On another approach to the analysis of the known problem of optimal stopping, p.3

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Abstact

Part 3 implements the idea mentioned earlier in part 1 in the case of the small and even horizon n = 6: Again, the desired relationship between the objective function of the problem and the optimal moment of stopping time was very interesting and simple.

Keywords: the known optimal stopping problem, the relationship between its objective function and the optimal stopping time

1 Introduction

In [2] it is shown that in the particular case $n = 5$, $p = q$, and also on the assumption that $f(s,t) > 0$ for $\forall(s,t) \in D$, there is a connection between the form of the function f, i.e. the surface $y = f(s,t)$, defined in the domain D of the integer lattice of the plane (s,t) and the optimal stopping time corresponding to it – an integer k, $0 \leq k \leq n$, in its simplified analogue of the known optimal stopping problem:

$$V = \max_{0 \leq k \leq n} V_k, \quad V_k = Ef(S_k, M_n).$$

In this paper it is shown in another particular case $n = 6$, $p = q$, and on the same assumptions related to the function f and the domain D.

2 Prolusion

As before, to write the required conditions, it suffices to write out all the expressions for V_k, $0 \leq k \leq n$. For this we use the formula of conditional expectation

$$V_k = \sum_{l,t} f(2l-k,t)P(S_k = 2l-k, M_n = t).$$

For extreme k the formulas for V_0 and V_n are easily written:

$$n = 2m+1\colon\ 2^n V_0 = \sum_{t=0}^{n} C_n^{m+u+1} f(0,t); \quad n = 2m\colon\ 2^n V_0 = \sum_{t=0}^{n} C_n^{m+v} f(0,t),$$

where $u = [t/2]$, $v = [(t+1)/2]$ (these formulas are given in [1]), and

$$2^n V_n = \sum_{l=0}^{n} \sum_{t=(2l-n)^+}^{l} f(2l-n,t)[C_n^{n-l+t} - C_n^{n-l+t+1}].$$

For other k in the case $n = 6$ it is easily done using the motion tree of particle or the table in [1]. In the following the expressions V_k are written out for $1 \le k \le 5$, because for $k = 0$ or n they are given higher:

$$V_1 = 2^{-6}\{[20f(-1,0) + 5f(-1,1) + 5f(-1,2) + f(-1,3) + f(-1,4)]$$

$$+[10f(1,1) + 10f(1,2) + 5f(1,3) + 5f(1,4) + f(1,5) + f(1,6)]\},$$

$$V_2 = 2^{-6}\{[14f(-2,0) + f(-2,1) + f(-2,2)] + [6f(0,0) + 14f(0,1) + 8f(0,2) +$$

$$+2f(0,3) + 2f(0,4) + 6f(2,2) + 4f(2,3) + 4f(2,4) + f(2,5) + f(2,6)]\},$$

$$V_3 = 2^{-6}\{[8f(-3,0) + 12f(-1,0) + 9f(-1,1) + 3f(-1,2)] + [6f(1,1) + 12f(1,2) +$$

$$+3f(1,2) + 3f(1,4) + 3f(3,3) + 3f(3,4) + f(3,5) + f(3,6)]\},$$

$$V_4 = 2^{-6}\{[4f(-4,0) + 12f(-2,0) + 4f(-2,1)] + [4f(0,0) + 11f(0,1) + 9f(0,2) +$$

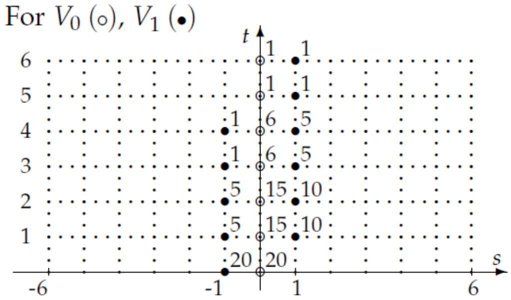
$$+6f(2,2) + 6f(2,3) + 4f(2,4) + 2f(4,4) + f(4,5) + f(4,6)]\},$$

$$V_5 = 2^{-6}\{[2f(-5,0) + 8f(-3,0) + 2f(-3,1) + 10f(-1,0) + 8f(-1,1) + 2f(-1,2)] +$$

