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The Problem of large deviations. Comparison of the classical and alternative representations, p.1

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Abstract

This year marks the 80th anniversary of the origin of the research problem (see [1]), later called the problem of large deviations. And after the appearance in early 2000 of its alternative (see [4]-[6]), the original version it was natural to call it a classic. In this work, it is proposed to resume the study of both options in the simplest, one-dimensional case, i.e. take the first step in a certain direction. More precisely, In this and subsequent work of M. V. Maslikhin, a comparison of representations of large deviations obtained in the classical (in the style [3]) and alternative (in the style [6]) cases for the normalized sums of the i.i.d.r.v's. and 10 (5 in each work) of different distributions of the summands of these sums is carried out. To conduct this analysis has proven difficult, but the conclusions that they allowed us to make were very interesting.

Keywords: classical and alternative versions of representations of large deviations, the deviation function and its "analogue".

1 Introduction

Among the limit theorems of the probability theory the problem of large deviations (LD) occupies a prominent place. The beginning of its research was laid in the article [1]. And over the past 40 years, by the works of Borovkov A.A. and his disciples (mainly) it has moved forward very much. Moreover, as noted on page IX in [2], interest in LD resumed in the late 90-ies of the 20th century and on two grounds. I will note the second: LD estimates have proved their ability to solve many problems in statistics, statistical mechanics and in the field of applied probability theory. But, of course, there were also the simplest statements of this problem for students, as in [3]. On the other hand, for a long time there was no alternative approach to its analysis, which is not always good. And if I'm not mistaken, it was first proposed in articles [4]-[6].

In the present article (and another article of this collection) an attempt is made to compare the representations of LD for the normalised sums of i.i.d.r.v's. obtained in the classical version from [3] and in the alternative from [6] for several specific one-dimentional distributions of the summands of these sums. Their appearance is specified below, and for the 5 distributions in each article. And at the same time we wanted to understand not only how much easier the representation is in one case compared to the other, which is due to its practical usefulness, but also the complexity of the process of obtaining it. We express our opinion about these sides of ideas not only here, in conclusion, but also in the above article of Maslikhin M. V..

On the view of representations in both cases. First, let's clarify what type of views we will compare, and what elements of these views will have to be searched in each of the 5 cases.

1. Let's start by the representation of LD from theorem 1 in [6]

$$P(S_n > xs_n) = \frac{\varphi^n(z)e^{-\lambda x}}{c\sigma(z)\sqrt{2\pi}}\,(1+\delta_n(\lambda)), \tag{1}$$

$$where\ \ \delta_n(\lambda) = 2\theta\alpha(z)c\sigma(z)\sqrt{2\pi} - \left[1 - J\big(c\sigma(z)\big)\right],\ J(t) = \frac{t\big(1-\Phi(t)\big)}{\Phi'(t)}, \tag{2}$$

$S_n = \sum_1^n X_l$, $s_n^2 = DS_n$, $c = \lambda/d$, θ is such number, for which $|\theta| \le 1$ (the remaining elements are defined below).

It is performed under assumpsions

$$EX_l \equiv 0, \qquad d^2 \equiv DX_l > 0, \tag{3}$$

$$\varphi(\lambda) \equiv Ee^{\lambda X_1} < \infty, \quad 0 \le \lambda < \Delta, \tag{4}$$

under any $0 < \lambda < s_n\Delta = d\Delta\sqrt{n}$.

The connection of the auxiliary parametre λ with the main one x is given by equivalent equations.

$$x = E\zeta\left(\lambda; \frac{S_n}{s_n}\right) \ \Leftrightarrow\ \ x = \frac{n}{s_n}\,m(z),\ \ z = \frac{\lambda}{s_n}. \tag{5}$$

In the first, which determines this relationship, the so-called Cramer transformation $\zeta = \zeta(\lambda;\eta)$ of a r.v. η is used (instead of the deviation function), i.e. a r.v., the distribution of which is determined by the equality

$$dP(\zeta < y) = \frac{e^{\lambda y}}{\varphi(\lambda)}\,dP(\eta < y).$$

However, any specific distribution further, i.e. it's representing r.v. ξ is connected with X_1 as follows: $X_1 = \xi - E\xi$.

For each ξ we will look for the following elements of the representation (1):

$$\varphi(\lambda),\ \ m(\lambda) = E\zeta = \frac{\varphi'}{\varphi},\ \sigma^2(\lambda) = D\zeta = \frac{\varphi''\varphi - \varphi'^2}{\varphi^2},\ \alpha = \alpha(\lambda, X_1) = \tag{6}$$

$$\sup_t|H(t) - \Phi(t)|,\ \ H(t) = P(\zeta < x + \sigma t),\ \Phi(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{t} e^{-y^2/2}dy.$$

In them, 3 functions are defined by the value $\zeta = \zeta(\lambda; X_1)$, and the connection of parameters x and λ of (5) is proved in theorem 1.

