Generalization and Extension of Burke Theorem

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Abstract

A simplification of Burke theorem proof [1] and its generalizations for queuing systems and networks are considered. The proof simplification is based on the fact that points in output flow take place in moments when Markov process of customers number in queuing system has jumps down. In such way it is possible to obtain a property of the mutual independence of the flow into disjoint periods of time and to calculate intensity of output flow.

Keywords: an output Poisson flow, the Jackson network, the Burke theorem

1 New proof of Burke theorem

In [1] we prove the following statement: in queung system $M|M|n|\infty$ in stationary state, the output flow has the same distribution as the input flow. Recently, however, interest in the study of flows in queuing systems is increased. Now it is necessary to give a more compact and convenient for generalizations proof of this theorem.

A random sequence of points will be called a Poisson flow with continuously differentiable intensity $\lambda(t),\ t \geq 0$, if the following conditions are satisfied [2, page 12, 13], [3, page 20, 35 -- 37]:

a) the probability of the existence of the point of flow on the time interval $[t, t+h)$ does not depend on the location of the points of the stream up to the time t (this property is called lack of follow-through and expresses the mutual independence of the flow stream into disjoint periods of time);

b) the probability that a flow point appears in the semi-interval $[t, t+h)$ is $\lambda(t)h + o(h)$, $h \to 0$;

c) the probability of occurrence of two or more flow points in the range $[t, t+h)$ is $o(h)$, $h \to 0$.

Let the system $A_n = M|M|n|\infty$ of the Poisson input flow has an intensity $\lambda > 0$, and the service time has an exponential distribution with the parameter $\mu > 0,\ 1 \leq n < \infty$. Denote $P_{k,n}(t)$, $k \geq 0$, distribution of the number of customers in the system at the time t.

Theorem 1. *The output flow in queuing system A_n is Poisson with intensity $a(t) = \sum_{0<k} \mu P_{k,n}(t) min(k, n)$.* Let the output flow $T_n = \{0 \leq t_1 < t_2 < \cdots\}$ be A_n described by a random function $y_n(t)$ equal to the number of points of this stream on the segment $([0, t)$. Denote $x_n(t)$ the number of customers in the system A_n at the time t. It is known that a random process $x_n(t)$ is Markov process (of death and birth of [3, Chapter II, $$]), with each point of the T_n flow corresponding to the time of the jump down process $x_n(t)$. Therefore, the output flow T_n satisfies the condition a). In turn, the condition b) follows from the equalities:

$$P(y_n(t+h)=y_n(t)+1)=\sum_{k=1}^{n} P(y_n(t+h)=y_n(t)+1/x_n(t)=k)P_{k,n}(t)+$$

$$+P(y_n(t+h=y_n(t)+1/x_n(t)>n)\sum_{k>n} P_{k,n}(t)=$$

$$=\sum_{k=1}^{n} P_{k,n}(t)(k\mu h+o_k(h))+\sum_{k>n} P_{k,n}(t)(n\mu h+o_0(h))=a(t)\mu h+o(h),$$

where for $h\to 0$ we have $o_k(h)/h\to 0,\ k=0,\dots,n,\ o_0(h)/h\to 0$,

$$o(h)=\sum_{k=1}^{n} P_{k,n}(t)o_k(h)+\sum_{k>n} P_{k,n}(t)o_0(h),\ o_0(h)/h\to 0.$$

Thus, the output flow T_n satisfies the condition b). Check of condition c) is quite obvious.

Theorem 2. *In queuing system A_n, when the ergodicity condition $\lambda<\mu$ is satisfied and the process $x_n(t)$ is stationary, the output flow is Poisson with intensity λ.* Due to the steady state of the process $x_n(t)$ the distribution of the number of customers $P_{k,n}(t)\equiv P_{k,n}$ in the queuing system A_n at $\rho=\lambda/\mu$ satisfies the equations [3, page 93]:

$$P_{k,n}=P_{0,n}k!\,\rho^k,\ 0\le k\le n,\ P_{k,n}=P_{0,n}n^n n!\,(\rho n)^k,\ k>n,$$

$$P_{0,n}^{-1}=\sum_{k=0}^{n} 1k!\,\rho^k+\sum_{k>n} n^n n!\,(\rho n)^k.$$

It is not difficult to obtain the following equality by simple algebraic calculations: $a(t)\equiv\sum_{k\ge 0}\mu P_{k,n}\min(k,n)=\mu\rho=\lambda$. Therefore, Theorem 1 follows the validity of Theorem 2.

