Reliability Function of Renewable System under Marshall-Olkin Failure Model

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Abstract

In this note we obtain reliability function of two-component system under the Marshall-Olkin failure model in terms of Laplace transform. The problem of its sensitivity to the shape of the system components repair times is investigated as well.
Keywords: Heterogeneous reliability systems, Laplace transform, Marshall-Olkin bivariate failure model, reliability function, sensitivity analysis.

1 Introduction and Motivation

The stability of system characteristics with respect to the changes in initial states or external factors are the key problems for all natural sciences. For stochastic systems stability is often identified by insensitivity or low sensitivity of their output characteristics to the shapes of some input distributions.

One of the earliest results concerning insensitivity of system characteristics to the shape of service time distribution has been obtained in 1957 by Sevast'yanov [1], who established the insensitivity of Erlang formulas to the shape of service time distribution with fixed mean value for loss queueing systems with Poisson input flow. In 1976, Kovalenko [2] found necessary and sufficient conditions for insensitivity of stationary probabilities of redundant renewable systems, whose components have exponential life time and repair time distributions of general type. These conditions consist in a huge amount of repairing facilities. The sufficiency of immediate start to repair any failed element in the case of general life and repair time distributions has been found in 2013 by Rykov [3] with the help of multi-dimensional alternative processes theory. However, in the case of limited possibilities for recovering these results do not hold, as it was shown in [4].

On the other hand, in series of works Gnedenko (1964) and Solov'ev (1970) (see, e.g. [5, 6, 7]) show that under "quick" restoration the reliability function of a cold standby double redundant homogeneous system tends to the exponential one for any life and repair time distributions of its components. These results also imply the asymptotic insensitivity of the reliability characteristics of such system to the shape of their components life and repair times distributions. An alternative approach based on system states merging has been proposed by V. Korolyuk, see [8] and references therein.

Very recently, the problem of asymptotic insensitivity of reliability function for redundant systems to the shape of their components repair time distribution under condition of rare failures has been considered by Rykov and Kozyrev in [9, 10, 11, 12] using the Markovization method. All these studies describe the system with independently functioning components. We will relax this assumption in the present paper since the common environment implies some kind of dependence between elements of the system.

In 1967 Marshall and Olkin [13] proposed a bivariate distribution that can be used as a failure model for two-component reliability system with dependent components. The Marshall-Olkin (MO hereafter) model is specified by the stochastic representation

$$(T_1, T_2) = (\min(A_1, A_3), \min(A_2, A_3)), \quad (1)$$

where non-negative continuous random variables (r.v.) A_1 and A_2 represent times to occurrence of independent "individual shocks" affecting two devices and A_3 represents time to their "common shock" under assumption that the times to all shocks are independent and exponentially distributed. The joint distribution of random vector (T_1, T_2) can be characterized by the bivariate lack of memory property (BLMP) defined by the functional equation

$$S(x+t, y+t) = S(x,y)S(t,t), \quad for\ all \quad x, y, t \geq 0,$$

where $S(x,y)$ is the joint survival function of the pair (T_1, T_2). Many textbooks give a special attention to the BLMP and related MO bivariate exponential distribution exhibiting singularity along the main diagonal in R_+^2, see Barlow and Proschan [14] (1981), Singpurwalla [15] (2006), Balakrishnan and Lai [16] (2009), Gupta et al. [17] (2010), McNeil et al. [18] (2015) among others. Many articles complement and extend the MO model, justifying advantages in analysis of various data sets from engineering, medicine, insurance, finance, biology, etc. For example, Li and Pellerey [19] (2011) launched the Generalized MO model considering non-exponential independent random variables A_i in (1), $i = 1,2,3$. The corresponding joint distributions do not possess BLMP, i.e., are "aging". In 2014 the model is extended to the multidimensional case by Lin and Li [20]. As a further step, in 2015 Pinto and Kolev [21] introduced the Extended MO model assuming dependence between variables A_1 and A_2, but keeping A_3 independent of them in (1). The motivation is that the individual shocks might be dependent if the items share a common environment. In this case however, BLMP may be fulfilled or not depending on parameters of joint distribution of (A_1, A_2) and distribution of A_3.

