# Stability of Discrete Multi-Server Queueing Systems with Heterogeneous Servers, Interruptions and Regenerative Input Flow

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## Abstract

In the paper we study a discrete-time multichannel queueing system with heterogeneous servers, regenerative input flow, and interruptions. The breakdowns of servers may occur at any time even if they are not occupied by customers. Consecutive moments of breakdowns are defined by a renewal process, but we do not assume blocked and available periods to be independent. We consider the preemptive repeat different service discipline as well as preemptive resume service discipline. We exploit the regeneration property of the input flow and renewal structure of the processes describing the servers' breakdowns to organise synchronisation of the input and service flows. This approach helps to establish the necessary and sufficient stability condition of the system. Generally, for preemptive repeat different service discipline this stability condition can not be expressed it terms of moments of service and interruption processes. Therefore, we derive the sufficient but not necessary condition, which can be expressed through these moments, and show that it coincides with condition obtained in existing literature for simpler queueing systems.

**Keywords:** Multichannel system, Regenerative input flow, Ergodicity, Interruption, Vacation, Unreliable servers

## 1 Introduction

Queueing systems with unavailable servers can be a useful abstraction in modelling of some real-life service operation. Such models may arise naturally as models of many computer, communication and manufacturing systems. Servers interruptions may result from resource sharing, server breakdowns, priority assignment, vacations, some external events, and others. For instance, if we concern the system with customers priority, the service of secondary customers is equivalent to the service interruption of the primary customers during this period.

Systems with unreliable servers have been intensively investigated for a long time. The main point was focused on the single-server case. There are some review papers, that cover most of the literature in these sphere. Some of the important papers on the single-server case are presented in [12]. Concerning systems with servers vacations it should be mentioned [10] and [17].

The most exhaustive literature survey on systems with interruptions is in [19], where non-Markovian multichannel systems were also covered. The are some other articles with extensive literature survey as well [24], [22].

Synchronization method combined with the regenerative theory is one of the powerful approach to obtain stability results for multichannel systems with discrete-time and service interruptions. Basing on this method the multichannel queueing system with identical service distribution function for all servers, renewal input flow, alternating renewal-type servers' interruptions in the discrete-time case was considered in [22]. Authors established some sufficient conditions of stability for the preemptive repeat different and preemptive resume service disciplines. In paper [2] this approach helps to implement asymptotic analysis of the single-server system with a regenerative input flow. The similar approach was applied in [27] to study the stability condition of the multichannel system with heterogeneous servers and a regenerative input flow in a random environment, which breaks all the servers simultaneously.

In this paper we consider a discrete-time queueing system with regenerative input flow and heterogeneous servers that may suffer independent interruptions. Consecutive moments of breakdowns are defined by a renewal process and do not depend on the system state. The preemptive repeat different service discipline as well as preemptive resume service discipline [14] are considered. The former case implies that the service is repeated from the beginning with different independent service time after restoration of the server. In the latter case the service of a customer is continued after restoration. The necessary and sufficient stability condition is established. The key element of our analysis is synchronization of the processes under the consideration. This method is based on the regeneration property of the input flow and renewal structure of the processes describing the servers' breakdowns (see, e.g [2]).

Let us also mention the fluid approximation approach to the stability analysis of queueing systems [8], [7], [9]. See also [13] for a survey of various approaches to stability of queueing systems with a focus on the fluid approach. Nevertheless, in the present paper we do not rely on fluid approximation since ergodic conditions cannot be expressed in terms of expected values for preemptive repeat different service discipline and regenerative method turns out to be suitable to obtain complete and transparent proofs.

The model under consideration is similar to the system that was investigated in [22]. However, there is essential generalisation that lead us to consider different processes. Firstly, we employ the regenerative flow as an input flow. Secondly, service distribution functions may differ for different servers. In this paper the necessary and sufficient condition for stability of the queue-length process is established, whereas in [22] only sufficient conditions for these service disciplines are proved.

The article is organised as follows. In the next section the model is described in detail. In the third section auxiliary service flows are introduced and the traffic rate is defined. In the next two sections we conduct the synchronization of the input and service flows. The sections 6-8 are devoted to the (in)stability problem. In the ninth section we provide some comments and make conclusion in the final section.

## 2 Model description

We consider a system with $m$ heterogeneous servers and a common queue. Service times of customers by the $i$th server constitute a sequence $\{\eta_{i,n}\}_{n=1}^{\infty}$ of independent identically distributed (iid) random variables that does not depend on input flow and service times by other servers. Let $B_i(t)$ be a distribution function (d.f.) of $\eta_{i,n}$ and $b_i = \mathbf{E}\eta_{i,n} < \infty$ $(i = \overline{1,m})$. We assume that the servers may be unavailable for service from time to time. The breakdowns of the servers may occur at any time even if they are not occupied by customers. Let $\{s_{i,n}^{(2)}\}_{n=0}^{\infty}$ be moments of breakdowns and $\{s_{i,n}^{(1)}\}_{n=1}^{\infty}$ be moments of restoration for the $i$th server. Here $0 = s_{i,0}^{(2)} < s_{i,1}^{(1)} < s_{i,1}^{(2)} < s_{i,2}^{(1)}$ .... Then

$u_{i,n}^{(1)} = s_{i,n}^{(1)} - s_{i,n-1}^{(2)}$ and $u_{i,n}^{(2)} = s_{i,n}^{(2)} - s_{i,n}^{(1)}$ denote the length of the $n$th blocked and $n$th available period of the $i$th server respectively ($i = \overline{1,m}$). The sequence $\{u_{i,n}^{(1)}, u_{i,n}^{(2)}\}_{n=1}^{\infty}$ consists of iid random vectors (for all $i = \overline{1,m}$) that do not depend on the input flow and service times. However, for each $n$ and $i$,random variables $u_{i,n}^{(1)}$ and $u_{i,n}^{(2)}$ are not assumed to be independent. Let $u_{i,n} = u_{i,n}^{(1)} + u_{i,n}^{(2)}$ be the length of the $n$th cycle for server $i$. A cycle consists of a blocked period followed by an available period. We assume that $\mathbf{E}u_{i,n}^{(1)} = a_i^{(1)} < \infty$, $\mathbf{E}u_{i,n}^{(2)} = a_i^{(2)} < \infty$, $a_i = a_i^{(1)} + a_i^{(2)}$ $(i = \overline{1,m})$. Server is free if it is neither serving a customer nor interrupted. If server becomes free and there are customers in the queue, a new customer enters the server. It is possible that more than one server becomes free simultaneously. Then customer in the queue chooses an idle server according to some algorithm, possibly random. For definiteness we assume that a customer chooses a free server with the least number. It is possible that an unavailable period starts while a customer is receiving service. Then service of the customer is immediately interrupted. There are various disciplines for continuation of the service after server restoration [14]. Here we consider the preemptive repeat different service discipline ($D_1$) and preemptive resume service discipline ($D_2$). In the former case service is repeated from the start and the service time after restoration is independent of the original service time. In the latter case service continues after restoration. In the both cases, customers remain with the same server until service completion. For the service discipline $D_1$ in order to ensure the service process for the $i$th server we have to assume that

$$\mathbf{P}(\eta_{i,1} \leq u_{i,1}^{(2)}) > 0 \quad \text{forall} \quad i = \overline{1,m}. \tag{1}$$

