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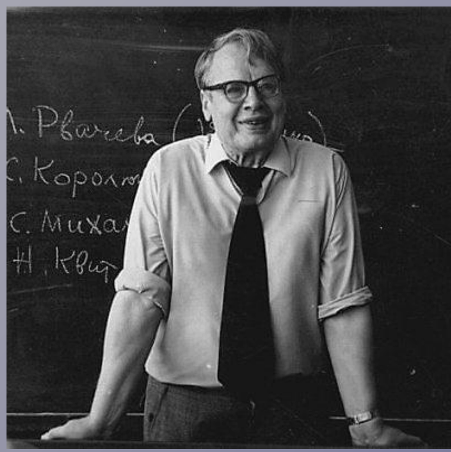

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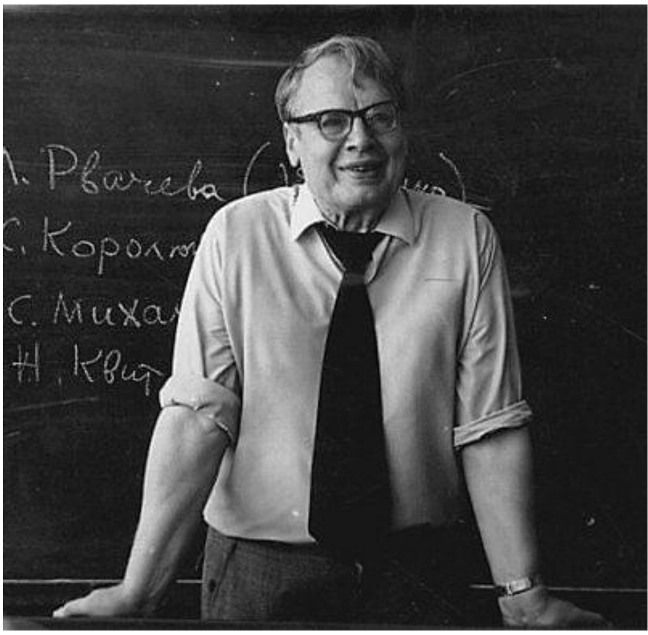

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Classical and Bayes Analysis of A Competing Risk Model Based on Two Weibull Distributions with Increasing and Decreasing Hazards

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Abstract

The paper considers a competing risk model based on two Weibull distributions, one with increasing and the other with decreasing hazard rate. It then considers both classical and Bayesian analysis of the model, the later development utilizes the informative but weak priors for the parameters. The analysis is facilitated by the fact that a competing risk model can be considered as an incomplete data model even if the situation allows all the observations on the test to be made available although the results are extended for censored data cases as well. The paper uses the expectation-maximization algorithm for classical maximum likelihood estimation and Gibbs sampler algorithm for posterior based inferences. It is shown that the likelihood function offers unique and consistent maximum likelihood estimates. The results are illustrated based on a real data example. Finally, the compatibility of the model is examined for the considered real data set using some standard tools of Bayesian paradigm.

Keywords: Competing risk, Weibull distribution, Right censoring, Expectation-maximization algorithm, Gibbs sampler, Predictive p-value.

1 Introduction

In real life situations, it is quite frequent that an item is exposed to experience failure due to more than one cause at the same point of time. Say for instance, in reliability experiment, an item may fail due to one of several possible causes, such as breakdown, manufacturing defects, etc. Similarly, in medical experiment, a patient suffering from several diseases may die because of the one that relapses first. Such situations usually come under the purview of competing risk scenario where an item or organism is subject to several competing causes and the failure may occur because of any cause that arises first. A traditional approach for modelling failure time data in presence of competing risks is to assume that there is a latent failure time associated with each of the causes to which the item is exposed and the realized failure time of the item is lowest among these latent failure times. Moreover, these latent failure times are assumed to be independent of each other following some distributions, either same or different (see [5]).

The simplest case of competing risk model arises when failure of an item is subject to two possible causes. In this case, a competing risk model is defined considering the distribution of minimum of two different failure times. The analysis of such competing risk models based on two failure time distributions is considered by several authors. [10] studied a competing risk model defined on the basis of exponential and Weibull failures to model failures due to shock and wear out, respectively, and discussed the properties of maximum likelihood (ML) estimates of resulting model parameters. [21] proposed both parametric and non-parametric estimation techniques for a two-component competing risk model under the assumption that the component failure times are exponentially distributed. [4] proposed a competing risk model for a situation where the population is exposed to wear out failures but a fraction of population is also exposed to early failures. The authors obtained ML estimates of model parameters under the assumption that failure modes follow either Weibull or lognormal distributions. [2] considered both classical and Bayesian analyses of a model based on minimum of Weibull and exponential failures assuming that former results in failures due to ageing and the latter results in accidental failures as the two competing causes. More recently, [22] suggested an alternative competing risk model based on gamma and exponential distributions to model ageing and accidental failures, respectively, and provided complete classical as well as Bayesian analyses of the resulting competing risk model.

This paper models a situation where infancy and ageing work together to induce failures. Such a situation may occur where an item having an initial birth defect is also subject to failure due to ageing. Many real life situations can be found in practice that may include items from automobile segment, high-power transmitting tubes and computer disk-drives, etc. Obviously, to deal with such situations, one may consider, among various other choices, a model based on two distributions, one with decreasing hazard rate and other with increasing hazard rate. The resulting failure time can then be considered as the minimum of two failure times, one corresponding to decreasing hazard rate distribution and the other corresponding to increasing hazard rate distribution. One can, of course, consider a number of models to define decreasing and increasing hazard rate behaviour. We, however, consider a competing risk model defined on the basis of two Weibull distributions, one corresponding to decreasing hazard rate situation and the other corresponding to increasing hazard rate situation.

The Weibull distribution is an important failure time distribution that encompasses both increasing and decreasing hazard rate behaviour and, perhaps because of its enormous scope and flexibility, it has been used to describe both initial as well as ageing failures (see, for example, Lawless (2002)). The probability density function (pdf) of its simplest two-parameter form ($\mathcal{W}(\theta,\beta)$) can be written as

$$f_W(t|\theta,\beta)=\frac{\beta}{\theta}\left(\frac{t}{\theta}\right)^{\beta-1}\exp\left[-\left(\frac{t}{\theta}\right)^{\beta}\right];\quad t>0,\quad \theta>0,\quad \beta>0, \tag{1}$$

where θ and β are scale and shape parameters, respectively. It is actually the shape parameter that results in different characteristics of the model. Say, for instance, $\beta<1.0$ defines a decreasing hazard rate distribution that can be considered to model early birth defects. Similarly, $\beta>1.0$ defines increasing hazard rate behavior, a situation that can be very well used for defining failures due to ageing. Although, not of importance in the present work, the distribution reduces to constant hazard-rate exponential distribution when $\beta=1.0$. The important reliability characteristics such as the reliability at time t, the hazard rate and the mean time to failure (MTF) for $\mathcal{W}(\theta,\beta)$ can be written as

$$R_{\mathcal{W}}(t)=\exp\left[-\left(\frac{t}{\theta}\right)^{\beta}\right], \tag{2}$$

$$h_{\mathcal{W}}(t)=\frac{\beta}{\theta}\left(\frac{t}{\theta}\right)^{\beta-1}, \tag{3}$$

and

$$MTF_{\mathcal{W}} = \theta\Gamma(1 + 1/\beta), \quad (4)$$

respectively.

As mentioned, the proposed model is defined on the basis of two Weibull distributions, one with $\beta < 1.0$ and the other with $\beta > 1.0$. A similar model, named as Bi-Weibull ($\mathcal{BW}$) distribution, was also entertained by [1] but they did not impose any restriction on the corresponding shape parameters. The authors considered the model as a particular case of poly-Weibull model and analyzed in a Bayesian framework using Gibbs sampler algorithm. The illustration was, however, based on a simulated data set. The unrestricted model was later analyzed by a number of authors including [16], [6] and [17]). Whereas [16] provided complete parametric characterization of the model in a three dimensional parameter space, [6] analyzed the model for real as well as generated data sets using both classical and Bayesian tools. In fact, [6] mentioned that the likelihood of the model could be obtained in a simple manner and there was no need of using Gibbs sampler algorithm. The authors rather used standard likelihood method to obtain classical inferences and both Laplace's method and sampling-importance resampling technique for Bayesian inferences.

Recently [17] considered various characterizations of ($\mathcal{BW}$) model in its four-parameter setup and obtained ML estimates of model parameters with observed information matrix. The authors then used the results on asymptotic normality of ML estimators to derive approximate confidence intervals and confidence regions for the model parameters. The results obtained by [17] are certainly extensive but inferential aspects are comparatively meagre as compared to various characterizations of the model. Moreover, the model considered by the authors is a simplified version that separates the two parameters of the Weibull model and it is often considered for mathematical convenience.

The present paper can be considered as an extension of previous work where two Weibull distributions with form given in (1) are used to define the competing risk model. Since the shape parameters of the corresponding components are restricted, we shall call the resulting competing risk model as restricted Bi-Weibull ($\mathcal{BW}_{\mathcal{R}}$) model. The paper then considers not only the complete Bayes analysis using weak proper priors but also the ML estimation of model parameters using expectation-maximization (EM) algorithm. It is shown that there exists a unique consistent solution of the likelihood function, a result that may be considered significant for the likelihood form arising from $\mathcal{BW}_{\mathcal{R}}$ model. Throughout the inferential developments are done for both complete and censored data cases, the latter situation is, of course, important in failure time data analysis but not considered in any of the previous references.

The plan of the paper is as follows. The next section introduces the proposed $\mathcal{BW}_{\mathcal{R}}$ model and provides a few important characteristics for the same. Some of the results provided in this section are given in slightly different forms in [17] but we have reproduced them for a ready reference and also because of the fact that the model form for the Weibull distribution used by [17] is different from the one considered by us in this paper. Section 3 details the ML estimation of the model parameters for the considered $\mathcal{BW}_{\mathcal{R}}$ model using EM algorithm. Since the competing risk model as considered in the paper is an incomplete data model in the sense that we do not have the actual cause of failures, the EM algorithm happens to be an important choice for such situations. Section 4 provides the Bayesian model formulation for the considered likelihood and prior combinations. The resulting posterior is not easy to deal with and, therefore, we have considered the use of Gibbs sampler algorithm for drawing the posterior based inferences. The section also provides a brief discussion of the Gibbs sampler algorithm and its implementation details for the model in hand. Throughout the censored data cases are also considered and appropriate implementation details for the same are given. Section 5 is given for completeness that provides a few important tools for model compatibility study in Bayesian paradigm. These tools are used in the next section where we have considered a real data set for numerical illustration and provided the compatibility of the data with $\mathcal{BW}_{\mathcal{R}}$ to justify our analysis. Finally, a conclusion is given in the last section along with the proof of theorems in the Appendix.

