On another approach to the analysis of the known problem of optimal stopping, p.2

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Abstract

Part 2 implements the idea mentioned earlier in part 1 in the case of the small and odd horizon n = 5. Again, the desired relationship between the objective function of the roblem and the optimal moment of stopping time was very interesting and simple.

Keywords: the known optimal stopping problem, the relationship between its objective function and the optimal stopping time

1 Introduction

In part 1 of the work it was said that for any particular n there is a connection between the form of the function f, i.e. the surface $y = f(s,t)$, defined in the region D of the integer lattice of the plane (s,t), and the optimal stopping moment corresponding to it is an integer k, $0 \le k \le n$, in its a simplified analogue of the known optimal stopping problem (look [1]):

$$V = \max_{0 \le k \le n} V_k, \quad V_k = Ef(S_k, M_n).$$

Here this is shown in the special case of $n = 5$, $p = q$, under the assumption that $f(s,t) > 0$ at $\forall\ (s,t) \in D$, ie discharged condition for f to guarantee the optimality of any integer k, $0 \le k \le 5$. Moreover, the change of the specified surface depending on the change of the optimal k is quite eloquent and this is also illustrated at the end of the article.

2 Preliminary observations

For recording the desired conditions sufficient to write all 6 expressions V_k, $0 \le k \le 5$. Let's use the formula for the conditional mathematical expectation

$$V_k = \sum_{l=0}^{k} f(2l-k,t) P(S_k = 2l-k, M_n = t).$$

But the event $(S_k = 2l-k, M_n = t)$ in the General case when $0 < k < n$ record is not easy. And for extreme k, i.e. formulas for V_0 and V_n are easy to write:

$$n = 2m+1\colon\ 2^n V_0 = \sum_{t=0}^{n} C_n^{m+u+1} f(0,t); \quad n = 2m\colon\ 2^n V_0 = \sum_{t=0}^{n} C_n^{m+v} f(0,t),$$

where $u = [t/2]$, $v = [(t+1)/2]$ (these formulas are given in p.1), and

$$2^n V_n = \sum_{l=0}^{n} \sum_{t=(2l-n)^+}^{l} f(2l-n,t)[C_n^{n-l+t} - C_n^{n-l+t+1}].$$

In the case of $n = 5$, this can be done easily for any k, using the particle motion tree or table of part 1. In addition, the expressions V_k are written here for $1 \le k \le 4$, because for $k = 0$ or n they are higher:

$$V_1 = 2^{-5}\{[f(-1,0)(C_4^4 + C_4^3 + (C_4^2 - C_4^4)) + f(-1,1)(C_4^4 + (C_4^3 - C_4^4))$$

$$+f(-1,2) + f(-1,3)] + [f(1,1)(C_4^4 + (C_4^3 - C_4^4) + (C_4^2 - C_4^3))$$

$$+(f(1,2) + f(1,3))((C_4^4 + (C_4^3 - C_4^4)) + f(1,4) + f(1,5)]\},$$

$$V_2 = 2^{-5}\{[f(-2,0)(C_3^3 + C_3^2 + C_3^1) + f(-2,1)] + [f(0,0)(C_3^3 + (C_3^2 - C_3^3))$$

$$+f(0,1)(C_3^3 + C_3^2 + (C_3^2 - C_3^3) + C_3^3 + (C_3^2 - C_3^3)) + (f(0,2) + f(0,3))C_2^1]$$

$$+[(f(2,2) + f(2,3))(C_3^3 + (C_3^2 - C_3^3)) + f(2,4) + f(2,5)]\},$$

$$V_3 = 2^{-5}\{f(-3,0)2^2 + [f(-1,0)(C_3^2 - C_3^3)(C_2^2 + C_2^1) + f(-1,1)((C_3^2 - C_3^3)$$

$$+2^2)] + [f(1,1)(C_3^2 - C_3^3)(C_2^2 + (C_2^1 - C_2^2)) + f(1,2)((C_2^2 + C_2^1) +$$

$$(C_3^2 - C_3^3)) + f(1,3)C_3^1] + [f(3,3)(C_2^2 + (C_2^1 - C_2^2)) + f(3,4) + f(3,5)]\},$$