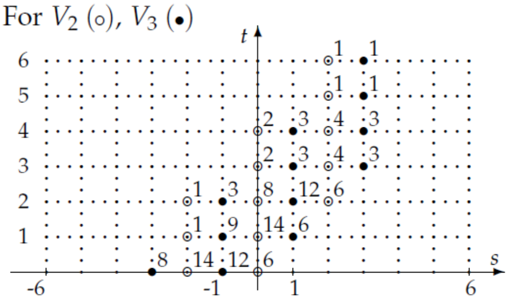
$$+[5f(1,1) + 13f(1,2) + 2f(1,3) + 4f(3,3) + 6f(3,4) + f(5,5) + f(5,6)]\}.$$

As in [2], each of V_k, $0 \le k \le 6$, can be represented as a set of nodes of the integer lattice of the plane. Recall that, for example, the bold node $(s,t) = (1,3)$ of the diagram with the number 2 next, corresponding to the expression V_5, has the following context: in the expression for V_5 there is a term $2f(1,3)$. The remaining diagrams are related to the corresponding V_k in a similar way.

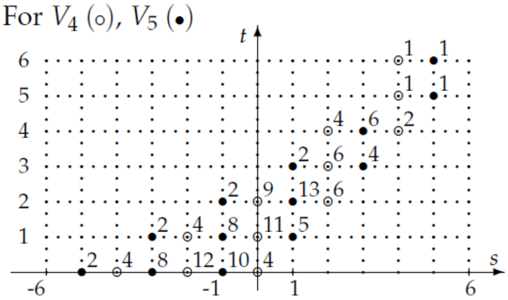
For V_0 (○), V_1 (•)



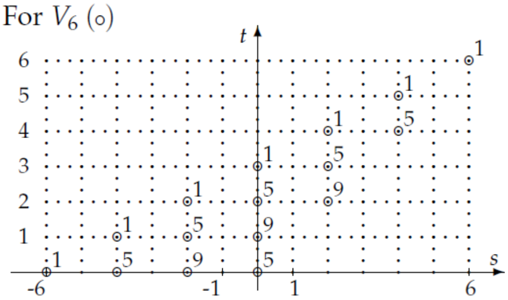
For V_2 (○), V_3 (•)



For V_4 (○), V_5 (•)



For V_6 (○)



Remarks. 1. It can be seen in the diagrams shown above that for any k the total number of trajectories determining the value of the price V_k and passing through the nodes of the level t,

$0 \le t \le 6$, is equal to

$$C_6^m, \quad m = [(6-t)/2].$$

So it is defined by the numbers 20, 15, 6, 1 for the levels 0; 1,2; 3,4; 5,6.

2. As in the case $n = 5$ the nodes of all 7 sets of nodes "run through" the domain D of the definition of the function $f(s,t)$ from [1]. This fact is clearly visible in the diagrams.

3 Optimality conditions

The stopping time $\tau = k$ is optimal if in some conditions for f

$$V_k > max_{l \neq k} V_l.$$

To substantiate 7 such statements, we will present prices in the form of sums

$$\begin{aligned}
2^6 V_6 &= V_{60} + V_{61} + V_{62} + V_{63} + V_{64} + V_{65} + V_{66}\\
2^6 V_5 &= V_{50} + V_{51} + V_{52} + V_{53} + V_{54} + V_{55} + V_{56}\\
2^6 V_4 &= V_{40} + V_{41} + V_{42} + V_{43} + V_{44} + V_{45} + V_{46}\\
2^6 V_3 &= V_{30} + V_{31} + V_{32} + V_{33} + V_{34} + V_{35} + V_{36}\\
2^6 V_2 &= V_{20} + V_{21} + V_{22} + V_{23} + V_{24} + V_{25} + V_{26}\\
2^6 V_1 &= V_{10} + V_{11} + V_{12} + V_{13} + V_{14} + V_{15} + V_{16}\\
2^6 V_0 &= V_{00} + V_{01} + V_{02} + V_{03} + V_{04} + V_{05} + V_{06}
\end{aligned}$$

where V_{lt} is the sum of the functions $f(s,t)$ with the coefficients are available for them from the expression of $2^6 V_l$ for all s, corresponding to the given t. Moreover, all 49 elements V_{lt} are positive, because earlier we assumed that the function $f(s,t)$ is positive. The matrix $V = (V_{lt})$ of these terms, given below and divided into 2 parts,