2 The Proposed $\mathcal{BW}_{\mathcal{R}}$ Model

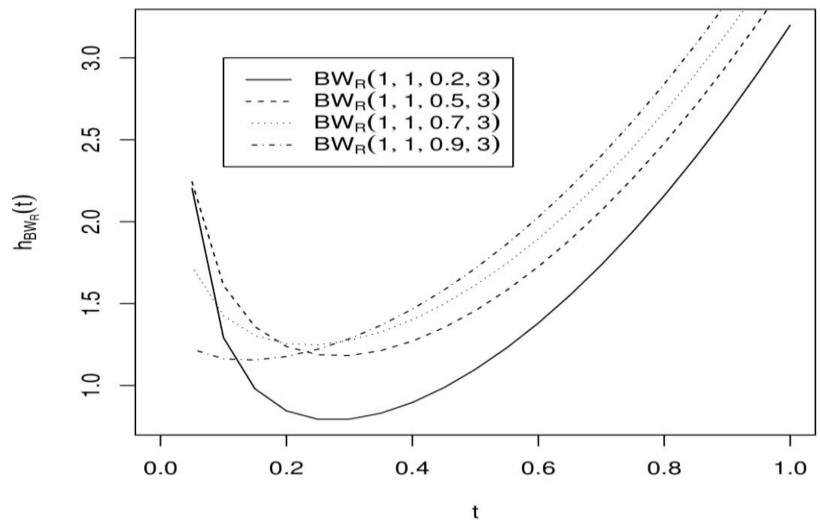
Let T_1 be a random variable following $\mathcal{W}(\theta_1, \beta_1)$, $\beta_1 < 1.0$, and T_2 be another random variable following $\mathcal{W}(\theta_2, \beta_2)$, $\beta_2 > 1.0$. A competing risk $\mathcal{BW}_{\mathcal{R}}$ model based on random variable $T = \min(T_1, T_2)$ can be characterized by four parameters θ_1, θ_2, β_1 and β_2 where $\theta_1 > 0, \theta_2 > 0, \beta_1 < 1.0$ and $\beta_2 > 1.0$. The hazard rate of $\mathcal{BW}_{\mathcal{R}}(\theta_1, \theta_2, \beta_1, \beta_2)$ model can be expressed as sum of hazard rates of its components, which can be given as

$$h_{BW_R}(t) = \frac{\beta_1}{\theta_1}\left(\frac{t}{\theta_1}\right)^{\beta_1 - 1} + \frac{\beta_2}{\theta_2}\left(\frac{t}{\theta_2}\right)^{\beta_2 - 1}. \tag{5}$$

The hazard rate of general $\mathcal{BW}$ model is discussed by several authors. A few important references include [16], [6] and [17]. Truly speaking, the hazard rate of $\mathcal{BW}$ model may be characterized by its shape parameters β_1 and β_2. If $\min(\beta_1, \beta_2) > 1.0$, the hazard rate is increasing; if $\max(\beta_1, \beta_2) < 1.0$, the hazard rate is decreasing; and for $\beta_1 < 1.0$ and $\beta_2 > 1.0$, the hazard rate is bathtub shaped. On the other hand, the hazard rate curve for $\mathcal{BW}_{\mathcal{R}}$ model is always bathtub shaped since one of its shape parameters is less than unity. The hazard rate curve, however, changes its shape with different choices of β_1 and β_2. The change-point for bathtub shaped hazard rate for the model $\mathcal{BW}_{\mathcal{R}}$ can be obtained as

$$t^* = \left[\frac{\beta_1(1-\beta_1)\times\theta_2^{\beta_2}}{\beta_2(\beta_2-1)\times\theta_1^{\beta_1}}\right]^{\frac{1}{\beta_2-\beta_1}}. \tag{6}$$

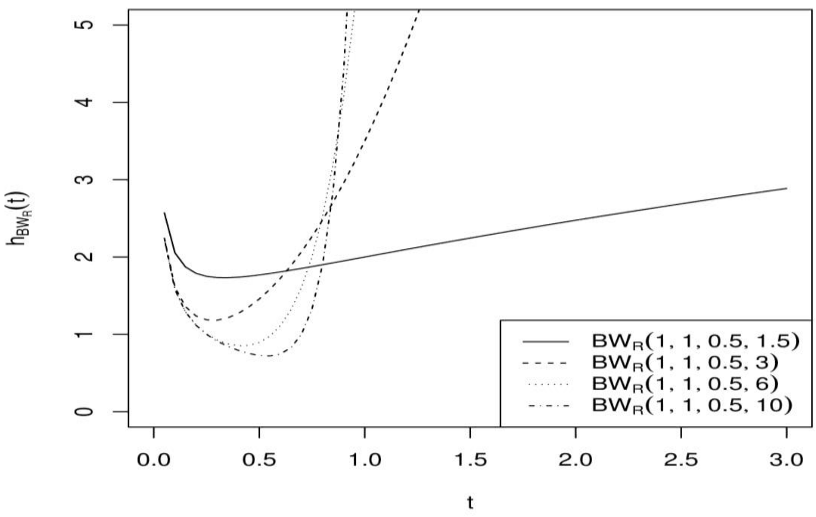
Figure 1: Hazard rate curves corresponding to $\mathcal{BW}_{\mathcal{R}}$ model for different values of β_1.



It can be seen that t^* is always positive since β_1 is less than unity and the other shape parameter β_2 is greater than unity. Moreover, it can be seen that the change point increases as β_1 approaches towards zero. The first derivative of h(t) can be shown to be negative(positive) for values of t less(greater) than t^* and, therefore, the hazard rate is always decreasing(increasing) for values of t less(greater) than t^* and finally approaching to infinity both when t approaches to zero or infinity.

The hazard rate curves corresponding to $\mathcal{BW}_{\mathcal{R}}$ model are shown in Figures 1-2 for some arbitrary choices of the parameters. Since the shape parameters are only responsible for changing the hazard rates shapes, both the scale parameters θ_1 and θ_2 are fixed at unity. It is obvious from the figures that the curves exhibit convex shapes in every case with a unique minimum given by (6). Rest of the conclusions are same that have been detailed in the preceding paragraph.

Figure 2: Hazard rate curves corresponding to $\mathcal{BW}_{\mathcal{R}}$ model for different values of $\boldsymbol{\beta_2}$.



The reliability function corresponding to $\mathcal{BW}_{\mathcal{R}}$ model can be obtained as product of reliability functions of its component models. The expression for the same can be written as

$$R_{\mathcal{BW}_{\mathcal{R}}}(t) = \exp\left[-\left(\frac{t}{\theta_1}\right)^{\beta_1} - \left(\frac{t}{\theta_2}\right)^{\beta_2}\right]. \tag{7}$$

Using (5) and (7), one can easily obtain the pdf corresponding to $\mathcal{BW}_{\mathcal{R}}$ model. The same can be written as

$$f_{\mathcal{BW}_{\mathcal{R}}}(t) = \left[\frac{\beta_1}{\theta_1}\left(\frac{t}{\theta_1}\right)^{\beta_1-1} + \frac{\beta_2}{\theta_2}\left(\frac{t}{\theta_2}\right)^{\beta_2-1}\right] \times \exp\left[-\left(\frac{t}{\theta_1}\right)^{\beta_1} - \left(\frac{t}{\theta_2}\right)^{\beta_2}\right]. \tag{8}$$

Fortunately, the moments of $\mathcal{BW}_{\mathcal{R}}$ distribution exist in closed forms and the corresponding expression for the s^{th} moment (see also [17]) can be written as

$$\begin{aligned}\mu'_s = {} & \frac{\beta_1\theta_2^s}{\beta_2}\sum_{m=0}^{\infty}(-1)^m\left(\frac{\theta_2}{\theta_1}\right)^{(m+1)\beta_1}\Gamma\left(\frac{s+(m+1)\beta_1}{\beta_2}\right) \\ & +\theta_2^s\sum_{m=0}^{\infty}(-1)^m\left(\frac{\theta_2}{\theta_1}\right)^{m\beta_1}\Gamma\left(\frac{s+m\beta_1}{\beta_2}+1\right).\end{aligned} \tag{9}$$

Another important characteristic of interest is the probability of failure due to one of the causes. The corresponding expression for the probability of failure due to early birth defect can be written as

$$P_r(t = t_1) = \int_0^{\infty}\frac{\beta_1}{\theta_1}\left(\frac{t}{\theta_1}\right)^{\beta_1-1}\exp\left[-\left(\frac{t}{\theta_1}\right)^{\beta_1} - \left(\frac{t}{\theta_2}\right)^{\beta_2}\right]dt,$$

which, on simplification, reduces to

$$P_r(t = t_1) = \frac{\beta_1}{\beta_2}\sum_{m=0}^{\infty}(-1)^m\left(\frac{\theta_2}{\theta_1}\right)^{(m+1)\beta_1}\Gamma\left(\frac{(m+1)\beta_1}{\beta_2}\right). \tag{10}$$

Obviously, the probability of failure due to ageing can be written as complimentary probability of (10).

3 The ML Estimation

Let n items having failure time distribution given in (8) be put on test and let $y: (y_i = (t_i, \delta_i); i = 1,2,\dots,n)$ denote the corresponding observations. We use t_i to denote the failure time of i^{th} unit and δ_i as an associated censoring indicator such that $\delta_i = 0$ indicates that i^{th} observation is right censored at time t_i and $\delta_i = 1$ indicates that t_i is the observed failure time of i^{th} item. The corresponding likelihood function (LF) can be written as

$$L(y|\boldsymbol{\theta}) = \prod_{i=1}^{n} [f_{\mathcal{BW}_{\mathcal{R}}}(t_i)]^{\delta_i}[R_{\mathcal{BW}_{\mathcal{R}}}(t_i)]^{1-\delta_i},$$

which, using (7) and (8), becomes

$$L(y|\boldsymbol{\theta}) = \prod_{i=1}^{n} \left\{\frac{\beta_1}{\theta_1}\left(\frac{t_i}{\theta_1}\right)^{\beta_1-1} + \frac{\beta_2}{\theta_2}\left(\frac{t_i}{\theta_2}\right)^{\beta_2-1}\right\}^{\delta_i} \exp\left[-\left(\frac{t_i}{\theta_1}\right)^{\beta_1} - \left(\frac{t_i}{\theta_2}\right)^{\beta_2}\right], \tag{11}$$

where $\boldsymbol{\theta}$ is used to denote the parameter vector $(\theta_1, \theta_2, \beta_1, \beta_2)$. It is important to mention here that the LF corresponding to complete data case can be written as a special case of (11) when all δ_is are taken to be unity. The LF for complete data case can, therefore, be written as

$$L(y|\boldsymbol{\theta}) = \prod_{i=1}^{n} \left[\frac{\beta_1}{\theta_1}\left(\frac{t_i}{\theta_1}\right)^{\beta_1-1} + \frac{\beta_2}{\theta_2}\left(\frac{t_i}{\theta_2}\right)^{\beta_2-1}\right] \exp\left[-\left(\frac{t_i}{\theta_1}\right)^{\beta_1} - \left(\frac{t_i}{\theta_2}\right)^{\beta_2}\right]. \tag{12}$$

The ML estimates are usually obtained by maximizing the LF, but in high dimensional case direct maximization of LF may be often difficult and, sometimes, may also lead to unstable results. Fortunately, in our case, we are in a position to verify that the corresponding likelihood equations offer unique and consistent solutions both for complete and censored data cases (the details of our proof are given in the Appendix). Instead of direct maximization of the LF, we propose to obtain ML estimates of model parameters using EM algorithm. It may be noted that the EM algorithm is an iterative method proposed by [7] for computing ML estimates of model parameters, especially in the situations where data or model can be viewed as incomplete. This is perhaps the reason that the implementation of EM algorithm is facilitated in a competing risk scenario simply because the data corresponding to a competing risk model are always incompletely specified in the sense that we do not know exact cause of failure for a particular item (see also [23]). The algorithm is lucid and simple in the sense that it does not require calculation of the Jacobian matrix which is normally required in other optimization techniques like Newton-Raphson, etc. (see also [20] and [18]).

