefficient μ two models are employed – those of Ozmidov [6] and Smagorinsky [7]. In the Ozmidov model the turbulent diffusion coefficient is ascribed to the whole plume taking into account its dimensions. In the Smagorinsky model this coefficient is determined through calculation of the local value of the shear stress tensor thereby being different for various computational cells. At any moment the maximum of the two magnitudes is ascribed to every particle.

The distribution of concentration in the Eulerian coordinates is integrated as contribution from the clouds of particles. The horizontal dimension of the cloud of a particle is assumed to be finite and is equal to three radii of the particle (“the three-sigma rule”).

The application of Gauss particles with finite dimensions, as opposed to a classical Lagrangian stochastic model, significantly loosens restriction imposed on the total number of particles in order to obtain "smoothed" solutions. The choice of the numerical value of the parameter β be equal to 0,9 is justified in the article [8].

The integration time step is limited by the maximum value of the flow velocity since the change in the displacement a particle must not exceed the dimensions of an Eulerian cell: $U_{\max}\tau \le \frac{L}{2}$, where L is the horizontal dimension of the Eulerian cell.

The effect of large-scale downwelling at the lower border of the mixed layer is introduced in the model. The corresponding vertical current W conveys contamination to the lower cold layers. The decrease of the total activity in a fluid column of the height H and of the surface S is given by the following equation:

$$\frac{dA}{dt} = -\rho WS \qquad (5)$$

where $\rho = A/\left(S \cdot H\right)$ is the specific volume activity.

Thus, over the period of the duration τ concentration is diminished by a value given by $\Delta A = A_0\left(1 - e^{-\frac{W\tau}{H}}\right)$.

III. Interaction with the coastline

The way the distribution of concentration is calculated in the vicinity of the coastline is different as opposed to the other parts of computational domain. Hence, all cells are split into computational cells which are occupied by fluid and ghost cells where no motion of fluid occur. Thus, the coastline is represented by a number of computational cells that adjacent to ghost cells. The horizontal dimension of a particle in the vicinity of the coastline is adjusted so as to guarantee intersection of the particle only with computational cells. At a distance from the coastline, which is equal to 3σ , the particle dimension is strictly determined by turbulent diffusion. If 3σ is larger than a distance from the coastline, then 3σ is limited by that distance. Thus, the increase of concentration within a particle approaching to the coastline is modelled as a consequence of the law of conservation of mass.

In Fig. 1 results obtained using a classical Lagrangian approach and a puff Lagrangian stochastic approach are presented. In the latter case one need to filter out artificial values of concentration occurring in the ghost cells, while in the former one particles do not intercept ghost cells due to the fact that their radii are equal to zero. In order to obtain a smooth distribution of concentration in the case of a classical Lagrangian approach a significant increase in number of particles is required. The usage of Gauss particles allows for a smooth distribution of concentration to be obtained while putting a reasonable upper limit on the total number of particles.

Trajectories of Lagrangian particles are calculated regardless of computational mesh. Hence, the motion of particles in the vicinity of the coastline has to be corrected in order to avoid intersection of the center of a particle and the coastline. If the displacement vector of a particle occurring over the time step intersects the coastline, then the method of images is used. Thus, the part of the segment of the displacement vector lying outside the coastline is reflected. Having made a few such iterations, one can force the center of the particle to be in a computational cell. Then, the minimal distance from the center of the particle to the coastline is determined with 3σ of the particle being equal to that distance.

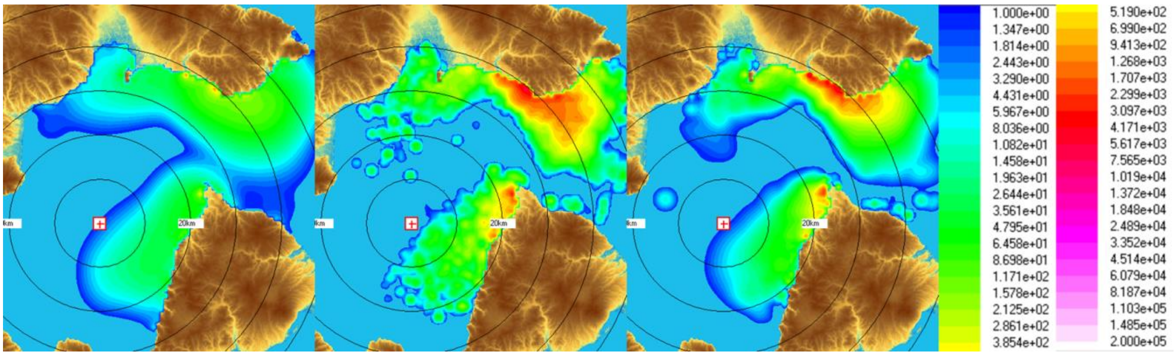

Figure 1: *Illustration of different approaches to modelling of interaction between the plume of contamination and the coastline*

IV. Input data

In order to perform a calculation, the following parameters are required:

- a two-dimensional array where each element stands either for land or water;
- a relief map for postprocessing;
- a multy-year database of the zonal and meridional components of velocity in the upper layer in the Arctic and the far-East-regions;
- the mixed layer depth distribution in a given region.

The two-dimensional array used to represent the coastline in the model is interpolated from Earth relief data which is quite accessible in the present [9-12]. Databases of ocean currents are generated as a result of simulations of various coupled ocean-atmosphere general circulation models. In the last years, such data have been provided for free access by the largest research centers [13-16]. In this article only free access databases were used.

V. Parallelization method

The accuracy of modelling strongly depends on the total number of particles. In order to obtain statistically significant results, the number of particles is requiered to be large enough. Hence, the efficiency of a method is determined by the parallelization technique applied which in terms of hardware is the way how a whole task is divided into processes. Trajectories of the particles being independent, the solution of the problem can be obtained through performing N independent calculations. The whole release is divided equally between N processes which are carried out independently from each other. Thus, summing results obtained in each process one can obtain the solution of the whole task.

In order to avoid occurrence of identical trajectories in different processes, a unique statistic is ascribed to each process via utilizing different subsequences of a whole sequence of pseudorandom numbers with a long period. In this article all results are obtained using the pseudorandom generator (PNG) [17].

That PNG constitutes a multiplicative congruent sequence with a very long period:

$$\begin{aligned} &u_1 = 0, \\ &u_{k+1} = u_k A \bmod M, \\ &a_k = u_k / M. \end{aligned} \tag{6}$$

where the modulus $M = 2^{128}$, the multiplier $A = 5^{100109} \bmod M$ and the period $P = 2^{126}$. The values thus obtained conform to the uniform distribution, which in turn can be transformed into values described by the standard normal distribution using the Box–Muller transform. A whole sequence of the length P generated by the PNG can be arbitrary divided into a number of subsequences of equal lengths nested into the whole sequence. In order to discern such a subsequence, one needs to know the first element of each subsequence, which is determined for the m-th subsequence of the length n as follows:

$$\begin{aligned} &\hat{u}_0 = 1, \\ &\hat{u}_{m+1} = \hat{u}_m A(n) \ \text{mod} \ M, \\ &\hat{a}_m = \hat{u}_m / M, \\ &A(n) = A^n \,\text{mod}\, M. \end{aligned} \tag{7}$$

Therefore, one does not need to recalculate the values generated before given subsequences in order to skip those subsequences, which is very important in parallel calculations.

Two levels of nesting are used: the first one is a subsequence applied for calculation of a given scenario of emergency situation, the second one constitutes a set of subsequences for running different processes.

The peculiarity of the PNG used in this article lies in its program realization due to the fact that it operates with very large numbers which cannot be processed without overflow using even 64-bit computing. In the model a version of the PNG [18] written in the C language is used.

III. Simulation results

As an example of hypothetical emergency situation various scenarios of releases into the ocean near Razboynik Bay (one the largest repositories of radioactive waste) were considered. The source of releases is located at 132,36°E, 42,89°N. To determine a typical pattern of propagation of contamination in that water area a number of calculation with varying parameters of the release were conducted.

In the first calculation a release of 90Sr and 137Cs was simulated with the total activity of the source as follows: 90Sr -$1{,}87 \cdot 10^{15}$ Bq, 137Cs – $6{,}22 \cdot 10^{16}$ Bq. The source rate was kept constant during the calculation which covered a period from 1 January 2001 to 31 December 2004.

In the figures 2 - 4 instant fields of concentration (Bq/l) are presented for some dates within the aforementioned period. The legend used in these figures coincides with that applied for the figure 5.

Knowing a series of values of concentration in every point of the water area obtained for different dates, one can determine a field of statistical characteristics. In figure 5 maximum levels of concentration with a 95% confidence interval are plotted.

This probabilistic estimate is most applicable for the case of a long-term release in the environment. In case of emergency situation when the duration of the release is much less than the characteristic time scale of propagation of contamination, ensemble calculations are used. It is assumed that the release might occur at an arbitrary moment within a given time interval. For every such a moment corresponding simulation of propagation of contamination with fixed parameters of the source is performed. Thus, the empirical distribution function in a given point of the water area is obtained. Given a value of the confidence interval, seldom peaks of concentration can be cut off.

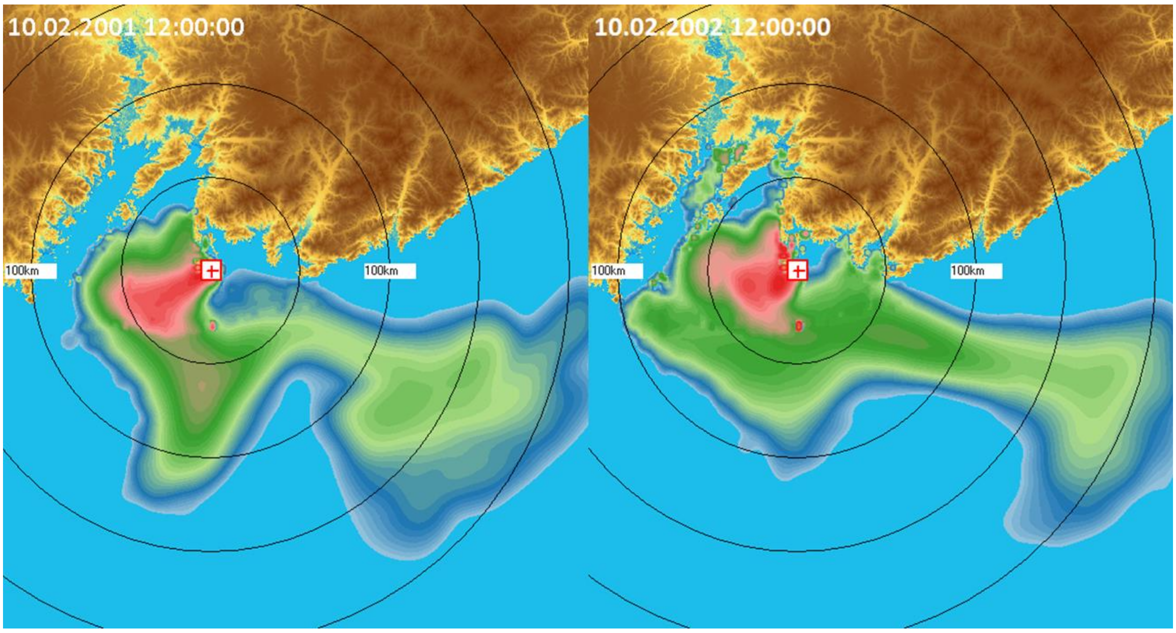

Figure 2: *Instant fields of concentration on 10 February 2001 (left) and 2002 (right)*

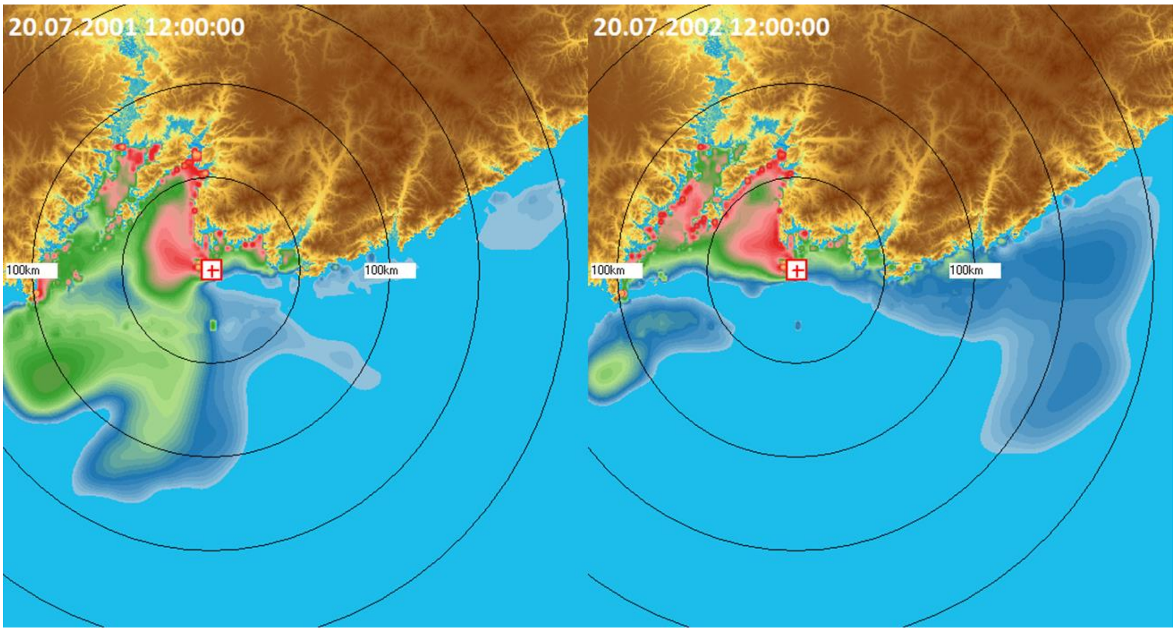

Figure 3 *Instant fields of concentration on 20 July 2001 (left) and 2002 (right)*

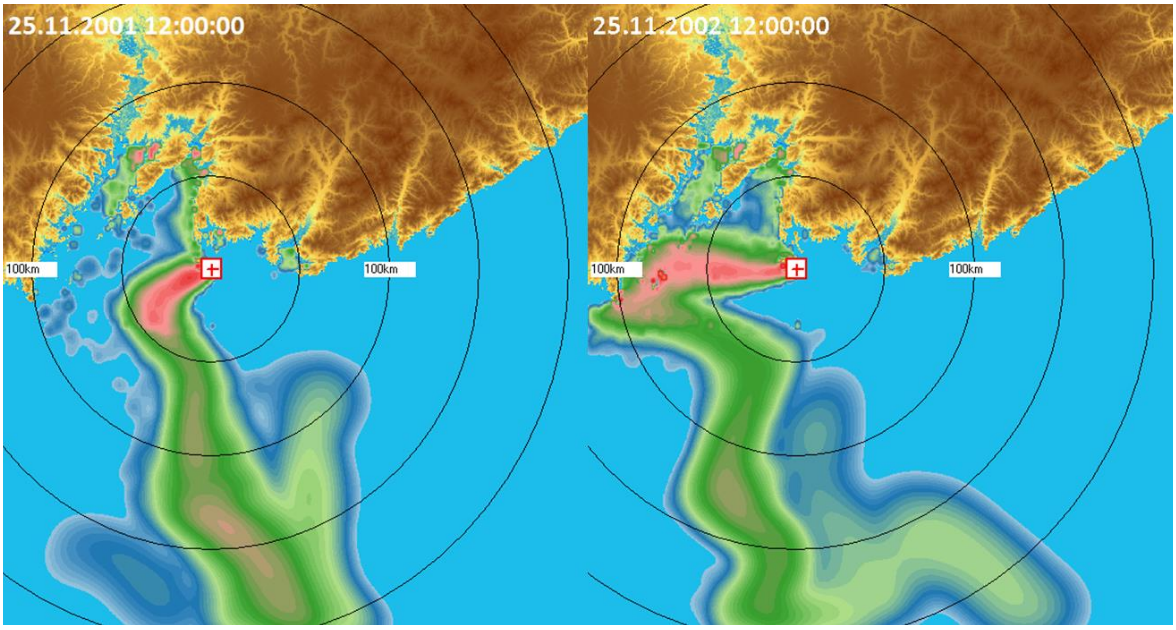

Figure 4: *Instant fields of concentration on 25 November 2001 (left) and 2002 (right)*