Remark. Using the scheme of the proof of Theorem 2, it is possible to extend the results to output flows of systems with limited queue, with priority service, with unreliable servers [3, $$].

Theorem 3. *When the $\lambda>n\mu$ inequality is executed, the output flow T_n in the system A_n is Poisson with intensity $a(t)\to n\mu,\ t\to\infty$.* We introduce independent random variables $u_n(t),\ v_n(t)$, having Poisson distributions with parameters $\lambda t,\ \mu t$, respectively. It is easy to establish that with probability unit the inequality $x_n(t)\ge w_n(t)=u_n(t)-v_n(t)$ is valid, hence the following inequalities are fulfilled:

$$P(x_n(t)\le n)\le P(w_n(t)\le n)=P(w_n(t)-Mw_n(t)\le n-Mw_n(t))=$$

$$=P(Mw_n(t)-w_n(t)\ge Mw_n(t)-n)\le P(|Mw_n(t)-w_n(t)|\ge Mw_n(t)-n),$$

where $Mw_n(t)=(\lambda-\mu)t>0,\ t>0,\ Dw_n(t)=(\lambda+\mu)t$. Assume that $(\lambda-\mu)t-n>0$, then, due to Chebyshev's inequality, we have:

$$P(|Mw_n(t)-w_n(t)|\ge Mw_n(t)-n)\le Dw_n(t)(Mw_n(t)-n)^2=(\lambda+\mu)t((\lambda-\mu)t-n)^2\to 0,$$
$t\to\infty$.

Finally we get the ratio:

$$P(x_n(t)\le n)\le(\lambda+\mu)t((\lambda-\mu)t-n)^2\to 0,\ t\to\infty,$$

and then $\sum_{0<k\le n} P_{k,n}(t)\to 0,\ t\to\infty$. It follows that the limit ratio is met:

$$a(t)=\sum_{0<k\le n} P_{k,n}(t)k\mu+\left(1-\sum_{0<k\le n} P_{k,n}(t)\right)n\mu\to n\mu,\ t\to\infty.$$

From Theorem 1 and the last relation we obtain the statement of Theorem 3.

2 Poisson flows in stationary queuing networks

Consider an open queuing network (Jackson network) S with a Poisson input flow of intensity λ_0, consisting of a finite number of nodes $k=0,1,\dots,m$ with exponentially distributed service times. The dynamics of the movement of customers in the network is set by the route matrix $\Theta=||\theta_{i,j}||_{i,j=0}^{m}$, where $\theta_{i,j}$ is the probability of customer transition after service in the i-th node to j-th node, $\theta_{0,0}=0$, where the node 0 is an external source and at the same time a drain for customers leaving the network. The i node contains $l_i<\infty$ servers, the service time of which has an exponential distribution with the parameter $\mu_i,\ i=1,\dots,m$.

Assume that route matrix $\Theta=||\theta_{i,j}||_{i,j=0}^{m}$ is indecomposable, i.e.

$$\forall\ i,\ j\in\{0,\dots,m\}\ \exists\ i_1,\dots,i_r\in\{0,\dots,m\}:\ \theta_{i,i_1}>0,\theta_{i_1,i_2}>0,\dots,\theta_{i_r,j}>0.$$

Then for a fixed $\lambda_0 > 0$, the system of linear algebraic equations for intensities of fluxes coming from nodes of S

$$\lambda_k = \lambda_0\theta_{0,k} + \sum_{t=1}^{m} \lambda_t\theta_{t,k},\ k = 1, \dots, m \quad (1)$$

has the only solution $(\lambda_1, \dots, \lambda_m)$ with $\lambda_1 > 0, \dots, \lambda_m > 0$, [4, p. 13].

The system (1) is called the system of balance relations and plays an important role in the formulation and the proof of the product Jackson theorem [5], widely used in queuing theory. If $\lambda_i < l_i\mu_i,\ i = 1, \dots, m$, then the discrete Markov process $(n_1(t), \dots, n_m(t)),\ t \geq 0$, describing the number of customers in the network nodes has a limiting distribution $P_S(k_1, \dots, k_m)$, independent of initial conditions and representable in the form $P_S(k_1, \dots, k_m) = \prod_{i=1}^{m} P_i(k_i)$, where $P_i(k_i)$ is the limiting distribution of the number of customers in a stand-alone l_i - channel queuing system with Poisson input flow of intensity $\lambda_i,\ i = 1, \dots, m$.