For implementing the EM algorithm, we rather work with an alternative formulation of the LF by introducing missing observations in the form of indicator variables (see also [2] since each observation t_i has a missing component in the sense that it is not known exactly which of the two causes, early birth defect or ageing, is responsible for producing t_i. Similarly, each observation t_i can either be a failure time or censoring time as governed by the censoring indicator δ_i. Thus to provide an alternative formulation of the LF, we associate with each t_i a missing cause indicator z_i with components of z_i as (z_i^1, z_i^2), $i = 1,2,\dots,n$. We may then define z_i^1=1 and z_i^2=0 as an indication that t_i is an observed failure time arising from $\mathcal{W}(\theta_1, \beta_1)$ and, similarly, $z_i^1 = 0$ and $z_i^2 = 1$ as an indication that t_i is an observed failure time arising from $\mathcal{W}(\theta_2, \beta_2)$. Obviously, for each censored observation t_i $(\delta_i = 0)$, we can define associated z_i^1 and z_i^2 both equal to zero, $i = 1,2,\dots,n$. Moreover, since failure can arise due to only one of the two competing causes, both z_i^1 and z_i^2 cannot take value unity simultaneously.

With the assumptions as given above, the pdf for each observation can be written as

$$\begin{aligned} f(x_i) &= [f_{\mathcal{W}}(t_i, \theta_1, \beta_1)]^{z_i^1}[R_{\mathcal{W}}(t_i, \theta_1, \beta_1)]^{1-z_i^1}[f_{\mathcal{W}}(t_i, \theta_2, \beta_2)]^{z_i^2}[R_{\mathcal{W}}(t_i, \theta_2, \beta_2)]^{1-z_i^2} \\ &= [h_{\mathcal{W}}(t_i, \theta_1, \beta_1)]^{z_i^1}[h_{\mathcal{W}}(t_i, \theta_2, \beta_2)]^{z_i^2} R_{\mathcal{W}}(t_i, \theta_1, \beta_1) R_{\mathcal{W}}(t_i, \theta_2, \beta_2), \end{aligned} \tag{13}$$

where $f_{\mathcal{W}}$, $h_{\mathcal{W}}$ and $R_{\mathcal{W}}$ denote the pdf, hazard function and the reliability function of Weibull model, respectively, and $x_i = (t_i, z_i)$, $i = 1,2,\dots,n$. Obviously, the corresponding LF based on a set of n observations can be written as

$$L(x|\boldsymbol{\theta}) \propto \prod_{i=1}^{n} [h_{\mathcal{W}}(t_i, \theta_1, \beta_1)]^{z_i^1}[h_{\mathcal{W}}(t_i, \theta_2, \beta_2)]^{z_i^2} R_{\mathcal{W}}(t_i, \theta_1, \beta_1) R_{\mathcal{W}}(t_i, \theta_2, \beta_2), \tag{14}$$

and the associated log likelihood can be written as

$$l(x|\boldsymbol{\theta}) = \sum_{i=1}^{n} \{z_i^1 \ln[h_{\mathcal{W}}(t_i,\theta_1,\beta_1)] + z_i^2 \ln[h_{\mathcal{W}}(t_i,\theta_2,\beta_2)] + \ln[R_{\mathcal{W}}(t_i,\theta_1,\beta_1)] + \ln[R_{\mathcal{W}}(t_i,\theta_2,\beta_2)]\}. \quad (15)$$

where $x = (x_1, x_2, \ldots, x_n)$. Once the log likelihood is specified as in (15), the EM algorithm can proceed as usual in two steps, the expectation (E) step and the maximization (M) step. The E step may involve the calculation of expected value of the log likelihood based on the complete data and the current realization of model parameters. The M step, on the other hand, intends to find the values of model parameters that maximizes the expected log likelihood function evaluated at E step.

To clarify, suppose $\widetilde{\boldsymbol{\theta}} = (\tilde{\theta}_1, \tilde{\theta}_2, \tilde{\beta}_1, \tilde{\beta}_2)$ is the current realization of parameter vector $\boldsymbol{\theta}$ at a particular iteration then the expected log LF $\{(= E[l(x|\boldsymbol{\theta})|t, \widetilde{\boldsymbol{\theta}}]), t = (t_1, t_2, \ldots, t_n)\}$ denoted by $A_l(\boldsymbol{\theta}|t, \widetilde{\boldsymbol{\theta}})$, say, can be obtained as

$$A_l(\boldsymbol{\theta}|t,\widetilde{\boldsymbol{\theta}}) = \sum_{i=1}^{n} \{E(z_i^1|t_i,\widetilde{\boldsymbol{\theta}})\ln[h_{\mathcal{W}}(t_i,\theta_1,\beta_1)] + E(z_i^2|t_i,\widetilde{\boldsymbol{\theta}})\ln[h_{\mathcal{W}}(t_i,\theta_2,\beta_2)] + \ln[R_{\mathcal{W}}(t_i,\theta_1,\beta_1)] + \ln[R_{\mathcal{W}}(t_i,\theta_2,\beta_2)]\}, \quad (16)$$

where $E(z_i^1|t_i,\widetilde{\boldsymbol{\theta}}) = \tilde{p}(t_i, \mathcal{W}(\tilde{\theta}_1,\tilde{\beta}_1))$ and $E(z_i^2|t_i,\widetilde{\boldsymbol{\theta}}) = \tilde{p}(t_i, \mathcal{W}(\tilde{\theta}_2,\tilde{\beta}_2)$. $\tilde{p}(t_i, \mathcal{W}(\tilde{\theta}_1,\tilde{\beta}_1))$ $[\tilde{p}(t_i, \mathcal{W}(\tilde{\theta}_2,\tilde{\beta}_2))]$ gives the probability that the observation t_i is arising from $\mathcal{W}(\tilde{\theta}_1,\tilde{\beta}_1)$ $[\mathcal{W}(\tilde{\theta}_2,\tilde{\beta}_2)]$ given the current realization of parameters. These probabilities can be obtained as

$$\tilde{p}(t_i, \mathcal{W}(\tilde{\theta}_1,\tilde{\beta}_1)) = P_r(z_i^1 = 1|t_i,\widetilde{\boldsymbol{\theta}}) = \begin{cases} 0 & if \quad \delta_i = 0 \\ \frac{h_{\mathcal{W}}(t_i,\tilde{\theta}_1,\tilde{\beta}_1)}{h_{\mathcal{W}}(t_i,\tilde{\theta}_1,\tilde{\beta}_1)+h_{\mathcal{W}}(t_i,\tilde{\theta}_2,\tilde{\beta}_2)} & if \quad \delta_i = 1, \end{cases} \quad (17)$$

and

$$\tilde{p}(t_i, \mathcal{W}(\tilde{\theta}_2,\tilde{\beta}_2)) = P_r(z_i^2 = 1|t_i,\widetilde{\boldsymbol{\theta}}) = \begin{cases} 0 & if \quad \delta_i = 0 \\ \frac{h_{\mathcal{W}}(t_i,\tilde{\theta}_2,\tilde{\beta}_2)}{h_{\mathcal{W}}(t_i,\tilde{\theta}_1,\tilde{\beta}_1)+h_{\mathcal{W}}(t_i,\tilde{\theta}_2,\tilde{\beta}_2)} & if \quad \delta_i = 1 \end{cases} \quad (18)$$

respectively. From (16), one can see that $A_l(\boldsymbol{\theta}|t,\widetilde{\boldsymbol{\theta}})$ can be split into two parts as given below

$$A_l(\boldsymbol{\theta}|t,\widetilde{\boldsymbol{\theta}}) = A_l^1(\boldsymbol{\theta}|t,\widetilde{\boldsymbol{\theta}})) + A_l^2(\boldsymbol{\theta}|t,\widetilde{\boldsymbol{\theta}}), \quad (19)$$

where

$$A_l^1(\boldsymbol{\theta}|t,\widetilde{\boldsymbol{\theta}}) = \sum_{i=1}^{n} \{\tilde{p}(t_i, \mathcal{W}(\tilde{\theta}_1,\tilde{\beta}_1))\ln[h_{\mathcal{W}}(t_i,\theta_1,\beta_1)] + \ln[R_{\mathcal{W}}(t_i,\theta_1,\beta_1)]\}, \quad (20)$$

and

$$A_l^2(\boldsymbol{\theta}|t,\widetilde{\boldsymbol{\theta}})) = \sum_{i=1}^{n} \{\tilde{p}(t_i, \mathcal{W}(\tilde{\theta}_2,\tilde{\beta}_2))\ln[h_{\mathcal{W}}(t_i,\theta_2,\beta_2)] + \ln[R_{\mathcal{W}}(t_i,\theta_2,\beta_2)]\}. \quad (21)$$