If this condition does not hold for some server $i$, then for discipline $D_1$ the $i$th server has to be excluded since it is always busy by service of the single customer. Without loss of generality in the rest of the paper we also assume

$$\mathbf{P}(\eta_{i,1} = 0) = 0 \quad \text{and} \quad \mathbf{P}(u_{i,1}^{(2)} = 0) = 0 \quad \text{forall} \quad i = \overline{1,m}. \tag{2}$$

We consider a discrete-time system, i.e. time is divided into fixed length intervals or slots and all arrivals, departures, interruptions (restorations) are synchronized with respect to slot boundaries. Moreover, in the case of synchronization of some events at one slot these events are ordered as follows: arrival, departure, and interruption (restoration). System is observed at the end of a slot, when all events of the slot are realized.

We assume that the input flow $X(t)$ is a regenerative one [2]. Suppose an integer-valued stochastic process $\{X(t), t \geq 0\}$ is defined on some probability space $(\Omega, \mathcal{F}, P)$ and $X(t)$ has nondecreasing right-continuous sample paths and $X(0) = 0$. Assume that there exists a filtration $\{\mathcal{F}_{\leq t}^X\}_{t\geq 0}$, ($\mathcal{F}_{\leq t}^X \subseteq \mathcal{F}$ for all $t \geq 0$) such that $X(t)$ is measurable with respect to $\{\mathcal{F}_{\leq t}^X\}_{t\geq 0}$.

**Definition 1** *The stochastic flow $X(t)$ is called regenerative if there is an increasing sequence of Markov moments $\{\theta_i,\ i \geq 0\}, \theta_0 = 0$ (with respect to $\{\mathcal{F}_{\leq t}^X\}_{t\geq 0}$) such that the sequence*

$$\{\grave{u}_i\}_{i=1}^{\infty} = \{X(\theta_{i-1} + t) - X(\theta_{i-1}), \theta_i - \theta_{i-1}, t \in (0, \theta_i - \theta_{i-1}]\}_{i=1}^{\infty}$$

consists of independent identically distributed random elements on $(\Omega, \mathcal{F}, P)$.

The random variable $\theta_i$ is said to be the $i$th regeneration point of $X(t)$ and $\tau_i = \theta_i - \theta_{i-1}$ is the $i$th regeneration period ($i = 1,2, \ldots$). Let $\xi_i = X(\theta_i) - X(\theta_{i-1})$ be the number of arrived customers during the $i$th regeneration period. Assume that $E\tau_1 < \infty$, $E\xi_1 < \infty$. The limit $\lambda_X = \lim_{t\to\infty} \frac{X(t)}{t}$ with probability one (w.p.1) is called the *intensity* of $X(t)$. It is easy to prove that $\lambda_X = \frac{E\xi_1}{E\tau_1}$ (e.g., see [2]). Class of regenerative flows contains most of fundamental flows that are exploited in the queueing theory [4]. Firstly, the doubly stochastic Poisson process [15], where random intensity is a regenerative process [25]. There are many other examples of regenerative flows, for instance, semi-markovian, Markov-modulated, Markov-arrival, and other processes [1]. Important properties of regenerative flows are given in [2].

## 3 Auxiliary processes

In this section we define auxiliary processes $Y_i^{(d)}(t)$ $(i=\overline{1,m},\ d=1,2)$ that will be used later. Here $d=1$ for the service discipline $D_1$ and $d=2$ for the service discipline $D_2$. We think of $Y_i^{(d)}(t)$ $(i=\overline{1,m})$ as the number of customers, that can be served by the $i$th server during its available period within interval $[0,t]$, if there are always customers for service. In order to construct the processes $Y_i^{(1)}(t)$ we introduce the collection $\left\{\{\eta_{i,n}^{(j)}\}_{n=1}^{\infty}\right\}_{j=1}^{m}$ of independent sequences $\{\eta_{i,n}^{(j)}\}_{n=1}^{\infty}$ consisting of iid random variables with d.f. $B_i(x)$. Let $K_{i,j}(t)$ be the counting process associated with the sequence $\{\eta_{i,n}^{(j)}\}_{n=1}^{\infty}$, i.e. $K_{i,j}(t)=\max\{k:\sum_{n=1}^{k}\eta_{i,n}^{(j)}\le t\}$ $(K_{i,j}(0)=0)$ and $N_i(t)$ be the number of cycles for the $i$th server during $[0,t]$, i.e.

$$N_i(t)=\max\{j:\sum_{n=1}^{j}u_{i,n}\le t\}\quad (N_i(0)=0). \tag{3}$$

Then the processes $Y_i^{(1)}(t)$ and $Y_i^{(2)}(t)$ are defined by the relations

$$Y_i^{(1)}(t)=\sum_{j=1}^{N_i(t)}K_{i,j}(u_{i,j}^{(2)})+K_{i,N_i(t)+1}\left(\max\left[0,t-s_{i,N_i(t)+1}^{(1)}\right]\right). \tag{4}$$

$$Y_i^{(2)}(t)=K_{i,1}\left(\sum_{j=1}^{N_i(t)}u_{i,j}^{(2)}+\max\left[0,t-s_{i,N_i(t)+1}^{(1)}\right]\right). \tag{5}$$

By $H_i(t)$ denote the renewal function for $K_{i,j}(t)$, i.e. $H_i(t)=\mathbf{E}K_{i,j}(t)$.