2. We also consider the representation of LD from theorem 13 on page 183 of [3],

$$P(S_n \ge y) \sim \frac{1}{\sigma_\alpha\lambda(\alpha)\sqrt{2\pi n}}\,\exp\{-n\Lambda(\alpha)\}, \tag{7}$$

where $S_n = \Sigma_1^n X_l$, $s_n^2 = DS_n$. Only Borovkov uses ξ_l insted of X_l we have, but in any case, the i.i.d.r.v. X_l satisfy conditions

$$EX_l \equiv 0, \qquad DX_l = 1, \tag{8}$$

$$\exists\lambda > 0: \quad \psi(\lambda) = Ee^{\lambda X_1} < \infty \tag{9}$$

and (7) is fulfilled if, in addition to (9) (i.e. Cramer's condition), there are

$$\frac{y}{\sqrt{n}} \to \infty, \qquad \limsup_{n\to\infty}\frac{y}{n} < \alpha_+ = \frac{\psi'(\lambda_+)}{\psi(\lambda_+)}, \tag{10}$$

and also h.f. $\psi^m(t)$, for example, is integrable for some whole $m \ge 1$. And recall, that $\lambda_+ = \sup\{\lambda\colon\ \psi(\lambda) < \infty\}$.

But in addition to the values $\lambda_+,\ \alpha_+$, we will look for functions first

$$\Lambda(\alpha), \quad \lambda(\alpha), \quad \sigma_\alpha, \tag{11}$$

defined below. And, as before, it will have to start with figuring out the functions (in paragraph 1 all the same, except $\alpha(\lambda, X_1)$)

$$\psi(\lambda), \quad \psi'(\lambda), \quad m(\lambda) = \frac{\psi'(\lambda)}{\psi(\lambda)}, \quad m'(\lambda),$$

that is, with the ch.f. of summands in the sum of S_n and their simplest transformations.

The *deviation function* determines the first equality and the second is true:

$$\Lambda(\alpha) = \sup_\lambda\big(\alpha\lambda - \ln\psi(\lambda)\big) = \alpha\lambda(\alpha) - \ln\psi\big(\lambda(\alpha)\big). \tag{12}$$

Point of $\lambda(\alpha)$, in which is achieved the value of $\Lambda(\alpha)$, is determined from

$$\frac{\psi'(\lambda(\alpha))}{\psi(\lambda(\alpha))} = \alpha, \tag{13}$$

since the derivative of the function $\alpha\lambda - \ln\psi(\lambda)$ in this point is 0, and $(\ln\psi(\lambda))' = \psi'(\lambda)/\psi(\lambda)$. But the relation between Λ and λ can be set by equality

$$\Lambda(\alpha) = \int_0^\alpha \lambda(u)du \tag{14}$$

also, since $\Lambda(0) = \lambda(0) = 0$ in force (12), and when using (13) we have

$$\Lambda'(\alpha) = \lambda(\alpha) + \alpha\lambda'(\alpha) - \psi'(\lambda(\alpha))/\psi(\lambda(\alpha))\lambda'(\alpha) = \lambda(\alpha).$$

It remains to clarify the search of σ_α and how it is related to the parameters α and α_+.

First, about the parameter α. In theorems 12 and 13 of [3], p. 182, $\alpha = y/n$. Allocated interval $0 < \alpha < \ alpha_+$ is simply a gap containing the maximum point of the deviation function. And the parameter σ_α is determined by the Cramer transformation $\xi^{(\alpha)}$ of the initial value X_1, i.e. r.v. (for $\lambda = \lambda(\alpha)$), which has the following distribution function

$$G(y) = \frac{1}{\psi(\lambda)} \int_{-\infty}^{y} e^{\lambda t} dP(X_1 < t): \tag{15}$$

$$E\xi^{(\alpha)} = \frac{\psi'(\lambda(\alpha))}{\psi(\lambda(\alpha))} = \alpha, \quad D\xi^{(\alpha)} = \frac{\psi''(\lambda(\alpha))}{\psi(\lambda(\alpha))} - \alpha^2 \equiv \sigma_\alpha^2 > 0,$$

$$in\ other\ words\ \ \sigma_\alpha^2 = m'(\lambda(\alpha)). (16)$$

In this case, any specific distribution further, i.e. representing it r.v. ξ, will have to be linked with X_1 here a bit differently in effect (8):

$$X_1 = \frac{\xi - E\xi}{\sqrt{D\xi}},$$

compared to the alternative approach, it is higher.

And then we consider 5 concrete distributions of r.v. ξ and in each of them we distinguish 3 parts: 1) introduction, in which we recall the known and necessary characteristics of the distribution, and then obtaining the elements of the representation 2) in (1) and 3) in (7). In conclusion, we give a table of (pairwise) representations (1) and (7) in each of 5 cases for comparison.

2 Results

1. Geometric distribution

In this case $P(\xi = k) = pq^k,\ k = 0,1,2,\cdots,\ q = 1 - p,\ 0 < p < 1,$
and it is known that

$$E\xi = \frac{q}{p}, \quad d^2 = D\xi = \frac{q}{p^2}, \quad f(t) = Ee^{t\xi} = \frac{p}{1-qe^t}.$$