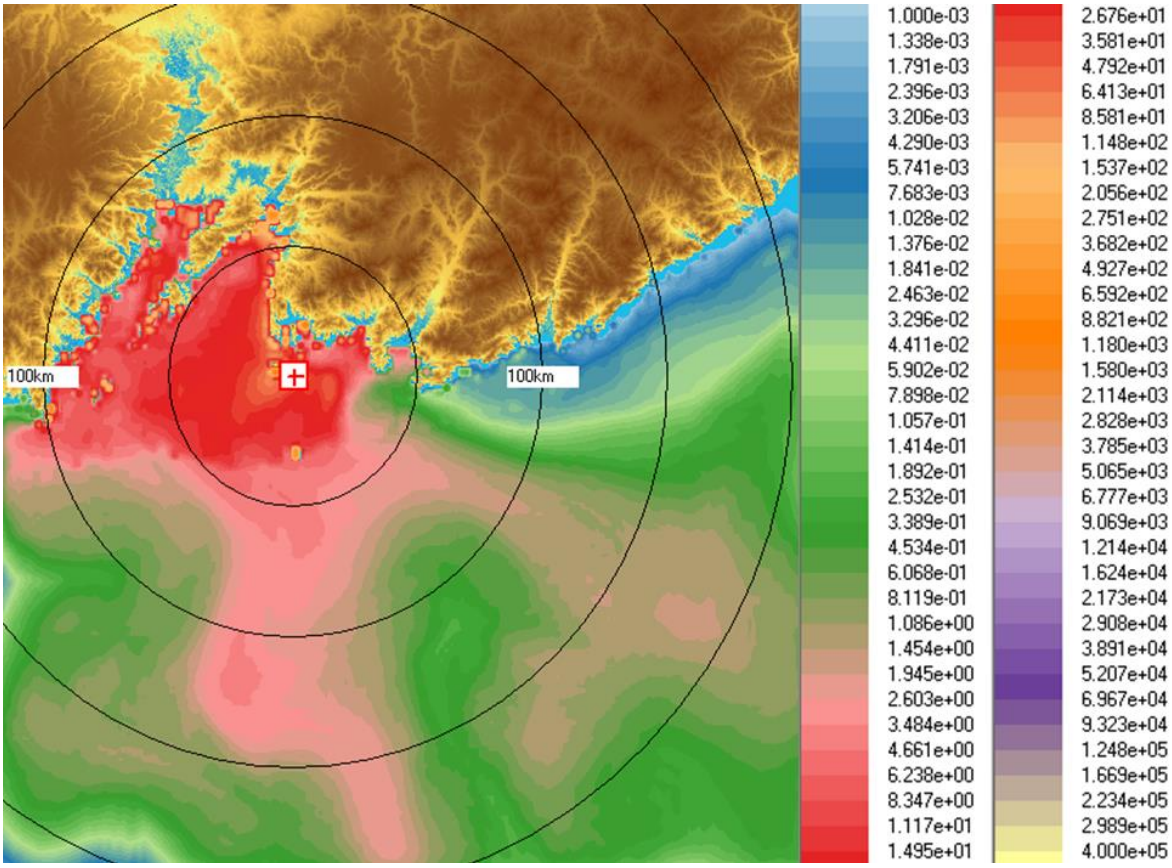

Figure 5: *Maximum levels of instant concentration with a 95% confidence interval*

The aforementioned approach was tested against the releases in the same water area (Razboynik bay) with the same localization of the source whereas the discharge was supposed to occur instantly. The date of the release varied from 1 to 31 January 2008. Each simulation covered a period of three months so as to allow contamination to be removed from the calculation domain.

A series of time-integrate concentration in every point of the water area is obtained. It represents the impact of the release in an arbitrary point of the water area. In the same way as described before a probabilistic field with a given confidence interval is determined. In the figure 6 maximum levels of time-integrated concentration (Bq*s/l) in every point of the water area with a 95% confidence interval are plotted.

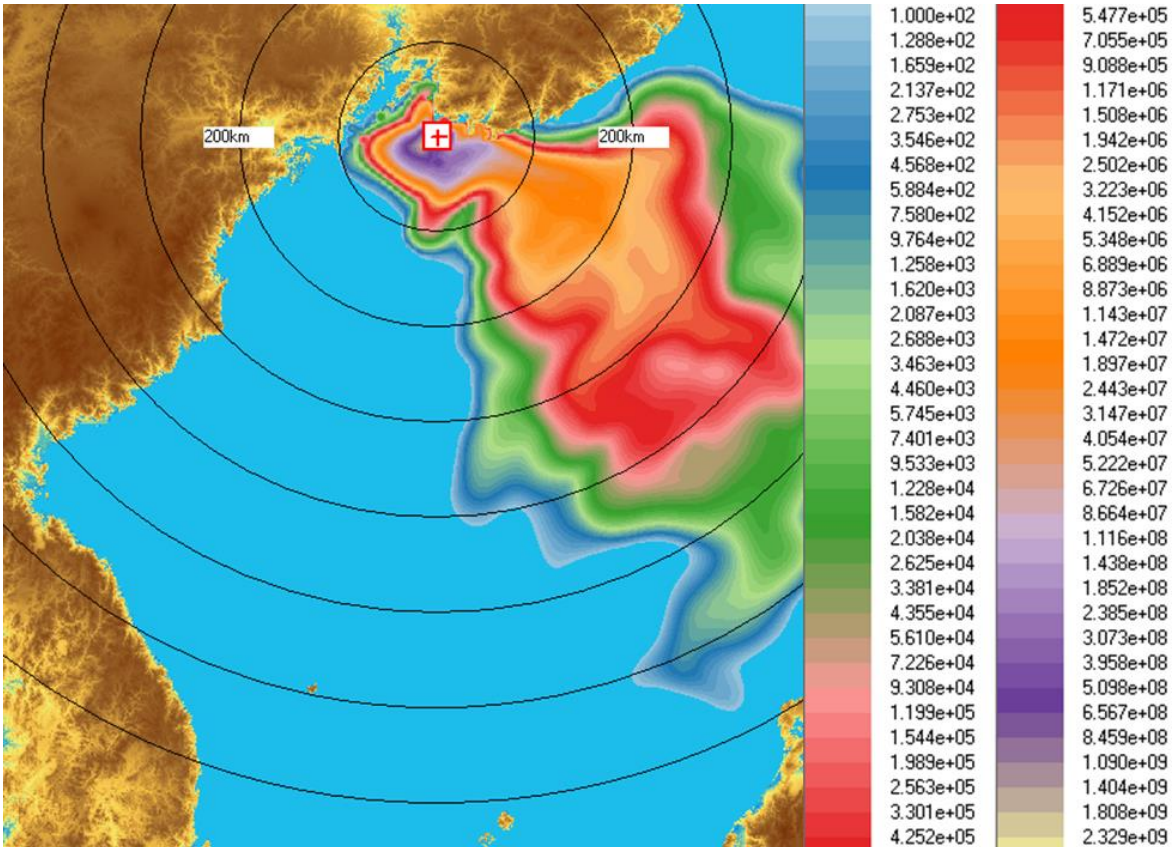

Figure 6: *Maximum levels of time-integrated concentration with a 95% confidence interval*

IV. Conclusion

In this article a Lagrangian stochastic model for calculation of ocean dispersion and probabilistic approaches to estimations of simulation results are presented. It was demonstrated in the case of a hypothetical emergency situation at Razboynik Bay that stable distribution patterns of contamination may occur in a large water area. Therefore, it is confirmed that an analysis utilizing the methods described in this article can be used for optimization of sampling strategy near coastal nuclear legacy repositories under assumption of the seasonal variability of ocean currents.

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References

[1] Problems of the nuclear heritage and ways to solve them. Development of the radioactive waste management system in Russia. – Ed. by Bolshov L.A., Laverov N.P., Linge I.I. – 2013. – 392 p. – V 2.

[2] The Cold War's legacy at the bottom of the Arctic. Radioecological, technical and economic problems of radiation rehabilitation of the seas / Sarkisov A.A., Sivintsev V.L., Vysotsky V.L., Nikitin V.S.; Nuclear Safety Institute (IBRAE) RAS. – Moscow, 2015. – 699 p.: ill. – ISBN 978-5-9907220-0-2 (bound).

[3] Semenov V.N., Sorokovikov A.V., Sorokovikova O.S. Conservative models for assessing the pollution of surface waters in the event of extreme hypothetical accidents and utilization of nuclear submarines in the Kamchatka Peninsula. Proceedings of IBRAE RAS / Ed. by L.A. Bolshov; Nuclear Safety Institute (IBRAE) RAS. – Moscow: Nauka, 2007 – Issue 9: Modeling of Radionuclide Transport in the Environment [in Russian] / Ed. by R.V. Arutyunyan. – 2008. – 229 p.: ill. – ISBN 978-5-02-036954-2 (bound).

[4] Safety report series N19. Generic models for use in assessing the impact of discharges of radioactive substances to the environment. IAEA, Vienna, 2001.

[5] Sofiev M., Siljamo, P., Valkama, I., Ilvonen, M., Kukkonen, J., A dispersion modelling system SILAM and its evaluation against ETEX data. Atmosph. Environ., Vol. 40, pp. 674-685, 2006, DOI:10.1016/j.atmosenv.2005.09.069.

[6] Monin A.S., Ozmidov R.V., Turbulence in the Ocean, Environmental Fluid Mechanics Vol. 3, 1985, Springer Science & Business Media, 248 p.

[7] Smagorinsky, J., General Circulation Experiments with the Primitive Equations. Monthly Weather Review, 1963. 91(3): p. 99-164.

[8] Belikov V.V., Goloviznin V.M., Katyshkov Yu.V., Semenov V.N., Starodubtseva L.P., Sorokovikova O.S., Fokine A.L. NOSTRADAMUS – computer system for predicting the radiation situation. Verification of the model of contamination transport in the atmosphere. Proceedings of IBRAE RAS / Ed. by L.A. Bolshov; Nuclear Safety Institute (IBRAE) RAS. – Moscow: Nauka, 2007 – Issue 9: Modeling of Radionuclide Transport in the Environment [in Russian] / Ed. by R.V. Arutyunyan. – 2008. – 229 p.: ill. – ISBN 978-5-02-036954-2 (bound).

[9] NOAA. ETOPO1 Global Relief Model [Web resource] https://www.ngdc.noaa.gov/mgg/global/global.html.

[10] Walter H. F. Smith and David T. Sandwell. Global Sea Floor Topography from Satellite Altimetry and Ship Depth Soundings. Science 26 Sep 1997. − Vol. 277, − Issue 5334, − P. 1956-1962. − DOI: 10.1126/science.277.5334.1956.

[11] Farr T. G. The Shuttle Radar Topography Mission / T.G. Farr [et al.]// Rev. Geophys. −2007. − Vol. 45. − RG2004, doi:10.1029/2005RG000183.

[12] ASTER global digital elevation map announcement [Web resource] https://asterweb.jpl.nasa.gov/gdem.asp.

[13] Geophysical Fluid Dynamics Laboratory [Web resource] https://www.gfdl.noaa.gov/ocean-data-assimilation/.

[14] Climate Data Guide. ORAS4: ECMWF OCEAN REANALYSIS AND DERIVED OCEAN HEAT CONTENT. [Web resource] https://climatedataguide.ucar.edu/climate-data/oras4-ecmwf-ocean-reanalysis-and-derived-ocean-heat-content.

[15] APDRC of the IPRC. The FRA-JCOPE2 17 years NW Pacific Ocean reanalysis data. [Web

resource] http://apdrc.soest.hawaii.edu/datadoc/fra-jcope2.php.

[16] APDRC of the IPRC. NRL NLOM 1/32° Nowcast. [Web resource] http://apdrc.soest.hawaii.edu/datadoc/nlom_30day.php.

[17] Mikhailov G.A., Marchenko M.A. Parallel realization of statistical simulation and random number generators. // Rus. J. Num. Anal. Math. Model- 2002. - Vol. 17. - No. 1. pp.113-124.

[18] Institute of Computational Mathematics and Mathematical Geophysics RAS, Laboratory of Monte Carlo Methods. [Web resource] http://osmf.sscc.ru/~mam/index_rus.html

On Solution of Renewal Equation in the Weibull-Gnedenko Model

Vladimir Rusev, Alexander Skorikov

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Gubkin Russian State University of Oil and Gas (National Research University)
mailto: vnrusev@yandex.ru, skorikov.a@gubkin.ruvnrusev@yandex.ru,
skorikov.a@gubkin.ru

Abstract

Renewal density of restorable systems and their components which depends on statistical estimates based on real operational data is studied. It is assumed that objects' entire life cycle is described by the Weibull-Gnedenko distribution. Analytical and discrete approaches for the solution to the renewal equation are proposed. New calculation schemes of the renewal density of restorable systems and their components are presented. Equivalence of suggested approaches is illustrated by numerical examples.

Keywords: the Weibull-Gnedenko distribution, reliability theory, renewal density (intensity), numerical methods, collocation knots, moments generating function

1 Introduction

All mechanisms, engineering constructions, operational systems are subjected to the processes of aging, degradation or failures in work. The renewal of normal operation mode possesses doubtless economical and sometimes vital importance. Construction of suitable for applications mathematical models of the renewal processes is thus far an actual challenge in the reliability theory, as existing models involve cumbersome calculations, while analytical solutions are not available in general case. This paper is devoted to the development of analytical and simple discrete schemes for the solution of the renewal density equation. Renewal functions have wide variety of applications in warranty analysis, inventory theory, supplies planning [1]. Examples of processing of real statistical data on refusals of technologically active elements of gas supply systems are considered in [2].

The scheme of a simple renewal process is the following. Let component (or system) failures occur at time moments t_1, t_2, ... , t_n, ... and it is assumed that replacement time is negligible relative to the operational time. Then t_n represents the operational time until the n-th failure takes place. And it is supposed the time intervals between failures $T_n = t_n - t_{n-1}$ are independent and identically distributed. In this case T_n is the random life time of the n-th item with cumulative distribution function $F(t)$ and probability density function (PDF) $f(t)$, and $N(t)$ is the number of renewals in the time interval $(0, t)$. The renewal function $H(t)$ is the expected value of renewals in that interval $H(t) = E(N(t))$. The renewal density (intensity) by definition is given by the equality $h(t) = H'(t)$. The fundamental renewal density equation has the form (e.g. [3]):

$$h(t) = f(t) + \int_0^t h(\tau) f(t - \tau)\, d\tau. \qquad (1)$$

Solution of this equation does not have explicit form, except for some cases when the renewal process is driven by the exponential and the Erlang distributions. In this paper new analytical and discrete methods of calculating of $h(t)$ are presented for the Weibull-Gnedenko probability density function which depends on two parameters: α, named "scale" parameter and β, called "shape" parameter:

$$f(x) = \begin{cases} \alpha^{\beta}\beta x^{\beta-1}e^{-(\alpha x)^{\beta}}, & x > 0 \\ 0, & x \leq 0. \end{cases} \quad (2)$$

This distribution was chosen for the study, because it allows to capture all life cycle of systems investigated in the reliability theory, that makes it one of key distributions. By results of many studies the typical curve of the hazard rate ([4]) usually is U-shaped, thus, it contains three main periods of life cycle: initial burn-in, normal operation and degradation. All these periods of the system functioning can be modelled by the Weibull-Gnedenko distribution with different shape parameter ([4], [5]). In particular, the first period corresponds to the Weibull-Gnedenko distribution with parameter $\beta \in (0; 1)$, the second period — with parameter $\beta \approx 1$ and the third period — with parameter $\beta > 2$. It should be noticed that upon transition from the second to the third stage the value of shape parameter jumps from 1 to the value more than 2. And this property thus far remains the opened question for discussion.

For large t it is well-known the asymptotic result for the renewal density function

$$h(t) \sim \frac{1}{\mu}, (t \to \infty),$$

where $\mu = E(T_n)$. But the values of $h(t)$ can oscillate (see Fig. 3) about the asymptotic value, thus it is important to have opportunity to calculate values of $h(t)$ more accurately.

W.L. Smith and M.R. Leadbetter [3] developed the method for computation of the renewal function for the Weibull-Gnedenko distribution by using power series expansion of t^{β}, where β is the shape parameter of the Weibull-Gnedenko PDF. However, for $\beta > 1$, the numerical computation of this series is limited to the small range of t: $0 < t < 2{,}5$. A. G. Constantine and N.I. Robinson [6] presented estimation method of $H(t)$ (and automatically $h(t)$) by residue calculations of the Laplace transform of the renewal integral equation to form uniformly convergent series of damped exponential terms. There are also many other approximations of the renewal function explored for the Normal, Gamma, Uniform underlying lifetime distributions developed by L. Cui and M. Xie [7], E. Smeltink and R. Dekker [8], S. Maghsoodloo and D. Helvaci [9].

2 Methods

2.1 The moment problem

In the present paper the analytical solution of the renewal density equation is closely connected with the moment problem or the problem of unique determination of a distribution of a nonnegative random variable by its moments. Consider this problem for basic distributions used in the reliability theory. As for the background the problem of unique determination of a distribution by the sequence of its moments was first investigated by P. L. Chebyshev back in 1874 in connection with research on limit theorems of probability theory.

