On another approach to the analysis of the known problem of optimal stopping, p.1

S.V. Zhulenev
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Lomonosov Moscow State University
mailto: szhulenev@yandex.ru

Abstract

In the well-known optimal stopping problem, it was always clear that there must be a connection between the type of the objective function or, in other words, the type of surface in three-dimensional space and the specific optimum stopping time. But it was unclear how this relationship discover. In this work and the following two, a simple idea is realized to establish this connection. It comes down to replacing of the initial and very large stop set consisting of Markov moments with respect to the flow of Sigma algebras generated by the random walk under consideration to a simplified stop set consisting of integer random variables. Moreover, in this part 1, the domain of the new objective function definition on the integer lattice of the plane is specified, the condition is given, when the optimal moment is 0, and also mention the known results from the combinatorics used in other parts. The following two parts explain what this relationship is for small horizons n.

Keywords: the known optimal stopping problem, the relationship between its objective function and the optimal stopping time

1 Introduction

In this paper we propose to implement a simple idea with respect to the well-known optimal stopping problem for the standard random walks of a particle on an integer lattice of a plane and on a finite time interval $0,1,\cdots,n$. Sets this walk sequence of r.v. $S=(S_k)_{0\le k\le n}$:

$$S_0=0,\quad S_k=\sum_{l=1}^{k}X_l,\quad X_1,\cdots,X_n - \text{i.e.d.r.v's with distribution}$$
$$P(X_l=1)=p,\ P(X_l=-1)=q,\quad p+q=1,\quad 1\le l\le n.$$

The probability of p is considered to be an arbitrary real number from the interval $0<p<1$. And mentioned the problem of optimal stopping

$$V=\sup_{\tau\in\mathcal{M}}Ef(S_\tau,M_n) \tag{1}$$

is determined by some function $f(s,t)$, i.e. surface over a plane (s,t), the maximum of the total trajectory $M_n=\sup_{0\le k\le n}S_k$ on the segment $[0,n]$ and the set $\mathcal{M}$ of Markov moments τ (with respect to the flow of σ −algebras generated by the sequence S) with possible values $0\le\tau\le n$, and

$$S_\tau=\sum_0^n S_k I(\tau=k). \tag{2}$$

The very idea, if briefly, is reduced to a significant reduction of the set $\mathcal{M}$ of Markov moments in (1) to the set $\mathcal{M}_1$ of integer and constant r.v. $\tau \equiv k,\ 0 \le k \le n$. The goal is due to the certainty that in this case it will be possible to establish a rigid and simple connection of the surface type $f(s,t)$ with the optimal moment (OM) τ_*. And then return to the problem (1) with the baggage of the information obtained in the analysis and in some sense "complete" the study of the initial task (1). After all, if you believe, say, the research works [1]-[3]), and I am not mistaken, it is far from "completion". So the proposed idea boils down to a desire to better understand the problem (1), as it were

going into it from the back door

2 Action plan to implement the idea

We turned to combinatorics, which is considered a "heavy tool". But we offer to facilitate the analysis, acting with the help of its well-known means. In addition, we started the analysis so as to quickly verify the validity of the idea, for which we checked it at small values of horizon n. Thus, in our analysis it is natural to single out the first step and make it from 3 parts, which are it says below. Let's finish it on the 2nd step.

To briefly describe the first step, we first indicate a simplified mathematics in accordance with the idea of the problem. As follows from (2), if $\tau \equiv k$, then $S_\tau = S_k$ and therefore (1) turns into equality

$$V = \max_{0 \le k \le n} V_k, \quad where \quad V_k = Ef(S_k, M_n). \tag{3}$$

So, our idea is that any integer $k,\ 0 \le k \le n$, not only can be OM, but also that any such moment itself is rigidly connected with some form of the surface $y = f(s,t)$. So our goal is to answer the question: what is the shape (there are infinitely many surfaces, but the shape is the only one!), corresponds to a particular OM for an arbitrary but finite n? In this case, the surface $y = f(s,t)$ we call a canvas stretched over a collection of points in three-dimensional space (s,t,y) over a certain area D of the integer lattice of the plane (s,t).

What goals were achieved at the first step? This part 1 clarifies the meaning of the idea, determines the above domain D, and also offers a simple optimality condition $k = 0$. In addition, it has a section of auxiliary results used in the future. The following parts 2 and 3 show that the idea is true for small n. But part 2 considers the case of odd $n = 5$, a part 3 of the case of even $n = 6$, because the calculations and analysis are markedly different.