**Lemma 1** *There exist the limits*

$$\lim_{t\to\infty}\frac{Y_i^{(1)}(t)}{t}=\frac{\mathbf{E}H_i(u_{i,n}^{(2)})}{a_i}=\lambda_{Y_i^{(1)}}\ \text{w.p.}\,1,$$
$$\lim_{t\to\infty}\frac{Y_i^{(2)}(t)}{t}=\frac{a_i^{(2)}}{b_ia_i}=\lambda_{Y_i^{(2)}}\ \text{w.p.}\,1.$$

*Proof.* We start with the discipline $D_1$. Let $g_i^{(1)}(n)=\sum_{j=1}^{n}K_{i,j}(u_{i,j}^{(2)})$. From (4) we get the inequalities

$$g_i^{(1)}(N_i(t))\le Y_i^{(1)}(t)\le g_i^{(1)}(N_i(t)+1). \tag{6}$$

Since $\{K_{i,j}(u_{i,j}^{(2)})\}_{j\ge1}$ is a sequence of iid random variables with a finite mean by the strong law of large numbers (SLLN) we have $n^{-1}g_i^{(1)}(n)\underset{n\to\infty}{\longrightarrow}\mathbf{E}H_i(u_{i,j}^{(2)})$ w.p.1. In view of independence $\{\{\eta_{i,n}^{(j)}\}_{n=1}^{\infty}\}_{j=1}^{m}$ and $\{u_{i,n}^{(1)},u_{i,n}^{(2)}\}_{n=1}^{\infty}$ one can obtain the convergence

$$\frac{g_i^{(1)}(N_i(t))}{N_i(t)}\underset{t\to\infty}{\longrightarrow}\mathbf{E}H_i(u_{i,1}^{(2)})\quad\text{w.p.}\,1.$$

It follows from the renewal theory that

$$t^{-1}N_i(t)\underset{t\to\infty}{\longrightarrow}a_i^{-1}\quad\text{w.p.}\,1. \tag{7}$$

Now the proof of the lemma for $D_1$ follows from (6).

Consider the discipline $D_2$. Let $g_i^{(2)}(n)=K_{i,1}(\sum_{j=1}^{n}u_{i,j}^{(2)})$. From (5) we have the inequalities

$$g_i^{(2)}(N_i(t))\le Y_i^{(2)}(t)\le g_i^{(2)}(N_i(t)+1). \tag{8}$$

Since $t^{-1}K_{i,1}(t)\underset{t\to\infty}{\longrightarrow}b_i^{-1}$ w.p.1, we get $n^{-1}g_i^{(2)}(n)\underset{n\to\infty}{\longrightarrow}b_i^{-1}a_i^{(2)}$ w.p.1. Thus (7) and (8) conclude the proof for the discipline $D_2$. +

Let $Y^{(d)}(t)=\sum_{i=1}^{m}Y_i^{(d)}(t)$ $(d=1,2)$. From Lemma 1 we have

$$\lambda_{Y^{(d)}}=\lim_{t\to\infty}\frac{Y^{(d)}(t)}{t}=\sum_{i=1}^{m}\lambda_{Y_i^{(d)}}\quad\text{w.p.}\,1\quad(d=1,2). \tag{9}$$

We think of $\lambda_X$ and $\lambda_{Y^{(d)}}$ as the arrival and service rate respectively. Intuitively, it is clear that traffic rates $\rho^{(d)}$ of the system have to be determined as

$$\rho^{(1)}=\frac{\lambda_X}{\sum_{i=1}^{m}\frac{\mathbf{E}H_i(u_{i,n}^{(2)})}{a_i}},\qquad \rho^{(2)}=\frac{\lambda_X}{\sum_{i=1}^{m}\frac{a_i^{(2)}}{b_ia_i}}. \tag{10}$$

At first sight the traffic rate $\rho^{(1)}$ for discipline $D_1$ can not be expressed in terms of the first moments of random variables defining the model. It is true if we consider the means of service times, available and block periods only. However, we may introduce random variables $\zeta_i^{(n)}$ which is the number of served customers by the $i$th server during the $n$th cycle $(i = \overline{1,m}, n = 1,2,\dots)$ under condition that there are always customers on the server. Putting $\alpha_i = \mathbf{E}\xi_i^{(n)}$ we get from (10)

$$\rho^{(1)} = \lambda_X [\sum_{i=1}^{m} \frac{\alpha_i}{a_i}]^{-1}.$$

In applications one may estimate these parameters basing on statistical data.

## 4 Synchronization of regenerative flows

First we obtain the result concerning synchronization of general regenerative aperiodic flows in a discrete-time case. Let $Z_1(t)$ and $Z_2(t)$ be independent regenerative flows with regeneration points $\{\theta_{1,j}\}_{j=1}^{\infty}$ and $\{\theta_{2,j}\}_{j=1}^{\infty}$ respectively $(\theta_{i,0} = 0; i = 1,2)$. As usually, aperiodicity means that the greatest common divisor (GCD)

$$GCD\{k: \mathbf{P}(\theta_{i,1} = k) > 0\} = 1, \quad i = 1,2. \tag{11}$$

Define common points of regeneration for $Z_1(t)$ and $Z_2(t)$ by the relation

$$T_k = \min\{\theta_{1,j} > T_{k-1}: \cup_{l=1}^{\infty} \{\theta_{2,l} = \theta_{1,j}\}\}, \quad T_0 = 0. \tag{12}$$

**Lemma 2** *Let condition (11) be fulfilled and* $\mathbf{E}(\theta_{i,1}) < \infty$ *$(i = 1,2)$. Then the sequence* $\{T_k\}_{k=1}^{\infty}$ *consists of regeneration points for* $Z_1(t)$ *and* $Z_2(t)$ *and*

$$\mathbf{E}T_1 = \mathbf{E}\theta_{1,1} \cdot \mathbf{E}\theta_{2,1} < \infty. \tag{13}$$

*Proof.* The first statement of the Lemma follows from the definition of $T_k$. To prove the second statement we put

$$\nu_k = \min\{j > \nu_{k-1}: \cup_{l=1}^{\infty} \{\theta_{1,j} = \theta_{2,l}\}\}, \quad \nu_0 = 0,$$

so that $T_k = \theta_{1,\nu_k}$. Then $\{\nu_k - \nu_{k-1}\}_{k=1}^{\infty}$ is a sequence of iid random variables. In accordance with Wald's identity [11] we get $\mathbf{E}T_1 = \mathbf{E}\theta_{1,1} \cdot \mathbf{E}\nu_1$. Therefore, we need to prove the finiteness of $\mathbf{E}\nu_1$. Let $h_2(t)$ $(h(t))$ be the mean of the number of renewals at time $t$ for the renewal process $\{\theta_{2,n}\}_{n=1}^{\infty}$ $(\{\nu_k\}_{k=1}^{\infty})$, so that $h_2(t) = \sum_{l=0}^{\infty} \mathbf{P}(\theta_{2,l} = t)$ and $h(t) = \sum_{k=0}^{\infty} \mathbf{P}(\nu_k = t)$. Taking into account (11) from Blackwell's theorem [26] we get

$$h_2(t) \underset{t\to\infty}{\longrightarrow} \frac{1}{\mathbf{E}\theta_{2,1}}, \quad h(t) \underset{t\to\infty}{\longrightarrow} \frac{1}{\mathbf{E}\nu_1}. \tag{14}$$

In view of independence $Z_1(t)$ and $Z_2(t)$

$$h(j) = \mathbf{P}\{\cup_{l=0}^{\infty} \{\theta_{1,j} = \theta_{2,l}\}\} = \mathbf{E}\big(\sum_{l=0}^{\infty} \mathbf{P}\{\theta_{1,j} = \theta_{2,l} | \theta_{1,j}\}\big) = \mathbf{E}h_2(\theta_{1,j}). \tag{15}$$

Since $\theta_{1,j} \underset{j\to\infty}{\longrightarrow} \infty$ w.p.1, then $h_2(\theta_{1,j}) \underset{j\to\infty}{\longrightarrow} \frac{1}{\mathbf{E}\theta_{2,1}}$ w.p.1. Thus from (14), (15) and Lebesgue's dominated convergence theorem we obtain $\mathbf{E}\nu_1 = \mathbf{E}\theta_{2,1} < \infty$. +

## 5 Synchronization of renewal points for input and service flows

To exploit Lemma 2 for synchronization of flows $X(t)$ and $Y_i^{(d)}(t)$ we consider the counting processes $N_i(t)$ $(i = \overline{1,m})$ defined by (3) and introduce a counting process $N_0(t)$ for the input flow by the relation

$$N_0(t) = max\{k: \theta_k \le t\}.$$

We assume throughout the following condition to be fulfilled.