It can be shown that such distributions as exponential, normal, truncated normal, gamma distribution and the Weibull-Gnedenko distribution (with the shape parameter $\beta \geq \frac{1}{2}$) are uniquely determined by their moments, and the log-normal distribution, Student's t-distribution and the Pareto distribution can not be determined uniquely. The solution of this problem can be verified by the necessary and sufficient criterion of Krein M. (1944) (or the Krein condition [10]). The distribution with the PDF $f(x)$ is determined uniquely if

$$\int_0^{+\infty} \frac{\ln f(x^2)}{1+x^2}\, dx = \infty.$$

2.2 Analytical solution of the integral renewal equation

To solve equation (1), we use the method of moments generating function [11] under the assumption of the two-parameter Weibull-Gnedenko distribution. The Laplace transform (or moments generating function) for the given distribution has the form:

$$\tilde{f}(s) = \int_0^{+\infty} e^{-st} f(t)\,dt = \int_0^{+\infty} \sum_{n=0}^{+\infty} \frac{(-st)^n}{n!} f(t)\,dt = \sum_{n=0}^{+\infty} (-1)^n \frac{s^n}{n!} \nu_n\,,$$

where ν_n is the n-th initial moment of a random variable ξ with PDF $f(t)$:

$$\nu_n = \int_0^{+\infty} t^n f(t)\,dt,$$

which for the Weibull-Gnedenko distribution has the form

$$\nu_n = \frac{1}{\alpha^n} \cdot \Gamma\left(1 + \frac{n}{\beta}\right)$$

and $\Gamma(s) = \int_0^{+\infty} x^{s-1} e^{-x} dx$ is the Euler gamma function, $s \in \mathbb{C}$. It should be noted that this series is absolutely convergent only for $\beta > 1$. Equation (1) in the Laplace transform domain has the form

$$\tilde{h}(s) = \tilde{f}(s) + \tilde{h}(s)\tilde{f}(s).$$

Consequently

$$\tilde{h}(s) = \frac{\tilde{f}(s)}{1-\tilde{f}(s)} = \frac{\sum_{n=0}^{+\infty} \frac{(-1)^n}{n!} \cdot \Gamma\left(1+\frac{n}{\beta}\right) \cdot \left(\frac{s}{\alpha}\right)^n}{-\sum_{n=1}^{+\infty} \frac{(-1)^n}{n!} \cdot \Gamma\left(1+\frac{n}{\beta}\right) \cdot \left(\frac{s}{\alpha}\right)^n} = \frac{\sum_{n=0}^{+\infty} (-1)^n \frac{\nu_n}{n!} s^n}{\sum_{n=1}^{+\infty} (-1)^{n+1} \frac{\nu_n}{n!} s^n}.$$

Applying the well-known in calculus technique of dividing infinite series, one can obtain the following expansion:

$$h(t) = \sum_{k=0}^{+\infty} \frac{c_k}{\nu_1^{k+1}} \cdot F^{(k)}(t), \tag{3}$$

where $F^{(0)}(t) = F(t)$ is the cumulative distribution function and $F^{(k)}(t)$ designates the k-th derivative of $F(t)$. The coefficients of the expansion have the following form:

$$c_0 = -1, \qquad c_1 = -m_0,$$

$$c_k = \begin{vmatrix} 1 & 0 & \dots & 0 & -m_0 \\ -m_0 & 1 & \dots & 0 & m_1 \\ m_1 & -m_0 & \dots & 0 & -m_2 \\ -m_2 & m_1 & \dots & 0 & m_3 \\ \dots & \dots & \dots & 1 & \dots \\ (-1)^{k+1} m_{k-2} & (-1)^k m_{k-3} & \dots & -m_0 & (-1)^k m_{k-1} \end{vmatrix}, \quad k = 2,\ 3, \dots$$

where

$$m_k = \frac{{\nu_1}^k \nu_{k+2}}{(k+2)!}, \quad k = 0,\ 1,\ 2, \dots$$

It was proved that

$$\sum_{k=0}^{+\infty} c_k = \frac{1}{-1+\sum_{k=0}^{+\infty} (-1)^k m_k} < +\infty,$$

consequently

$$\lim_{k\to\infty} c_k = 0.$$

Thus, the solution (3) of equation (1) is obtained in terms of the probability moments of the initial distribution of a nonnegative random variable. Moreover, one can notice that the expansion (3) is true not only for the Weibull-Gnedenko distribution (2) under some conditions, not discussed in the present paper.

A good estimate of exact solution is found (Fig. 2) by taking only 7 leading terms of the expansion (3) unlike the series of damped exponential terms in [6] when it is necessary to compute 500 or more coefficients of the given series [4].

The offered solution (3) is applicable only for the values of $\beta > 1$, characteristic for the degradation processes [5]. The start of system operation corresponds to the case of $0 < \beta < 1$ and its description is sometimes actual. Numerical methods can work with any values of β. So, we proceed to the description of discrete methods.

3.3 The discrete methods

A numerical method which generates a cubic spline approximation of the renewal function by the Galerkin technique for solving the renewal equation was proposed by Z.S. Deligonul and S. Bilgen [12]. The discretizing time method has been used by M.Xie [13] to approximate the renewal equation. Numerical algorithm in papers of M.Xie was based on the definition of the Riemann - Stieltjes integral (RS-method). T.K. Boehme, W.Preuss, V. van der Wall also used the similar method [14]. M. Tortorella [15] presented a paper describing analysis of the method based on quadrature schemes for Stieltjes integrals. Some numerical procedures, when the time scale is discrete, can be found in the books on reliability theory by E.A. Elsayed [1], A. K. S. Jardine, A. H. C. Tsang [16].

In this paper the function $h(t)$ is approximated by step functions or linear functions. The accuracy of this discretization was checked in three different ways. It is obvious that approximation error can be diminished by increasing of the number of collocation knots. All calculations presented below were done by using Wolfram Mathematica software including calculation of integrals (5), (8) without algorithm of numerical quadrature schemes.

The first step in the discretization scheme is the division of the specified time interval [0, t] into n equal-length subintervals by points (collocation knots)

$$t_0 = 0,\ t_1 = t_0 + \Delta, \dots ,\ t_n = t_0 + n\,\Delta,$$

where $\Delta = t/n$ is the length of each subinterval.

The method of right knots. Approximate solution $u_n(t)$ for function $h(t)$ can be found in the form of a linear combination $u_n(t) = \sum_{k=1}^{n} u_k I_k(t)$, where $u_k = h(t_k)$ and the so-called coordinate functions $I_k(t)$ which are equal to zero outside the interval $(t_{k-1},\ t_k]$ and $I_k(t) = 1$, $t_{k-1} < t \le t_k$. Thus, the approximate solution is determined from the conditions in right collocation knots. Similar approximation was proposed by A. Brezavscek in [17].

The midpoint method. Let the value of the approximate solution in the k-th segment be the average value (Fig. 1)

$$\tilde{u}_k = \frac{u_{k-1}+u_k}{2},\quad u_k = h(t_k).$$

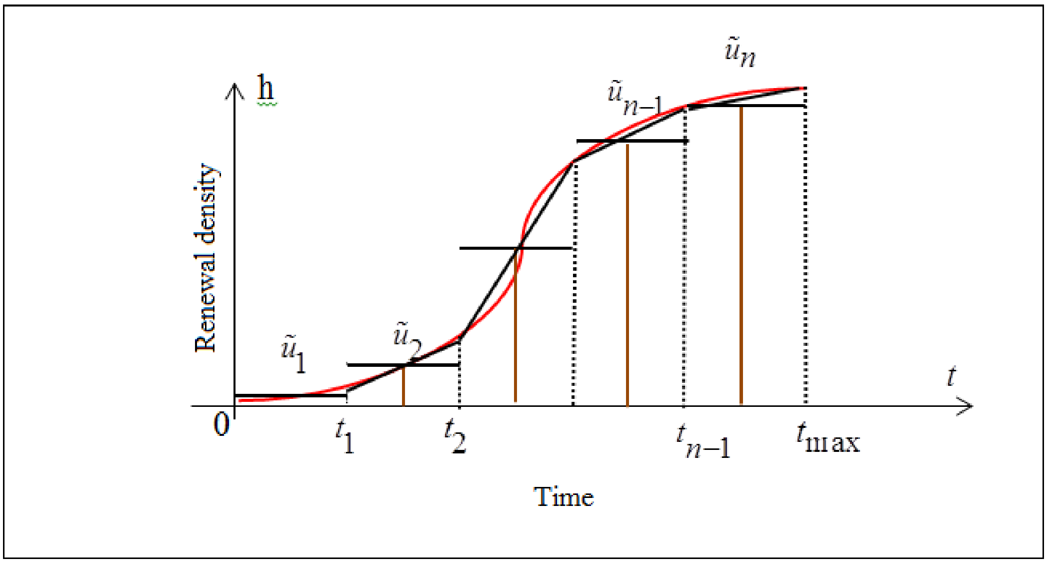

Figure 1: The discrete methods

The approximate solution, respectively, has the form $u_n(t) = \sum_{k=1}^{n} \tilde{u}_k I_k(t)$.

The Line Spline Finite Element solution. The function $h(t)$ in $[t_{k-1},\ t_k]$ can be approximated by the Lagrange polynomials as follows

$$\tilde{\tilde{u}}_k(t) = u_{k-1}\frac{t_k-t}{t_k-t_{k-1}} + u_k\frac{t-t_{k-1}}{t_k-t_{k-1}},$$

where $u_k = \ h(t_k)$.

Let us consider calculation schemes for the approximate solution of the renewal equation (1). The first method solution is defined by the following recurrent formulas:

$$\begin{cases} u_0 = f_0, \\ u_1 = f_1/(1-F_1), \\ u_2 = (f_2 + u_1F_2)/(1-F_1), \\ \cdots \\ u_n = (f_n + \sum_{j=1}^{n-1} u_{n-j}\, F_{j+1})/(1-F_1). \end{cases} \tag{4}$$

where

$$f_0 = f(0), \cdots, f_n = f(t_n),$$

$$F_1 = \int_0^{t_1} f(r)dr, \cdots, \quad F_n = \int_{t_{n-1}}^{t_n} f(r)dr. \tag{5}$$

The second method gives the next formulas:

$$\begin{cases} u_0 = f_0, \\ u_1 = \left(f_1 + \frac{u_0}{2}F_1\right)/\left(1-\frac{F_1}{2}\right), \\ u_2 = \left(f_2 + \frac{u_1}{2}F_2 + \frac{u_0}{2}F_2 + \frac{u_1}{2}F_1\right)/\left(1-\frac{F_1}{2}\right), \\ \cdots \\ u_n = \left(f_n + \frac{u_0}{2}F_n + \sum_{j=1}^{n-1}\frac{u_{n-j}}{2}\left(F_{j+1}+F_j\right)\right)/\left(1-\frac{F_1}{2}\right), \end{cases} \tag{6}$$

where f_k and $F_k, k = 0,1,\ldots,n$ are defined by formulas (5)

The recurrent formulas for approximate solutions obtained by linear splines have the form:

$$\begin{cases} u_0 = f_0, \\ u_1 = (f_1 - u_0G_{10})/(1-G_{11}), \\ u_2 = (f_2 - u_0G_{21} + u_1(G_{22}-G_{10}))/(1-G_{11}) \\ \cdots \\ u_n = (f_n - u_0G_{n\,n-1} + \sum_{j=1}^{n-1} u_{n-j}(G_{j+1j+1} - G_{j\,j-1}))/(1-G_{11}), \end{cases} \tag{7}$$

where

$$G_{kj} = \int_{t_{k-1}}^{t_k} f(r)l_j(r)dr, \tag{8}$$

$$l_k(t) = \frac{t_k-t}{t_k-t_{k-1}} = \frac{t_k-t}{\Delta}.$$

It should be noted that the first and the third methods were considered earlier [18]. And large computing capacities were required for the solution of the corresponding linear systems. Formulas (4), (7) introduced in the present paper dramatically diminished the computational complexity of the algorithms. So, mentioned algorithms are important if the calculation involves large number of knots.

The second method of discretization, that is, the midpoints method (6) gives the best agreement with analytical solution (3). The third method is time consuming for large values of $n >$ 40.

The discrete methods presented here can be used where the underlying lifetime distribution has the PDF with a singularity at the origin, such as the Weibull - Gnedenko distribution (2) with shape parameter less than unity. It is applied in the modelling of gas supply systems [19].

Fig. 2 illustrates the results of numerical computations of the renewal density performed according to all four methods of this paper. The solutions of equation (1) are presented for the Weibull-Gnedenko distribution with $\alpha = 1$, $\beta = 2$ which coincides the Rayleigh distribution. Red curve for the analytical solution was plotted using seven terms of the expansion (3). There were used twenty collocation knots ($n = 20$) in calculations by the discrete methods. It is worth noticing here that presented methods allow to carry out accurate calculation of the renewal function oscillations unlike the asymptotic formula mentioned above. The corresponding asymptotic value is shown by the straight green line. The curve of the PDF for the Rayleigh distribution is shown with green dotted line.

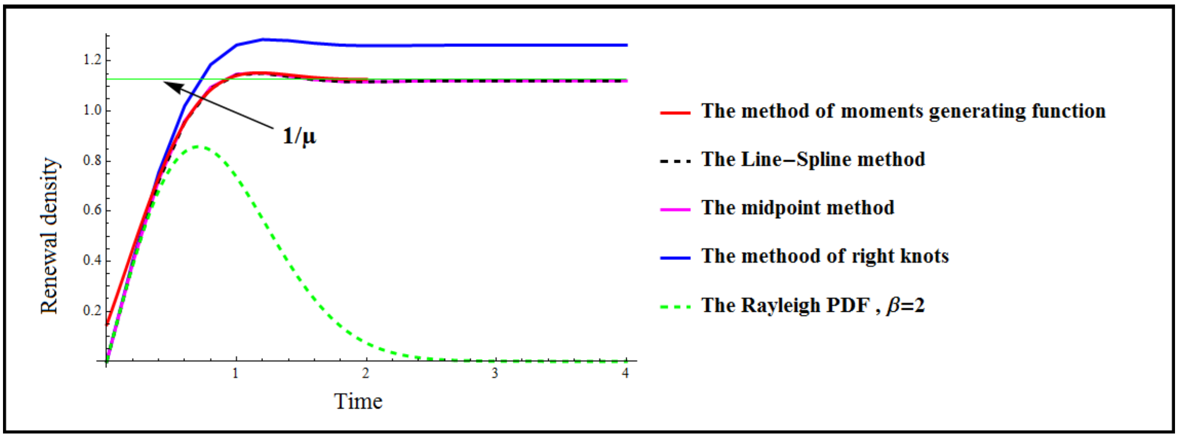

Figure 2: The Rayleigh renewal density

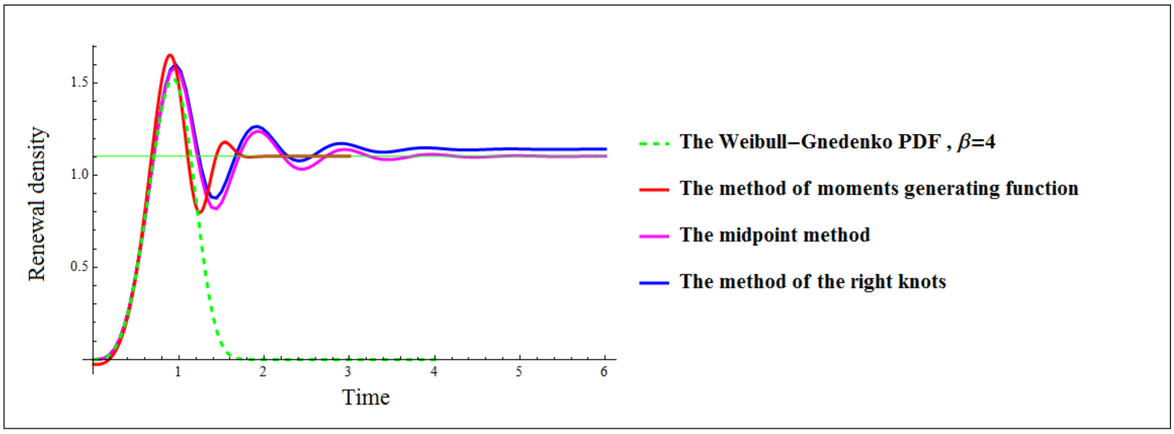

Figure 3: The Weibull-Gnedenko renewal density

In (Fig. 3) the curves for the analytical and all discrete solutions are shown for the Weibull-Gnedenko distribution with $\alpha = 1$, $\beta = 4$ and the number of knots $n = 100$. This increase of collocation knots gives a practical coincidence of approximate solutions found using all the above methods. While for less amount of collocation knots (Fig. 2) the agreement between the results is not ideal. One can see, that all presented methods give oscillations of the renewal density which tends to the asymptotics for big values of t. This can be explained by the specifics of the renewal process, when the underlying lifetime distribution has rather big shape parameter β. The higher the value of β, the more often failures of the system take place. So, the operation and renewal process become unstable. The curve of the corresponding PDF is also shown in Fig. 3. As it can be seen from the formulas (4), (6), (7) the first member of the series representing the solution is equal to the value of the PDF for $t = 0$. It explains the proximity of the presented solutions to the PDF curve in the vicinity of $t = 0$. So, if one needs to know the system behavior in the initial period of time, it is sufficient to evaluate the PDF of the underlying lifetime distribution. The parameters of this distribution can be obtained from the statistical data processing.