1.1 It follows that

$$\varphi(\lambda) = Ee^{\lambda X_1} = Ee^{\lambda(\xi - q/p)} = \frac{pe^{-\lambda q/p}}{1-qe^\lambda} < \infty, \quad 0 \le \lambda < \ln(1/q) = \Delta,$$

$$\varphi'(\lambda) = \frac{-qe^{-\lambda q/p}(1-qe^\lambda) + qe^\lambda pe^{-\lambda q/p}}{(1-qe^\lambda)^2} = \frac{qe^{-\lambda q/p}(e^\lambda - 1)}{(1-qe^\lambda)^2},$$

$$m(\lambda) = E\zeta = \frac{\varphi'(\lambda)}{\varphi(\lambda)} = \frac{q(e^\lambda - 1)}{p(1-qe^\lambda)} > 0, \quad 0 < \lambda < \Delta,\ \zeta = \zeta(\lambda, X_1),$$

$$\sigma^2(\lambda) = D\zeta = m'(\lambda) = \frac{qe^\lambda p(1-qe^\lambda) + qe^\lambda pq(e^\lambda - 1)}{p^2(1-qe^\lambda)^2} = \frac{qe^\lambda}{(1-qe^\lambda)^2}.$$

Thus,

$$\sigma(z) = \frac{e^{z/2}\sqrt{q}}{1-qe^z}, \quad \varphi(z) = \frac{pe^{-zq/p}}{1-qe^z}, \quad (z) = \frac{q(e^z-1)}{p(1-qe^z)}, \quad 0 < z < \Delta.$$

Note that $\alpha(z)$ is well-defined in(19) from [5], and $c\sigma(z)\sqrt{2\pi} = z\sigma(z)\sqrt{2\pi n}$.

1.2 In this case

$$\psi(\lambda)=Ee^{\lambda X_1}=Ee^{\lambda(p\xi-q)/\sqrt{q}}=\frac{pe^{-\lambda\sqrt{q}}}{1-qe^{\lambda p/\sqrt{q}}}<\infty,\quad 0\le\lambda<\frac{\sqrt{q}}{p}\ln(1/q)=\lambda_+,$$

$$\psi'(\lambda)=\frac{-p\sqrt{q}e^{-\lambda\sqrt{q}}(1-qe^{\lambda p/\sqrt{q}})+p\sqrt{q}e^{\lambda p/\sqrt{q}}pe^{-\lambda\sqrt{q}}}{(1-qe^{\lambda p/\sqrt{q}})^2}=\frac{p\sqrt{q}e^{-\lambda\sqrt{q}}(e^{\lambda p/\sqrt{q}}-1)}{(1-qe^{\lambda p/\sqrt{q}})^2},$$

$$m(\lambda)=\frac{\psi'(\lambda)}{\psi(\lambda)}=\frac{p\sqrt{q}e^{-\lambda\sqrt{q}}(e^{\lambda p/\sqrt{q}}-1)}{(1-qe^{\lambda p/\sqrt{q}})^2}\frac{(1-qe^{\lambda p/\sqrt{q}})}{pe^{-\lambda\sqrt{q}}}=\frac{\sqrt{q}(e^{\lambda p/\sqrt{q}}-1)}{1-qe^{\lambda p/\sqrt{q}}},$$

$$\alpha_+=\frac{\psi'(\lambda_+)}{\psi(\lambda_+)}=\frac{\sqrt{q}(1/q-1)}{1-q(1/q)}=\infty.$$

Let's start further with obtaining $\lambda(\alpha)$ using the equation (13):

$$\frac{\psi'(\lambda(\alpha))}{\psi(\lambda(\alpha))}=\alpha\quad\Leftrightarrow\quad\frac{\sqrt{q}(e^{\lambda p/\sqrt{q}}-1)}{1-qe^{\lambda p/\sqrt{q}}}=\alpha.$$

$$\lambda(\alpha)=\frac{\sqrt{q}}{p}\ln\left(\frac{\alpha+\sqrt{q}}{\alpha q+\sqrt{q}}\right).$$

Then use (12) to get $\Lambda(\alpha)$:

$$e^{-\lambda(\alpha)\sqrt{q}}=\frac{\sqrt{q}}{p}\ln\left(\frac{\alpha+\sqrt{q}}{\alpha q+\sqrt{q}}\right)^{-\frac{q}{p}},\ e^{\lambda(\alpha)\frac{\sqrt{q}}{p}}=\frac{\alpha+\sqrt{q}}{\alpha q+\sqrt{q}},\ 1-qe^{\lambda(\alpha)\frac{\sqrt{q}}{p}}=\frac{p\sqrt{q}}{\alpha q+\sqrt{q}},$$

$$\psi(\lambda(\alpha))=p\left(\frac{\alpha+\sqrt{q}}{\alpha q+\sqrt{q}}\right)^{-\frac{q}{p}}\frac{\alpha q+\sqrt{q}}{p\sqrt{q}}=\frac{1}{\sqrt{q}}(\alpha+\sqrt{q})^{-q/p}(\alpha q+\sqrt{q})^{1/p},$$

$$\Lambda(\alpha)=\frac{\alpha\sqrt{q}}{p}\ln\left(\frac{\alpha+\sqrt{q}}{\alpha q+\sqrt{q}}\right)-\ln\left\{\left(\frac{\alpha+\sqrt{q}}{\alpha q+\sqrt{q}}\right)^{-\frac{q}{p}}(\alpha q+\sqrt{q})\right\}+\ln\sqrt{q}=$$

$$\frac{\sqrt{q}}{p}(\alpha+\sqrt{q})\ln(\alpha+\sqrt{q})-\frac{1}{p\sqrt{q}}(\alpha q+\sqrt{q})\ln(\alpha q+\sqrt{q})+\ln\sqrt{q}.$$