**Condition 1** *The distributions of* $\theta_1$, $u_{i,1}$, $\eta_{i,1}$ $(i = \overline{1,m})$ *are aperiodic ones, i.e. (11)are fulfilled for these random variables.*

Let us define subsequence $\{T_k^{(1)}\}_{k=0}^{\infty}$ of the sequence $\{\theta_j\}_{j=1}^{\infty}$ by the recurrent relation

$$T_k^{(1)} = min\{\theta_j > T_{k-1}^{(1)}: \cap_{i=1}^m \{N_i(\theta_j) - N_i(\theta_j - 1) = 1\}\}, \quad (T_0^{(1)} = 0). \tag{16}$$

In other words $T_k^{(1)}$ is a point of regeneration of $X(t)$ such that all the servers get out of the order simultaneously at this moment. This means that $\{T_k^{(1)}\}_{k\geq 0}$ are the common regeneration points for the input flow $X(t)$ and $N_i(t)$ $(i = \overline{1,m})$. For $D_1$ service discipline $\{T_k^{(1)}\}_{k\geq 0}$ constitutes the sequence of regeneration points for $Y_i^{(1)}(t)$ $(i = \overline{1,m})$ and $Y^{(1)}(t)$ as well, but for $D_2$ this is not the case. For $D_2$ we introduce a subsequence $\{T_n^{(2)}\}_{n\geq 0}$ of the sequence $\{T_k^{(1)}\}_{k\geq 0}$ as follows

$$T_n^{(2)} = \min\left\{T_j^{(1)} > T_{n-1}^{(2)}:\ \cap_{i=1}^m \{Y_i^{(2)}(T_j^{(1)}) - Y_i^{(2)}(T_j^{(1)} - 1) = 1\}\right\},\ (T_0^{(2)} = 0). \tag{17}$$

Thus, $T_n^{(2)}$ is a general point of regeneration for $X(t)$ and $Y_i^{(2)}(t)$ $(i = \overline{1,m})$ since the following conditions are fulfilled:

- $T_n^{(2)}$ is a regeneration point for $X(t)$;
- at time $T_n^{(2)}$ all the servers complete the service;
- at time $T_n^{(2)}$ all the servers become unavailable.

**Lemma 3** *Let Condition 1 be fulfilled. Moreover, assume relations (1) for $D_1$ and (2) for the both disciplines take place. Then*

$$\mathbf{E}T_1^{(d)} < \infty, \quad (d = 1,2). \tag{18}$$

*Proof.* Since $\mathbf{E}\tau_1 < \infty$ and $\mathbf{E}(u_{i,1}) = a_i < \infty$, it follows that for discipline $D_1$ this lemma is the consequence of Lemma 2 and therefore

$$\mathbf{E}T_1^{(1)} = \frac{1}{\mathbf{E}\tau_1}\prod_{i=1}^m a_i^{-1} < \infty.$$

Let us consider discipline $D_2$. For server $i$ $(i = \overline{1,m})$ we introduce the sequence

$$s_{i,k} = \min\left\{s_{i,j}^{(2)} > s_{i,k-1}: Y_i^{(2)}(s_{i,j}^{(2)}) - Y_i^{(2)}(s_{i,j}^{(2)} - 1) = 1\right\}, s_{i,0} = 0. \tag{19}$$

Note that $\{s_{i,k}\}_{k\geq 0}$ is the sequence of breakdown moments for the $i$th server such that the flow $Y_i^{(2)}(t)$ has a jump at every of these moments. Hence $\{s_{i,k}\}_{k\geq 0}$ are regeneration points for $Y_i^{(2)}(t)$. If we show that $\mathbf{E}(s_{i,1}) < \infty$ for any $i = \overline{1,m}$, then Lemma 2 provides the finiteness of $\mathbf{E}T_1^{(2)}$.

Let

$$\nu_{i,k} = \min\{j > \nu_{i,k-1}: Y_i^{(2)}(s_{i,j}^{(2)}) - Y_i^{(2)}(s_{i,j}^{(2)} - 1) = 1\}, \quad \nu_{i,0} = 0, \quad (i = \overline{1,m}).$$

Then $\{\nu_{i,k} - \nu_{i,k-1}\}_{k=1}^\infty$ is a sequence of iid random variables and

$$\mathbf{E}(s_{i,k} - s_{i,k-1}) = \mathbf{E}u_{i,1} \cdot \mathbf{E}\nu_{i,1}. \tag{20}$$

Since $\{s_{i,j}\}_{j\geq 1}$ does not depend on service times we have from the key renewal theorem [26]

$$\lim_{j\to\infty} \mathbf{P}\{Y_i^{(2)}(s_{i,j}^{(2)}) - Y_i^{(2)}(s_{i,j}^{(2)} - 1) = 1\} = \frac{1}{E\nu_{i,1}}, \tag{21}$$

where $\frac{1}{\mathbf{E}\nu_{i,1}} = 0$ if $\mathbf{E}\nu_{i,1} = \infty$.

