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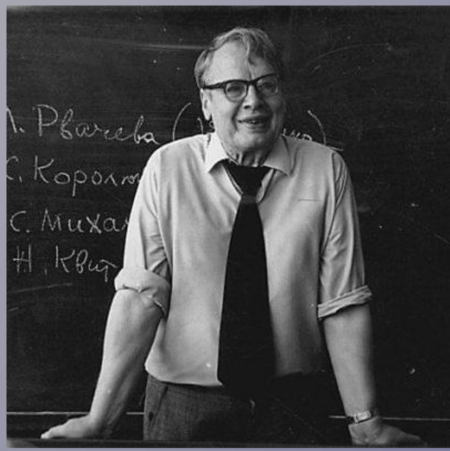

RELIABILITY:
THEORY&APPLICATIONS

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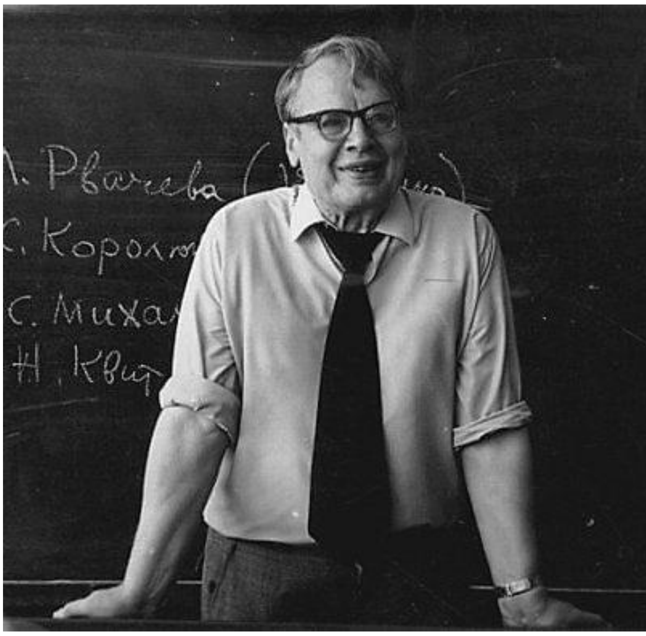

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The 80th Birthday of Vladimir Rykov

The Editor-in-chief of our journal and full member of the Informatization Academy, Professor Vladimir Rykov became eighty on January 2018. Rykov got his bachelor and master degree in 1960 from the Lomonosov Moscow State University (MSU). In his diploma project, supervised by A.N. Kolmogorov, Rykov studied the properties of trajectories of random processes and simultaneously with R. Dobrushin found the condition for absence of jumps in them. After the graduation in 1960 Rykov has begun his career in the MSU Computer Center, then he has worked for ten years at the Central Research and Development Institute of Complex Automation (CRICA), and for more than 35 year he is Professor at the department of Applied Mathematics and Computer Modeling in Gubkin State University of Oil and Gas. At the same time, he gives lectures at the Department of Applied Probability and Informatics of the Peoples' Friendship University of Russia and has actively collaborated before with the RAS Kharkevich Institute for Information Transmission Problems. During two years he has been teaching as invited professor at the mathematical department in Kettering University (USA).

Scientific interests of Vladimir Rykov are related mostly to studies on the theory of controllable queuing systems, stochastic networks and reliability. During the period of his work at CRICA (1961–1969) he has actively participated in research into the issues of reliability and in the development of theoretical methods of industrial control. He has authored one of the first publications on choosing the optimal time between the maintenance repairs [1]. Applied problems led him to studying the controllable queuing systems [2]. The conditions for optimality of priorities in queuing systems were established in collaboration with O. Bronshtein [3]. For the queuing systems, this rule was published in monographs, and in the English-language literature it is known as the $c\mu$ -rule. The interest in theoretical studies manifested itself in the works on the controllable Markov processes with finite state space for which it was shown that Markov strategies provide optimal control with respect to long-run and discounted cost criteria for Markov process with additive functionals [4]. Various applied aspects of control over the queuing systems were reported at different conferences and discussed in numerous publications. The proof of validity of the rule of optimal priority service in a wider class of all Markov strategies [5] and extension of this rule to the case of optimal customer service in the system with branching flows of customers [6] have made a valuable contribution to the theory of controllable systems. All these investigations have formed the basis of his PhD thesis, that was defended in Central Economics and Mathematics Institute (CEMI) of Russian Academy of Science in 1966. These results were published in the review "Controllable queuing systems" in "Itogi nauki i techniki, Teorija Verojatn. Math. Statist. Teoretich. Cybern" [7] in 1975 and further in1995 jointly with M.Yu.Kitaev in CRC Press as a monograph "Controlled queuing systems" [8].

By extending these investigations Rykov has established the general conditions for monotonicity of optimal policies for controllable queuing systems [9]. The continuation of studies in this direction have enabled to find the solution of the so-called slow server problem in a general case of arbitrary number of heterogeneous servers [10, 11]. This allows to compare the performance of systems with heterogeneous servers with respect to optimal and various heuristic service disciplines [12], which underlie the design of protocols of information and communication networks and these studies are still carried out.

Another field of Rykov's studies is represented by the theory of regenerative processes. In Russia he was the first who introduced and considered the regenerative processes with several types of regeneration points and jointly with M.A. Yastrebenetsky proved the ergodicity theorem for them [13] and found use in the studies of complex systems of various classes. Later these processes in English papers were called as the semi-regenerative processes. The development of this theory and introduction of the theory of decomposable semi-regenerative processes extended further this direction of research [14]. Their analysis and application were reflected in some of the subsequent works. For example, the $GI/GI/1/\infty$ system was studied using methods of the theory of decomposable semi-regenerative processes [15, 16]. These methods were then used for decomposition and analysis of complex hierarchical systems [17, 18], as well as polling systems [19]. Based on series of these investigations Rykov represented in 1990 the doctoral thesis "Decomposable semi-regenerative processes and their application for queuing problems investigation", for which he got the doctor of sciences degree in physics and mathematics from Lomonosov Moscow State University.

Later queuing systems were studied in a series of works, where for the open and closed networks the well-known multiplicative representation of stationary probabilities was extended to the case of networks with dependent service times [20, 21]. For the systems with dynamic multi-address connection the model of transparent customers has been proposed in 2001 and multiplicative representation for its steady state probabilities has been done in [22].

Other interests of prof. Rykov reflected themselves in a series of publications devoted to the relation between the periodic Poisson processes and the almost lack of memory distributions and to their statistical analysis [23, 24].

In the last time prof. Rykov turns to the investigation of reliability of complex systems. Series of works jointly with his students and colleagues has been devoted to the study of the problem of systems' reliability characteristics' sensitivity to the shapes of some initial distributions see [25, 26] ans the bibliography therein).

Vladimir Rykov has been member of Organizing and Program Committees of many conferences, including almost all conferences of Mathematical methods in Reliability (MMR) and many others. Rykov is the author of more than 260 publications in scientific journals, conference proceedings, a series of reviews, teaching books, and the aforementioned monograph. He also edited conference proceedings and translated from English and edited five monographs.

The Editorial Board of "Reliability: Theory and Applications " congratulates Vladimir Rykov on his eighty anniversary and wishes him good health and further creative advances.

RT&A EditorIAl Board Members

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Two A. N. Kolmogorov's Early Letters About Mathematics Education

B. V. Gnedenko
Publication and notes by D. B. Gnedenko

Abstract

It is well known that A.N. Kolmogorov made a great contribution to the development of Russian mathematics pedagogy. Mr. Kolmogorov became increasingly interested in pedagogical issues in early 1962, when he came up with many new ideas that encouraged teachers and mathematicians to use their creativity in search of ways to solve these issues. Those ideas were set forth in the letters I received from Mr. Kolmogorov in 1962. My son came across two of them while sorting out books and papers, and both of them deserve publication. We will present the part of the letters devoted to the topic of our discussion in full. The publication takes the reader to the early 1960s and introduces him or her to Andrey Nikolaevich Kolmogorov's vision of mathematics education at schools and universities (particularly at the faculties of mathematics, physics, chemistry, and at physics and technology institutes, as well as at engineering and physics institutes).

§ 1. Issues of secondary education

It was long ago that Mr. Kolmogorov showed interest to mathematics education. The first Moscow School Mathematics Competition was held in 1935. Since then mathematics competitions became an integral part of this country's cultural life. Mr. Kolmogorov actively participated in the first competition and all those that followed it: he took part in analyzing problems, gave lectures to schoolchildren, was a Chairman of the Organizing Committee of the Moscow Mathematics Competition (in 1937, 1963 and 1975). But in the late 50s - early 60s he took interest in a broad range of aspects of this issue. He noted that the country that prioritized the scientific development sought to improve mathematics education. At that time he took great interest in the progress of physics, mathematics, and technology, and noticed that mathematics school education did not meet new requirements.

I remember a meeting that was devoted to probability theory and took place in Uzhgorod in August 1959. When the meeting was over all the participants who lived in Moscow and Kiev, including Mr. Kolmogorov, went back by train. They talked about mathematics school education.

Since autumn 1959 special mathematics courses were introduced in some schools. Postgraduates and young teachers who graduated from Moscow State University did lecturing in Moscow, Moscow region and in Ivanovo. Mr. Kolmogorov engaged his students into these activities.

In the letters, which were mentioned in the beginning, Mr. Kolmogorov set forth interesting ideas that explained the need to renounce a unified approach to teaching mathematics at different schools.

We will begin with a letter that is devoted exclusively to school and higher mathematics education[1].

"1. The transition to universal secondary education is one of the fundamental measures needed to overcome the opposition between physical and mental labor. It will undoubtedly lead

[1] Boris Vladimirovich did not mention in the text of the letter when he wrote it, and I could not find the original letter so far. (note D. B.)

to a profound change in the structure of our secondary school, that is gradually taking shape. However narrower special needs of this country should be factored in while changing the methods of upbringing and education of tens of millions of our youth.

2. Accelerated development of scientific research in physics, mathematics and chemistry, and modern technology based on them is one of the urgent needs of this country. Today expenditures on scientific and technical researches account for a considerable share of about 10-15% of the total national revenue. Thus, it is clear that research activities have become an integral part of production. There are practically no boundaries between research institutes and industrial enterprises in many branches of modern technology.

3. There may be many different ways of how to engage in research activities, however successful development of physics and mathematics and modern technology is certainly impossible without involving a lot of young people aged 17 to 19 in a close study of physics and mathematics and in the atmosphere of scientific research activities.

4. Faculties of physics and mathematics, mechanics and mathematics, physics, and chemistry in universities and technical universities, such as physics and technology institutes and engineering and physics institutes should be open to young people of various background. Young people, whose life journey began with simple physical labour, should get access to universities of this type in an organized manner. However, in order to achieve scientific and technological development it is necessary to recruit the majority of youth, rather than its significant part, aged 17 to 19 studying at such universities. It is essential to find ways to provide this part of youth with good vocational raining before university entrance, as well as during their studies.

5. First-year students should gain profound knowledge of the basics of physics and mathematics at the aforementioned universities. Extra-mural forms of studies and on-the-job training may be regarded only as an additional form of work, rather than training of highly qualified workers. It will be appropriate only for undergraduates studying at such universities to have a closer connection with the production sector, and work at enterprises and scientific institutes while studying.

6. The aforementioned universities need tens of thousand of new students annually. Technical universities which are not part of this group and faculties of physics and mathematics and chemistry in pedagogical institutes should also enroll a number of talented young people aged 17 to 19 who have good knowledge of physics and mathematics. Naturally, young people aged 17 to 19 who are to be enrolled in the universities right after they finish secondary school should satisfy high demands with regard to their capacities and knowledge of mathematics, physics and chemistry, and their serious approach to labour. Underestimating such career management scheme applied to mathematics and physics, modern technical sciences could have led to the most deplorable consequences.

7. Thus, the demand for personnel consisting of young people well trained in physics and mathematics who enter in universities at the age of 17-19 amounts to 50,000 if not 100,000.

8. It appears inappropriate to establish secondary schools with a focused specialization for outstandingly gifted young people (separate classes for would-be mathematicians, physicists, chemists) collaborating with universities and at universities that young people think they will definitely enter (Academy of pedagogy project). Indeed a group of students who are good at and interested in mathematics, physics and chemistry, enjoy making research, constructing apparatuses is formed when they reach 15, i.e. by the end of 8-year studies at school. A narrower specialization is usually defined later.

9. The most serious part of the aforementioned group of high-school students of a 8-year school can be prepared for studying at universities and technical institutes in two or three years.

Less serious ones can find application for their propensities and interests after undergoing the necessary training and work as laboratory assistants at scientific and industrial laboratories, as qualified workers in such industrial sectors as radio-technology, as computists and operators at computation centres and enterprises, as draughtsman at design engineering bureaus and the like.

Tens of thousands of such employees are needed now.

10. From this one can obviously conclude that it is appropriate for 8-year schools of various types to have special physics and mathematics classes or physics and technology classes which will produce approximately 100,000-150,000 students a year.

11. If principles of vocational training are to be introduced in schools in a reasonable manner and one bears in mind that since these schools are situated primarily in cities they should be available to talented 8-year school graduates from villages and towns, it is sensible to let them study for three years (from 15 to 18). It will let enroll young people who finished 2-year training courses in secondary schools of different type for the second year. This is what we adopt from the Academy of pedagogy project.

12. Young people should enter in physics and mathematics schools by competition, and to a large extent (but not exclusively) upon the recommendation obtained from 8-year schools. The majority of students should receive scholarships and places in dormitories.

§ 2. Higher mathematics education

We will address the issues related to higher mathematics education which were outlined in this letter.

"13. Teachers of mathematics, physics, chemistry who work at physics and mathematics schools should study mainly at universities. As far as mathematics is concerned, there should be about 3,000-5,000 teachers, thus universities training 300-500 teachers per year will not be overstretched. On the contrary, universities will benefit from such forms of interaction with secondary school.

14. About half of physics and mathematics school graduates should apply for universities right after finishing schools. Still they should not regard admission to universities as something self-evident and guaranteed. Naturally, they will gain access by a competition.

15. Despite such broad horizons physics and mathematics schools will not look like closed schools, "cultivating" talents. Their name itself should not suggest any "specially gifted" students (which certainly does not prevent such students from applying for them sometimes).

16. As it was noted in paragraph 4, physics and mathematics schools should not be the only normal way to get access to universities we are interested in (physics and mathematics faculties of universities and technical universities of a new format). If it turns out that young people studying at physics and mathematics schools for three years are better trained than at other secondary schools, it is likely to be more reasonable to let first-year students learn a wider variety of subjects (for example, without separating mathematicians, mechanics and physicists at universities) and to let physics and mathematics school graduates apply for the second year of studies (the third year of studies at physics and mathematics schools will probably include big courses of analytical geometry and elements of analysis).

Mr. Kolmogorov always took great interest in university education. He systematically monitored the content of the courses and formulated wishes about changes in curricula and making sure that both mandatory and special classes have a practical aspect. Mr. Kolmogorov was a true think tank of mechanics and mathematics faculty of Moscow State University. He was well aware of both advantages and disadvantages of courses lectured to students. This allowed him to suggest new ideas every year and carry them out into practice. During the war he suggested that mathematical analysis should be lectured for third-year students. This part of the course was based on functional analysis. Parts of contemporary mathematics was presented coherently, rather than separately to students. Moreover, he proposed to introduce a workshop on mathematical analysis, which implied fundamental problems to be solved on the basis of all the material that has been studied. Many students found the tasks difficult, but their solution facilitated rapid development of mathematical skills. No wonder those years brought him many talented students. We would like to note today the faculty mainly comprises Mr. Kolmogorov's students, who stepped into the scientific field with the help of a broad programme that made them get in-depth knowledge of its parts and constituents.

§ 3. The contents of the letter dated from 28 September 1962

In the letter dated from 28 September 1962 Andrey elaborates on the ideas he has just outlined.

"It is common knowledge that people should manage their own lives rather go with the tide. Now you're engaging me in all sorts of new long-term undertakings, which means that if I take a serious approach to their implementation, I will spend on them the rest of my active life. I will not start with these undertakings, but with your plans of work, then I will turn to mine, after that I will touch upon different institutes, faculties, laboratories to be established.

About you

When you moved to Moscow, you intended to give your work a new, interesting and productive direction. I highly value you as a mathematician who at the same time educates young people and can organize scientific work.

a) You will PROBABLY bring something new to mathematics when you are between 50 and 60 years old that will be as good as the series of your works I was honoured to write about in the article on the occasion of your fiftieth birthday (I have deliberately stressed what I consider essential) or may be something more valuable? But I'm afraid that for the past few years you haven't even had a chance to take any visible effort to learn other people's ideas that are SIGNIFICANTLY NEW to you, or at least to place understanding and presentation (in lectures, books) of a sphere that you used to have a profound knowledge of about twenty years ago (then were aware of the most recent data!) to a significantly new level to .

b) Hence the result: it is still unclear what will encourage young people in Moscow who might one day turn into someone similar to your Korolyuk[2] or Skorokhodov living in Kiev to attend your classes[3]. And you would deserve[4] that success. I cannot see from what you have said

---

[2] Vladimir Semenovich Korolyuk (born 1925) – Candidate of Physical and Mathematical Sciences (1954), Doctor of Physical and Mathematical Sciences (1964), Professor (1965), Corresponding Member of the Academy of Sciences of the Ukrainian SSR (Academy of Sciences of Ukraine since 1991, the National Academy of Sciences of Ukraine 1994) (1967), Member of the Academy of Sciences of the Ukrainian SSR (1976). (D. B.)

[3] Anatoliy Vladimirovich Skorokhod (1930-2011) – Candidate of Physical and Mathematical Sciences (1957), Doctor of Physical and Mathematical Sciences (1962), Professor (1964), Corresponding Member of the Academy of Sciences of the Ukrainian SSR (1967), Member of the Academy of Sciences of the Ukrainian SSR (1985). (D. B.)

[4] At that time I got such a student as I. N. Kovalenko who made a major contribution to the development of the theory of waiting lines.

/Boris Vladimirovich was wrong. Igor Nikolaevich Kovalenko (born 1935) embarked on postgraduate studies to B.V. in 1957 ("...to study at postgraduate classes <...> it was only I. N. Kovalenko who I took, he was one of the most capable of my students," B. V. writes in his memoirs "My Life in Mathematics and Mathematics in My Life"), became Candidate of Physical and Mathematical Sciences in 1960, Doctor of Engineering in 1964, Doctor of Physical and Mathematical Sciences in 1970, Corresponding Member of the Academy of Sciences of the Ukrainian SSR in 1972, Member of the Academy of Sciences of the Ukrainian SSR in 1978.

In 1962 Boris Vladimirovich had such postgraduate students as Bronyus Igno Grigelionis (1935-2014) (would-be Corresponding Member of the Academy of Sciences of the Lithuanian SSR (Academy of Sciences of Lithuania since 1990) (1972), member of the Academy of Sciences of the that you have students for whom you could have similar hopes. It is natural that you do not have them yet, what is important though is that there should be conditions for them to appear.

c) You have enthusiastically spoken about Keldysh[5] worrying about the fact that we have no serious practical statistics. The well-known message you try to deliver through "reliability" classes is evidently just as good: it is a desire to introduce in this country something that others have already had. It is in balance with works demanding great "creative" efforts I have already mentioned in paragraph a). These undertakings are good and they do not necessarily prevent that deeper scientific work (that is inevitably more individual). But, if one does not create (in statistics or reliability theory) something radically new from the international point of view, one can take on a serious and interesting task of dramatically improving the work culture in this country. And that starts with big critical "sanitary" work that aims at exposing any false pseudoscience, introducing serious practical training for young people (my last-year workshop for the staff was certainly just a tiny haphazard attack, but still I had that goal in mind). Our young people who are as old as Mr. Meshalkin[6] or Mr. Belyaev[7] also need primarily the examples of work on the spot that will be more substantive than what they can do themselves. Unfortunately, the number of any more superficial undertakings devoted to the reliability theory that you are now dealing with has a priori exceeded the one that is needed for such an approach.

d) However, I do not deny that it may be exciting to be engaged in a broader organizational work, which implies that a mentor does not show others a good example but takes advantage of his ability to select people. Our "laboratory" is certainly too small for such interesting work. You are a perfect director of an institution of any size devoted to the development of statistics without additional training (in terms of traditional significance tests, experiment design with the least required number of tests etc.), who is aware of everything.

As far as cybernetics is concerned, in the broader sense, at the moment you would lack understanding of overall development prospects. This new science requires a mentor who has a keen scent for the prospects of new emerging areas. You would possibly get such skills in the future but under the current circumstances in case the authorities wanted to see someone as a director of a future institute you would have the advantage of an honest decent man who is not obsessed with any far-fetched ideas (it is the combination of honesty and the absence of too narrow personal interests that is rare in this case).

About me

I intended to limit my activities to paragraphs a) and b), and a little bit of c)[8]. I have succeeded so far only in the organizational work that implied my authority should be based on the ability to show a personal example. Between 1933 and 1939 I took care of practically all the post-graduate students of mathematics institute, selecting mentors for them, giving advice to them together with their mentors, replacing a mentor in case of a failure, in any case, I got to the core of their post-graduate programmes. On the contrary, my activities in the capacity of the secretary of Physics, Mathematics Department, or Dean was not HIGHLY successful: here the ability to assess people and assign functions to them that a mentor is unable to carry out is at the forefront. These are, in fact, administrative talents. I have no intention to belittle them by characterizing.

Lithuanian SSR (1987)) and Tadeush Pavlovich Marjanovich (1932-2014) (would-be Corresponding Member of the National Academy of Sciences of Ukraine (1992))/. (D. B.)

[5] Mstislav Vsevolodovich Keldysh (1911-1978) – member of the Academy of Sciences of the USSR (1946), President of the USSR Academy of Sciences (1961-1975). (D. B.)

[6] Lev Dmitrievich Meshalkin (1934-2000) – A. N. Kolmogorov's student, Candidate of Physical and Mathematical Sciences (1979), Professor (1991). (D. B.)

[7] Yuriy Konstantinovich Belyaev (born in 1932) – A. N. Kolmogorov's student, Candidate of Physical and Mathematical Sciences (1960), Doctor of Physical and Mathematical Sciences (1970). (D. B.)

[8] See in the text below. (D. B.)

It was that idea that stipulated my lengthy evasion from all proposals to be a director of an institution.

However, certainly, self-consistency has never been one of my traits of character. And I immediately become enthusiastic if there is a chance to facilitate the creation of active and interesting groups of young mathematicians. Especially due to the fact that over the past years, despite some very talented students, I still lack the usual environment I used to enjoy back then when I was surrounded by a group of young colleagues who I found nice (a friendly generation of my students, or just colleagues in charge of mathematics workshop at the faculty, or even a group of students at Humboldt University, etc.).

However, I needed some staff positions, at most, that I have and will get without taking an administrative position to realize my personal research plans, or to write books (the activity I should resume). In addition to that I am interested in an opportunity to have more freedom when it comes to spending a semester in Leningrad, Novosibirsk, Calcutta, Paris, or even Vilnius, or Dresden, where I could be not only a guest of honor, but rather take part in productive teamwork.

Indeed, I even dreamed about giving our laboratory and chair to you, for example.

Due to various circumstances I am not in a very good shape now and I am still trying to pave the way for such a specifically organized productive work beyond the limits that are too restraining.

3. Now, about a new mathematics institute and faculty

Today these ideas have suddenly taken the shape of a "small Novosibirsk" in Fryazino that still looks more like a dream. Now the talks about this may be more interesting for you to understand the nature of ideas I am passionate about which have changed little in the course of time.

a) This year I am once again in charge of another Moscow competition[9], but I am not contented with that either. It means that I like my closest assisstants (Sasha Kirillov[10] and Andrey Yegorov[11] and Kolya Vasiliev[12] who have just entered a PhD programme), but I do not enjoy the style of "the club members"[13] created by Mr.Kronrod, and it got deeply rooted. There is a good secondary school in Fryazino (which produces sometimes Moscow Competition winners, although they are not among the first ones) and there are definitely many children whose parents, engineers and qualified workers, are extremely interested making secondary education more physics-and-

[9] Andrey Nikolaevich is referring to the current (1962-1963) academic year (Moscow mathematics competitions are held in the spring). (D. B.)

[10] Alexander Aleksandrovich Kirillov (born 1936) – student of I. M. Gelfand, Doctor of Physical and Mathematical Sciences (1962), Professor (1965). (D. B.)

[11]Andrey Aleksandrovich Yegorov – Candidate of Physical and Mathematical Sciences, since 1963he has been working at Moscow Physics and Mathematics Boarding School №18 named after A. N. Kolmogorov at Moscow State University, Senior Professor of the Mathematics Chair of AESC MSU. Member of the organizing committee and the jury of all-Russian and all-Union competitions between1961 and 1979. (D. B.)

[12] Nikolai Borisovich Vasiliev (1940-1998) – graduated from MSU Mechanics and Mathematics Faculty in 1962 and ambarked on post-graduate studies in this faculty. After finishing post-graduate studies, he spent the rest of his life working at MSU Interfaculty Laboratory of Mathematical Methods in Biology. He spent many years dealing with Moscow, all-Russian and then all-Union mathematics competitions for schoolchildren. (D. B.)

[13] Alexander Semenovich Kronrod (1921-1986) – Doctor of Physical and Mathematical Sciences (1949). In the 1950s he was the head of the laboratory at the Institute for Theoretical and Experimental Physics, the main purpose of which was to solve issues related to the development of nuclear weapons. Stalin prize recipient. He started to work in School №7 in 1961. Professor (1966). He worked at the Patent Information Institute, since 1974 he worked in the Central Geophysical Expedition of the USSR Ministry of Oil Industry. (D. B.)

mathematics-oriented, just as directors of institutes and enterprises. It will be natural to launch the corresponging work parallel to School № 7 headed by Landis[14] and Kronrod and situated at the end of Leninsky Prospekt in the view of moving a score of our faculty graduates and post-graduates to Fryazino.

b) The establishment of a faculty (say, physics and mathematics faculty) that is cybernetics-oriented and has the chairs of 1) analysis and function theory, 2) differential equations and vibration theory, 3) equations of mathematical physics and functional analysis, 4) geometry and algebra complemented by scientific work on the theory of continuous groups, bundle spaces, etc., 5) theoretical physics, 6) electronics, 7) logic and discrete automatons, etc. (It is not a very big contribution, though the names reflect a trend, to forming[15] a robust basis for recruiting not only those who specialize in cybernetics, probability theory, but in topology, number theory who are capable of giving lectures on new "combinatorics" who are interested in automatons (like Tolya Karatsuba) to work as members of the faculty and Cybernetics Institute that is subordinate to the Academy of Sciences[16]).

c) It would be natural to find a dean who specializes in physics and radio electronics, or the like, and I could be a Director of the Mathematics and Cybernetics Institute for the first five years, in case the dean (who is a party member) is very young, I could assume the functions of Academic Council Chairman. Anyway, these are just possible examples, it is not a plan.

d) If you have no intention to move to Fryazino, then you will, presumably, take over my Chair in Moscow and will be invited as an advisor – hopefully, the salary will adequate reckoning with the novelty of the work you will do.

Yours Andrey

If we take it more seriously: such a plan suggests at least three people involved who are Members or Corresponding Members of the Academy of Sciences mathematician (there is one), physicist and radio engineer, ready to take up the initiative.

But seriously, could you, please, learn who is the head of research institutes that already exist in Fryazino.

Meantime, the details are fantastic: the faculty that can accept 200-300 people a year and specializes in pure mathematics, computational mathematics, cybernetics, vibration theory, radioelectronics, automation, cybernetics, and theoretical physics. Thus, these are physical professions, that require only advanced mathematical training! General training courses are provided for all first-year students".

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[14] Evgenii Mikhailovich Landis (1921-1997) – Candidate of Physical and Mathematical Sciences (1953), Doctor of Physical and Mathematical Sciences (1957), Professor (1961). (D. B.)

[15] It is what is written in the original letter. (D. B.)

[16] Anatoly Alekseevich Karatsuba (1937-2008) – Candidate of Physical and Mathematical Sciences (1962), Doctor of Physical and Mathematical Sciences (1966). He was a head of the Number Theory Department at the Steklov Institute of Mathematics and a Professor of Number Theory Chair (1970), Mathematical Analysis Chair (since 1980), Mechanics and Mathematics Faculty of MSU. (D. B.)

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Memorial board

On December 5, at 12:00, with the participation of representatives of the Kyiv City State Administration, the memorial plaque was unveiled to academician of the National Academy of Sciences of Ukraine, Director of the Institute of Mathematics of the National Academy of Sciences of Ukraine Gnedenko Boris Vladimirovich.

This was reported in the Kiev Scientific and Methodological Center for the Protection, Restoration and Use of Historical, Cultural and Protected Land Monuments of the Department of Culture of the Kyiv City State Administration. Location: Proreznaya street, 10.