You can check the expression found by using (14):

$$\Lambda(\alpha)=\int_0^\alpha\lambda(u)du=\frac{\sqrt{q}}{p}\left(\int_0^\alpha\ln(u+\sqrt{q})du-\int_0^\alpha\ln(uq+\sqrt{q})du\right),$$

$$\int_0^\alpha\ln(u+\sqrt{q})du=\alpha\ln(\alpha+\sqrt{q})-\alpha+\sqrt{q}(\ln(\alpha+\sqrt{q})-\ln\sqrt{q}),$$

$$\int_0^\alpha\ln(uq+\sqrt{q})du=\alpha\ln(\alpha q+\sqrt{q})-\alpha+\frac{1}{\sqrt{q}}(\ln(\alpha q+\sqrt{q})-\ln\sqrt{q}),$$

$$\Lambda(\alpha)=\frac{\sqrt{q}}{p}\{\alpha\ln(\alpha+\sqrt{q})-\alpha+\sqrt{q}(\ln(\alpha+\sqrt{q})-\ln\sqrt{q})-$$

$$[\alpha\ln(\alpha q+\sqrt{q})-\alpha+\frac{1}{\sqrt{q}}(\ln(\alpha q+\sqrt{q})-\ln\sqrt{q})]\},$$

$$\Lambda(\alpha)=\frac{\sqrt{q}}{p}(\alpha+\sqrt{q})\ln(\alpha+\sqrt{q})-\frac{1}{p\sqrt{q}}(\alpha q+\sqrt{q})\ln(\alpha q+\sqrt{q})+\ln\sqrt{q}.$$

It remains to find σ_α^2. But

$$m'(\lambda)=\left(\frac{\sqrt{q}(e^{\lambda p/\sqrt{q}}-1)}{1-qe^{\lambda p/\sqrt{q}}}\right)=\frac{pe^{\lambda p/\sqrt{q}}(1-qe^{\lambda p/\sqrt{q}})+qpe^{\lambda p/\sqrt{q}}(e^{\lambda p/\sqrt{q}}-1)}{(1-qe^{\lambda p/\sqrt{q}})^2}$$

$$=\frac{p^2e^{\lambda p/\sqrt{q}}}{(1-qe^{\lambda p/\sqrt{q}})^2},$$

$$\sigma_\alpha^2=m'(\lambda(\alpha))=\frac{p^2(\alpha+\sqrt{q})}{\alpha q+\sqrt{q}}\left(\frac{\alpha q+\sqrt{q}}{p\sqrt{q}}\right)^2=\frac{1}{q}(\alpha+\sqrt{q})(\alpha q+\sqrt{q})=$$

$$(\alpha/\sqrt{q}+1)(\alpha\sqrt{q}+1)=\alpha^2+\alpha(\sqrt{q}+1/\sqrt{q})+1>(\alpha+1)^2,\quad\forall q,\ 0<q<1.$$

2. Poisson distribution

In this case $P(\xi = k) = \frac{\nu^k}{k!} e^{\nu}, \quad k = 0,1,2,\cdots, \quad nu > 0$, and

$$E\xi = \nu, \quad d^2 = D\xi = \nu, \quad f(t) = Ee^{t\xi} = e^{\nu(e^t - 1)} < \infty, \quad > 0.$$

2.1 It follows that

$$\varphi(\lambda) = Ee^{\lambda X_1} = Ee^{\lambda(\xi - \nu)} = e^{\nu(e^{\lambda} - \lambda - 1)} < \infty, \quad 0 \le \lambda < \infty = \Delta,$$

$$\varphi'(\lambda) = \nu(e^{\lambda} - 1)e^{\nu(e^{\lambda} - \lambda - 1)},$$

$$m(\lambda) = E\zeta = \frac{\varphi'(\lambda)}{\varphi(\lambda)} = \nu(e^{\lambda} - 1) > 0, \quad \lambda > 0,$$

$$\sigma^2(\lambda) = D\zeta = m'(\lambda) = \nu e^{\lambda}.$$

Thus

$$\sigma(z) = \sqrt{\nu}\, e^{z/2}, \quad \varphi(z) = e^{\nu(e^z - z - 1)}, \quad (z) = \nu(e^z - 1), \quad 0 < z < \infty.$$

2.2 In this case

$$\psi(\lambda) = Ee^{\lambda X_1} = Ee^{\lambda(\xi - \nu)/\sqrt{\nu}} = e^{\nu(e^{\lambda/\sqrt{\nu}} - \lambda/\sqrt{\nu} - 1)} < \infty, \quad 0 \le \lambda < \infty = \lambda_+,$$

$$\psi'(\lambda) = \sqrt{\nu}(e^{\lambda/\sqrt{\nu}} - 1)e^{\nu(e^{\lambda/\sqrt{\nu}} - \lambda/\sqrt{\nu} - 1)},$$

$$m(\lambda) = \frac{\psi'(\lambda)}{\psi(\lambda)} = \sqrt{\nu}(e^{\lambda/\sqrt{\nu}} - 1), \quad \alpha_+ = m(\lambda_+) = \infty.$$