3 Results & Discussion

The research techniques of restorable systems and their components renewal density are presented. They are based both on analytical and discrete methods and they consider the dependence of the renewal density on time. Using the Weibull-Gnedenko distribution some peculiarities of the renewal density behavior can be investigated.

The variation of the shape parameter β allows observation of the restorable system properties. As it was mentioned the higher the value of β the more often failures of the system take place, consequently, the less stable its behavior. For relatively small values of β the renewal density does not oscillate at all, that is the system operates in normal mode. This can be seen in Fig. 2. For larger values of β (Fig. 3) the oscillations are observed for some period of time, after which the renewal density goes to the asymptotics, corresponding to normal operation mode of the system. In [6] it was shown that for even higher values of β the oscillations period is longer. And this fact can be easily checked using the approaches of the present paper. Also the influence of the scale parameter α on the solution can be studied. Its variation gives the expansion or contraction of the curves over the time axis. So, a sort of criterion for restorable systems analysis can be developed if the failures statistics is approximated by the Weilbull-Gnedenko distribution. Unstable oscillating period is for sure undesirable for any application. The length and shape of this period are regulated by the parameters of the Weibull-Gnedenko distribution. The knowledge of these parameters values for a given restorable system can give a recommendation on its exploitation. It worth mentioning also, that the time scale in the renewal function dependences shown in Fig. 2 and Fig. 3 is conditional and should be adopted for each application separately. Practical significance of research results was demonstrated on several examples of processing of real statistical data on technologically active elements in gas supply systems failures [2, 19].

The advantages of the considered methods include their simplicity of algorithms and calculations. Nevertheless, one should keep in mind that the analytical approach (3) is valid only for the values of β higher than unity, which correspond to the most actual degradation period of restorable systems life cycles. Moreover, expansion (3) represents the asymptotic series, which does not converge uniformly. The summation of series (3) should be restricted by several members, giving the least relative error. This property was taken into account in curves plotting for Fig. 2 and Fig. 3. The presented discrete methods are more universal, thought their application is more time consuming.

References

[1] *Elsayed E. A.* Reliability Engineering. — John Wiley and Sons, Hoboken, 2012.

[2] *Rusev V. N., Skorikov A. V.* Analysis of elements of gas supply systems by method of moment generating functions. — Proceeding of Gubkin Russian State University of Oil and Gaz. — 2016. — no. 1(282). — P. 68–79. (In Russian).

[3] *Smith W. L., Leadbetter M. R.* On the Renewal Function for the Weibull Distribution. – 1963. –Technometrics. – Vol. 5. — P. 393–396.

[4] *Rinne H.* The Weibull distribution. A handbook. — London, New York: CRC press, Taylor and Francis Group, 2009.

[5] *Grigoriev L., Kucheryavy V., Rusev V., Sedyh I.* Formation of estimates of reliability indicators for active elements in gas transport systems on the basis of refusals statistics. — Journal of Polish Safety and Reliability Association, Summer Safety and Reliability Seminars. — 2014. — Vol. 5, no. 1-2. — P. 41-47

[6] *Constantine A. G., Robinson N. I.* The Weibull renewal function for moderate to large arguments. — Computational Statistics and Data Analysis. — 1997. — Vol. 24. — P. 9–27.

[7] *Cui L., Xie M.* Some normal approximations for renewal function if large Weibull shape parameter. — Communications in statistics – -2003. — Vol. 32, no.1. —P. 1–16.

[8] *Smeitink E., Dekker R.* A Simple Approximation to the Renewal Function. — IEEE Transactions on Reliability — 1990. — Vol. 39, no.1. — P. 71–75.

[9] *Maghsoodloo S., Helvaci D.* Renewal and Renewal-Intensity Functions with Minimal Repair —Hindawi Publishing Corporation. Journal of Quality and Reliability Engineering — 2014. — Vol. 2014, Article ID 857437. —10 Pages. http://dx.doi.org/10.1155/2014/857437.

[10] *Krein M. G.* On one extrapolation problem of A. N. Kolmogorov. —Doklady Akad. Nauk SSSR. — 1944. — no. 46(8). — P. 339–342. (In Russian).

[11] *Sukharev M. G.* Modeli nadezhnosti markovskogo tipa s prilozheniyami k neftegazovomu delu. — Moscow, Izdatel'skiy tsentr RGU nefti i gaza imeni I.M. Gubkina, 2012, — 132 p. (In Russian).

[12] *Deligonul Z. S., Bilgen S.* Solution of the Volterra Equation of Renewal Theory with the Galerkin Technique using Cubic Spline. –Journal of Statistical Computation and Simulation — 1984. — Vol. 20, no.1 —P. 37–45.

[13] *Xie M.* On the solution of the renewal-type integral equations. —Communications in Statistics. —1989. — Vol. 18, no.1. — P. 281–283.

[14] *Boehme T. K., Preuss W., van der Wall V.* On a simple numerical method for computing Stielltjes integrals in reliability theory. — Probability in the Engineering and Informational Sciences — 1991. — Vol. 5, no.1. — P. 113–128.

[15] *Tortorella M.* Numerical solutions of renewal-type integral equations. —Informs Journal on Computing. — 2005. — Vol. 17, no.1. — P. 66–74.

[16] *Jardine A. K. S., Tsang A. H. C.* Maintenance, Replacement, and Reliability: Theory and Applications. Second Edition. — Boca Raton, CRC/Taylor & Francis. — 2013. — 330 P.

[17] *Brezavscek A.* A simple discrete approximation for the renewal function. —Business systems research. — 2013. — Vol. 4, no.1. — P. 65–75.

[18] *Strelayev Y. M., Klimenko M. I.* Primenenie metoda konechnyh elementov k resheniyu integralnyh uravneniy Volterra vtorogo roda. —Visnik ZNU. Fiziko - matematichni nauki. —2011. — Vol. 18, no.2. —P. 131–135. (In Russian).

[19] *Rusev V. N., Skorikov A. V.* Analytical and discrete methods of research of failure intensity in gas transport. — Proceeding of Gubkin Russian State University of Oil and Gaz. — 2016. — no. 3(284). — P.104-117. (In Russian)

Flow Thinning With Limited Aftereffect: Differently Distributed Intervals Between the Moments of Customers Entrance

Viktor Ivnitskiy

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Railway Research Institute, Moscow
Ivnitsky.viktor@vniizht.ru

Abstract

We present some analytical results obtained for probability characteristics of flow thinning with limited aftereffect. The thinning is processed according to a given function which depends on the evolution time and on the number of customers in the thinned flow and the number of lost customers in the original flow. The characteristics are obtained in the form of Laplace-Stieltjes transforms which are defined by the system of recurrence equations using the inverse Laplace-Stieltjes transform.

Keywords: flow with limited aftereffect, thinning of the flow, time-dependent function of thinning, Laplace-Stieltjes transform, inverse Laplace-Stieltjes transform.

1. Introduction

Below we present some results on flow thinning . Renyi, A. [1] in 1956 proved the first theorem on thinning of renewal flow. The customer remains in the thinned flow with constant probability q and is lost with probability 1- q . By changing the time scale, the flow rate remains constant. Let the thinning is performed n times with different probabilities $q_1,\dots,q_n$. Then, provided that $n \to \infty$ and $q_1,\dots,q_n \to 0$, the thinned flow converges to Poisson flow. In his review of "Random threads and theory of recovery" of the book by D. Cox and V. Smith [2],
Yu.K. Belyaev [2] investigated the preservation properties of Poisson flow in the thinning of the original Poisson streams.

Belyaev Yu. K. [3] generalized this fact to an arbitrary stream. In the book Gnedenko B. V. and I.N. Kovalenko I. N. [4], Belyaev's theorem was generalized to the case of non-stationary limit flow. A. D. Solov'ev [1] in 1971 proved that asymptotically the time of the first occurrence of a rare event in a regenerative process with appropriate normalization tends to an exponential random variable with parameter 1. Some other results about thinned flows can be found in [8-12].

For all of these works, the aim was to produce the ultimate results in the infinite thinning under appropriate normalization. Common for the above works was the fact that the thinning was carried out according to rules which were not time-dependent.

The outstanding feature of this paper is that the thinning is performed according to a set of time dependent procedures,

2. Statement of the problem

V. Smith [6] studied the flow of customers with differently distributed intervals between the moments of customer appearance. A. J. Khinchin [7] called these flows by flows with a limited aftereffect. This article considers such flows with thinning. The first customer of this flow enters at the random time having distribution $F_1(x)$. The time interval from the arrival of the first and second customer has a distribution $F_2(x)$. The interval time between the i - 1-st customer to the and i -st customer has a distribution $F_i(x)$, etc.

The thinning goes on as follows. If the customer was received at time t and the number of received customers of the thinned flow up to this point in time is equal i - 1, and the number of lost customersin the original flow is equal j, then the customer joins the customers of the thinned flow with a probability of $P_{i-1+j}(t)$, where the functions $P_{i-1+j}(t)$ are assumed to be known, and the time before admission of the followed customer of flow has distribution function $F_{i+j}(x)$. Otherwise it is lost. It is necessary to find the distribution of the number of received requirements of the thinned stream to an arbitrary point in time ,t under the condition that at t = 0 the number of acted customers of thinned flow was equal zero.

3. Problem solution

We introduce the following notation: $\nu(t)-$ the number of received customers of the thinned stream, $\nu_0(t)-$ the number of lost customers from the initial flow with limited aftereffect, $\xi(t)-$ time prior to *t* of the receipt of the following customer of the flow with limited aftereffect.

First, consider the process $\zeta(t)=(\nu(t),\ \xi(t))$. This process will not be Markovian random process, since its development after the time t will depend not only on $\nu(t)$ and on $\xi(t)$, but and will also depend on the number of lost customers to the time *t* of the initial flow with limited aftereffect. This is because lost customers shift the points in time of receiving of customers of the thinned stream on the time axis.

Indeed, consider two consecutive time ξ_0 and ξ_0 $+\xi$. Let both times received customers original flow with limited aftereffect has not joined the thinned stream, the probability of this event equals (1 - $P_0(\ \xi_0$))(1 - $P_0(\ \xi_0+\xi$)). If we consider the process $\zeta(t)=$ $(\nu(t),\ \xi(t))$, the probability of this event is equal to
(1 - $P_0(\ \xi_0\ ))^2$, as the shift on the time axis by the amount ξ will not be considered because the value $\xi(t)$ it does not take into account.

Let us now consider the process $\zeta(t)=(\nu(t),\ \nu_0(t),\ \xi(t))$. This process already takes into account the fact that the lost customers shift points in time of receipt of customers of the thinned stream on the time axis. Therefore, the process
$\zeta(t)=(\nu(t),\ \nu_0(t),\ \xi(t))$ will already be a Markov random process, its development after the time t will depend on $\nu(t)$, $\nu_0(t)$ and $\xi(t)$, i.e. will not depend on its states before time *t*. We introduce the notation
$\varphi_{i,j}(t,x)$ = $P(\ \nu(t)=i,\ \nu_0(t)$=j, $\xi(t)<x)$, $\varphi_{i,j}(t)$ = $\varphi_{i,j}(t,\infty)$, $i=0,1,2,\ldots$, $j=0,1,2,\ldots$.

The problem of finding the distribution of the number of received customers of the thinned flow $\varphi_{i,j}(t)$ to a fixed point in time t is placed.

First we find the distribution of the number of received requirements of the thinned stream together with an additional variable x to a fixed point t, i.e. $\varphi_{i,j}(t,x)$. At the initial time of number customers is zero.

For desired quantities $\varphi_{i,j}(t,x)$ we derive the corresponding system of differential equations. We have the following system of difference equations

$$\varphi_{0,0}(t+\Delta t, x-\Delta t) = \varphi_{0,0}(t,x) - \varphi_{0,0}(t,\Delta t),$$

$$\dots$$

$$\varphi_{0,j}(t+\Delta t, x-\Delta t) = \varphi_{0,j}(t,x) - \varphi_{0,j}(t,\Delta t) + \varphi_{0,j-1}(t,\Delta t)(1-P_{0,j-1}(t))F_j(x),\ j>0,$$

$$\dots$$

$$\varphi_{i,0}(t+\Delta t, x-\Delta t) = \varphi_{i,0}(t,x) - \varphi_{i,0}(t,\Delta t) + \varphi_{i-1,0}(t,\Delta t)P_{i-1,0}(t)F_i(x), i>0,$$

$$\varphi_{i,1}(t+\Delta t, x-\Delta t) = \varphi_{i,1}(t,x) - \varphi_{i,1}(t,\Delta t) + \varphi_{i-1,1}(t,\Delta t)P_{i-1,1}(t)F_{i+1}(x) +$$

$$\varphi_{i,0}(t,\Delta t)(1-P_{i,0}(t))F_{i+1}(x), i>0,$$

$$\dots$$

$$\varphi_{i,j}(t+\Delta t, x-\Delta t) = \varphi_{i,j}(t,x) - \varphi_{i,j}(t,\Delta t) + \varphi_{i-1,j}(t,\Delta t)\ P_{i-1,j}(t)\ F_{i+j}(x) +$$

$$\varphi_{i,j-1}(t,\Delta t)(1-P_{i,j-1}(t))F_{i+j}(x),\ i>0. \qquad (1)$$

This yields the following system of differential equations for $\varphi_{i,j}(t,x)$:

$$\frac{\partial}{\partial t}\varphi_{0,0}(t,x) - \frac{\partial}{\partial x}\varphi_{0,0}(t,x) = -\frac{\partial}{\partial x}\varphi_{0,0}(t,0),$$

$$\dots$$

$$\frac{\partial}{\partial t}\varphi_{0,j}(t,x) - \frac{\partial}{\partial x}\varphi_{0,j}(t,x) = -\frac{\partial}{\partial x}\varphi_{0,j}(t,0) + \frac{\partial}{\partial x}\varphi_{0,j-1}(t,0)(1-P_{0,j-1}(t))F_j(x),\ j>0,$$

$$\frac{\partial}{\partial t}\varphi_{i,0}(t,x) - \frac{\partial}{\partial x}\varphi_{i,0}(t,x) = -\frac{\partial}{\partial x}\varphi_{i,0}(t,0) + \frac{\partial}{\partial x}\varphi_{i-1,0}(t,0)\ P_{i-1,0}(t)\ F_i(x),\ i>0,$$

$$\frac{\partial}{\partial t}\varphi_{i,1}(t,x) - \frac{\partial}{\partial x}\varphi_{i,1}(t,x) = -\frac{\partial}{\partial x}\varphi_{i,1}(t,0) + \frac{\partial}{\partial x}\varphi_{i-1,1}(t,0)\ P_{i-1,1}(t)\ F_{i+1}(x)$$

$$+\frac{\partial}{\partial x}\varphi_{i,0}(t,0)(1-P_{i,0}(t))F_{i+1}(x),\ i>0,$$

$$\dots$$

$$\frac{\partial}{\partial t}\varphi_{i,j}(t,x) - \frac{\partial}{\partial x}\varphi_{i,j}(t,x) = -\frac{\partial}{\partial x}\varphi_{i,j}(t,0) + \frac{\partial}{\partial x}\varphi_{i-1,j}(t,0)\ P_{i-1,j}(t)\ F_{i+j}(x)$$

$$+\frac{\partial}{\partial x}\varphi_{i,j-1}(t,0)(1-P_{i,j-1}(t))F_{i+j}(x),\ i>0, j>1. \qquad (2)$$