Decree of the Kyiv City State Administration on the memorial plaque in memory of Boris Vladimirovich was adopted in 2011. The installation was carried out by the Kiev Scientific and Methodological Center for Protection, restoration and use of historical, cultural and conservation monuments Territories of the Department of Culture of the Kyiv City State Administration. Sculptor - Ivan Melnik.

The board is installed on the house in which Gnedenko lived in Kiev. In the photo, the middle balcony in the middle row of balconies is the balcony of his apartment. Two hundred meters further down the street - Khreshchatyk. Another two hundred meters to the left on Khreshchatyk - Maidan.

Source: КИЇВСЬКА МІСЬКА ДЕРЖАВНА АДМІНІСТРАЦІЯ. ОФІЦІЙНИЙ ІНТЕРНЕТ-ПОРТАЛ

Application of SMED (Single Minutes Exchange of Die) for Production Optimization

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Abstract

For optimization of engineering production, it is proposed to use a system of quick changeover (SMED). The introduction of this approach will enhance, in particular, the increment of the coefficient of technical use of equipment A (availability factor) by reducing non-productive time downtime. In this case, A is used as the optimization parameter. Examples of technical decisions applying SMED are presented.

Keywords: Single minutes exchange of die (SMED), maintenance and repair system, availability coefficient, series volume

I. Introduction

The increasing the reliability of plant equipment can be carried out in several directions. Engineers and developers traditionally primarily pay attention to improving the strength and durability of products [1], which is directly related to manufacturing quality and control. But we cannot ignore such an important part of how the organization of production is performed.

Maintenance and Repair works maintain equipment in working condition [2]. The works of the Maintenance ensure the implementation of some of the most important stages in the life cycle of a product (PLM).

In [3], it is shown how changing the maintenance strategy over the past half-century in connection with the existing realities and their tasks that were designed to perform the technique corresponding stage of development were developed. The engineers considered gradual transition from strategies from "repair on failure" to "service status" and then to automated systems Maintenance and repair, "CMM – Computer Maintenance Management System. The latter involves the use of programs to build network graphs (e.g., PROJECT MANAGER, which is the part of MICROSOFT OFFICE).

II. Methods

As the method we selected the analysis of machine availability factor A. Therefore, the service is focused on reliability, ultimately aims is to improve integrated indices of quality of equipment, namely, availability factor A and the coefficient of the technical use the A_H [4].

$$A = \frac{E(Uptime)}{E(Uptime) + E(Downtime)} \quad (1)$$

$$A_H = \frac{E(Uptime)}{E(Uptime) + E(Downtime) + E(Hangover)} \tag{2}$$

In formulas (1) and (2): $E(Uptime)$ - the mean time of the product in working condition (Uptime); $E(Downtime)$ - the mean time of the product in the failed state (Downtime); $E(Hangover)$ – the mean time required for changeovers. Indicators A and A_H dimensionless, and the values $E(Uptime)$, $E(Downtime) and\ E(Hangover)$ have the dimension of time (e.g., hours).

The structures of the formulas show that A<1, AH<1 and Ah<A. The degree of closeness increased A to unity indicates a high reliability of the equipment and the condition AH→1 characterizes the compliance of the level of production to modern standards.

Mathematical aspects related to the interpretation of complex indicator A were considered in [5]. Previously, we developed a method that allows to build the confidence interval of values A [6].

Important resource for increasing the A_H is the reduction of exchange time, in connection with which we propose to discuss an approach which is one of the methods for lean production, namely Single Minute Exchange of dies (SMED).

The Toyota company in 1969 made the first drastic steps to reduce the time of changeover, preceded by a 19-year-old practice that allowed Japanese scientist S. Singo to make his discovery. The author of the concept "single minute exchange of die" Shigeo Shingo describes the essential principles of it approach [7]:

- The distinction between internal and external operations;
- Replacement internal operations to external;
- Standardize functions, not forms;
- Use functional clamps. Possible avoid the fastener;
- Maximum use of intermediate devices;
- Operations should be performed in parallel manner;
- Operations without further adjustment;
- The use of mechanization.

In the base of the method the division of operations of the readjustment into two categories lies 1) Internal, which are executed ONLY when the equipment is stopped. For example, the mold can be replaced only when the press is turned off; 2) External actions, on the other hand, can be performed during operation of the equipment. For example, it is possible to pick and sort the bolts of the press molds when the press is running.

The idea of acceleration is to replace as many internal activities into external. This reduces changeover by several times and thereby increases A_H.

III. Results

In the Fig.1 the distributions of A for one of the subsystems of the coal-mining excavator [6] are shown: (a) original; (b) after the hypothetical event SMED, which halved the non-productive loss of time. The distribution is constructed using a method developed by the authors [6], based on the statistical bootstrap [8].

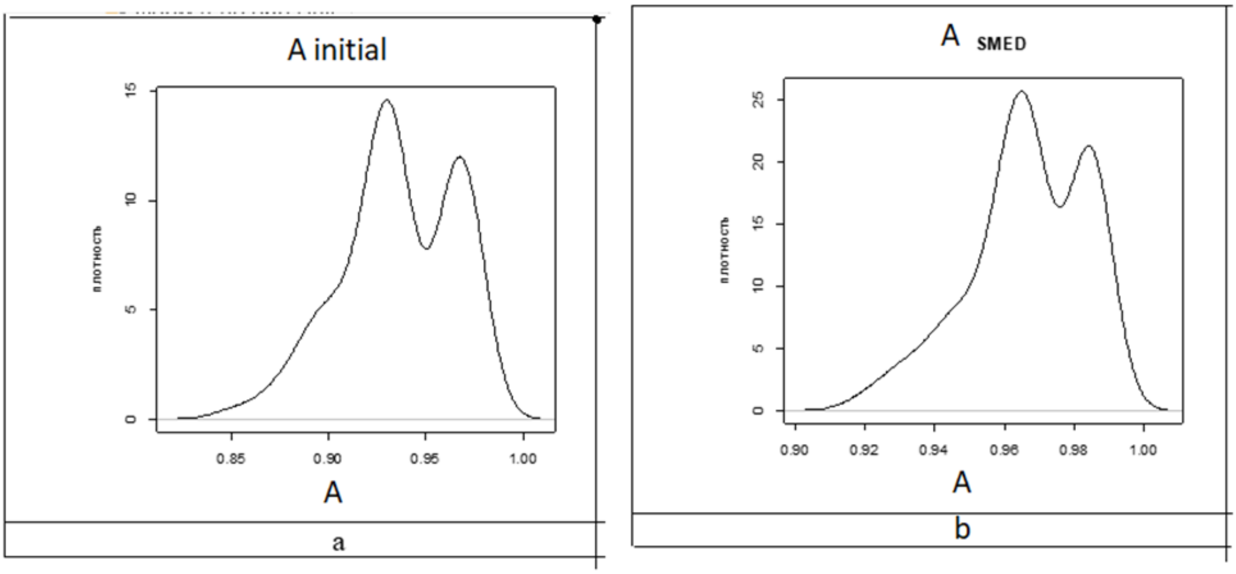

Figure 1. The graph of the probability density of the sub-system A of coal mining excavator: a - initial; b - after the SMED events

In the Table the summarizes statistical characteristics (“SUMMARY” function in R) distributions of the A values shown in Pic.1 is presented. Table data and graphs in Fig.1 are performed in the programming environment R [9].

Table. Statistical summary of the bootstrap distributions of A before (initial data) and after SMED events

	Minimum value min(A)	First quantile Q_1	Median	Mean value	Third quantile Q_3	Maximum value max(A)
Initial data	0.7850	0.8972	0.9302	0.9244	0.9626	0.9776
After SMED	0.9191	0.9591	0.9670	0.9665	0.9828	0.9889

From Figure 1 and the Table it might be seen that the median value of the reliability index in the hypothetical operations of SMED median A is shifting to larger values, namely, it increases from 0.9302 to 0.967.

The concept of SMED is one of the directions of development of "lean production" approach. Under the latter, for example, combating against non-productive storage is being made by switching to smaller series. In the economy of the future client - oriented production is performed, i.e. it is produced the exact amount of goods what the customer needs, and in the desired quantity.

For the small parties have become economically viable, it is necessary to reduce the changeover time, as mentioned above. In [10] the formulas, allowing to estimate the economic feasibility of a certain size party are presented.

First, the specific time of manufacture of details taking into account setup time is calculated:

$$t = (pK_1 + S_1)/K_1 \tag{3}$$

In the formula (3) p - machining time; S_1 – the set-up time prior to the event; K_1 is the size of the party prior to the event.

When calculating the new size of the party it is necessary to take into account that the specific time of manufacture remains unchanged:

$$K_2=S_2/(t-p) \tag{4}$$

The ratio K_1/K_2 shows how many times you can reduce the batch run, and it depends on the ratio S_1/S_2.

The average level of finished goods reserve:

$$C=(K+d)/2 \tag{5}$$

Here d is the minimum size of reserve of output production

As the size of the party depends on the time of changeover, then the reserve will depend on changeover times. Through the necessary transformation, the authors of [10] obtained the dependence of the reduction in reserve levels, resulting from the reduction in changeover times:

$$\Delta C=(S_2+d_2\, s +2Ds)/(S_1 +d_1 s +2Ds) \tag{6}$$

From the formula (6) it follows that the decrease of changeover times, increased reserve turnover and materials. This reduces the need for storage space and reduces the level of storage costs.

The Fig.2 schematically shows the impact of the costs of adjustment to the economically justified size of the party. The figure shows that the decrease of changeover time is greatly reduced economically reasonable series size.

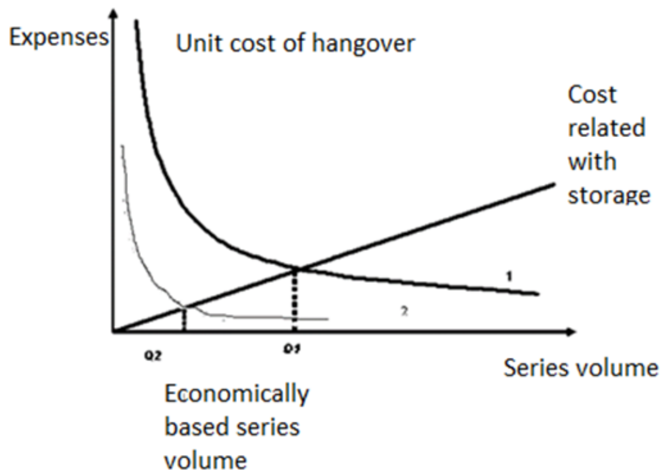

Figure. 2. Schematic representation based on the economically justified size of the party while reducing the changeover time (thin line shows the graph after the transition to SMED) [11].

IV. Examples of application

Techniques SMED applied in different fields of industry (not only engineering). In [7] it is described, in particular, examples of applications to optimize the change of cartridges in semi-automatic lathe, automatic screw machine, with the installation of replacement gears.

Techniques to implement fast changeovers is complicated and require special knowledge. To begin with, it is necessary to consider the specifics feature of the equipment and processes of its conversion. For example, for precision machining equipment, the most difficult is the elimination of the adjustment, without which under the deficit of qualified installers it is impossible to drastically reduce the changeover. For businesses related to the production of wire or cable production, the most important and significant technical solutions lie at the junction of the ends and wire.

In many industrial enterprises, the problems of installation the press molds (to ensure speed and accuracy of positioning) are being solved. In cases where changeover is conducted not on individual pieces of equipment and automated production line, the first-priority task is solving the task of the team of operators, technicians-mechanics and engineers of industrial automation. Only at the expense of the competent organization of their transitions from one equipment to another, it is sometimes possible to reduce the changeover time by 3-4 times, and it provides solid gain in uptime of the line.

The Fig. 3 shows two schemes to illustrate the technical solution for implementing a quick exchange of dies [7]. In practice the question of the stamps needs to be polished often raises. Thus, it is necessary to insert spacers to adjust the height of the stamps. One way to solve this problem is to replace the block with the thicker one on the value that needed to be removed by polish. Blocks used with this method of adjustment will usually be attached to the lower surface of the lower half of the stamp. In some cases, they can be attached to the top of the upper half of the stamp (Fig.3). This method can be considered as one of the applications of transfer functions from internal to the external operations.

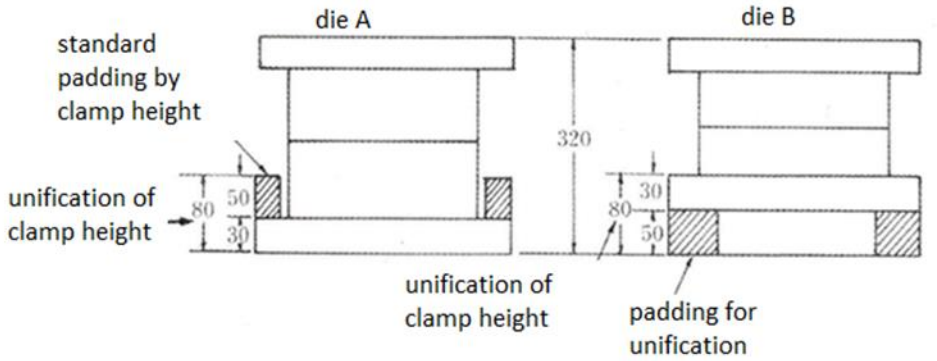

Figure 3. Standardization of the height of stamps clamps

Significant time reduction is the replacement of fasteners by the clamps. The direct attachment requires a large number of turns of the screw. The key to the development of the method in accordance with SMED is the understanding of the role of the number of threads that provide the necessary friction for reliable operation of the mechanism. It is necessary to revise the approach using only threaded connections. The Fig.4 shows an example of a spring clip to secure the gear on the shaft. The elastic energy of the spring provides a change of gears "one-touch". The mechanisms serving this purpose also include wedges, taper pins, ejectors. Promising are also the vacuum and magnetic methods for details installation.

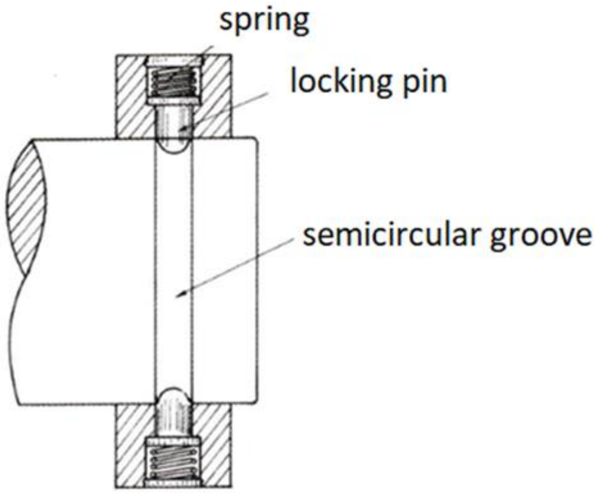

Figure 4. Spring clip for installation of a replacement gear

Conclusion

The basic principles of one of the tools of "lean production", namely, single minute exchange of die, SMED are considered. The importance of this approach at the present stage of technology development is shown. Examples of engineering solutions are presented. As an optimization parameter, we use the complex index of reliability, namely availability coefficient A and of technical use factor A_H. Some technical solutions are shown.

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Interval Dependence Structures of Two Bivariate Distributions in Risk and Reliability

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Abstract

We follow the ideas of measuring strength of dependence between random events, presented at two previous MMR conferences in South Africa and Tokyo. In our work here we apply it for analyzing local dependence structure of some popular bivariate distributions. At the Grenoble conference presentation we focus on the Bivariate Normal distributions with various correlation coefficients, and on the Marshal-Olkin distribution with various parameter's combinations. We draw the surface $z = g_{ii}(x,y)$, $i=1,2$ *of dependence of i-th component on the other component* $j \neq i$ *within the squares [x, x +1]x[y,y+1], and [x, x +.5]x[y,y+.5]. The points (x,y) run within the square [-3.5, 3.5]x[-3.5, 3.5] for Bivariate Normal distribution, and in [0.10]x[0,10] for the Marshal-Olkin distribution.*

Keywords: Local dependence, local regression coefficients, strength of dependence, strength of dependence, surface of dependence, Bivariate normal, Marshal-Olkin distributions

I. Introduction

In several previous publications [1-6] we developed an idea how probability tools can be used to measure strength of dependence between random events. More details contain articles [1] and [2]. In the present article we propose to use it for measuring magnitude of local dependences between random variables. Such dependence is completely different from the global measure of dependence, measured usually by the correlation coefficient. As illustration, we demonstrate how it works in measuring local dependence inside the jointly distributed pairs of random variables, using the regression coefficients between random events. Short illustrations (graphics and tables) are showing the use of these measures in already known popular Bivariate Normal distribution with different correlation values, and inside the popular in reliability Marshal-Olkin distribution.

II. How people indicate dependence

The dependence in uncertainty is a complex concept. In the classical approach conditional probability is used to determine if two events are dependent, or not: A and B are independent when the probability for their joint occurrence equals to the product of the probabilities for their individual appearance, i.e. when

$$P(A \cap B) = P(A).P(B)$$

Otherwise, the two events are dependent.

In courses on Probability the independence for random events is always introduced simultaneously with conditional probability. Where independence does not hold, events are dependent, but more the dependence is never discussed. There are ways to go deeply in the analysis of dependence, to see some detailed pictures inside the global pictures, and use it in the studies of uncertainty. This matter is discussed in our previous articles ([1] and [2]). Some particular situations are analyzed in [3] to [6]. We refer to these articles for making a quick passage to the essentials.

First we notice here that the most informative measures of dependence between random events are the two *regression coefficients*. Their definition is given here:

Definition. Regression coefficient $r_B(A)$ of the event *A* with respect to the event *B* is called the difference between the conditional probability for the event *A* given the event *B,* and the conditional probability for the event *A* given the complementary event $\overline{B}$, namely

$$r_B(A) = P(A \mid B) - P(A \mid \overline{B}). \qquad \textbf{(1)}$$

This *measure of the dependence of the event A on the event B,* is *directed dependence.*

The regression coefficient $r_A(B)$ of the event *B* with respect to the event *A, is defined analogously,*

$$r_A(B) = P(B \mid A) - P(B \mid \overline{A}).$$

From the many interesting properties of the regression coefficients we would like to point out here just few:

(r1) The equality to zero $r_B(A) = r_A(B)$ =0 takes place if and only if the two events are independent.

(r2) The regression coefficients $r_B(A)$ and $r_A(B)$ are numbers with equal signs and this is the sign of their *connection* $\delta(A,B)$ =P(A∩B)-P(A)P(B). The relationships

$$r_B(A) = \frac{P(A\cap B) - P(A)P(B)}{P(B)[1-P(B)]}, \ \textit{and}\ r_A(B) = \frac{P(A\cap B) - P(A)P(B)}{P(A)[1-P(A)]}.$$

The numerical values of $r_B(A)$ and $r_A(B)$ may not always be equal. There **exists an asymmetry in the dependence between random events, and this reflects the nature of real life.**

(r3) The regression coefficients $r_B(A)$ and $r_A(B)$ are numbers between –1 and 1, i.e. they satisfy the inequalities $-1 \le r_B(A) \le 1; \qquad -1 \le r_A(B) \le 1.$

(r4.1) The equality $r_B(A) = 1$ holds only when the random event *A* coincides with (or is equivalent to) the event *B*. Then is also valid the equality $r_A(B)$ =1;

(r4.2) The equality $r_B(A) = -1$ holds only when the random event *A* coincides with (or is equivalent to) the event $\overline{B}$ - the complement of the event *B*. Then is also valid $r_A(B)$ = - 1, and respectively.

(r5) The name regression coefficient of the random event *A* with respect to the event *B* comes from the following fact: If $I_A(\omega)$ and $I_B(\omega)$ are the random indicator variables, related to the two events *A* and *B*, then the best linear regression between $I_A(\omega)$ and $I_B(\omega)$ is expressed by the equation

$$I_B(\omega) = P(B \mid \overline{A}) + r_B(A)\, I_A(\omega) + \epsilon(\omega),$$

where $\epsilon(\omega)$is a r.v. with zero expectation and minimum variance.

We interpret the properties (**r4**) of the regression coefficients in the following way**:** As closer is the numerical value of $r_B(A)$ to 1, "as denser inside within each other are the events A and B, considered as sets of outcomes of the experiment". In a similar way we interpret also the negative values of the regression coefficient.

The regression coefficient is always defined, for any pair of events A and B (zero, sure, arbitrary).

In our opinion, it is possible one event to have stronger dependence magnitude on the other than the reverse. This measure suits for measuring the magnitude of dependence between events. The distance of the regression coefficient from the zero (where the independence holds) could be used to classify the strength of dependence, e,g. as in some interpretations of the regression coefficient measuring the global dependence:

- almost independent (when $|R_A(B)| < .05$) ;
- weakly dependent (when $.05 < |R_A(B)| < .2$) ;
- moderately dependent (when $.2 < |R_A(B)| < .45$) ;
- in average dependent (when $.45 < |R_A(B)| < .8$) ;
- strongly dependent (when $|R_A(B)| > .8$) .

Predictions using Regression coefficients

One serious advantage of the Regression coefficients is that its magnitude can be used to evaluate the posterior probability of one event when information that the other event occurred is available. We have the following relation fulfilled:

$$P(A \mid B) = P(A) + R_B(A)[1-P(B)]. \tag{2}$$

This formula competes with the BAYES RULE, that requires joint probability $P(A\cap B)$. We offer to use the strength of dependence $R_B(A)$ instead of the Bayes rule. It seems much more natural for applications, since it uses long run experience.

III. Transfer rules: From events to random variables and distributions

The above measures allow studying the behavior of interaction between any pair of numeric r.v.'s (X,Y) throughout the sample space, and better understanding and use of dependence.

Let the joint cumulative distribution function (c.d.f.) of the pair (X,Y) be $F(x,y)=P(X \le x,\ Y \le y)$, with marginal c.d.f.'s $F(x)=P(X \le x)$, $G(y)=P(Y \le y)$. Let introduce the events

$$A_x=\{x\le X\le x+\Delta_1 x\}; \qquad B_y=\{y\le Y\le y+\Delta_2 y\}, \text{ for any } x,\ y \in (-\infty, \infty).$$

Then the measures of dependence between events A_x and B_y turn into ***a measure of local dependence between the pair of r.v.'s X and Y on the rectangle*** $D=[x,\ x+\Delta_1 x]\times[y,\ y+\Delta_2 y]$. Naturally, they can be named and calculated as follows:

Regression coefficient of X with respect to Y, and of Y with respect to X on the rectangle $[x, x+\Delta_1 x]\times[y, y+\Delta_2 y]$ can be introduced in analogy to considerations in previous section. By the use of Definition 1 we get

$$R_Y((X,Y)\in D) = \frac{\Delta_D F(x,y) - [F(x+\Delta_1 x)-F(x)][G(y+\Delta_2 y)-G(y)]}{[F(x+\Delta_1 x)-F(x)]\{1-[F(x+\Delta_1 x)-F(x)]\}}\ . \tag{3}$$

Here $\Delta_D F(x,y)$ denotes the two dimensional finite difference for the function $F(x,y)$ on rectangle $D=[x, x+\Delta_1 x]\times[y, y+\Delta_2 y]$. Namely

$$\Delta_D F(x,y) = F(x+\Delta_1 x, y+\Delta_2 y) - F(x+\Delta_1 x, y) - F(x, y+\Delta_2 y) + F(x, y). \tag{4}$$

In an analogous way is defined $R_X((X,Y)\in D)$. Just denominator in the above expression is changed (symbol F to symbol G) respectively.

Using these rules one can see and visualize the local dependence between every pair of two r.v.'s X and Y with given joint distribution $F(x,y)$ and marginal s $F(x)$ and $G(y)$.

The biggest advantage of the *Regression Coefficients as* measures of the magnitude of dependence is their easy interpretation, described above, and the fact that they come available from the knowledge of the probabilities of the respective events, or proportional number of individuals in the sets of subpopulations of interests.

In Probability modeling which use Multivariate Distribution we see GREAT Advantages: knowing that one component falls within an interval, then we can predict everything that may happen with the other component. For instance, when we know that $X\in[a,b]$, we can predict (by use of measure the strength) how likely is that $Y\in[c,d]$, for any choice of c and d.

Next we illustrate specific rules in calculation of Regression Coefficients as measures of dependence to analyze the local dependence structure in Bivariate Normal Distribution, and in the Marshal-Olkin Distribution. We end our theoretical background of the general local dependence structural studies. Next we illustrate its application on the two selected qualitative and quantitative probability models.

IV. Correlated Bivariate Normal distribution

I. Analytical expressions

The random vector $(X\ Y)$ has Bivariate Normal probability distribution if its probability density function is given by the expression:

$$f(x,y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho_2}} e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\nu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x-\nu_1}{\sigma_1}\right)\left(\frac{y-\nu_2}{\sigma_2}\right) + \left(\frac{y-\nu_2}{\sigma_2}\right)^2\right]}$$

where ρ is the correlation coefficient between X and Y; μ_1, μ_2 are the expected values, and σ_1, σ_2 are the standard deviations of the components X and Y correspondingly. We analyze how the magnitude of the correlation between components influences this local dependence structure, assuming $\mu_1=\mu_2=0$, and $\sigma_1=1$, σ_2 any. The functions

$$g1(x,y) = \frac{\int_x^{x+1}\int_y^{y+1} e^{-\frac{1}{2(1-\rho^2)}\left[u^2-2\rho uv/\sigma_2+v^2/\sigma_2^2\right]} du\, dv - \sqrt{1-\rho^2}\int_x^{x+1} e^{-u^2/2}du . \int_y^{y+1} e^{-v^2/(2\sigma_2^2)}dv}{\sqrt{2\pi}\sqrt{1-\rho^2}\int_x^{x+1} e^{-u^2/2}du\left(1-(1/\sqrt{2\pi})\int_x^{x+1} e^{-u^2/2}du\right)} \tag{5}$$

Here we consider the symmetric case $\sigma_2=1$, and then

$$g2(x,y) = g1(y,x) \tag{6}$$

identify the magnitude of dependence between the two components X and Y on the square $[x,x+1]\text{x}[y,y+1]$ with the lower left vertex (x,y) and side lengths equal to 1.

- The marginals $F_X(x)$ and $G_Y(y)$ are Normal Distributions of means μ_i and st. deviations σ_i *i=1,2*.
- *We use standard normal marginals* $\mu_i = 0$, $\sigma_i = 1$ *and correlated components with different numeric values of the Correlation Coefficient* $\varrho_{X,Y}$ in our illustrations.
- Our goals are to study local dependence between *X* and *Y* as functions of the values of the Correlation Coefficient ϱ, and of the width *a* of the rectangle (square) $[x,x+a]\mathrm{x}[y,y+a]$.

The predictions one can make by use of the Global Correlation Coefficient ρ are through the regression equation

$$Y = \mu_Y + \rho \frac{\sigma_Y}{\sigma_X}(X - \mu_X) + \varepsilon$$

Here ε is a r.v. with Standard Normal Distribution.

Our approach suggests to use INTEVAL Regression Coefficients, based on the alternative to the Bayes Formula for poster for probabilities cited above.
For any pair of variables (X,Y) it looks like this:

$$P\{Y\in[c,d] \mid X\in[a,b]\} = P\{Y\in[c,d]) + R_{X\in[a,b]}(Y\in[c,d])[1-P\{ X\in[a,b]\}].$$

It is easily presented in terms of any particular Bivariate Distribution, including the Normal one.

Brief analytical and algorithmic discussion, and more graphics and numeric illustrations are used in our concluding observations. Here we give just one of many, illustrating global and local dependence on squares of sides lengths equal to 1 and .5, for values of the correlation between *X* and *Y* equal to ± .95, ± .5 and ± .15:

I. Graphical illustrations

All graphing and numeric illustrations are made by program system Maple. Since the symmetry we show only one of the two options.

Correlated Bivariate Normal –Local Dependence Function g2(x,y): ρ = + - .95

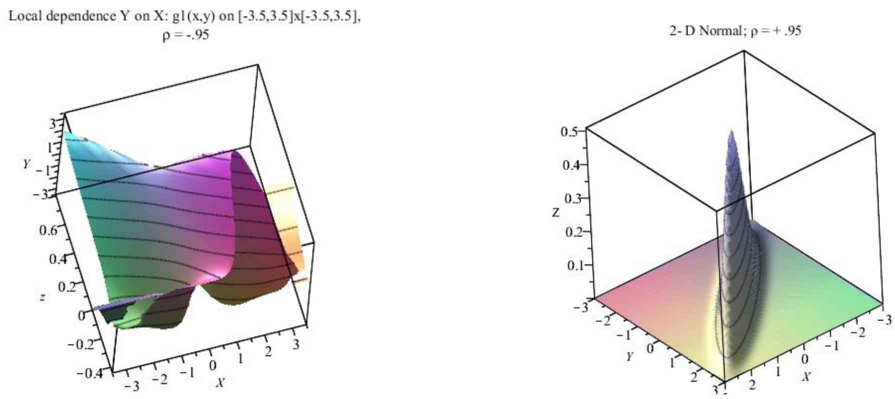

Figure 1. Correlated Bivariate Normal –Local Dependence Function g1(x,y): ϱ = -.95, and squares of length 1 and .5, and the global distribution pdf.

