Om Distribution With Properties And Applications

Rama Shanker & Kamlesh Kumar Shukla*
•
Department of Statistics
College of Science, EIT, Asmara, Eritrea
kkshukla22@gmail.com , shankerrama2009@gmail.com
* Corresponding Author

Abstract

A new one parameter lifetime distribution named, 'Om distribution' has been proposed and studied. Its various statistical properties including shapes for probability density, moments based measures, hazard rate function, mean residual life function, stochastic ordering, mean deviations, Bonferroni and Lorenz curves, distribution of order statistics, and stress-strength reliability have been discussed. Estimation of parameter has been discussed with the method of maximum likelihood. Applications of the distribution have been explained through two examples of real lifetime data from engineering and the goodness of fit found to be quite satisfactory over several one parameter lifetime distributions.

Keywords: Lifetime distributions, Statistical Properties, Maximum likelihood estimation, Applications

I. Introduction

In the present world, the time to the occurrence of some event is of interest for some populations of individuals in almost every field of knowledge. The event may be death of a person or any living creature, failure of a piece of electronic equipment, development (or remission) of symptoms. In reliability analysis, the time to the occurrences of events are known as "lifetimes" or "survival times" or "failure times" according to the event of interest in the fields of study. The modeling and statistical analysis of lifetime data has been a topic of considerable interest to statisticians and research workers in engineering, biomedical science, insurance, finance, amongst others. Applications of lifetime distributions range from investigations into the endurance of manufactured items in engineering to research involving human diseases in biomedical sciences.

During recent decades, a number of one parameter and two-parameter lifetime distributions for modeling lifetime data have been introduced by different researchers in statistics. The popular one parameter lifetime distributions available in statistics literature are exponential distribution and Lindley distribution introduced by Lindley (1958). Recently Shanker (2015 a, 2015 b, 2016 a, 2016 b, 2017 a, 2017 b, 2017 c, 2017 d) has proposed several one parameter lifetime distributions, namely Shanker, Akash, Sujatha, Aradhana, Rama, Akshaya, Amarendra and Devya, and it has been showed by Shanker that these distributions have advantages and disadvantages over the others. The probability density function (pdf) and the cumulative distribution function (cdf) of exponential, Lindley, Shanker, Akash, Sujatha, Aradhana, Rama, Akshaya, Amarendra and Devya distributions along with their introducers and year have been presented in table 1.

Table 1: pdf and cdf of one parameter lifetime distributions

Distributions	pdf and cdf	Introducer(Year)
Devya	$f(x;\theta)=\dfrac{\theta^5}{\theta^4+\theta^3+2\theta^2+6\theta+24}\left(1+x+x^2+x^3+x^4\right)e^{-\theta x}$	Shanker (2016 d)
	$F(x,\theta)=1-\left[1+\dfrac{\theta^4\left(x^4+x^3+x^2+x\right)+\theta^3\left(4x^3+3x^2+2x\right)+6\theta^2\left(2x^2+x\right)+24\theta x}{\theta^4+\theta^3+2\theta^2+6\theta+24}\right]e^{-\theta x}$	
Amarendra	$f(x;\theta)=\dfrac{\theta^4}{\theta^3+\theta^2+2\theta+6}\left(1+x+x^2+x^3\right)e^{-\theta x}$	Shanker (2016 c)
	$F(x,\theta)=1-\left[1+\dfrac{\theta^3x^3+\theta^2(\theta+3)x^2+\theta\left(\theta^2+2\theta+6\right)x}{\theta^3+\theta^2+2\theta+6}\right]e^{-\theta x}$	
Akshaya	$f(x;\theta)=\dfrac{\theta^4}{\theta^3+3\theta^2+6\theta+6}(1+x)^3e^{-\theta x}\quad;x>0,\ \theta>0$	Shanker (2017 b)
	$F(x;\theta)=1-\left[1+\dfrac{\theta^3x^3+3\theta^2(\theta+1)x^2+3\theta\left(\theta^2+2\theta+2\right)x}{\theta^3+3\theta^2+6\theta+6}\right]e^{-\theta x}$	
Rama	$f(x;\theta)=\dfrac{\theta^4}{\theta^3+6}\left(1+x^3\right)e^{-\theta x}$	Shanker (2017 a)
	$F(x,\theta)=1-\left[1+\dfrac{\theta^3x^3+3\theta^2x^2+6\theta x}{\theta^3+6}\right]e^{-\theta x}$	
Aradhana	$f(x;\theta)=\dfrac{\theta^3}{\theta^2+2\theta+2}(1+x)^2e^{-\theta x}\quad;x>0,\ \theta>0$	Shanker (2016 b)
	$F(x;\theta)=1-\left[1+\dfrac{\theta x(\theta x+2\theta+2)}{\theta^2+2\theta+2}\right]e^{-\theta x}\quad;x>0,\theta>0$	
Sujatha	$f(x;\theta)=\dfrac{\theta^3}{\theta^2+\theta+2}\left(1+x+x^2\right)e^{-\theta x}\quad;x>0,\ \theta>0$	Shanker (2016 a)
	$F(x,\theta)=1-\left[1+\dfrac{\theta x(\theta x+\theta+2)}{\theta^2+\theta+2}\right]e^{-\theta x}$	
Akash	$f(x;\theta)=\dfrac{\theta^3}{\theta^2+2}\left(1+x^2\right)e^{-\theta x}\quad;x>0,\ \theta>0$	Shanker (2015 b)
	$F(x;\theta)=1-\left[1+\dfrac{\theta x(\theta x+2)}{\theta^2+2}\right]e^{-\theta x}\quad;x>0,\theta>0$	
Shanker	$f(x;\theta)=\dfrac{\theta^2}{\theta^2+1}(\theta+x)e^{-\theta x}\quad;x>0,\ \theta>0$	Shanker (2015 a)
	$F(x,\theta)=1-\left[1+\dfrac{\theta x}{\theta^2+1}\right]e^{-\theta x}\quad;x>0,\theta>0$	
Lindley	$f(x;\theta)=\dfrac{\theta^2}{\theta+1}(1+x)e^{-\theta x}\quad;x>0,\ \theta>0$	Lindley (1958)
	$F(x;\theta)=1-\left[1+\dfrac{\theta x}{\theta+1}\right]e^{-\theta x}\quad;x>0,\theta>0$	
Exponentia	$f(x;\theta)=\theta e^{-\theta x}\quad;x>0,\ \theta>0$	

1	$F_1(x;\theta)=1-e^{-\theta x}\ ;x>0,\theta>0$	

Ghitany *et al* (2008) have discussed various statistical properties, estimation of parameter and application of Lindley distribution for modeling waiting time data in a bank. It has been observed that these lifetime distributions are not always suitable for modeling lifetime data from biomedical sciences and engineering. In the present paper an attempt has been made to propose a one parameter lifetime distribution named 'Om distribution' which gives better fit than all one parameter lifetime distributions. Its various statistical properties, estimations of parameter and applications for modeling two real lifetime data from engineering have been discussed.