Most of these investigations deal with bivariate distributions and their properties and use the MO model for the case of explicit failure. So far, the MO model has been not applied in the context of system reliability. In the present paper we consider a renewable heterogeneous double redundant standby renewable systems, where the failures of elements follow the MO model. The reliability function in terms of its Laplace transforms will be calculated. In this case the renovation procedure after the system components failures is very important and it will be included into the model.

The paper is organized as follows. In the next section the problem setting and some notations will be introduced. In the section 3 the reliability function is calculated in terms of its Laplace transforms, and in the next 4-th section its asymptotic insensitivity to the shape of the system components repair time distributions will be considered. The paper ends with conclusions.

2 Problem setting and notations

Consider a heterogeneous hot double redundant repairable reliability system, graphically represented on figure 1.

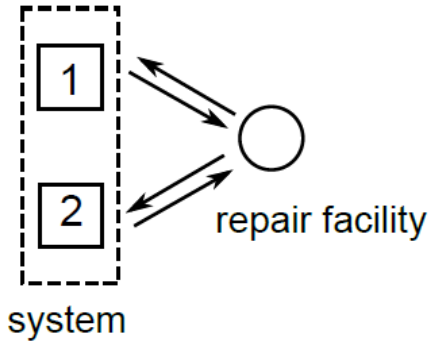

Figure 1: 2-unit hot-standby repairable system with one repair facility

We assume that component failures follow the MO model. This means that there exists three sources of shocks, which lead to the system failure. The first shock act only to the first component (identified by r.v. A_1), the second one act only to the second one (identified by r.v. A_2), while the third one (represented by r.v. A_3) act to both components and provokes a system failure. Thus, accordingly to the MO failure model (1), the system lifetime is determined by the joint distribution of (T_1, T_2), where A_1, A_2 and A_3 are independent r.v.'s.

Dealing with reparable model we need to propose some procedure of recovering. Let the repair time B_i of i-th component has absolute continuous distribution with cumulative distribution function $B_i(x)$ and probability density functions $b_i(x)$, correspondingly, $i = 1{,}2$. All repair times are assumed to be independent.

In order to describe the system behavior after its partial failure, when only one of components fails it is necessary to generalize the MO model. Note that there are at least two scenarios. The first one supposes that if one component fails and during its repair a non-fatal shock can arise leading to failure of another component which results in system breakdown. The second option is that a common shock also can arise, and it leads to the full system failure.

We will use the following notations. $\alpha = \alpha_1 + \alpha_2 + \alpha_3$ the summary intensity of failures; $\bar{\alpha}_i = \alpha_i + \alpha_3, (i = 1{,}2)$; $b_i = \int_0^\infty (1 - B_i(x))dx$ the i-th r.v. B_i $(i = 1{,}2{,}3)$ expectations; $\rho_i = \alpha_i b_i, i = 1{,}2{,}3$; $\beta_i(x) = (1 - B_i(x))^{-1} b_i(x)$ the i-th r.v. conditional repair intensity given elapsed repair time is x for $(i = 1{,}2{,}3)$; $\tilde{b}_i(s) = \int_0^\infty e^{-sx} b_i(x)dx$ the Laplace transform (LT) of the i-th component repair time distribution $(i = 1{,}2)$.

Under considered assumptions the state space of the system can be represented as $E = \{0, 1, 2, 3\}$, which means: 0 — both components are working, 1 — the first component has failed and is being repaired while the second one is working, 2 — the second component has failed and is being repaired while the first one is working, 3 — both components are in failure (down) states, system has failed and is being repaired.

In this paper we are interested in the *reliability function*

$$R(t) = P\{T > t\},$$

where T denotes the system life time.

3 Reliability Function

We will use the so-called Markovization method to calculate the system reliability function. Specifically, let us consider two-dimensional absorbing Markov process $Z = \{Z(t), t \geq 0)\}$, with $Z(t) = (J(t), X(t))$ where $J(t)$ represents the system state, and $X(t)$ is an additional variable, which means the elapsed repair time of $J(t)$-th component at time t. The process phase space is given by $E = \{0, (1, x), (2, x), 3\}$, which mean: 0 – both components are working, $(1, x)$ – the second component is working, the first one is failed and repairing, and its elapsed repair time equal to x, $(2, x)$ – the first component is working, the second one is failed and repairing, and its elapsed repair time equal to x, 3 – both components are failed, and therefore the system is failed. Corresponding probabilities are denoted by $\pi_0(t), \pi_1(t; x), \pi_2(t; x), \pi_3(t)$. The state transition graph of the system is represented on figure 2.