$l \backslash t$	0	1
6	$f(-6{,}0) + 5f(-4{,}0) + 9f(-2{,}0) + 5f(0{,}0)$	$f(-4{,}1) + 5f(-2{,}1) + 9f(0{,}1)$
5	$2f(-5{,}0) + 8f(-3{,}0) + 10f(-1{,}0)$	$2f(-3{,}1) + 8f(-1{,}1) + 5f(1{,}1)$
4	$4f(-4{,}0) + 12f(-2{,}0) + 4f(0{,}0)$	$4f(-2{,}1) + 11f(0{,}1)$
3	$8f(-3{,}0) + 12f(-1{,}0)$	$9f(-1{,}1) + 6f(1{,}1)$
2	$14f(-2{,}0) + 6f(0{,}0)$	$f(-2{,}1) + 14f(0{,}1)$
1	$20f(-1{,}0)$	$5f(-1{,}1) + 10f(1{,}1)$
0	$20f(0{,}0)$	$15f(0{,}1)$

$l \backslash t$	2	3	4	5	6
6	$f(-2{,}2) + 5f(0{,}2) + 9f(2{,}2)$	$f(0{,}3) + 5f(2{,}3)$	$f(2{,}4) + 5f(4{,}4)$	$f(4{,}5)$	$f(6{,}6)$
5	$2f(-1{,}2) + 13f(1{,}2)$	$2f(1{,}3) + 4f(3{,}3)$	$6f(3{,}4)$	$f(5{,}5)$	$f(5{,}6)$
4	$9f(0{,}2) + 6f(2{,}2)$	$6f(2{,}3)$	$4f(2{,}4) + 2f(4{,}4)$	$f(4{,}5)$	$f(4{,}6)$
3	$3f(-1{,}2) + 12f(1{,}2)$	$3f(1{,}3) + 3f(3{,}3)$	$3f(1{,}4) + 3f(3{,}4)$	$f(3{,}5)$	$f(3{,}6)$
2	$f(-2{,}2) + 8f(0{,}2) + 6f(2{,}2)$	$2f(0{,}3) + 4f(2{,}3)$	$2f(0{,}4) + 4f(2{,}4)$	$f(2{,}5)$	$f(2{,}6)$
1	$5f(-1{,}2) + 10f(1{,}2)$	$f(-1{,}3) + 5f(1{,}3)$	$f(-1{,}4) + 5f(1{,}4)$	$f(1{,}5)$	$f(1{,}6)$
0	$15f(0{,}2)$	$6f(0{,}3)$	$6f(0{,}4)$	$f(0{,}5)$	$f(0{,}6)$

we use to determine the connection between the form of the surface $y = f(s,t)$ and optimality of the stopping time k. The following lemmas are given for proving this statement.

As in [2], we assume everywhere below that the function $f(s,t)$ in the domain D takes on two values, and we use the diagrams of the second page for proving the lemmas, $\Delta > 0$.

5.3mmLemma 1 $V_6 > max_{l \neq 6} V_l$ *if*

$$f(s,t) \equiv a, \ (s,t) \in \{s + 6 = 2t\}; \ f(s,t) \equiv a - \Delta \text{ out straight line.}$$

Proof. Inside the table the numbers $b_{lt} = V_{6t} - V_{lt}$, $l \neq 6$, $0 \le t \le 6$.

$l\backslash t$	0	1	2	3	4	5	6
6	Δ	Δ	Δ	Δ	Δ	Δ	Δ
5	Δ	Δ	Δ	Δ	-3Δ	0	Δ
4	Δ	Δ	Δ	Δ	Δ	Δ	Δ
3	Δ	Δ	0	-Δ	-3Δ	Δ	Δ
2	Δ	Δ	Δ	Δ	Δ	Δ	Δ
0	Δ	Δ	Δ	-5Δ	Δ	Δ	Δ

From here for $b_l = \sum_{t=0}^{6} b_{lt}$, $0 \le l \le 5$, we have

$b_l\backslash l$	0	1	2	3	4	5
b_l	Δ	7Δ	0	7Δ	2Δ	7Δ

This calculations show that on the assumptions of the lemma $V_6 = V_2 > \max_{l\neq 2,6} V_l$. However, it is easy to slightly change the conditions and get the declared result. Actually, under the concerned conditions, the function $f(s,t)$ has 2 levels: a and $a-\Delta$. So, it's enough to increase the values in some nodes (s,t) a little at the lower level $a-\Delta$. For example, in 3 nodes let $f(2,3) = f(0,4) = f(4,4) = a - 0.9\Delta$. In this case it is easy to show that earlier $b_{23} = -\Delta$, $b_{24} = -3\Delta$, and on the new assumptions

$$b_{23} = a + 5(a - 0.9\Delta) - [2a + 4(a - 0.9\Delta)] = -0.9\Delta,\ b_{24} = a + 5(a - 0.9\Delta) - [2(a - 0.9\Delta) + 4a] = -2.7\Delta.$$