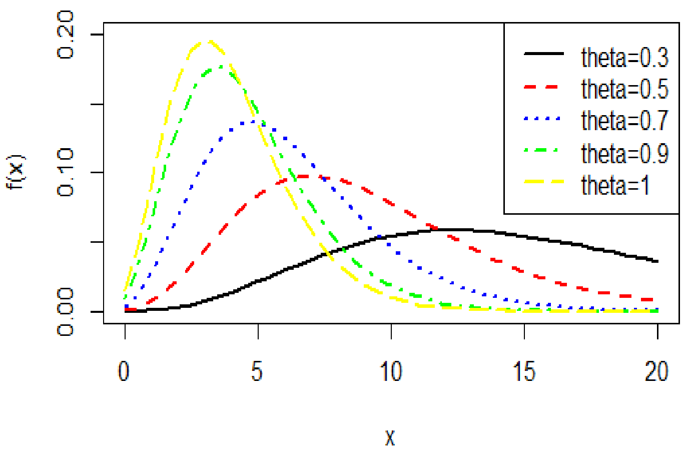
II. Om Distribution

A new one parameter lifetime distribution named Om distribution can be defined by its pdf and cdf

$$f(x;\theta)=\frac{\theta^5}{\theta^4+4\theta^3+12\theta^2+24\theta+24}(1+x)^4 e^{-\theta x} \quad ;x>0,\ \theta>0 \tag{2.1}$$

$$F(x;\theta)=1-\left[\frac{(1+x)^4\theta^4+4(1+x)^3\theta^3+12(1+x)^2\theta^2+24(1+x)\theta+24}{\theta^4+4\theta^3+12\theta^2+24\theta+24}\right]e^{-\theta x} \quad ;x>0,\theta>0 \tag{2.2}$$

The nature of the pdf and cdf of Om distribution for varying values of parameter θ have been shown graphically in figures 1 and 2 respectively.



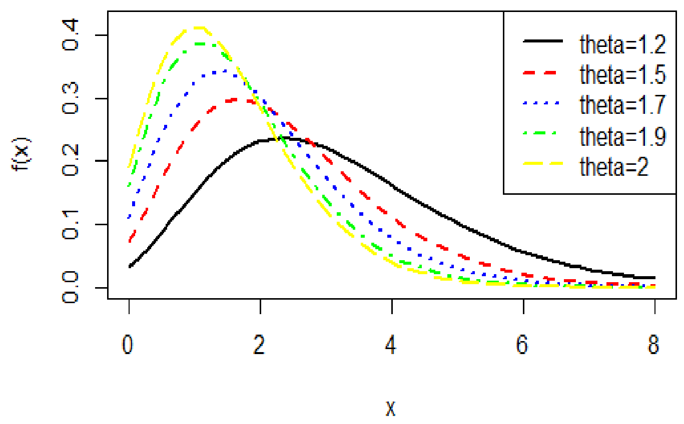
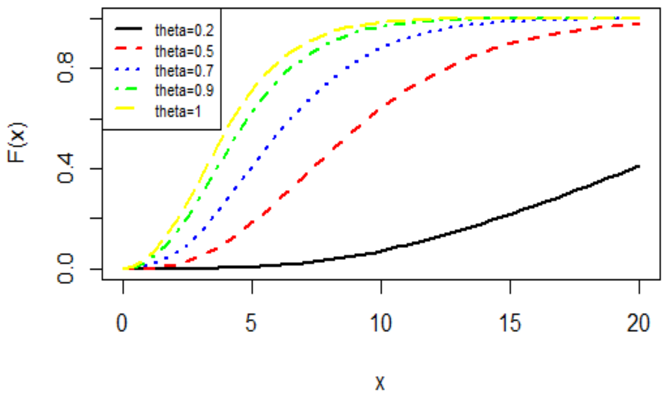


Fig. 1: Nature of the pdf of Om distribution for varying values of parameter θ



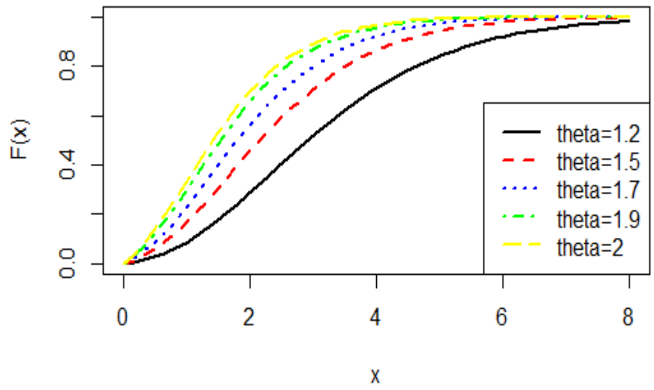


Fig. 2: Nature of the cdf of Om distribution for varying values of parameter θ

3. Moments and Associated measures

The r th moment about origin of Om distributon (2.1) can be obtained as

$$\mu_r' = \frac{r!\left\{\begin{matrix}\theta^4+4(r+1)\theta^3+6(r+1)(r+2)\theta^2+4(r+1)(r+2)(r+3)\theta \\ +(r+1)(r+2)(r+3)(r+4)\end{matrix}\right\}}{\theta^r\left(\theta^4+4\theta^3+12\theta^2+24\theta+24\right)}\;;r=1,2,3,... \tag{3.1}$$

The first four moments about origin of Om distribution can be given as

$$\mu_1' = \frac{\theta^4+8\theta^3+36\theta^2+96\theta+120}{\theta\left(\theta^4+4\theta^3+12\theta^2+24\theta+24\right)}$$

$$\mu_2' = \frac{2\left(\theta^4+12\theta^3+72\theta^2+240\theta+360\right)}{\theta^2\left(\theta^4+4\theta^3+12\theta^2+24\theta+24\right)}$$

$$\mu_3' = \frac{6\left(\theta^4+16\theta^3+120\theta^2+480\theta+840\right)}{\theta^3\left(\theta^4+4\theta^3+12\theta^2+24\theta+24\right)}$$

$$\mu_4' = \frac{24\left(\theta^4+20\theta^3+180\theta^2+840\theta+1680\right)}{\theta^4\left(\theta^4+4\theta^3+12\theta^2+24\theta+24\right)}$$

Thus the moments about mean of the Om distribution (2.1) are obtained as

$$\mu_2 = \frac{\theta^8 + 16\theta^7 + 128\theta^6 + 624\theta^5 + 1920\theta^4 + 3840\theta^3 + 5760\theta^2 + 5760\theta + 2880}{\theta^2\left(\theta^4 + 4\theta^3 + 12\theta^2 + 24\theta + 24\right)^2}$$

$$\mu_3 = \frac{2\begin{pmatrix}\theta^{12} + 24\theta^{11} + 276\theta^{10} + 1928\theta^9 + 8856\theta^8 + 28512\theta^7 + 70848\theta^6 + 141696\theta^5 + 233280\theta^4 \\ +311040\theta^3 + 311040\theta^2 + 207360\theta + 69120\end{pmatrix}}{\theta^3\left(\theta^4 + 4\theta^3 + 12\theta^2 + 24\theta + 24\right)^3}$$

$$\mu_4 = \frac{3\begin{pmatrix}3\theta^{16} + 96\theta^{15} + 1472\theta^{14} + 14048\theta^{13} + 92672\theta^{12} + 454656\theta^{11} + 1767936\theta^{10} + 5640960\theta^9 \\ +15034752\theta^8 + 33675264\theta^7 + 63148032\theta^6 + 98039808\theta^5 + 123863040\theta^4 \\ +123863040\theta^3 + 92897280\theta^2 + 46448640\theta + 11612160\end{pmatrix}}{\theta^4\left(\theta^4 + 4\theta^3 + 12\theta^2 + 24\theta + 24\right)^4}$$

