The Necessary Stability Conditions of a Tandem System With Feedback

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Abstract

In this paper, we consider Markovian model of a two-station tandem network with the following feedback admission control policy: the first station rejects new arrivals when the queue size in the second station exceeds a certain threshold N. We provide necessary stability conditions of this model. Each station operates as a multiserver queuieng system, and thus work in part generalizes the results from the paper [1] in which single-server stations have been considered. The analysis is based on the Burke's theorem and stochastic monotonicity of the Birth-Death process describing the number of customers in the second station.

Keywords: queuing system, ergodicity, input flow, feedback

I Introduction

We consider the following two-station queueing system with a feedback admission control policy. The input flow in this system is Poisson with the parameter λ. Station i has N_i servers, and the service time of each server in station i is exponentially distributed with parameter $\mu_i, i = 1,2$.

We consider a *feedback admission control* when the 1st station closes the admission gate provided the *queue size* (number of customers) in the 2nd station exceeds a fixed threshold $N \geq 1$. When the queue length of the 2nd station falls below the threshold, admission gate opens again. With this non-idling control policy, the system losses arrivals during the period when the gate is closed. We assume the FIFO service discipline at both stations. (In general, under the same conditions, stability of the system holds true for any work-conserving service discipline.) The detailed motivation of this model can be found in [1].

Our analysis is based on the dependencies between the rates of the flows, in particular, input rate and output rate from the first station, in stationary regime. Also the analysis is heavily based on the Burke's theorem stating the equality of the input and output rates in the stationary (non-overloaded) multiserver first station. Finally, we apply stochastic monotonicity of the Birth-Death (BD) process, describing the multiserver queuing system.

II Stability Conditions

In this section, we establish the necessary stability conditions of the basic model described shortly above.

First of all, we give more detailed description of the model. We consider the described above two-station tandem system with Piosson input with rate λ and feedback admission control, assuming that the first station operates as a queueing system $M|M|N_1$ with N_1 identical servers and infinite buffer. The second station is the system $M|M|N_2$, also with infinity capacity buffer. The service rate is μ_i at each server of station $i = 1,2$. Because all governing distributions are exponential, this feedback system is completely defined by the parameters $\lambda,\ \mu_i,\ N_i, N$.

The dynamics of this model can be described by a continuous-time discrete-valued Markov process $Z(t) =: (z_1(t), z_2(t)),\ t \geq 0$, where component $z_i(t)$ is the number of customers at station i at instant $t, i = 1,2$. Denote $y(t)$ the number of arrivals in the interval $(0, t],\ y(0) = 0$, in the Poisson input flow (with the intensity λ), and define $x(t)$, the *actual* number of arrivals to the 1st station in interval $(0, t],\ x(0) = 0$.

The following statement generalizes the necessary stability conditions found in [1] for the single-server stations.

Theorem 1. *Assume the Markov process Z is ergodic. If i) $N_1\mu_1 < N_2\mu_2$, then $\lambda < F_N(N_1\mu_1)$;*
ii) otherwise, if $N_1\mu_1 \geq N_2\mu_2$, that there are no other restrictions except $\lambda < \infty$.

Proof. Assume that the Markov process Z is in steady state, and denote $P_N = P(z_2(t) > N)$ the stationary probability that there are at least N customers in the 2nd station. The Poisson arrivals with the intensity λ enter the 1st station. Then, at an arrival instant a transition $y(t) \to y(t) + 1$ happens , and moreover, transition $x(t) \to x(t) + 1$ happens if and only if $z_2(t) \leq N$. Thus, the transition rate $x(t) \to x(t) + 1$ equals $\nu := \lambda P_N$.

Therefore, for each t and constant T, the number of customers entering the 1st station in interval $[t, t + T)$ does not depend on the number of customers arriving in interval $(0, t],\ t > 0$. Then it follows from [2], [3] that the rate of the arrivals entering the 1st station equals $\nu = \lambda P_N$ as well. Since the flow of arrivals entering the 1st station is Poisson with rate ν and the process Z is ergodic, then the process $z_1(t),\ t \geq 0$, turns out to be ergodic also. As a result , the process $z_1(t)$ is distributed as a BD process with the birth rate ν and the death rates $\mu_k = \min(k, N_1)\mu_1$ [§ 1.2][4]. It then follows from Karlin – McGregor criterion [6], we obtain the inequality $\nu < N_1\mu_1$. Because the stationary output from the 1st station is also Poisson process with the rate $\nu = \lambda P_N$, then we may notation $P_N = P_N(\nu)$ which is heavily used below.