We introduce the notation:

$$\tilde{\varphi}_0^{(0)}(s) = \int_0^\infty e^{-sx}dF_1(x) = \tilde{\varphi}_1(s),,\ \tilde{\varphi}_i(s) = \int_0^\infty e^{-sx}dF_i(x),\ i>0,$$

$$\tilde{\varphi}_{i,j}(u,s) = \int_0^\infty\int_0^\infty e^{-sx-ut}d_x\varphi_{i,j}(t,x)dt,\ \tilde{\varphi}_{i,j}(u) = \int_0^\infty e^{-ut}\varphi_{i,j}(t)dt,\ \frac{\partial}{\partial x}\tilde{\varphi}_{i,j}(u,0) = \int_0^\infty e^{-ut}\frac{\partial}{\partial x}$$

$$\varphi_{i,j}(t,0)dt,\ \frac{\partial}{\partial x}\tilde{\tilde{\varphi}}_{i,j}(u,0) = \int_0^\infty e^{-ut}\frac{\partial}{\partial x}\varphi_{i,j}(t,0)\ P_{i,j}(t)dt,\ i=0,1,2,\dots,\ j=0,1,2,\dots.$$

Then we have the following theorem:

Theorem 1. *For the Laplace-Stieltjes* $\tilde{\varphi}_{i,j}(u,s)$ *of function* $\varphi_{i,j}(t,x)$ *fair following formulas*

$$\tilde{\varphi}_{0,0}(u,s) = (u-s)^{-1}(\tilde{\varphi}_1(s) - \tilde{\varphi}_1(u)), \tag{3}$$

$$\dots$$

$$\tilde{\varphi}_{i,j}(u,s) = (u-s)^{-1}\Big(-\frac{\partial}{\partial x}\tilde{\varphi}_{i,j}(u,0) + \frac{\partial}{\partial x}\tilde{\tilde{\varphi}}_{i-1,j}(u,0)\,\tilde{\varphi}_{i+j}(s)$$

$$+\frac{\partial}{\partial x}\tilde{\varphi}_{i,j-1}(u,0)\,\tilde{\varphi}_{i+j}(s) - \frac{\partial}{\partial x}\tilde{\tilde{\varphi}}_{i,j-1}(u,0)\,\tilde{\varphi}_{i+j}(s),\ i>0,\ j>1.$$

where $\frac{\partial}{\partial x}\tilde{\varphi}_{i,j}(u,0)$ *are determined sequentially from the following recurrent equations*

$$\frac{\partial}{\partial x}\tilde{\varphi}_{i,j}(u,0) = \tilde{\varphi}_{i+j}(u)\Big(\frac{\partial}{\partial x}\tilde{\tilde{\varphi}}_{i-1,j}(u,0) + \frac{\partial}{\partial x}\tilde{\varphi}_{i,j-1}(u,0) - \frac{\partial}{\partial x}\tilde{\tilde{\varphi}}_{i,j-1}(u,0)\Big) =$$

$$\tilde{\varphi}_{i+j}(u)\Big(\int_0^\infty e^{-ut}\frac{\partial}{\partial x}\varphi_{i-1,j}(t,0)\,P_{i-1,j}(t)dt + \int_0^\infty e^{-ut}\frac{\partial}{\partial x}\varphi_{i,j-1}(t,0)\,(1 - P_{i,j-1},(t))dt\Big), \tag{4}$$

Consistent application of recurrent equations (4) given in proof of this theorem ".

Proof. Applying to (2) transform of the Laplace-Stieltjes obtained

$$\tilde{\varphi}_{0,0}(u,s)(u-s) = -\frac{\partial}{\partial x}\tilde{\varphi}_{0,0}(u,0) + \tilde{\varphi}_1(s),$$

$$\dots$$

$$\tilde{\varphi}_{0,j}(u,s)\,(u-s) = -\frac{\partial}{\partial x}\tilde{\varphi}_{0,j}(u,0) + \frac{\partial}{\partial x}\tilde{\varphi}_{0,j-1}(u,0)\,\tilde{\varphi}_j(s) - \frac{\partial}{\partial x}\tilde{\tilde{\varphi}}_{0,j-1}(u,0)\,\tilde{\varphi}_j(s),\ j>0,$$

$$\tilde{\varphi}_{i,0}(u,s)\,(u-s) = -\frac{\partial}{\partial x}\tilde{\varphi}_{i,0}(u,0) + \frac{\partial}{\partial x}\tilde{\tilde{\varphi}}_{i-1,0}(u,0)\,\tilde{\varphi}_i(s),\ i>0,$$

$$\tilde{\varphi}_{i,1}(u,s)\,(u-s) = -\frac{\partial}{\partial x}\tilde{\varphi}_{i,1}(u,0) + \frac{\partial}{\partial x}\tilde{\tilde{\varphi}}_{i-1,1}(u,0)\,\tilde{\varphi}_{i+1}(s)$$

$$+\frac{\partial}{\partial x}\tilde{\varphi}_{i,0}(u,0)\,\tilde{\varphi}_{i+1}(s) - \frac{\partial}{\partial x}\tilde{\tilde{\varphi}}_{i,0}(u,0)\,\tilde{\varphi}_{i+1}(s),\ i>0,$$

$$\dots$$

$$\tilde{\varphi}_{i,j}(u,s)\,(u-s) = -\frac{\partial}{\partial x}\tilde{\varphi}_{i,j}(u,0) + \frac{\partial}{\partial x}\tilde{\tilde{\varphi}}_{i-1,j}(u,0)\,\tilde{\varphi}_{i+j}(s)$$

$$+\frac{\partial}{\partial x}\tilde{\varphi}_{i,j-1}(u,0)\,\tilde{\varphi}_{i+j}(s) - \frac{\partial}{\partial x}\tilde{\tilde{\varphi}}_{i,j-1}(u,0)\,\tilde{\varphi}_{i+j}(s),\ i>0,\ j>1. \tag{5}$$

Assuming in (5) $u = s$, get

$$\frac{\partial}{\partial x}\tilde{\varphi}_{0,0}(u,0) = \tilde{\varphi}_1(u),$$

$$\frac{\partial}{\partial x}\tilde{\varphi}_{0,j}(u,0) = \frac{\partial}{\partial x}\tilde{\varphi}_{0,j-1}(u,0)\,\tilde{\varphi}_j(u) - \frac{\partial}{\partial x}\tilde{\tilde{\varphi}}_{0,j-1}(u,0)\,\tilde{\varphi}_j(u),\ j>0,$$

$$\dots$$

$$\frac{\partial}{\partial x}\tilde{\varphi}_{i,0}(u,0) = \frac{\partial}{\partial x}\tilde{\tilde{\varphi}}_{i-1,0}(u,0)\,\tilde{\varphi}_i(u),\ i>0,$$

$$\frac{\partial}{\partial x}\tilde{\varphi}_{i,1}(u,0) = \frac{\partial}{\partial x}\tilde{\tilde{\varphi}}_{i-1,1}(u,0)\,\tilde{\varphi}_{i+1}(u) + \frac{\partial}{\partial x}\tilde{\varphi}_{i,0}(u,0)\,\tilde{\varphi}_{i+1}(u) - \frac{\partial}{\partial x}\tilde{\tilde{\varphi}}_{i,0}(u,0)\,\tilde{\varphi}_{i+1}(u),\ i>0,$$

$$\frac{\partial}{\partial x}\tilde{\varphi}_{i,j}(u,0) = \frac{\partial}{\partial x}\tilde{\tilde{\varphi}}_{i-1,j}(u,0)\,\tilde{\varphi}_{i+j}(u) + \frac{\partial}{\partial x}\tilde{\varphi}_{i,j-1}(u,0)\,\tilde{\varphi}_{i+j}(u) - \frac{\partial}{\partial x}\tilde{\tilde{\varphi}}_{i,j-1}(u,0)\,\tilde{\varphi}_{i+j}(u),$$

$$i>0,\ j>1. \tag{6}$$

Inversing the first equation (6) we obtain the following expression

$$\frac{\partial}{\partial x}\varphi_{0,0}(t,0) = F_1(t)\,. \tag{7}$$

From the second equation (6) have at $j = 1$

$$\frac{\partial}{\partial x}\tilde{\varphi}_{0,1}(u,0)=\tilde{\varphi}_1(u)\,\tilde{\varphi}_1(u)-\frac{\partial}{\partial x}\tilde{\tilde{\varphi}}_{0,0}(u,0)\,\tilde{\varphi}_1(u)=\tilde{\varphi}_1(u)(\tilde{\varphi}_1(u)-\int\limits_0^\infty e^{-ut}F_1(t)\,P_{0,0}(t)dt)). \tag{8}$$

Inversing (8), we can find the unknown function $\frac{\partial}{\partial x}\varphi_{0,1}(t,0)$.

Then from the second equation (6) have at $j = 2$

$$\frac{\partial}{\partial x}\tilde{\varphi}_{0,2}(u,0)=\tilde{\varphi}_2(u)(\frac{\partial}{\partial x}\tilde{\varphi}_{0,1}(u,0)-\int\limits_0^\infty e^{-ut}\frac{\partial}{\partial x}\varphi_{0,1}(t,0)\,P_{0,1}(t)dt)). \tag{9}$$

Inversing (9), we can find the unknown function $\frac{\partial}{\partial x}\varphi_{0,2}(t,0)$.

Then from the second equation (6) with arbitrary *j*>0,we have the following recursive sequence completely determines $\frac{\partial}{\partial x}\varphi_{0,j}(t,0)$, namely, from the following expression

$$\frac{\partial}{\partial x}\tilde{\varphi}_{0,j}(u,0)=\tilde{\varphi}_j(u)(\frac{\partial}{\partial x}\tilde{\varphi}_{0,j-1}(u,0)-\int\limits_0^\infty e^{-ut}\frac{\partial}{\partial x}\varphi_{0,j-1}(t,0)\,P_{0,j-1}(t)dt)) \tag{10}$$

by his conversion it is possible to find the unknown function $\frac{\partial}{\partial x}\varphi_{0,j}(t,0)$.

From the third equation of (6) obtained by $i = 1$

$$\frac{\partial}{\partial x}\tilde{\varphi}_{1,0}(u,0) = \frac{\partial}{\partial x}\tilde{\tilde{\varphi}}_{0,0}(u,0)\,\tilde{\varphi}_1(u) = \tilde{\varphi}_1(u)\int\limits_0^\infty e^{-ut}\frac{\partial}{\partial x}\varphi_{0,0}(t,0)\,P_{0,0}(t)dt =$$

$$\tilde{\varphi}_1(u)\int\limits_0^\infty e^{-ut}F_1(t)\,P_{0,0}(t)dt\,. \tag{11}$$

Inversing (11), we can find the unknown function $\frac{\partial}{\partial x}\varphi_{1,0}(t,0)$.

Further, from the third equation of (6) at $i > 1$ have

$$\frac{\partial}{\partial x}\tilde{\varphi}_{i,0}(u,0) = \frac{\partial}{\partial x}\tilde{\tilde{\varphi}}_{i-1,0}(u,0)\,\tilde{\varphi}_i(u)= \tilde{\varphi}_i(u)\int\limits_0^\infty e^{-ut}\frac{\partial}{\partial x}\varphi_{i-1,0}(t,0)\,P_{i-1,0}(t)dt,\ i > 1. \tag{12}$$

Reversing (12), we can find the unknown function $\frac{\partial}{\partial x}\varphi_{i,0}(t,0)$, since (12) together with (11) is a recurrence formula for finding $\frac{\partial}{\partial x}\varphi_{i,0}(t,0)$ at $i > 1$.

Let us consider the fourth equation of (6) at $i > 1,\ j = 1$. It can be converted to the form

$$\frac{\partial}{\partial x}\tilde{\varphi}_{i,1}(u,0) = \tilde{\varphi}_{i+1}(u)(\frac{\partial}{\partial x}\tilde{\tilde{\varphi}}_{i-1,1}(u,0)+\frac{\partial}{\partial x}\tilde{\varphi}_{i,0}(u,0) - \frac{\partial}{\partial x}\tilde{\tilde{\varphi}}_{i,0}(u,0)) = \tilde{\varphi}_{i+1}(u)\times$$

$$(\int\limits_0^\infty e^{-ut}\frac{\partial}{\partial x}\varphi_{i-1,1}(t,0)\,P_{i-1,1}(t)dt + \tilde{\varphi}_i(u)\int\limits_0^\infty e^{-ut}\frac{\partial}{\partial x}\varphi_{i-1,0}(t,0)\,P_{i-1,0}(t)dt\ -$$

$$\tilde{\varphi}_i(u)\int\limits_0^\infty e^{-ut}\frac{\partial}{\partial x}\varphi_{i,0}(t,0)\,P_{i,0}(t)dt),\ i > 0,\ j = 1. \tag{13}$$

Reversing (13), we can find the unknown function $\frac{\partial}{\partial x}\varphi_{i,1}(t,0)$, since (13) together with (12) is a recurrence formula for finding $\frac{\partial}{\partial x}\varphi_{i,1}(t,0)$ at $i>1$, $j=1$.

Let us consider the last fifth of equation (6) at $i>1$, $j>1$. It can be transform to the form

$$\frac{\partial}{\partial x}\tilde{\varphi}_{i,j}(u,0) = \tilde{\varphi}_{i+j}(u)\,(\frac{\partial}{\partial x}\tilde{\tilde{\varphi}}_{i-1,j}(u,0)+\frac{\partial}{\partial x}\tilde{\varphi}_{i,j-1}(u,0) - \frac{\partial}{\partial x}\tilde{\tilde{\varphi}}_{i,j-1}(u,0)\,) =$$

$$\tilde{\varphi}_{i+j}(u)\,(\int_0^\infty e^{-ut}\frac{\partial}{\partial x}\varphi_{i-1,j}(t,0)\;P_{i-1,j}(t)dt+\int_0^\infty e^{-ut}\frac{\partial}{\partial x}\varphi_{i,j-1}(t,0)\,(1 - P_{i,j-1}(t))dt. \tag{14}$$

Equation (14) is a recurrence relation expressed $\frac{\partial}{\partial x}\varphi_{i,j}(u,0)$ through $\frac{\partial}{\partial x}\varphi_{i-1,j}(t,0)$ and $\frac{\partial}{\partial x}\varphi_{i,j-1}(t,0)$. The beginning of this recurrence relation initiated by formulas (12) together with (11) and formulas (13) together with (12).

Thus, the expression for $\frac{\partial}{\partial x}\varphi_{i,j}(u,0)$ it is possible to obtain by the above method. Substituting these expressions into the formula (5), we can obtain expressions for the desired quantities $\tilde{\varphi}_{i,j}(u,s)$.

Corollary 1. "*For the Laplace-Stieltjes* $\tilde{\varphi}_{i,j}(u)$ *function* $\varphi_{i,j}(t)$ *fair following formulas*

$$\tilde{\varphi}_{0,0}(u) = u^{-1}(1 - \tilde{\varphi}_1(u))), \tag{15}$$

$$\ldots$$

$$\tilde{\varphi}_{i,j}(u) = (u-s)^{-1}\,(- \frac{\partial}{\partial x}\tilde{\varphi}_{i,j}(u,0) + \frac{\partial}{\partial x}\tilde{\tilde{\varphi}}_{i-1,j}(u,0)\;\tilde{\varphi}_{i+j}(s)$$

$$+\frac{\partial}{\partial x}\tilde{\varphi}_{i,j-1}(u,0)\;\tilde{\varphi}_{i+j}(s) - \frac{\partial}{\partial x}\tilde{\tilde{\varphi}}_{i,j-1}(u,0)\;\tilde{\varphi}_{i+j}(s),\; i>0,\; j>1,$$

where $\frac{\partial}{\partial x}\tilde{\varphi}_{i,j}(u,0)$ *consistently determined from recurrent equations* (4)."

4. Special case

We now turn to the consideration of the problem of thinning of the flow, when the probability of thinning of this thread $P_{i,j}(t)$ not time-dependent, and depend only on the received number of customers thinned flow and the number of lost customers of the initial flow, i.e. have the form $P_{i,j}$. This gives the following results. Function $\frac{\partial}{\partial x}\tilde{\tilde{\varphi}}_{i,j}(u,0)$ takes the form

$$\frac{\partial}{\partial x}\tilde{\tilde{\varphi}}_{i,j}(u,0) = P_{i,j}\int_0^\infty e^{-ut}\frac{\partial}{\partial x}\varphi_{i,j}(t,0)\;dt = P_{i,j}\frac{\partial}{\partial x}\tilde{\varphi}_{i,j}(u,0),\; i=0,1,2,\ldots,\; j=0,1,2,\ldots.$$

Theorem 1 transforms into theorem 2, which has the following form.