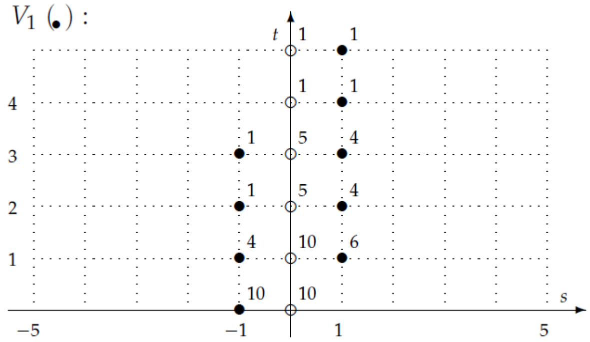
$$V_4 = 2^{-5}\{f(-4,0)2 + [f(-2,0)(C_4^3 - C_4^4)2 + f(-2,1)2] +$$

$$+[f(0,0)2 + f(0,1)((C_4^3 - C_4^4)2 + (C_4^2 - C_4^3)) + f(0,2)2]$$

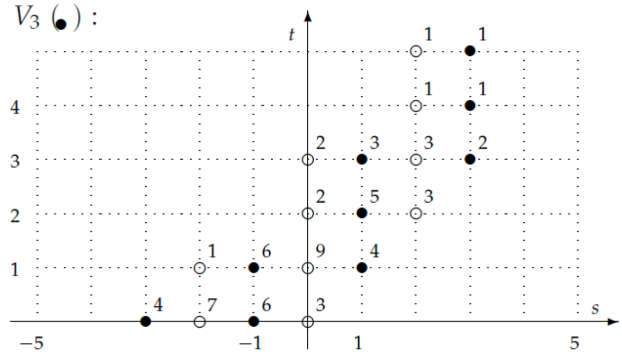
$$+[f(2,2)(C_4^3 - C_4^4) + f(2,3)((C_4^3 - C_4^4) + 2)] + f(4,4) + f(4,5)\},$$

But each of the 6 representations of V_k for our purposes is symbolic let us present below as a set of integer nodes lattice plane (3 diagrams indicate 2 of these sets).

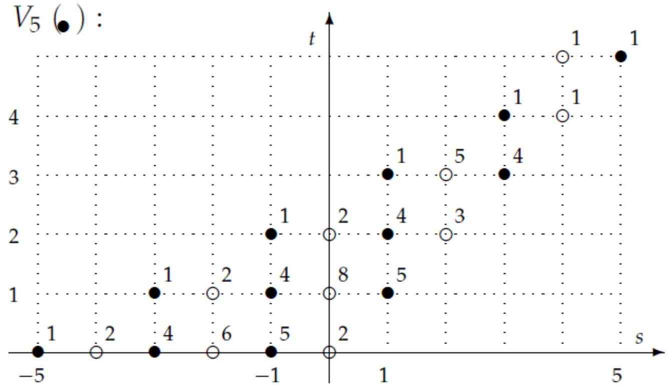
For V_0 (○), V_1 (●) :



For V_2 (○), V_3 (●) :



For V_4 (○), V_5 (●) :



It is convenient first of all because such representation written out expressions equivalent, but much easier and clearer. And, means, will help deal with optimality. Take, for example, the bold nodes-the points of the first chart, representing V_1, and of them a node $(s,t) = (1,3)$, with number 4 next for example. The meaning of this representation is simple: in the expression for V_1 there is a summand $4f(1,3)$. Other nodes with such or other numbers next to each other are associated with expressions for V_k similarly.

Remarks. 1. As it is easy to see, for any k the total number of trajectories defining price value V_k and passing through nodes of level t, $0 \leq t \leq 5$ (in number from 1 to 3), equals to the value of

$$C_5^m, \qquad m = [(5-t)/2].$$

In other words, the numbers 10, 5, 1 are given for the levels 0,1; 2,3; 4,5. This fact can be be generalized to any horizon of n using the table part 1. But here it just flows out of expressions for V_k above.

2. A total of 6 nodes for all sets of nodes "sweep" region D definitions of the function $f(s,t)$ from part 1, which is clearly seen in the diagrams.