Other b_{lt} remain the same. Therefore, instead of $b_2 = 0$ we get $b_2 = 0{,}4\Delta > 0$, which proves the Lemma ∎

Thus, Lemma 1 establishes the required inequality $V_6 > \max_{l\neq 6} V_l$ if the old surface $y = f(s,t)$ with 2 levels is replaced by a surface with 3 levels – the third in the three above mentioned nodes. In all the following lemmas it suffices to have a 2-level surface $y = f(s,t)$ to obtain the desired result.

5.3mmLemma 2 $V_5 > max_{l\neq 5} V_l$ *if* $f(s,t) \equiv a,\ (s,t) \in \{s+5 = 2t,\ t \le 5\}$;
$f(5,6) \equiv a;\quad f(s,t) \equiv a - \Delta$ in other nodes.

Proof. Inside the table the numbers $b_{lt} = V_{5t} - V_{lt}$, $l \neq 5$, $0 \le t \le 6$.

$l\backslash t$	0	1	2	3	4	5	6
6	2Δ	2Δ	2Δ	2Δ	6Δ	Δ	Δ
5	2Δ	2Δ	2Δ	2Δ	6Δ	Δ	Δ
4	2Δ	2Δ	-Δ	-Δ	3Δ	Δ	Δ
3	2Δ	2Δ	2Δ	2Δ	6Δ	Δ	Δ
2	2Δ	2Δ	-3Δ	-3Δ	6Δ	Δ	Δ
0	2Δ	2Δ	2Δ	6Δ	6Δ	Δ	Δ

Therefore for $b_l = \sum_{t=0}^{6} b_{lt}$, $l \neq 5$, we have

$b_l\backslash l$	0	1	2	3	4	6
b_l	16Δ	6Δ	16Δ	7Δ	16Δ	16Δ

And this proves the Lemma ∎

5.3mmLemma 3 $V_4 > max_{l \neq 4} V_l$ *if* $f(s,t) \equiv a - \Delta$ *in nodes D, different* from $\{(-4,0), (-2,1), (0,2), (2,3), (2,4), (4,5), (4,6)\}$, where $f(s,t) \equiv a$.

Proof. Inside the table the numbers $b_{lt} = V_{4t} - V_{lt}$, $l \neq 4$, $0 \leq t \leq 6$.

$l\backslash t$	0	1	2	3	4	5	6
6	-Δ	-Δ	4Δ	Δ	3Δ	0	Δ
5	4Δ	4Δ	9Δ	6Δ	4Δ	Δ	Δ
4	4Δ	4Δ	9Δ	6Δ	4Δ	Δ	Δ
3	4Δ	3Δ	Δ	2Δ	0	Δ	Δ
2	4Δ	4Δ	9Δ	6Δ	4Δ	Δ	Δ
0	4Δ	4Δ	-6Δ	6Δ	4Δ	Δ	Δ

From here it follows that for $b_l = \sum_{t=0}^{6} b_{lt}$, $l \neq 4$, we have

$b_l\backslash l$	0	1	2	3	5	6
b_l	14Δ	29Δ	12Δ	29Δ	29Δ	7Δ

This proves the Lemma ■

Lemma 4 $V_3 > max_{l \neq 3} V_l$ *if* $f(s,t) \equiv a - \Delta$ *in nodes D, different* from $\{(-3,0), (-1,1), (-1,2), (1,3), (1,4), (3,5), (3,6)\}$, where $f(s,t) \equiv a$.

Proof. Inside the table the numbers $b_{lt} = V_{3t} - V_{lt}$, $l \neq 3$, $0 \leq t \leq 6$.