There are some lines of discontinuity on both surfaces. We observe these in a table of selected values of the local correlation function given below. In our opinion, this defect is due to the counting/graphing program, which in this case is MAPLE.

Table 1. *Numeric values of the function g1(x,y) on integer points in the square [-3,3]x[-3,3], **ρ=.95***

X\Y	-3	-2	-1	0	1	2	3
-3	.6531	.1818	-.3884		-.1389	-.0219	-.0013
-2	.0324	.6289	-.1354	-.3948		-.0248	-.0015
-1	-.0325	-0707	.6396	-.2937	-.2062		-0020
0		-.2062	-.2937	.6396	-.0707	-.0325	
1	-.0248		-.3948	-.1354	‘6829	.0324	-.0015
2	-.0219	-.1389		-.3484	.1818	.6531	.0244
3	-.0214	-.1361	-.3418		-.1353	.4884	.5773

In both cases ***ρ=±.95*** we observe some lines of discontinuity in the surface functions $z=g1(x,y)$ and $z=g2(x,y)$; Analytic reason for us is unclear for now. We think the deficiency is in the used program. However, the symmetry between local dependence magnitudes are seen from the table.

Another interesting fact is, that no matter if the global correlation ρ between X and Y is negative or positive, their local Regression Coefficients are positive near the lines $y=x$ (for positive global regression), and the line $y=-x$ (for negative global correlation). Also local regression dependence measure becomes negative (drops quickly) not so far from these lines, and goes to indication of independence with the growth of the distance driven from those lines.

One more interesting fact is, that the local regression magnitudes do nor exceed the global correlation magnitude, but vary with the location of the square within the considered range.

Speaking of predictions, if $X\in[x, x+1]$ it is most likely that $Y\in[x, x+1]$ when ρ *is positive, or* $Y\in[-x, -x-1]$ when ρ *is negative, based on the rule* (2)

The function $g2(x,y)$ exhibits similar behavior and it is symmetric to $g1(x,y)$ with respect to the line $y=x$. We omit the show of these details.

Also we observe high positive local dependence close to the line $y=x$, and negative local dependences, also of relatively high magnitude, about the opposite signs $y= - x$. This magnitude vanishes as long the points become far from the origin (0,0). Notice reduction of magnitude on smaller square.

Correlated Bivariate Normal –Local Dependence Function g2(x,y): ρ = + - .5

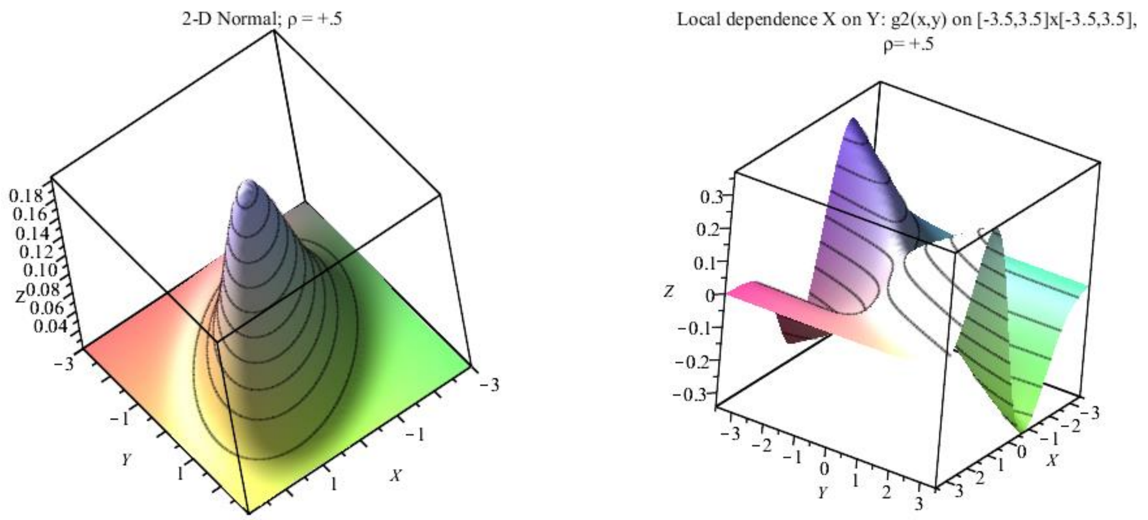

Figure 2. Bivariate Normal density functions f(x,y): ϱ = ±.5 : Global and local Interval dependence

In both cases ***ρ* = + - .5** we observe saddle points on the surface functions z=g1(x,y) and z=g2(x,y) in the region of the origin (0,0), showing slight positive local dependence;

Interesting fact is, that the Regression Coefficients between X and Y still behave negative or positive, not exceeding in absolute value the global correlation $\boldsymbol{\rho}$ **.**

The Local Regression Coefficients are positive near the lines y=x (for positive global regression), and the line y=-x (for negative global correlation). Local regression dependence measure becomes negative (drops) near these lines, and goes to indication of independence with the growth of the distance driven from those lines. Interesting level curve is at level L=0. There the two variables X and Y are independent on the square $\{x, x+1]$x$[y, y+1]$.

The local regression magnitudes do not exceed the global correlation magnitude, and vary with the location of the square.

Speaking of predictions, if $X\in[x, x+1]$ it is most likely that $Y\in[x, x+1]$ when ϱ is positive, or $Y\in[-x, -x+1]$ when ϱ is negative, based on the rule (2)

Here we just observe the dependence of the graphs of local dependences how are these influenced by the sign of the Correlation Coefficient. Obviously, one is symmetric with respect to the ordinate axes compare to the other. The level curves are shown on most graphs. They show points of the same magnitude of local dependence.

Correlated Bivariate Normal Density: when ρ = + - .1

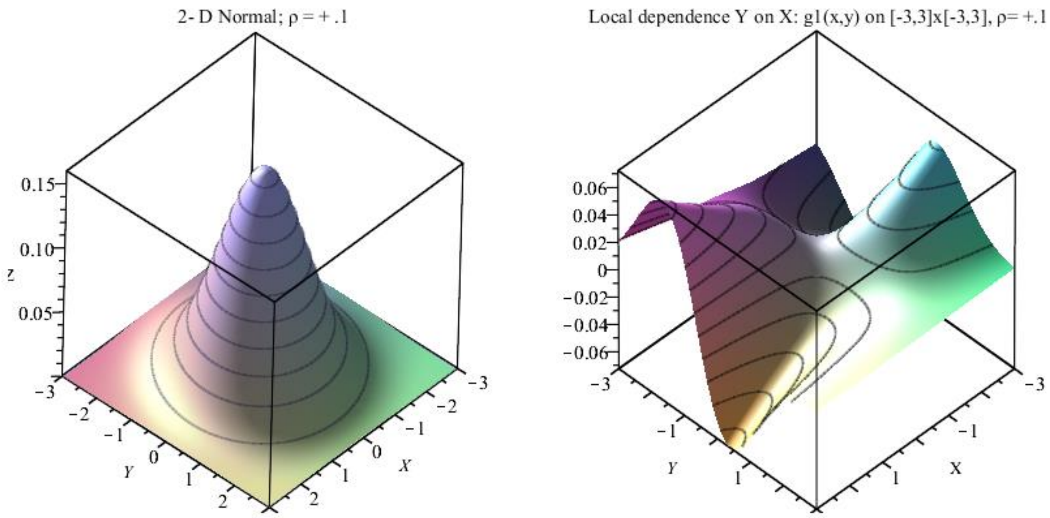

Figure 3. Bivariate Normal density functions f(x,y): ϱ = ±.1 : Global and local Interval dependence

The original Distribution is almost symmetric. No significant global correlation. However, we observe differences in the graphs of local dependences, and how are these influenced by the sign of the Regression Coefficient.

Obviously, the magnitudes do not exceed 60% of the correlation, but go up and down.

The level curves are shown on most graphs. They show points of the same magnitude of local dependence.

In general, we observe again high positive local dependence close to the line y=x, and negative local dependences, also of relatively high magnitude, about the opposite signs y= - x. This magnitude vanishes as long the points become far from the origin (0,0) and away from the lines y=x, or y-x in case of negative global correlation. Notice reduction of magnitude in half on smaller square.

Something similar we observe and in the case of low correlations ϱ = -.10 and ϱ = -.15. As the ancient Greeks used to say, when you have a graph, "Just seat, watch, and make your own conclusions".

V. The Bivariate Marshal-Olkin Distribution

This distribution is well known in reliability from the "fatal shock models", where two exponentially distributed life times X and Y interact so their residual life times have the joint distribution

$$P\{X>x,\ Y>y\}=e^{-\lambda x-\mu y-\nu.max(x,y)},\quad x,y\geq 0. \tag{7}$$

The marginal residual life times are

$$P\{X>x\}=e^{-(\lambda+\nu)x},\ x\geq 0,\ \text{and } P\{Y>y\}=e^{-(\mu+\nu)y},\ y\geq 0. \tag{8}$$

Applying these functions into the rules (3) (4) with $\Delta x=\Delta y = a$, we get the expressions for the regression coefficients on the squares $[x, x+a]$x$[y, y+a]$, namely

$R1(x,y)$ =

$$\Big(e^{-\mu\,y+\nu\,x}\Big(e^{-\nu\,\max(x,y)}-e^{-\lambda\,a-\nu\,\max(x+a,y)}-e^{-\mu\,a-\nu\,\max(x,y+a)}+e^{-\lambda\,a-\mu\,a-\nu\,\max(x+a,y+a)}\Big)-e^{-(\mu+\nu)\,y}\Big(1-e^{-(\lambda+\nu)\,a}\Big)\Big(1-e^{-(\mu+\nu)\,a}\Big)\Big)\Big/\Big(\Big(1-e^{-(\lambda+\nu)\,a}\Big)\Big(1-e^{-(\lambda+\nu)\,x}+e^{-(\lambda+\nu)\,(x+a)}\Big)\Big)$$

Respectively, $R2(x,y)$ is given by a similar expression with the change of μ by λ and of x by y. The two functions are symmetric with respect to the line y=x only when $\mu = \lambda$. Due to the limited space, we present our graphs of the local dependence surfaces between the components X and Y on the squares of size a=.5 and 1 on the square [0,3]x[0,3], and for values of the parameters λ=1, μ=2 and ν=3. We observe high positive dependence in a neighborhood of the origin, negative dependence of the small values of dependent variable, positive dependence along the line y=x, and vanishing dependence on the large values of the dependent variable.

Respectively, R2(x,y) is given by a similar expression with the change of μ by λ and of x *by y*.

The two functions representing Regression Coefficients surfaces are symmetric with respect to the line y=x only when μ = λ.

Our graphs show the local dependence surfaces between the components X and Y on the squares of size a=.5 and a =1 on the square [0,3]x[0,3], and for values of the parameters λ=1, μ=2 and ν=3. They are presented on the next figures 4 and 5.

On both we observe high positive dependence in a neighborhood of the origin, negative dependence around the small values of dependent variable, positive dependence along the line y=x, and vanishing dependence on the large values of the dependent variable.

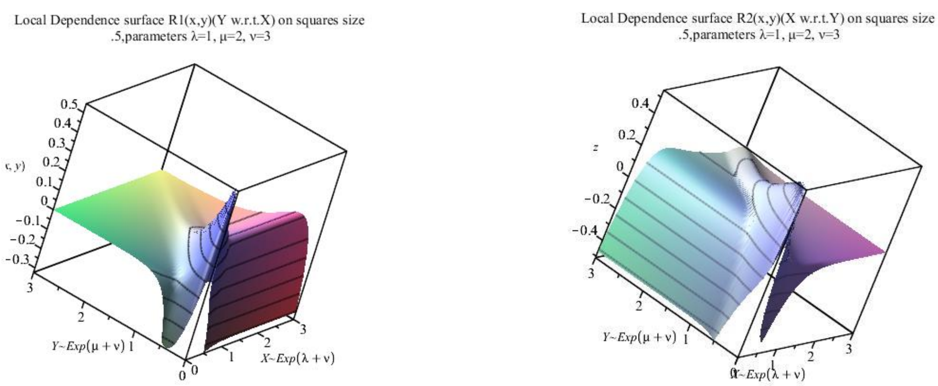

Figure 4. Marshal-Olkin λ=1, μ=2, ν=3 – Local Dependence Functions R1(x,y) and R2(x,y): on squares of length 5

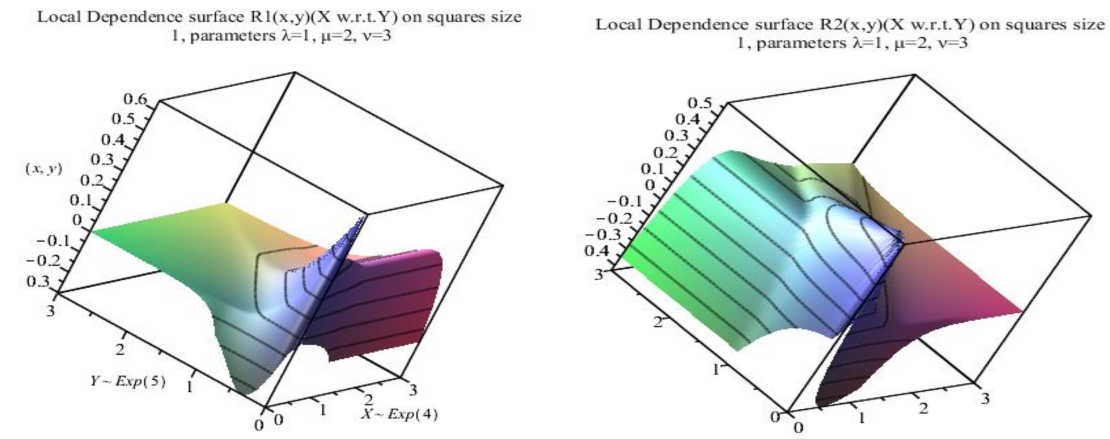

Figure 5. Marshal-Olkin λ=1, μ=2, ν=3 – Local Dependence Functions R1(x,y) and R2(x,y): on squares of length 1.

These graphs and numeric values on the axes of the boxes indicate that the magnitude of dependence is slightly affected on the size of the squares.

There is a negative dependence near the lines y=0, and x=0 depending with respect to which variable dependence is measured. Then this magnitude quickly rises and keeps a positive value along lines parallel to the respective axis. Then magnitude of mutual local dependence drops for a while, and rises again near the line y=x.

On the opposite direction dependence vanishes with the growth of the distance from the line y=x.

The local dependence functions R1(x,y) and R2(x,y) on squares length a=1 and a=.5 for combination λ=3, μ=2, ν=1 between parameters in the Marshal-Olkin distribution is shown on Fig.6.

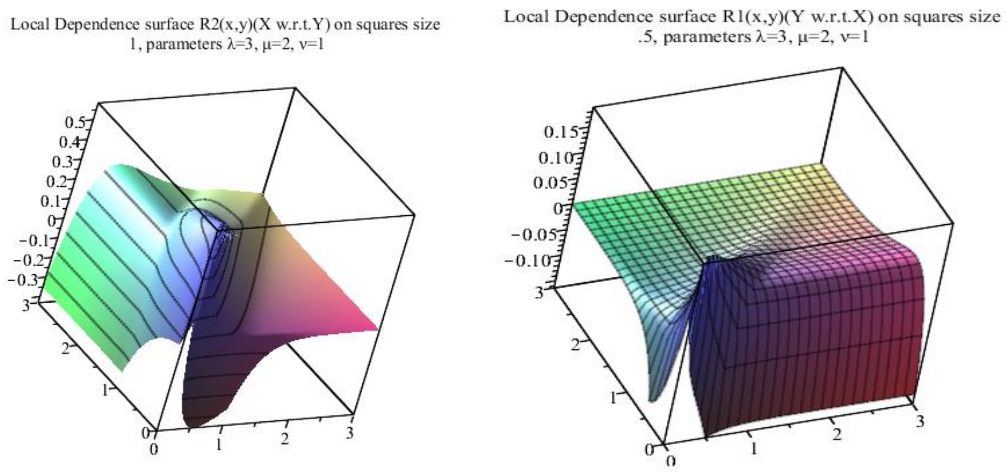

Figure 6. Marshal-Olkin local dependence functions R1(x,y) and R2(x,y) on squares of length 1 and .5 fpr parameters λ=3, μ=2, ν=1.

The graphs and numeric values on the coordinate axes of the box indicate that the magnitude of dependence is now affected on the sides of the square. One is of magnitude .6, the other one – of magnitude .15. However, the shape of the surface of dependence is similar to others.

The negative dependence near the lines y=0, and x=0 depending with respect to which variable dependence is measured still saves behavior, and keep stable value along the variable with smaller value parameter.

The magnitude of this dependence is now higher compare to the case where the interacting component has had higher intensity (compare ν=3 to ν=1).

The magnitude near the line *y=x keeps its ridge kind of shape*.

On the opposite directions dependence vanishes with the growth of distance from the line y=x.

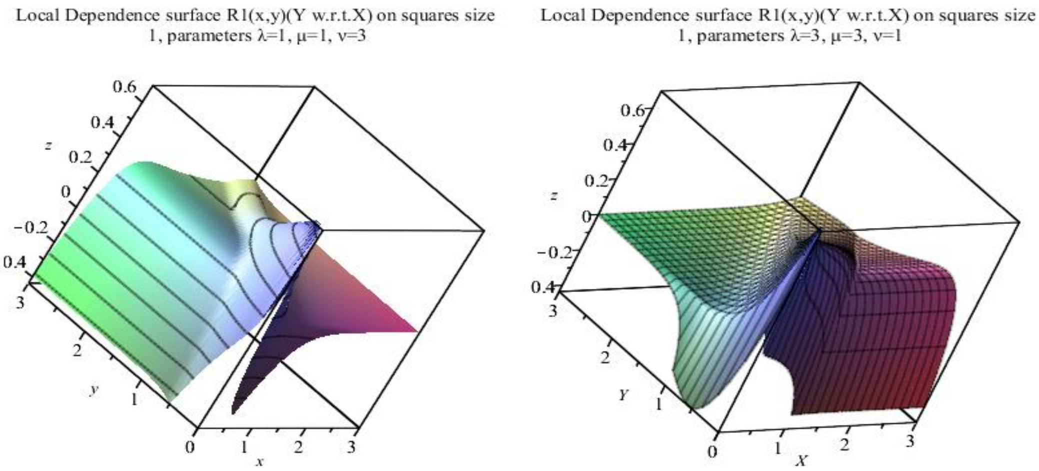

Figure 7. Marshal-Olkin local dependence functions R1(x,y) and R2(x,y) on squares of length 1 and .5 for parameters λ=1, μ=1, ν=3, and λ=3, μ=3, ν=1.

The last combinations of numeric values between parameters in Marshal-Olkin distribution show that the magnitude of dependence is not affected on the values of parameters of the marginals (as soon as the two marginals are equal λ=μ), no matter the value ν of the interaction component is.

The overall shape of the graphs of local mutual dependence is similar to others. The magnitude of the dependence rises near the origin (up to .6, compare to .5 in other numeric combinations between parameters)

On the opposite direction of the pair (Y w.r.t. X, or X w.r.t. Y) dependence vanishes with the distance from the line *y=x*.

The ridge local dependence along the line y=x stays steady positive near the origin and slowly vanishes away from the origin.

VI. Conclusions

- We discussed Regression Coefficients as measures of dependence between two random events. These measures are asymmetric, and exhibit natural properties.
- Their numerical values serve as indication for the magnitude of dependence between random events.
- These measures provide simple ways to detect independence, coincidence, degree of dependence.
- If either measure of dependence is known, it allows better prediction of the chance for occurrence of one event, given that the other one occurs.
- We observe unexpected behavior of the Regression Coefficients between the two components of symmetric Bivariate normal distribution with different magnitude of the correlation coefficient.
- These measures are examined by the 3-D surface of dependence on squares [x, $x+a$]x[y,$y+a$] with a=.5; a=1.0 and $(x,y)\in$[-3.5, 3.5]x[-3.5,3.5]
- There is some high positive local dependence close to the lines *y=x or y=-x*; negative local dependences also is present. The magnitudes of dependence vanish as long the points become far from the origin (0,0) or from the lines of correlation..
- Notice reduction of magnitude on smaller square.

- We observe unexpected behavior of the Regression Coefficients between the two components of the M-O Distribution.
- These measures are examined by the 3-D surface of dependence on squares $[x, x+a]\mathrm{x}[y,y+a]$ with a=.5; a=1.0 and $(x,y)\in[0, 3]\mathrm{x}[0,3]$
- There is some high positive local dependence close to the origin and along the line $y=x$
- Negative local dependences is presented near the axis of the variable w.r.t. which Regression Coefficient is considered.
- The magnitudes of dependence vanish at the opposite side, as the points get far from the origin (0,0).
- Notice reduction of magnitude on smaller square for interacting component with lower parameter' values.
- The magnitudes of dependence do not change with the value interacting component parameter as long the two components have same distributions.

As possible future investigation we challenge the readers of this article to compare the local dependence structures in the asymmetric bivariate normal distribution. Also look and compare our findings to the local dependence structures in the copulas of the bivariate inside any of the known bivariate distribution normal and the Mashal-Olkin distribution. Open is the local structure with correlated components.

VII. Acknowledgements

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Reliability Function of Renewable System under Marshall-Olkin Failure Model

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Abstract

In this note we obtain reliability function of two-component system under the Marshall-Olkin failure model in terms of Laplace transform. The problem of its sensitivity to the shape of the system components repair times is investigated as well.

Keywords: Heterogeneous reliability systems, Laplace transform, Marshall-Olkin bivariate failure model, reliability function, sensitivity analysis.

1 Introduction and Motivation

The stability of system characteristics with respect to the changes in initial states or external factors are the key problems for all natural sciences. For stochastic systems stability is often identified by insensitivity or low sensitivity of their output characteristics to the shapes of some input distributions.

One of the earliest results concerning insensitivity of system characteristics to the shape of service time distribution has been obtained in 1957 by Sevast'yanov [1], who established the insensitivity of Erlang formulas to the shape of service time distribution with fixed mean value for loss queueing systems with Poisson input flow. In 1976, Kovalenko [2] found necessary and sufficient conditions for insensitivity of stationary probabilities of redundant renewable systems, whose components have exponential life time and repair time distributions of general type. These conditions consist in a huge amount of repairing facilities. The sufficiency of immediate start to repair any failed element in the case of general life and repair time distributions has been found in 2013 by Rykov [3] with the help of multi-dimensional alternative processes theory. However, in the case of limited possibilities for recovering these results do not hold, as it was shown in [4].

On the other hand, in series of works Gnedenko (1964) and Solov'ev (1970) (see, e.g. [5, 6, 7]) show that under "quick" restoration the reliability function of a cold standby double redundant homogeneous system tends to the exponential one for any life and repair time distributions of its components. These results also imply the asymptotic insensitivity of the reliability characteristics of such system to the shape of their components life and repair times distributions. An alternative approach based on system states merging has been proposed by V. Korolyuk, see [8] and references therein.

Very recently, the problem of asymptotic insensitivity of reliability function for redundant systems to the shape of their components repair time distribution under condition of rare failures has been considered by Rykov and Kozyrev in [9, 10, 11, 12] using the Markovization method. All these studies describe the system with independently functioning components. We will relax this assumption in the present paper since the common environment implies some kind of dependence between elements of the system.

In 1967 Marshall and Olkin [13] proposed a bivariate distribution that can be used as a failure model for two-component reliability system with dependent components. The Marshall-Olkin (MO hereafter) model is specified by the stochastic representation

$$(T_1, T_2) = (\min(A_1, A_3), \min(A_2, A_3)), \tag{1}$$

where non-negative continuous random variables (r.v.) A_1 and A_2 represent times to occurrence of independent "individual shocks" affecting two devices and A_3 represents time to their "common shock" under assumption that the times to all shocks are independent and exponentially distributed. The joint distribution of random vector (T_1, T_2) can be characterized by the bivariate lack of memory property (BLMP) defined by the functional equation

$$S(x+t, y+t) = S(x,y)S(t,t), \quad for\ all \quad x, y, t \geq 0,$$

where $S(x,y)$ is the joint survival function of the pair (T_1, T_2). Many textbooks give a special attention to the BLMP and related MO bivariate exponential distribution exhibiting singularity along the main diagonal in R_+^2, see Barlow and Proschan [14] (1981), Singpurwalla [15] (2006), Balakrishnan and Lai [16] (2009), Gupta et al. [17] (2010), McNeil et al. [18] (2015) among others. Many articles complement and extend the MO model, justifying advantages in analysis of various data sets from engineering, medicine, insurance, finance, biology, etc. For example, Li and Pellerey [19] (2011) launched the Generalized MO model considering non-exponential independent random variables A_i in (1), $i = 1,2,3$. The corresponding joint distributions do not possess BLMP, i.e., are "aging". In 2014 the model is extended to the multidimensional case by Lin and Li [20]. As a further step, in 2015 Pinto and Kolev [21] introduced the Extended MO model assuming dependence between variables A_1 and A_2, but keeping A_3 independent of them in (1). The motivation is that the individual shocks might be dependent if the items share a common environment. In this case however, BLMP may be fulfilled or not depending on parameters of joint distribution of (A_1, A_2) and distribution of A_3.

Most of these investigations deal with bivariate distributions and their properties and use the MO model for the case of explicit failure. So far, the MO model has been not applied in the context of system reliability. In the present paper we consider a renewable heterogeneous double redundant standby renewable systems, where the failures of elements follow the MO model. The reliability function in terms of its Laplace transforms will be calculated. In this case the renovation procedure after the system components failures is very important and it will be included into the model.

The paper is organized as follows. In the next section the problem setting and some notations will be introduced. In the section 3 the reliability function is calculated in terms of its Laplace transforms, and in the next 4-th section its asymptotic insensitivity to the shape of the system components repair time distributions will be considered. The paper ends with conclusions.

2 Problem setting and notations

Consider a heterogeneous hot double redundant repairable reliability system, graphically represented on figure 1.

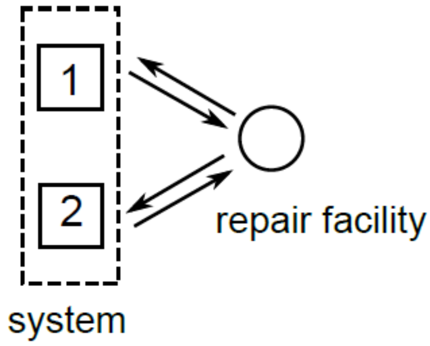

Figure 1: 2-unit hot-standby repairable system with one repair facility

We assume that component failures follow the MO model. This means that there exists three sources of shocks, which lead to the system failure. The first shock act only to the first component (identified by r.v. A_1), the second one act only to the second one (identified by r.v. A_2), while the third one (represented by r.v. A_3) act to both components and provokes a system failure. Thus, accordingly to the MO failure model (1), the system lifetime is determined by the joint distribution of (T_1, T_2), where A_1, A_2 and A_3 are independent r.v.'s.

Dealing with reparable model we need to propose some procedure of recovering. Let the repair time B_i of i-th component has absolute continuous distribution with cumulative distribution function $B_i(x)$ and probability density functions $b_i(x)$, correspondingly, $i = 1,2$. All repair times are assumed to be independent.

In order to describe the system behavior after its partial failure, when only one of components fails it is necessary to generalize the MO model. Note that there are at least two scenarios. The first one supposes that if one component fails and during its repair a non-fatal shock can arise leading to failure of another component which results in system breakdown. The second option is that a common shock also can arise, and it leads to the full system failure.

We will use the following notations. $\alpha = \alpha_1 + \alpha_2 + \alpha_3$ the summary intensity of failures; $\bar{\alpha}_i = \alpha_i + \alpha_3, (i = 1,2)$; $b_i = \int_0^\infty (1 - B_i(x))dx$ the i-th r.v. B_i $(i = 1,2,3)$ expectations; $\rho_i = \alpha_i b_i, i = 1,2,3$; $\beta_i(x) = (1 - B_i(x))^{-1} b_i(x)$ the i-th r.v. conditional repair intensity given elapsed repair time is x for $(i = 1,2,3)$; $\tilde{b}_i(s) = \int_0^\infty e^{-sx} b_i(x)dx$ the Laplace transform (LT) of the i-th component repair time distribution $(i = 1,2)$.