3 Optimality condition

Of course, the time of a stop of $\tau = k$ is optimal if in some conditions on f

$$V_k > \max_{l \neq k} V_l.$$

To justify 6 of these statements, we present the prices in the form of sums

$$\begin{array}{lcllllll} 2^5V_5 & = & V_{50}+ & V_{51}+ & V_{52}+ & V_{53}+ & V_{54}+ & V_{55} \\ 2^5V_4 & = & V_{40}+ & V_{41}+ & V_{42}+ & V_{43}+ & V_{44}+ & V_{45} \\ 2^5V_3 & = & V_{30}+ & V_{31}+ & V_{32}+ & V_{33}+ & V_{34}+ & V_{35} \\ 2^5V_2 & = & V_{20}+ & V_{21}+ & V_{22}+ & V_{23}+ & V_{24}+ & V_{25} \\ 2^5V_1 & = & V_{10}+ & V_{11}+ & V_{12}+ & V_{13}+ & V_{14}+ & V_{15} \\ 2^5V_0 & = & V_{00}+ & V_{01}+ & V_{02}+ & V_{03}+ & V_{04}+ & V_{05} \end{array} \qquad (1)$$

where V_{lt} is the sum of functions $f(2s-l,t)$ with the coefficients from the expression for 2^5V_l for all s corresponding to a given t. In this case, all 36 items V_{lt} is positive as above, we we assumed that the function $f(s,t)$ is positive. The matrix $V = (V_{lt})$ of these terms given below and divided into 2 parts forcedly

$l\backslash t$	0	1
5	$f(-5{,}0)+4f(-3{,}0)+5f(-1{,}0)$	$f(-3{,}1)+4f(-1{,}1)+5f(1{,}1)$
4	$2f(-4{,}0)+6f(-2{,}0)+2f(0{,}0)$	$2f(-2{,}1)+8f(0{,}1)$
3	$4f(-3{,}0)+6f(-1{,}0)$	$6f(-1{,}1)+4f(1{,}1)$
2	$7f(-2{,}0)+3f(0{,}0)$	$f(-2{,}1)+9f(0{,}1)$
1	$10f(-1{,}0)$	$4f(-1{,}1)+6f(1{,}1)$
0	$10f(0{,}0)$	$10f(0{,}1)$

$l\backslash t$	2	3	4	5
5	$f(-1{,}2)+4f(1{,}2)$	$f(1{,}3)+4f(3{,}3)$	$f(3{,}4)$	$f(5{,}5)$
4	$2f(0{,}2)+3f(2{,}2)$	$5f(2{,}3)$	$f(4{,}4)$	$f(4{,}5)$
3	$5f(1{,}2)$	$3f(1{,}3)+2f(3{,}3)$	$f(3{,}4)$	$f(3{,}5)$
2	$2f(0{,}2)+3f(2{,}2)$	$2f(0{,}3)+3f(2{,}3)$	$f(2{,}4)$	$f(2{,}5)$
1	$f(-1{,}2)+4f(1{,}2)$	$f(-1{,}3)+4f(1{,}3)$	$f(1{,}4)$	$f(1{,}5)$
0	$5f(0{,}2)$	$5f(0{,}3)$	$f(0{,}4)$	$f(0{,}5)$

(this matrix is easier to obtain using 3 diagrams with nodes higher than the expressions for v_l) we will use this matrix to determine the relationship between the surface type $y=f(s,t)$ and the optimality the stop point of k. Proof the following lemmas say about this. In them for write reductions are always assumed to be $(s,t)\in D$.

Everywhere below, the function $f(s,t)$ in the D region takes 2 values, when justifying lemmas the second page charts are used, and $a>\Delta>0$.

Lemma 1. $V_5>\max_{l\neq 5}V_l$ *if*

$$f(s,t)\equiv a,\ (s,t)\in\{s+5=2t\};\quad f(s,t)\equiv a-\Delta\quad out\ \ straight\ \ line\,.$$

Proof. In the table numbers inside $b_{lt}=V_{5t}-V_{lt},\ l\neq 5,\ 0\le t\le 5.$

$l\backslash t$	0	1	2	3	4	5
4	Δ	Δ	Δ	Δ	Δ	Δ
3	Δ	Δ	Δ	-2Δ	Δ	Δ
2	Δ	Δ	Δ	Δ	Δ	Δ
1	Δ	Δ	Δ	-3Δ	Δ	Δ
0	Δ	Δ	Δ	Δ	Δ	Δ

From here for $b_l=\sum_{t=0}^{5}b_{lt},\ 0\le l\le 4$, we have:

$b_l\backslash l$	0	1	2	3	4
b_l	6Δ	Δ	6Δ	2Δ	6Δ

But this proves the Lemma ∎

Lemma 2. $V_4>\ max_{l\neq 4}V_l$ *if* $f(s,t)\equiv a,\ (s,t)\in\{s+4=2t,\ t\le 4\};$

$$f(4{,}5)=a;\quad f(s,t)\equiv a-\Delta\quad in\ \ other\ \ \ nodes\ \ D.$$

Proof. In the table numbers inside $b_{lt}=V_{4t}-V_{lt},\ l\neq 4,\ 0\le t\le 5.$

$l\backslash t$	0	1	2	3	4	5
5	2Δ	2Δ	2Δ	5Δ	Δ	Δ
3	2Δ	2Δ	2Δ	5Δ	Δ	Δ
2	2Δ	Δ	0	2Δ	Δ	Δ
1	2Δ	2Δ	2Δ	5Δ	Δ	Δ
0	2Δ	2Δ	-3Δ	5Δ	Δ	Δ

From here for $b_l = \sum_{t=0}^{5} b_{lt},\ l \neq 4$, we have:

$b_l\backslash l$	0	1	2	3	5
b_l	8Δ	13Δ	7Δ	13Δ	13Δ

And this proves the Lemma ■

Lemma 3. $V_3 > \ max_{l\neq 3} V_l \ \ if\, f(s,t) \equiv a - \Delta \quad in\ nodes\ \ D,\ distinct$

$from\ \{(-3{,}0), (-1{,}1), (1{,}2), (1{,}3), (3{,}4), (3{,}5)\},\ in\ which\ f(s,t) \equiv a.$

Proof. In the table numbers inside $b_{lt} = V_{3t} - V_{lt},\ l \neq 3,\ 0 \leq t \leq 5.$

$l\backslash t$	0	1	2	3	4	5
5	0	2Δ	Δ	2Δ	0	Δ
4	4Δ	6Δ	5Δ	3Δ	Δ	Δ
2	4Δ	6Δ	5Δ	4Δ	Δ	Δ
1	4Δ	2Δ	Δ	$-\Delta$	Δ	Δ
0	4Δ	6Δ	5Δ	3Δ	Δ	Δ

From here for $b_l = \sum_{t=0}^{5} b_{lt},\ l \neq 3$, we have:

$b_l\backslash l$	0	1	2	4	5
b_l	20Δ	8Δ	20Δ	20Δ	6Δ

And this proves the Lemma ■

Lemma 4. $V_2 > \max_{l\neq 2} V_l \ \ if\, f(s,t) \equiv a - \ \ Delta \quad in\ nodes\ \ D,\ distinct$

$from\ \{(-2{,}0), (-2{,}1), (0{,}2), (0{,}3), (2{,}4), (2{,}5)\},\ in\ which\ f(s,t) \equiv a.$

Proof. In the table numbers inside $b_{lt} = V_{2t} - V_{lt},\ l \neq 2,\ 0 \leq t \leq 5.$

$l\backslash t$	0	1	2	3	4	5
5	7Δ	Δ	2Δ	2Δ	Δ	Δ
4	Δ	$-\Delta$	0	2Δ	Δ	Δ
3	7Δ	Δ	2Δ	2Δ	Δ	Δ
1	7Δ	Δ	2Δ	2Δ	Δ	Δ
0	7Δ	Δ	-3Δ	-3Δ	Δ	Δ

From here for $b_l = \sum_{t=0}^{5} b_{lt},\ l \neq 2$, we have:

$b_l \backslash l$	0	1	3	4	5
b_l	4Δ	14Δ	14Δ	4Δ	14Δ

And this proves the Lemma ■

Lemma 5. $V_1 > \max_{l \neq 1} V_l$ *if* $f(s,t) \equiv a - \Delta$ *in nodes* D, *distinct from* $\{(-1,0),(-1,1),(-1,2),(1,3),(1,4),(1,5)\}$, *in which* $f(s,t) \equiv a$.

Proof. In the table numbers inside $b_{lt} = V_{1t} - V_{lt}$, $l \neq 1$, $0 \leq t \leq 5$.