Let's start further with obtaining $\lambda(\alpha)$ using the equation (13):

$$\frac{\psi'(\lambda(\alpha))}{\psi(\lambda(\alpha))} = \alpha \quad \Leftrightarrow \quad \sqrt{\nu}(e^{\lambda(\alpha)/\sqrt{\nu}} - 1) = \alpha.$$

$$\lambda(\alpha) = \sqrt{\nu}\ln(1 + \alpha/\sqrt{\nu}).$$

Then use (12) to get $\Lambda(\alpha)$:

$$\psi(\lambda(\alpha)) = e^{\nu(1 + \alpha/\sqrt{\nu} - \ln(1 + \alpha/\sqrt{\nu}) - 1)} = \frac{e^{\alpha\sqrt{\nu}}}{(1 + \alpha/\sqrt{\nu})^{\nu}},$$

$$\ln\psi(\lambda(\alpha)) = \nu(\alpha/\sqrt{\nu} - \ln(1 + \alpha/\sqrt{\nu})),$$

$$\Lambda(\alpha) = \alpha\sqrt{\nu}\ln(1 + \alpha/\sqrt{\nu}) - \alpha\sqrt{\nu} + \nu\ln(1 + \alpha/\sqrt{\nu}) =$$

$$\nu\left[(1 + \alpha/\sqrt{\nu})\ln(1 + \alpha/\sqrt{\nu}) - \alpha/\sqrt{\nu}\right].$$

Finding σ_α^2 is easier this time:

$$m'(\lambda) = e^{\lambda/\sqrt{\nu}} \quad \Rightarrow \quad \sigma_\alpha^2 = m'(\lambda(\alpha)) = e^{\lambda(\alpha)/\sqrt{\nu}} = 1 + \alpha/\sqrt{\nu}.$$

3. Exponential distribution

The density of this distribution is $p(x) = \mu e^{-\mu x}, \; x \ge 0 \; (\mu > 0)$, with

$$E\xi = 1/\mu, \quad d^2 = D\xi = 1/\mu^2, \quad f(t) = Ee^{t\xi} = \frac{\mu}{\mu - t} < \infty, \quad 0 < t < \mu.$$

3.1 It follows that

$$\varphi(\lambda) = Ee^{\lambda X_1} = Ee^{\lambda(\xi - 1/\mu)} = \frac{\mu e^{-\lambda/\mu}}{\mu - \lambda} < \infty, \quad 0 < \lambda < \mu = \Delta,$$

$$\varphi'(\lambda) = \frac{-e^{-\lambda/\mu}(\mu - \lambda) + \mu e^{-\lambda/\mu}}{(\mu - \lambda)^2} = \frac{\lambda e^{-\lambda/\mu}}{(\mu - \lambda)^2},$$

$$m(\lambda) = E\zeta = \frac{\varphi\prime(\lambda)}{\varphi(\lambda)} = \frac{\lambda}{\mu(\mu-\lambda)},$$

$$\sigma^2(\lambda) = D\zeta = m'(\lambda) = \frac{\mu(\mu-\lambda)+\lambda\mu}{\mu^2(\mu-\lambda)^2} = \frac{1}{(\mu-\lambda)^2}.$$

Thus

$$\sigma(z) = \frac{1}{\mu-z}, \quad \varphi(z) = \frac{\mu e^{-z/\mu}}{\mu-z}, \quad (z) = \frac{z}{\mu(\mu-z)}.$$

3.2 In this case

$$\psi(\lambda) = Ee^{\lambda X_1} = Ee^{\lambda(\mu\xi-1)} = \frac{\mu e^{-\lambda}}{\mu-\lambda\mu} = \frac{e^{-\lambda}}{1-\lambda} < \infty, \quad 0 < \lambda < 1 = \lambda_+,$$

$$\psi'(\lambda) = \frac{-e^{-\lambda}(1-\lambda)+e^{-\lambda}}{(1-\lambda)^2} = \frac{\lambda e^{-\lambda}}{(1-\lambda)^2},$$

$$m(\lambda) = \frac{\lambda}{1-\lambda}, \qquad \alpha_+ = m(\lambda_+) = \infty.$$

Let's start further with obtaining $\lambda(\alpha)$ using the equation (13):

$$\frac{\psi\prime(\lambda(\alpha))}{\psi(\lambda(\alpha))} = \alpha \quad \Leftrightarrow \quad \frac{\lambda}{1-\lambda} = \alpha \ \Rightarrow \ \lambda(\alpha) = \frac{\alpha}{1+\alpha}.$$

Then use (12) to get $\Lambda(\alpha)$:

$$\psi(\lambda(\alpha)) = \frac{e^{-\frac{\alpha}{1+\alpha}}}{1-\frac{\alpha}{1+\alpha}} = (1+\alpha)e^{-\frac{\alpha}{1+\alpha}}, \ \ \ln\psi(\lambda(\alpha)) = \ln(1+\alpha) - \frac{\alpha}{1+\alpha},$$

$$\Lambda(\alpha) = \alpha\frac{\alpha}{1+\alpha} + \frac{\alpha}{1+\alpha} - \ln(1+\alpha) = \alpha - \ln(1+\alpha).$$

Finding σ_α^2 is even easier this time: $\sigma_\alpha^2 = m'(\lambda(\alpha)) = (1+\alpha)^2$.

4. χ^2 −distribution with k degrees of freedom

A random variable ξ has such a distribution if its density

$$p(x) = \frac{e^{-x/2}}{2^{k/2}\Gamma(k/2)}\, x^{k/2-1}, \ \ x > 0.$$

In this case $E\xi = k, \quad d^2 = D\xi = 2k, \ \ f(t) = Ee^{t\xi} = (1-2t)^{-k/2}$.