Theorem 2. "*For the Laplace-Stieltjes* $\tilde{\varphi}_{i,j}(u,s)$ *function* $\varphi_{i,j}(t,x)$ fair following formulas

$$\tilde{\varphi}_{0,0}(u,s) = (u-s)^{-1}(\tilde{\varphi}_1(s) - \tilde{\varphi}_1(u)), \tag{16}$$

$$\ldots$$

$$\tilde{\varphi}_{i,j}(u,s) = (u-s)^{-1}\,(- \frac{\partial}{\partial x}\tilde{\varphi}_{i,j}(u,0) + \tilde{\varphi}_{i+j}(s)\,(P_{i-1,j}\frac{\partial}{\partial x}\tilde{\varphi}_{i-1,j}(u,0)$$

$$+(1-\ P_{i,j-1})\frac{\partial}{\partial x}\ \tilde{\varphi}_{i,j-1}(u,0)\ )),\ i>0,\ j>1.$$

where $\frac{\partial}{\partial x}\ \tilde{\varphi}_{i,j}(u,0)$ are determined sequentially from the following recurrent formulas

$$\frac{\partial}{\partial x}\ \tilde{\varphi}_{i,j}(u,0)=\ \tilde{\varphi}_{i+j}(u)\ (P_{i-1,j}\ \frac{\partial}{\partial x}\ \tilde{\varphi}_{i-1,j}(u,0)\ +(1\ -\ P_{i,j-1})\ \frac{\partial}{\partial x}\ \tilde{\varphi}_{i,j-1}(u,0)\ ), \tag{17}$$

and $\frac{\partial}{\partial x}\ \tilde{\varphi}_{0,0}(u,0)=\ \tilde{\varphi}_1(u),\ \frac{\partial}{\partial x}\ \tilde{\varphi}_{0,1}(u,0)\ =\tilde{\varphi}_1^2(u)(1\ -\ P_{0,0}\ ),$

$$\frac{\partial}{\partial x}\ \tilde{\varphi}_{0,j}(u,0)=\tilde{\varphi}_j(u)(1-\ P_{0,j-1})\ \frac{\partial}{\partial x}\ \tilde{\varphi}_{0,j-1}(u,0)\ =\ \tilde{\varphi}_1(u)\prod_{l=1}^{j}\ \tilde{\varphi}_l(u)(1\ -\ P_{0,l-1}),\ j>1,$$

$$\frac{\partial}{\partial x}\ \tilde{\varphi}_{1,0}(u,0)\ =\ \tilde{\varphi}_1^2(u)\ P_{0,0}\ ,$$

$$\frac{\partial}{\partial x}\ \tilde{\varphi}_{i,0}(u,0)\ =\ \tilde{\varphi}_i(u)\ P_{i-1,0}\ \frac{\partial}{\partial x}\ \tilde{\varphi}_{i-1,0}(u,0)\ =\ \tilde{\varphi}_1(u)\prod_{l=1}^{i}\ \tilde{\varphi}_l(u)\ P_{l-1,0},\ i>1.''$$

Here are a few of the subsequent formulas. Have

$$\frac{\partial}{\partial x}\ \tilde{\varphi}_{1,1}(u,0)=\ \tilde{\varphi}_2(u)\ (P_{0,1}\frac{\partial}{\partial x}\ \tilde{\varphi}_{0,1}(u,0)\ +(1\ -\ P_{1,0})\ \frac{\partial}{\partial x}\ \tilde{\varphi}_{1,0}(u,0)\ )=$$

$$\tilde{\varphi}_2(u)\ \tilde{\varphi}_1^2(u)(P_{0,1}\ (1\ -P_{0,0}\ )+(1\ -\ P_{1,0})\ P_{0,0}\ ),$$

$$\frac{\partial}{\partial x}\ \tilde{\varphi}_{2,1}(u,0)\ =\ \tilde{\varphi}_3(u)(P_{1,1}\frac{\partial}{\partial x}\ \tilde{\varphi}_{1,1}(u,0)\ +(1\ -\ P_{2,0})\frac{\partial}{\partial x}\ \tilde{\varphi}_{2,0}(u,0)\ )=$$

$$\tilde{\varphi}_3(u)(P_{1,1}(\tilde{\varphi}_2(u)\ \tilde{\varphi}_1^2(u)(P_{0,1}\ (1\ -P_{0,0}\ )+(1\ -\ P_{1,0})\ P_{0,0}\ ))+$$

$$(1\ -\ P_{2,0})\ \tilde{\varphi}_1(u)\prod_{l=1}^{2}\ \tilde{\varphi}_l(u)\ P_{l-1,0}),$$

$$\frac{\partial}{\partial x}\ \tilde{\varphi}_{1,2}(u,0)\ =\ \tilde{\varphi}_3(u)(\ P_{0,2}\ \frac{\partial}{\partial x}\ \tilde{\varphi}_{0,2}(u,0)\ +(1\ -\ P_{1,1})\ \frac{\partial}{\partial x}\ \tilde{\varphi}_{1,1}(u,0)\ )=$$

$$\tilde{\varphi}_3(u)(\ P_{0,2}\ \tilde{\varphi}_1(u)\prod_{l=1}^{2}\ \tilde{\varphi}_l(u)(1\ -\ P_{0,l-1})+$$

$$(1\ -\ P_{1,1})\ \tilde{\varphi}_2(u)\ \tilde{\varphi}_1^2(u)(P_{0,1}\ (1\ -P_{0,0}\ )+(1\ -\ P_{1,0})\ P_{0,0}\ )).$$

Thus, in the article, obtained some analytical results for probability characteristics of a thinning of the flow with different-distributed intervals between the moments of customers entrance (flow with limited aftereffect). The thinning is processed according to a given function which depends on evolution time and on the number customers of the thinned flow and the number of lost customers in theoriginal flow. The characteristics are obtained in the form of Laplace-Stieltjes transforms which are defined by the system of recurrence equations with using inversion of Laplace-Stieltjes transforms.

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References

1. Renyi, A.: A Characterization of Poisson Processes. *Magyar Tud. Akad. Mam. Kutato Int. Kozl*. v. 1, N 4, 519-527 (1956) (in Hungarian).

2.Cox, D.R., Smith W.L.: *Renewal Theory*. Soviet Radio, Moscow (1967) (in Russian), p. 300.

3. Belyaev, Yu.K.: Limit theorems for Rearing Flows. *Probability Theory and its Applications*. v. 8, N. 2, 175-184 (1963) (in Russian).

4. Gnedenko, B.V., Kovalenko, I.N.: *Introduction to Queueing Theory*, Second Edition. Birkhauser, 1989.

5. Soloviev, A.D.: Asymptotic Behavior of the First occurrence of a Rare Eevent in a Regenerating Process. *Engineering Cybernetics,* N 6, 79-89 (1971) (in Russian)/

Изв. АН СССР. Техн. кибернетика, 1971, № 6, с. 79-89.

6. Smith W.L. On some general renewal theorems for non-identically distributed Variables. *Proc. 4-th Berkeley Symposium*, 2, 1961, 467-514.

7. Khintchine, A.Y., *Mathematical Methods in the Theory of Queueing*, Charles Griffin and Co., London, 1960 (translation of 1955 Russian book).

8. Streit, R.l.: *Poisson Point Processes: Imaging, Tracking, and Sensing*. Springer, New York (2010).

9. Serfozo,R.: *A Course in Applied Stochastic Processes*. Springer, New York (2009).

10. Assuncao R.M., Ferrari, P.A.: Independence of Trinned Processes Characteristics the Poisson Process: an Elementary Proof and a Statistical Application. Test, 16, Iss. 2, 333-345 (2007).

11. Kushnir, A.O.: Asymptotic Behavior of a Renewal Process Trinned by an Alternating Process. *Cybernetics and Systems Analysis*, 29, Iss. 1, 20-25 (1993).

12. Gurel-Gurevich, O., Peled, R.: Poisson Thickening. *Israel J. of Math.* 196, Iss. 1, 215-234 (2013).

Accelerated Life Testing Design Using Geometric Process for Generalized Rayleigh Distribution with Complete Data

Kamal Ullah, Inthekhab Alam, Showkat Ahmad Lone*
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Department of Statistics & Operations Research,
Aligarh Muslim University, Aligarh.
Email: showkatmaths25@gmail.com

Abstract

The log-linear function between life and stress which is just a simple re-parameterization of the original parameter of the life distribution is used to obtain the estimates of original parameters in many of the studies concerning Accelerated life testing (ALT). But from the statistical point of view, it is preferable to work with the original parameters instead of developing inferences for the parameters of the log-linear link function. In this study we introduce the geometric process for the analysis of accelerated life testing with Generalized Rayleigh Distribution for constant stress. Assuming that the lifetimes of units under increasing stress levels form a geometric process, the maximum likelihood estimation approach is used for the estimation of parameters. The confidence intervals (CIs) of the model parameters are derived. A Simulation study is also performed to check the statistical properties of estimates of the parameters and the confidence intervals.

Keywords: Geometric process, Generalized Rayleigh Distribution, *Maximum* Likelihood Estimator, Fisher Information Matrix, Asymptotic Confidence Interval, Simulation Study.

1. Introduction

Accelerated life testing is the process of testing a product by subjecting it to conditions (stress, strain, temperatures, voltage, vibration rate, pressure etc.) in excess of its normal service parameters in an effort to uncover faults and potential modes of failure in a short amount of time. By analyzing the product's response to such tests, statisticians can make predictions about the service life and maintenance intervals of a product.

In general, ALT deals with three types of stress patterns: constant stress, step stress and Progressive stress. In the former case, each unit is run at a pre-specified constant stress level which does not vary with time. This means that every item is subjected to only one stress level until the item fails or the test is stopped for other reasons. In use, most products such as semiconductors and microelectronics, capacitors, lamps …etc, run at a constant stress. This type of stress is widely used and preferred because the stress is constant in most applications, it is much easier to apply and quantify constant stress and models for constant stress are available, widely publicized and empirically verified.

There is a lot of literature on constant stress accelerated life testing, for example, Ahmad et al. [1], Islam and Ahmad [2], Ahmad and Islam [3], Ahmad et al.[4] and Ahmad [5] discuss the optimal constant stress accelerated life test designs under periodic inspection and Type-I censoring. Yang [6] proposed an optimal design of 4-level constant stress ALT plans considering different censoring times. Pan et al. [7] proposed a Bivariate constant stress accelerated degradation test model by assuming that the copula parameter is a function of the stress level that can be described by the logistic function. Wilkins and johns [8] considered constant stress accelerated life test based on Weibull distribution with constant shape and a log-linear link between scale the stress factor which is terminated by a Type-II censoring regime at one of the stress levels.

The concept of geometric process in accelerated life testing was first introduced by Lam [9] in the problems of repair replacement. Lam [10] studied the geometric process model for a multistate system and concluded a replacement policy to minimize the long run average cost per unit time. Since then a lot of studies in maintenance problems and system reliability have been shown that a GP model is a good and simple model for analysis of data with a single trend or multiple trends, for example, Lam and Zhang [11], Lam [12] and Zhang [13]. Huang [14] introduced the GP model for the analysis of constant stress ALT with complete and censored exponential samples. Kamal et al. [15] extended the GP model for the analysis of complete Weibull failure data in constant stress ALT. Zhou et al. [16] implement the GP in ALT based on the progressive Type-I hybrid censored Rayleigh failure data. Kamal et al. [17] used the geometric process for the analysis of constant stress accelerated life testing for Pareto Distribution with complete data. S. Saxena [18] introduces the Rayleigh geometric process model for the analysis of accelerated life testing under constant stress. Sadia Anwar et al. [19] presented the mathematical model of accelerated life testing for Marshall-Olkin extended exponential distribution using geometric process and extended her work using type I censored data [20]. Recently Kamal [21] presented an application of the geometric process in accelerated life testing analysis on type-I censored Weibull failure data.

In the present study, the GP model is implemented in the analysis of ALT for the Generalized Rayleigh life distribution under constant stress with complete data. Maximum likelihood (ML) estimates of parameters and their asymptotic confidence intervals (CIs) are obtained. The performance of the estimates is evaluated by a simulation study.

2. The Model and Test Procedure

2.1. The Geometric Process

A geometric process describes a stochastic process $\{X_n, n = 1,2,...\}$, where there exists a real-valued $\lambda > 0$ such that $\{\lambda^{n-1} X_n, n = 1,2,...\}$ forms a renewal process. It can be shown that if $\{X_n, n = 1,2,...\}$ is a GP and the probability density function of X_1 is $f(x)$ with mean μ and variance σ^2 then the probability density function of X_n will be $\lambda^{n-1} f(\lambda^{n-1} x)$ with $E(X_n) = \frac{\mu}{\lambda^{n-1}}$ and $\text{var}(X_n) = \frac{\sigma^2}{\lambda^{2(n-1)}}$. Thus λ , μ and σ^2 are three important parameters of GP.

2.2. The Generalized Rayleigh Distribution

The probability density function (pdf) of a generalized Rayleigh distribution is given by

$$f(x/\alpha,\beta) = \begin{cases} (1-e^{-\beta x^2})^{\alpha} & x > 0, \alpha > 0, \beta > 0 \\ 0, elsewhere \end{cases}$$

where, $\alpha > 0$ is the shape parameter and $\beta > 0$, is the scale parameter of the distribution. Generalized Rayleigh distribution is a member of the family of Burr distributions which was appeared since 1942. It is known also Burr type X distribution. The cumulative distribution function (cdf) of generalized Rayleigh distribution is

$$F(x/\alpha,\beta)=\begin{cases}\left(1-e^{-\beta x^2}\right)^{\alpha} & ,x>0\,,\alpha>0\,,\beta>0\\ 0\ ,elsewhere & \end{cases}$$

The Hazard function of the Generalized Rayleigh distribution takes the following form

$$S(x/\alpha,\beta)=1-\left(1-e^{-\beta x^2}\right)^{\alpha}$$

The failure rate (or hazard rate) for the Generalized Rayleigh distribution is given by

$$h(x/\alpha,\beta)=\frac{2\alpha\beta xe^{x^2}\left(1-e^{-\beta x^2}\right)^{\alpha-1}}{1-\left(1-e^{-\beta x^2}\right)^{\alpha}}$$

The two-parameter Generalized Rayleigh distribution was first proposed by (Raqab and Kundu; 2003) [22] and is denoted by $GR(\alpha,\beta)$. It is observed that the hazard function of a Generalized Rayleigh distribution can be either bathtub type or increasing function, depending on the shape parameter α . For $\alpha\leq\frac{1}{2}$, the hazard function is bathtub type and for $\alpha>\frac{1}{2}$, it has an increasing hazard function. Surles and Padgett (2001) [23] showed that the two-parameter GR distribution can be used quite effectively in modelling strength data and also modelling general lifetime data.

2.3. Assumptions and test procedure

1. Under any constant stress, the time to failure of test unit follows Generalized Rayleigh distribution.

2. The Generalized Rayleigh shape parameter α is constant, i.e. independent of stress.

3. Let the sequence of random variables $X_0, X_1,...X_s$ denote the lifetimes under each stress level, where X_0 denotes lifetime of an item under the design stress. We assume $\{X_k, k=1,2,...,s\}$ is a geometric process with ratio $\lambda>0$.

4. Suppose that an ALT under $z_k, k=1,2,...,s$, arithmetically increasing stress levels is performed. A random sample of $N_i, i=1,2,...,n$, identical items are placed under each stress level and start to operate at the same time. Whenever an item fails, it is removed from the test and its observed failure time x_{ki} is recorded.

5. The scale parameter is a log-linear function of stress that is $\log\beta_i=a+bS_i$, here *a* and *b* are unknown parameters depending on the nature of the product and the test method.

Theorem: *If the stress level in an ALT is increasing with a constant difference then under each stress level the lifetimes of items forms a GP. That is, If* $S_{k+1}-S_k$ *is constant for* $k=1,2,...,s-1$, *then* $\{X_k, k=1,2,...,s\}$ *forms a GP.*

Proof: From assumption (5), we get

$$\log\left(\frac{\beta_{k+1}}{\beta_k}\right)=b\left(S_{k+1}-S_k\right)=b\Delta S$$

This shows that the increased stress levels form an arithmetic sequence with a constant difference ΔS .