$l \backslash t$	0	1	2	3	4	5
5	5Δ	0	0	3Δ	Δ	Δ
4	10Δ	4Δ	Δ	4Δ	Δ	Δ
3	4Δ	-2Δ	Δ	Δ	Δ	Δ
2	10Δ	4Δ	Δ	4Δ	Δ	Δ
0	10Δ	4Δ	Δ	4Δ	Δ	Δ

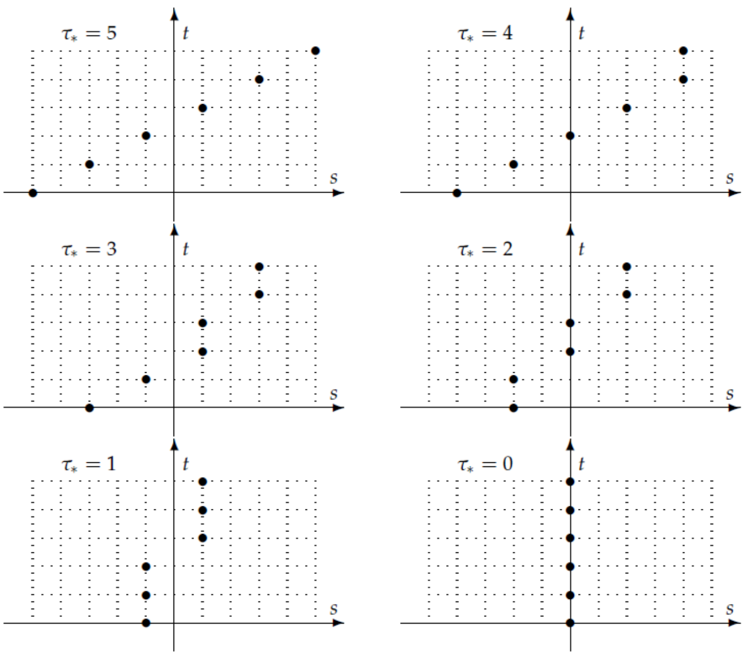
From here for $b_l = \sum_{t=0}^{5} b_{lt}$, $l \neq 1$, we have:

$b_l \backslash l$	0	2	3	4	5
b_l	21Δ	21Δ	6Δ	21Δ	10Δ

The latter Lemma can be justified similarly. But we will not prove it, because it - a corollary of theorem 2 of [1]. Therefore, we will only give its wording, considering it still, that $\Delta > 0$, $a - \Delta > 0$.

Lemma 6. $V_0 > \max_{l \neq 0} V_l$ *if*

$$\forall t,\ 0 \leq t \leq 5:\quad a - \Delta = f(s,t) < f(0,t) = a, \qquad for \quad s \neq 0,\ (s,t) \in D.$$



The results of the lemmas are illustrated in the diagrams below. In them $\tau_* -$ the optimal moment (OM) and black highlighted nodes areas D, in which the values of $f(s,t)$ are maximal, i.e. equal to a. In this connection between OM and the surface shape $y = f(s,t)$ emphasize at least 2 points. First, the length between projections of extreme black nodes on the horizontal axis equals $5k,\ k = \tau_*$. And secondly, the tangent of the slope angle of the line passing through the extreme nodes, $tg\varphi = 5/2k$.

4 Conclusion

The idea expressed in [1] found its confirmation in this article for particular and small the odd value of $n = 5$. But it is also confirmed in the case of a small even value $n = 6$ in [2]. In this very manner the proofs allow us to hope for a relatively easy generalization to even and odd values of n around $20 - 30$. And this will already talk about the practical usefulness of the implemented idea.

It seems that the used version of the Lemma justification turned out to be promising for a simple reason: it was decided use the two-digit target function $f(s,t)$. Although, of course, there are infinitely many different "suitable" forms. Two factors played a significant role in the proofs of [2] and here: 1) the geometric representation of the values of V_k on the second page here and 2) remark 1, which characterizes one of the key properties of the situations in question.

Finally, if we try to smooth in some way the surface $y = f(s,t)$

considered in the proofs of lemmas, the resulting "mountain ranges" should have approximately the same height everywhere. But this and a number of other things, of interest, for example, $p \neq q$, naturally attributed to the next stage of research.

References

[1] Zhulenev S.V. On another approach to the analysis of the known problem of optimal stopping, p.1, this collection, pp. 71-76.

[2] Filatov A.S. On another approach to the analysis of the known problem of optimal stopping, p.3, present collection, pp. 84-91.