The coefficient of variation $\left(C.V\right)$, coefficient of skewness $\left(\sqrt{\beta_1}\right)$, coefficient of kurtosis $\left(\beta_2\right)$ and Index of dispersion $\left(\gamma\right)$ of Om distribution (2.1) are thus obtained as

$$C.V = \frac{\sigma}{\mu_1'} = \frac{\sqrt{\theta^8 + 16\theta^7 + 128\theta^6 + 624\theta^5 + 1920\theta^4 + 3840\theta^3 + 5760\theta^2 + 5760\theta + 2880}}{\theta^4 + 8\theta^3 + 36\theta^2 + 96\theta + 120}$$

$$\sqrt{\beta_1} = \frac{\mu_3}{\mu_2^{3/2}} = \frac{2\begin{pmatrix}\theta^{12} + 24\theta^{11} + 276\theta^{10} + 1928\theta^9 + 8856\theta^8 + 28512\theta^7 + 70848\theta^6 + 141696\theta^5 + 233280\theta^4 \\ +311040\theta^3 + 311040\theta^2 + 207360\theta + 69120\end{pmatrix}}{\left(\theta^8 + 16\theta^7 + 128\theta^6 + 624\theta^5 + 1920\theta^4 + 3840\theta^3 + 5760\theta^2 + 5760\theta + 2880\right)^{3/2}}$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{3\begin{pmatrix}3\theta^{16} + 96\theta^{15} + 1472\theta^{14} + 14048\theta^{13} + 92672\theta^{12} + 454656\theta^{11} + 1767936\theta^{10} + 5640960\theta^9 \\ +15034752\theta^8 + 33675264\theta^7 + 63148032\theta^6 + 98039808\theta^5 + 123863040\theta^4 \\ +123863040\theta^3 + 92897280\theta^2 + 46448640\theta + 11612160\end{pmatrix}}{\left(\theta^8 + 16\theta^7 + 128\theta^6 + 624\theta^5 + 1920\theta^4 + 3840\theta^3 + 5760\theta^2 + 5760\theta + 2880\right)^2}$$

$$\gamma = \frac{\sigma^2}{\mu_1'} = \frac{\theta^8 + 16\theta^7 + 128\theta^6 + 624\theta^5 + 1920\theta^4 + 3840\theta^3 + 5760\theta^2 + 5760\theta + 2880}{\theta\left(\theta^4 + 4\theta^3 + 12\theta^2 + 24\theta + 24\right)\left(\theta^4 + 8\theta^3 + 36\theta^2 + 96\theta + 120\right)}.$$

The behaviors of coefficient of variation, skewness, kurtosis and index of dispersion of Om distribution have been shown graphically for varying values of parameter θ in figure 3.

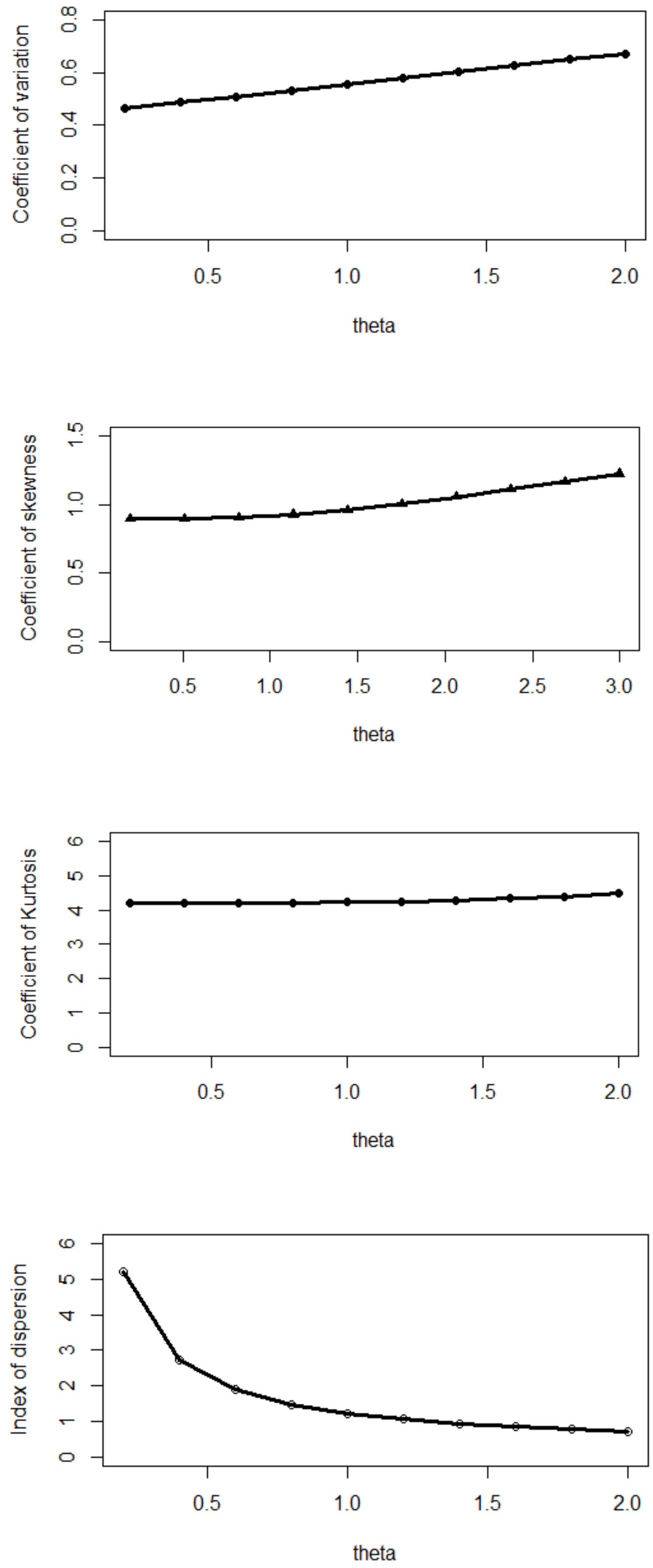


Fig. 3: Behavior of coefficient of variation, skewness, kurtosis and index of dispersion of Om distribution for varying values of parameter θ

The conditions of dispersion of Om distribution along with other one parameter lifetime distribution for values of the parameter θ have been presented in table 2.