From (20) and (21), it is obvious that both $A_l^1(\boldsymbol{\theta}|t,\widetilde{\boldsymbol{\theta}})$ and $A_l^2(\boldsymbol{\theta}|t,\widetilde{\boldsymbol{\theta}})$ depend separately on (θ_1,β_1) and (θ_2,β_2), respectively, and, therefore, $A_l(\boldsymbol{\theta}|t,\widetilde{\boldsymbol{\theta}})$ in (19) can be maximized by maximizing separately the two terms $A_l^1(\boldsymbol{\theta}|t,\widetilde{\boldsymbol{\theta}})$ and $A_l^2(\boldsymbol{\theta}|t,\widetilde{\boldsymbol{\theta}})$. This partitioning has an obvious advantage in the sense that it reduces a four-dimensional optimization problem into a two-dimensional optimization problem and thereby resulting into an increased efficiency of the algorithm. Thus differentiating (20) with respect to θ_1 and β_1 and (21) with respect to θ_2 and β_2 and equating all the derivatives to zero, we get the following set of equations

$$\frac{1}{\hat{\beta}_1} + \frac{\sum_{i=1}^{n} \tilde{p}(t_i,\mathcal{W}(\tilde{\theta}_1,\tilde{\beta}_1))\ln(t_i)}{\sum_{i=1}^{n} \tilde{p}(t_i,\mathcal{W}(\tilde{\theta}_1,\tilde{\beta}_1))} - \frac{\sum_{i=1}^{n} t_i^{\hat{\beta}_1}\ln(t_i)}{\sum_{i=1}^{n} t_i^{\hat{\beta}_1}} = 0, \quad (22)$$

$$\hat{\theta}_1 = \left(\frac{\sum_{i=1}^{n} t_i^{\hat{\beta}_1}}{\sum_{i=1}^{n} \tilde{p}(t_i,\mathcal{W}(\tilde{\theta}_1,\tilde{\beta}_1))}\right)^{\frac{1}{\hat{\beta}_1}}, \quad (23)$$

$$\frac{1}{\hat{\beta}_2}+\frac{\sum_{i=1}^{n}\tilde{p}(t_i,\mathcal{W}(\tilde{\theta}_2,\tilde{\beta}_2))\ln(t_i)}{\sum_{i=1}^{n}\tilde{p}(t_i,\mathcal{W}(\tilde{\theta}_2,\tilde{\beta}_2))}-\frac{\sum_{i=1}^{n}t_i^{\hat{\beta}_2}\ln(t_i)}{\sum_{i=1}^{n}t_i^{\hat{\beta}_2}}=0, \tag{24}$$

$$\hat{\theta}_2=\left(\frac{\sum_{i=1}^{n}t_i^{\hat{\beta}_2}}{\sum_{i=1}^{n}\tilde{p}(t_i,\mathcal{W}(\tilde{\theta}_2,\tilde{\beta}_2))}\right)^{\frac{1}{\hat{\beta}_2}}. \tag{25}$$

The value of $\widehat{\boldsymbol{\theta}}=(\hat{\theta}_1,\hat{\theta}_2,\hat{\beta}_1,\hat{\beta}_2)$ can be obtained by solving the above system of equations. Moreover, once $\widehat{\boldsymbol{\theta}}$ is obtained, we update $\widetilde{\boldsymbol{\theta}}$ by $\widehat{\boldsymbol{\theta}}$ for the next iterations. The process is repeated unless the systematic pattern of convergence or desired level of accuracy is achieved.

4 Bayesian Model Formulation

In order to provide the Bayesian model formulation for analyzing the model in (8), let us begin with specifying the prior distributions for the four parameters involved in the model. We consider independent uniform priors for the two shape parameters β_1 and β_2 and independent conditional priors for θ_1 and θ_2 such that for given β_1 and β_2, the parameters $\theta_1^{\beta_1}$ and $\theta_2^{\beta_2}$ have inverted gamma $\mathcal{IG}(a,b)$ and $\mathcal{IG}(c,d)$ distributions, respectively (see also [1]). The first term in the parenthesis represents the shape parameter of $\mathcal{IG}$ distribution while the second term is the scale parameter. The considered priors for the parameters can be written as

$$\begin{aligned}
&\pi_1(\beta_1)\propto 1; && 0<\beta_1\le 1\\
&\pi_2(\beta_2)\propto \frac{1}{\beta_{2u}-1}; && 1\le\beta_2\le\beta_{2u}\\
&\pi_3(\theta_1|\beta_1)\propto \frac{1}{\theta_1^{\beta_1 a+1}}\exp\left[\frac{-b}{\theta_1^{\beta_1}}\right]; && \theta_1,a,b>0\\
&\pi_4(\theta_2|\beta_2)\propto \frac{1}{\theta_2^{\beta_2 c+1}}\exp\left[\frac{-d}{\theta_2^{\beta_2}}\right]; && \theta_2,c,d>0.
\end{aligned} \tag{26}$$

where a, b, c, $d$and β_{2u} are the hyperparameters. The choice of prior hyperparameters is always crucial and in case of non-availability of any specific information, either subjective or objective, one can consider vague or weak priors for the parameters and allow inferences to remain data dependent. This can be done, for instance, by taking a large choice of β_{2u} in case of β_2. Similarly, in case of θ_1 and θ_2, one can consider choices of a, b, c and d in such a way that resulting $\mathcal{IG}$ distribution becomes more or less flat. It is important to mention here that the standard non-informative priors suggested for the parameters of Weibull distribution cannot be used here as such priors lead to improper posterior in a situation where all the observations arise from one of the two Weibull components in the model (8) (see also [1]).

The prior hyperparameters can also be specified on the basis of experts' opinion if the same are available. Say, for instance, while specifying β_{2u} associated with β_2 of $\mathcal{W}(\theta_2,\beta_2)$ model, the expert conveys that the hazard rate is not increasing abruptly with a steep rise in its behaviour rather rises in such a way that the slope of the hazard rate curve increases from the beginning itself, say at a constant rate. The expert also conveys that the choice of β_2 is such that the corresponding Weibull distribution is exhibiting skewed behaviour with slightly high variability. It is to be noted that for large values of shape parameter, the Weibull distribution approaches to symmetry (close to normality) with very low variability. As such, the expert feels that β_2 cannot be too large and in any case cannot go beyond (say) 10.0, that is, a value suggested for β_{2u}. Similarly, for the $\mathcal{IG}$ prior, the expert can be asked to specify at least two characteristics of the prior model so that two of its hyperparameters can be made known using these two specified characteristics. The characteristics can be simply mean and variance of the $\mathcal{IG}$ distribution or two of its quantiles, etc. Finally, it is essential to mention that the prior of β_1 does not involve any hyperparameter so we do not need any opinion from the expert in this case.

To proceed further, let n items with failure time distribution given in (8) be put on test and let

$t: t_1, t_2, \ldots, t_n$ be the observed failure times. Combining the priors in (26) with the LF in (12) via Bayes theorem yields the corresponding posterior distribution and the same can be written up to proportionality as

$$p(\theta_1, \theta_2, \beta_1, \beta_2 | t) \propto \left\{ \prod_{i=1}^{n} \left[\frac{\beta_1}{\theta_1} \left(\frac{t_i}{\theta_1} \right)^{\beta_1 - 1} + \frac{\beta_2}{\theta_2} \left(\frac{t_i}{\theta_2} \right)^{\beta_2 - 1} \right] \right\} \exp\left[-\sum_{i=1}^{n} \left(\frac{t_i}{\theta_1} \right)^{\beta_1} - \sum_{i=1}^{n} \left(\frac{t_i}{\theta_2} \right)^{\beta_2} \right]$$
$$\times \frac{1}{\theta_1^{\beta_1 a+1}} \exp\left[\frac{-b}{\theta_1^{\beta_1}} \right] \times \frac{1}{\theta_1^{\beta_1 a+1}} \exp\left[\frac{-b}{\theta_1^{\beta_1}} \right] \times I_{(0,1)}(\beta_1) \times I_{(1,\beta_{2u})}(\beta_2); \quad (27)$$
$$\theta_1 > 0,\ \theta_2 > 0,$$

where $I_{(v1,v2)}(\xi)$ is an indicator function that takes value unity if ξ lies in the interval $(v1, v2)$ and zero otherwise. It can be easily seen that the form of the posterior given in (27) is difficult to offer closed form solution and, therefore, one can proceed with sample based approaches. The Gibbs sampler is, however, difficult since the corresponding full conditionals from (27) are not available for easy sample generation. We may, therefore, recommend Metropolis-Hastings algorithm by defining a four-dimensional appropriately centred and scaled candidate generating density (see, for example, [25]) and proceed with ML estimates and the corresponding Hessian based approximation as the initial values for running the chain.

Alternatively, one can use the augmented data structure as given in Section 3 and resort to implementation of the Gibbs sampler algorithm in a routine manner. Following (14), the corresponding likelihood for augmented data structure can be written as

$$L(x|\boldsymbol{\theta}) \propto \frac{\beta_1^{\sum_{i=1}^{n} z_i^1} \beta_2^{\sum_{i=1}^{n} z_i^2}}{\theta_1^{\beta_1 \sum_{i=1}^{n} z_i^1} \theta_2^{\beta_2 \sum_{i=1}^{n} z_i^2}} \prod_{i=1}^{n} \left[t_i^{\beta_1 - 1} \right]^{z_i^1} \prod_{i=1}^{n} \left[t_i^{\beta_2 - 1} \right]^{z_i^2} \exp\left[-\frac{\sum_{i=1}^{n} t_i^{\beta_1}}{\theta_1^{\beta_1}} - \frac{\sum_{i=1}^{n} t_i^{\beta_2}}{\theta_2^{\beta_2}} \right], \quad (28)$$

where z_i $(= (z_i^1, z_i^2))$ is a binary indicator vector associated with t_i, $i = 1,2,\ldots,n$, such that $z_i = (1,0)$ conveys that associated t_i is coming from $\mathcal{W}(\theta_1, \beta_1)$ and $z_i = (0,1)$ results in t_i coming from $\mathcal{W}(\theta_2, \beta_2)$. Also, in case of no censoring z_i^1 and z_i^2 are related as $z_i^1 = 1 - z_i^2$ since our assumption confirms that an observed t_i will certainly come from one of the two Weibull distributions. Next, let us consider the same priors as given in (26) and obtain the joint posterior up to proportionality by combining (26) with (28) via Bayes theorem. This posterior can be written as

$$p_1(\theta_1, \theta_2, \beta_1, \beta_2 | t, z) \propto \frac{\beta_1^{\sum_{i=1}^{n} z_i^1}}{\theta_1^{(\sum_{i=1}^{n} z_i^1 + a)\beta_1 + 1}} \exp\left[-\frac{\sum_{i=1}^{n} t_i^{\beta_1} + b}{\theta_1^{\beta_1}} \right] \prod_{i=1}^{n} \left[t_i^{\beta_1 - 1} \right]^{z_i^1}$$
$$\times \frac{\beta_2^{\sum_{i=1}^{n} z_i^2}}{\theta_2^{(\sum_{i=1}^{n} z_i^2 + c)\beta_2 + 1}} \exp\left[-\frac{\sum_{i=1}^{n} t_i^{\beta_2} + d}{\theta_2^{\beta_2}} \right] \prod_{i=1}^{n} \left[t_i^{\beta_2 - 1} \right]^{z_i^2}; \quad (29)$$
$$\theta_1 > 0, \theta_2 > 0, 0 \le \beta_1 \le 1,\ 1 \le \beta_2 \le \beta_{2u},$$

where $z = (z_1, z_2, \ldots, z_n)$.