4.1 It follows that

$$\varphi(\lambda) = Ee^{\lambda X_1} = Ee^{\lambda(\xi-k)} = \frac{e^{-\lambda k}}{(1-2\lambda)^{k/2}} < \infty, \quad 0 \le \lambda < 1/2 = \Delta,$$

$$\varphi'(\lambda) = \frac{-ke^{-\lambda k}(1-2\lambda)^{k/2} + k(1-2\lambda)^{k/2-1}\, e^{-\lambda k}}{(1-2\lambda)^k} = \frac{2k\lambda e^{-\lambda k}}{(1-2\lambda)^{k/2+1}}.$$

$$m(\lambda) = \frac{\varphi\prime(\lambda)}{\varphi(\lambda)} = \frac{2k\lambda}{1-2\lambda}, \ \sigma^2(\lambda) = m'(\lambda) = \frac{2k(1-2\lambda)+2[2k\lambda]}{(1-2\lambda)^2} = \frac{2k}{(1-2\lambda)^2}.$$

Thus

$$\sigma(z) = \frac{\sqrt{2k}}{1-2z}, \qquad \varphi(z) = \frac{e^{-kz}}{(1-2z)^{k/2}}, \qquad m(z) = \frac{2kz}{1-2z}.$$

4.2 In this case $b = \sqrt{k/2}$

$$\psi(\lambda) = Ee^{\lambda X_1} = Ee^{\lambda(\xi-k)/\sqrt{2k}} = \frac{e^{-\lambda b}}{(1-\lambda/b)^{k/2}} < \infty, \quad 0 \le \lambda < b = \lambda_+,$$

$$\psi'(\lambda) = \frac{-be^{-\lambda b}(1-\lambda/b)^{b^2} + b(1-\lambda/b)^{b^2-1}e^{-\lambda b}}{(1-\lambda/b)^k} = \frac{\lambda e^{-\lambda b}}{(1-\lambda/b)^{k/2+1}},$$

$$m(\lambda) = E\zeta = \frac{\varphi\prime(\lambda)}{\varphi(\lambda)} = \frac{\lambda}{1-\lambda/b}, \qquad \alpha_+ = m(\lambda_+) = \infty.$$

Let's start further with obtaining $\lambda(\alpha)$ using the equation (13):

$$\frac{\psi\prime(\lambda(\alpha))}{\psi(\lambda(\alpha))} = \alpha \quad \Leftrightarrow \quad \frac{\lambda}{1-\lambda/b} = \alpha \ \Rightarrow \ \lambda(\alpha) = \frac{\alpha b}{\alpha+b}.$$

Then use (12) to get $\Lambda(\alpha)$:

$$\psi(\lambda(\alpha)) = \left(\frac{\alpha+b}{b}\right)^{b^2} e^{-\frac{b^2\alpha}{\alpha+b}}, \ \ \ln\psi(\lambda(\alpha)) = b^2[\ln(\alpha+b) - \ln b] - \frac{b^2\alpha}{\alpha+b},$$

$$\Lambda(\alpha)=\frac{\alpha^2 b+\alpha b^2}{\alpha+b}-b^2[\ln(\alpha+b)-\ln b]=b\alpha-b^2[\ln(\alpha+b)-\ln b].$$

Search σ_α^2 and this time simple:

$$m'(\lambda)=\frac{1-\lambda/b+\lambda/b}{(1-\lambda/b)^2}=\left(\frac{b}{b-\lambda}\right)^2 \quad\Rightarrow\quad \sigma_\alpha^2=m'(\lambda(\alpha))=\left(\frac{\alpha+b}{b}\right)^2.$$

5. Triangular distribution on the segment $[\boldsymbol{a},\boldsymbol{b}]$

In this case, the density of r.v. ξ and other indicators are

$$p(x)=\frac{2}{b-a}\left(1-\frac{|a+b-2x|}{b-a}\right),\quad x\in\ [a,b],$$

$$E\xi=\frac{a+b}{2},\quad D\xi=\frac{(b-a)^2}{24}=d^2,\ (t)=Ee^{t\xi}=\left[\frac{2(e^{tb/2}-e^{ta/2})}{t(b-a)}\right]^2.$$

5.1 At this time $(c=(a+b)/2,\ \delta=(b-a)/2)$

$$\varphi(\lambda)=Ee^{\lambda(\xi-c)}=\left[\frac{2(e^{\lambda\delta/2}-e^{-\lambda\delta/2})}{\lambda(b-a)}\right]^2=\left[\frac{(e^{\lambda\delta/2}-e^{-\lambda\delta/2})}{\lambda\delta}\right]^2,$$

$$\varphi'(\lambda)=2\left(\frac{e^{\frac{\lambda\delta}{2}}-e^{-\frac{\lambda\delta}{2}}}{\lambda\delta}\right)\frac{\frac{\delta}{2}(e^{\frac{\lambda\delta}{2}}+e^{-\frac{\lambda\delta}{2}})\lambda\delta-\delta(e^{\frac{\lambda\delta}{2}}-e^{-\frac{\lambda\delta}{2}})}{\lambda^2\delta^2}=$$

$$\frac{e^{\lambda\delta}-e^{-\lambda\delta}}{\lambda^2\delta}-\frac{2(e^{\lambda\delta/2}-e^{-\lambda\delta/2})^2}{\lambda^3\delta^2},\ (\lambda)=\delta\left(\frac{e^{\lambda\delta/2}+e^{-\lambda\delta/2}}{e^{\lambda\delta/2}-e^{-\lambda\delta/2}}\right)-\frac{2}{\lambda},$$

$$\sigma^2(\lambda)=D\zeta=m'(\lambda)=\frac{\delta^2}{2}\left[1-\left(\frac{e^{\lambda\delta/2}+e^{-\lambda\delta/2}}{e^{\lambda\delta/2}-e^{-\lambda\delta/2}}\right)^2\right]+\frac{2}{\lambda^2},$$