Table 2. Over-dispersion, equi-dispersion and under-dispersion of Om distribution and other one parameter lifetime distributions for varying values of their parameter θ

Distributions	Over-dispersion $(\mu < \sigma^2)$	Equi-dispersion $(\mu = \sigma^2)$	Under-dispersion $(\mu > \sigma^2)$
Om	$\theta < 1.306113562$	$\theta = 1.306113562$	$\theta > 1.306113562$
Devya	$\theta < 1.451669994$	$\theta = 1.451669994$	$\theta > 1.451669994$
Amarendra	$\theta < 1.525763580$	$\theta = 1.525763580$	$\theta > 1.525763580$
Akshaya	$\theta < 1.327527885$	$\theta = 1.327527885$	$\theta > 1.327527885$
Rama	$\theta < 1.950164618$	$\theta = 1.950164618$	$\theta > 1.950164618$
Aradhana	$\theta < 1.283826505$	$\theta = 1.283826505$	$\theta > 1.283826505$
Sujatha	$\theta < 1.364271174$	$\theta = 1.364271174$	$\theta > 1.364271174$
Akash	$\theta < 1.515400063$	$\theta = 1.515400063$	$\theta > 1.515400063$
Shanker	$\theta < 1.171535555$	$\theta = 1.171535555$	$\theta > 1.171535555$
Lindley	$\theta < 1.170086487$	$\theta = 1.170086487$	$\theta > 1.170086487$
Exponential	$\theta < 1$	$\theta = 1$	$\theta > 1$

IV. Statistical Properties

I. Survival function, Hazard rate function and Mean Residual life function

Suppose $f(x)$ and $F(x)$ be the pdf and cdf of a continuous random variable X. The survival function, $S(x)$, hazard rate function $h(x)$ (also known as the failure rate function) and the mean residual life function $m(x)$ of X are respectively defined as

$$S(x) = P(X > x) = 1 - F(x)$$

$$h(x) = \lim_{\Delta x \to 0} \frac{P(X < x + \Delta x \mid X > x)}{\Delta x} = \frac{f(x)}{1 - F(x)}$$

and $m(x) = E\left[X - x \middle| X > x\right] = \dfrac{1}{1 - F(x)} \int_x^\infty \left[1 - F(t)\right]\, dt$

The corresponding survival function $S(x)$, hazard rate function, $h(x)$ and the mean residual life function, $m(x)$ of Om distribution are thus obtained as

$$S(x) = \left[\frac{(1+x)^4\theta^4 + 4(1+x)^3\theta^3 + 12(1+x)^2\theta^2 + 24(1+x)\theta + 24}{\theta^4 + 4\theta^3 + 12\theta^2 + 24\theta + 24}\right] e^{-\theta x} \quad ; x > 0, \theta > 0$$

$$h(x) = \frac{\theta^5 (1+x)^4}{(1+x)^4\theta^4 + 4(1+x)^3\theta^3 + 12(1+x)^2\theta^2 + 24(1+x)\theta + 24} \quad ; x > 0, \theta > 0$$

and $$m(x) = \frac{\theta^4 + 4\theta^3 + 12\theta^2 + 24\theta + 24}{\left[(1+x)^4\theta^4 + 4(1+x)^3\theta^3 + 12(1+x)^2\theta^2 + 24(1+x)\theta + 24\right] e^{-\theta x}} \times$$

$$\int_x^\infty \left[(1+t)^4\theta^4 + 4(1+t)^3\theta^3 + 12(1+t)^2\theta^2 + 24(1+t)\theta + 24\right] e^{-\theta t} dt$$

$$=\frac{(1+x)^4\theta^4+8(1+x)^3\theta^3+36(1+x)^2\theta^2+96(1+x)\theta+120}{\theta\left[(1+x)^4\theta^4+4(1+x)^3\theta^3+12(1+x)^2\theta^2+24(1+x)\theta+24\right]}$$

It can be easily verified that $h(0)=\dfrac{\theta^5}{\theta^4+4\theta^3+12\theta^2+24\theta+24}=f(0)$ and $m(0)=\dfrac{\theta^4+8\theta^3+36\theta^2+96\theta+120}{\theta\left(\theta^4+4\theta^3+12\theta^2+24\theta+24\right)}=\mu_1'$. The behaviors of $h(x)$ and $m(x)$ of Om distribution have been shown in figures 4 and 5 respectively.

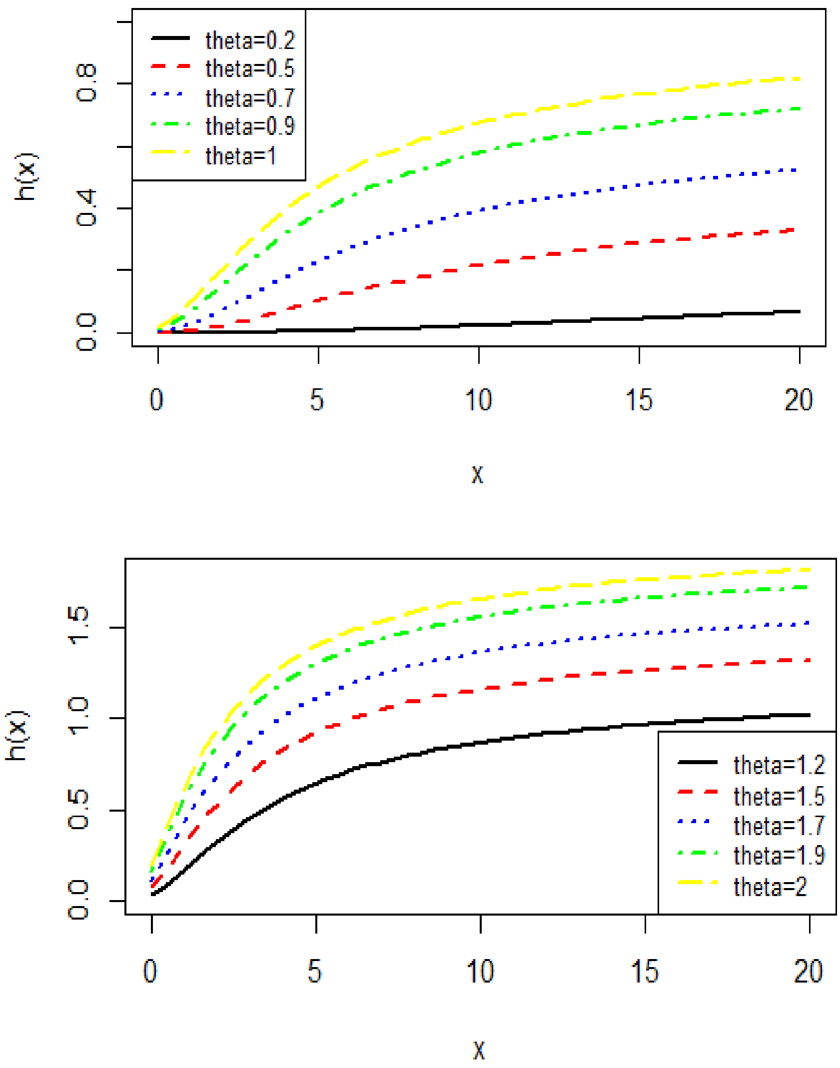
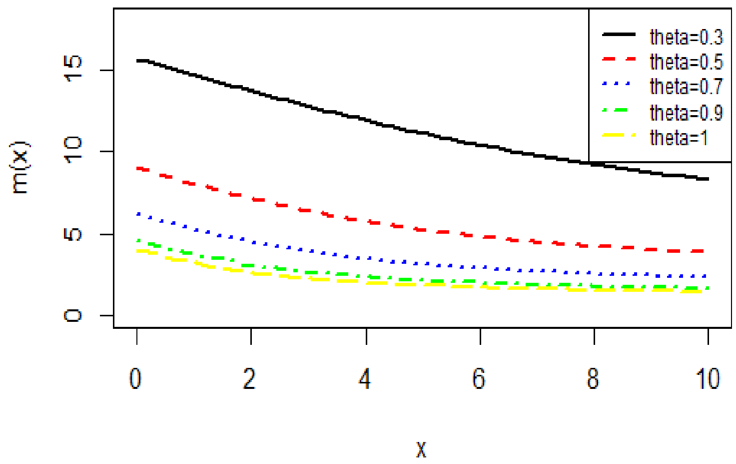