Before we proceed further, let us briefly comment on the Gibbs sampler algorithm. Gibbs sampler is a Markovian updating mechanism for extracting samples from (often) high-dimensional posteriors specified up to proportionality by extracting samples from all univariate (or lower dimensional) full conditionals. The generation begins with some appropriately chosen initial values and proceeds in a cyclic frame-work covering all the full conditionals and each time using the most recent values of conditioning variates. The values obtained after one complete cycle represent a state of a Markov chain. This process is repeated until a systematic pattern of convergence is achieved by the generating chain. Once the convergence is achieved, one can either pick up equidistant observations in a single long run of the chain or pick up observations from various parallel chains to form independent and identically distributed samples from the concerned posterior. For details about the Gibbs sampler algorithm, its implementation and convergence diagnostic issues, one can refer to [12], [24] and [25], etc.

Thus in order to apply the Gibbs sampler algorithm, we need to specify all possible full conditionals corresponding to posterior (29). The incomplete specification of data can be resolved by using additional full conditionals corresponding to different indicator variables. We shall begin with a comment on this latter full conditionals and the corresponding generations. Since z_i^1, $i = 1,2,\ldots,n$, is a binary indicator variable, it can be considered to follow a Bernoulli distribution with parameter $\widetilde{p_\iota}$, where

$$\widetilde{p_\iota} = P_r[z_i^1 = 1] = \frac{h_{\mathcal{W}}(t_i,\theta_1,\beta_1)}{h_{\mathcal{W}}(t_i,\theta_1,\beta_1)+h_{\mathcal{W}}(t_i,\theta_2,\beta_2)}.$$

Moreover, using the fact that each z_i^1 is independent of all other z_i^1s, we can generate it independently corresponding to each t_i at the current realization of parameters $(\theta_1,\theta_2,\beta_1,\beta_2)$ and for each generated z_i^1, we can obtain corresponding z_i^2 using the relationship $z_i^2 = 1 - z_i^1$. Rest of the full conditionals corresponding to (29) can be obtained up to proportionality as

$$p_2^*(\theta_1|\theta_2,\beta_1,\beta_2,t,z) \propto \frac{1}{\theta_1^{\left(\sum_{i=1}^n z_i^1+a\right)\beta_1+1}} \exp\left[-\frac{\sum_{i=1}^n t_i^{\beta_1}+b}{\theta_1^{\beta_1}}\right]; \quad \theta_1 > 0 \tag{30}$$

$$p_3^*(\theta_2|\theta_1,\beta_1,\beta_2,t,z) \propto \frac{1}{\theta_2^{\left(\sum_{i=1}^n z_i^2+c\right)\beta_2+1}} \exp\left[-\frac{\sum_{i=1}^n t_i^{\beta_2}+d}{\theta_2^{\beta_2}}\right]; \quad \theta_2 > 0 \tag{31}$$

$$p_4^*(\beta_1|\theta_1,\theta_2,\beta_2,t,z) \propto \frac{\beta_1^{\sum_{i=1}^n z_i^1}}{\theta_1^{\left(\sum_{i=1}^n z_i^1+a\right)\beta_1}} \exp\left[-\frac{\sum_{i=1}^n t_i^{\beta_1}+b}{\theta_1^{\beta_1}}\right] \prod_{i=1}^n \left[t_i^{\beta_1}\right]^{z_i^1}; \; 0 < \beta_1 \le 1 \tag{32}$$

$$p_5^*(\beta_2|\theta_1,\theta_2,\beta_1,t,z) \propto \frac{\beta_2^{\sum_{i=1}^n z_i^2}}{\theta_2^{\left(\sum_{i=1}^n z_i^2+c\right)\beta_2}} \exp\left[-\frac{\sum_{i=1}^n t_i^{\beta_2}+d}{\theta_2^{\beta_2}}\right] \prod_{i=1}^n \left[t_i^{\beta_2}\right]^{z_i^2}; \; 1 < \beta_1 \le \beta_{2u}. \tag{33}$$

Full conditionals (30) and (31) appear to be the kernels of $\mathcal{IG}$ distribution. Hence by considering transformation $\lambda_1 = \theta_1^{\beta_1}$, one can show that $\lambda_1 \sim \mathcal{IG}(\sum_{i=1}^n z_i^1 + a, \sum_{i=1}^n t_i^{\beta_1} + b)$. Similarly, $\lambda_2 = \theta_2^{\beta_2}$ can be shown to follow $\mathcal{IG}(\sum_{i=1}^n z_i^2 + c, \sum_{i=1}^n t_i^{\beta_2} + d)$. Hence both θ_1 and θ_2 can be generated using any standard routine for inverse gamma generator (see, for example,[8]). The full conditionals (32) and (33) can be shown to be logconcave hence both β_1 and β_2 can be simulated easily using (say) adaptive rejection sampling (ARS) algorithm proposed by [14].

4.1 Implementation of Gibbs sampler in case of censored data

An appreciable property of Gibbs sampler algorithm is that it can be easily extended to deal with censored data situations. The idea is very simple in the sense that the algorithm proceeds with exactly the same posterior as specified for the complete data case but assumes censored observations as further unknowns. Thus, in censored data case, the Gibbs sampler algorithm has additional full conditionals corresponding to censored observations. Obviously, the full conditionals for the other parameters will remain same to those obtained for the complete data case whereas the full conditionals for the independent censored observations will be the parent sampling distributions truncated in the appropriate regions (for details, see [26]).

As mentioned in Section 3, each t_i is associated with an indicator variable δ_i such that if $\delta_i = 0$, t_i is the right censoring time and if $\delta_i = 1$, t_i is the observed failure time corresponding to the item i, $i = 1,2,\ldots,n$. Obviously, the full conditionals corresponding to $z, \theta_1, \theta_2, \beta_1$ and β_2 are same that were obtained earlier for complete data case. The additional full conditionals correspond to censored observations that can be generated from the left truncated $\mathcal{BW}_{\mathcal{R}}$ distribution. The generation can be simple and the variate value t_i can be retained if it lies in the constrained region (t_i, ∞). For the generation from truncated $\mathcal{BW}_{\mathcal{R}}$ model, a simple two-step algorithm can be designed based on the following theorem.

Theorem 1 *Suppose T_1 and T_2 be two random variables following $\mathcal{W}(\theta_1, \beta_1)$, $\beta_1 < 1.0$ and $\mathcal{W}(\theta_2, \beta_2)$, $\beta_2 > 1.0$, respectively, and suppose both are truncated from left at the same point c. If we define a random variable $T = min(T_1, T_2)$, then T will follow left truncated $\mathcal{BW}_{\mathcal{R}}$ distribution with parameters $(\theta_1, \theta_2, \beta_1, \beta_2)$, the point of truncation from left is at c.*

Proof. The proof of the Theorem (1) is added to Appendix given at the end.

The generation of random variable T from truncated $\mathcal{BW}_{\mathcal{R}}$ distribution involves the following two steps. First, generate T_1 and T_2 from left truncated $\mathcal{W}(\theta_1, \beta_1)$, $\beta_1 < 1.0$ and $\mathcal{W}(\theta_2, \beta_2)$, $\beta_2 > 1.0$, respectively. Second, take $T = \min(T_1, T_2)$. The resulting T will follow left truncated $\mathcal{BW}_{\mathcal{R}}$ distribution.

5 Model Compatibility

Model compatibility study is an important concept in any statistical data analysis which provides an assurance that the entertained model is rightly used for the data in hand. If the model is compatible with the data, the analysis is of course justified. In case of poor resemblance with the data, it is desired to consider an alternative model that best represents the entertained data. The model compatibility study can be performed in a variety of ways. The classical statisticians, of course, use tail area probabilities based on a discrepancy measuring statistic to provide agreement or disagreement of data with the model.

Bayesian statistics offers a number of tools for studying model compatibility. The simplest and an informal approach may involve checking predictive capability of the model with regard to some of its important characteristics possibly using graphical tools (see, for example,[19]). A practical approach to implement this idea in reliability studies may entail investigating the empirical plots of observed data based and some of the posterior predictive data based reliability characteristics on the same graphical scale. Some of the important reliability characteristics in this context may be considered as hazard rate function, reliability function, mean time to failure, etc. Thus one can consider plotting the observed data based entertained characteristic and correspondingly the predictive data based same characteristic where predictive data are generated from the posited model. Such a graphical tool will not only provide an informal assessment of discrepancy between the model and the data but also sometimes help in improving the model (see also Upadhyay et al. (2001)).

Bayesian study on model compatibility can also be extended in an objective manner using tail area probability or the p-value based on a discrepancy measuring statistic under the assumption that the considered model is true for data. A number of versions of Bayesian p-values are defined in the literature based on several considerations, each having its own merit or demerit. We shall not discuss these details here due to space restriction rather refer to Bayarri and Berger (1998) for a systematic accountability. In this paper, we shall use posterior predictive p-value based on chi-square discrepancy measure (see also Upadhyay et al. (2001)). Although the posterior predictive p-value (PPV) has its own disadvantages, the most important being double use of data, we shall use it for its inherent simplicity and also because of the fact that it is easily computable for any choice of prior. A brief review about the PPV is given in the next subsection.

5.1 Posterior predictive p-value

[15] proposed the use of p-value based on the posterior predictive distribution of model

departure statistics (see also [26]). The idea is very simple. To begin with, let us define the chi-square statistic as a measure of discrepancy given by

$$\chi^2 = \frac{\sum_{i=1}^{n}(t_i - E(t_i|\boldsymbol{\theta}))^2}{V(t_i|\boldsymbol{\theta})}, \tag{34}$$

where $E(.)$ denoets the operation of taking expectation and $V(.)$ denotes the variance. Accordingly, the PPV based on χ^2 discrepancy measure can be obtained as

$$PPV = \int P_r[\chi_2^2 > \chi_1^2|f, \boldsymbol{\theta}]p(\boldsymbol{\theta}|t)d\boldsymbol{\theta}, \tag{35}$$

where χ_1^2 and χ_2^2 are the calculated values of χ^2 for the observed and predictive data sets, respectively, $p(\boldsymbol{\theta}|t)$ is posterior distribution of $\boldsymbol{\theta}$ and f is the entertained model. For complete data case, PPV can be calculated using the procedure suggested by [26] (see also [13]) which consists of two steps. The first step is to draw $\boldsymbol{\theta}$ from $p(\boldsymbol{\theta}|t)$ and calculate χ_1^2 based on the given data set. The second step is to extract predictive data sets each of same size as that of given data from the model f using the simulated $\boldsymbol{\theta}$ and calculate χ_2^2 based on these predictive data sets. We then calculate $P_r[\chi_2^2 > \chi_1^2|f, \boldsymbol{\theta}]$ as the number of times χ_2^2 exceeds χ_1^2. These steps are repeated a number of times with different simulated $\boldsymbol{\theta}$ and PPV is estimated as the posterior expectation of $P_r[\chi_2^2 > \chi_1^2|f, \boldsymbol{\theta}]$.