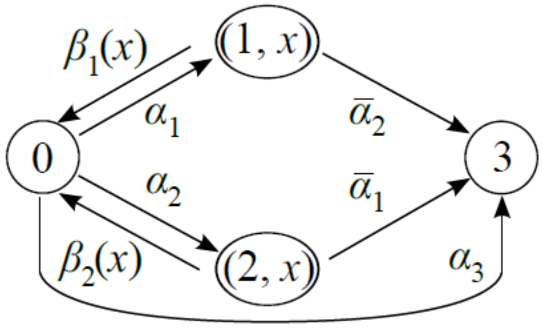

Figure 2: Absorbing system transition graph.

Under the above assumptions, the following statement is true.

Theorem 1 *The system reliability function Laplace transform is given by*

$$\tilde{R}(s) = \frac{(s+\bar{\alpha}_1)(s+\bar{\alpha}_2)+(s+\bar{\alpha}_1)\phi_1(s)+(s+\bar{\alpha}_2)\phi_2(s)}{(s+\bar{\alpha}_1)(s+\bar{\alpha}_2)[s+\phi_1(s)+\phi_2(s)+\alpha_3]} \quad (2)$$

where $s > 0$ and

$$\phi_i(s) = \alpha_i(1 - \tilde{b}_i(s + \bar{\alpha}_{i*})), \quad i = 1,2 \quad (3)$$

with $i^* = 2$ if $i = 1$ and vice versa.

Proof. Applying the usual method of comparing the process probabilities at closed times t and $t + \Delta$ the system of Kolmogorov forward partial differential equations can be written as follows

$$\begin{aligned} &ddt\pi_0(t) = -\alpha\pi_0(t) + \int_0^t \pi_1(t, x)\beta_1(x)dx + \int_0^t \pi_2(t, x)\beta_2(x)dx; \\ &(\partial\,\partial t + \partial\,\partial x)\pi_1(t; x) = -(\bar{\alpha}_2 + \beta_1(x))\pi_1(t; x); \\ &(\partial\,\partial t + \partial\,\partial x)\pi_2(t; x) = -(\bar{\alpha}_1 + \beta_2(x))\pi_2(t; x); \\ &ddt\pi_3(t) = \alpha_3\pi_0(t) + \bar{\alpha}_1 \int_0^t \pi_2(t; x)dx + \bar{\alpha}_2 \int_0^t \pi_1(t; x)dx, \end{aligned} \quad (4)$$

taking into account the initial $\pi_0(0) = 1$ and boundary conditions

$$\pi_1(t, 0) = \alpha_1\pi_0(t), \quad \pi_2(t, 0) = \alpha_2\pi_0(t). \quad (5)$$

To solve this system we use the method of characteristics for solving first-order partial differential equations, consult [22]. According to this method we obtain[17]

$$\begin{aligned} &\pi_1(t; x) = h_1(t - x)e^{-\bar{\alpha}_2 x}(1 - B_1(x)), \quad x \le t; \\ &\pi_2(t; x) = h_2(t - x)e^{-\bar{\alpha}_1 x}(1 - B_2(x)), \quad x \le t, \end{aligned} \quad (6)$$

[17] The represented below solutions have a nice probabilistic interpretation. The functions $h_i(\cdot)$ can be considered as renewal densities of the process returning to the states with zero elapsed times, and two other multipliers show that during the time x neither failure, nor repair occurs.

and from boundary conditions (5) it holds

$$\pi_1(t;0) = h_1(t) = \alpha_1\pi_0(t), \quad \pi_2(t;0) = h_2(t) = \alpha_1\pi_0(t). \tag{7}$$

Substitution of these solutions to the first equation in (7) gives

$$ddt\pi_0(t) = -\alpha\pi_0(t) + \int_0^t \, h_1(t-x)e^{-\bar{\alpha}_2 x}b_1(x)dx +$$
$$+\int_0^t \, h_2(t-x)e^{-\bar{\alpha}_1 x}b_1(x)dx.$$

In terms of Laplace transform with $\pi_0(0) = 1$ we have

$$(s+\alpha)\tilde{\pi}_0(s) - 1 = \tilde{h}_1(s)\tilde{b}_1(s+\bar{\alpha}_2) + \tilde{h}_2(s)\tilde{b}_2(s+\bar{\alpha}_1).$$