Fig. 4: Behavior of $h(x)$ of Om distribution for varying values of parameter θ



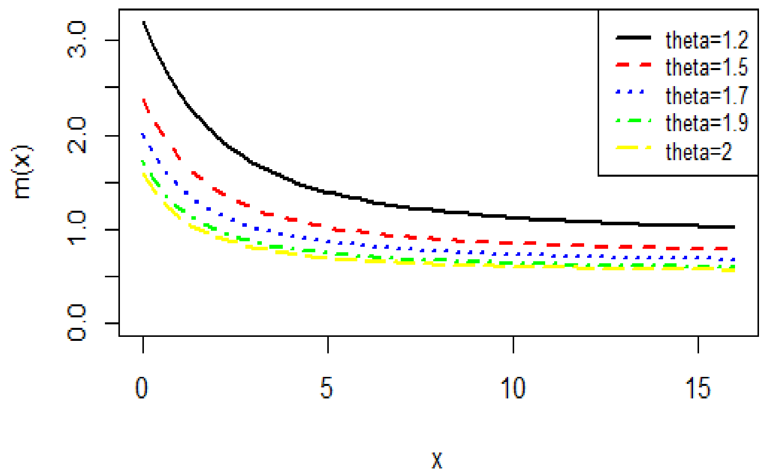


Fig. 5: Behavior of $m(x)$ of Om distribution for varying values of parameter θ

II. Mean deviations from the mean and the Median

The amount of scatter in a population is measured to some extent by the totality of deviations usually from their mean and median. These are known as the mean deviation about the mean and the mean deviation about the median and are defined as

$\delta_1(X)=\int_0^\infty |x-\mu|\, f(x)\,dx$ and $\delta_2(X)=\int_0^\infty |x-M|\, f(x)\,dx$, respectively, where $\mu=E(X)$ and $M=\text{Median}\ (X)$. The measures $\delta_1(X)$ and $\delta_2(X)$ can be computed using the following simplified relationships

$$\delta_1(X)=2\mu F(\mu)-2\int_0^\mu x\, f(x)\,dx \tag{4.2.1}$$

and

$$\delta_2(X)=\mu-2\int_0^M x\, f(x)\,dx \tag{4.2.2}$$

Using pdf (2.1) and expression for the mean of Om distribution, we get

$$\int_0^{\mu} x\, f(x;\theta)\,dx = \mu - \frac{\left\{\begin{array}{l}\left(\mu^5+4\mu^4+6\mu^3+4\mu^2+\mu\right)\theta^5+\left(5\mu^4+16\mu^3+18\mu^2+8\mu+1\right)\theta^4 \\ +\left(20\mu^3+48\mu^2+36\mu+8\right)\theta^3+\left(60\mu^2+96\mu+36\right)\theta^2+\left(120\mu+96\right)\theta+120\end{array}\right\} e^{-\theta\mu}}{\theta\left(\theta^4+4\theta^3+12\theta^2+24\theta+24\right)}$$

(4.2.3)

$$\int_0^{M} x\, f(x;\theta)\,dx = \mu - \frac{\left\{\begin{array}{l}\left(M^5+4M^4+6M^3+4M^2+M\right)\theta^5+\left(5M^4+16M^3+18M^2+8M+1\right)\theta^4 \\ +\left(20M^3+48M^2+36M+8\right)\theta^3+\left(60M^2+96M+36\right)\theta^2+\left(120M+96\right)\theta+120\end{array}\right\} e^{-\theta M}}{\theta\left(\theta^4+4\theta^3+12\theta^2+24\theta+24\right)}$$

(4.2.4)

Using expressions (4.2.1), (4.2.2), (4.2.3) and (4.2.4), the mean deviation about mean, $\delta_1(X)$ and the mean deviation about median, $\delta_2(X)$ of Om distribution (2.1),, after tedious algebraic simplification are obtained as

$$\delta_1(X) = 2\left[\frac{(1+\mu)^4\theta^4+8(1+\mu)^3\theta^3+36(1+\mu)^2\theta^2+96(1+\mu)\theta+120}{\theta\left(\theta^4+4\theta^3+12\theta^2+24\theta+24\right)}\right]e^{-\theta\mu}$$

(4.2.5)

$$\partial_2(X) = \frac{\left\{\begin{array}{l}\left(M^5+4M^4+6M^3+4M^2+M\right)\theta^5+\left(5M^4+16M^3+18M^2+8M+1\right)\theta^4 \\ +\left(20M^3+48M^2+36M+8\right)\theta^3+\left(60M^2+96M+36\right)\theta^2+\left(120M+96\right)\theta+120\end{array}\right\} e^{-\theta M}}{\theta\left(\theta^4+4\theta^3+12\theta^2+24\theta+24\right)} - \mu$$

III. Bonferroni and Lorenz Curves

The Bonferroni and Lorenz curves (Bonferroni, 1930) and Bonferroni and Gini indices have applications not only in economics to study income and poverty, but also in other fields like reliability, demography, insurance and medicine. The Bonferroni and Lorenz curves are defined as

$$B(p) = \frac{1}{p\mu}\int_0^q x\, f(x)\,dx = \frac{1}{p\mu}\left[\int_0^\infty x\, f(x)\,dx - \int_q^\infty x\, f(x)\,dx\right] = \frac{1}{p\mu}\left[\mu - \int_q^\infty x\, f(x)\,dx\right] \quad (4.3.1)$$

and

$$L(p) = \frac{1}{\mu}\int_0^q x\, f(x)\,dx = \frac{1}{\mu}\left[\int_0^\infty x\, f(x)\,dx - \int_q^\infty x\, f(x)\,dx\right] = \frac{1}{\mu}\left[\mu - \int_q^\infty x\, f(x)\,dx\right] \quad (4.3.2)$$

respectively or equivalently

$$B(p) = \frac{1}{p\mu}\int_0^p F^{-1}(x)\,dx \quad (4.3.3)$$

and

$$L(p) = \frac{1}{\mu}\int_0^p F^{-1}(x)\,dx \quad (4.3.4)$$

respectively, where $\mu = E(X)$ and $q = F^{-1}(p)$.