Under considered assumptions the state space of the system can be represented as $E = \{0, 1, 2, 3\}$, which means: 0 — both components are working, 1 — the first component has failed and is being repaired while the second one is working, 2 — the second component has failed and is being repaired while the first one is working, 3 — both components are in failure (down) states, system has failed and is being repaired.

In this paper we are interested in the *reliability function*

$$R(t) = P\{T > t\},$$

where T denotes the system life time.

3 Reliability Function

We will use the so-called Markovization method to calculate the system reliability function. Specifically, let us consider two-dimensional absorbing Markov process $Z = \{Z(t), t \geq 0)\}$, with $Z(t) = (J(t), X(t))$ where $J(t)$ represents the system state, and $X(t)$ is an additional variable, which means the elapsed repair time of $J(t)$-th component at time t. The process phase space is given by $E = \{0, (1,x), (2,x), 3\}$, which mean: 0 – both components are working, $(1,x)$ – the second component is working, the first one is failed and repairing, and its elapsed repair time equal to x, $(2,x)$ – the first component is working, the second one is failed and repairing, and its elapsed repair time equal to x, 3 – both components are failed, and therefore the system is failed. Corresponding probabilities are denoted by $\pi_0(t), \pi_1(t;x), \pi_2(t;x), \pi_3(t)$. The state transition graph of the system is represented on figure 2.

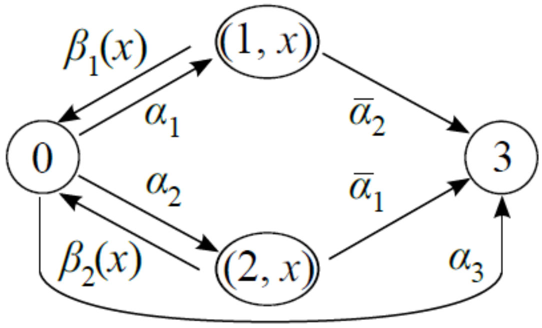

Figure 2: Absorbing system transition graph.

Under the above assumptions, the following statement is true.

Theorem 1 *The system reliability function Laplace transform is given by*

$$\tilde{R}(s) = \frac{(s+\bar{\alpha}_1)(s+\bar{\alpha}_2)+(s+\bar{\alpha}_1)\phi_1(s)+(s+\bar{\alpha}_2)\phi_2(s)}{(s+\bar{\alpha}_1)(s+\bar{\alpha}_2)[s+\phi_1(s)+\phi_2(s)+\alpha_3]} \tag{2}$$

where $s > 0$ and

$$\phi_i(s) = \alpha_i(1 - \tilde{b}_i(s + \bar{\alpha}_{i*})), \quad i = 1,2 \tag{3}$$

with $i^* = 2$ if $i = 1$ and vice versa.

Proof. Applying the usual method of comparing the process probabilities at closed times t and $t + \Delta$ the system of Kolmogorov forward partial differential equations can be written as follows

$$\begin{aligned}
&ddt\pi_0(t) = -\alpha\pi_0(t) + \int_0^t \pi_1(t,x)\beta_1(x)dx + \int_0^t \pi_2(t,x)\beta_2(x)dx;\\
&(\partial\,\partial t + \partial\,\partial x)\pi_1(t;x) = -(\bar{\alpha}_2 + \beta_1(x))\pi_1(t;x);\\
&(\partial\,\partial t + \partial\,\partial x)\pi_2(t;x) = -(\bar{\alpha}_1 + \beta_2(x))\pi_2(t;x);\\
&ddt\pi_3(t) = \alpha_3\pi_0(t) + \bar{\alpha}_1 \int_0^t \pi_2(t;x)dx + \bar{\alpha}_2 \int_0^t \pi_1(t;x)dx,
\end{aligned} \tag{4}$$

taking into account the initial $\pi_0(0) = 1$ and boundary conditions

$$\pi_1(t,0) = \alpha_1\pi_0(t), \quad \pi_2(t,0) = \alpha_2\pi_0(t). \tag{5}$$

To solve this system we use the method of characteristics for solving first-order partial differential equations, consult [22]. According to this method we obtain[17]

$$\begin{aligned}
&\pi_1(t;x) = h_1(t-x)e^{-\bar{\alpha}_2 x}(1 - B_1(x)), \quad x \le t;\\
&\pi_2(t;x) = h_2(t-x)e^{-\bar{\alpha}_1 x}(1 - B_2(x)), \quad x \le t,
\end{aligned} \tag{6}$$

[17] The represented below solutions have a nice probabilistic interpretation. The functions $h_i(\cdot)$ can be considered as renewal densities of the process returning to the states with zero elapsed times, and two other multipliers show that during the time x neither failure, nor repair occurs.

and from boundary conditions (5) it holds

$$\pi_1(t;0) = h_1(t) = \alpha_1\pi_0(t), \quad \pi_2(t;0) = h_2(t) = \alpha_1\pi_0(t). \tag{7}$$

Substitution of these solutions to the first equation in (7) gives

$$ddt\pi_0(t) = -\alpha\pi_0(t) + \int_0^t h_1(t-x)e^{-\bar{\alpha}_2 x}b_1(x)dx +$$
$$+\int_0^t h_2(t-x)e^{-\bar{\alpha}_1 x}b_1(x)dx.$$

In terms of Laplace transform with $\pi_0(0) = 1$ we have

$$(s+\alpha)\tilde{\pi}_0(s) - 1 = \tilde{h}_1(s)\tilde{b}_1(s+\bar{\alpha}_2) + \tilde{h}_2(s)\tilde{b}_2(s+\bar{\alpha}_1).$$

Substitution into this equation the Laplace transform of the boundary conditions (7)

$$\tilde{h}_1(s) = \alpha_1\tilde{\pi}_0(s), \;\; \tilde{h}_2(s) = \alpha_2\tilde{\pi}_0(s),$$

after some algebra we get

$$(s+\alpha)\tilde{\pi}_0(s) - \alpha_1\tilde{b}_1(s+\bar{\alpha}_2)\tilde{\pi}_0(s) - \alpha_2\tilde{b}_2(s+\bar{\alpha}_1)\tilde{\pi}_0(s) = 1.$$

From this equality one can find $\tilde{\pi}_0(s)$ in the following form:

$$\tilde{\pi}_0(s) = [s + \phi_1(s) + \phi_2(s) + \alpha_3]^{-1}, \tag{8}$$

where for simplicity the notations (3) are used.

To find $\tilde{\pi}_3(s)$ we apply the Laplace transform (8) in the last equation of the system (??). Taking into account the expressions (??) for probabilities $\pi_i(t;x)$ for $i = 1,2$ we obtain

$$s\tilde{\pi}_3(s) = \alpha_3\tilde{\pi}_0(s) + \bar{\alpha}_2\tilde{h}_1(s)\frac{1-\tilde{b}_1(s+\bar{\alpha}_2)}{s+\bar{\alpha}_2} +$$
$$+\bar{\alpha}_1\tilde{h}_2(s)\frac{1-\tilde{b}_2(s+\bar{\alpha}_1)}{s+\bar{\alpha}_1}.$$

By substituting instead of $\tilde{h}_i(s)$ its representation in terms of $\tilde{\pi}_0(s)$ we get

$$s\pi_3(s) = \tilde{\pi}_0(s)\left(\frac{\bar{\alpha}_2}{s+\bar{\alpha}_2}\phi_1(s) + \frac{\bar{\alpha}_1}{s+\bar{\alpha}_1}\phi_2(s) + \alpha_3\right).$$

Finally, since

$$\tilde{R}(s) = 1s - \tilde{\pi}_3(s),$$

we arrive to

$$\tilde{R}(s) = 1s\left(1 - \frac{\bar{\alpha}_2 s + \bar{\alpha}_2\phi_1(s) + \bar{\alpha}_1 s + \bar{\alpha}_1\phi_2(s) + \alpha_3}{[s+\phi_1(s)+\phi_2(s)+\alpha_3]}\right)$$
$$= \frac{(s+\bar{\alpha}_1)(s+\bar{\alpha}_2) + (s+\bar{\alpha}_1)\phi_1(s) + (s+\bar{\alpha}_2)\phi_2(s)}{(s+\bar{\alpha}_1)(s+\bar{\alpha}_2)[s+\phi_1(s)+\phi_2(s)+\alpha_3]},$$

which ends the proof.

As a corollary, by a substitution $s = 0$ we find the mean time to the system failure.

Corollary 1 *The mean system life time is given by*

$$E[T] = \tilde{R}(0) = \frac{\bar{\alpha}_1\bar{\alpha}_2 + \bar{\alpha}_1\alpha_1(1-\tilde{b}_1(\bar{\alpha}_2)) + \bar{\alpha}_2\alpha_2(1-\tilde{b}_2(\bar{\alpha}_1))}{\bar{\alpha}_1\bar{\alpha}_2[\alpha_1(1-\tilde{b}_1(\bar{\alpha}_2)) + \alpha_2(1-\tilde{b}_2(\bar{\alpha}_1))]} \tag{9}$$

Remark 1 *Note that in homogeneous case, when all failure parameters are equal ($\alpha_1 = \alpha_2 = \alpha_3 = \alpha$) the system mean time to failure simplifies to*

$$m_T = E[T] \approx 1\alpha.$$

4 Rare failures

The above formulas demonstrate the evident dependence of the reliability function on the shape of repair time distribution. It is expressed in the form of Laplace transform of the repair time distribution at points of elements' failure intensities.

On the other side, as it was mentioned in the introduction, for systems with independent component failures in case of quick restoration of components the system reliability function tends to the exponential one for any repair time distribution. Here we consider the behavior of the considered system reliability function under MO failure model with condition of "rare" failures instead of "quick" restorations.

For the considered model the rare failures should be understood as the slow intensity of failures with respect to the fixed repair times. Thus we will suppose that $q = \max\{\alpha_1, \alpha_2, \alpha_3\} \to 0$. Naturally the asymptotic analysis should be done with respect to a certain scale parameter. In the place of such a parameter the asymptotic mean lifetime value will be considered.

Using (3) and relations $\rho_i = \alpha_i b_i$ one can find that

$$\phi_i(0) = \alpha_i(1 - \tilde{b}_i(\bar{\alpha}_{i^*})) \approx \rho_i \bar{\alpha}_{i*}$$

as $q \to 0$, and therefore, the mean value of the system time to failure from (9) is

$$m = E[T] = \tilde{R}(0) = \frac{\bar{\alpha}_1\bar{\alpha}_2 + \bar{\alpha}_1\phi_1(\bar{\alpha}_2) + \bar{\alpha}_2\phi_2(\bar{\alpha}_1)}{\alpha_1\alpha_2(\phi_1(\bar{\alpha}_2) + \phi_2(\bar{\alpha}_1)) + \alpha_3} = \\ = \frac{1+\rho_1+\rho_2}{\bar{\alpha}_1\rho_2 + \bar{\alpha}_2\rho_1 + \alpha_3}.$$

Theorem 2 *Under rare components' failures the system reliability function becomes asymptotically insensitive to the shapes of their repair distributions. Moreover the reliability function for the considered model in scale of $m = E[T]$ has unit exponential distribution, i.e.,*

$$\lim_{q\to 0} P\{Tm > t\} = e^{-t}.$$

Proof. Instead of the large parameter m we consider the small parameter $\gamma = m^{-1}$. We are interested on the asymptotic behavior of the reliability function of the system

$$R(t\gamma) = P\{\gamma T > t\}$$

when $\gamma \to 0$. To do that, we investigate the asymptotic behavior of its Laplace transform

$$\gamma\tilde{R}(\gamma s) = \gamma\frac{(\gamma s+\bar{\alpha}_1)(\gamma s+\bar{\alpha}_2) + (\gamma s+\bar{\alpha}_1)\phi_1(\gamma s) + (\gamma s+\bar{\alpha}_2)\phi_2(\gamma s)}{(\gamma s+\bar{\alpha}_1)(\gamma s+\bar{\alpha}_2)[\gamma s+\phi_1(\gamma s)+\phi_2(\gamma s)+\alpha_3]} = \\ = \frac{1+\phi_1(\gamma s)\gamma s+\bar{\alpha}_2+\phi_2(\gamma s)\gamma s+\bar{\alpha}_1}{\gamma s+\phi_1(\gamma s)+\phi_2(\gamma s)+\alpha_3}.$$

When $\gamma \to 0$, it holds that

$$\phi_i(\gamma s) = \alpha_i(1 - \tilde{b}_i(\gamma s + \bar{\alpha}_i)) \approx \alpha_i b_i(\gamma s + \bar{\alpha}_{i*}) = \rho_i(\gamma s + \alpha_{i*}).$$

Therefore, $\phi_i(\gamma s)\gamma s + \bar{\alpha}_{i*} \approx \rho_i$ and the last relation yields

$$\gamma\tilde{R}(\gamma s) = \gamma \frac{1+\rho_1+\rho_2}{\gamma s(1+\rho_1+\rho_2)+\rho_1\bar{\alpha}_2+\rho_2\bar{\alpha}_1+\alpha_3} = \gamma\gamma(s+1) = 1s+1.$$

So, when $\gamma \to 0$ it follows that

$$P\{\gamma T > t\} = R(t\gamma) \to e^{-t}.$$

5 Conclusions

We focus on assessing and study of the system-level reliability of a heterogeneous double redundant renewable system under Marshall-Olkin failure model in the case when repair times of its components have a general continuous distribution. The proposed mathematical model allows to obtain the explicit expression in terms of Laplace transform for the system reliability function. The produced analytical results reveal asymptotic insensitivity of the reliability function of the system under the 'rare' failures of its elements to the shape of their repair time distribution. In addition, we showed that when the scale parameter is mean time to failure, the system's reliability function converge to the unit exponential law.

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Research of a Multidimensional Markov Chain as a Model for the Class of Queueing Systems Controlled by a Threshold Priority Algorithm

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Abstract

A class of controlled queueing systems with several heterogeneous conflicting input flows is investigated. A model of such systems is a time-homogeneous multidimensional Markov chain with a countable state space. Classification of the chain states is made: a closed set of recurrent aperiodic states and a set of transient states are determined. An ergodic theorem for the Markov chain is formulated and proved.

Keywords: controlled queueing system, threshold priority, multidimensional Markov chain, recurrent state, stationary distribution

1 Introduction

Nowadays, there is a great amount of works that deal with the problems of controlled queueing systems [1, 2]. Many of similar researches have high applied value since they concern real biological, logistical, engineering and technical objects (e. g. [3, 4, 5]). Various quality characteristics and scores for such systems are investigated. One of the important goals of these works is system optimization. In order to synthesize an optimal system, it is usually necessary to study its asymptotic behaviour [4, 6, 7, 8, 9]. In particular, a number of works are devoted to obtaining the limit theorems and searching for the conditions of stationarity existence. Mainly such investigations are based on the mathematical and imitation modeling. If a quite simple reliable mathematical model is constructed, it becomes possible to study limiting dynamics and to determine existence of a stable stationary mode.

The work [10] studies a system with several stochastic independent conflicting input flows of customers (demands). In this work a specific model of input flows constructed in [11] is considered. The system carries not only service functionality for the customers but also control functions for the flows. It is supposed that the flows are controlled by a cyclic algorithm. As a rule, such control algorithm is applied if the input flows are regarded to be homogeneous, which means no preference is given to any of the flows. A case of heterogeneous input flows is considered in [12]. Such heterogeneity may imply, for example, different probabilistic structure of the flows, substantially different arrival intensities, different priority of the flows, etc. It is usually assumed in such case that a complicated adaptive feedback control algorithm is used. The present work is a continuation and expansion of [12] and it mainly focuses on the limiting behavior of the system.

2 System description

A system that controls $m \geq 2$ independent conflicting flows Π_1, Π_2, ..., Π_m and serves their customers is studied. It is assumed that the input flows are formed under the influence of the similar external environments. This means they can be approximated by the non-ordinary Poisson flows. For example, it is shown in [11] that a non-ordinary Poisson flow can be an adequate model for a traffic flow under certain external conditions. If the weather or the roadbed is quite bad, the heterogeneity of the vehicles becomes clear: some slow vehicles become a trouble for the fast ones. That is why vehicles start to gather into groups – traffic bathes. There is a certain dependency between the vehicles inside a batch while the different batches can be considered as independent. Similarly, it is supposed in this work that each input flow can be approximated by non-ordinary Poisson flow Π_j (henceforth, $j \in J = \{1,2,\dots,m\}$) with the following parameters: $\lambda_j > 0$ – arrival intensity for the batches (groups of customers), p_j, q_j and s_j – probabilities of batches of one, two and three customers in a batch correspondingly ($p_j + q_j + s_j = 1$). The expressions for the one-dimensional distributions of the non-ordinary Poisson flow of this kind are derived in [11]. Probability $\varphi_j(n;t)$ that $n \in X = \{0,1,\dots\}$ customers of the flow Π_j arrive to the system during the interval $[0,t)$, $t > 0$, is given by formula

$$\varphi_j(n;t) = e^{-\lambda_j t} \sum_{u=0}^{[\frac{n}{2}]} \sum_{v=0}^{[\frac{n-2u}{3}]} p_j^{n-2u-3v} q_j^u s_j^v \frac{(\lambda_j t)^{n-u-2v}}{u!v!(n-2u-3v)!}.$$

Though the flows have the same probabilistic structure, they differ in priority and arrival intensity. The flows Π_1, Π_2, ..., Π_{m-1} are the low-intensity flows, the flow Π_m has the highest intensity. At the same time, the customers of the flow Π_1 have the highest priority. The considered queueing system is a lossless system. The customers of the flow Π_j that arrive to the system and cannot be served at once are forced to wait in the queue O_j. The service device that also perform control functions for the flows can be in one of the states of the set $\Gamma = \{\Gamma^{(1)}, \Gamma^{(2)}, \dots, \Gamma^{(2m+1)}\}$. The service device stays in each state $\Gamma^{(k)}$, $k \in \{1,2,\dots,2m+1\}$, during a time period of duration T_k. The state $\Gamma^{(2j-1)}$, where $j \in \{1,2,\dots,m-1\}$, is reserved for servicing the flow Π_j. A service intensity in this case is $\mu_j > 0$. Since the flow Π_m has the highest intensity, there are two service states for this flow: $\Gamma^{(2m-1)}$ and $\Gamma^{(2m)}$. The service intensity is the same for both of these states and it equals $\mu_m > 0$. It is also assumed that $T_{2m} < T_{2m-1}$. The input flows are conflicting which means no two of them can be served simultaneously. Moreover, for the sake of safety, it is recommended to have certain adjusting states between the service states for different flows. Therefore, the intermediate readjusting state $\Gamma^{(2j)}$, $j \in \{1,2,\dots,m-1\}$ is allocated for safe switching between service of the flow Π_j and Π_{j+1}. The readjusting state after the flow Π_m is $\Gamma^{(2m+1)}$. The variables $l_j = [\mu_j T_{2j-1}]$, $j \in J$, and the variable $l'_m = [\mu_m T_{2m}]$ characterize the capacity of the service device in the corresponding state. It is supposed that the system always functions in the emergency mode [10]. This means there is no unmotivated downtimes. Each time the service device is in a service state for certain flow Π_j, as many customers waiting in the queue O_j as possible are served.

At the same time, the number of served customers cannot exceed the service capacity in this state. When the service period for certain flow ends, either the current state switches to the next one according a certain control algorithm $s(\Gamma)$ or the decision to prolong service is made. The control algorithm $s(\Gamma)$ is to be described later. The served customers of the flow Π_j compose the output flow Π'_j. A general scheme of the considered class of the systems is presented in Figure. 1.

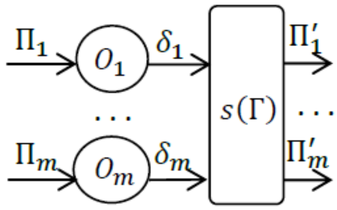

Figure 1: General scheme of the considered class of systems

Let τ_i, $i \in I = \{0,1,\dots\}$, denote the moments in which the decisions about state switching or prolongations are made. Such moments are the random variables since the initial state of the service device is unknown, an initial distribution of the state in moment $t = 0$ may be set and, generally speaking, the durations T_1, T_2, ..., T_{2m+1} are different. The time axis $0t$ is divided by these moments into the intervals $\Delta_{-1} = [0,\tau_0)$, $\Delta_i = [\tau_i, \tau_{i+1})$, $i \in I$. The following random variables and elements characterize the system in the interval Δ_i for $i \in I$: 1) $\Gamma_i \in \Gamma$ – state of the service device; 2) $\eta_{j,i} \in X$ – number of demands of the flow Π_j arrived to the system; 3) $\xi_{j,i}$ – maximum number of demands of the flow Π_j that can be served; 4) $\xi'_{j,i}$ – number of customers of the flow Π_j that are actually served during this interval. Here for any $j \in \{1,2,\dots,m-1\}$ we have $\xi_{j,i} \in B_j = \{0, l_j\}$ and $\xi'_{j,i} \in Y_j = \{0,1,\dots,l_j\}$ and $\xi_{m,i} \in B_m = \{0, l'_m, l_m\}$, $\xi'_{m,i} \in Y_m = \{0,1,\dots,l_m\}$. Apart from this, let the variable æ $_{j,i} \in X$ count the random number of customers waiting in the queue O_j at the moment τ_i. For each flow Π_j it is also necessary to introduce the random variable $\xi'_{j,-1} \in \{0,1,\dots\}$ – number of demands of the flow Π_j that are really served during the interval Δ_{-1}. Now the control algorithm is to be introduced. Decision about the next state of the service device is made according to the following rule:

$$\Gamma_{i+1} = u(\Gamma_i, æ_{1,i}, \eta_{1,i}),$$

where the control function $u: \Gamma \times X \times X \to \Gamma$ is given point-wise:

$$u(\Gamma^{(k)}, x_1, n_1) = \begin{cases} \Gamma^{(k+1)}, \ k \in M \backslash \{2m-2, 2m, 2m+1\}; \\ \Gamma^{(2m-1)}, \ k = 2m-2, \ x_1 + n_1 < h_1; \\ \Gamma^{(2m)}, \ k = 2m-2, \ x_1 + n_1 \geq h_1; \\ \Gamma^{(2m)}, \ k = 2m, \ x_1 + n_1 < h_1; \\ \Gamma^{(2m+1)}, \ k = 2m, \ x_1 + n_1 \geq h_1; \\ \Gamma^{(1)}, \ k = 2m+1. \end{cases} \quad (1)$$

Such algorithm has several peculiarities. Firstly, it implements feedback on the number of waiting customers in the queue for the high-priority flow. Secondly, the service device may prolong service for the flow with high intensity. Thirdly, the described algorithm is an anticipatory algorithm, since it takes into account data about the number $\eta_{1,i}$ of customers that are to arrive to the system during the succeeding time period. It should be noted that a conflict of interests between high-intensity and high-priority flows is resolved with the help of a threshold priority variable $h_1 \in \{0,1,\dots\}$. The service device state is switched from the service of high-intensity flow to the service of the high-priority flow only if the number of waiting customers in the high-priority queue reaches the threshold value. A graph of the described control algorithm is shown in Figure 2.

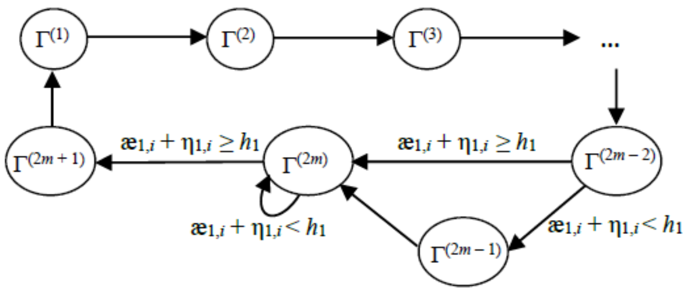

Figure 2: Graph of the control algorithm $s(\Gamma)$

The variables $\eta_{j,i}$ and $\xi_{j,i}$ are defined by their conditional distributions

$$\mathbf{P}(\eta_{j,i} = n \mid \Gamma_i = \Gamma^{(k)}) = \varphi_j(n; T_k),$$

$$\mathbf{P}(\xi_{j,i} = b \mid \Gamma_i = \Gamma^{(k)}) = \beta_j(b; \Gamma^{(k)}),$$

where function $\beta_j : B_j \times \Gamma \to \{0,1\}$ is given point-wise:

$$\beta_j(b; \Gamma^{(k)}) = \begin{cases} 1, \ b = 0, \ k \in M \backslash \{2j-1\}, \ j \in J \backslash \{m\}; \\ 1, \ b = 0, \ k \in M \backslash \{2m-1, 2m\}, \ j = m; \\ 1, \ b = l_j, \ k = 2j-1, \ j \in J; \\ 1, \ b = l'_m, \ k = 2m, \ j = m; \\ 0, \ otherwise. \end{cases} \tag{2}$$

Moreover, these variables are conditionally independent.

3 Problem statement

It is proposed in [12] to consider a random vector

$$\chi_i = (\Gamma_i, æ_{1,i}, æ_{m,i}, \xi'_{1,i-1}, \xi'_{m,i-1}) \in \Gamma \times X \times X \times Y_1 \times Y_m \tag{3}$$

as a system state at the moment τ_i, $i \in I$. This approach allows one to study the system dynamics from the point of view of two flows: Π_1 and Π_m. The following recurrent relations are given in [12]:

$$\Gamma_{i+1} = u(\Gamma_i, æ_{1,i}, \eta_{1,i}), æ_{1,i+1} = \max\{0, æ_{1,i} + \eta_{1,i} - \xi_{1,i}\}, \ \ æ_{m,i+1} = \max\{0, æ_{m,i} + \eta_{m,i} - \xi_{m,i}\},$$

$$\xi'_{1,i} = \min\{æ_{1,i} + \eta_{1,i}, \xi_{1,i}\}, \ \ \xi'_{m,i} = \min\{æ_{m,i} + \eta_{m,i}, \xi_{m,i}\}.$$

They describe changes in the system state for any $i \in I$. The work [12] contains a proof of the following theorem.

Theorem 1 *Vector sequence*

$$\{(\Gamma_i, æ_{1,i}, æ_{m,i}, \xi'_{1,i-1}, \xi'_{m,i-1}); i \in I\} \tag{4}$$

with initial distribution of vector $(\Gamma_0, æ_{1,0}, æ_{m,0}, \xi'_{1,-1}, \xi'_{m,-1})$ is a multidimensional time-homogeneous controlled Markov chain.

A purpose of this work is to study the state space of the Markov chain (4) and to research its limiting behaviour.