In order to evaluate PPV in situation where data set has some right censored observations, one can first complete the data set by replacing all censored observations with the maximum of their respective censoring times and predictive means (see [11]). The PPV can then be calculated using this completed data set in the same way as described above for complete data case.

6 Numerical Illustration

For analyzing the proposed $\mathcal{BW}_{\mathcal{R}}$ model, we considered a real data set reported initially by [9]. The dataset consists of failure times of 58 electrodes (segments cut from bars) which were put on a high-stress voltage endurance life test. Observations on failure time from voltage endurance test are given in Table 1. First, fourth and seventh columns of the table list the observations on failure times of electrodes in hours whereas second, fifth and eighth column represent failure modes, that is, the causes of failures. The failures were attributed to one of two modes (causes) which are as under. The first cause is the insulation defect due to a processing problem (mode E) which tends to occur early in life. The second cause, on the other hand, is degradation of the organic material (mode D) which typically occurs at a later stage. Third, sixth and ninth columns of the table indicate completely observed or censored failure times. $\delta = 0$ indicates observed failure times while $\delta = 1$ indicates censored failure times. Since, for censored observations, failure cause is unknown, we have denoted the missing cause by '$*$' (see Table 1).

In order to analyze the model for the assumed dataset, we first considered only those observations which were completely observed and left those observations which were censored. As such, we formed a new sample where all the failure times are completely observed. Besides, we also omitted the two causes of failures (E and D) so that the observations can be treated appropriate for the considered competing risk model with latent (unknown) causes of failures. We next considered the entire sample treating it as a case of censored data problem but omitted the two causes of failures (E and D) for the appropriateness of proposed competing risk model with latent causes of failures.

In the first part of our analysis, we considered obtaining ML estimates of $\mathcal{BW}_{\mathcal{R}}$ parameters using EM algorithm after visualizing the model as an incomplete data model and introducing missing indicator variables z^1 for failure mode E and z^2 for failure mode D as described in Section 3. The ML estimates for the parameters θ_1, θ_2, β_1 and β_2 were found to be 885.030(1209.506), 341.553(343.841), 0.613(0.629) and 5.545(5.592), respectively. The values in parenthesis correspond to the estimates of parameters for censored data case.

Table 1: Voltage endurance life test results of 58 electrodes.

Hours	Failure Mode	δ	Hours	Failure Mode	δ	Hours	Failure Mode	δ
2	E	0	144	E	0	303	E	0
3	E	0	157	*	1	314	D	0
5	E	0	160	E	0	317	D	0
8	E	0	168	D	0	318	D	0
13	*	1	179	*	1	320	D	0
21	E	0	191	D	0	327	D	0
28	E	0	203	D	0	328	D	0
31	E	0	211	D	0	328	D	0
31	*	1	221	E	0	348	*	1
52	*	1	226	D	0	348	D	0
53	*	1	236	E	0	350	D	0
64	E	0	241	*	1	360	D	0
67	*	1	257	*	1	369	D	0
69	E	0	261	D	0	377	D	0
76	E	0	264	D	0	387	D	0
78	*	1	278	D	0	392	D	0
104	E	0	282	E	0	412	D	0
113	*	1	284	D	0	446	D	0
119	E	0	286	D	0			
135	*	1	298	D	0			

For Bayesian analysis of the model, we first consider specification of prior hyperparameters of the considered prior distributions. In this context, we begin with the choice of $\beta_{2u} = 7.0$ following the argument given in Section 4 so that the shape of the hazard rate curve is increasing at a constant rate from the beginning itself and the Weibull distribution does not approach to normality in a real sense. We next assume that given β_1, the expert has provided two characteristics of $\mathcal{IG}(a, b)$ distribution, that is, the expert specifies the mean and variance of $\theta_1^{\beta_1}$ to be 200 and 1000, respectively. Similarly, we assume that given β_2, the expert specifies both mean and variance of $\theta_2^{\beta_2}$ to be very large, say of order 1.0 e+10. It may be noted that since the expert is not available in a real sense when specifying mean and variance of $\theta_1^{\beta_1}$ and $\theta_2^{\beta_2}$, we have taken help of ML estimates of various parameters as well. Utilizing ML estimates is certainly an objective consideration but there is no harm if data based information is used in forming the appropriate priors. Also, the large variability for both $\theta_1^{\beta_1}$ and $\theta_2^{\beta_2}$ convey a kind of vagueness in the choice of priors. Based on these choices, the prior hyperparameters a and b were evaluated to be 5.0 and 600.0, respectively. Similarly, the prior hyperparameters c and d were found to be 6.0 and 5.0e+08, respectively.

In order to provide the posterior based inferences of model (8), the posterior samples were extracted using the Gibbs sampler algorithm as described in Section (4) for complete data case and Subsection 4.1 for censored data case. In each case, we considered a single long run of Gibbs chain using the ML estimates of the parameters as the starting values for running the chain. For censored case, we also used known censoring time for each censored observation as the initial value of the corresponding censored observation. Convergence of the chain was monitored using the ergodic averages at about 20K iterations in each case. Once the convergence was achieved a sample of size 2K was taken from the corresponding posterior distribution by picking up equidistant observations (at a gap of 10). The gap was chosen to make the serial correlation among the generating variates negligibly small (see also [26]). It may be noted that the assessment of a specific risk in the presence of other risk factors is of particular interest in a competing risk scenario and, therefore, we also obtained a sample of size 2K for the probability of failure due to insulation defects (mode E) at each iterated value of posterior variates (see also (10)).

Some of the important sample based estimated posterior characteristics were obtained based on the final sample of size 2K. We do not report here all the characteristics except the estimated posterior modes and the highest posterior density (HPD) intervals with coverage probability 0.95 for all the parameters considering both complete and censored data cases (see Table 2). However, while writing the conclusion, some other characteristics, not reported in the paper, may be taken into account as well. Posterior modes are given because these are the most probable values and can be reasonably considered to be the Bayes estimates. Similarly, HPD limits will provide to a large extent the overall idea of the estimated posterior densities. Table 2 also provides the corresponding estimates for censored data case and these estimates are shown in parentheses. Besides the model parameters, the table also exhibits the estimated posterior modes and HPD limits for the probability of failure due to mode E (see (10)).

Table 2: **Estimated posterior modes and HPD limits with coverage probability 0.95 for $\mathcal{BW}_{\mathcal{R}}$ parameters**

Parameters	Estimated posterior mode	HPD limits	
		lower	upper
θ_1	702.131 (950.466)	289.561 (345.505)	3819.390 (5734.279)
β_1	0.663 (0.610)	0.497 (0.500)	0.823 (0.845)
θ_2	340.758 (342.718)	317.392 (319.954)	375.069 (372.495)
β_2	5.311 (5.594)	5.138 (5.415)	5.562 (5.824)
$P_r(t = t_1)$	0.349 (0.310)	0.189 (0.199)	0.546 (0.557)

Values in parentheses correspond to censored data case.

It can be seen that the Bayes estimates in both complete and censored data cases are more or less similar to the classical ML estimates except in case of θ_1 for censored data case where ML estimate appears to be slightly overestimated value. The finding, therefore, confirms the vague consideration of priors with choice of hyperparameters guided to some extent by ML estimates. Based on the HPD limits and the estimated posterior modes, it can be concluded that the parameters θ_2, β_1 and β_2 are more or less symmetric with small variability whereas the parameter θ_1 is highly positively skewed with very large variability (see Table 2). This last conclusion was also confirmed by the characteristics such as estimated posterior means and medians, the values of which are not shown in the paper. A word of remark: since the posterior variability of θ_1 is quite large and the variability is appreciably increased for censored data case, the difference between Bayes and ML estimates, especially for censored data case, can be considered marginal only. Another important and striking conclusion is that the censoring does not cause appreciable loss of information as the estimates corresponding to complete and censored data case are quite close to each other although we have considered only 22 % observations to be censored. The Table 2 also shows the estimated posterior mode and HPD limits with coverage probability 0.95 for $P_r(t = t_1)$. It can be seen that nearly 35% of the observations (31% for censored data case) are failed due to initial birth defect, a conclusion that appears to be quite close to the true entertained values reported in Table 1.

Figure 3: Estimated posterior densities and bi-variate characteristics for complete data case.

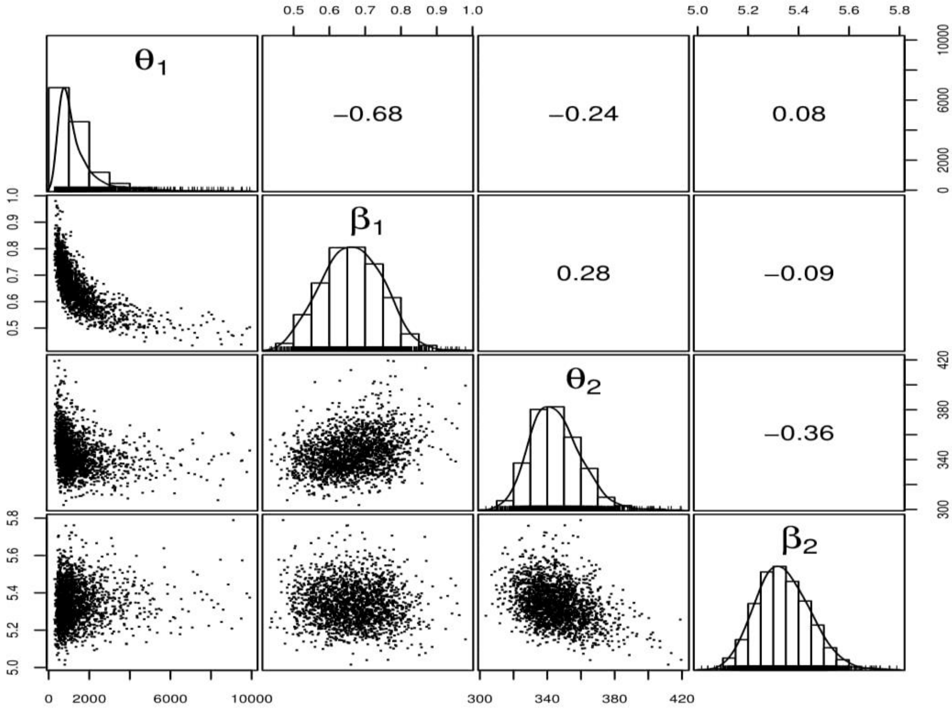
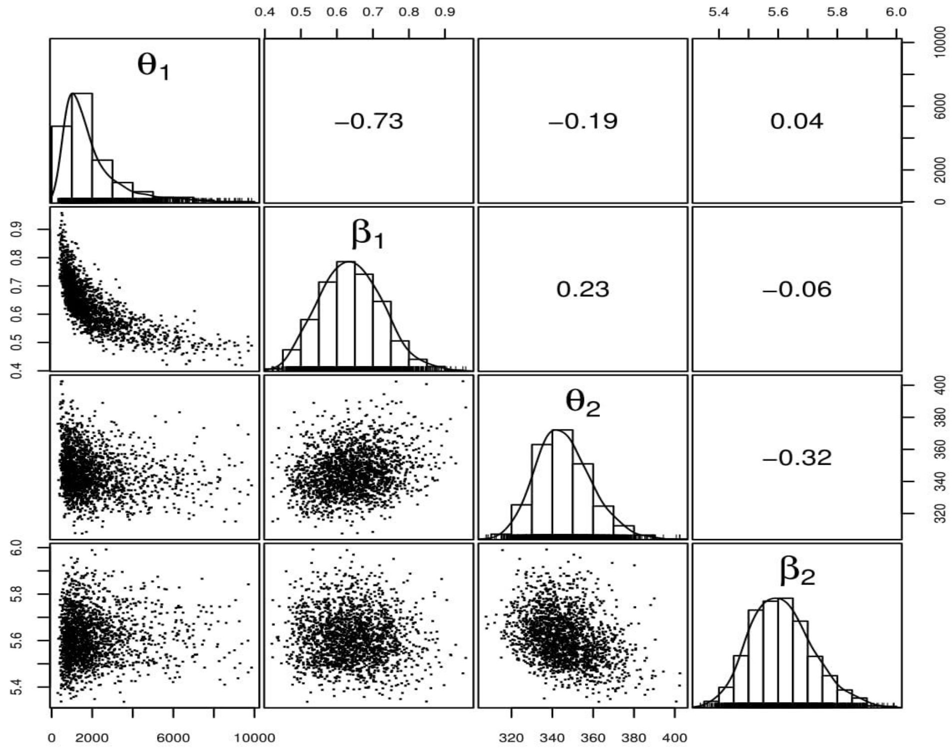