The Bonferroni and Gini indices are thus define

$$B = 1 - \int_0^1 B(p)\,dp \tag{4.3.5}$$

$$G = 1 - 2\int_0^1 L(p)\,dp \tag{4.3.6}$$

respectively
Using pdf (2.1), we get

$$\int_q^\infty x\, f(x;\theta)\,dx = \frac{\left\{\begin{array}{l}\left(q^5+4q^4+6q^3+4q^2+q\right)\theta^5+\left(5q^4+16q^3+18q^2+8q+1\right)\theta^4\\ +\left(20q^3+48q^2+36q+8\right)\theta^3+\left(60q^2+96q+36\right)\theta^2+\left(120q+96\right)\theta+120\end{array}\right\}e^{-\theta q}}{\theta\left(\theta^4+4\theta^3+12\theta^2+24\theta+24\right)} \tag{4.3.7}$$

Now using equation (4.3.7) in (4.3.1) and (4.3.2), we get

$$B(p) = \frac{1}{p}\left[1 - \frac{\left\{\begin{array}{l}\left(q^5+4q^4+6q^3+4q^2+q\right)\theta^5+\left(5q^4+16q^3+18q^2+8q+1\right)\theta^4\\ +\left(20q^3+48q^2+36q+8\right)\theta^3+\left(60q^2+96q+36\right)\theta^2+\left(120q+96\right)\theta+120\end{array}\right\}e^{-\theta q}}{\theta^4+8\theta^3+36\theta^2+96\theta+120}\right] \tag{4.3.8}$$

$$L(p) = 1 - \frac{\left\{\begin{array}{l}\left(q^5+4q^4+6q^3+4q^2+q\right)\theta^5+\left(5q^4+16q^3+18q^2+8q+1\right)\theta^4\\ +\left(20q^3+48q^2+36q+8\right)\theta^3+\left(60q^2+96q+36\right)\theta^2+\left(120q+96\right)\theta+120\end{array}\right\}e^{-\theta q}}{\theta^4+8\theta^3+36\theta^2+96\theta+120} \tag{4.3.9}$$

Now using equations (4.3.8) and (4.3.9) in (4.3.5) and (4.3.6), the Bonferroni and Gini indices of Om distribution (2.1) are obtained as

$$B = 1 - \frac{\left\{\begin{array}{l}\left(q^5+4q^4+6q^3+4q^2+q\right)\theta^5+\left(5q^4+16q^3+18q^2+8q+1\right)\theta^4\\ +\left(20q^3+48q^2+36q+8\right)\theta^3+\left(60q^2+96q+36\right)\theta^2+\left(120q+96\right)\theta+120\end{array}\right\}e^{-\theta q}}{\theta^4+8\theta^3+36\theta^2+96\theta+120} \tag{4.3.9}$$

$$G = \frac{\left\{\begin{array}{l}\left(q^5+4q^4+6q^3+4q^2+q\right)\theta^5+\left(5q^4+16q^3+18q^2+8q+1\right)\theta^4\\ +\left(20q^3+48q^2+36q+8\right)\theta^3+\left(60q^2+96q+36\right)\theta^2+\left(120q+96\right)\theta+120\end{array}\right\}e^{-\theta q}}{\theta^4+8\theta^3+36\theta^2+96\theta+120} - 1 \tag{4.3.10}$$

IV. Stochastic ordering

Stochastic ordering of positive continuous random variables is an important tool for judging their comparative behavior. A random variable X is said to be smaller than a random variable Y in the

(i) stochastic order $\left(X \le_{st} Y\right)$ if $F_X\left(x\right) \ge F_Y\left(x\right)$ for all x

(ii) hazard rate order $\left(X \le_{hr} Y\right)$ if $h_X\left(x\right) \ge h_Y\left(x\right)$ for all x

(iii) mean residual life order $\left(X \le_{mrl} Y\right)$ if $m_X\left(x\right) \le m_Y\left(x\right)$ for all x

(iv) likelihood ratio order $\left(X \le_{lr} Y\right)$ if $\dfrac{f_X\left(x\right)}{f_Y\left(x\right)}$ decreases in x.

The following results due to Shaked and Shanthikumar (1994) are well known for establishing stochastic ordering of distributions

$$\begin{array}{c} X \le_{lr} Y \Rightarrow X \le_{hr} Y \Rightarrow X \le_{mrl} Y \\ \Downarrow \\ X \le_{st} Y \end{array}$$

The Om distribution is ordered with respect to the strongest 'likelihood ratio' ordering as shown in the following theorem:

Theorem: Let X ~ Om distribution$\left(\theta_1\right)$ and Y ~ Om distribution$\left(\theta_2\right)$. If $\theta_1 \ge \theta_2$, then $X \le_{lr} Y$ and hence $X \le_{hr} Y$, $X \le_{mrl} Y$ and $X \le_{st} Y$.

Proof: We have

$$\frac{f_X\left(x;\theta_1\right)}{f_Y\left(x;\theta_2\right)} = \frac{\theta_1^5\left(\theta_2^4 + 4\theta_2^3 + 12\theta_2^2 + 24\theta_2 + 24\right)}{\theta_2^5\left(\theta_1^4 + 4\theta_1^3 + 12\theta_1^2 + 24\theta_1 + 24\right)} e^{-\left(\theta_1 - \theta_2\right)x}; x > 0$$

Now

$$\ln \frac{f_X\left(x;\theta_1\right)}{f_Y\left(x;\theta_2\right)} = \ln\left[\frac{\theta_1^5\left(\theta_2^4 + 4\theta_2^3 + 12\theta_2^2 + 24\theta_2 + 24\right)}{\theta_2^5\left(\theta_1^4 + 4\theta_1^3 + 12\theta_1^2 + 24\theta_1 + 24\right)}\right] - \left(\theta_1 - \theta_2\right)x$$

This gives $\dfrac{d}{dx}\ln \dfrac{f_X\left(x;\theta_1\right)}{f_Y\left(x;\theta_2\right)} = -\left(\theta_1 - \theta_2\right)$. Thus for $\theta_1 \ge \theta_2$, $\dfrac{d}{dx}\ln \dfrac{f_X\left(x;\theta_1\right)}{f_Y\left(x;\theta_2\right)} \le 0$. This means that $X \le_{lr} Y$ and hence $X \le_{hr} Y$, $X \le_{mrl} Y$ and $X \le_{st} Y$.