4 Markov chain state space classification

Let $Q_i(\Gamma^{(k)}, x_1, x_m, y_1, y_m)$ for any $\Gamma^{(k)} \in \Gamma$, $x_1, x_m \in X$, $y_1 \in Y_1$, $y_m \in Y_m$ be the following probabilities:

$$Q_i(\Gamma^{(k)}, x_1, x_m, y_1, y_m) = \mathbf{P}(\Gamma_i = \Gamma^{(k)}, æ_{1,i} = x_1, æ_{m,i} = x_m, \xi'_{1,i-1} = y_1, \xi'_{m,i-1} = y_m).$$

The proof of Theorem 1 from [12] contains the foundation for the following relation:

$$Q_{i+1}(\Gamma^{(k)}, x_1, x_m, y_1, y_m) = \sum_{r=1}^{2m+1} \sum_{v_1=0}^{\infty} \sum_{v_m=0}^{\infty} \sum_{w_1=0}^{l_1} \sum_{w_m=0}^{l_m} Q_i(\Gamma^{(r)}, v_1, v_m, w_1, w_m) \times$$

$$\times \sum_{n_1=0}^{\infty} \sum_{b_1 \in B_1} \sum_{n_m=0}^{\infty} \sum_{b_m \in B_m} \varphi_1(n_1; T_r) \beta_1(b_1; \Gamma^{(r)}) \varphi_m(n_m; T_r) \beta_m(b_m; \Gamma^{(r)}) \times$$

$$\times \mathbf{P}(u(\Gamma^{(r)}, v_1, n_1) = \Gamma^{(k)}, \max\{0, v_1 + n_1 - b_1\} = x_1,$$

$$\max\{0, v_m + n_m - b_m\} = x_m, \min\{v_1 + n_1, b_1\} = y_1, \min\{v_m + n_m, b_m\} = y_m).$$

Note that taking into account (1) and (2) this relation is transformed into several special-case relations:

$$Q_{i+1}(\Gamma^{(1)}, x_1, x_m, y_1, y_m) = \sum_{v_1=0}^{x_1} \varphi_1(x_1 - v_1; T_{2m+1}) \sum_{v_m=0}^{x_m} \varphi_m(x_m - v_m; T_{2m+1}) \times \\ \times \sum_{w_1=0}^{l_1} \sum_{w_m=0}^{l_m} Q_i(\Gamma^{(2m+1)}, v_1, v_m, w_1, w_m) \mathbf{P}(y_1 = 0, y_m = 0). \quad (5)$$

$$Q_{i+1}(\Gamma^{(2)}, x_1, x_m, y_1, y_m) = \sum_{v_1=0}^{y_1} \varphi_1(y_1 - v_1; T_1) \sum_{v_m=0}^{x_m} \varphi_m(x_m - v_m; T_1) \times \\ \times \sum_{w_1=0}^{l_1} \sum_{w_m=0}^{l_m} Q_i(\Gamma^{(1)}, v_1, v_m, w_1, w_m) \mathbf{P}(x_1 = 0, y_1 < l_1, y_m = 0) + \\ + \sum_{v_1=0}^{x_1+l_1} \varphi_1(x_1 + l_1 - v_1; T_1) \sum_{v_m=0}^{x_m} \varphi_m(x_m - v_m; T_1) \times \\ \times \sum_{w_1=0}^{l_1} \sum_{w_m=0}^{l_m} Q_i(\Gamma^{(1)}, v_1, v_m, w_1, w_m) \mathbf{P}(y_1 = l_1, y_m = 0); \quad (6)$$

$$Q_{i+1}(\Gamma^{(k)}, x_1, x_m, y_1, y_m) = \sum_{v_1=0}^{x_1} \varphi_1(x_1 - v_1; T_{k-1}) \sum_{v_m=0}^{x_m} \varphi_m(x_m - v_m; T_{k-1}) \times \\ \times \sum_{w_1=0}^{l_1} \sum_{w_m=0}^{l_m} Q_i(\Gamma^{(k-1)}, v_1, v_m, w_1, w_m) \mathbf{P}(y_1 = 0, y_m = 0), \quad k \in \{3,4, \dots, 2m-2\}; \quad (7)$$

$$Q_{i+1}(\Gamma^{(2m-1)}, x_1, x_m, y_1, y_m) = \sum_{v_1=0}^{x_1} \varphi_1(x_1 - v_1; T_{2m-2}) \sum_{v_m=0}^{x_m} \varphi_m(x_m - v_m; T_{2m-2}) \times \\ \times \sum_{w_1=0}^{l_1} \sum_{w_m=0}^{l_m} Q_i(\Gamma^{(2m-2)}, v_1, v_m, w_1, w_m) \mathbf{P}(x_1 < h_1, y_1 = 0, y_m = 0); \quad (8)$$

$$Q_{i+1}(\Gamma^{(2m)}, x_1, x_m, y_1, y_m) = \sum_{v_1=0}^{x_1} \varphi_1(x_1 - v_1; T_{2m-2}) \sum_{v_m=0}^{x_m} \varphi_m(x_m - v_m; T_{2m-2}) \times \\ \times \sum_{w_1=0}^{l_1} \sum_{w_m=0}^{l_m} Q_i(\Gamma^{(2m-2)}, v_1, v_m, w_1, w_m) \mathbf{P}(x_1 \geq h_1, y_1 = 0, y_m = 0) + \\ + \sum_{v_1=0}^{x_1} \varphi_1(x_1 - v_1; T_{2m-1}) \sum_{c_m=0}^{\infty} \sum_{v_m=0}^{c_m} \varphi_m(c_m - v_m; T_{2m-1}) \times \\ \times \sum_{w_1=0}^{l_1} \sum_{w_m=0}^{l_m} Q_i(\Gamma^{(2m-1)}, v_1, v_m, w_1, w_m) \times \\ \times \mathbf{P}(\max\{0, c_m - l_m\} = x_m, y_1 = 0, \min\{c_m, l_m\} = y_m) + \\ + \sum_{v_1=0}^{x_1} \varphi_1(x_1 - v_1; T_{2m}) \sum_{c_m=0}^{\infty} \sum_{v_m=0}^{c_m} \varphi_m(c_m - v_m; T_{2m}) \sum_{w_1=0}^{l_1} \sum_{w_m=0}^{l_m} Q_i(\Gamma^{(2m)}, v_1, v_m, w_1, w_m) \times \\ \times \mathbf{P}(x_1 < h_1, \max\{0, c_m - l'_m\} = x_m, y_1 = 0, \min\{c_m, l'_m\} = y_m); \quad (9)$$

$$Q_{i+1}(\Gamma^{(2m+1)}, x_1, x_m, y_1, y_m) = \sum_{v_1=0}^{x_1} \varphi_1(x_1 - v_1; T_{2m}) \times \\ \times \sum_{c_m=0}^{\infty} \sum_{v_m=0}^{c_m} \varphi_m(c_m - v_m; T_{2m}) \sum_{w_1=0}^{l_1} \sum_{w_m=0}^{l_m} Q_i(\Gamma^{(2m)}, v_1, v_m, w_1, w_m) \times \\ \times \mathbf{P}(x_1 \geq h_1, \max\{0, c_m - l'_m\} = x_m, y_1 = 0, \min\{c_m, l'_m\} = y_m). \quad (10)$$

Theorem 2 *State space $S = \Gamma \times X \times X \times Y_1 \times Y_m$ of the Markov chain (4) consists of set D of transient states and minimal closed set E of recurrent aperiodic states:*

$$D = \{(\Gamma^{(1)}, x_1, x_m, y_1, y_m) \in S : x_1 \in \{0,1, \dots, h_1 - 1\}\} \cup \\ \cup \{(\Gamma^{(k)}, x_1, x_m, y_1, y_m) \in S : k \in M \backslash \{2\}, y_1 \in Y_1 \backslash \{0\}\} \cup \\ \cup \{(\Gamma^{(k)}, x_1, x_m, y_1, y_m) \in S : k \in M \backslash \{2m, 2m+1\}, y_m \in Y_m \backslash \{0\}\} \cup$$

$$\cup \{(\Gamma^{(2)}, x_1, x_m, y_1, y_m) \in S: x_1 \in X \backslash \{0\}, y_1 \in Y_1 \backslash \{l_1\}\} \cup$$
$$\cup \{(\Gamma^{(2)}, x_1, x_m, y_1, y_m) \in S: y_1 \in \{0,1, \dots, h_1 - 1\}\} \cup$$
$$\cup \{(\Gamma^{(2m-1)}, x_1, x_m, y_1, y_m) \in S: x_1 \in \{h_1, h_1 + 1, \dots\}\} \cup$$
$$\cup \{(\Gamma^{(2m)}, x_1, x_m, y_1, y_m) \in S: x_1 \in \{0,1, \dots, h_1 - 1\}, x_m \in X \backslash \{0\}, y_m \in Y_m \backslash \{l'_m, l_m\}\} \cup$$
$$\cup \{(\Gamma^{(2m)}, x_1, x_m, y_1, y_m) \in S: x_1 \in \{h_1, h_1 + 1, \dots\}, x_m \in X \backslash \{0\}, y_m \in Y_m \backslash \{0, l_m\}\} \cup$$
$$\cup \{(\Gamma^{(2m+1)}, x_1, x_m, y_1, y_m) \in S: x_1 \in \{0,1, \dots, h_1 - 1\}\} \cup$$
$$\cup \{(\Gamma^{(2m+1)}, x_1, x_m, y_1, y_m) \in S: x_1 \in \{h_1, h_1 + 1, \dots\}, x_m \in X \backslash \{0\}, y_m \in \{0,1, \dots, l'_m - 1\}\} \cup$$
$$\cup \{(\Gamma^{(2m+1)}, x_1, x_m, y_1, y_m) \in S: x_1 \in \{h_1, h_1 + 1, \dots\}, y_m \in \{l'_m + 1, l'_m + 2, \dots, l_m\}\};$$

$$E(\Gamma^{(1)}) = \{(\Gamma^{(1)}, x_1, x_m, 0,0): x_1 \in \{h_1, h_1 + 1, \dots\}, x_m \in X\};$$
$$E(\Gamma^{(2)}) = \{(\Gamma^{(2)}, x_1, x_m, l_1, 0): x_1 \in X, x_m \in X\} \cup$$
$$\cup \{(\Gamma^{(2)}, 0, x_m, y_1, 0): x_m \in X, y_1 \in \{h_1, h_1 + 1, \dots, l_1 - 1\}\};$$
$$E(\Gamma^{(k)}) = \{(\Gamma^{(k)}, x_1, x_m, 0,0): x_1 \in X, x_m \in X\}, \ k \in \{3,4, \dots, 2m - 2\};$$
$$E(\Gamma^{(2m-1)}) = \{(\Gamma^{(2m-1)}, x_1, x_m, 0,0): x_1 \in \{0,1, \dots, h_1 - 1\}, x_m \in X\};$$
$$E(\Gamma^{(2m)}) = \{(\Gamma^{(2m)}, x_1, x_m, 0,0): x_1 \in \{h_1, h_1 + 1, \dots\}, x_m \in X \backslash \{0\}\} \cup$$
$$\cup \{(\Gamma^{(2m)}, x_1, x_m, 0, l_m): x_1 \in X, x_m \in X \backslash \{0\}\} \cup$$
$$\cup \{(\Gamma^{(2m)}, x_1, 0,0, y_m): x_1 \in X, y_m \in Y_m\} \cup$$
$$\cup \{(\Gamma^{(2m)}, x_1, x_m, 0, l'_m): x_1 \in \{0,1, \dots, h_1 - 1\}, x_m \in X \backslash \{0\}\};$$
$$E(\Gamma^{(2m+1)}) = \{(\Gamma^{(2m+1)}, x_1, x_m, 0, l'_m): x_1 \in \{h_1, h_1 + 1, \dots\}, x_m \in X \backslash \{0\}\} \cup$$
$$\cup \{(\Gamma^{(2m+1)}, x_1, 0,0, y_m): x_1 \in \{h_1, h_1 + 1, \dots\}, y_m \in \{0,1, \dots, l'_m\}\},$$
$$E = \cup_{k=1}^{2m+1} E(\Gamma^{(k)}).$$

Proof. First of all, it is necessary to determine the states of the Markov chain such that probabilities of being in them are equal to zero for any moment starting with τ_1. Based on relation (5) and control algorithm function (1), it follows that the Markov chain can move with positive probability to the state of the form $(\Gamma^{(1)}, x_1, x_m, y_1, y_m) \in S$ only from a state of the form $(\Gamma^{(2m+1)}, v_1, v_m, w_1, w_m) \in S$. The probability $\mathbf{P}(y_1 = 0, y_m = 0)$ in the right side of equation (5) equals zero if at least one of the equalities $y_1 = 0$ and $y_m = 0$ does not take place. That is why a probability of being in any state from the set

$$D(\Gamma^{(1)}) = \{(\Gamma^{(1)}, x_1, x_m, y_1, y_m) \in S: y_1 \in Y_1 \backslash \{0\}\} \cup \{(\Gamma^{(1)}, x_1, x_m, y_1, y_m) \in S: y_m \in Y_m \backslash \{0\}\}$$

at the moment τ_i equals zero for any $i \in I \backslash \{0\}$. According to relation (6), if the Markov chain initially starts from any state from the set $D(\Gamma^{(1)})$, it moves with a positive probability to the state $(\Gamma^{(2)}, 0, x_m, l_1, 0) \in S$. Following the algorithm $s(\Gamma)$ the chain further comes with a positive probability to the state of the form $(\Gamma^{(2m+1)}, x_1, x_m, y_1, y_m) \in S$, leaving which the Markov chain has the zero probability of moving to any state from the set $D(\Gamma^{(1)})$. Therefore, the states from the set $D(\Gamma^{(1)})$ are transient by definition.

Similarly, based on recurrent relations (6)–(10), it can be derived that the probabilities of being in any state from the sets

$$D(\Gamma^{(2)}) = \{(\Gamma^{(2)}, x_1, x_m, y_1, y_m) \in S: y_m \in Y_m \backslash \{0\}\} \cup$$
$$\cup \{(\Gamma^{(2)}, x_1, x_m, y_1, y_m) \in S: x_1 \in X \backslash \{0\}, y_1 \in Y_1 \backslash \{l_1\}\},$$
$$D(\Gamma^{(k)}) = \{(\Gamma^{(k)}, x_1, x_m, y_1, y_m) \in S: y_1 \in Y_1 \backslash \{0\}\} \cup$$
$$\cup \{(\Gamma^{(k)}, x_1, x_m, y_1, y_m) \in S: y_m \in Y_m \backslash \{0\}\}, \quad k \in \{3,4, \dots, 2m - 2\},$$
$$D(\Gamma^{(2m-1)}) = \{(\Gamma^{(2m-1)}, x_1, x_m, y_1, y_m) \in S: y_1 \in Y_1 \backslash \{0\}\} \cup$$
$$\cup \{(\Gamma^{(2m-1)}, x_1, x_m, y_1, y_m) \in S: y_m \in Y_m \backslash \{0\}\} \cup$$
$$\cup \{(\Gamma^{(2m-1)}, x_1, x_m, y_1, y_m) \in S: x_1 \in \{h_1, h_1 + 1, \dots\}\},$$
$$D(\Gamma^{(2m)}) = \{(\Gamma^{(2m)}, x_1, x_m, y_1, y_m) \in S: y_1 \in Y_1 \backslash \{0\}\} \cup$$
$$\cup \{(\Gamma^{(2m)}, x_1, x_m, y_1, y_m) \in S: x_1 \in \{0,1, \dots, h_1 - 1\}, x_m \in X \backslash \{0\},$$
$$y_m \in Y_m \backslash \{l'_m, l_m\}\} \cup \{(\Gamma^{(2m)}, x_1, x_m, y_1, y_m) \in S: x_1 \in \{h_1, h_1 + 1, \dots\},$$
$$x_m \in X \backslash \{0\}, y_m \in Y_m \backslash \{0, l'_m\}\},$$
$$D(\Gamma^{(2m+1)}) = \{(\Gamma^{(2m+1)}, x_1, x_m, y_1, y_m) \in S: y_1 \in Y_1 \backslash \{0\}\} \cup$$
$$\cup \{(\Gamma^{(2m+1)}, x_1, x_m, y_1, y_m) \in S: x_1 \in \{0,1, \dots, h_1 - 1\}\} \cup$$
$$\cup \{(\Gamma^{(2m+1)}, x_1, x_m, y_1, y_m) \in S: x_1 \in \{h_1, h_1 + 1, \dots\},$$
$$y_m \in \{l'_m + 1, l'_m + 2, \dots, l_m\}\} \cup$$
$$\cup \{(\Gamma^{(2m+1)}, x_1, x_m, y_1, y_m) \in S: x_1 \in \{h_1, h_1 + 1, \dots\},$$
$$x_m \in X \backslash \{0\}, y_m \in \{0,1, \dots, l'_m - 1\}\}$$

at the moment τ_i for $i \in I\backslash\{0\}$ equal zero, which means all of the states mentioned above are transient.

Secondly, consider one more of the subsets of the state space S. According to (5), the following equation takes place for any $x_1, x_m \in X$:

$$Q_{i+1}(\Gamma^{(1)}, x_1, x_m, 0,0) = \sum_{v_1=0}^{x_1} \varphi_1(x_1 - v_1; T_{2m+1}) \times$$
$$\times \sum_{v_m=0}^{x_m} \varphi_m(x_m - v_m; T_{2m+1}) \sum_{w_1=0}^{l_1} \sum_{w_m=0}^{l_m} Q_i(\Gamma^{(2m+1)}, v_1, v_m, w_1, w_m).$$

This means a state of the form $(\Gamma^{(1)}, x_1, x_m, 0,0)$, $x_1 \in \{0,1,\dots,h_1-1\}$, $x_m \in X$, is achievable only from the set $D(\Gamma^{(2m+1)})$, namely from the states $(\Gamma^{(2m+1)}, x_1, x_m, y_1, y_m) \in S$ for any $x_1 \in \{0,1,\dots,h_1 - 1\}$. Therefore, at any moment τ_i, starting from $i = 2$, a probability of being in any state of the set

$$D^*(\Gamma^{(1)}) = \{(\Gamma^{(1)}, x_1, x_m, 0,0) \in S: x_1 \in \{0,1,\dots,h_1-1\}\}$$

is equal to zero and $D^*(\Gamma^{(1)})$ also contains only transient states. In its turn, according to (6), the Markov chain can move with a positive probability to the states of the set

$$D^*(\Gamma^{(2)}) = \{(\Gamma^{(2)}, x_1, x_m, y_1, y_m) \in S: y_1 \in \{0,1,\dots,h_1-1\}\}$$

only from the states of the set $D^*(\Gamma^{(1)})$. This means at any moment τ_i for $i \in I\backslash\{0,1,2\}$ the Markov chain (4) has the zero probabilities of being in states of the set $D^*(\Gamma^{(2)})$. Therefore, $D^*(\Gamma^{(2)})$ is a set of transient states as well. Note that

$$D = \bigcup_{k=1}^{2m+1} D(\Gamma^{(k)}) \cup D^*(\Gamma^{(1)}) \cup D^*(\Gamma^{(2)}).$$

The set D is an open set which contains only transient states of the chain (4).

It can be easily verified that $E = S\backslash D$. Let us show that all states from the set E communicate with each other. At first, consider the state $(\Gamma^{(2m-2)}, 0,0,0,0) \in E$. For any $(\Gamma^{(k)}, x_1, x_m, y_1, y_m) \in E$ let us demonstrate that it is possible to get to this state from $(\Gamma^{(2m-2)}, 0,0,0,0)$ and get back with a positive probability and finite-step transition. Such transition between states will be further illustrated with the help of the arrows directed to the final state. If one-step transition is considered, the arrow will also be marked with the probability of such transition.

1. For any $x_1, x_m \in X$ transition $(\Gamma^{(2m-2)}, x_1, x_m, 0,0) \to (\Gamma^{(2m-2)}, 0,0,0,0)$ may be performed as follows.

1.1. In case $0 \le x_1 < h_1$:

$$(\Gamma^{(2m-2)}, x_1, x_m, 0,0) \xrightarrow{\varphi_1(h_1 - x_1; T_{2m-2})\varphi_m(0;T_{2m-2})} (\Gamma^{(2m)}, h_1, x_m, 0,0) \xrightarrow{\varphi_1(0;T_{2m})\varphi_m(0;T_{2m})}$$
$$\to (\Gamma^{(2m+1)}, h_1, \max\{0, x_m - l'_m\}, 0, \min\{x_m, l'_m\}) \xrightarrow{\varphi_1(0;T_{2m+1})\varphi_m(0;T_{2m+1})}$$
$$\to (\Gamma^{(1)}, h_1, \max\{0, x_m - l'_m\}, 0,0) \xrightarrow{\varphi_1(0;T_1)\varphi_m(0;T_1)}$$
$$\to (\Gamma^{(2)}, 0, \max\{0, x_m - l'_m\}, h_1, 0) \xrightarrow{\varphi_1(0;T_2)\varphi_m(0;T_2)} (\Gamma^{(3)}, 0, \max\{0, x_m - l'_m\}, 0,0) \to$$
$$\to \cdots \xrightarrow{\varphi_1(0;T_{2m-3})\varphi_m(0;T_{2m-3})} (\Gamma^{(2m-2)}, 0, \max\{0, x_m - l'_m\}, 0,0).$$

Such procedure should be repeated $[\frac{x_m}{l'_m}] + 1$ times untill the Markov chain gets to the state $(\Gamma^{(2m-2)}, 0,0,0,0)$.

1.2. In case $x_1 \ge h_1$:

$$(\Gamma^{(2m-2)}, x_1, x_m, 0,0) \xrightarrow{\varphi_1(0;T_{2m-2})\varphi_m(0;T_{2m-2})} (\Gamma^{(2m)}, x_1, x_m, 0,0) \xrightarrow{\varphi_1(0;T_{2m})\varphi_m(0;T_{2m})}$$
$$\to (\Gamma^{(2m+1)}, x_1, \max\{0, x_m - l'_m\}, 0, \min\{x_m, l'_m\}) \to$$
$$\xrightarrow{\varphi_1(0;T_{2m+1})\varphi_m(0;T_{2m+1})} (\Gamma^{(1)}, x_1, \max\{0, x_m - l'_m\}, 0,0) \to$$
$$\xrightarrow{\varphi_1(0;T_1)\varphi_m(0;T_1)} (\Gamma^{(2)}, \max\{0, x_1 - l_1\}, \max\{0, x_m - l'_m\}, \min\{x_1, l_1\}, 0) \to$$
$$\xrightarrow{\varphi_1(0;T_2)\varphi_m(0;T_2)} (\Gamma^{(3)}, \max\{0, x_1 - l_1\}, \max\{0, x_m - l'_m\}, 0,0) \to \cdots \to$$
$$\xrightarrow{\varphi_1(0;T_{2m-3})\varphi_m(0;T_{2m-3})} (\Gamma^{(2m-2)}, \max\{0, x_1 - l_1\}, \max\{0, x_m - l'_m\}, 0,0).$$

Such combination of transitions is repeated $[\frac{x_1}{l_1}]$ times. If any of the inequalities

$$\max\{0, x_1 - [\tfrac{x_1}{l_1}] \times l_1\} > 0$$

$$\max\{0, x_m - [\tfrac{x_1}{l_1}] \times l'_m\} > 0$$

take place, then proceed with 1.1 after performing such combination.

2. Consider the states $(\Gamma^{(2m-1)}, x_1, x_m, 0,0) \in E(\Gamma^{(2m-1)})$ for $x_1 \in \{0,1,\dots,h_1 - 1\}$. The

transition

$$(\Gamma^{(2m-2)},0{,}0{,}0{,}0)\xrightarrow{\varphi_1(x_1;T_{2m-2})\varphi_m(x_m;T_{2m-2})}(\Gamma^{(2m-1)},x_1,x_m,0{,}0)$$

is possible. Backward transition may be as follows:

$$(\Gamma^{(2m-1)},x_1,x_m,0{,}0)\xrightarrow{\varphi_1(h_1-x_1;T_{2m-1})\varphi_m(0;T_{2m-1})}$$
$$\to(\Gamma^{(2m)},h_1,\max\{0,x_m-l_m\},0,\min\{x_m,l_m\})\xrightarrow{\varphi_1(0;T_{2m})\varphi_m(0;T_{2m})}$$
$$\to(\Gamma^{(2m+1)},h_1,\max\{0,x_m-l_m-l'_m\},0,\min\{\max\{0,x_m-l_m\},l'_m\})\to$$
$$\xrightarrow{\varphi_1(0;T_{2m+1})\varphi_m(0;T_{2m+1})}(\Gamma^{(1)},h_1,\max\{0,x_m-l_m-l'_m\},0{,}0)\to$$
$$\xrightarrow{\varphi_1(0;T_1)\varphi_m(0;T_1)}(\Gamma^{(2)},0,\max\{0,x_m-l_m-l'_m\},h_1,0)\to$$
$$\xrightarrow{\varphi_1(0;T_2)\varphi_m(0;T_2)}(\Gamma^{(3)},0,\max\{0,x_m-l_m-l'_m\},0{,}0)\to\cdots\to$$
$$\xrightarrow{\varphi_1(0;T_{2m-3})\varphi_m(0;T_{2m-3})}(\Gamma^{(2m-2)},0,\max\{0,x_m-l_m-l'_m\},0{,}0).$$

If $\max\{0,x_m-l_m-l'_m\}\neq 0$, go to step 1.1 after completing such procedure.

3. Consider states $(\Gamma^{(k)},x_1,x_m,y_1,y_m)\in E(\Gamma^{(2m)})$.

3.1. Let $k=2m$, $x_1\in\{h_1,h_1+1,\dots\}$, $x_m\in X$, $y_1=y_m=0$. Then the transition

$$(\Gamma^{(2m-2)},0{,}0{,}0{,}0)\xrightarrow{\varphi_1(x_1;T_{2m-2})\varphi_m(x_m;T_{2m-2})}(\Gamma^{(2m)},x_1,x_m,0{,}0)$$

takes place. The backward transition starts with

$$(\Gamma^{(2m)},x_1,x_m,0{,}0)\xrightarrow{\varphi_1(0;T_{2m})\varphi_m(0;T_{2m})}$$
$$\to(\Gamma^{(2m+1)},x_1,\max\{0,x_m-l'_m\},0,\min\{x_m,l'_m\})\to$$
$$\xrightarrow{\varphi_1(0;T_{2m+1})\varphi_m(0;T_{2m+1})}(\Gamma^{(1)},x_1,\max\{0,x_m-l'_m\},0{,}0)\to$$
$$\xrightarrow{\varphi_1(0;T_1)\varphi_m(0;T_1)}(\Gamma^{(2)},\max\{0,x_1-l_1\},\max\{0,x_m-l'_m\},\min\{x_1,l_1\},0)\to$$
$$\xrightarrow{\varphi_1(0;T_2)\varphi_m(0;T_2)}(\Gamma^{(3)},\max\{0,x_1-l_1\},\max\{0,x_m-l'_m\},0{,}0)\to\cdots\to$$
$$\xrightarrow{\varphi_1(0;T_{2m-3})\varphi_m(0;T_{2m-3})}(\Gamma^{(2m-2)},\max\{0,x_1-l_1\},\max\{0,x_m-l'_m\},0{,}0).$$

If $\max\{0,x_m-l'_m\}\neq 0$ or $\max\{0,x_1-l_1\}\neq 0$, proceed with step 1.

3.2. Let $k=2m$, $x_1\in X$, $x_m\in X\backslash\{0\}$, $y_1=0$, $y_m=l_m$. Then it is possible to perform the following transition:

$$(\Gamma^{(2m-2)},0{,}0{,}0{,}0)\xrightarrow{\varphi_1(0;T_{2m-2})\varphi_m(x_m+l_m;T_{2m-2})}(\Gamma^{(2m-1)},0,x_m+l_m,0{,}0)\to$$
$$\xrightarrow{\varphi_1(x_1;T_{2m-1})\varphi_m(0;T_{2m-1})}(\Gamma^{(2m)},x_1,x_m,0,l_m).$$

The backward transition starts with

$$(\Gamma^{(2m)},x_1,x_m,0,l_m)\xrightarrow{\varphi_1(\max\{0;h_1-x_1\};T_{2m})\varphi_m(0;T_{2m})}(\Gamma^{(2m)},\max\{x_1,h_1\},x_m,0{,}0),$$

and continues with transition in analogy with case 3.1.

3.3. In case $k=2m$, $x_1\in X$, $x_m=0$, $y_1=0$, $y_m\in Y_m$ the forward transition can be performed as follows:

$$(\Gamma^{(2m-2)},0{,}0{,}0{,}0)\xrightarrow{\varphi_1(0;T_{2m-2})\varphi_m(y_m;T_{2m-2})}(\Gamma^{(2m-1)},0,y_m,0{,}0)\to$$
$$\xrightarrow{\varphi_1(x_1;T_{2m-1})\varphi_m(0;T_{2m-1})}(\Gamma^{(2m)},x_1,0{,}0,y_m).$$

In its turn, the backward transition is

$$(\Gamma^{(2m)},x_1,0{,}0,y_m)\xrightarrow{\varphi_1(\max\{0;h_1-x_1\};T_{2m})\varphi_m(0;T_{2m})}(\Gamma^{(2m)},\max\{x_1,h_1\},0{,}0{,}0),$$

and after that proceed with transition 3.1.

3.4. Let $k=2m$, $x_1\in\{0,1,\dots,h_1-1\}$, $x_m\in X\backslash\{0\}$, $y_1=0$, $y_m=l'_m$. There is a positive probability that the transition

$$(\Gamma^{(2m-2)},0{,}0{,}0{,}0)\xrightarrow{\varphi_1(0;T_{2m-2})\varphi_m(0;T_{2m-2})}(\Gamma^{(2m-1)},0{,}0{,}0{,}0)\to$$
$$\xrightarrow{\varphi_1(x_1;T_{2m-1})\varphi_m(0;T_{2m-1})}(\Gamma^{(2m)},x_1,0{,}0{,}0)\xrightarrow{\varphi_1(0;T_{2m})\varphi_m(x_m+l'_m;T_{2m})}(\Gamma^{(2m)},x_1,x_m,0,l'_m)$$

takes place. The backward transition starts with transition

$$(\Gamma^{(2m)},x_1,x_m,0,l'_m)\xrightarrow{\varphi_1(0;T_{2m})\varphi_m(0;T_{2m})}(\Gamma^{(2m)},x_1,\max\{0,x_m-l'_m\},0,\min\{x_m,l'_m\}),$$

which is repeated $[\frac{x_m}{l'_m}]+1$ times, untill the state $(\Gamma^{(2m)},x_1,0{,}0{,}0)$ is reached. After that, the transition continues with

$$(\Gamma^{(2m)},x_1,0{,}0{,}0)\xrightarrow{\varphi_1(\max\{0;h_1-x_1\};T_{2m})\varphi_m(0;T_{2m})}(\Gamma^{(2m)},\max\{x_1,h_1\},0{,}0{,}0),$$

and, then, proceed with case 3.1.