Apply now a similar analysis to the 2nd station. Since the input to the 2nd station (output from the 1st station) is Poisson with rate ν, and the process Z is ergodic then the process $z_2(t),\ t \geq 0$, is ergodic also.

As above then the process $z_2(t)$ is distributed as a BD process with the birth rate ν and the death rates $\psi_k = \min(k, N_2)\mu_2$. Then, as above it follows from Karlin – McGregor criterion, that the inequality $\nu < N_2\mu_2$ holds. Thus, we obtain the following relations:

$$\nu = \lambda P_N,\ \nu < N_1\mu_1,\ \nu < N_2\mu_2. \tag{1}$$

Consider another BD process $z_2'(t),\ t \geq 0$, with the same death rates $\{\psi_k\}$ and a birth rate $\nu' > \nu$. Moreover, we assume the same initial state in both processes, that is $z_2(0) = z_2'(0)$. Then it follows from Theorem 4.2.1 in [8], that the following inequality holds:

$$\lim_{t\to\infty} P(z_2(t) > N) = P_N(\nu) \geq \lim_{t\to\infty} P(z_2' t) > N) =: P_N(\nu'). \tag{2}$$

Because $\psi_j = \min(j, N_2)\mu_2,\ j \geq 1$, then it follows from [5] (Chapter 2, Section 3), that for each fixed $N > 0$ and for all $\nu,\ 0 < \nu < N_2\mu_2$, the function $P_N(\nu)$ has the following explicit expression

$$P_N(\nu) = 1 + \sum_{k=1}^{N} \nu^k / \prod_{j=1}^{k} \psi_j 1 + \sum_{k=1}^{\infty} \nu^k / \prod_{j=1}^{k} \psi_j,$$

and moreover, is monotonically decreasing (2) and continuous in ν. Because, under condition $\nu \geq N_2\mu_2$, the process $z_2(t)$ is not ergodic, then we obtain $P_N(\nu) = 0$ for $\nu \geq N_2\mu_2$. Therefore, for the fixed $N > 0$, the function

$$F_N(\nu) = \frac{\nu}{P_N(\nu)} \tag{3}$$

is continuous and monotonically increases in ν, as long as $0 < \nu < N_2\mu_2$, while we put $F_N(\nu) = \infty$ if $\nu \geq N_2\mu_2$. Then the equality

$$\nu = \lambda P_N(\nu) = F_N(\nu)P_N(\nu)$$

in (1) can be rewritten as $\nu = F^{-1}(\lambda)$, where F^{-1} is the inverse function to function F. Hence, by the monotonicity, we obtain from (1) that, for $N_1\mu_1 < N_2\mu_2$,

$$\lambda < F_N(N_1\mu_1). \quad (4)$$

Assume that $N_1\mu_1 \geq N_2\mu_2$. Take an arbitrary $\varepsilon \in (0, N_2\mu_2)$. Then, by the ergodicity of the Markov process $Z(t),\ t \geq 0$, the inequality $\nu < N_2\mu_2 - \varepsilon < N_1\mu_1$ follows, which in turn, is equivalent to the inequality $\nu < N_2\mu_2 - \varepsilon$. The latter inequality implies $\lambda < F_N(N_2\mu_2 - \varepsilon)$ by the monotonicity of function F_N. Because ε is arbitrary and

$$F_N(N_2\mu_2 - \varepsilon) \to F_N(N_2\mu_2) = \infty,\ \varepsilon \to 0,$$

then (4) becomes $\boldsymbol{\lambda} < \infty$, and the proof is completed.