Now the above equation can be written as

$$\frac{\beta_{k+1}}{\beta_k} = e^{b\Delta S} = \lambda(say) \tag{1}$$

It is clear from (1) that

$$\beta_k = \lambda\beta_{k-1} = \lambda^2\beta_{k-2} = \ldots = \lambda^k\beta$$

The lifetime PDF of an item at the kth stress level is

$$f_{X_k}(x) = 2\alpha\beta_k^2 xe^{-(\beta_k x)^2}\left(1-e^{-(\beta_k x)^2}\right)^{\alpha-1}$$

$$= 2\alpha\left(\lambda^k\beta\right)^2 xe^{-\left(\lambda^k\beta x\right)^2}\left(1-e^{-\left(\lambda^k\beta x\right)^2}\right)^{\alpha-1} \tag{2}$$

This implies that

$$f_{X_k}(x) = \lambda^k f_{X_0}\left(\lambda^k x\right) \tag{3}$$

Now, from the definition of GP and from expression (3) it is clear that, if density function of X_0 is $f_{X_0}(x)$, then the pdf of X_k will be given by $\lambda^k f_{X_0}\left(\lambda^k x\right)$, $k = 1,2,\ldots,s$. Therefore, it is clear that lifetimes under a sequence of arithmetically increasing stress levels form a GP with ratio λ . Now, the pdf of a lifetime of an item at the k^{th} stress level is

$$f_{X_k}(x/\alpha,\beta,\lambda) = \; = 2\alpha\left(\lambda^k\beta\right)^2 xe^{-\left(\lambda^k\beta x\right)^2}\left(1-e^{-\left(\lambda^k\beta x\right)^2}\right)^{\alpha-1} \tag{4}$$

It is clear from above expression that if lifetimes of items under a sequence of increasing stress level form a geometric process with ratio λ and if the life distribution at design stress level is generalized Rayleigh with characteristic β , then the life distribution at k^{th} stress level will also be generalized Rayleigh with characteristic life $\beta\lambda^k$.

2.4. Maximum likelihood Estimation

The likelihood function for constant stress ALT for complete case generalized Rayleigh distribution failure data using GP for s stress levels is given by:

$$L(\lambda,\alpha,\beta) = \prod_{k=1}^{s}\prod_{i=1}^{n} 2\alpha\left(\lambda^k\beta\right)^2 x_{k_i} e^{-\left(\lambda^k\beta x_{k_i}\right)^2}\left(1-e^{-\left(\lambda^k\beta x_{k_i}\right)^2}\right)^{\alpha-1} \tag{5}$$

The log likelihood of (5) can be written a

$$l(\lambda,\alpha,\beta) = \sum_{k=1}^{s}\sum_{i=1}^{n}\left\{\begin{matrix}\log 2\alpha + 2k\log\lambda + 2\log\beta + \log x_{k_i} - \left(\lambda^k\beta x_{k_i}\right)^2 + \\ (\alpha-1)\log\left(1-e^{-\left(\lambda^k\beta x_{k_i}\right)^2}\right)\end{matrix}\right\}$$

Partial derivatives of above equation with respect to λ and β are:

$$\frac{\partial l}{\partial\lambda} = \sum_{k=1}^{s}\sum_{i=1}^{n}\left\{\frac{2k}{\lambda} - 2k\lambda^{2k-1}\left(\beta x_{k_i}\right)^2 + 2k\lambda^{2k-1}(\alpha-1)\left(\beta x_{k_i}\right)^2\frac{A}{D}\right\} \tag{6}$$

$$\frac{\partial l}{\partial \beta} = \sum_{k=1}^{s} \sum_{i=1}^{n} \left\{ \frac{2}{\beta} - 2\beta\left(\lambda^{k} x_{k_i}\right)^{2} + 2\beta(\alpha - 1)\left(\lambda^{k} x_{k_i}\right)^{2} \frac{A}{D} \right\} \tag{7}$$

Where

$$\text{A=} e^{-\left(\lambda^{k} \beta x_{k_i}\right)^{2}} \quad \text{and} \quad \text{D=}\left(1 - e^{-\left(\lambda^{k} \beta x_{k_i}\right)^{2}}\right)$$

From equations (6) and (7), it is observed that these equations are non-linear. Therefore, the closed forms of MLEs of λ and β do not exist. So, Newton-Raphson method must be used to solve these equations simultaneously to obtain the MLEs of λ and β .

3. Asymptotic Confidence Interval

Let $I(\lambda, \beta)$ denotes the Fisher Information matrix, then observed Information matrix of $I(\lambda, \beta)$ is given as

$$I(\lambda, \beta) = \begin{bmatrix} \hat{I}_{11} & \hat{I}_{12} \\ \hat{I}_{21} & \hat{I}_{22} \end{bmatrix}$$

Where

$$\widehat{I}_{11} = -\left(\frac{\partial^2 l}{\partial \lambda^2}\right)$$

$$= \sum_{k=1}^{s} \sum_{i=1}^{n} \left[\begin{array}{l} \frac{2k}{\lambda^2} + 2k(2k-1)Z^{2}\lambda^{2(k-1)} + (\alpha - 1) \\ \times \left\{ \frac{4ADZ^{4}k^{2}\lambda^{2(k-1)} - 2kAD(2k-1)Z^{2}\lambda^{2(k-1)} + 4k^{2}A^{2}Z^{4}\lambda^{2(2k-1)}}{D^{2}} \right\} \end{array} \right]$$

$$\widehat{I}_{22} = -\left(\frac{\partial^2 l}{\partial \beta^2}\right) = \sum_{k=1}^{s} \sum_{i=1}^{n} \left[\frac{2}{\beta^2} + 2Z^{2} + (\alpha - 1)\left\{ \frac{4AD\beta^{2}Z^{4} - 2ADZ^{2} + 4A^{2}\beta^{2}Z^{4}}{D^{2}} \right\} \right]$$

$$\widehat{I}_{12} = -\left(\frac{\partial^2 l}{\partial \lambda \partial \beta}\right) = \widehat{I}_{21} = -\left(\frac{\partial^2 l}{\partial \beta \partial \lambda}\right)$$

$$= \sum_{k=1}^{s} \sum_{i=1}^{n} \left[\frac{4\beta k Z^{2}}{\lambda} + 4\left(\frac{\alpha - 1}{\lambda}\right)\left\{ \frac{ADk\beta^{3}Z^{4} - ADk\beta Z^{2} + Z^{4}A^{2}\beta^{3}k}{D^{2}} \right\} \right]$$

Where

$$Z = \left(\lambda^{k} x_{k_i}\right)$$

Now, the variance-covariance matrix can be written as

$$\begin{bmatrix} \operatorname{var}(\hat{\lambda}) \\ \operatorname{var}(\hat{\beta}) \end{bmatrix} = \begin{bmatrix} \hat{I}_{11} \\ \hat{I}_{22} \end{bmatrix}^{-1}$$

The $100(1-\theta)\%$ asymptotic confidence interval for λ and β are then given respectively as

$$\left[\hat{\lambda} \pm z_{1-\frac{\vartheta}{2}} \sqrt{\operatorname{var}(\hat{\lambda})}\right]$$

And

$$\left[\hat{\beta} \pm z_{1-\frac{\vartheta}{2}} \sqrt{\operatorname{var}(\hat{\beta})}\right]$$

4. Simulation Studies:

Simulation of data is the initial task for studying different properties of parameters. It is an attempt to model an assumed condition to study the behaviour of the function.

1. First, to perform the simulation study, first, a random sample is generated from Uniform distribution by using R software.
2. Now, we use inverse cdf method to transform the cdf at kth stress level in terms of u and get the expression of $X_{ki}, k = 1,2,...,s;\ i = 1,2,...,n.$

$$X_{ki} = -\frac{In\,(1-u)^{\frac{1}{2\alpha}}}{\beta\lambda^{k}}, \qquad k = 1\,,2,...,s; \quad i = 1,\,2,...,n.$$

Where X_{ki} is obtained for n=20, 40 and 60.

3. The values of parameters and numbers of the stress levels are chosen to be $\alpha = 1, \beta = 2.8, \lambda = 1.1$ and $s = 4\ or\ 6$.
4. By using optim() function, we obtain ML estimates, the mean squared error (MSE), relative absolute bias (RAB), relative error (RE) and lower and upper bound of 95% and 99% confidence intervals for different sample sizes n=20,40 and 60. The results obtained in the above simulation study are summarized in Table1 & 2.

Table 1: Simulation results of Generalized Rayleigh distribution using GP at $\alpha = 1, \beta = 2.8, \lambda = 1.1$ and $s = 4$.

Sample	Estimate	Mean	SE	√MSE	LCL	UCL
20	β	3.078	0.319	0.095	2.452 2.254	3.703 3.901
	λ	1.107	0.103	0.099	0.797 0.732	1.202 1.267
40	β	3.039	0.256	0.061	2.536 2.377	3.542 3.701
	λ	1.081	0.103	0.100	0.797 0.732	1.202 1.267
60	β	3.003	0.241	0.054	2.529 2.380	3.477 3.627
	λ	1.072	0.103	0.099	0.797 0.733	1.202 1.267

Table 2: Simulation results of Generalized Rayleigh distribution using GP at $\alpha = 1, \beta = 2.8, \lambda = 1.1$ and $s = 4$.

Sample	Estimate	Mean	SE	√MSE	LCL	UCL
20	β	3.078	0.218	0.078	2.491 2.311	3.628 3.807
	λ	0.977	0.112	0.103	0.797 0.732	1.202 1.267
40	β	3.039	0.209	0.074	2.520 2.348	3.609 3.781
	λ	0.981	0.103	0.101	0.797 0.732	1.202 1.267
60	β	2.953	0.192	0.044	2.551 2.416	3.407 3.543
	λ	0.992	0.020	0.100	0.784 0.732	1.202 1.267

5. Conclusions

In this study, the geometric process is introduced for the analysis of accelerated life testing under constant stress when the life data are from a generalized Rayleigh distribution. It is a better choice for life testing because of its simplicity in nature. The Mean, SE and RMSE of the parameters are obtained. Based on the asymptotic normality, the 95% and 99% confidence intervals of the parameters are also obtained.

The results show in Table 1 and Table 2 that the estimated values of β and λ are very close to true (or initial) value with very small SE and RMSE. As sample size increases, the value of SE and RMSE decreases and the confidence interval become narrower. For the Table 2, the maximum likelihood estimators have good statistical properties than the Table 1 for all sample sizes.

References

[1] Ahmad, R. K. (1994). Optimal design of accelerated life test plans under periodic inspection and type I censoring: the case of Rayleigh failure law. *South African Statistical Journal*, *28*(2), 93-101.

[2] Islam, A., & Ahmad, N. (1994). Optimal design of accelerated life tests for the Weibull distribution under periodic inspection and type I censoring. *Microelectronics Reliability*, *34*(9), 1459-1468.

[3 Ahmad, N., & Islam, A. (1996). Optimal accelerated life test designs for Burr type XII distributions under periodic inspection and type I censoring. *Naval Research Logistics*, *43*(8), 1049-1077.

[4] Ahmad, N., Islam, A., & Salam, A. (2006). Analysis of optimal accelerated life test plans for periodic inspection: The case of exponentiated Weibull failure model. *International Journal of Quality & Reliability Management*, *23*(8), 1019-1046.

[5] Ahmad, N. (2010). Designing accelerated life tests for generalised exponential distribution with log-linear model. *International Journal of Reliability and Safety*, *4*(2-3), 238-264.

[6] Yang, G. B. (1994). Optimum constant-stress accelerated life-test plans. *IEEE Transactions on Reliability*, *43*(4), 575-581.

[7] Pan, Z., Balakrishnan*, N., & Sun, Q. (2011). Bivariate constant-stress accelerated degradation model and inference. *Communications in Statistics—Simulation and Computation®*, *40*(2), 247-257.

[8] Watkins, A. J., & John, A. M. (2008). On constant stress accelerated life tests terminated by Type II censoring at one of the stress levels. *Journal of Statistical Planning and Inference*, *138*(3), 768-786.

[9] Lin, Y. L. Y. (1988). Geometric processes and replacement problem. *Acta Mathematicae Applicatae Sinica*, *4*, 366-377.

[10] Yeh, L. (2005). A monotone process maintenance model for a multistate system. *Journal of Applied Probability*, *42*(1), 1-14.

[11] Lam, Y., & Zhang, Y. L. (1996). Analysis of a two-component series system with a geometric process model. *Naval Research Logistics (NRL)*, *43*(4), 491-502.

[12] Yeh, L. (2005). A monotone process maintenance model for a multistate system. *Journal of Applied Probability*, *42*(1), 1-14.

[13] Zhang, Y. L. (2008). A geometrical process repair model for a repairable system with delayed repair. *Computers & Mathematics with Applications*, *55*(8), 1629-1643.

[14] Huang, S. (1911). *Statistical inference in accelerated life testing with geometric process model* (Doctoral dissertation, Sciences).

[15] Kamal, M., Zarrin, S., & Saxena, S. (2012). Weibull Geometric Process Model for the Analysis of Accelerated Life Testing with Complete Data. *International Journal of Statistics and Applications*, *2*(5), 60-66.

[16] Zhou, K., Shi, Y. M., & Sun, T. Y. (2012). Reliability analysis for accelerated life-test with progressive hybrid censored data using geometric process.

[17] Kamal, M., Zarrin, S., & Saxena, S. (2012). Weibull Geometric Process Model for the Analysis of Accelerated Life Testing with Complete Data. *International Journal of Statistics and Applications*, *2*(5), 60-66.

[18] Saxena, S., Zarrin, S., & Kamal, M. (2012). Optimum Step Stress Accelerated Life Testing for Rayleigh Distribution. *International journal of statistics and applications*, *2*(6), 120-125.

[19] S. Anwar, M. Kamal and A. Islam, "Mathematical model of accelerated life testing using geometric process for Marshall-Olkin extended exponential distribution" , *International Journal of Innovative Research in Science , Engineering and Technology,* vol.2, no.12, pp. 7382-7390, 2013

[20] Anwar, S., Shahab, S., & Islam, A. U. (2014). Accelerated Life Testing Design Using Geometric Process For Marshall-Olkin Extended Exponential Distribution With Type I Censored Data. *International Journal of Scientific & Technology Research*, *3*(1), 179-186.

[21] Kamal, M. (2013). Application Of Geometric Process in Accelerated Life Testing Analysis With Type-I Censored Weibull Failure Data. *Reliability: Theory & Applications*, *8*(3).

[22] Kundu, D., & Raqab, M. Z. (2005). Generalized Rayleigh distribution: different methods of estimations. *Computational statistics & data analysis*, *49*(1), 187-200.

[23] Surles, J. G., & Padgett, W. J. (2001). Inference for reliability and stress-strength for a scaled Burr Type X distribution. *Lifetime Data Analysis*, *7*(2), 187-200.

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Time-Dependent Analysis of a Single-Server Queuing Model with Discouraged Arrivals and Retention of Reneging Customers

Rakesh Kumar∗ & Sapana Sharma
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Department of Mathematics, Shri Mata Vaishno Devi University,
Katra Jammu and Kashmir, India-182320
Email: rakesh stat kuk@yahoo.co.in,
sapanasharma736@gmail.com
∗ Corresponding Author

Abstract

In this paper, a finite capacity Markovian single-server queuing system with discouraged arrivals, reneging, and retention of reneging customers is studied. The time-dependent probabilities of the queuing system are ob- tained by using a computational technique based on the 4th order Runge- Kutta method. With the help of the time-dependent probabilities, we develop some important measures of performance of the system, such as expected system size, expected reneging rate, and expected retention rate. The time-dependent behavior of the system size probabilities and the ex- pected system size is also studied. Further, the variations in the expected system size, the expected reneging rate, and the expected retention rate with respect to the probability of retaining a reneging customer are also studied. Finally, the effect of discouragement in the same model is ana- lyzed.

Keywords: time-dependent analysis, single server queuing system, discouraged arrivals, reneging, Runge-Kutta method, retention

Introduction

Queuing systems are used in the design and analysis of computer-communication networks, production systems, surface and air traffic systems, service systems etc. The study of queueing systems help to manage waiting lines and to construct an optimal system for balancing customer waiting time with the idle time of the server Gnedenko and Kovalenko (1989). The enormous literature in queuing theory is available where the customers always wait in the queue until their service is completed. But in many practical situations customers become impatient and leave the systems without getting service. Therefore, queuing systems with customers' impatience have attracted a lot of attention. The study of customers' impatience in queueing theory is started in the early 1950's. Haight (1959) studies a single-server queue in steady-state with a Poisson input and exponential holding time, for various reneging distributions. Ancker and Gafarian [(1963a), (1963b)] analyze an $M/M/1/N$ queuing system with balking and reneging. In addition, the effect of reneging on an $M/M/1/N$ queue is investigated in the works of Abou El-Ata (1991), Zhang et al. (2006), Al Seddy et al. (2009), and Wang and Chang (2002). Kovalenko (1961) discusses some queuing systems with restrictions.