4. Consider now the states $(\Gamma^{(k)}, x_1, x_m, y_1, y_m) \in E(\Gamma^{(2m+1)})$.

4.1. Let $k = 2m + 1$, $x_1 \in \{h_1, h_1 + 1, \dots\}$, $x_m \in X \backslash \{0\}$, $y_1 = 0$, $y_m = l'_m$. In this case it is possible to perform transition

$$(\Gamma^{(2m-2)}, 0,0,0,0) \xrightarrow{\varphi_1(x_1;T_{2m-2})\varphi_m(x_m+l'_m;T_{2m-2})} (\Gamma^{(2m)}, x_1, x_m + l'_m, 0,0) \to \xrightarrow{\varphi_1(0;T_{2m})\varphi_m(0;T_{2m})} (\Gamma^{(2m+1)}, x_1, x_m, 0, l'_m)$$

and backward transition

$$(\Gamma^{(2m+1)}, x_1, x_m, 0, l'_m) \xrightarrow{\varphi_1(0;T_{2m+1})\varphi_m(0;T_{2m+1})} (\Gamma^{(1)}, x_1, x_m, 0,0) \to \xrightarrow{\varphi_1(0;T_1)\varphi_m(0;T_1)} (\Gamma^{(2)}, \max\{0, x_1 - l_1\}, x_m, \min\{x_1, l_1\},0) \to \xrightarrow{\varphi_1(0;T_2)\varphi_m(0;T_2)} (\Gamma^{(3)}, \max\{0, x_1 - l_1\}, x_m, 0,0) \to \dots \to \xrightarrow{\varphi_1(0;T_{2m-3})\varphi_m(0;T_{2m-3})} (\Gamma^{(2m-2)}, \max\{0, x_1 - l_1\}, x_m, 0,0),$$

which is continued with case 1.

4.2. Let now $k = 2m + 1$, $x_1 \in \{h_1, h_1 + 1, \dots\}$, $x_m = 0$, and also $y_1 = 0$, $y_m \in \{0,1, \dots, l'_m\}$. The forward transition

$$(\Gamma^{(2m-2)}, 0,0,0,0) \xrightarrow{\varphi_1(x_1;T_{2m-2})\varphi_m(y_m;T_{2m-2})} (\Gamma^{(2m)}, x_1, y_m, 0,0) \to \xrightarrow{\varphi_1(0;T_{2m})\varphi_m(0;T_{2m})} (\Gamma^{(2m+1)}, x_1, 0,0, y_m)$$

is possible. The backward transition may, for example, be as follows:

$$(\Gamma^{(2m+1)}, x_1, 0,0, y_m) \xrightarrow{\varphi_1(0;T_{2m+1})\varphi_m(0;T_{2m+1})} (\Gamma^{(1)}, x_1, 0,0,0) \to \xrightarrow{\varphi_1(0;T_1)\varphi_m(0;T_1)} (\Gamma^{(2)}, \max\{0, x_1 - l_1\},0, \min\{x_1, l_1\},0) \to \xrightarrow{\varphi_1(0;T_2)\varphi_m(0;T_2)} (\Gamma^{(3)}, \max\{0, x_1 - l_1\},0,0,0) \to \dots \to \xrightarrow{\varphi_1(0;T_{2m-3})\varphi_m(0;T_{2m-3})} (\Gamma^{(2m-2)}, \max\{0, x_1 - l_1\},0,0,0).$$

In case $\max\{0, x_1 - l_1\} \neq 0$ proceed with transition 1.

5. For any state $(\Gamma^{(k)}, x_1, x_m, y_1, y_m)$ from the set $E(\Gamma^{(1)})$ equalities $k = 1$, $x_1 \in \{h_1, h_1 + 1, \dots\}$, $x_m \in X$, $y_1 = 0$, $y_m = 0$ take place. Therefore, transition

$$(\Gamma^{(2m-2)}, 0,0,0,0) \xrightarrow{\varphi_1(x_1;T_{2m-2})\varphi_m(x_m+l'_m;T_{2m-2})} (\Gamma^{(2m)}, x_1, x_m + l'_m, 0,0) \to \xrightarrow{\varphi_1(0;T_{2m})\varphi_m(0;T_{2m})} (\Gamma^{(2m+1)}, x_1, x_m, 0, l'_m) \xrightarrow{\varphi_1(0;T_{2m+1})\varphi_m(0;T_{2m+1})} (\Gamma^{(1)}, x_1, x_m, 0,0)$$

is possible. The backward transition starts with

$$(\Gamma^{(1)}, x_1, x_m, 0,0) \xrightarrow{\varphi_1(0;T_1)\varphi_m(0;T_1)} (\Gamma^{(2)}, \max\{0, x_1 - l_1\}, x_m, \min\{x_1, l_1\},0) \to \xrightarrow{\varphi_1(0;T_2)\varphi_m(0;T_2)} (\Gamma^{(3)}, \max\{0, x_1 - l_1\}, x_m, 0,0) \to \dots \to \xrightarrow{\varphi_1(0;T_{2m-3})\varphi_m(0;T_{2m-3})} (\Gamma^{(2m-2)}, \max\{0, x_1 - l_1\}, x_m, 0,0).$$

Then, it continues with case 1 if $x_m \neq 0$ or $\max\{0, x_1 - l_1\} \neq 0$.

6. Consider a state of the form $(\Gamma^{(k)}, x_1, x_m, y_1, y_m) \in E(\Gamma^{(2)})$.

6.1. If $k = 2$, $x_1, x_m \in X$, $y_1 = l_1$, $y_m = 0$, the forward transition may be as follows:

$$(\Gamma^{(2m-2)}, 0,0,0,0) \xrightarrow{\varphi_1(h_1;T_{2m-2})\varphi_m(x_m+l'_m;T_{2m-2})} (\Gamma^{(2m)}, h_1, x_m + l'_m, 0,0) \xrightarrow{\varphi_1(0;T_{2m})\varphi_m(0;T_{2m})} (\Gamma^{(2m+1)}, h_1, x_m, 0, l'_m) \to \xrightarrow{\varphi_1(x_1+l_1-h_1;T_{2m+1})\varphi_m(0;T_{2m+1})} (\Gamma^{(1)}, x_1 + l_1, x_m, 0,0) \xrightarrow{\varphi_1(0;T_1)\varphi_m(0;T_1)} (\Gamma^{(2)}, x_1, x_m, l_1, 0).$$

The backward transition contains

$$(\Gamma^{(2)}, x_1, x_m, l_1, 0) \xrightarrow{\varphi_1(0;T_2)\varphi_m(0;T_2)} (\Gamma^{(3)}, x_1, x_m, 0,0) \to \dots \to \xrightarrow{\varphi_1(0;T_{2m-3})\varphi_m(0;T_{2m-3})} (\Gamma^{(2m-2)}, x_1, x_m, 0,0).$$

In case $x_1 \neq 0$ or $x_m \neq 0$, it is then necessary to go on with case 1.

6.2. Consider the case $k = 2$, $x_1 = 0$, $x_m \in X$, $y_1 \in \{h_1, h_1 + 1, \dots, l_1 - 1\}$, $y_m = 0$. The feasible forward transition is

$$(\Gamma^{(2m-2)}, 0,0,0,0) \xrightarrow{\varphi_1(h_1;T_{2m-2})\varphi_m(x_m+l'_m;T_{2m-2})} (\Gamma^{(2m)}, h_1, x_m + l'_m, 0,0) \to \xrightarrow{\varphi_1(0;T_{2m})\varphi_m(0;T_{2m})} (\Gamma^{(2m+1)}, h_1, x_m, 0, l'_m) \to \xrightarrow{\varphi_1(y_1-h_1;T_{2m+1})\varphi_m(0;T_{2m+1})} (\Gamma^{(1)}, y_1, x_m, 0,0) \xrightarrow{\varphi_1(0;T_1)\varphi_m(0;T_1)} (\Gamma^{(2)}, 0, x_m, y_1, 0).$$

And a probable backward transition has the following form:

$$(\Gamma^{(2)},0,x_m,y_1,0)\xrightarrow{\varphi_1(0;T_2)\varphi_m(0;T_2)}(\Gamma^{(3)},0,x_m,0,0)\to\cdots\to$$
$$\xrightarrow{\varphi_1(0;T_{2m-3})\varphi_m(0;T_{2m-3})}(\Gamma^{(2m-2)},0,x_m,0,0).$$

If $x_m\neq 0$, proceed with case 1.

7. Finally, let $(\Gamma^{(k)},x_1,x_m,y_1,y_m)\in E(\Gamma^{(r)})$, $r\in\{3,4,\dots,2m-2\}$. Then $k\in\{3,4,\dots,2m-2\}$, $x_1,x_m\in X$, $y_1=y_m=0$ and the following transition take place:

$$(\Gamma^{(2m-2)},0,0,0,0)\xrightarrow{\varphi_1(h_1;T_{2m-2})\varphi_m(x_m+l'_m;T_{2m-2})}(\Gamma^{(2m)},h_1,x_m+l'_m,0,0)\to$$
$$\xrightarrow{\varphi_1(0;T_{2m})\varphi_m(0;T_{2m})}(\Gamma^{(2m+1)},h_1,x_m,0,l'_m)\to$$
$$\xrightarrow{\varphi_1(x_1+l_1-h_1;T_{2m+1})\varphi_m(0;T_{2m+1})}(\Gamma^{(1)},x_1+l_1,x_m,0,0)\to$$
$$\xrightarrow{\varphi_1(0;T_1)\varphi_m(0;T_1)}(\Gamma^{(2)},x_1,x_m,l_1,0)\xrightarrow{\varphi_1(0;T_2)\varphi_m(0;T_2)}(\Gamma^{(3)},x_1,x_m,0,0)\to$$
$$\to\cdots\xrightarrow{\varphi_1(0;T_{k-1})\varphi_m(0;T_{k-1})}(\Gamma^{(k)},x_1,x_m,0,0)$$

and

$$(\Gamma^{(k)},x_1,x_m,0,0)\xrightarrow{\varphi_1(0;T_k)\varphi_m(0;T_k)}\dots\xrightarrow{\varphi_1(0;T_{2m-3})\varphi_m(0;T_{2m-3})}(\Gamma^{(2m-2)},x_1,x_m,0,0).$$

Note that if $x_1\neq 0$ or $x_m\neq 0$, it is necessary to continue with case 1.

Now it is seen that every two states of the set E communicate with each other – at least, across the state $(\Gamma^{(2m-2)},0,0,0,0)$. Therefore, the set E is an indecomposable class of recurrent communicating states, i. e. a minimal closed set (see [13]). Moreover, this class contains the state $(\Gamma^{(2m)},0,0,0,l'_m)$ for which a loop-transition

$$(\Gamma^{(2m)},0,0,0,l'_m)\xrightarrow{\varphi_1(0;T_{2m})\varphi_m(l'_m;T_{2m})}(\Gamma^{(2m)},0,0,0,l'_m)$$

is possible. Thus, this state has period 1, which means the class E is a class of aperiodic states (see [14]).

5 Ergodic theorem

Theorem 3 *For any initial distribution*

$$\{Q_0(\Gamma^{(k)},x_1,x_m,y_1,y_m):(\Gamma^{(k)},x_1,x_m,y_1,y_m)\in S\}$$

of the multidimensional Markov chain (4) two limiting options are possible: either 1) for any $(\Gamma^{(k)},x_1,x_m,y_1,y_m)\in S$ the limiting equality

$$\lim_{i\to\infty}Q_i(\Gamma^{(k)},x_1,x_m,y_1,y_m)=0$$

takes place and there is no stationary distribution, or 2) the limits

$$\lim_{i\to\infty}Q_i(\Gamma^{(k)},x_1,x_m,y_1,y_m)=Q(\Gamma^{(k)},x_1,x_m,y_1,y_m)$$

exist, where

$$Q(\Gamma^{(k)},x_1,x_m,y_1,y_m)>0\text{for}(\Gamma^{(k)},x_1,x_m,y_1,y_m)\in E,$$

$$Q(\Gamma^{(k)},x_1,x_m,y_1,y_m)=0\text{for}(\Gamma^{(k)},x_1,x_m,y_1,y_m)\in D,$$

equality

$$\sum_{(\Gamma^{(k)},x_1,x_m,y_1,y_m)\in S}Q(\Gamma^{(k)},x_1,x_m,y_1,y_m)=1$$

takes place and there is one and only stationary distribution.

Proof. Since set D is countable, a situation may take place when the Markov chain with an initial distribution given in the set of transient states may walk in this set indefinitely. Demonstrate that such behaviour is not possible for the Markov chain (4). According to notation (3) and relations (5)–(10), the probability

$$P_0^1(\Gamma^{(k)}, x_1, x_m, y_1, y_m) = \mathbf{P}(\chi_1 \in E \mid \chi_0 = (\Gamma^{(k)}, x_1, x_m, y_1, y_m))$$

that the Markov chain (4) moves from $(\Gamma^{(k)}, x_1, x_m, y_1, y_m) \in D$ to any state from the set E is positive. Moreover, for any state $(\Gamma^{(k)}, x_1, x_m, y_1, y_m) \in D$ estimation

$$P_0^1(\Gamma^{(k)}, x_1, x_m, y_1, y_m) > \min\{\varphi_1(h_1; T_1)\varphi_m(0; T_1), \varphi_1(h_1; T_{2m+1})\varphi_m(0; T_{2m+1})\} > 0 \quad (11)$$

takes place. Let $P_0^{å}(\Gamma^{(k)}, x_1, x_m, y_1, y_m)$ be a probability that the chain (4) ever comes to the class E starting from a transient state $(\Gamma^{(k)}, x_1, x_m, y_1, y_m) \in D$, i. e.

$$P_0^{å}(\Gamma^{(k)}, x_1, x_m, y_1, y_m) =$$
$$= \sum_{n=1}^{\infty} \mathbf{P}(\chi_n \in E, \chi_i \in D, i = 0,1, \ldots, n-1 \mid \chi_0 = (\Gamma^{(k)}, x_1, x_m, y_1, y_m)).$$

Then, according to [15] these probabilities meet the system of linear equalities

$$\begin{aligned} &P_0^{å}(\Gamma^{(k)}, x_1, x_m, y_1, y_m) = \sum_{(\Gamma^{(r)}, v_1, v_m, w_1, w_m) \in D} P_0^{å}(\Gamma^{(r)}, v_1, v_m, w_1, w_m) \times \\ &\times \mathbf{P}(\chi_1 = (\Gamma^{(r)}, v_1, v_m, w_1, w_m) \mid \chi_0 = (\Gamma^{(k)}, x_1, x_m, y_1, y_m)) + \\ &+ P_0^1(\Gamma^{(k)}, x_1, x_m, y_1, y_m), \quad (\Gamma^{(k)}, x_1, x_m, y_1, y_m) \in D. \end{aligned} \quad (12)$$

Inequality (11) for any $(\Gamma^{(k)}, x_1, x_m, y_1, y_m) \in D$ allows one to prove estimation

$$\begin{aligned} &\sum_{(\Gamma^{(r)}, v_1, v_m, w_1, w_m) \in D} \mathbf{P}(\chi_1 = (\Gamma^{(r)}, v_1, v_m, w_1, w_m) \mid \chi_0 = (\Gamma^{(k)}, x_1, x_m, y_1, y_m)) = \\ &= 1 - P_0^1(\Gamma^{(k)}, x_1, x_m, y_1, y_m) < \\ &< 1 - \min\{\varphi_1(h_1; T_1)\varphi_m(h_1; T_1), \varphi_1(h_1; T_{2m+1})\varphi_m(h_1; T_{2m+1})\} < 1. \end{aligned}$$

Thus, the system (12) is a completely regular system. Then, according to [16], this system has an only limited solution. It can be easily verified that such solution is $P_0^{å}(\Gamma^{(k)}, x_1, x_m, y_1, y_m) = 1$ for any $(\Gamma^{(k)}, x_1, x_m, y_1, y_m) \in D$. Therefore, the Markov chain leaves the set D of transient states with probability one.

If the initial distribution is given only in the closed set E of recurrent states, the Markov chain (4) becomes an irreducible aperiodic Markov chain. In such case the statement of the theorem follows from the ergodic theorem in [15].

Note that the reasonings above demonstrate a general method for proving similar statements for the systems with scheme in Figure 1. Such method is especially useful if it becomes difficult to determine based on recurrent relations the finite number of steps that a Markov chain needs to leave the set of transient states. However, it was proved in Theorem 2 that in case of the algorithm $s(\Gamma)$ with graph in Figure 2 it is enough to make three steps in order to leave the set D. In other words, for any initial distribution

$$Q_i(\Gamma^{(k)}, x_1, x_m, y_1, y_m) = 0$$

for any $(\Gamma^{(k)}, x_1, x_m, y_1, y_m) \in D$ and $i \in I \backslash \{0,1,2\}$.

6 Conclusion

The multidimensional Markov chain which is a model of controlled queueing systems is researched. The structure of the Markov chain state space is investigated. It is proved that for any initial distribution, the Markov chain leaves the set of transient states for the finite number of steps. Thereby, it is further recommended to chose initial distribution only in set of recurrent states. Ergodic theorem is formulated and proved. The results form the basis for the further investigations concerning conditions of stationary mode existence and synthesis of optimal control.

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Generalization and Extension of Burke Theorem

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Abstract

A simplification of Burke theorem proof [1] and its generalizations for queuing systems and networks are considered. The proof simplification is based on the fact that points in output flow take place in moments when Markov process of customers number in queuing system has jumps down. In such way it is possible to obtain a property of the mutual independence of the flow into disjoint periods of time and to calculate intensity of output flow.

Keywords: an output Poisson flow, the Jackson network, the Burke theorem

1 New proof of Burke theorem

In [1] we prove the following statement: in queung system $M|M|n|\infty$ in stationary state, the output flow has the same distribution as the input flow. Recently, however, interest in the study of flows in queuing systems is increased. Now it is necessary to give a more compact and convenient for generalizations proof of this theorem.

A random sequence of points will be called a Poisson flow with continuously differentiable intensity $\lambda(t),\ t \geq 0$, if the following conditions are satisfied [2, page 12, 13], [3, page 20, 35 -- 37]:

a) the probability of the existence of the point of flow on the time interval $[t, t+h)$ does not depend on the location of the points of the stream up to the time t (this property is called lack of follow-through and expresses the mutual independence of the flow stream into disjoint periods of time);

b) the probability that a flow point appears in the semi-interval $[t, t+h)$ is $\lambda(t)h + o(h)$, $h \to 0$;

c) the probability of occurrence of two or more flow points in the range $[t, t+h)$ is $o(h)$, $h \to 0$.

Let the system $A_n = M|M|n|\infty$ of the Poisson input flow has an intensity $\lambda > 0$, and the service time has an exponential distribution with the parameter $\mu > 0,\ 1 \leq n < \infty$. Denote $P_{k,n}(t)$, $k \geq 0$, distribution of the number of customers in the system at the time t.

Theorem 1. *The output flow in queuing system A_n is Poisson with intensity $a(t) = \sum_{0<k} \mu P_{k,n}(t) min(k,n)$.* Let the output flow $T_n = \{0 \leq t_1 < t_2 < \cdots\}$ be A_n described by a random function $y_n(t)$ equal to the number of points of this stream on the segment $([0, t)$. Denote $x_n(t)$ the number of customers in the system A_n at the time t. It is known that a random process $x_n(t)$ is Markov process (of death and birth of [3, Chapter II, $$]), with each point of the T_n flow corresponding to the time of the jump down process $x_n(t)$. Therefore, the output flow T_n satisfies the condition a). In turn, the condition b) follows from the equalities:

$$P(y_n(t+h) = y_n(t)+1) = \sum_{k=1}^{n} P(y_n(t+h) = y_n(t)+1/x_n(t)=k)P_{k,n}(t) +$$

$$+P(y_n(t+h = y_n(t)+1/x_n(t)>n)\sum_{k>n} P_{k,n}(t) =$$

$$= \sum_{k=1}^{n} P_{k,n}(t)(k\mu h + o_k(h)) + \sum_{k>n} P_{k,n}(t)(n\mu h + o_0(h)) = a(t)\mu h + o(h),$$

where for $h \to 0$ we have $o_k(h)/h \to 0,\ k = 0, \dots, n,\ o_0(h)/h \to 0,$

$$o(h) = \sum_{k=1}^{n} P_{k,n}(t)o_k(h) + \sum_{k>n} P_{k,n}(t)o_0(h),\ o_0(h)/h \to 0.$$

Thus, the output flow T_n satisfies the condition b). Check of condition c) is quite obvious.

Theorem 2. *In queuing system A_n, when the ergodicity condition $\lambda < \mu$ is satisfied and the process $x_n(t)$ is stationary, the output flow is Poisson with intensity λ.* Due to the steady state of the process $x_n(t)$ the distribution of the number of customers $P_{k,n}(t) \equiv P_{k,n}$ in the queuing system A_n at $\rho = \lambda/\mu$ satisfies the equations [3, page 93]:

$$P_{k,n} = P_{0,n}k!\,\rho^k,\ 0 \le k \le n,\ P_{k,n} = P_{0,n}n^n n!\,(\rho n)^k,\ k > n,$$

$$P_{0,n}^{-1} = \sum_{k=0}^{n} 1k!\,\rho^k + \sum_{k>n} n^n n!\,(\rho n)^k.$$

It is not difficult to obtain the following equality by simple algebraic calculations: $a(t) \equiv \sum_{k\ge 0} \mu P_{k,n}\min(k,n) = \mu\rho = \lambda$. Therefore, Theorem 1 follows the validity of Theorem 2.

Remark. Using the scheme of the proof of Theorem 2, it is possible to extend the results to output flows of systems with limited queue, with priority service, with unreliable servers [3, $$].

Theorem 3. *When the $\lambda > n\mu$ inequality is executed, the output flow T_n in the system A_n is Poisson with intensity $a(t) \to n\mu,\ t \to \infty$.* We introduce independent random variables $u_n(t),\ v_n(t)$, having Poisson distributions with parameters $\lambda t,\ \mu t$, respectively. It is easy to establish that with probability unit the inequality $x_n(t) \ge w_n(t) = u_n(t) - v_n(t)$ is valid, hence the following inequalities are fulfilled:

$$P(x_n(t) \le n) \le P(w_n(t) \le n) = P(w_n(t) - Mw_n(t) \le n - Mw_n(t)) =$$

$$= P(Mw_n(t) - w_n(t) \ge Mw_n(t) - n) \le P(|Mw_n(t) - w_n(t)| \ge Mw_n(t) - n),$$

where $Mw_n(t) = (\lambda-\mu)t > 0,\ t > 0,\ Dw_n(t) = (\lambda+\mu)t$. Assume that $(\lambda-\mu)t - n > 0$, then, due to Chebyshev's inequality, we have:

$$P(|Mw_n(t) - w_n(t)| \ge Mw_n(t) - n) \le Dw_n(t)(Mw_n(t) - n)^2 = (\lambda+\mu)t((\lambda-\mu)t - n)^2 \to 0,$$
$t \to \infty$.

Finally we get the ratio:

$$P(x_n(t) \le n) \le (\lambda+\mu)t((\lambda-\mu)t - n)^2 \to 0,\ t \to \infty,$$

and then $\sum_{0<k\le n} P_{k,n}(t) \to 0,\ t \to \infty$. It follows that the limit ratio is met:

$$a(t) = \sum_{0<k\le n} P_{k,n}(t)k\mu + \left(1 - \sum_{0<k\le n} P_{k,n}(t)\right)n\mu \to n\mu,\ t \to \infty.$$

From Theorem 1 and the last relation we obtain the statement of Theorem 3.

2 Poisson flows in stationary queuing networks

Consider an open queuing network (Jackson network) S with a Poisson input flow of intensity λ_0, consisting of a finite number of nodes $k = 0,1,\dots,m$ with exponentially distributed service times. The dynamics of the movement of customers in the network is set by the route matrix $\Theta = ||\theta_{i,j}||_{i,j=0}^{m}$, where $\theta_{i,j}$ is the probability of customer transition after service in the i-th node to j-th node, $\theta_{0,0} = 0$, where the node 0 is an external source and at the same time a drain for customers leaving the network. The i node contains $l_i < \infty$ servers, the service time of which has an exponential distribution with the parameter $\mu_i,\ i = 1,\dots,m$.

Assume that route matrix $\Theta = ||\theta_{i,j}||_{i,j=0}^{m}$ is indecomposable, i.e.

$$\forall\ i,\ j \in \{0,\dots,m\}\ \exists\ i_1,\dots,i_r \in \{0,\dots,m\}:\ \theta_{i,i_1} > 0, \theta_{i_1,i_2} > 0, \dots, \theta_{i_r,j} > 0.$$

Then for a fixed $\lambda_0 > 0$, the system of linear algebraic equations for intensities of fluxes coming from nodes of S

$$\lambda_k = \lambda_0\theta_{0,k} + \sum_{t=1}^{m} \lambda_t\theta_{t,k},\ k = 1,\dots,m \tag{1}$$

has the only solution $(\lambda_1,\dots,\lambda_m)$ with $\lambda_1 > 0,\dots,\lambda_m > 0$, [4, p. 13].

The system (1) is called the system of balance relations and plays an important role in the formulation and the proof of the product Jackson theorem [5], widely used in queuing theory. If $\lambda_i < l_i\mu_i,\ i = 1,\dots,m$, then the discrete Markov process $(n_1(t),\dots,n_m(t)),\ t \geq 0$, describing the number of customers in the network nodes has a limiting distribution $P_S(k_1,\dots,k_m)$, independent of initial conditions and representable in the form $P_S(k_1,\dots,k_m) = \prod_{i=1}^{m} P_i(k_i)$, where $P_i(k_i)$ is the limiting distribution of the number of customers in a stand-alone l_i - channel queuing system with Poisson input flow of intensity $\lambda_i,\ i = 1,\dots,m$.

In [6] network S is mapped to a directed graph G with edges corresponding to positive elements of the route matrix. Let's call the vertex set $U \subseteq \{0,1,\dots,m\}$ irrevocable if from any node not included in U, there is no edge to the node belonging to U. Then all flows passing through the edges from the node set U to the node set $\{0,1,\dots,m\}\backslash U$, are independent and Poisson.

Theorem 4. *Flow T_S^i, coming out of node i of open queuing network S, with stationary process $(n_1(t),\dots,n_m(t)),\ t \geq 0$, is Poisson with intensity $\lambda_i,\ i = 1,\dots,m$.* Indeed, the points of the flow T_S^i, exiting the i, node are the moments of jumps down the $n_i(t)$ component of the discrete Markov process $(n_1(t),\dots,n_m(t),\ t \geq 0$. Hence the flow T_S^i satisfies the condition a). Conditions b), c) are checked similarly to the proof of Theorem 1. Note that the limit probability that the i node contains k_i of customers is $P_i(k_i)$, and the flow rate T_S^i is $\lambda_i,\ i = 1,\dots,m$.

Theorem 5. *Flows $T_S^i,\ i = 1,\dots,m$, are independent.* From Theorem 4 and independence of stationary random variables $n_j(t),\ j = 1,\dots,m$, it follows that the union $T_S = \cup_{j=1}^{m} t_S^j$ of flows leaving the nodes of open queuing network S is also Poisson flow with intensity $\lambda_\Sigma = \sum_{j=1}^{m} \lambda_j$. And each point of the combined flow T_S belongs to the flow T_S^i with probability $p_i = \lambda_i\lambda_\Sigma$.

Lemma 1. *Let $\Lambda = \{0 \leq t_1 \leq t_2 \leq \cdots\}$ is a Poisson flow of intensity λ_Σ, each point of which, regardless of other points with probability p_i becomes a flow Λ_i point, $i = 1,\dots,m$. Then flows $\Lambda_1,\dots,\Lambda_m$ are Poisson with intensities $\lambda_\Sigma p_1,\dots,\lambda_\Sigma p_m$ and independent.* Without limitation of generality it is enough to limit ourselves to the case of $m = 2$. Take an arbitrary segment $[t, t+T],\ 0 \leq t,\ 0 < T$ and denote $n,\ n_1,\ n_2$ the number of flow points $\Lambda,\ \Lambda_1,\ \Lambda_2$ on this segment, respectively. Calculate the probability

$$\begin{aligned} P(n_1 = k_1, \quad n_2 = k_2) &= P(n = k_1 + k_2)C_{k_1+k_2}^{k_1} p_1^{k_1} p_2^{k_2} \\ &= e^{-\lambda T}(\lambda T)^{k_1+k_2}(k_1+k_2)!\,(k_1+k_2)!\,k_1!\,k_2!\,p_1^{k_1}p_2^{k_2} = \\ &= e^{-\lambda T p_1}(\lambda T p_1)^{k_1}k_1! \cdot e^{-\lambda T p_2}(\lambda T p_2)^{k_2}k_2! = P(n_1 = k_1) \cdot P(n_2 = k_2). \end{aligned} \tag{2}$$

Let now segments $[t^{(1)}, t^{(1)} + T^{(1)}],\dots,[t^{(k)}, t^{(k)} + T^{(k)}]$ of the time axis don't intersect. Similarly to (2) we prove the independence of the random vectors $(n_1^{(1)},\dots,n_1^{(k)})$, $(n_2^{(1)},\dots,n_2^{(k)})$ regarding the respective intervals, and independence of their components. Hence the flows $T_S^i,\ i = 1,\dots,m$, are independent.