5.2 In this case $(\gamma=\delta/2d=\sqrt{3/2})$

$$\psi(\lambda)=Ee^{\lambda X_1}=Ee^{\lambda(\xi-c)/d}=\left(\frac{e^{\lambda\gamma}-e^{-\lambda\gamma}}{2\lambda\gamma}\right)^2<\infty,\quad 0\le\lambda<\infty=\lambda_+,$$

$$\psi'(\lambda)=2\left(\frac{e^{\lambda\gamma}-e^{-\lambda\gamma}}{2\lambda\gamma}\right)\left[\frac{\gamma(e^{\lambda\gamma}+e^{-\lambda\gamma})2\lambda\gamma-2\gamma(e^{\lambda\gamma}-e^{-\lambda\gamma})}{(2\lambda\gamma)^2}\right]=$$

$$\frac{2}{\lambda}\left[\frac{e^{2\lambda\gamma}-e^{-2\lambda\gamma}}{4\lambda\gamma}-\left(\frac{e^{\lambda\gamma}-e^{-\lambda\gamma}}{2\lambda\gamma}\right)^2\right],$$

$$m(\lambda)=\frac{\psi\prime(\lambda)}{\psi(\lambda)}=\frac{2}{\lambda}\left(\frac{\lambda\gamma(e^{\lambda\gamma}+e^{-\lambda\gamma})}{e^{\lambda\gamma}-e^{-\lambda\gamma}}-1\right),\ \alpha_+=m'(\lambda_+)=\infty.$$

$$m'(\lambda)=2\gamma^2\left[1-\frac{(e^{\lambda\gamma}+e^{-\lambda\gamma})^2}{(e^{\lambda\gamma}-e^{-\lambda\gamma})^2}+\frac{1}{(\lambda\gamma)^2}\right],$$

Basic rates $\lambda(\alpha),\ \Lambda(\alpha),\ \sigma_\alpha^2$ are defined from the equality:

$$m(\lambda(\alpha))=\alpha,\quad \sigma_\alpha^2=m'(\lambda(\alpha)),\quad \Lambda(\alpha)=\alpha\lambda(\alpha)-\ln\psi(\lambda(\alpha)).$$

As you can see, the main thing is the definition of $\lambda(\alpha)$ from the first equation. We haven't decided yet.

Two types of representations that arise

1. Representation

$$P(S_n>xs_n)=\frac{\varphi^n(z)e^{-\lambda x}}{c\sigma(z)\sqrt{2\pi}}(1+\delta_n(\lambda)),(1)$$

in which the main parameter x is associated with the auxiliary λ by equality

$$x = \frac{n}{s_n} m(z), \ z = \lambda / s_n,$$

is true under (3), (4) for any $0 < \lambda < s_n \Delta = d\Delta\sqrt{n}$.

By virtue of theorem 1 of [6], the first term $\delta_n(\lambda)$ is $O(1/\sqrt{n})$, and the smallness of the second determines fast convergence $J(x) \uparrow 1, \ x \mapsto \infty$. So below in (1) we can put $\delta_n(\lambda) = 0$,

2. Asymptotic equivalence

$$P(S_n \geq y) \sim \frac{1}{\sigma_\alpha \lambda(\alpha)\sqrt{2\pi n}} \exp\{-n\Lambda(\alpha)\}, \quad \alpha = y/n, (7)$$

occurs if the conditions (8)-(10) are met and also ch.f $\varphi^m(t)$, for example ($\varphi(t) = Ee^{it\xi}$), is integrable for some whole $m \geq 1$.

And now let us present the representations of (1) and (7) in each of 5 cases, and indicate the form $\delta_n(\lambda)$) in (1) we will not, assuming it is zero and thus turning into "similar" (7) equivalence.

1. Geometric distribution

$$1.1 \ P(S_n > xs_n) \sim \left(\frac{pe^{-qz/p}}{1-qe^z}\right)^n \frac{d(1-qe^z)e^{-\lambda x}}{\lambda e^{z/2}\sqrt{2\pi q}}, \quad \Delta = \ln(1/q),$$

$$x = \frac{q(e^z-1)\sqrt{n}}{dp(1-qe^z)}.$$

$$1.2 \ (\lambda_+ = \frac{\sqrt{q}}{p}\ln(1/q), \ \alpha_+ = \infty) \quad - \quad P(S_n \geq y) \sim$$

$$\frac{\exp\{-n[\frac{\sqrt{q}}{p}(\alpha+\sqrt{q})\ln(\alpha+\sqrt{q}) - \frac{1}{p\sqrt{q}}(\alpha q+\sqrt{q})\ln(\alpha q+\sqrt{q})+\ln\sqrt{q}]\}}{(\alpha/\sqrt{q}+1)(\alpha\sqrt{q}+1)\frac{\sqrt{q}}{p}\ln\left(\frac{\alpha+\sqrt{q}}{\alpha q+\sqrt{q}}\right)\sqrt{2\pi n}}.$$