3 The domain of the function $f(s,t)$

Let's call the set D of possible values of the vector (S_k, M_n) on the considered random tree- i.e. on each of its 2^n trajectories - and for any k. Of all such integer points, the domain d takes place

Theorem 1. *Area D belongs to " parallelogram"*
$$G = \{(s,t):\ t - n \le s \le t,\quad 0 \le t \le n\}, \qquad n \ge 1,$$
on the integer lattice of the plane with number of points $(n+1)^2$. More precisely, $D = G$ for $n = 1$, a for $n > 1$ the bases and the right side are the same, and the left side D is concave inside the parallelogram. If we define the area D as some set of nodes (s,t), whatever $n = 3d - 1$, $3d,\ or\ \ 3d + 1$ C some $d \ge 1$ and $s_t = [(n-t)/2],\ where\ \ [a]\ -\ mboxinteger\ part\ numbers\ a \ge 0$,

$$D = \{(s,t):\ 2t - n \le s \le t,\ 0 \le t \le d;\ -s_t \le s \le t,\ d < t \le n\}.$$

Proof. To support the above General presentation make it clear, just indicate it first in a particular case, with maximum visibility, for which we will present two diagrams below the binomial tree and the D and G areas at $n = 5 = 2m + 1$.

In this case, the binomial tree of the random process the walks $S = (S_k)_{0 \le k \le 5 = n}$ above allows you to do the following to associate the desired set of D (31 bold point) parallelogram G below (36 points; empty circles – nodes $(s, t) \in G$, not owned by D).

Demonstration. To support the above General presentation make it clear, just indicate it first in a particular case, with maximum visibility, for which we will present two diagrams below the binomial tree and the D and G areas at $n = 5 = 2m + 1$.

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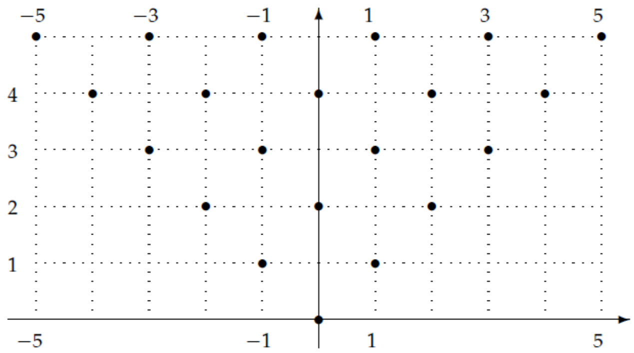

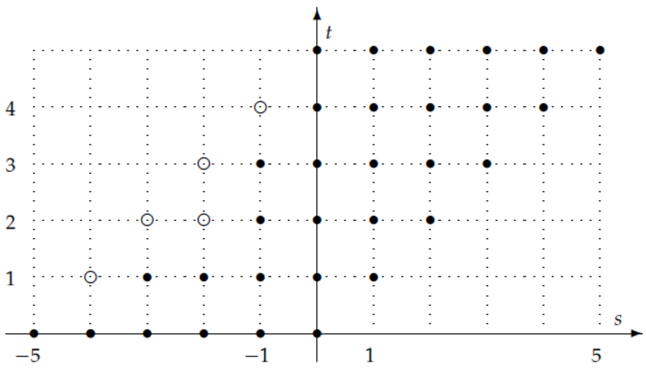

And then explain why it turns out about the same in General if through t denote the possible value of the maximum of the total trajectories M_n, and through s – possible values on the same trajectory C. V. S_k, $0 \le k \le n$. So, first, obviously, that under any $t \ge 0$ the set s, $0 \le s \le t$, necessarily belongs to the set D_t −horizontal cross section of the sought area D at t. So ... that the problem is only in the negative values of s in the set D_t. Second, there are only two variants of the trajectories of the particle with the maximum of t, which can be called extreme or left and right, because the rest lead to intermediate results: 1) first $t$is made movements to the right, then all of the remaining $n - 2K$ steps to the left, 2) first, it is s_t moves to the left, then the same amount to the right (when $2s_t + t \le n$), and then the remaining $n - 2s_t$ steps t to the right and possibly another one to the left.