V.Distribution of Order Statistics

Let $X_1, X_2, ..., X_n$ be a random sample of size n from Om distribution (2.1). Let $X_{(1)} < X_{(2)} < ... < X_{(n)}$ denote the corresponding order statistics. The pdf and the cdf of the k th order statistic, say $Y = X_{(k)}$ are given by

$$f_Y\left(y\right) = \frac{n!}{\left(k-1\right)!\left(n-k\right)!} F^{k-1}\left(y\right)\left\{1 - F\left(y\right)\right\}^{n-k} f\left(y\right)$$

$$= \frac{n!}{\left(k-1\right)!\left(n-k\right)!} \sum_{l=0}^{n-k} \binom{n-k}{l} \left(-1\right)^l F^{k+l-1}\left(y\right) f\left(y\right)$$

and

$$F_Y\left(y\right) = \sum_{j=k}^{n} \binom{n}{j} F^j\left(y\right)\left\{1 - F\left(y\right)\right\}^{n-j}$$

$$=\sum_{j=k}^{n}\sum_{l=0}^{n-j}\binom{n}{j}\binom{n-j}{l}(-1)^{l}F^{j+l}(y),$$

respectively, for $k=1,2,3,...,n$.

Thus, the pdf and the cdf of k th order statistics of Om distribution (2.1) are obtained as

$$f_Y(y)=\frac{n!\theta^5(1+x)^4e^{-\theta x}}{(\theta^4+4\theta^3+12\theta^2+24\theta+24)(k-1)!(n-k)!}$$
$$\times\sum_{l=0}^{n-k}\binom{n-k}{l}(-1)^{l}\left[1-\left\{\frac{(1+x)^4\theta^4+4(1+x)^3\theta^3+12(1+x)^2\theta^2+24(1+x)\theta+24}{\theta^4+4\theta^3+12\theta^2+24\theta+24}\right\}e^{-\theta x}\right]^{k+l-1}$$

and

$$F_Y(y)=\sum_{j=k}^{n}\sum_{l=0}^{n-j}\binom{n}{j}\binom{n-j}{l}(-1)^{l}\left[1-\left\{\frac{(1+x)^4\theta^4+4(1+x)^3\theta^3+12(1+x)^2\theta^2+24(1+x)\theta+24}{\theta^4+4\theta^3+12\theta^2+24\theta+24}\right\}e^{-\theta x}\right]^{j+l}$$

VI. Stress-Strength Reliability

The stress- strength reliability describes the life of a component which has random strength X that is subjected to a random stress Y. When the stress Y applied to it exceeds the strength X, the component fails instantly and the component will function satisfactorily till $X>Y$. Therefore, $R=P(Y<X)$ is a measure of component reliability and is known as stress-strength reliability in statistical literature. It has wide applications in almost all areas of knowledge especially in engineering such as structures, deterioration of rocket motors, static fatigue of ceramic components, aging of concrete pressure vessels etc.

Let X and Y be independent strength and stress random variables having Om distribution (2.1) with parameter θ_1 and θ_2 respectively. Then the stress-strength reliability R of Om distribution can be obtained as

$$R=P(Y<X)=\int_0^\infty P(Y<X\mid X=x)f_X(x)dx$$
$$=\int_0^\infty f(x;\theta_1)\,F(x;\theta_2)dx$$

$$=1-\frac{\theta_1^5\left[\begin{array}{l}(\theta_2^4+4\theta_2^3+12\theta_2^2+24\theta_2+24)(\theta_1+\theta_2)^8+(8\theta_2^4+28\theta_2^3+72\theta_2^2+120\theta_2+96)(\theta_1+\theta_2)^7\\+(56\theta_2^4+168\theta_2^3+360\theta_2^2+480\theta_2+288)(\theta_1+\theta_2)^6\\+(336\theta_2^4+840\theta_2^3+1440\theta_2^2+1440\theta_2+576)(\theta_1+\theta_2)^5\\+(1680\theta_2^4+3360\theta_2^3+4320\theta_2^2+2880\theta_2+576)(\theta_1+\theta_2)^4\\+(6720\theta_2^4+10080\theta_2^3+8640\theta_2^2+2880\theta_2)(\theta_1+\theta_2)^3\\+(20160\theta_2^4+20160\theta_2^3+8640\theta_2^2)(\theta_1+\theta_2)^2+(40320\theta_2^4+20160\theta_2^3)(\theta_1+\theta_2)+40320\theta_2^4\end{array}\right]}{(\theta_1^4+4\theta_1^3+12\theta_1^2+24\theta_1+24)(\theta_2^4+4\theta_2^3+12\theta_2^2+24\theta_2+24)(\theta_1+\theta_2)^9}$$

V. Maximum likelihood estimation

Let $(x_1, x_2, ..., x_n)$ be a random sample of size n from Om distribution (2.1). The likelihood function L of Om distribution can be expressed as

$$L = \left(\frac{\theta^5}{\theta^4 + 4\theta^3 + 12\theta^2 + 24\theta + 24} \right)^n \prod_{i=1}^{n} (1 + x_i)^4 e^{-n\theta \bar{x}}$$

$$= \left(\frac{\theta^5}{\theta^4 + 4\theta^3 + 12\theta^2 + 24\theta + 24} \right)^n \prod_{i=1}^{n} \left(1 + 4x_i + 6x_i^2 + 4x_i^3 + x_i^4\right) e^{-n\theta \bar{x}}$$

The log likelihood function is thus given by

$$\log L = n \log\left(\frac{\theta^5}{\theta^4 + 4\theta^3 + 12\theta^2 + 24\theta + 24} \right) \sum_{i=1}^{n} \log\left(1 + 4x_i + 6x_i^2 + 4x_i^3 + x_i^4\right) - n\theta \bar{x} \, .$$

The maximum likelihood estimates (MLE) $\hat{\theta}$ of parameter θ is the solution of the log-likelihood equation $\frac{d \log L}{d\theta} = 0$ and is given by

$$\frac{d \log L}{d\theta} = \frac{5n}{\theta} - \frac{n\left(4\theta^3 + 12\theta^2 + 24\theta + 24\right)}{\theta^4 + 4\theta^3 + 12\theta^2 + 24\theta + 24} - n\,\bar{x} = 0 \, .$$

This gives a fifth degree polynomial equation in θ as

$$\bar{x}\,\theta^5 + \left(4\bar{x} - 1\right)\theta^4 + 4\left(3\bar{x} - 2\right)\theta^3 + 12\left(2\,\bar{x} - 3\right)\theta^2 + 24\left(\bar{x} - 4\right)\theta - 120 = 0 \, .$$

This equation can be easily solved using any numerical iterationmethod namely, Newton-Raphson method, Regula Falsi method or Bisection method. In this paper Newton-Raphson method has been used to estimate the parameter θ from above equation. It should be noted that equating the population mean to the corresponding sample mean, the method of moment estimate is the same as method of maximum likelihood.

VI. Data analysis

In this section the goodness of fit of Om distribution has been discussed with following two real lifetime datasets from engineering.