2. Poisson distribution

$$2.1 \ P(S_n > xs_n) \sim \frac{d\exp(\mu n(e^z-z-1)-\lambda x)}{\lambda e^{z/2}\sqrt{2\pi\mu}}, \ x = \frac{\sqrt{n}}{d}\mu(e^z-1), \ \Delta = \infty.$$

$$2.2 \ P(S_n \geq y) \sim \frac{\exp(-n\mu[(1+\alpha/\sqrt{\mu})\ln(1+\alpha/\sqrt{\mu})-\alpha\sqrt{\mu}])}{(1+\alpha/\sqrt{\mu})(\sqrt{\mu}\ln(1+\alpha\sqrt{\mu})\sqrt{2\pi n})},$$

$$\lambda_+ = \infty = \alpha_+.$$

3. Exponential distribution

$$3.1 \ P(S_n > xs_n) \sim \frac{d\mu^n e^{zn/\mu-\lambda x}}{\lambda(\mu-z)^{n-1}\sqrt{2\pi}}, \quad x = \frac{\lambda}{\mu d^2(\mu-z)}, \quad \Delta = \mu.$$

$$3.2 \ P(S_n \geq y) \sim \frac{(1+\alpha)^n e^{-n\alpha}}{\alpha\sqrt{2\pi n}}, \quad \lambda_+ = 1, \ \alpha_+ = \infty$$

4. Chi-square distribution

$$4.1 \ P(S_n > xs_n) \sim \frac{de^{-(nkz+x\lambda)}}{2\lambda(1-2z)^{nk/2-1}\sqrt{\pi k}}; \quad x = \frac{2kz\sqrt{n}}{d(1-2z)}, \ \Delta = 1/2.$$

$$4.2 \ P(S_n \geq y) \sim \frac{e^{-nb\alpha}}{\alpha\sqrt{2\pi n}}\left(\frac{\alpha+b}{b}\right)^{nb^2}, \quad \lambda_+ = b = \sqrt{k/2}, \quad \alpha_+ = \infty.$$

5. Triangular distribution

$$5.1 \ (x = \frac{\sqrt{n}}{d}\left\{\delta\left(\frac{e^{z\delta/2}+e^{-z\delta/2}}{e^{z\delta/2}-e^{-z\delta/2}}\right) - \frac{2}{z}\right\}, \ \Delta = \infty) \ - \ P(S_n > xs_n) \sim$$

$$\frac{de^{-\lambda x}}{\lambda}\left(\frac{e^{z\delta/2}-e^{-z\delta/2}}{z\delta}\right)^{2n}\left\{\left(\frac{\delta^2}{2}\left[1-\left(\frac{e^{z\delta/2}+e^{-z\delta/2}}{e^{z\delta/2}-e^{-\delta/2}}\right)^2\right]+\frac{2}{z^2}\right)2\pi n\right\}^{-1/2}$$

5.2 In this case, $\lambda_+ = \alpha_+ = \infty$, and the values $\Lambda(\alpha)$ and σ_α^2 are easily found when $\lambda(\alpha)$ is known. But it is determined from the equation

$$\frac{2}{\lambda}\left(\frac{\lambda\gamma(e^{\lambda\gamma}+e^{-\lambda\gamma})}{e^{\lambda\gamma}-e^{-\lambda\gamma}}-1\right)=\alpha \qquad (17)$$

regarding λ, which cannot be solved yet.

3 Conclusion

The aim of this work was only to compare the representations of the two types and the difficulty of obtaining them, and not to consider other issues, the occurrence of which is natural, especially since the comparison itself proved to be very cumbersome. Thus, it was possible

· to transform the equation (17) in some way to obtain its approximate solution in some area of the parameter α,

· or to try to simplify the expressions themselves in the views, for example, replacing z by λ or α by y,

· or to bring a table for better comparison of views, etc.

	1	2	3	4	5
Δ	$\ln(1/q)$	∞	μ	$1/2$	∞
λ_+	$\frac{\sqrt{q}}{p}\ln(1/q)$	∞	1	$\sqrt{k/2}$	∞
α_+	∞	∞	∞	∞	∞

(in it, the numbers at the top represent the numbers of the distributions in the text).

It must be admitted, however, that after receiving the first submissions, the duality became clear set goal. On the one hand, complicated calculations led to certain expressions, but on the other the parties are immediately found not only their "uncircumcised" form, i.e. there is a problem of optimizing the type of representations, but also one more and more important point: to understand the practical usefulness, it will be necessary to carry out their serious analysis, and in other words, time. Therefore, we will limit ourselves to the simplest conclusions.

The receipt of all submissions (1) was notably simpler than all submissions (7). On the other hand, the right side of the submissions is approximately the same in cases 3. and 4. for both types, is much more cumbersome in cases 1. and 2. for submission (7) and finally, an alternative submission in case 5. managed to get (unlike the classic), although it turned out to be a bulky slick.

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