Substitution into this equation the Laplace transform of the boundary conditions (7)

$$\tilde{h}_1(s) = \alpha_1\tilde{\pi}_0(s), \;\; \tilde{h}_2(s) = \alpha_2\tilde{\pi}_0(s),$$

after some algebra we get

$$(s+\alpha)\tilde{\pi}_0(s) - \alpha_1\tilde{b}_1(s+\bar{\alpha}_2)\tilde{\pi}_0(s) - \alpha_2\tilde{b}_2(s+\bar{\alpha}_1)\tilde{\pi}_0(s) = 1.$$

From this equality one can find $\tilde{\pi}_0(s)$ in the following form:

$$\tilde{\pi}_0(s) = [s + \phi_1(s) + \phi_2(s) + \alpha_3]^{-1}, \tag{8}$$

where for simplicity the notations (3) are used.

To find $\tilde{\pi}_3(s)$ we apply the Laplace transform (8) in the last equation of the system (??). Taking into account the expressions (??) for probabilities $\pi_i(t;x)$ for $i = 1,2$ we obtain

$$s\tilde{\pi}_3(s) = \alpha_3\tilde{\pi}_0(s) + \bar{\alpha}_2\tilde{h}_1(s)\frac{1-\tilde{b}_1(s+\bar{\alpha}_2)}{s+\bar{\alpha}_2} +$$
$$+\bar{\alpha}_1\tilde{h}_2(s)\frac{1-\tilde{b}_2(s+\bar{\alpha}_1)}{s+\bar{\alpha}_1}.$$

By substituting instead of $\tilde{h}_i(s)$ its representation in terms of $\tilde{\pi}_0(s)$ we get

$$s\pi_3(s) = \tilde{\pi}_0(s)\left(\frac{\bar{\alpha}_2}{s+\bar{\alpha}_2}\phi_1(s) + \frac{\bar{\alpha}_1}{s+\bar{\alpha}_1}\phi_2(s) + \alpha_3\right).$$

Finally, since

$$\tilde{R}(s) = 1s - \tilde{\pi}_3(s),$$

we arrive to

$$\tilde{R}(s) = 1s\left(1 - \frac{\bar{\alpha}_2 s + \bar{\alpha}_2\phi_1(s) + \bar{\alpha}_1 s + \bar{\alpha}_1\phi_2(s) + \alpha_3}{[s+\phi_1(s)+\phi_2(s)+\alpha_3]}\right)$$
$$= \frac{(s+\bar{\alpha}_1)(s+\bar{\alpha}_2) + (s+\bar{\alpha}_1)\phi_1(s) + (s+\bar{\alpha}_2)\phi_2(s)}{(s+\bar{\alpha}_1)(s+\bar{\alpha}_2)[s+\phi_1(s)+\phi_2(s)+\alpha_3]},$$

which ends the proof.

As a corollary, by a substitution $s = 0$ we find the mean time to the system failure.

Corollary 1 *The mean system life time is given by*

$$E[T] = \tilde{R}(0) = \frac{\bar{\alpha}_1\bar{\alpha}_2 + \bar{\alpha}_1\alpha_1(1-\tilde{b}_1(\bar{\alpha}_2)) + \bar{\alpha}_2\alpha_2(1-\tilde{b}_2(\bar{\alpha}_1))}{\bar{\alpha}_1\bar{\alpha}_2[\alpha_1(1-\tilde{b}_1(\bar{\alpha}_2)) + \alpha_2(1-\tilde{b}_2(\bar{\alpha}_1))]} \tag{9}$$

Remark 1 *Note that in homogeneous case, when all failure parameters are equal ($\alpha_1 = \alpha_2 = \alpha_3 = \alpha$) the system mean time to failure simplifies to*

$$m_T = E[T] \approx 1\alpha.$$

4 Rare failures

The above formulas demonstrate the evident dependence of the reliability function on the shape of repair time distribution. It is expressed in the form of Laplace transform of the repair time distribution at points of elements' failure intensities.

On the other side, as it was mentioned in the introduction, for systems with independent component failures in case of quick restoration of components the system reliability function tends to the exponential one for any repair time distribution. Here we consider the behavior of the considered system reliability function under MO failure model with condition of "rare" failures instead of "quick" restorations.