Let $U_{i,j}^{(2)} = \sum_{n=1}^j u_{i,n}^{(2)}$ be the total available time of the $i$th server during $n$ cycles. Then

$$\delta_{i,j} = \mathbf{P}\{Y_i^{(2)}(s_{i,j}^{(2)}) - Y_i^{(2)}(s_{i,j}^{(2)} - 1) = 1\} = \mathbf{E}\left(\mathbf{P}\left(\cup_{k=1}^\infty \left\{\sum_{l=1}^k \eta_{i,l} = U_{i,j}^{(2)}\right\} | U_{i,j}^{(2)}\right)\right) =$$

$$= \mathbf{E}\left(\sum_{k=1}^\infty \mathbf{P}\left(\sum_{l=1}^k \eta_{i,l} = U_{i,j}^{(2)} | U_{i,j}^{(2)}\right)\right) = \sum_{n=1}^\infty \sum_{k=1}^\infty \mathbf{P}\left(\sum_{l=1}^k \eta_{i,l} = n\right)\mathbf{P}\left(U_{i,j}^{(2)} = n\right).$$

In the last equality we used independence of $\{\eta_{i,k}\}_{k=1}^\infty$ and $\{U_{i,n}^{(2)}\}$. From Blackwell's theorem [26] we have

$$\sum_{k=1}^\infty \mathbf{P}\left(\sum_{l=1}^k \eta_{i,l} = n\right) \underset{n\to\infty}{\longrightarrow} \frac{1}{b_i},$$

so one may easily verify that $\delta_{i,j} \underset{j\to\infty}{\longrightarrow} \frac{1}{b_i}$. Thus, from (21) we have $\mathbf{E}\nu_{i,1} = b_i$ and from (20) we get $\mathbf{E}s_{i,1} = a_i b_i < \infty$. +

So, we have constructed the sequence $\{T_k^{(d)}\}_{k\geq 0}$ of common regeneration points for the processes $X(t)$ and $Y^{(d)}(t)$. Denote by $\Delta_{X,k}^{(d)} = X(T_k^{(d)}) - X(T_{k-1}^{(d)})$, $\Delta_{Y_i,k}^{(d)} = Y_i^{(d)}(T_k^{(d)}) - Y_i^{(d)}(T_{k-1}^{(d)})$, and $\Delta_{Y,k}^{(d)} = \sum_{i=1}^m \Delta_{Y_i,k}^{(d)}$ $(d = 1,2)$.

**Lemma 4** *Let conditions of Lemma 3 take place. Then the traffic rate of the system defined by (10) is equal to*

$$\rho^{(d)} = \frac{\mathbf{E}\Delta_{X,k}^{(d)}}{\mathbf{E}\Delta_{Y,k}^{(d)}}, \quad (d = 1,2).$$

The proof follows from renewal theory and SLLN.

## 6 Instability results for $\rho^{(d)} \geq 1$

Let $Q^{(d)}(t)$ be the number of customers in the system with the service discipline $D_d$ (including customers in the servers) at instant $t$ ($d = 1,2$).

**Theorem 1** *Let conditions of Lemma 3 take place. Then*

- $Q^{(d)}(t) \underset{t\to\infty}{\longrightarrow} \infty$ w.p.1 if $\rho^{(d)} > 1$,
- $Q^{(d)}(t) \overset{P}{\to} \infty$ if $\rho^{(d)} = 1$.

Here $\rho^{(d)}$ is defined by (10) ($d = 1,2$).

*Proof.* Denote by $\tilde{Y}_i^{(d)}(t)$ the number of customers really served by the $i$th server during the interval $[0, t]$ and $\tilde{Y}^{(d)}(t) = \sum_{i=1}^{m} \tilde{Y}_i^{(d)}(t)$, $\widetilde{\Delta}_{Y,n}^{(d)} = \tilde{Y}^{(d)}(T_n^{(d)}) - \tilde{Y}^{(d)}(T_{n-1}^{(d)})$. Note that $\tilde{Y}_i^{(d)}(t)$ may contain idle periods, so it is not the same like $Y_i^{(d)}(t)$. Employing the approach proposed in [5] and developed in [16] we can choose service times from the collection $\{\{\eta_{i,n}^{(j)}\}_{n=1}^{\infty}\}_{j\geq 1}$ by such a way that

$$\tilde{Y}^{(d)}(t) \leq Y^{(d)}(t) \quad \text{w.p.1}, \tag{22}$$

$$\widetilde{\Delta}_{Y,n}^{(d)} \leq \Delta_{Y,n}^{(d)} \quad \text{w.p.1}. \tag{23}$$

Consider the case $\rho^{(d)} > 1$. Taking into account (22) we have

$$Q^{(d)}(t) = Q^{(d)}(0) - \tilde{Y}^{(d)}(t) + X(t) \geq Q^{(d)}(0) - Y^{(d)}(t) + X(t), \; t \geq 0 \; w.p.1. \tag{24}$$

From (10) and (24) we obtain

$$\lim_{t\to\infty} \frac{Q^{(d)}(t)}{t} \geq \lambda_X - \lambda_{Y^{(d)}} > 0, \; w.p.1,$$

which concludes the first statement of this Theorem.

Let $\rho^{(d)} = 1$. Consider the embedded process $Q_n^{(d)} = Q^{(d)}(T_n^{(d)})$ and denote $Z_k^{(d)} = \sum_{j=1}^{k} (\Delta_{X,j}^{(d)} - \Delta_{Y,j}^{(d)})$ ($Z_0^{(d)} = 0$). We define the auxiliary sequence $\{\hat{Q}_k^{(d)}\}_{k\geq 0}$ by the recursive relation

$$\hat{Q}_k^{(d)} = \max[0, \; \hat{Q}_{k-1}^{(d)} + \Delta_{X,k}^{(d)} - \Delta_{Y,k}^{(d)}], \; \hat{Q}_0^{(d)} = 0.$$

Since $Q_k^{(d)} = Q_{k-1}^{(d)} + \Delta_{X,k}^{(d)} - \widetilde{\Delta}_{Y,k}^{(d)}$ from (23) we get $Q_k^{(d)} \geq \hat{Q}_k^{(d)}$ w.p.1 and in distribution the following equality is fulfilled [20].

$$\hat{Q}_k^{(d)} = \max_{0\leq j\leq k} Z_j^{(d)}.$$

If $\rho^{(d)} = 1$, it follows from Lemma 4 that $\mathbf{E}\Delta_{X,j}^{(d)} = \mathbf{E}\Delta_{Y,j}^{(d)}$. Therefore, $\{Z_k^{(d)}\}_{k\geq 0}$ is a random walk with zero drift. Hence, except when $\Delta_{X,j}^{(d)} = \Delta_{Y,j}^{(d)} = c$ w.p.1 ($c$ is a constant) $\max_{0\leq j\leq k} Z_j^{(d)} \overset{P}{\to} \infty$ (see, e.g. [11]). It means that $Q_k^{(d)} \overset{P}{\to} \infty$ and the second statement of this Theorem holds.+

## 7 Stability theorem for the preemptive repeat different service discipline

In this section we consider the preemptive repeat different service discipline ($D_1$). So index (1) will be omitted during the rest of the section, if it does not make confusion. We start with definitions.

**Definition 2** *The process $\{Q(t), t \geq 0\}$ is called stochastically bounded if for any $\varepsilon > 0$ there exists $y < \infty$ such that for any $t > 0$*

$$\mathbf{P}\{Q(t) < y\} > 1 - \varepsilon.$$

Otherwise we say that $Q(t)$ is stochastically unbounded. This definition is close to the notion of *tightness* [23].