III A Generalization

In the paper [1], also the following more general m-station system, $m \geq 2$, is considered: the external input (with rate λ) is rejected at the first station, if the number of customers $z_k(t)$ in each remaining station k exceeds a given threshold $N^{(k)}$. Moreover, the output from station k is the input to station $k+1, k = 1, \dots, m-1$. Denote $z_k(t)$ the number of customers at station k at instant t. In more detail, keeping other notation, consider an m - station exponential queueing system, in which station k has N_k (stochastically equivalent) servers with exponential service time with rate $\mu_k,\ k = 1, \dots, m$. It is assumed that a customer of the external Poisson input is rejected if the following inequalities hold true:

$$z_2(t) > N^{(2)}, \dots, z_m(t) > N^{(m)}.$$

The dynamics of this system is described by the following m-dimensional Markov process

$$Z = (z_1(t), \dots, z_m(t)),\ t \geq 0.$$

Theorem 2. *Assume the process Z is ergodic. If*

$$N_1\mu_1 < \min_{2\leq k\leq m} N_k\mu_k,$$

then $\lambda < F_N(N_1\mu_1)$. Otherwise, if

$$N_1\mu_1 \geq \min_{2\leq k\leq m} N_k\mu_k,$$

that only requirement is $\lambda < \infty$.

Proof. Denote ν the output rate of the (Poisson) flow of each station $1, \dots, m$. (This rate is the same for all stations by the ergodicity.) By the product-form theorem for stationary regime [9], the joint stationary distribution of the basic process satisfies

$$P(z_2(t) > N^{(2)}, \dots, z_m(t) > N^{(m)}) = \prod_{k=2}^{m} P(z_k(t) > N^{(k)}) =: P_{N^{(2)},\dots,N^{(m)}}(\nu). \quad (5)$$

The component processes $z_2(t), \dots, z_m(t)$ are the BD processes. Moreover, the process $z_k(t)$ has the birth rate ν and, if $z_k(t) = i$, the death rate $\mu_{k,i} = \min(i, N_k)\mu_k,\ k = 2, \dots, m$. It follows by Theorem 4.2.1 [8] and from analysis of the proof of Theorem 1 above, that the kth multiplier $P(z_k(t) > N^{(k)})$ in (5) (as function of ν) is continuous and decreases for all $\nu,\ 0 < \nu < N_k\mu_k,\ k = 2, \dots, m$. Thus, function $P_{N^{(2)},\dots,N^{(m)}}(\nu)$ is monotonically decreasing (and continuous) in ν as long as

$$0 < \nu < \min_{2\leq k\leq m} N_k\mu_k.$$

Because the process Z is ergodic, then the rate of the (Poisson) process entering the 1st station is $\nu = \lambda P_{N^{(2)},\dots,N^{(m)}}$. Furthermore, the output flows of all stations in the system are Poisson with the same rate ν. Now, repeating the arguments used in the proof of Theorem 1, we obtain the following relations

$$\nu = \lambda P_{N^{(2)},\dots,N^{(m)}}(\nu),\ \nu < N_1\mu_1, \dots, \nu < N_m\mu_m. \quad (6)$$

At that, the equality

$$\nu = \lambda P_{N^{(2)},\dots,N^{(m)}}(\nu) =: F_{N^{(2)},\dots,N^{(m)}}(\nu)$$

in (6) can be rewritten as

$$\nu = F^{-1}_{N^{(2)},\dots,N^{(m)}}(\lambda),$$

where $F^{-1}_{N^{(2)},\dots,N^{(m)}}$ is the inverse function to function $F_{N^{(2)},\dots,N^{(m)}}$. Now, by the monotonicity, we obtain from (6), for $N_1\mu_1 < \min_{2\le k\le m} N_k\mu_k$, the following inequality

$$\lambda < F_{N^{(2)},\dots,N^{(m)}}(N_1\mu_1). \qquad (7)$$

If $N_1\mu_1 \ge \min_{2\le k\le m} N_k\mu_k$, then again repeating arguments used in the proof of Theorem 1, we obtain finally the inequality $\lambda < \infty$, which completes the proof.

IV Conclusion

The necessary stability conditions of the Markovian model of a two-station tandem queueing network with a special type of feedback are found. Under this feedback, the input to the first station is rejected as long as the queue size in the second station exceeds a predefined fixed level. The analysis is based on the introduction of a function expressing the dependence between the rates of input and output at the first station. We apply stochastic monotonicity of the Birth-Death process describing the dynamics of the system, to obtain the necessary conditions in an explicit form. Analysis of the two-station system is then generalized to multi-station system.

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