In [6] network S is mapped to a directed graph G with edges corresponding to positive elements of the route matrix. Let's call the vertex set $U \subseteq \{0,1, \dots, m\}$ irrevocable if from any node not included in U, there is no edge to the node belonging to U. Then all flows passing through the edges from the node set U to the node set $\{0,1, \dots, m\}\backslash U$, are independent and Poisson.

Theorem 4. *Flow T_S^i, coming out of node i of open queuing network S, with stationary process $(n_1(t), \dots, n_m(t)),\ t \geq 0$, is Poisson with intensity $\lambda_i,\ i = 1, \dots, m$.* Indeed, the points of the flow T_S^i, exiting the i, node are the moments of jumps down the $n_i(t)$ component of the discrete Markov process $(n_1(t), \dots, n_m(t),\ t \geq 0$. Hence the flow T_S^i satisfies the condition a). Conditions b), c) are checked similarly to the proof of Theorem 1. Note that the limit probability that the i node contains k_i of customers is $P_i(k_i)$, and the flow rate T_S^i is $\lambda_i,\ i = 1, \dots, m$.

Theorem 5. *Flows $T_S^i,\ i = 1, \dots, m$, are independent.* From Theorem 4 and independence of stationary random variables $n_j(t),\ j = 1, \dots, m$, it follows that the union $T_S = \cup_{j=1}^{m} t_S^j$ of flows leaving the nodes of open queuing network S is also Poisson flow with intensity $\lambda_\Sigma = \sum_{j=1}^{m} \lambda_j$. And each point of the combined flow T_S belongs to the flow T_S^i with probability $p_i = \lambda_i\lambda_\Sigma$.

Lemma 1. *Let $\Lambda = \{0 \leq t_1 \leq t_2 \leq \cdots\}$ is a Poisson flow of intensity λ_Σ, each point of which, regardless of other points with probability p_i becomes a flow Λ_i point, $i = 1, \dots, m$. Then flows $\Lambda_1, \dots, \Lambda_m$ are Poisson with intensities $\lambda_\Sigma p_1, \dots, \lambda_\Sigma p_m$ and independent.* Without limitation of generality it is enough to limit ourselves to the case of $m = 2$. Take an arbitrary segment $[t, t + T],\ 0 \leq t,\ 0 < T$ and denote $n,\ n_1,\ n_2$ the number of flow points $\Lambda,\ \Lambda_1,\ \Lambda_2$ on this segment, respectively. Calculate the probability

$$\begin{aligned} P(n_1 = k_1, \quad n_2 = k_2) &= P(n = k_1 + k_2)C_{k_1+k_2}^{k_1}p_1^{k_1}p_2^{k_2} \\ &= e^{-\lambda T}(\lambda T)^{k_1+k_2}(k_1 + k_2)!\,(k_1 + k_2)!\,k_1!\,k_2!\,p_1^{k_1}p_2^{k_2} = \\ &= e^{-\lambda T p_1}(\lambda T p_1)^{k_1}k_1! \cdot e^{-\lambda T p_2}(\lambda T p_2)^{k_2}k_2! = P(n_1 = k_1) \cdot P(n_2 = k_2). \end{aligned} \quad (2)$$

Let now segments $[t^{(1)}, t^{(1)} + T^{(1)}], \dots, [t^{(k)}, t^{(k)} + T^{(k)}]$ of the time axis don't intersect. Similarly to (2) we prove the independence of the random vectors $(n_1^{(1)}, \dots, n_1^{(k)})$, $(n_2^{(1)}, \dots, n_2^{(k)})$ regarding the respective intervals, and independence of their components. Hence the flows $T_S^i,\ i = 1, \dots, m$, are independent.

Remark. *Theorems 4, 5 enhance the results of the article [6], removing restrictions on the independent Poisson flows considered in it.*

Consider now a closed queueing network $\overline{S}$, consisting of a finite number of nodes $i = 1, \dots, m$. The i node contains $l_i < \infty$ servers, the service time on which has an exponential distribution with the parameter $\mu_i,\ i = 1, \dots, m$. A finite number N of customers move along network $\overline{S}$. The dynamics of the customers movement in the network is specified by the matrix $\overline{\Theta} = ||\overline{\theta}_{i,j}||_{i,j=1}^{m}$, where $\overline{\theta}_{i,j}$ is the probability of transition after service of customer in the ith node to j-th one.