Figure 4: Estimated posterior densities and bi-variate characteristics for censored data case.



We also worked out for the bivariate posterior characteristics for both complete and censored data cases, which are shown in Figures 3 and 4 in the form of scatter plots and estimated posterior correlations although the figures also display the estimated marginal posterior densities. As far as the marginal densities are concerned, the conclusions are same that are discussed in the previous paragraph and, therefore, we do not need to interpret these estimated densities anymore. Based on the scatter plots and the estimated correlations, it can be concluded that the parameters θ_1 and β_1 are highly correlated *a posteriori*, the estimated correlation being -0.68. Similarly, the parameters θ_2 and β_2 are also exhibiting good correlation, the estimated value for the same being -0.33. In this very sense the conditional priors for θ_1 and θ_2 given in (26) are justified to some extent. The remaining parameter combinations (θ_1, θ_2), (θ_1, β_2), (β_1, θ_2) and (β_1, β_2) are, however, having relatively low estimated correlations, the estimated values being -0.21(-0.19), 0.03(0.04), 0.26(0.23) and -0.03(-0.06), respectively. Again the values within parentheses correspond to censored data case.

To proceed further, we estimated the change point as given in (6) based on the final posterior samples of size 2K. The change point is certainly an important characteristic which differentiates between the two modes of failure. That is, the units having failure times less than the change point can be said to have failed due to early birth defect and the failure of units with failure times exceeding it can be attributed to degradation failure. The estimated posterior mode and HPD limits with coverage probability 0.95 for t^* were found to be 107.240{109.467} and [92.269, 125.235]{[95.320, 125.764]}, respectively, where the values in curly parentheses correspond to censored data case. In addition to these estimates, we also obtained the posterior estimates of hazard rate at different time points including at the change point. The estimates in the form of posterior modes based on a sample of size 2K are given in Table 3. It can be seen that the estimated hazard rate is least at the change point time and increasing as we move away from the change point time in either direction, a conclusion that was expected too.

Table 3: **Posterior estimates of hazard rate at different times**

t=80	t=100	t=t^*	t=150	t=200
0.121e-02 (0.147e-02)	0.115e-02 (0.141e-02)	0.113e-02 (0.139e-02)	0.125e-02 (0.147e-02)	0.224e-02 (0.254e-02)

Values in parentheses correspond to censored data case.

Before we end the section, let us examine the compatibility of the model with the entertained data based on the ideas discussed in Section 5. For this purpose, we first generated a posterior sample of size 20 using the Gibbs sampler algorithm (subsection 4.1) and correspondingly obtained 20 predictive samples each of same size as that of informative data. We then considered the non-parametric empirical hazard rate estimates for both observed and predictive data sets at the time corresponding to the two data sets. Since some of the observations in the original data are censored, we implemented our procedure after completing the data by replacing the censored observations by the maximum of their predictive means and censoring times. The corresponding plots are shown in Figure 5 where solid line represents the informative data based estimated hazard rate and the dotted lines represent the predictive data based estimated hazard rate.

A similar strategy was used to draw observed data and correspondingly predictive data based estimates of reliability function. This latter plot is shown in Figure 6. It can be seen that in both the cases the solid line is well superimposed by the dotted lines (see Figures 5-6) giving us a clear cut conclusion that the model is justified for the data in hand.

We next obtained the numerical summary of model compatibility study in the form of PPV discussed in Section 5. To calculate the same, we first considered 100 posterior samples corresponding to the model $\mathcal{BW}_{\mathcal{R}}$ using the Gibbs sampler algorithm (subsection 4.1) and then obtained χ_1^2 for the given data set for each generated posterior sample of $\boldsymbol{\theta}$. As a second step, we simulated 1K predictive samples from (8), each of size exactly similar to that of the observed data, for each value of $\boldsymbol{\theta}$ and calculated χ_2^2 based on these predictive data sets. Our next step calculated $P_r[\chi_2^2 > \chi_1^2]$ for each given $\boldsymbol{\theta}$. Finally, the above steps were repeated to calculate PPV as described in Section 5. The estimated PPV was found to be 0.331, a value that again confirms the compatibility of the model (8) with the observed data set. A word of remark: for the compatibility study considered here, we do not report the results corresponding to complete data case by omitting the censored observations as it was done earlier in obtaining other estimates. The results for the complete data case were more or less similar and, therefore, avoided due to space restriction.

Figure 5: Estimated hazard rate plots corresponding to observed (solid line) and predictive data sets.

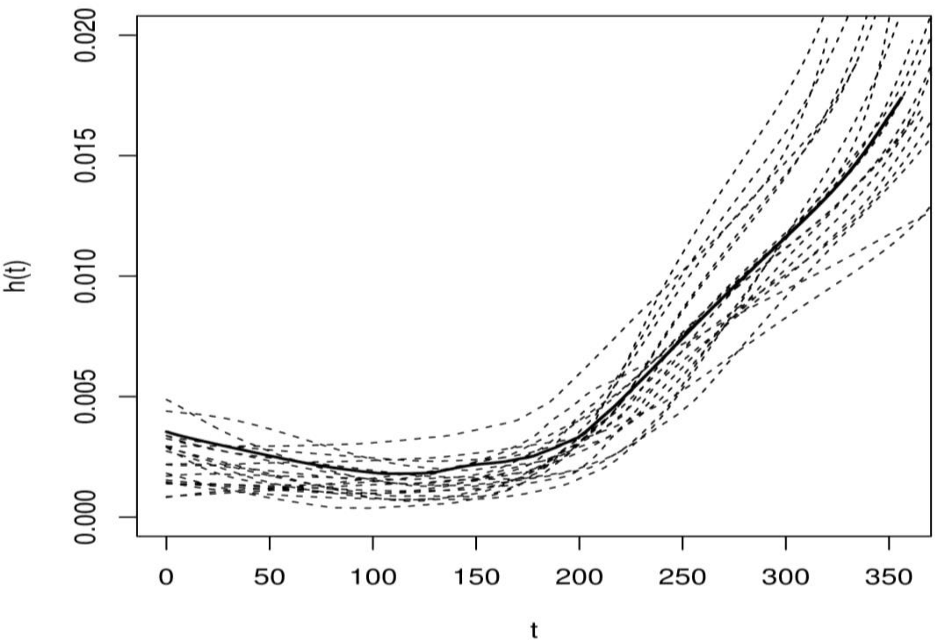
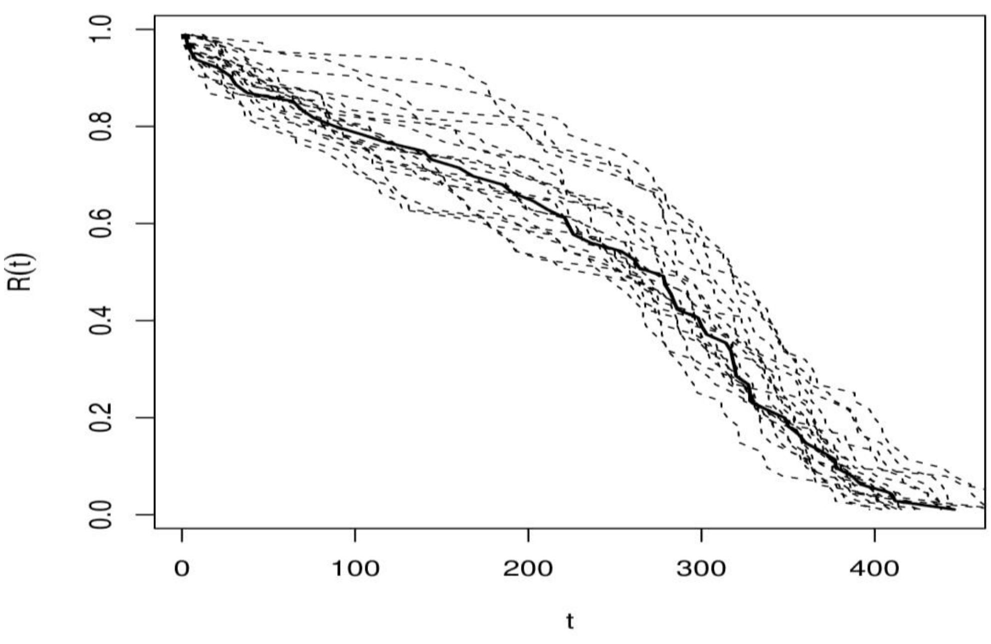


Figure 6: Estimated reliability plots corresponding to observed (solid line) and predictive data sets.



7 Conclusion

The paper considers a $\mathcal{BW}_{\mathcal{R}}$ model, a model that is quite popular in a competing risk scenario. Our assumption, however, considers two competing causes, the initial birth defect and ageing effect, that work together to induce failure of a unit. The considered model is analyzed in both classical and Bayesian frameworks although the classical analysis focuses on ML estimates only. The important feature of the corresponding likelihood function is that it offers unique consistent solution in the form of ML estimates. The paper finally shows how the idea of visualizing the competing risk model as an incomplete data model facilitates both classical and Bayesian analyses of the considered model not only for complete case but also for censored data situation. The applicability of model is also justified by conducting a Bayesian compatibility study of the model with a real data set involving a voltage endurance life test with two different failure modes.