$l\backslash t$	0	1	2	3	4	5	6
6	8Δ	9Δ	3Δ	3Δ	3Δ	Δ	Δ
5	0	Δ	Δ	Δ	3Δ	Δ	Δ
4	8Δ	9Δ	3Δ	3Δ	3Δ	Δ	Δ
3	8Δ	9Δ	3Δ	3Δ	3Δ	Δ	Δ
2	8Δ	4Δ	-2Δ	-2Δ	-2Δ	Δ	Δ
0	8Δ	9Δ	3Δ	3Δ	3Δ	Δ	Δ

Therefore for $b_l = \sum_{t=0}^{6} b_{lt}$, $l \neq 3$, we have

$b_l\backslash l$	0	1	2	4	5	6
b_l	28Δ	8Δ	28Δ	28Δ	8Δ	28Δ

As was to be proved ■

5.3mmLemma 5 $V_2 > max_{l \neq 2} V_l$ *if* $f(s,t) \equiv a - \Delta$ *in nodes D, different* from $\{(-2,0), (-2,1), (0,2), (0,3), (0,4), (2,5), (2,6)\}$, where $f(s,t) \equiv a$.

Proof. Inside the table the numbers $b_{lt} = V_{2t} - V_{lt}$, $l \neq 2$, $0 \leq t \leq 6$.

$l \backslash t$	0	1	2	3	4	5	6
6	5Δ	-4Δ	3Δ	Δ	2Δ	Δ	Δ
5	14Δ	Δ	8Δ	2Δ	2Δ	Δ	Δ
4	2Δ	-3Δ	-Δ	2Δ	2Δ	Δ	Δ
3	14Δ	Δ	8Δ	2Δ	2Δ	Δ	Δ
2	14Δ	Δ	8Δ	2Δ	2Δ	Δ	Δ
0	14Δ	Δ	-7Δ	-4Δ	-4Δ	Δ	Δ

From here for $b_l = \sum_{t=0}^{6} b_{lt}$, $l \neq 2$, we have

$b_l \backslash l$	0	1	3	4	5	6
b_l	2Δ	29Δ	29Δ	4Δ	29Δ	9Δ

As was to be proved ■

5.3mmLemma 6 $V_1 > max_{l \neq 1} V_l$ *if* $f(s,t) \equiv a - \Delta$ *in nodes* D, *different* from $\{(-1,0), (-1,1), (-1,2), (1,3), (1,4), (1,5), (1,6)\}$, where $f(s,t) \equiv a$.

Proof. Inside the table the numbers $b_{lt} = V_{1t} - V_{lt}$, $l \neq 1$, $0 \leq t \leq 6$.

$l \backslash t$	0	1	2	3	4	5	6
6	20Δ	5Δ	5Δ	5Δ	5Δ	Δ	Δ
5	10Δ	-3Δ	3Δ	3Δ	5Δ	Δ	Δ
4	20Δ	5Δ	5Δ	5Δ	5Δ	Δ	Δ
3	8Δ	-4Δ	2Δ	2Δ	2Δ	Δ	Δ
2	20Δ	5Δ	5Δ	5Δ	5Δ	Δ	Δ
0	20Δ	5Δ	5Δ	5Δ	5Δ	Δ	Δ