Let the route matrix $\overline{\Theta}$ be indecomposable, i.e.

$$\forall\ i,\ j \in \{1, \dots, m\}\ \exists\ i_1, \dots, i_r \in \{1, \dots, m\}:\ \overline{\theta}_{i,i_1} > 0, \overline{\theta}_{i_1,i_2} > 0, \dots, \overline{\theta}_{i_r,j} > 0.$$

Then for a fixed $\lambda_1 > 0$, the system of linear algebraic equations

$$\lambda_k = \sum_{t=1}^{m} \lambda_t\overline{\theta}_{t,k},\ k = 1, \dots, m \quad (3)$$

has a unique solution of $(\lambda_1,\dots,\lambda_m)$ with $\lambda_1>0,\dots,\lambda_m>0$, [4, p. 13].

For a closed queueing network $\overline{S}$ with N customers discrete Markov process $(\overline{n}_1(t),\dots,\overline{n}_m(t)),\ t\ge 0$, describing the number of customers in the network nodes has a limit distribution of $P_{\overline{S}}(k_1,\dots,k_m)$, independent of the initial conditions and presented in the form

$$P_{\overline{S}}(k_1,\dots,k_m)=\prod_{i=1}^m P_i(k_i)\sum_{k_1,\dots,k_m:\ k_1+\cdots+k_m=N}\prod_{i=1}^m P_i(k_i),\ k_1+\cdots+k_m=N.$$

Hence, the stationary probability $\pi_i(k_i)$ that in a node i of the network $\overline{S}$ there is k_i customers satisfies the equality

$$\pi_i(k_i)=\sum_{k_j,1\le j\ne i\le m,\sum_{1\le j\ne i\le m}k_j=N-k_i}P_{\overline{S}}(k_1,\dots,k_m),\ k_i=0,\dots,N.$$

Theorem 6. *The flow $T_{\overline{S}}^i$, leaving the i node of the closed queueing network $\overline{S}$ with the total number of customers N, being in a stationary state, is Poisson with intensity $\sum_{k_i=1}^N min(k_i,l_i)\mu_i\pi_i(k_i),\ i=1,\dots,m$.*
Indeed, the points of the flow $T_{\overline{S}}^i$, exiting the node i, are the moments of jumps down the components $\overline{n}_i(t)$ of the discrete Markov process $(\overline{n}_1(t),\dots,\overline{n}_m(t)),\ t\ge 0$. Consequently, the flow $t_{\overline{S}}^i$ satisfies condition a). Conditions b), c) are proved similarly to the proof of Theorem 1.
Example. Theorem 6 allows us to consider flows in systems backup with recovery. For example, for the simplest $\overline{S}$ system, a restore reservation consisting of one workstation (work phase 1), one repair location (repair phase 2), and one item. Let the random time before the failure of the item in the workplace has an exponential distribution with the parameter α, the random time to restore the item in the repair location has an exponential distribution with the parameter β. Denote $n_1(t)$ the number of elements in the working phase, $n_2(t)$ - the number of elements in the repair phase of $\overline{s}$. Then $n_1(1)=0{,}1,\ n_2(t)=0{,}1,\ n_1(t)+n_2(t)=n=1$, and therefore the route matrix of the system $\overline{on}$ has the form $\overline{\theta}=||\overline{\theta}_{i,j}||_{i,j=0}^1$, where $\overline{\theta}_{1,1}=\overline{\theta}_{2,2}=0,\ \overline{\theta}_{1,2}=\overline{\theta}_{2,1}=1$. Then equalities are just

$$P_{\overline{S}}(1{,}0)=\beta\alpha+\beta,\ P_{\overline{S}}(0{,}1)=\alpha\alpha+\beta,\ P_{\overline{S}}(0{,}0)=P_{\overline{S}}(1{,}1)=0,$$
$$\pi_1=\beta\alpha+\beta,\ \pi_2=\alpha\alpha+\beta,\ \nu_1=\nu_2=\alpha\beta\alpha+\beta,$$

where $\nu_1,\ \nu_2$ are the intensities of stationary Poisson flows leaving phases 1, 2, respectively.

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