For the considered model the rare failures should be understood as the slow intensity of failures with respect to the fixed repair times. Thus we will suppose that $q = \max\{\alpha_1, \alpha_2, \alpha_3\} \to 0$. Naturally the asymptotic analysis should be done with respect to a certain scale parameter. In the place of such a parameter the asymptotic mean lifetime value will be considered.

Using (3) and relations $\rho_i = \alpha_i b_i$ one can find that

$$\phi_i(0) = \alpha_i(1 - \tilde{b}_i(\bar{\alpha}_{i^*})) \approx \rho_i \bar{\alpha}_{i*}$$

as $q \to 0$, and therefore, the mean value of the system time to failure from (9) is

$$m = E[T] = \tilde{R}(0) = \frac{\bar{\alpha}_1\bar{\alpha}_2 + \bar{\alpha}_1\phi_1(\bar{\alpha}_2) + \bar{\alpha}_2\phi_2(\bar{\alpha}_1)}{\alpha_1\alpha_2(\phi_1(\bar{\alpha}_2) + \phi_2(\bar{\alpha}_1)) + \alpha_3} =$$
$$= \frac{1 + \rho_1 + \rho_2}{\bar{\alpha}_1\rho_2 + \bar{\alpha}_2\rho_1 + \alpha_3}.$$

Theorem 2 *Under rare components' failures the system reliability function becomes asymptotically insensitive to the shapes of their repair distributions. Moreover the reliability function for the considered model in scale of $m = E[T]$ has unit exponential distribution, i.e.,*

$$\lim_{q \to 0} P\{Tm > t\} = e^{-t}.$$

Proof. Instead of the large parameter m we consider the small parameter $\gamma = m^{-1}$. We are interested on the asymptotic behavior of the reliability function of the system

$$R(t\gamma) = P\{\gamma T > t\}$$

when $\gamma \to 0$. To do that, we investigate the asymptotic behavior of its Laplace transform

$$\gamma\tilde{R}(\gamma s) = \gamma \frac{(\gamma s + \bar{\alpha}_1)(\gamma s + \bar{\alpha}_2) + (\gamma s + \bar{\alpha}_1)\phi_1(\gamma s) + (\gamma s + \bar{\alpha}_2)\phi_2(\gamma s)}{(\gamma s + \bar{\alpha}_1)(\gamma s + \bar{\alpha}_2)[\gamma s + \phi_1(\gamma s) + \phi_2(\gamma s) + \alpha_3]} =$$
$$= \frac{1 + \phi_1(\gamma s)\gamma s + \bar{\alpha}_2 + \phi_2(\gamma s)\gamma s + \bar{\alpha}_1}{\gamma s + \phi_1(\gamma s) + \phi_2(\gamma s) + \alpha_3}.$$

When $\gamma \to 0$, it holds that

$$\phi_i(\gamma s) = \alpha_i(1 - \tilde{b}_i(\gamma s + \bar{\alpha}_i)) \approx \alpha_i b_i(\gamma s + \bar{\alpha}_{i*}) = \rho_i(\gamma s + \alpha_{i*}).$$

Therefore, $\phi_i(\gamma s)\gamma s + \bar{\alpha}_{i*} \approx \rho_i$ and the last relation yields

$$\gamma\tilde{R}(\gamma s) = \gamma \frac{1+\rho_1+\rho_2}{\gamma s(1+\rho_1+\rho_2)+\rho_1\bar{\alpha}_2+\rho_2\bar{\alpha}_1+\alpha_3} = \gamma\gamma(s+1) = 1s+1.$$

So, when $\gamma \to 0$ it follows that

$$P\{\gamma T > t\} = R(t\gamma) \to e^{-t}.$$

5 Conclusions

We focus on assessing and study of the system-level reliability of a heterogeneous double redundant renewable system under Marshall-Olkin failure model in the case when repair times of its components have a general continuous distribution. The proposed mathematical model allows to obtain the explicit expression in terms of Laplace transform for the system reliability function. The produced analytical results reveal asymptotic insensitivity of the reliability function of the system under the 'rare' failures of its elements to the shape of their repair time distribution. In addition, we showed that when the scale parameter is mean time to failure, the system's reliability function converge to the unit exponential law.

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