**Definition 3** *Process $\{Q(t),\ t \geq 0\}$ is ergodic one if for any initial state $Q(0)$ there exists*

$$\lim_{t\to\infty} \mathbf{P}\{Q(t) \leq x\} = F(x),$$

where $F(x)$ is a distribution function and it does not depend on $Q(0)$.

Denote $Q_n = Q(T_n^{(1)})$. Introduce the process

$$x_n = \begin{cases} (Q_n, e_1(n), \dots, e_m(n)) & \text{if } \ 0 < Q_n < m, \\ Q_n \ \text{ if } \ Q_n = 0 \ \text{ or } \ Q_n \geq m, \end{cases} \quad n \geq 0,$$

where $e_i(n) = 1$ if there is a customer in the $i$th server at time $T_n^{(1)}$ and $e_i(n) = 0$ otherwise. In view of interruption discipline $D_1$ and properties of the moments of synchronization $\{T_n^{(1)}\}_{n\geq 1}$ the process $\{x_n\}_{n\geq 1}$ is a Markov chain with countable set of states.

**Theorem 2** *Let conditions of Lemma 3 take place and $\mathbf{P}(Q(0) < \infty) = 1$. If $\rho^{(1)} < 1$, then $Q(t)$ is stochastically bounded. If, in addition, $\{x_n\}_{n\geq 1}$ is an irreducible and aperiodic Markov chain, then $Q(t)$ is ergodic.*

*Proof.* Consider the $i$th server. We assume that service times of the customers processing during the $k$th available period $[s_{i,k}^{(1)}, s_{i,k}^{(2)}]$ ($k = 1,2, \dots$) are consequently selected from the sequence of iid random variables $\{\eta_{i,n}^{(k)}\}_{n\geq 1}$. Let us recall that process $Y_i(t)$ is defined by the same sequence on the $k$th cycle with the help of (4). Introduce the event

$$A_n = \left\{Q(t) \geq m \text{forall} t \in [T_{n-1}^{(1)}, T_n^{(1)}]\right\}. \quad (25)$$

Then

$$\Delta_{Y,n}\mathbf{I}(A_n) = \widetilde{\Delta}_{Y,n}\mathbf{I}(A_n) \quad \text{w.p.}1, \quad (26)$$

where $\mathbf{I}(A)$ is an indicator of the event $A$. By $\mathfrak{K}$ denote the set of states for $\{x_n\}_{n\geq 0}$. Let $\mathfrak{K}_0$ be the set of unessential states and $\mathfrak{K}_l$ ($l = \overline{1,r}$) irreducible classes of communicating states. Since $\rho^{(1)} < 1$ from Lemma 4 we have $\mathbf{E}\Delta_{X,1} < \mathbf{E}\Delta_{Y,1}$. It yields that there exists $k_0$ such that for any essential state $x \in \mathfrak{K}_l$ one can find $n(x)$ so that

$$\mathbf{P}(Q_{n(x)} < m + k_0 | Q_0 = x) > 0. \quad (27)$$

It provides the finiteness of the number of classes $r$, so $\mathfrak{K} = \cup_{l=0}^{r} \mathfrak{K}_l$. Consider the first class $\mathfrak{K}_1$. Assume that it is aperiodic. Then for any $x \in \mathfrak{K}_1$, $y \in \mathfrak{K}_1$ there exists

$$\lim_{n\to\infty} \mathbf{P}(x_n = x | x_0 = y) = \pi_x^{(1)}. \quad (28)$$

If

$$\textstyle\sum_{x\in\mathfrak{K}_1} \pi_x^{(1)} = 1 \quad (29)$$

then $Q_n$ is stochastically bounded under condition that $Q_0 \in \mathfrak{K}_1$. Let us show that (29) is fulfilled employing Foster's criterion [21]. For any $x \in \mathfrak{K}_1$ we define the test function $f(x) = q$, where $q$ is the first coordinate of $x$. It is sufficient to show that for some $\varepsilon_1 > 0$ there exists $M_{\varepsilon_1}$ such that

$$\mathbf{E}(f(x_n) - f(x_{n-1}) | x_{n-1} = x) < -\varepsilon_1 \quad (30)$$

for all $x \in \mathfrak{K}_1$ with $q > M_{\varepsilon_1}$. Taking into account (??) we get

$$\begin{aligned} Q_n = Q_{n-1} + \Delta_n^X - \widetilde{\Delta}_n^Y = Q_{n-1} + \Delta_n^X - \widetilde{\Delta}_n^Y\mathbf{I}(A_n) - \widetilde{\Delta}_n^Y\mathbf{I}(\overline{A}_n) \leq \\ \leq Q_{n-1} + \Delta_n^X - \Delta_n^Y\mathbf{I}(A_n) = Q_{n-1} + \Delta_n^X - \Delta_n^Y + \Delta_n^Y\mathbf{I}(\overline{A}_n). \end{aligned} \quad (31)$$

From the assumption $\rho^{(1)} < 1$ we have

$$E\Delta_{X,k} - E\Delta_{Y,k} = -\delta < 0. \quad (32)$$

Note, firstly, that for any $\varepsilon > 0$ there exists $M_\varepsilon$ such that $\mathbf{P}(\overline{A}_n) < \varepsilon$ if $Q_{n-1} > M_\varepsilon$. Therefore, in view of integrability of random variables $\Delta_{Y,n}$ one may choose $M_\delta \geq m$ such that $\mathbf{E}\Delta_{Y,n}\mathbf{I}(\overline{A}_n) < \frac{\delta}{2}$ if $Q_{n-1} > M_\delta$. Thus, we obtain from (31) and (32)

$$\mathbf{E}(f(x_n) - f(x_{n-1}) | x_{n-1} = x) < E\Delta_{X,n} - E\Delta_{Y,n} + \frac{\delta}{2} = -\frac{\delta}{2}$$

if $x_{n-1} > M_\delta$ that proves (30).

Let $\mathfrak{K}_1$ be a periodic class with a period $h$. Then we consider a sequence $\{\tilde{x}_n^{(l)}\}_{n\geq 1}$ ($l = \overline{0, h-1}$), where $\tilde{x}_n^{(l)} = x_{nh+l}$. It is well-known [11] that Markov chain $\{\tilde{x}_n^{(l)}\}_{n\geq 1}$ is irreducible and aperiodic. Arguing as above we prove that $\tilde{Q}_n^{(l)} = Q_{nh+l}$ is stochastically bounded as $n \to \infty$.

Stochastic boundedness of $Q_n$ for initial state $x_0 = (Q_0, e_1(0), \dots, e_m(0)$ from other classes $\mathfrak{K}_i$ ($i = \overline{2, r}$) can be similarly proved. Since the number of classes $r$ is finite we conclude that $Q_n$ is stochastically bounded as $n \to \infty$ for any initial state of Markov chain $x_0 \in \mathfrak{K}$. Hence, the process $Q(t)$ is also stochastically bounded.