Data Set 1: The data is given by Birnbaum and Saunders (1969) on the fatigue life of 6061 – T6 aluminum coupons cut parallel to the direction of rolling and oscillated at 18 cycles per second. The data set consists of 101 observations with maximum stress per cycle 31,000 psi. The data ($\times 10^{-3}$) are presented below (after subtracting 65**).**

5	25	31	32	34	35	38	39	39	40	42	43
43	43	44	44	47	47	48	49	49	49	51	54
55	55	55	56	56	56	58	59	59	59	59	59
63	63	64	64	65	65	65	66	66	66	66	66
67	67	67	68	69	69	69	69	71	71	72	73
73	73	74	74	76	76	77	77	77	77	77	77
79	79	80	81	83	83	84	86	86	87	90	91
92	92	92	92	93	94	97	98	98	99	101	103
105	109	136	147								

Data Set 2: This data set is the strength data of glass of the aircraft window reported by Fuller *et al* (1994)

18.83	20.8	21.657	23.03	23.23	24.05	24.321	25.5	25.52	25.8	26.69	26.77
26.78	27.05	27.67	29.9	31.11	33.2	33.73	33.76	33.89	34.76	35.75	35.91
36.98	37.08	37.09	39.58	44.045	45.29	45.381					

For these two datasets, Om distribution has been fitted along with other one parameter lifetime distributions. The ML estimate, value of $-2\ln L$, Akaike Information criteria (AIC), K-S statistics and p-value of the fitted distributions are presented in tables 3 and 4.. The AIC and K-S Statistics are computed using the following formulae: $AIC = -2\ln L + 2k$ and $\text{K-S} = \underset{x}{\text{Sup}}\left|F_n(x) - F_0(x)\right|$, where k = the number of parameters, n = the sample size , $F_n(x)$ is the empirical (sample) cumulative distribution function, and $F_0(x)$ is the theoretical cumulative distribution function. The best distribution is the distribution corresponding to lower values of $-2\ln L$, AIC, and K-S statistics and higher p-value

Table 3: MLE's, - 2ln L, AIC, K-S and p-values of the fitted distributions for dataset 1

Distributions	$\text{MLE}(\hat{\theta})$	$\text{S.E}(\hat{\theta})$	$-2\log L$	AIC	K-S	P-Value
Om	0.07211	0.00322	924.64	926.64	0.138	0.043
Shambhu	0.08755	0.00357	918.61	920.61	0.117	0.131
Devya	0.07289	0.00326	924.26	926.26	0.333	0.000
Amarendra	0.05824	0.00213	934.38	936.38	0.163	0.010
Suja	0.07317	0.00327	924.21	926.21	0.136	0.049
Akshaya	0.05769	0.00288	935.11	937.11	0.164	0.008
Rama	0.05854	0.00293	934.05	934.05	0.162	0.012
Aradhana	0.04327	0.00249	952.58	954.58	0.196	0.001
Sujatha	0.04356	0.00251	951.78	953.78	0.195	0.001
Akash	0.04387	0.00253	950.97	952.97	0.194	0.001
Shanker	0.02925	0.00206	980.97	982.97	0.248	0.000
Lindley	0.02887	0.00204	983.11	985.11	0.252	0.000
Exponential	0.01463	0.00145	1044.87	1046.87	0.366	0.000

Table 4: MLE's, - 2ln L, AIC, K-S and p-values of the fitted distributions for dataset 2

Distributions	$\text{MLE}(\hat{\theta})$	$\text{S.E}(\hat{\theta})$	$-2\log L$	AIC	K-S	P-Value
Om	0.15718	0.01262	228.81	230.81	0.230	0.061
Shambhu	0.19339	0.01417	223.40	225.40	0.199	0.148
Devya	0.16087	0.01292	227.68	229.68	0.422	0.000
Amarendra	0.12829	0.01210	233.41	235.41	0.257	0.027
Suja	0.16227	0.01303	227.25	229.25	0.223	0.077
Akshaya	0.12574	0.01129	234.44	236.44	0.263	0.022
Rama	0.12978	0.01165	232.79	234.79	0.253	0.030
Aradhana	0.09432	0.00978	242.22	244.22	0.306	0.004
Sujatha	0.09561	0.00990	241.50	243.50	0.303	0.005
Akash	0.09706	0.01005	240.68	242.68	0.298	0.006
Shanker	0.64716	0.00820	252.35	254.35	0.358	0.000
Lindley	0.06299	0.00800	253.98	255.98	0.365	0.000
Exponential	0.03245	0.00582	274.53	276.53	0.458	0.000

VII. Concluding remarks and future works

This paper proposes a new one parameter lifetime distribution named, 'Om distribution'. Statistical properties including shapes for probability density, moments based measures, hazard rate function, mean residual life function, stochastic ordering, mean deviations, Bonferroni and Lorenz curves, distribution of order statistics, and stress-strength reliability have been discussed. Method of maximum likelihood has been discussed for estimating the parameter of the distribution. Applications of the distribution have been explained through two examples of real lifetime data from engineering and the goodness of fit has been found to be quite satisfactory over several one parameter lifetime distributions.

Since the present distribution is a new distribution in statistics literature, a lot of works can be done on the distribution. The future works to be done on the distribution includes Poisson mixture of the distribution, weighted version of the distribution, Power version of the distribution, discretization of the distribution using infinite series and survival function approach, exponentiation of the distribution, Bayesian method of estimation, some among others. All these works will appear in statistics literature with passage of time.

References

[1] Birnbaum, Z.W. and Saunders, S.C. (1969): Estimation for a family of life distributions with applications to fatigue, *Journal of Applied Probability*, 6, 328-347.

[2] Bonferroni, C.E. (1930): Elementi di Statistca generale, Seeber, Firenze

[3] Ghitany, M.E., Atieh, B. and Nadarajah, S. (2008): Lindley distribution and its Application, *Mathematics Computing and Simulation*, 78, 493 – 506

[4] Fuller, E.J., Frieman, S., Quinn, J., Quinn, G., and Carter, W.(1994): Fracture mechanics approach to the design of glass aircraft windows: A case study, SPIE Proc 2286, 419-430.

[5] Lindley, D.V. (1958): Fiducial distributions and Bayes' theorem, *Journal of the Royal Statistical Society, Series B*, 20, 102- 107.

[6] Shaked, M. and Shanthikumar, J.G. (1994): Stochastic Orders and Their Applications, Academic Press, New York.

[7] Shanker, R. (2015 a): Shanker Distribution and Its Applications, *International Journal of Statistics and Applications,* 5 (6), 338 – 348

[8] Shanker, R. (2015 b): Akash Distribution and Its Applications, *International Journal of Probability and Statistics,* 4 (3), 65 – 75

[9] Shanker, R. (2016 a): Sujatha Distribution and Its Applications, *Statistics in Transition-New series*, 17(3), 1- 20.

[10] Shanker, R. (2016 b): Aradhana Distribution and Its Applications, *International Journal of Statistics and Application*s, 6(1), 23 – 34.

[11] Shanker, R. (2017 c): Amarendra Distribution and Its Applications, *American Journal of Mathematics and Statistics*, 6(1), 44 – 56

[12] Shanker, R. (2016 d): Devya Distribution and Its Applications, *International Journal of Statistics and Applications*, 6(4), 189 – 202

[13] Shanker, R. (2017 a): Rama Distribution and its Application, *International Journal of Statistics and Applications*, 7(1), 26 – 35

[14] Shanker, R. (2017 b): Akshaya Distribution and its Application, *American Journal of Mathematics and Statistics*, 7(2), 51 – 59