Queuing systems with discouraged arrivals are widely studied due to their significant role in managing daily queueing situations. In many practical situations, the service facility possesses defense mechanisms against long waiting lines. For instance, the congestion control mechanism prevents the formation of long queues in computer and communication systems by controlling the transmission rates of packets based on the queue length(of packets) at source or destination. Moreover, a long waiting line may force the servers to increase their rate of service as well as discourage prospective customers which results in balking. Hence, one should study queueing systems by taking into consideration the state-dependent nature of the system. In state-dependent queues the arrival and service rates depend on the number of customers in the system. The discouragement affects the arrival rate of the queueing system. Customers arrive in a Poisson fashion with rate that depends on the number of customers present in the system at that time i.e. $\frac{\lambda}{n+1}$. Morse (1958) considers discouragement in which the arrival rate falls according to a negative exponential law. Natvig (1974), Van Doorn (1981), Sharma and Maheswar (1993), and Parthasarathy and Selvaraju (2001) have also studied the discouraged arrivals queuing systems. Ammar et. al (2012) derive the transient solution of an M/M/1/N queuing model with discouraged arrivals and reneging by employing matrix method. Abdul Rasheed and Manoharan (2016) study a Markovian queueing system with discouraged arrivals and self-regulatory servers. They discuss the steady-state behavior of the system. Rykov (2001) considers a multi-server controllable queuing systems with heterogeneous servers. He studies several monotonicity properties of optimal policies for this system. Koba and Kovalenko (2002) study retrial queuing systems which are used in the analysis of aircraft landing process. Efrosinin and Rykov (2008) study a multi-server heterogeneous queuing system and obtain its steady-state solution. They derive the waiting and sojourn time distributions. They also study the optimal control of the queuing system. Rykov (2013) generalizes the slow server problem to include additional cost structure. He finds that the optimal policy for the problem has a monotone property. Sani et al. (2017) perform the reliability analysis of a system subjected to deterioration before failure. They use system state transition probabilities to derive the Markov models of the system.

Queuing systems with customers' impatience have negative impact on the performance of the system, because it leads to the loss of potential customers. Kumar and Sharma (2012a) take this practically valid aspect into account and study an $M/M/1/N$ queuing system with reneging and retention of reneging customers. Kumar (2013) obtains the transient solution of an $M/M/c$ queue with balking, reneging and retention of reneging customers. Kumar and Sharma (2014) obtain the steady-state solution of a Markovian single server queueing system with discouraged arrivals and retention of reneging customers by using iterative method.

The steady-state results do not reveal the actual functioning of the system. Moreover, stationary results are mainly used within the system design process and it cannot give insight into the transient behavior of the system. That is why, we extend the work of Kumar and Sharma (2014) in the sense that the time-dependent analysis of the model is performed. The time-dependent numerical behavior is studied by using a numerical technique Runge-Kutta method.

1 Queuing Model Description

In this section, we describe the queueing model. The model is based on following assumptions:

1. We consider a single-server queueing system in which the customers arrive in a Poisson fashion with rate that depends on the number of customers present in the system at that time i.e. $\frac{\lambda}{n+1}$.
2. There is single server and the service time distribution is negative exponential with parameter μ.
3. Arriving customers form a single waiting line based on the order of their arrivals and are served according to the first-come, first-served (FCFS) discipline.
4. The capacity of the system is finite (say N).
5. A queue gets developed when the number of customers exceeds the number of

servers, that is, when $n > 1$. After joining the queue each customer will wait for a certain length of time T (say) for his service to begin. If it has not begun by then he may get renege with probability p and may remain in the queue for his service with probability q $(= 1 - p)$ if certain customer retention strategy is used. This time T is a random variable which follows negative exponential distribution with parameter ξ. The reneging rate is given by

$$\xi_n = \begin{cases} 0, & 0 < n \leq 1 \\ (n-1)\xi, & n \geq 2 \end{cases}$$

2 Mathematical Model

Let $\{X(t), t \geq 0\}$ be the number of customers present in the system at time t. Let $P_n(t) = P\{X(t) = n\}, n = 0,1,\ldots$ be the probability that there are n customers in the system at time t. We assume that there is no customer in the system at $t = 0$.

The differential-difference equations of the model are:

$$\frac{dP_0(t)}{dt} = -\lambda P_0(t) + \mu P_1(t), \tag{1}$$

$$\frac{dP_n(t)}{dt} = -\left[\left(\frac{\lambda}{n+1}\right) + \mu + (n-1)\xi p\right] P_n(t) + \left(\frac{\lambda}{n}\right) P_{n-1}(t) + (\mu + n\xi p)P_{n+1}(t), 1 \leq n < N \tag{2}$$

$$\frac{dP_N(t)}{dt} = \left(\frac{\lambda}{N}\right) P_{N-1}(t) - (\mu + (N-1)\xi p)P_N(t), \tag{3}$$

3 Transient analysis of the model

In this section, we perform the time-dependent analysis of a finite capacity single-server Markovian Queuing model with discouraged arrivals and retention of reneging customers using Runge-Kutta method of fourth order (R-K 4). The "ode45" function of MATLAB software is used to find the time-dependent numerical results corresponding to the differential-difference equations of the model.

We study the following performance measures in transient state:

1. Expected System Size ($L_s(t)$)

$$L_s(t) = \sum_{n=0}^{N} nP_n(t)$$

2. Average Reneging Rate ($R_r(t)$)

$$R_r(t) = \sum_{n=1}^{N} (n-1)\xi p P_n(t)$$

3.Average Retention Rate ($R_r(t)$)

$$R_R(t) = \sum_{n=1}^{N} (n-1)\xi q P_n(t)$$

Now, we perform the time-dependent numerical analysis of the model with the help of a numerical example. We take $N = 10$, $\lambda = 2, \mu = 3, \xi = 0.1$, and $p = 0.4$. The results are presented in the form of Figures 1-5. Following are the main observations:

In Figure 1, the probabilities of number of customers in the system at different time points are plotted. We observe that the probability values $P_1(t)$, $P_2(t)$, ..., $P_{10}(t)$ increase gradually until they reach stable values except the probability curve $P_0(t)$ which decreases rapidly in the beginning and then attains steady-state with the passage of time.

Figure 2 shows the effect of the probability of retaining a reneging customer on the expected system size in transient state. One can observe that as the probability of retaining a reneging customer increases, the expected system size also increases. This establishes the role of probability of retention associated with any customer retention strategy.

In Figure 3, the change in average reneging rate with the change in probability of retaining a reneging customer is shown. One can observe that there is a proportional decrease in average reneging rate with the increase in probability of retention, q.

The variation in average retention rate with probability of retention is shown in Figure 4. We can see that there is a proportional increase in $R_R(t)$ with increase in q, which justifies the functioning of the model.

In figure 5, the impact of discouraged arrivals on the performance of the system is shown. We compare two single server finite capacity Markovian queuing systems having retention of reneging customers with and without discouraged arrivals. One can see from Figure 5 that the expected system size is always lower in case of discouraged arrivals as compare to the queuing model without discouragement.

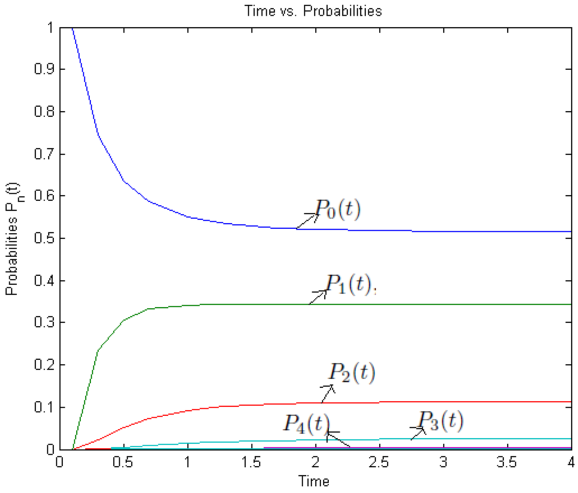

Figure 1: The probability values for different time points are plotted for the case $N = 10$, $\lambda = 2, \mu = 3, \xi = 0.1$, and $p = 0.4$

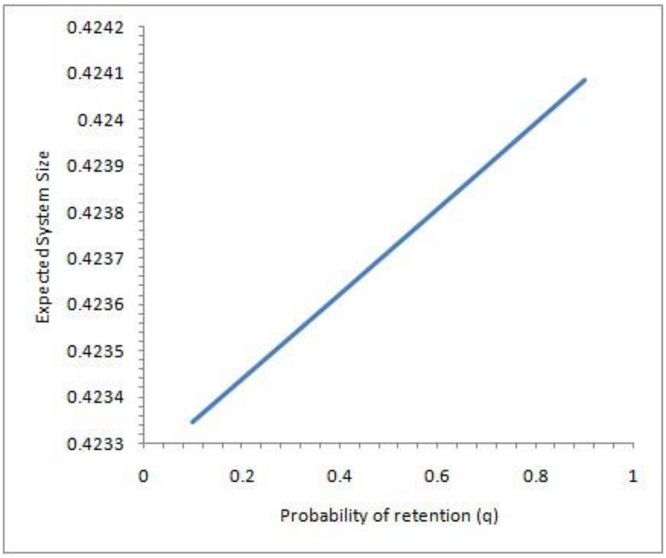

Figure 2: The expected system sizes versus probability of retention (q) are plotted for the case $N = 10$, $\lambda = 2, \mu = 3, \xi = 0.1, t = 0.5$, and $q = 0.1,0.2, \dots ,0.9$

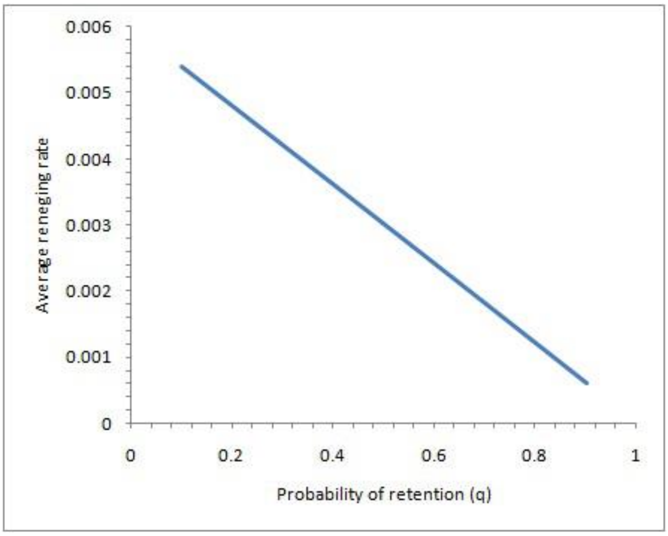

Figure 3: Variation of average reneging rate with the variation in probability of retention for the case $N = 10$, $\lambda = 2, \mu = 3, \xi = 0.1, t = 0.5$, and $q = 0.1,0.2, \dots ,0.9$

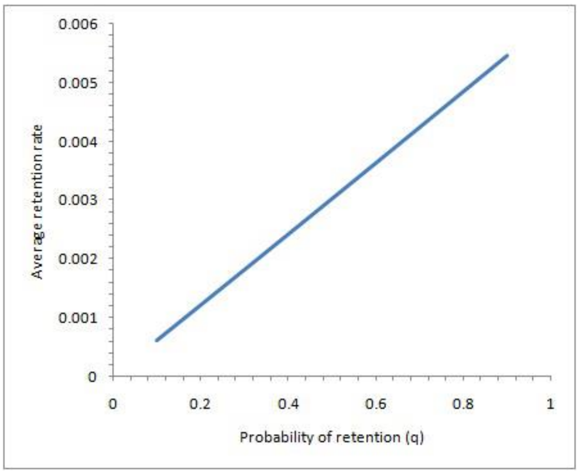

Figure 4: Variation of average retention rate with the variation in probability of retention for the case $N = 10$, $\lambda = 2, \mu = 3, \xi = 0.1, t = 0.5$, and $q = 0.1,0.2, \dots ,0.9$

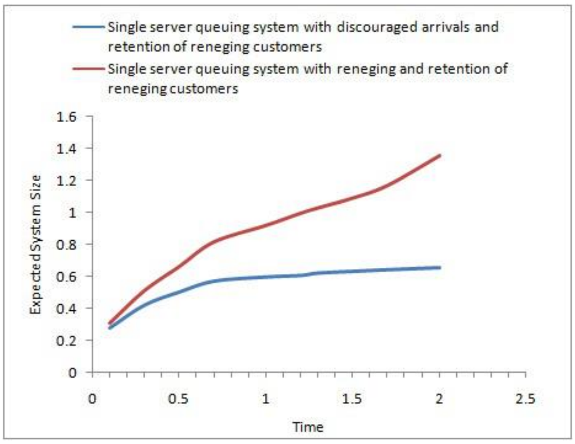

Figure 5: The impact of discouragement on expected system size

4 Conclusions

The time-dependent analysis of a single-server queuing system with discouraged arrivals, reneging and retention of reneging customers is performed by using Runge Kutta method. The numerical results are computed with the help of MATLAB software. The effect of probability of retaining a reneging customer on various performance measures is studied. We also study the impact of discouraged arrivals on the system performance.

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References

[1] Abdul Rasheed, K.V., Manoharan, M. (2016). Markovian queueing system with discouraged arrivals and self-regulatory servers. *Advances in Operations Research*, -Vol. 2016, Article ID 2456135, -P. 11 pages, doi:10.1155/2016/2456135.

[2] Abou-El-Ata, M.O. and Hariri, A.M.A. (1992). The M/M/c/N queue with balking and reneging. *Computers and Operations Research*, 19:713-716.

[3] Al-Seedy, R.O., El-Sherbiny, A.A., El-Shehawy, S.A, Ammar, S.I. (2009). Transient solution of the M/M/c queue with balking and reneging. *Computers and Mathematics with Applications*, 57:1280-1285.

[4] Ammar, S.I., El-Sherbiny, A.A., Al-Seedy, R.O. (2012). A matrix approach for the transient solution of an M/M/1/N queue with discouraged arrivals and reneging, *International Journal of Computer Mathematics*, 89:482-491.

[5] Ancker. Jr., C.J., Gafarian A.V. (1963a). Some queuing problems with balking and reneging I. *Operations Research*, 11:88-100

[6] Ancker. Jr., C.J., Gafarian A.V. (1963b). Some queuing problems with balking and reneging II. *Operations Research*, 11:928-937.

[7] Efrosinin, D., Rykov, V. (2008). On performance characteristics for queueing systems with heterogeneous servers. *Automation and Remote Control*, 69:61-75.

[8] Gnedenko, B.V., Kovalenko, I. N. (1989). Introduction to Queuing Theory. *Birkhäuser, 2nd Edition*.

[9] Haight, F. A. (1959). Queuing with reneging. *Metrika*, 2:186-197.

[10] Koba, E. V., Kovalenko, I. N. (2002). Three Retrial Queueing Systems Representing Some Special Features of Aircraft Landing. *Journal of Automation and Information Sciences*, 34:DOI · 10.1615/JAutomatInfScien.v34.i4.10, 4 pages.

[11] Kovalenko, I. N. (1961). Some Queuing Problems with Restrictions. *Theory of Probability and Its Applications*, 6:204-208.

[12] Kumar, R. and Sharma, S. K. (2012a). An M/M/1/N queuing system with retention of reneged customers. *Pakistan Journal of Statistic and Operation Research*, 8:859-866.

[13] Kumar, R. and Sharma, S. K. (2014). A single-server Markovian queuing system with discouraged arrivals and retention of reneged customers. *Yugoslav Journal of Operations Research*, 24:119-216.

[14] Kumar, R. (2013). Economic analysis of an M/M/c/N queuing model with balking, reneging and retention of reneged customers. *Opsearch*, 50:383-403.

[15] Morse, P.M. Queues, inventories and maintenance. Wiley, New York, 1958.

[16] Natvig, B. (1974). On the transient state probabilities for a queueing model where potential customers are discouraged by queue length. *Journal of Applied Probability*, 11:345-354.

[17] Parthasarathy, P.R. and Selvaraju, N. (2001). Transient analysis of a queue where potential customers are discouraged by queue length. *Mathematical Problems in Engineering*, 7:433-454.

[18] Rykov, V. (2001). Monotone Control of Queueing Systems with Heterogeneous Servers. *Queueing Systems*, 37391-403.

[19] Rykov, V. (2013). On a Slow Server Problem. *Stochastic Orders in Reliability and Risk*, 351-361.

[20] Sani, B., Gatawa, R. I. and Yusuf, I. (2017). Reliability Assessment of Deteriorating System. *Reliability: Theory and Applications*, 12:20-29.

[21] Sharma, O.P., Maheswar, M.V.R. (1993). Transient behavior of a simple queue with discouraged arrivals. *Optimization*, 27:283-291.

[22] Van Doorn, E. A. (1981). The transient state probabilities for a queueing model where potential customers are discouraged by queue length. *Journal of Applied Probability*, 18:499-506.

[23] Wang, K.H., Chang, Y.C. (2002). Cost analysis of a finite M/M/R queueing system with balking, reneging and server breakdowns. *Mathematical Methods of Operations Research* 56:169-180.

[24] Zhang, Y., Yue, D., Yue, W. (2006). Optimal performance analysis of an M/M/1/N queue system with balking, reneging and server vacation. *International Journal of Pure and Applied Mathematics*, 28:101-115.

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