To prove the second statement of the Theorem we have to assume that Markov chain $\{x_n\}_{n\geq 1}$ is an irreducible and aperiodic. The set of states $\mathfrak{K}$ of $\{x_n\}_{n\geq 1}$ may have some unessential states but all the essential states organize the unique class $\mathfrak{K}_1$ of communicating states. It follows from the first statement of the Theorem that there exists the limit (28), where $\pi_x^{(1)} > 0$ for $x \in \mathfrak{K}_1$ and (29) is fulfilled, i.e. the Markov chain $\{x_n\}_{n\geq 1}$ is ergodic. Let us take a state $j_0 \in \mathfrak{K}_1$, $j_0 \geq m$ and assume that $x_0 = j_0$. Denote $\nu_{j_0} = \min\{n > 0 : x_n = j_0\}$, so that $\nu_{j_0}$ is the time of the return to the state $j_0$. Since Markov chain $\{x_n\}_{n\geq 1}$ is ergodic, it follows that $\mathbf{E}\nu_{j_0} < \infty$. Now consider $Q(t)$. Subsequence $\{T_{n_k}^{(1)}\}_{k\geq 0}$ of $\{T_n^{(1)}\}_{n\geq 0}$ such that $Q(T_{n_k}^{(1)}) = j_0$ is a sequence of regeneration points for $Q(t)$. So $Q(t)$ is a regenerative process. Let $\tilde{\tau}_{j_0}$ be the time of return to the state $j_0$ for $Q(t)$, i.e.

$$\tilde{\tau}_{j_0} = \min\{t > 0 : Q(t) = j_0\},$$

under assumption that $Q(0) = j_0$. Since $\mathbf{E}(T_n^{(1)} - T_{n-1}^{(1)}) = \mathbf{E}T_1^{(1)} < \infty$ (Lemma 2) from Wald's identity we have $\mathbf{E}\tilde{\tau}_{j_0} = \mathbf{E}T_1^{(1)}\mathbf{E}\nu_{j_0} < \infty$. Also, from any initial state the process $Q(t)$ gets into $j_0$ in finite time w.p.1. We remark that condition (11) holds for $\tilde{\tau}_{j_0}$. Therefore, Smith's theorem [25] will conclude the proof. +

**Remark 1** *Let us note that in [22] the sufficient condition of stability for a system with a recurrent input flow, identically distributed service times ($b_i = b$, $i = \overline{1, m}$) and preemptive repeat different service discipline was obtained. This condition in our term has a form*

$$\lambda_X < \sum_{i=1}^{m} \frac{a_i^{(2)} - b}{a_i b}. \tag{33}$$

One can easily see that (33) is a corollary of the condition $\rho^{(1)} < 1$ where $\rho^{(1)}$ is defined by (10). Indeed, taking into account the well-known inequality (see, e.g. [11])

$$H_i(t) \geq \frac{t}{b_i} - 1, \quad t \geq 0$$

we get from (10) and Theorem 2 the following sufficient condition of stability

$$\lambda_X < \sum_{i=1}^{m} \frac{a_i^{(2)} - b_i}{a_i b_i}.$$

This condition is the same as (33) when $b_i = b$ ($i = \overline{1, m}$).

## 8 Stability theorem for the preemptive resume service discipline

We now touch upon the stability of the model with preemptive resume service discipline ($D_2$). As opposed to queues with preemptive repeat different interruptions, interrupted service continuous when the server returns from a blocked period. Since in this section we consider discipline $D_2$ only, index (2) will be omitted when it will not lead to misunderstanding.

Put $Q_n = Q(T_n^{(2)})$. Let $e_i(n) = 1$ if there is a customer in the $i$th server at time $T_n^{(2)}$ and $e_i(n) = 0$ otherwise. Note that under discipline $D_2$ the process $(Q_n, e_1(n), \dots, e_m(n))$ is not a Markov chain. So we will use another approach, which is based on Theorem 1 from [3].

Denote by $t_n$ ($n = 1, 2, \dots$) the arrival instant of the $n$th customer at the system. Let $q_{i,n}$ be the number of customers at moment $t_n$, which will be served by the $i$th server, $q_n = \sum_{i=1}^{m} q_{i,n} = Q(t_n)$. As before, $\tilde{Y}_i(t)$ is the number of customers that had been served by the $i$th server during time interval $[0, t]$.

**Lemma 5** *Assume that*

$$\mathbf{E}\eta_{i,1}^2 < \infty, \mathbf{E}u_{i,1}^2 < \infty. \tag{34}$$

If $q_{i,n} \xrightarrow{P} \infty$, then for any $\varepsilon > 0$ there exists $n_\varepsilon$, such that for $n > n_\varepsilon$

$$\mathbf{E}\big(\tilde{Y}_i(t_n) - \tilde{Y}_i(t_{n-1})\big) \geq \frac{\beta_i}{\lambda_X} - \varepsilon, \tag{35}$$

where $\beta_i = \frac{a_i^{(2)}}{a_i b_i}$.

*Proof.* This lemma can be proved similarly to Lemma 2 in [3]. The only difference is that in [3] system with reliable servers was considered. Therefore $Y_i(t)$ was a renewal process corresponding to $\{\eta_{i,n}\}_{n=1}^{\infty}$ of i.i.d random variables. The proof of Lemma 2 in [3] was based on Blackwell's theorem for $\mathbf{E}Y_i(t)$, i.e. on the following convergence for any $h > 0$

$$\mathbf{E}(Y_i(t+h) - Y_i(t)) \underset{t\to\infty}{\longrightarrow} \frac{h}{b_i}. \tag{36}$$

For the system under consideration time intervals between jumps of the process $Y_i(t)$ generally speaking are dependent random variables. Therefore, to apply Lemma 2 from [3] we have to obtain an analog of (36) in our case. Let

$$H_i(t) = \mathbf{E}K_{i,1}(t), \quad \zeta_i(t) = max(0, t - s_{N_i(t)}^{(1)} + 1), \quad V_i(t) = \textstyle\sum_{j=1}^{N_i(t)} u_{i,j}^{(2)} + \zeta_i(t),$$

where $N_i(t)$ and $K_{i,1}(t)$ are defined in Section 4. Then from (5) we get $U_i(t) = \mathbf{E}Y_i(t) = \mathbf{E}H_i(V_i(t))$. Therefore we need to show that for any $h > 0$

$$U_i(t+h) - U_i(t) \underset{t\to\infty}{\longrightarrow} \beta_i h. \tag{37}$$

Our proof is based on well-known expansion for renewal functions (see, e.g. [6]). Namely, under condition (34)

$$\begin{aligned} H_i(t) &= \frac{t}{b_i} + c_{i,1} + R_i(t), \\ \mathbf{E}N_i(t) &= \frac{t}{a_i} + c_{i,2} + D_i(t), \end{aligned} \tag{38}$$

where $c_{i,1}$ and $c_{i,2}$ are constants and $R_i(t) \to 0$, $D_i(t) \to 0$ as $t \to \infty$. Moreover,

$$\lim_{t\to\infty} \mathbf{E}\zeta_i(t) = \frac{\mathbf{E}u_{i,1}^2}{2a_i}. \tag{39}$$