Consider the following two lines on the plane (s, t)

$$1)\ t = 2s + n, \qquad 2)\ 2t = s + n$$

(the first passes through the points $(0, n)$ and $(-1, n - 2)$, and the second passes through

$(-n, 0)$ and $(2-n, 1)$) intersecting at $(-\frac{n}{3}, \frac{n}{3})$. It follows from the above that these direct limit the sought the area on the left (first from above!). More precisely, its boundary is determined by the following image:

$$s = -s_t \ \ when \ \ t \geq \frac{n}{3}, \qquad s = 2t - n, \ \ when \ \ t \leq \frac{n}{3}.$$

But this means that the theorem is proved by ■

4 On the connection of OM with the surface shape $y = f(s, t)$

We give one simple statement, for which we consider the condition:

$$\forall t, \ 0 \leq t \leq n: \ \ f(s,t) < f(0,t), \ \ mboxif \ \ \ s \neq 0, \ (s,t) \in D. \qquad (4)$$

We show that, for a finite $n \geq 1$, the Corollary 1 below implies

Theorem 2. *under condition (4) regardless of* p *and* n

$$V = V_0$$

Proof. On the one hand, this corollary says (although in it $p = 1/2$, $n = 2m+1$) that $P(f(S_0, M_n) = f(0,t)) = 2^{-n} C_n^{m+u+2}$. On the other hand, it is clear that the possible values C. V. $f(S_k, M_n)$ at $\forall k > 0$ are the values of $f(2l-k, t)$, $t \leq l \leq [(t+k)/2]$. However, despite the fact that there are several, the total probability of their adoption is higher probabilities $2^{-n} C_n^{m+u+2}$. So (4) $\Rightarrow$ $V = \max_{0 \leq k \leq n} V_k = v_0$. And clear, that other values of p, $n = 2m$ will not affect the outputs of ■

5 Auxiliary results

Let us recall some certain facts, but paying attention is paid to a number of necessary moments related to the considered process of random walk.

Lemma 1. *the Number of complete trajectories with a maximum of* t *is*

Proof. We can say that this result is known and is equivalent to the theorem 1, page 107 of [4]. But it is difficult for the author to refer to some source with him. And, in addition, the rationale it is advisable to spend not so much to establish the declared fact, how much for a certain detail, which will be required in the future. Therefore, we will conduct a justification based on the decision of the UPR. 13 Chapter 5 of [5] (p. 325), in which the number of trajectories connecting the points (0,0) and $(n, 2l-n)$, $0 \leq l \leq n$ with the max is exactly equal to t, $(2l-n)^+ \leq t \leq l$ is equal to

$$C_n^{n-l+t} - C_n^{n-l+t+1}. \qquad (5)$$

The justification of the Lemma will be quite simple, if in addition to (5) we use the obvious equation $u + v = t$, as well as the following two implications

$$n = 2m+1 \quad Rightarrow \ t \leq l \leq m+v, \ \ n = 2m \quad Rightarrow \ t \leq l \leq m+u, \quad eqno\,(6)$$

In fact, then we get the opportunity to record

$$n = 2m+1: \quad \sum_{l=t}^{m+v} (C_n^{n-l+t} - C_n^{n-l+t+1}) = C_n^{n-m-v+t} = C_n^{m+u+1},$$

$$n = 2m: \quad \sum_{l=t}^{m+u} (C_n^{n-l+t} - C_n^{n-l+t+1}) = C_n^{n-m-u+t} = C_n^{m+v}.$$

To explain (6), we note that (1) the left boundaries of the specified intervals are also obvious, and to understand the right, just look at the final tree of the trajectories of our wandering type from theorem 1 ■

But here is another version of the explanation of the Lemma, which will be required in the future, because it gives a different detail of our combinatorial considerations, based on the following two tables:

$n = 2m + 1$: $t \le l \le m + [(t+1)/2],\ 0 \le t \le n.$

$(n, 2l - n)$	t	l	$C_n^{n-l+t} - C_n^{n-l+t+1}$
$(n, 4u - n)$	$2u$	$2u$	$C_n^n - C_n^{n+1}$
$(n, 2 + 4u - n)$	$2u$	$2u + 1$	$C_n^{n-1} - C_n^n$
…	…	…	…
$(n, 2u - 1)$	$2u$	$m + u$	$C_n^{m+u+1} - C_n^{m+u+2}$
$(n, 2 + 4u - n)$	$2u + 1$	$2u + 1$	$C_n^n - C_n^{n+1}$
$(n, 4 + 4u - n)$	$2u + 1$	$2u + 2$	$C_n^{n-1} - C_n^n$
…	…	…	…
$(n, 2u + 1)$	$2u + 1$	$m + u + 1$	$C_n^{m+u+1} - C_n^{m+u+2}$