From here for $b_l = \sum_{t=0}^{6} b_{lt}$, $l \neq 1$, we have

$b_l \backslash l$	0	2	3	4	5	6
b_l	42Δ	42Δ	12Δ	42Δ	20Δ	42Δ

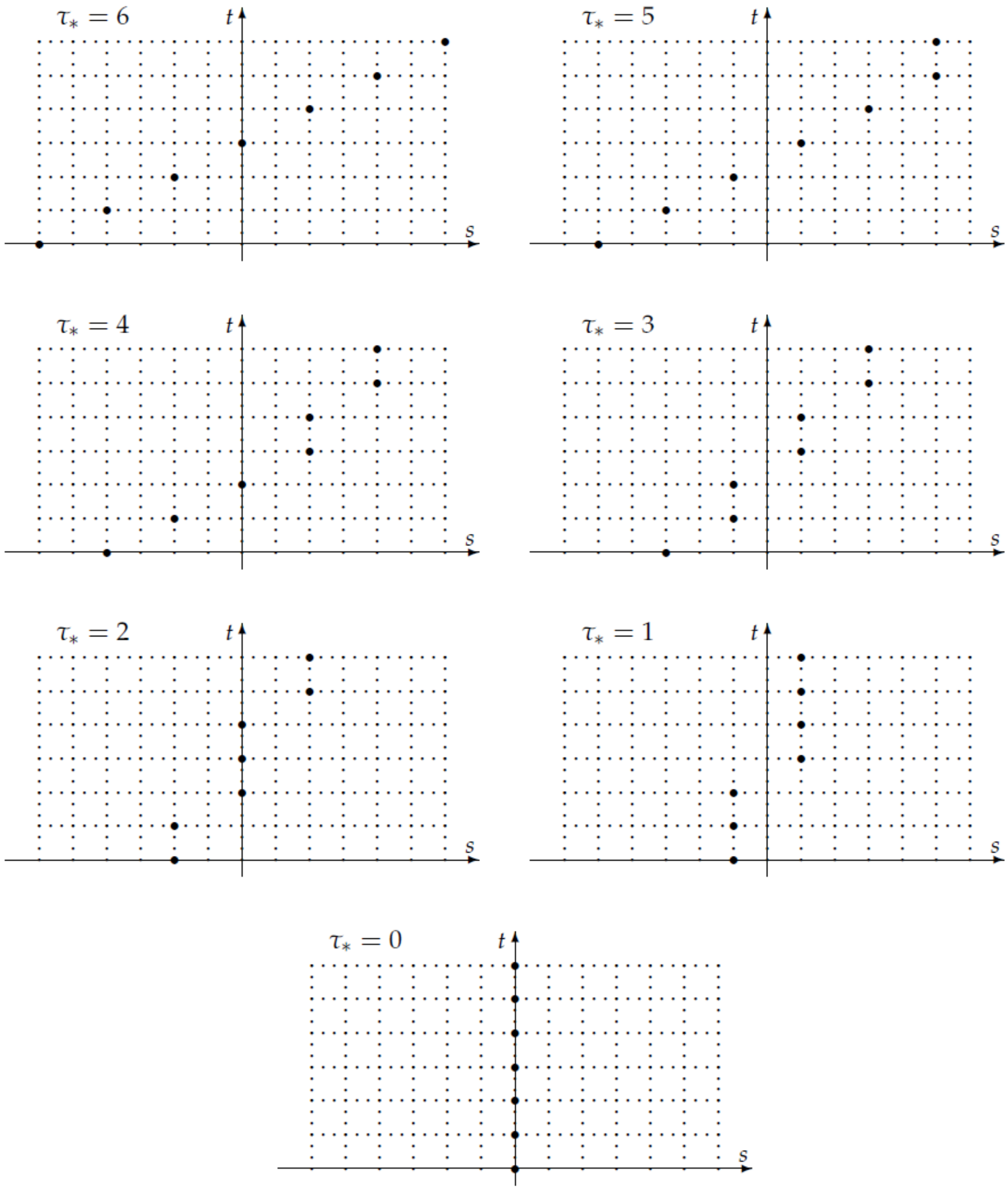
As was to be proved ■

The last Lemma is given without proof (it can be justified in a similar way), since it is a consequence of the theorem 2 in [1]. We still assume that $\Delta > 0$, $a - \Delta > 0$.

5.3mmLemma 7 $V_0 > max_{l \neq 0} V_l$ *if*
$\forall t$, $0 \leq t \leq 6$: $a - \Delta = f(s,t) < f(0,t) = a$ for $s \neq 0$, $(s,t) \in D$.

4 Results

The results of the lemmas are illustrated in the diagrams below. In them τ_* – optimal time (optimal moment – OM), and the nodes of the domain D, in which the values $f(s,t)$ are maximal, i.e. are equal a, are marked in black. With that, as in [2], at least 2 moments emphasize the connection between the OM and the shape of the surface $y = f(s,t)$. First, the length between the projections of the extreme black nodes to the abscissa axis is $6k$, $k = \tau_*$. And secondly, the tangent of the slope of the straight line passing through the extreme nodes, $tg\varphi = 6/2k$.



5 Conclusion

The idea expressed in [1] was confirmed in this article for the particular and small even value $n = 6$. But it is also confirmed in the case of the small odd value $n = 5$ in [2]. Moreover, as noted in [2], the manner of the proof allows us to hope for a relatively easy generalization to even and odd values of n about 20-30, that will speak of the practical usefulness of the realized idea.

References

[1] Zhulenev S.V. On another approach to the analysis of the known problem of optimal stopping, p.1, this collection, pp. 71-76.

[2] Zhulenev S.V. On another approach to the analysis of the known problem of optimal stopping, p.2, this collection, pp. 77-83.