From decomposition (38) we have

$$U_i(t) = \mathbf{E}\mathbf{E}(H_i(V_i(t)|V_i(t))) = \frac{\mathbf{E}V_i(t)}{b_i} + c_{i,1} + \mathbf{E}R_i(V_i(t)),$$

$$\mathbf{E}V_i(t) = \frac{a_i^{(2)}}{a_i} t + a_i^{(2)} c_{i,2} + a_i^{(2)} D_i(t) + \mathbf{E}\zeta_i(t).$$

Therefore,

$$U_i(t+h) - U_i(t) = \beta_i h + \frac{a_i^{(2)}}{b_i}(D_i(t+h) - D_i(t)) + b_i^{-1}(\mathbf{E}\zeta_i(t+h) - \mathbf{E}\zeta_i(t)) +$$

$$+\mathbf{E}R_i(V_i(t+h)) - \mathbf{E}R_i(V_i(t)). \tag{40}$$

Since $\lim_{t\to\infty} \mathbf{E}R_i(V_i(t)) = 0$ we get (37) from (39) and (40).

The rest of the proof of the Lemma is the same as the proof of Lemma 2 in [3]. It is based on limit theorems for renewal processes and some estimations that take place in our case. +

For the ergodic theorem we will need the following assumptions.

**Condition 2** *Let for some server $i$ the following inequalities be fulfilled*

- $\mathbf{P}\{u_{i,1}^{(2)} \geq x\} > 0$ for any $x < \infty$;
- $\mathbf{P}\{\xi_1 = 0\} + \mathbf{P}\{\xi_1 = 1, t_1 + \eta_{i,1} + l < \tau_1\} > 0$, where $t_1$ is a moment of the first customer arrival and $l = \min\{j \geq 1 : \mathbf{P}\{u_{i,1}^{(1)} = j\} > 0\}$.

**Theorem 3** *Let Conditions 1, 2 and relations (1), (2), (34) be fulfilled. Then the process $Q(t)$ is ergodic if and only if $\rho^{(2)} < 1$.*

*Proof.* In view of Theorem 1 we need to consider the case $\rho^{(2)} < 1$ only. Note that $Q(t)$ and $Q_n$ are regenerative processes and moments of regeneration are the subsequence $\{T_{n_k}^{(2)}\}_{k\geq 0}$ of the sequence $\{T_n^{(2)}\}_{n\geq 0}$ such that $Q(T_{n_k}^{(2)}) = Q_{n_k} = 0$. Let $y_i(n)$ be the residual service time of a customer in the $i$th server at moment $T_n^{(2)}$ if there is a customer ($y_i(n) = 0$ if there is no customer). Under

Condition 2 the process $Q_n$ has the following properties:

- $\mathbf{P}\{Q_{n+1} = 0 | Q_n = 0\} > 0$;
- for any $j > 0$ and $x > 0$ there exists $m_j(x) > 0$ such that $\mathbf{P}\{Q_{n+m_j(x)} = 0 | Q_n \le j,\ y_i(n) \le x,\ i = \overline{1,m}\} > 0$.

Under this conditions the circumstances of Theorem 1 from [3] are fulfilled, so $Q(t)$ is either ergodic or $Q(t) \xrightarrow{P} \infty$. Assume that $\rho^{(2)} < 1$ and $Q(t) \xrightarrow{P} \infty$. Then

$$q_n = \sum_{i=1}^{m} q_{i,n} \xrightarrow{P} \infty \quad \text{and} \quad q_{i,n} \xrightarrow{P} \infty \tag{41}$$

for all $i = \overline{1,m}$ (see, e.g. [18]). From Lemma 5 we get that for any $\varepsilon > 0$ there exists $n_\varepsilon$ such that for any $n > n_\varepsilon$

$$\mathbf{E}\big(\tilde{Y}(t_n) - \tilde{Y}(t_{n-1})\big) \ge \frac{\beta}{\lambda_X} - \varepsilon = \frac{1}{\rho^{(2)}} - \varepsilon, \quad \text{where} \quad \tilde{Y}(t) = \sum_{i=1}^{m} \tilde{Y}_i(t), \beta = \sum_{i=1}^{m} \beta_i.$$

Therefore,

$$\mathbf{E}q_{n+1} = \mathbf{E}q_n + 1 - \mathbf{E}(\tilde{Y}(t_n) - \tilde{Y}(t_{n-1}) \le \mathbf{E}q_n - \frac{1-\rho^{(2)}}{\rho^{(2)}} + \varepsilon$$

and

$$\mathbf{E}q_{n+1} \le \mathbf{E}q_n$$

if $\varepsilon < \frac{1-\rho^{(2)}}{\rho^{(2)}}$ and $n > n_\varepsilon$. It contradicts (41). Thus, $Q(t)$ is ergodic. +

Let us note that if all servers have the same distribution function of service times, then condition (34) in Theorem 3 can be omitted.

**Remark 2** *So far we consider zero-delayed regenerative flows $X(t)$ and $Y_i(t)$ assuming that*

$$\mathbf{P}(\theta_0 = 0) = \mathbf{P}(s_{i,0}^{(2)} = 0) = 1,\ (\overline{i = 1,m}).$$

Let this condition be not fulfilled and we have delayed regenerative flows. Note that results of Lemmas 1 — 5 on which our proofs of Theorems 2 and 3 are based hold for delayed regenerative flows. We only have to claim

$$\mathbf{P}(\theta_0 < \infty) = \mathbf{P}(s_{i,0}^{(2)} < \infty) = 1,\ (\overline{i = 1,m}).$$

## 9 Conclusion

This paper we are focused on the discrete-time multichannel queueing system with heterogeneous servers that may be unreliable and regenerative input flow. We considered preemptive repeat different service disciplines as well as preemptive resume service disciplines. Exploiting renewal technique we proved instability theorem for queue-length process when traffic coefficient greater or equal than one (Theorem 1). Under some assumptions, based on the regenerative structure of the queue-length process, we established stability theorem, when traffic coefficient less than one (Theorem 2 for preemptive repeat different service disciplines and Theorem 3 for preemptive resume service disciplines). Note that traffic coefficient for preemptive repeat different service disciplines cannot be expressed in terms of moments of service and interruption processes.

There are many further research topics worth conducting. First, the conjecture that diffusion scaled process $Q(t)$ converges to the Brownian motion (diffusion process) when traffic coefficient greater (less) than one remains to be proved for multichannel systems (see [2] for single server case). Second, steady-state distribution of scaled queue-length process is not investigated for multichannel systems. Third, the large deviation problem is also relevant for this system.

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