$n = 2m$: $t \le l \le m + [t/2],\ 0 \le t \le n.$

$(n, 2l - n)$	t	l	$C_n^{n-l+t} - C_n^{n-l+t+1}$
$(n, 4u - n)$	$2u$	$2u$	$C_n^n - C_n^{n+1}$
$(n, 2 + 4u - n)$	$2u$	$2u + 1$	$C_n^{n-1} - C_n^n$
…	…	…	…
$(n, 2u)$	$2u$	$m + u$	$C_n^{m+u} - C_n^{m+u+1}$
$(n, 2 + 4u - n)$	$2u + 1$	$2u + 1$	$C_n^n - C_n^{n+1}$
$(n, 4 + 4u - n)$	$2u + 1$	$2u + 2$	$C_n^{n-1} - C_n^n$
…	…	…	…
$(n, 2u)$	$2u + 1$	$m + u$	$C_n^{m+u+1} - C_n^{m+u+2}$

On the meaning of the tables: 1) the ranges of l values for odd and even n values were justified above, 2) take the second column of the top half of the 1st table - it is the same even (but any) value $t = 2u$ - and in the 3rd column increasing values l, determined by the range above at this t (they are associated with the first column.) Finally, the sum of the rows in the right column gives the desired result lemmas ■

The following 2 statements follow from Lemma, $u,\ v$ in them are the former. The first one is obvious, therefore, we will prove only the second.

Corollary 1. *At $p = q$ and $n = 2m + 1$ in (7) or $n = 2m$ in (8)*

$$2^n V_0 = \sum_{t=0}^{n} C_n^{m+u+1} f(0,t) = \sum_{u=0}^{m} C_n^{m+u+1}[f(0{,}2u) + f(0{,}2u+1)], \qquad (7)$$

$$2^n V_0 = \sum_{t=0}^{n} C_n^{m+v} f(0,t) = \sum_{u=0}^{m} C_n^{m+u} f(0{,}2u) + \sum_{u=0}^{m-1} C_n^{m+u+1} f(0{,}2u+1). \qquad (8)$$

Corollary 2. *In the General case of a fair presentation,*

$$V_0 = \sum_{u=0}^{m} \sum_{k=0}^{m-u} (C_n^{n-k} - C_n^{n-k+1}) p^{2u+k} q^{n-2u-k} [f(0.2u) + pq^{-1} f(0.2u+1)],$$

if $n = 2m + 1$, and if $n = 2m$, then

$$V_0 = \sum_{u=0}^{m} f(0{,}2u) \sum_{k=0}^{m-u} (C_n^{n-k} - C_n^{n-k+1}) p^{2u+k} q^{n-2u-k}$$

$$+ \sum_{u=0}^{m-1} f(0{,}2u+1) \sum_{k=0}^{m-u-1} (C_n^{n-k} - C_n^{n-k+1}) p^{2u+k+1} q^{n-2u-k-1}. \qquad (9)$$

Proof. It is clear that this time the probabilities 2^{-n} should replace with $p^a q^b$, $a+b=n$, differently depending on those or other groups of trajectories. And the tables of Lemma 1 actually suggest, how to make this replacement. In fact, consider for example the case of $n=2m+1$ and let $t=2u$. When moving along a trajectory with a finite node $(n, 2l-n)$ regardless of the maximum, the particle does exactly l moves to the right. Therefore, this trajectory has the probability $p^M Q^{n-l}$. But in this case $l=2u+k$, if $k=l-t$. And therefore,

$$2^{-n}C_n^{m+u+1} \quad Rightarrow \sum_{k=0}^{m-u} (C_n^{n-k} - C_n^{n-k+1}) p^{2u+k} q^{n-2u-k}, \quad 0 \le u \le m.$$

It is clear, however, that the same can be done in the other three cases of the specified tables. Therefore, the corollary is proved by ■

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