Remark. *Theorems 4, 5 enhance the results of the article [6], removing restrictions on the independent Poisson flows considered in it.*

Consider now a closed queueing network $\overline{S}$, consisting of a finite number of nodes $i = 1,\dots,m$. The i node contains $l_i < \infty$ servers, the service time on which has an exponential distribution with the parameter $\mu_i,\ i = 1,\dots,m$. A finite number N of customers move along network $\overline{S}$. The dynamics of the customers movement in the network is specified by the matrix $\overline{\Theta} = ||\overline{\theta}_{i,j}||_{i,j=1}^{m}$, where $\overline{\theta}_{i,j}$ is the probability of transition after service of customer in the ith node to j-th one.

Let the route matrix $\overline{\Theta}$ be indecomposable, i.e.

$$\forall\ i,\ j \in \{1,\dots,m\}\ \exists\ i_1,\dots,i_r \in \{1,\dots,m\}\text{: } \overline{\theta}_{i,i_1} > 0, \overline{\theta}_{i_1,i_2} > 0,\dots,\overline{\theta}_{i_r,j} > 0.$$

Then for a fixed $\lambda_1 > 0$, the system of linear algebraic equations

$$\lambda_k = \sum_{t=1}^{m} \lambda_t\overline{\theta}_{t,k},\ k = 1,\dots,m \tag{3}$$

has a unique solution of $(\lambda_1, \ldots, \lambda_m)$ with $\lambda_1 > 0, \ldots, \lambda_m > 0$, [4, p. 13].

For a closed queueing network $\overline{S}$ with N customers discrete Markov process $(\overline{n}_1(t), \ldots, \overline{n}_m(t)),\ t \geq 0$, describing the number of customers in the network nodes has a limit distribution of $P_{\overline{S}}(k_1, \ldots, k_m)$, independent of the initial conditions and presented in the form

$$P_{\overline{S}}(k_1, \ldots, k_m) = \prod_{i=1}^{m} P_i(k_i) \sum_{k_1, \ldots, k_m:\ k_1+\cdots+k_m=N} \prod_{i=1}^{m} P_i(k_i),\ k_1 + \cdots + k_m = N.$$

Hence, the stationary probability $\pi_i(k_i)$ that in a node i of the network $\overline{S}$ there is k_i customers satisfies the equality

$$\pi_i(k_i) = \sum_{k_j, 1 \leq j \neq i \leq m, \sum_{1 \leq j \neq i \leq m} k_j = N - k_i} P_{\overline{S}}(k_1, \ldots, k_m),\ k_i = 0, \ldots, N.$$

Theorem 6. *The flow $T_{\overline{S}}^{i}$, leaving the i node of the closed queueing network $\overline{S}$ with the total number of customers N, being in a stationary state, is Poisson with intensity $\sum_{k_i=1}^{N} min(k_i, l_i)\mu_i \pi_i(k_i),\ i = 1, \ldots, m$.*

Indeed, the points of the flow $T_{\overline{S}}^{i}$, exiting the node i, are the moments of jumps down the components $\overline{n}_i(t)$ of the discrete Markov process $(\overline{n}_1(t), \ldots, \overline{n}_m(t)),\ t \geq 0$. Consequently, the flow $t_{\overline{S}}^{i}$ satisfies condition a). Conditions b), c) are proved similarly to the proof of Theorem 1.

Example. Theorem 6 allows us to consider flows in systems backup with recovery. For example, for the simplest $\overline{S}$ system, a restore reservation consisting of one workstation (work phase 1), one repair location (repair phase 2), and one item. Let the random time before the failure of the item in the workplace has an exponential distribution with the parameter α, the random time to restore the item in the repair location has an exponential distribution with the parameter β. Denote $n_1(t)$ the number of elements in the working phase, $n_2(t)$ - the number of elements in the repair phase of $\overline{s}$. Then $n_1(1) = 0{,}1,\ n_2(t) = 0{,}1,\ n_1(t) + n_2(t) = n = 1$, and therefore the route matrix of the system $\overline{on}$ has the form $\overline{\theta} = ||\overline{\theta}_{i,j}||_{i,j=0}^{1}$, where $\overline{\theta}_{1,1} = \overline{\theta}_{2,2} = 0,\ \overline{\theta}_{1,2} = \overline{\theta}_{2,1} = 1$. Then equalities are just

$$P_{\overline{S}}(1{,}0) = \beta\alpha + \beta,\ P_{\overline{S}}(0{,}1) = \alpha\alpha + \beta,\ P_{\overline{S}}(0{,}0) = P_{\overline{S}}(1{,}1) = 0,$$
$$\pi_1 = \beta\alpha + \beta,\ \pi_2 = \alpha\alpha + \beta,\ \nu_1 = \nu_2 = \alpha\beta\alpha + \beta,$$

where $\nu_1,\ \nu_2$ are the intensities of stationary Poisson flows leaving phases 1, 2, respectively.

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Stability of Discrete Multi-Server Queueing Systems with Heterogeneous Servers, Interruptions and Regenerative Input Flow

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Abstract

In the paper we study a discrete-time multichannel queueing system with heterogeneous servers, regenerative input flow, and interruptions. The breakdowns of servers may occur at any time even if they are not occupied by customers. Consecutive moments of breakdowns are defined by a renewal process, but we do not assume blocked and available periods to be independent. We consider the preemptive repeat different service discipline as well as preemptive resume service discipline. We exploit the regeneration property of the input flow and renewal structure of the processes describing the servers' breakdowns to organise synchronisation of the input and service flows. This approach helps to establish the necessary and sufficient stability condition of the system. Generally, for preemptive repeat different service discipline this stability condition can not be expressed it terms of moments of service and interruption processes. Therefore, we derive the sufficient but not necessary condition, which can be expressed through these moments, and show that it coincides with condition obtained in existing literature for simpler queueing systems.

Keywords: Multichannel system, Regenerative input flow, Ergodicity, Interruption, Vacation, Unreliable servers

1 Introduction

Queueing systems with unavailable servers can be a useful abstraction in modelling of some real-life service operation. Such models may arise naturally as models of many computer, communication and manufacturing systems. Servers interruptions may result from resource sharing, server breakdowns, priority assignment, vacations, some external events, and others. For instance, if we concern the system with customers priority, the service of secondary customers is equivalent to the service interruption of the primary customers during this period.

Systems with unreliable servers have been intensively investigated for a long time. The main point was focused on the single-server case. There are some review papers, that cover most of the literature in these sphere. Some of the important papers on the single-server case are presented in [12]. Concerning systems with servers vacations it should be mentioned [10] and [17].

The most exhaustive literature survey on systems with interruptions is in [19], where non-Markovian multichannel systems were also covered. The are some other articles with extensive literature survey as well [24], [22].

Synchronization method combined with the regenerative theory is one of the powerful approach to obtain stability results for multichannel systems with discrete-time and service interruptions. Basing on this method the multichannel queueing system with identical service distribution function for all servers, renewal input flow, alternating renewal-type servers' interruptions in the discrete-time case was considered in [22]. Authors established some sufficient conditions of stability for the preemptive repeat different and preemptive resume service disciplines. In paper [2] this approach helps to implement asymptotic analysis of the single-server system with a regenerative input flow. The similar approach was applied in [27] to study the stability condition of the multichannel system with heterogeneous servers and a regenerative input flow in a random environment, which breaks all the servers simultaneously.

In this paper we consider a discrete-time queueing system with regenerative input flow and heterogeneous servers that may suffer independent interruptions. Consecutive moments of breakdowns are defined by a renewal process and do not depend on the system state. The preemptive repeat different service discipline as well as preemptive resume service discipline [14] are considered. The former case implies that the service is repeated from the beginning with different independent service time after restoration of the server. In the latter case the service of a customer is continued after restoration. The necessary and sufficient stability condition is established. The key element of our analysis is synchronization of the processes under the consideration. This method is based on the regeneration property of the input flow and renewal structure of the processes describing the servers' breakdowns (see, e.g [2]).

Let us also mention the fluid approximation approach to the stability analysis of queueing systems [8], [7], [9]. See also [13] for a survey of various approaches to stability of queueing systems with a focus on the fluid approach. Nevertheless, in the present paper we do not rely on fluid approximation since ergodic conditions cannot be expressed in terms of expected values for preemptive repeat different service discipline and regenerative method turns out to be suitable to obtain complete and transparent proofs.

The model under consideration is similar to the system that was investigated in [22]. However, there is essential generalisation that lead us to consider different processes. Firstly, we employ the regenerative flow as an input flow. Secondly, service distribution functions may differ for different servers. In this paper the necessary and sufficient condition for stability of the queue-length process is established, whereas in [22] only sufficient conditions for these service disciplines are proved.

The article is organised as follows. In the next section the model is described in detail. In the third section auxiliary service flows are introduced and the traffic rate is defined. In the next two sections we conduct the synchronization of the input and service flows. The sections 6-8 are devoted to the (in)stability problem. In the ninth section we provide some comments and make conclusion in the final section.

2 Model description

We consider a system with m heterogeneous servers and a common queue. Service times of customers by the ith server constitute a sequence $\{\eta_{i,n}\}_{n=1}^{\infty}$ of independent identically distributed (iid) random variables that does not depend on input flow and service times by other servers. Let $B_i(t)$ be a distribution function (d.f.) of $\eta_{i,n}$ and $b_i = \mathbf{E}\eta_{i,n} < \infty$ $(i = \overline{1,m})$. We assume that the servers may be unavailable for service from time to time. The breakdowns of the servers may occur at any time even if they are not occupied by customers. Let $\{s_{i,n}^{(2)}\}_{n=0}^{\infty}$ be moments of breakdowns and $\{s_{i,n}^{(1)}\}_{n=1}^{\infty}$ be moments of restoration for the ith server. Here $0 = s_{i,0}^{(2)} < s_{i,1}^{(1)} < s_{i,1}^{(2)} < s_{i,2}^{(1)}$ .... Then

$u_{i,n}^{(1)} = s_{i,n}^{(1)} - s_{i,n-1}^{(2)}$ and $u_{i,n}^{(2)} = s_{i,n}^{(2)} - s_{i,n}^{(1)}$ denote the length of the nth blocked and nth available period of the ith server respectively ($i = \overline{1,m}$). The sequence $\{u_{i,n}^{(1)}, u_{i,n}^{(2)}\}_{n=1}^{\infty}$ consists of iid random vectors (for all $i = \overline{1,m}$) that do not depend on the input flow and service times. However, for each n and i,random variables $u_{i,n}^{(1)}$ and $u_{i,n}^{(2)}$ are not assumed to be independent. Let $u_{i,n} = u_{i,n}^{(1)} + u_{i,n}^{(2)}$ be the length of the nth cycle for server i. A cycle consists of a blocked period followed by an available period. We assume that $\mathbf{E}u_{i,n}^{(1)} = a_i^{(1)} < \infty$, $\mathbf{E}u_{i,n}^{(2)} = a_i^{(2)} < \infty$, $a_i = a_i^{(1)} + a_i^{(2)}$ $(i = \overline{1,m})$. Server is free if it is neither serving a customer nor interrupted. If server becomes free and there are customers in the queue, a new customer enters the server. It is possible that more than one server becomes free simultaneously. Then customer in the queue chooses an idle server according to some algorithm, possibly random. For definiteness we assume that a customer chooses a free server with the least number. It is possible that an unavailable period starts while a customer is receiving service. Then service of the customer is immediately interrupted. There are various disciplines for continuation of the service after server restoration [14]. Here we consider the preemptive repeat different service discipline (D_1) and preemptive resume service discipline (D_2). In the former case service is repeated from the start and the service time after restoration is independent of the original service time. In the latter case service continues after restoration. In the both cases, customers remain with the same server until service completion. For the service discipline D_1 in order to ensure the service process for the ith server we have to assume that

$$\mathbf{P}(\eta_{i,1} \leq u_{i,1}^{(2)}) > 0 \quad \text{forall} \quad i = \overline{1,m}. \tag{1}$$

If this condition does not hold for some server i, then for discipline D_1 the ith server has to be excluded since it is always busy by service of the single customer. Without loss of generality in the rest of the paper we also assume

$$\mathbf{P}(\eta_{i,1} = 0) = 0 \quad \text{and} \quad \mathbf{P}(u_{i,1}^{(2)} = 0) = 0 \quad \text{forall} \quad i = \overline{1,m}. \tag{2}$$

We consider a discrete-time system, i.e. time is divided into fixed length intervals or slots and all arrivals, departures, interruptions (restorations) are synchronized with respect to slot boundaries. Moreover, in the case of synchronization of some events at one slot these events are ordered as follows: arrival, departure, and interruption (restoration). System is observed at the end of a slot, when all events of the slot are realized.

We assume that the input flow $X(t)$ is a regenerative one [2]. Suppose an integer-valued stochastic process $\{X(t), t \geq 0\}$ is defined on some probability space $(\Omega, \mathcal{F}, P)$ and $X(t)$ has nondecreasing right-continuous sample paths and $X(0) = 0$. Assume that there exists a filtration $\{\mathcal{F}_{\leq t}^X\}_{t\geq 0}$, ($\mathcal{F}_{\leq t}^X \subseteq \mathcal{F}$ for all $t \geq 0$) such that $X(t)$ is measurable with respect to $\{\mathcal{F}_{\leq t}^X\}_{t\geq 0}$.

Definition 1 *The stochastic flow $X(t)$ is called regenerative if there is an increasing sequence of Markov moments $\{\theta_i,\ i \geq 0\}, \theta_0 = 0$ (with respect to $\{\mathcal{F}_{\leq t}^X\}_{t\geq 0}$) such that the sequence*

$$\{\grave{u}_i\}_{i=1}^{\infty} = \{X(\theta_{i-1} + t) - X(\theta_{i-1}), \theta_i - \theta_{i-1}, t \in (0, \theta_i - \theta_{i-1}]\}_{i=1}^{\infty}$$

consists of independent identically distributed random elements on $(\Omega, \mathcal{F}, P)$.

The random variable θ_i is said to be the ith regeneration point of $X(t)$ and $\tau_i = \theta_i - \theta_{i-1}$ is the ith regeneration period ($i = 1,2, \ldots$). Let $\xi_i = X(\theta_i) - X(\theta_{i-1})$ be the number of arrived customers during the ith regeneration period. Assume that $E\tau_1 < \infty$, $E\xi_1 < \infty$. The limit $\lambda_X = \lim_{t\to\infty} \frac{X(t)}{t}$ with probability one (w.p.1) is called the *intensity* of $X(t)$. It is easy to prove that $\lambda_X = \frac{E\xi_1}{E\tau_1}$ (e.g., see [2]). Class of regenerative flows contains most of fundamental flows that are exploited in the queueing theory [4]. Firstly, the doubly stochastic Poisson process [15], where random intensity is a regenerative process [25]. There are many other examples of regenerative flows, for instance, semi-markovian, Markov-modulated, Markov-arrival, and other processes [1]. Important properties of regenerative flows are given in [2].

3 Auxiliary processes

In this section we define auxiliary processes $Y_i^{(d)}(t)$ $(i=\overline{1,m},\ d=1,2)$ that will be used later. Here $d=1$ for the service discipline D_1 and $d=2$ for the service discipline D_2. We think of $Y_i^{(d)}(t)$ $(i=\overline{1,m})$ as the number of customers, that can be served by the ith server during its available period within interval $[0,t]$, if there are always customers for service. In order to construct the processes $Y_i^{(1)}(t)$ we introduce the collection $\left\{\{\eta_{i,n}^{(j)}\}_{n=1}^{\infty}\right\}_{j=1}^{m}$ of independent sequences $\{\eta_{i,n}^{(j)}\}_{n=1}^{\infty}$ consisting of iid random variables with d.f. $B_i(x)$. Let $K_{i,j}(t)$ be the counting process associated with the sequence $\{\eta_{i,n}^{(j)}\}_{n=1}^{\infty}$, i.e. $K_{i,j}(t)=\max\{k: \sum_{n=1}^{k}\eta_{i,n}^{(j)}\le t\}$ $(K_{i,j}(0)=0)$ and $N_i(t)$ be the number of cycles for the ith server during $[0,t]$, i.e.

$$N_i(t)=\max\{j: \textstyle\sum_{n=1}^{j} u_{i,n}\le t\}\quad (N_i(0)=0). \tag{3}$$

Then the processes $Y_i^{(1)}(t)$ and $Y_i^{(2)}(t)$ are defined by the relations

$$Y_i^{(1)}(t)=\sum_{j=1}^{N_i(t)} K_{i,j}\big(u_{i,j}^{(2)}\big)+K_{i,N_i(t)+1}\left(\max\left[0,t-s_{i,N_i(t)+1}^{(1)}\right]\right). \tag{4}$$

$$Y_i^{(2)}(t)=K_{i,1}\left(\sum_{j=1}^{N_i(t)} u_{i,j}^{(2)}+\max\left[0,t-s_{i,N_i(t)+1}^{(1)}\right]\right). \tag{5}$$

By $H_i(t)$ denote the renewal function for $K_{i,j}(t)$, i.e. $H_i(t)=\mathbf{E}K_{i,j}(t)$.

Lemma 1 *There exist the limits*

$$\lim_{t\to\infty}\frac{Y_i^{(1)}(t)}{t}=\frac{\mathbf{E}H_i(u_{i,n}^{(2)})}{a_i}=\lambda_{Y_i^{(1)}}\ \text{ w.p.}\,1,$$
$$\lim_{t\to\infty}\frac{Y_i^{(2)}(t)}{t}=\frac{a_i^{(2)}}{b_i a_i}=\lambda_{Y_i^{(2)}}\ \text{ w.p.}\,1.$$

Proof. We start with the discipline D_1. Let $g_i^{(1)}(n)=\sum_{j=1}^{n} K_{i,j}(u_{i,j}^{(2)})$. From (4) we get the inequalities

$$g_i^{(1)}(N_i(t))\le Y_i^{(1)}(t)\le g_i^{(1)}(N_i(t)+1). \tag{6}$$

Since $\{K_{i,j}(u_{i,j}^{(2)})\}_{j\ge 1}$ is a sequence of iid random variables with a finite mean by the strong law of large numbers (SLLN) we have $n^{-1}g_i^{(1)}(n)\xrightarrow[n\to\infty]{}\mathbf{E}H_i(u_{i,j}^{(2)})$ w.p.1. In view of independence $\{\{\eta_{i,n}^{(j)}\}_{n=1}^{\infty}\}_{j=1}^{m}$ and $\{u_{i,n}^{(1)},u_{i,n}^{(2)}\}_{n=1}^{\infty}$ one can obtain the convergence

$$\frac{g_i^{(1)}(N_i(t))}{N_i(t)}\xrightarrow[t\to\infty]{}\mathbf{E}H_i(u_{i,1}^{(2)})\quad \text{w.p.}\,1.$$

It follows from the renewal theory that

$$t^{-1}N_i(t)\xrightarrow[t\to\infty]{}a_i^{-1}\quad \text{w.p.}\,1. \tag{7}$$

Now the proof of the lemma for D_1 follows from (6).

Consider the discipline D_2. Let $g_i^{(2)}(n)=K_{i,1}(\sum_{j=1}^{n} u_{i,j}^{(2)})$. From (5) we have the inequalities

$$g_i^{(2)}(N_i(t))\le Y_i^{(2)}(t)\le g_i^{(2)}(N_i(t)+1). \tag{8}$$

Since $t^{-1}K_{i,1}(t)\xrightarrow[t\to\infty]{}b_i^{-1}$ w.p.1, we get $n^{-1}g_i^{(2)}(n)\xrightarrow[n\to\infty]{}b_i^{-1}a_i^{(2)}$ w.p.1. Thus (7) and (8) conclude the proof for the discipline D_2. +

Let $Y^{(d)}(t)=\sum_{i=1}^{m} Y_i^{(d)}(t)$ $(d=1,2)$. From Lemma 1 we have

$$\lambda_{Y^{(d)}}=\lim_{t\to\infty}\frac{Y^{(d)}(t)}{t}=\sum_{i=1}^{m}\lambda_{Y_i^{(d)}}\quad \text{w.p.}\,1\quad (d=1,2). \tag{9}$$

We think of λ_X and $\lambda_{Y^{(d)}}$ as the arrival and service rate respectively. Intuitively, it is clear that traffic rates $\rho^{(d)}$ of the system have to be determined as

$$\rho^{(1)}=\frac{\lambda_X}{\sum_{i=1}^{m}\frac{\mathbf{E}H_i(u_{i,n}^{(2)})}{a_i}},\qquad \rho^{(2)}=\frac{\lambda_X}{\sum_{i=1}^{m}\frac{a_i^{(2)}}{b_i a_i}}. \tag{10}$$

At first sight the traffic rate $\rho^{(1)}$ for discipline D_1 can not be expressed in terms of the first moments of random variables defining the model. It is true if we consider the means of service times, available and block periods only. However, we may introduce random variables $\zeta_i^{(n)}$ which is the number of served customers by the ith server during the nth cycle $(i = \overline{1,m}, n = 1,2,\dots)$ under condition that there are always customers on the server. Putting $\alpha_i = \mathbf{E}\xi_i^{(n)}$ we get from (10)

$$\rho^{(1)} = \lambda_X [\textstyle\sum_{i=1}^m \frac{\alpha_i}{a_i}]^{-1}.$$

In applications one may estimate these parameters basing on statistical data.

4 Synchronization of regenerative flows

First we obtain the result concerning synchronization of general regenerative aperiodic flows in a discrete-time case. Let $Z_1(t)$ and $Z_2(t)$ be independent regenerative flows with regeneration points $\{\theta_{1,j}\}_{j=1}^\infty$ and $\{\theta_{2,j}\}_{j=1}^\infty$ respectively $(\theta_{i,0} = 0; i = 1,2)$. As usually, aperiodicity means that the greatest common divisor (GCD)

$$GCD\{k: \mathbf{P}(\theta_{i,1} = k) > 0\} = 1, \quad i = 1,2. \tag{11}$$

Define common points of regeneration for $Z_1(t)$ and $Z_2(t)$ by the relation

$$T_k = \min\{\theta_{1,j} > T_{k-1}: \textstyle\bigcup_{l=1}^\infty \{\theta_{2,l} = \theta_{1,j}\}\}, \quad T_0 = 0. \tag{12}$$

Lemma 2 *Let condition (11) be fulfilled and $\mathbf{E}(\theta_{i,1}) < \infty$ $(i = 1,2)$. Then the sequence $\{T_k\}_{k=1}^\infty$ consists of regeneration points for $Z_1(t)$ and $Z_2(t)$ and*

$$\mathbf{E}T_1 = \mathbf{E}\theta_{1,1} \cdot \mathbf{E}\theta_{2,1} < \infty. \tag{13}$$

Proof. The first statement of the Lemma follows from the definition of T_k. To prove the second statement we put

$$\nu_k = \min\{j > \nu_{k-1}: \textstyle\bigcup_{l=1}^\infty \{\theta_{1,j} = \theta_{2,l}\}\}, \quad \nu_0 = 0,$$

so that $T_k = \theta_{1,\nu_k}$. Then $\{\nu_k - \nu_{k-1}\}_{k=1}^\infty$ is a sequence of iid random variables. In accordance with Wald's identity [11] we get $\mathbf{E}T_1 = \mathbf{E}\theta_{1,1} \cdot \mathbf{E}\nu_1$. Therefore, we need to prove the finiteness of $\mathbf{E}\nu_1$. Let $h_2(t)$ $(h(t))$ be the mean of the number of renewals at time t for the renewal process $\{\theta_{2,n}\}_{n=1}^\infty$ $(\{\nu_k\}_{k=1}^\infty)$, so that $h_2(t) = \sum_{l=0}^\infty \mathbf{P}(\theta_{2,l} = t)$ and $h(t) = \sum_{k=0}^\infty \mathbf{P}(\nu_k = t)$. Taking into account (11) from Blackwell's theorem [26] we get

$$h_2(t) \underset{t\to\infty}{\longrightarrow} \frac{1}{\mathbf{E}\theta_{2,1}}, \quad h(t) \underset{t\to\infty}{\longrightarrow} \frac{1}{\mathbf{E}\nu_1}. \tag{14}$$

In view of independence $Z_1(t)$ and $Z_2(t)$

$$h(j) = \mathbf{P}\{\textstyle\bigcup_{l=0}^\infty \{\theta_{1,j} = \theta_{2,l}\}\} = \mathbf{E}\left(\sum_{l=0}^\infty \mathbf{P}\{\theta_{1,j} = \theta_{2,l} | \theta_{1,j}\}\right) = \mathbf{E}h_2(\theta_{1,j}). \tag{15}$$

Since $\theta_{1,j} \underset{j\to\infty}{\longrightarrow} \infty$ w.p.1, then $h_2(\theta_{1,j}) \underset{j\to\infty}{\longrightarrow} \frac{1}{\mathbf{E}\theta_{2,1}}$ w.p.1. Thus from (14), (15) and Lebesgue's dominated convergence theorem we obtain $\mathbf{E}\nu_1 = \mathbf{E}\theta_{2,1} < \infty$. +

5 Synchronization of renewal points for input and service flows

To exploit Lemma 2 for synchronization of flows $X(t)$ and $Y_i^{(d)}(t)$ we consider the counting processes $N_i(t)$ $(i = \overline{1,m})$ defined by (3) and introduce a counting process $N_0(t)$ for the input flow by the relation

$$N_0(t) = max\{k: \theta_k \le t\}.$$

We assume throughout the following condition to be fulfilled.

Condition 1 *The distributions of θ_1, $u_{i,1}$, $\eta_{i,1}$ $(i = \overline{1,m})$ are aperiodic ones, i.e. (11)are fulfilled for these random variables.*

Let us define subsequence $\{T_k^{(1)}\}_{k=0}^\infty$ of the sequence $\{\theta_j\}_{j=1}^\infty$ by the recurrent relation

$$T_k^{(1)} = min\{\theta_j > T_{k-1}^{(1)} : \cap_{i=1}^m \{N_i(\theta_j) - N_i(\theta_j - 1) = 1\}\}, \quad (T_0^{(1)} = 0). \tag{16}$$

In other words $T_k^{(1)}$ is a point of regeneration of $X(t)$ such that all the servers get out of the order simultaneously at this moment. This means that $\{T_k^{(1)}\}_{k\geq 0}$ are the common regeneration points for the input flow $X(t)$ and $N_i(t)$ $(i = \overline{1,m})$. For D_1 service discipline $\{T_k^{(1)}\}_{k\geq 0}$ constitutes the sequence of regeneration points for $Y_i^{(1)}(t)$ $(i = \overline{1,m})$ and $Y^{(1)}(t)$ as well, but for D_2 this is not the case. For D_2 we introduce a subsequence $\{T_n^{(2)}\}_{n\geq 0}$ of the sequence $\{T_k^{(1)}\}_{k\geq 0}$ as follows

$$T_n^{(2)} = \min\left\{T_j^{(1)} > T_{n-1}^{(2)} : \ \cap_{i=1}^m \{Y_i^{(2)}(T_j^{(1)}) - Y_i^{(2)}(T_j^{(1)} - 1) = 1\}\right\}, \ (T_0^{(2)} = 0). \tag{17}$$

Thus, $T_n^{(2)}$ is a general point of regeneration for $X(t)$ and $Y_i^{(2)}(t)$ $(i = \overline{1,m})$ since the following conditions are fulfilled:

- $T_n^{(2)}$ is a regeneration point for $X(t)$;
- at time $T_n^{(2)}$ all the servers complete the service;
- at time $T_n^{(2)}$ all the servers become unavailable.

Lemma 3 *Let Condition 1 be fulfilled. Moreover, assume relations (1) for D_1 and (2) for the both disciplines take place. Then*

$$\mathbf{E}T_1^{(d)} < \infty, \quad (d = 1,2). \tag{18}$$

Proof. Since $\mathbf{E}\tau_1 < \infty$ and $\mathbf{E}(u_{i,1}) = a_i < \infty$, it follows that for discipline D_1 this lemma is the consequence of Lemma 2 and therefore

$$\mathbf{E}T_1^{(1)} = \frac{1}{\mathbf{E}\tau_1} \prod_{i=1}^m a_i^{-1} < \infty.$$

Let us consider discipline D_2. For server i $(i = \overline{1,m})$ we introduce the sequence

$$s_{i,k} = \min\left\{s_{i,j}^{(2)} > s_{i,k-1} : Y_i^{(2)}(s_{i,j}^{(2)}) - Y_i^{(2)}(s_{i,j}^{(2)} - 1) = 1\right\}, s_{i,0} = 0. \tag{19}$$

Note that $\{s_{i,k}\}_{k\geq 0}$ is the sequence of breakdown moments for the ith server such that the flow $Y_i^{(2)}(t)$ has a jump at every of these moments. Hence $\{s_{i,k}\}_{k\geq 0}$ are regeneration points for $Y_i^{(2)}(t)$. If we show that $\mathbf{E}(s_{i,1}) < \infty$ for any $i = \overline{1,m}$, then Lemma 2 provides the finiteness of $\mathbf{E}T_1^{(2)}$.

Let

$$\nu_{i,k} = \min\{j > \nu_{i,k-1} : Y_i^{(2)}(s_{i,j}^{(2)}) - Y_i^{(2)}(s_{i,j}^{(2)} - 1) = 1\}, \quad \nu_{i,0} = 0, \quad (i = \overline{1,m}).$$

Then $\{\nu_{i,k} - \nu_{i,k-1}\}_{k=1}^\infty$ is a sequence of iid random variables and

$$\mathbf{E}(s_{i,k} - s_{i,k-1}) = \mathbf{E}u_{i,1} \cdot \mathbf{E}\nu_{i,1}. \tag{20}$$

Since $\{s_{i,j}\}_{j\geq 1}$ does not depend on service times we have from the key renewal theorem [26]

$$\lim_{j\to\infty} \mathbf{P}\{Y_i^{(2)}(s_{i,j}^{(2)}) - Y_i^{(2)}(s_{i,j}^{(2)} - 1) = 1\} = \frac{1}{E\nu_{i,1}}, \tag{21}$$

where $\frac{1}{\mathbf{E}\nu_{i,1}} = 0$ if $\mathbf{E}\nu_{i,1} = \infty$.

Let $U_{i,j}^{(2)} = \sum_{n=1}^j u_{i,n}^{(2)}$ be the total available time of the ith server during n cycles. Then

$$\delta_{i,j} = \mathbf{P}\{Y_i^{(2)}(s_{i,j}^{(2)}) - Y_i^{(2)}(s_{i,j}^{(2)} - 1) = 1\} = \mathbf{E}\left(\mathbf{P}\left(\cup_{k=1}^\infty \left\{\sum_{l=1}^k \eta_{i,l} = U_{i,j}^{(2)}\right\} | U_{i,j}^{(2)}\right)\right) =$$

$$= \mathbf{E}\left(\sum_{k=1}^\infty \mathbf{P}\left(\sum_{l=1}^k \eta_{i,l} = U_{i,j}^{(2)} | U_{i,j}^{(2)}\right)\right) = \sum_{n=1}^\infty \sum_{k=1}^\infty \mathbf{P}\left(\sum_{l=1}^k \eta_{i,l} = n\right) \mathbf{P}\left(U_{i,j}^{(2)} = n\right).$$

In the last equality we used independence of $\{\eta_{i,k}\}_{k=1}^\infty$ and $\{U_{i,n}^{(2)}\}$. From Blackwell's theorem [26] we have

$$\sum_{k=1}^\infty \mathbf{P}\left(\sum_{l=1}^k \eta_{i,l} = n\right) \xrightarrow[n\to\infty]{} \frac{1}{b_i},$$

so one may easily verify that $\delta_{i,j} \xrightarrow[j\to\infty]{} \frac{1}{b_i}$. Thus, from (21) we have $\mathbf{E}\nu_{i,1} = b_i$ and from (20) we get $\mathbf{E}s_{i,1} = a_i b_i < \infty$. +

So, we have constructed the sequence $\{T_k^{(d)}\}_{k\geq 0}$ of common regeneration points for the processes $X(t)$ and $Y^{(d)}(t)$. Denote by $\Delta_{X,k}^{(d)} = X(T_k^{(d)}) - X(T_{k-1}^{(d)})$, $\Delta_{Y_i,k}^{(d)} = Y_i^{(d)}(T_k^{(d)}) - Y_i^{(d)}(T_{k-1}^{(d)})$, and $\Delta_{Y,k}^{(d)} = \sum_{i=1}^m \Delta_{Y_i,k}^{(d)}$ $(d = 1,2)$.

Lemma 4 *Let conditions of Lemma 3 take place. Then the traffic rate of the system defined by (10) is equal to*

$$\rho^{(d)} = \frac{\mathbf{E}\Delta_{X,k}^{(d)}}{\mathbf{E}\Delta_{Y,k}^{(d)}}, \quad (d = 1,2).$$

The proof follows from renewal theory and SLLN.

6 Instability results for $\rho^{(d)} \geq 1$

Let $Q^{(d)}(t)$ be the number of customers in the system with the service discipline D_d (including customers in the servers) at instant t $(d = 1,2)$.

Theorem 1 *Let conditions of Lemma 3 take place. Then*

- $Q^{(d)}(t) \underset{t\to\infty}{\longrightarrow} \infty$ w.p.1 if $\rho^{(d)} > 1$,
- $Q^{(d)}(t) \xrightarrow{P} \infty$ if $\rho^{(d)} = 1$.

Here $\rho^{(d)}$ is defined by (10) $(d = 1,2)$.

Proof. Denote by $\tilde{Y}_i^{(d)}(t)$ the number of customers really served by the ith server during the interval $[0, t]$ and $\tilde{Y}^{(d)}(t) = \sum_{i=1}^{m} \tilde{Y}_i^{(d)}(t)$, $\widetilde{\Delta}_{Y,n}^{(d)} = \tilde{Y}^{(d)}(T_n^{(d)}) - \tilde{Y}^{(d)}(T_{n-1}^{(d)})$. Note that $\tilde{Y}_i^{(d)}(t)$ may contain idle periods, so it is not the same like $Y_i^{(d)}(t)$. Employing the approach proposed in [5] and developed in [16] we can choose service times from the collection $\{\{\eta_{i,n}^{(j)}\}_{n=1}^{\infty}\}_{j\geq 1}$ by such a way that

$$\tilde{Y}^{(d)}(t) \leq Y^{(d)}(t) \quad \text{w. p. 1}, \tag{22}$$

$$\widetilde{\Delta}_{Y,n}^{(d)} \leq \Delta_{Y,n}^{(d)} \quad \text{w. p. 1}. \tag{23}$$

Consider the case $\rho^{(d)} > 1$. Taking into account (22) we have

$$Q^{(d)}(t) = Q^{(d)}(0) - \tilde{Y}^{(d)}(t) + X(t) \geq Q^{(d)}(0) - Y^{(d)}(t) + X(t),\ t \geq 0\ w.p.1. \tag{24}$$

From (10) and (24) we obtain

$$\lim_{t\to\infty} \frac{Q^{(d)}(t)}{t} \geq \lambda_X - \lambda_{Y^{(d)}} > 0,\ w.p.1,$$

which concludes the first statement of this Theorem.

Let $\rho^{(d)} = 1$. Consider the embedded process $Q_n^{(d)} = Q^{(d)}(T_n^{(d)})$ and denote $Z_k^{(d)} = \sum_{j=1}^{k} (\Delta_{X,j}^{(d)} - \Delta_{Y,j}^{(d)})$ $(Z_0^{(d)} = 0)$. We define the auxiliary sequence $\{\hat{Q}_k^{(d)}\}_{k\geq 0}$ by the recursive relation

$$\hat{Q}_k^{(d)} = \max[0,\ \hat{Q}_{k-1}^{(d)} + \Delta_{X,k}^{(d)} - \Delta_{Y,k}^{(d)}],\ \hat{Q}_0^{(d)} = 0.$$

Since $Q_k^{(d)} = Q_{k-1}^{(d)} + \Delta_{X,k}^{(d)} - \widetilde{\Delta}_{Y,k}^{(d)}$ from (23) we get $Q_k^{(d)} \geq \hat{Q}_k^{(d)}$ w.p.1 and in distribution the following equality is fulfilled [20].

$$\hat{Q}_k^{(d)} = \max_{0\leq j\leq k} Z_j^{(d)}.$$

If $\rho^{(d)} = 1$, it follows from Lemma 4 that $\mathbf{E}\Delta_{X,j}^{(d)} = \mathbf{E}\Delta_{Y,j}^{(d)}$. Therefore, $\{Z_k^{(d)}\}_{k\geq 0}$ is a random walk with zero drift. Hence, except when $\Delta_{X,j}^{(d)} = \Delta_{Y,j}^{(d)} = c$ w.p.1 (c is a constant) $\max_{0\leq j\leq k} Z_j^{(d)} \xrightarrow{P} \infty$ (see, e.g. [11]). It means that $Q_k^{(d)} \xrightarrow{P} \infty$ and the second statement of this Theorem holds.+

7 Stability theorem for the preemptive repeat different service discipline

In this section we consider the preemptive repeat different service discipline (D_1). So index (1) will be omitted during the rest of the section, if it does not make confusion. We start with definitions.

Definition 2 *The process $\{Q(t), t \geq 0\}$ is called stochastically bounded if for any $\varepsilon > 0$ there exists $y < \infty$ such that for any $t > 0$*

$$\mathbf{P}\{Q(t) < y\} > 1 - \varepsilon.$$

Otherwise we say that $Q(t)$ is stochastically unbounded. This definition is close to the notion of *tightness* [23].

Definition 3 *Process $\{Q(t),\ t \geq 0\}$ is ergodic one if for any initial state $Q(0)$ there exists*

$$\lim_{t\to\infty} \mathbf{P}\{Q(t) \leq x\} = F(x),$$

where $F(x)$ is a distribution function and it does not depend on $Q(0)$.

Denote $Q_n = Q(T_n^{(1)})$. Introduce the process

$$x_n = \begin{cases} (Q_n, e_1(n), \ldots, e_m(n)) & \text{if } \ 0 < Q_n < m, \\ Q_n \ \text{ if } \ Q_n = 0 \ \text{ or } \ Q_n \geq m, \end{cases} \quad n \geq 0,$$

where $e_i(n) = 1$ if there is a customer in the ith server at time $T_n^{(1)}$ and $e_i(n) = 0$ otherwise. In view of interruption discipline D_1 and properties of the moments of synchronization $\{T_n^{(1)}\}_{n\geq 1}$ the process $\{x_n\}_{n\geq 1}$ is a Markov chain with countable set of states.

Theorem 2 *Let conditions of Lemma 3 take place and $\mathbf{P}(Q(0) < \infty) = 1$. If $\rho^{(1)} < 1$, then $Q(t)$ is stochastically bounded. If, in addition, $\{x_n\}_{n\geq 1}$ is an irreducible and aperiodic Markov chain, then $Q(t)$ is ergodic.*

Proof. Consider the ith server. We assume that service times of the customers processing during the kth available period $[s_{i,k}^{(1)}, s_{i,k}^{(2)}]$ $(k = 1,2, \ldots)$ are consequently selected from the sequence of iid random variables $\{\eta_{i,n}^{(k)}\}_{n\geq 1}$. Let us recall that process $Y_i(t)$ is defined by the same sequence on the kth cycle with the help of (4). Introduce the event

$$A_n = \left\{Q(t) \geq m \text{forall} t \in [T_{n-1}^{(1)}, T_n^{(1)}]\right\}. \tag{25}$$

Then

$$\Delta_{Y,n}\mathbf{I}(A_n) = \widetilde{\Delta}_{Y,n}\mathbf{I}(A_n) \quad \text{w.p.}\,1, \tag{26}$$

where $\mathbf{I}(A)$ is an indicator of the event A. By $\mathfrak{K}$ denote the set of states for $\{x_n\}_{n\geq 0}$. Let $\mathfrak{K}_0$ be the set of unessential states and $\mathfrak{K}_l$ $(l = \overline{1,r})$ irreducible classes of communicating states. Since $\rho^{(1)} < 1$ from Lemma 4 we have $\mathbf{E}\Delta_{X,1} < \mathbf{E}\Delta_{Y,1}$. It yields that there exists k_0 such that for any essential state $x \in \mathfrak{K}_l$ one can find $n(x)$ so that

$$\mathbf{P}(Q_{n(x)} < m + k_0 | Q_0 = x) > 0. \tag{27}$$

It provides the finiteness of the number of classes r, so $\mathfrak{K} = \cup_{l=0}^{r} \mathfrak{K}_l$. Consider the first class $\mathfrak{K}_1$. Assume that it is aperiodic. Then for any $x \in \mathfrak{K}_1$, $y \in \mathfrak{K}_1$ there exists

$$\lim_{n\to\infty} \mathbf{P}(x_n = x | x_0 = y) = \pi_x^{(1)}. \tag{28}$$

If

$$\textstyle\sum_{x\in\mathfrak{K}_1} \pi_x^{(1)} = 1 \tag{29}$$

then Q_n is stochastically bounded under condition that $Q_0 \in \mathfrak{K}_1$. Let us show that (29) is fulfilled employing Foster's criterion [21]. For any $x \in \mathfrak{K}_1$ we define the test function $f(x) = q$, where q is the first coordinate of x. It is sufficient to show that for some $\varepsilon_1 > 0$ there exists M_{ε_1} such that

$$\mathbf{E}(f(x_n) - f(x_{n-1}) | x_{n-1} = x) < -\varepsilon_1 \tag{30}$$

for all $x \in \mathfrak{K}_1$ with $q > M_{\varepsilon_1}$. Taking into account (??) we get

$$\begin{aligned} Q_n &= Q_{n-1} + \Delta_n^X - \widetilde{\Delta}_n^Y = Q_{n-1} + \Delta_n^X - \widetilde{\Delta}_n^Y\mathbf{I}(A_n) - \widetilde{\Delta}_n^Y\mathbf{I}(\overline{A}_n) \leq \\ &\leq Q_{n-1} + \Delta_n^X - \Delta_n^Y\mathbf{I}(A_n) = Q_{n-1} + \Delta_n^X - \Delta_n^Y + \Delta_n^Y\mathbf{I}(\overline{A}_n). \end{aligned} \tag{31}$$

From the assumption $\rho^{(1)} < 1$ we have

$$E\Delta_{X,k} - E\Delta_{Y,k} = -\delta < 0. \tag{32}$$

Note, firstly, that for any $\varepsilon > 0$ there exists M_ε such that $\mathbf{P}(\bar{A}_n) < \varepsilon$ if $Q_{n-1} > M_\varepsilon$. Therefore, in view of integrability of random variables $\Delta_{Y,n}$ one may choose $M_\delta \geq m$ such that $\mathbf{E}\Delta_{Y,n}\mathbf{I}(\overline{A}_n) < \frac{\delta}{2}$ if $Q_{n-1} > M_\delta$. Thus, we obtain from (31) and (32)

$$\mathbf{E}(f(x_n) - f(x_{n-1}) | x_{n-1} = x) < E\Delta_{X,n} - E\Delta_{Y,n} + \frac{\delta}{2} = -\frac{\delta}{2}$$

if $x_{n-1} > M_\delta$ that proves (30).

Let $\mathfrak{K}_1$ be a periodic class with a period h. Then we consider a sequence $\{\tilde{x}_n^{(l)}\}_{n\geq 1}$ ($l = \overline{0, h-1}$), where $\tilde{x}_n^{(l)} = x_{nh+l}$. It is well-known [11] that Markov chain $\{\tilde{x}_n^{(l)}\}_{n\geq 1}$ is irreducible and aperiodic. Arguing as above we prove that $\tilde{Q}_n^{(l)} = Q_{nh+l}$ is stochastically bounded as $n \to \infty$.

Stochastic boundedness of Q_n for initial state $x_0 = (Q_0, e_1(0), \dots, e_m(0)$ from other classes $\mathfrak{K}_i$ ($i = \overline{2, r}$) can be similarly proved. Since the number of classes r is finite we conclude that Q_n is stochastically bounded as $n \to \infty$ for any initial state of Markov chain $x_0 \in \mathfrak{K}$. Hence, the process $Q(t)$ is also stochastically bounded.

To prove the second statement of the Theorem we have to assume that Markov chain $\{x_n\}_{n\geq 1}$ is an irreducible and aperiodic. The set of states $\mathfrak{K}$ of $\{x_n\}_{n\geq 1}$ may have some unessential states but all the essential states organize the unique class $\mathfrak{K}_1$ of communicating states. It follows from the first statement of the Theorem that there exists the limit (28), where $\pi_x^{(1)} > 0$ for $x \in \mathfrak{K}_1$ and (29) is fulfilled, i.e. the Markov chain $\{x_n\}_{n\geq 1}$ is ergodic. Let us take a state $j_0 \in \mathfrak{K}_1$, $j_0 \geq m$ and assume that $x_0 = j_0$. Denote $\nu_{j_0} = \min\{n > 0: x_n = j_0\}$, so that ν_{j_0} is the time of the return to the state j_0. Since Markov chain $\{x_n\}_{n\geq 1}$ is ergodic, it follows that $\mathbf{E}\nu_{j_0} < \infty$. Now consider $Q(t)$. Subsequence $\{T_{n_k}^{(1)}\}_{k\geq 0}$ of $\{T_n^{(1)}\}_{n\geq 0}$ such that $Q(T_{n_k}^{(1)}) = j_0$ is a sequence of regeneration points for $Q(t)$. So $Q(t)$ is a regenerative process. Let $\tilde{\tau}_{j_0}$ be the time of return to the state j_0 for $Q(t)$, i.e.

$$\tilde{\tau}_{j_0} = \min\{t > 0: Q(t) = j_0\},$$

under assumption that $Q(0) = j_0$. Since $\mathbf{E}(T_n^{(1)} - T_{n-1}^{(1)}) = \mathbf{E}T_1^{(1)} < \infty$ (Lemma 2) from Wald's identity we have $\mathbf{E}\tilde{\tau}_{j_0} = \mathbf{E}T_1^{(1)}\mathbf{E}\nu_{j_0} < \infty$. Also, from any initial state the process $Q(t)$ gets into j_0 in finite time w.p.1. We remark that condition (11) holds for $\tilde{\tau}_{j_0}$. Therefore, Smith's theorem [25] will conclude the proof. +

Remark 1 *Let us note that in [22] the sufficient condition of stability for a system with a recurrent input flow, identically distributed service times ($b_i = b$, $i = \overline{1, m}$) and preemptive repeat different service discipline was obtained. This condition in our term has a form*

$$\lambda_X < \sum_{i=1}^{m} \frac{a_i^{(2)} - b}{a_i b}. \tag{33}$$

One can easily see that (33) is a corollary of the condition $\rho^{(1)} < 1$ where $\rho^{(1)}$ is defined by (10). Indeed, taking into account the well-known inequality (see, e.g. [11])

$$H_i(t) \geq \frac{t}{b_i} - 1, \quad t \geq 0$$

we get from (10) and Theorem 2 the following sufficient condition of stability

$$\lambda_X < \sum_{i=1}^{m} \frac{a_i^{(2)} - b_i}{a_i b_i}.$$

This condition is the same as (33) when $b_i = b$ ($i = \overline{1, m}$).

8 Stability theorem for the preemptive resume service discipline

We now touch upon the stability of the model with preemptive resume service discipline (D_2). As opposed to queues with preemptive repeat different interruptions, interrupted service continuous when the server returns from a blocked period. Since in this section we consider discipline D_2 only, index (2) will be omitted when it will not lead to misunderstanding.

Put $Q_n = Q(T_n^{(2)})$. Let $e_i(n) = 1$ if there is a customer in the ith server at time $T_n^{(2)}$ and $e_i(n) = 0$ otherwise. Note that under discipline D_2 the process $(Q_n, e_1(n), \dots, e_m(n))$ is not a Markov chain. So we will use another approach, which is based on Theorem 1 from [3].

Denote by t_n ($n = 1, 2, \dots$) the arrival instant of the nth customer at the system. Let $q_{i,n}$ be the number of customers at moment t_n, which will be served by the ith server, $q_n = \sum_{i=1}^{m} q_{i,n} = Q(t_n)$. As before, $\tilde{Y}_i(t)$ is the number of customers that had been served by the ith server during time interval $[0, t]$.

Lemma 5 *Assume that*

$$\mathbf{E}\eta_{i,1}^2 < \infty, \mathbf{E}u_{i,1}^2 < \infty. \tag{34}$$

If $q_{i,n} \xrightarrow{P} \infty$, then for any $\varepsilon > 0$ there exists n_ε, such that for $n > n_\varepsilon$

$$\mathbf{E}\big(\tilde{Y}_i(t_n) - \tilde{Y}_i(t_{n-1})\big) \geq \frac{\beta_i}{\lambda_X} - \varepsilon, \tag{35}$$

where $\beta_i = \frac{a_i^{(2)}}{a_i b_i}$.

Proof. This lemma can be proved similarly to Lemma 2 in [3]. The only difference is that in [3] system with reliable servers was considered. Therefore $Y_i(t)$ was a renewal process corresponding to $\{\eta_{i,n}\}_{n=1}^{\infty}$ of i.i.d random variables. The proof of Lemma 2 in [3] was based on Blackwell's theorem for $\mathbf{E}Y_i(t)$, i.e. on the following convergence for any $h > 0$

$$\mathbf{E}(Y_i(t+h) - Y_i(t)) \underset{t\to\infty}{\longrightarrow} \frac{h}{b_i}. \tag{36}$$

For the system under consideration time intervals between jumps of the process $Y_i(t)$ generally speaking are dependent random variables. Therefore, to apply Lemma 2 from [3] we have to obtain an analog of (36) in our case. Let

$$H_i(t) = \mathbf{E}K_{i,1}(t), \quad \zeta_i(t) = max(0, t - s_{N_i(t)}^{(1)} + 1), \quad V_i(t) = \sum_{j=1}^{N_i(t)} u_{i,j}^{(2)} + \zeta_i(t),$$

where $N_i(t)$ and $K_{i,1}(t)$ are defined in Section 4. Then from (5) we get $U_i(t) = \mathbf{E}Y_i(t) = \mathbf{E}H_i(V_i(t))$. Therefore we need to show that for any $h > 0$

$$U_i(t+h) - U_i(t) \underset{t\to\infty}{\longrightarrow} \beta_i h. \tag{37}$$

Our proof is based on well-known expansion for renewal functions (see, e.g. [6]). Namely, under condition (34)

$$\begin{aligned} H_i(t) &= \frac{t}{b_i} + c_{i,1} + R_i(t), \\ \mathbf{E}N_i(t) &= \frac{t}{a_i} + c_{i,2} + D_i(t), \end{aligned} \tag{38}$$

where $c_{i,1}$ and $c_{i,2}$ are constants and $R_i(t) \to 0$, $D_i(t) \to 0$ as $t \to \infty$. Moreover,

$$\lim_{t\to\infty} \mathbf{E}\zeta_i(t) = \frac{\mathbf{E}u_{i,1}^2}{2a_i}. \tag{39}$$

From decomposition (38) we have

$$U_i(t) = \mathbf{E}\mathbf{E}(H_i(V_i(t)|V_i(t))) = \frac{\mathbf{E}V_i(t)}{b_i} + c_{i,1} + \mathbf{E}R_i(V_i(t)),$$

$$\mathbf{E}V_i(t) = \frac{a_i^{(2)}}{a_i} t + a_i^{(2)} c_{i,2} + a_i^{(2)} D_i(t) + \mathbf{E}\zeta_i(t).$$

Therefore,

$$U_i(t+h) - U_i(t) = \beta_i h + \frac{a_i^{(2)}}{b_i}(D_i(t+h) - D_i(t)) + b_i^{-1}(\mathbf{E}\zeta_i(t+h) - \mathbf{E}\zeta_i(t)) +$$

$$+\mathbf{E}R_i(V_i(t+h)) - \mathbf{E}R_i(V_i(t)). \tag{40}$$

Since $\lim_{t\to\infty}\mathbf{E}R_i(V_i(t)) = 0$ we get (37) from (39) and (40).

The rest of the proof of the Lemma is the same as the proof of Lemma 2 in [3]. It is based on limit theorems for renewal processes and some estimations that take place in our case. +

For the ergodic theorem we will need the following assumptions.

Condition 2 *Let for some server i the following inequalities be fulfilled*

- $\mathbf{P}\{u_{i,1}^{(2)} \geq x\} > 0$ for any $x < \infty$;
- $\mathbf{P}\{\xi_1 = 0\} + \mathbf{P}\{\xi_1 = 1, t_1 + \eta_{i,1} + l < \tau_1\} > 0$, where t_1 is a moment of the first customer arrival and $l = \min\{j \geq 1: \mathbf{P}\{u_{i,1}^{(1)} = j\} > 0\}$.

Theorem 3 *Let Conditions 1, 2 and relations (1), (2), (34) be fulfilled. Then the process $Q(t)$ is ergodic if and only if $\rho^{(2)} < 1$.*

Proof. In view of Theorem 1 we need to consider the case $\rho^{(2)} < 1$ only. Note that $Q(t)$ and Q_n are regenerative processes and moments of regeneration are the subsequence $\{T_{n_k}^{(2)}\}_{k\geq 0}$ of the sequence $\{T_n^{(2)}\}_{n\geq 0}$ such that $Q(T_{n_k}^{(2)}) = Q_{n_k} = 0$. Let $y_i(n)$ be the residual service time of a customer in the ith server at moment $T_n^{(2)}$ if there is a customer ($y_i(n) = 0$ if there is no customer). Under

Condition 2 the process Q_n has the following properties:

- $\mathbf{P}\{Q_{n+1} = 0 | Q_n = 0\} > 0$;
- for any $j > 0$ and $x > 0$ there exists $m_j(x) > 0$ such that $\mathbf{P}\{Q_{n+m_j(x)} = 0 | Q_n \le j,\ y_i(n) \le x,\ i = \overline{1,m}\} > 0$.

Under this conditions the circumstances of Theorem 1 from [3] are fulfilled, so $Q(t)$ is either ergodic or $Q(t) \xrightarrow{P} \infty$. Assume that $\rho^{(2)} < 1$ and $Q(t) \xrightarrow{P} \infty$. Then

$$q_n = \sum_{i=1}^{m} q_{i,n} \xrightarrow{P} \infty \quad \text{and} \quad q_{i,n} \xrightarrow{P} \infty \tag{41}$$

for all $i = \overline{1,m}$ (see, e.g. [18]). From Lemma 5 we get that for any $\varepsilon > 0$ there exists n_ε such that for any $n > n_\varepsilon$

$$\mathbf{E}\big(\tilde{Y}(t_n) - \tilde{Y}(t_{n-1})\big) \ge \frac{\beta}{\lambda_X} - \varepsilon = \frac{1}{\rho^{(2)}} - \varepsilon, \quad \text{where} \quad \tilde{Y}(t) = \sum_{i=1}^{m} \tilde{Y}_i(t), \beta = \sum_{i=1}^{m} \beta_i.$$

Therefore,

$$\mathbf{E}q_{n+1} = \mathbf{E}q_n + 1 - \mathbf{E}(\tilde{Y}(t_n) - \tilde{Y}(t_{n-1}) \le \mathbf{E}q_n - \frac{1-\rho^{(2)}}{\rho^{(2)}} + \varepsilon$$

and

$$\mathbf{E}q_{n+1} \le \mathbf{E}q_n$$

if $\varepsilon < \frac{1-\rho^{(2)}}{\rho^{(2)}}$ and $n > n_\varepsilon$. It contradicts (41). Thus, $Q(t)$ is ergodic. +

Let us note that if all servers have the same distribution function of service times, then condition (34) in Theorem 3 can be omitted.

Remark 2 *So far we consider zero-delayed regenerative flows $X(t)$ and $Y_i(t)$ assuming that*

$$\mathbf{P}(\theta_0 = 0) = \mathbf{P}(s_{i,0}^{(2)} = 0) = 1,\ (\overline{i = 1,m}).$$

Let this condition be not fulfilled and we have delayed regenerative flows. Note that results of Lemmas 1 — 5 on which our proofs of Theorems 2 and 3 are based hold for delayed regenerative flows. We only have to claim

$$\mathbf{P}(\theta_0 < \infty) = \mathbf{P}(s_{i,0}^{(2)} < \infty) = 1,\ (\overline{i = 1,m}).$$

9 Conclusion

This paper we are focused on the discrete-time multichannel queueing system with heterogeneous servers that may be unreliable and regenerative input flow. We considered preemptive repeat different service disciplines as well as preemptive resume service disciplines. Exploiting renewal technique we proved instability theorem for queue-length process when traffic coefficient greater or equal than one (Theorem 1). Under some assumptions, based on the regenerative structure of the queue-length process, we established stability theorem, when traffic coefficient less than one (Theorem 2 for preemptive repeat different service disciplines and Theorem 3 for preemptive resume service disciplines). Note that traffic coefficient for preemptive repeat different service disciplines cannot be expressed in terms of moments of service and interruption processes.

There are many further research topics worth conducting. First, the conjecture that diffusion scaled process $Q(t)$ converges to the Brownian motion (diffusion process) when traffic coefficient greater (less) than one remains to be proved for multichannel systems (see [2] for single server case). Second, steady-state distribution of scaled queue-length process is not investigated for multichannel systems. Third, the large deviation problem is also relevant for this system.

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