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Appendix

Proof of Theorem 1

Proof. Let T_1 and T_2 follow $\mathcal{W}(\theta_1,\beta_1)$, $\beta_1 < 1.0$ and $\mathcal{W}(\theta_2,\beta_2)$, $\beta_2 > 1.0$, respectively, where both are truncated from left at a point c. The densities of T_1 and T_2 can be written as

$$f_{W_1}^{c}(t_1|t_1>c)=\frac{\beta_1}{\theta_1}\left(\frac{t_1}{\theta_1}\right)^{\beta_1-1}\exp\left[-\left(\frac{t_1}{\theta_1}\right)^{\beta_1}+\left(\frac{c}{\theta_1}\right)^{\beta_1}\right],$$

$$f_{W_2}^{c}(t_2|t_2>c)=\frac{\beta_2}{\theta_2}\left(\frac{t_2}{\theta_2}\right)^{\beta_2-1}\exp\left[-\left(\frac{t_2}{\theta_2}\right)^{\beta_2}+\left(\frac{c}{\theta_2}\right)^{\beta_2}\right],$$

respectively. Similarly, the cumulative distribution functions of T_1 and T_2 can be written as

$$F_{W_1}^{c}(t_1|t_1>c)=1-\exp\left[-\left(\frac{t_1}{\theta_1}\right)^{\beta_1}+\left(\frac{c}{\theta_1}\right)^{\beta_1}\right],$$

$$F_{W_2}^{c}(t_2|t_2>c)=1-\exp\left[-\left(\frac{t_2}{\theta_2}\right)^{\beta_2}+\left(\frac{c}{\theta_2}\right)^{\beta_2}\right].$$

If we define a random variable $T=min(T_1,T_2|T_1,T_2>c)$ then the cumulative distribution function of T can be obtained as

$$F_{\mathcal{BW}_{\mathcal{R}}}^{c}(t)=1-\left[1-F_{W_1}^{c}(t|t>c)\right]\left[1-F_{W_2}^{c}(t|t>c)\right],$$

which simlifies to

$$F_{\mathcal{BW}_{\mathcal{R}}}^{c}(t)=1-\exp\left[-\left(\frac{t}{\theta_1}\right)^{\beta_1}+\left(\frac{c}{\theta_1}\right)^{\beta_1}\right]\exp\left[-\left(\frac{t}{\theta_2}\right)^{\beta_2}+\left(\frac{c}{\theta_2}\right)^{\beta_2}\right].$$

Now differentiating $F_{\mathcal{BW}_{\mathcal{R}}}^{c}(t)$ with respect to t, we can obtain the density function of T as

$$f_{\mathcal{BW}_{\mathcal{R}}}^{c}(t)=\frac{\left[\frac{\beta_1}{\theta_1}\left(\frac{t}{\theta_1}\right)^{\beta_1-1}+\frac{\beta_2}{\theta_2}\left(\frac{t}{\theta_2}\right)^{\beta_2-1}\right]\exp\left[-\left(\frac{t}{\theta_1}\right)^{\beta_1}-\left(\frac{t}{\theta_2}\right)^{\beta_2}\right]}{\exp\left[-\left(\frac{c}{\theta_1}\right)^{\beta_1}-\left(\frac{c}{\theta_2}\right)^{\beta_2}\right]}.$$

which is the density of $\mathcal{BW}_{\mathcal{R}}$ distribution with parameters $(\theta_1,\theta_2,\ \beta_1,\beta_2)$, truncated from left at point c. Hence we can say that $T = min(T_1,T_2)$ follows $\mathcal{BW}_{\mathcal{R}}(\theta_1,\theta_2,\beta_1,\beta_2)$, left truncated at point c.

Existence of Unique and Consistent Roots of Likelihood Equations

Theorem 2 *(see [3]). Let $f(t|\nu)$ be the pdf with parameter vector $\nu=(\nu_1,\nu_2,\dots,\nu_k)$, then the solution of the likelihood equation for the observation vector t*

$$\frac{\partial\sum_{i=1}^{n}ln(f(t_i|\nu))}{\partial\nu_r}=0,$$

will be unique and consistent if the following three conditions hold. For the sake of brevity, we shall write f for $f(t|\nu)$ and f_i for $f(t_i|\nu)$.

(i). For almost all t and $\nu\in\Theta$, $\frac{\partial\ln(f)}{\partial\nu_r},\frac{\partial^2\ln(f)}{\partial\nu_r\,\partial\nu_s},\frac{\partial^3\ln(f)}{\partial\nu_r\,\partial\nu_s\,\partial\nu_w}$ exist for all r,s,w = 1,2,...,k.

(ii). For almost all t and $\nu\in\Theta$, $|\frac{\partial f}{\partial\nu_r}|<F_r(t)$, $|\frac{\partial^2 f}{\partial\nu_r\,\partial\nu_s}|<F_{rs}(t)$, $|\frac{\partial^3 f}{\partial\nu_r\,\partial\nu_s\,\partial\nu_w}|<H_{rsw}(t)$, where $H_{rsw}(t)$ is such that $\int_{-\infty}^{\infty}H_{rsw}(t)f dt<C_M<\infty$ and $F_r(t)$ and $F_{rs}(t)$ are bounded for all t.

(iii). For all $\nu\in\Theta$, the matrix J = $\left((J_{rs}(\nu))\right)$, where

$$J_{rs}(\nu)=\int_{-\infty}^{\infty}\frac{\partial\ln(f)}{\partial\nu_r}\frac{\partial\ln(f)}{\partial\nu_s}f dt$$

is positive-definite and that $|J|$ is finite.

Now the conditions of above theorem may be verified for the likelihood equations corresponding to $\mathcal{BW}_{\mathcal{R}}$ model as follows. (It may be noted that the following proof verifies

$Chanda$'s theorem for the likelihood equations corresponding to right censored data case as well since for the censored observation, the expression of density for $\mathcal{BW}_{\mathcal{R}}$ model becomes $f(c_1) = \exp\left[-(\frac{c_1}{\theta_1})^{\beta_1} - (\frac{c_1}{\theta_2})^{\beta_2}\right]$ instead of (8) where c_1 denotes the censoring time (see also [2])).

Let us denote f for $f_{\mathcal{BW}_{\mathcal{R}}}(t)$. The function f is differentiable with respect to all the parameters $\theta_1, \theta_2, \beta_1$ and β_2 any number of times. Under the assumption that all these parameters are positive and finite, any arbitrary ℓ^{th} order partial derivative of f (let it be denoted by g) can only produce term of the following form

$$g = [\sum_{j,k=0,j+k\leq\ell}^{\ell} \left(\ln\left(\frac{t}{\theta_1}\right)\right)^j \left(\ln\left(\frac{t}{\theta_2}\right)\right)^k \Phi_{jk}(t)]\exp[-\left(\frac{t}{\theta_1}\right)^{\beta_1} - \left(\frac{t}{\theta_2}\right)^{\beta_2}]; \quad j = 0,1,\ldots,\ell; k = 0,1,\ldots,\ell, \tag{36}$$

where $\Phi_{jk}(t)$s are polynomials in t, hence continuous and bounded in any closed interval. $\ln(t)$ is also continuous for $t > 0$, which indicates that the term $[\sum_{j,k=0,j+k\leq\ell}^{\ell} \left(\ln\left(\frac{t}{\theta_1}\right)\right)^j \left(\ln\left(\frac{t}{\theta_2}\right)\right)^k \Phi_{jk}(t)]$ is continuous and bounded for t in any closed interval. For large values of t, behavior of $[\sum_{j,k=0,j+k\leq\ell}^{\ell} \left(\ln\left(\frac{t}{\theta_1}\right)\right)^j \left(\ln\left(\frac{t}{\theta_2}\right)\right)^k \Phi_{jk}(t)]$ is dominated by the term $\exp[-\left(\frac{t}{\theta_1}\right)^{\beta_1} - \left(\frac{t}{\theta_2}\right)^{\beta_2}]$ and also $\exp\left[-\left(\frac{t}{\theta_1}\right)^{\beta_1} - \left(\frac{t}{\theta_2}\right)^{\beta_2}\right] \to 0$ as $t \to \infty$. Hence,

$$|[\sum_{j,k=0,j+k\leq\ell}^{\ell} \left(\ln\left(\frac{t}{\theta_1}\right)\right)^j \left(\ln\left(\frac{t}{\theta_2}\right)\right)^k \Phi_{jk}(t)]\exp[-\left(\frac{t}{\theta_1}\right)^{\beta_1} - \left(\frac{t}{\theta_2}\right)^{\beta_2}]| < \infty.$$

With this we can say that all partial derivatives of density f exist. Now for verifying condition (i) of Theorem 2, we have to show that all the first, second and third order partial derivatives of $\ln(f)$ exist. These partial derivatives can be written as

$$\frac{\partial \ln(f)}{\partial v_r} = \frac{1}{f}\frac{\partial f}{\partial v_r} \tag{37}$$

$$\frac{\partial^2 \ln(f)}{\partial v_r \partial v_s} = \frac{1}{f}\frac{\partial^2 f}{\partial v_r \partial v_s} - \frac{1}{f^2}\frac{\partial f}{\partial v_r}\frac{\partial f}{\partial v_s} \tag{38}$$

$$\begin{aligned}\frac{\partial^3 \ln(f)}{\partial v_r \partial v_s \partial v_w} &= 2\frac{1}{f^3}\frac{\partial f}{\partial v_r}\frac{\partial f}{\partial v_s}\frac{\partial f}{\partial v_w} - \frac{1}{f^2}\frac{\partial^2 f}{\partial v_r \partial v_s}\frac{\partial f}{\partial v_w} - \frac{1}{f^2}\frac{\partial^2 f}{\partial v_s \partial v_w}\frac{\partial f}{\partial v_r} \\ &- \frac{1}{f^2}\frac{\partial^2 f}{\partial v_r \partial v_w}\frac{\partial f}{\partial v_s} + \frac{1}{f}\frac{\partial^3 f}{\partial v_r \partial v_s \partial v_w}\end{aligned} \tag{39}$$

All the partial derivatives of f that are involved in the above expressions are earlier shown to exist and since $0 < f < \infty$ for $t > 0$. Hence the derivatives in (37)-(39) exist and the condition (i) is verified.

Since for $t > 0$, $\ln(t) < t$, hence by replacing $\ln(t)$ by t and negative signs in $\Phi_{jk}(t)$s by positive signs in expression (36), we can always find a function $A(t) = [a_1 t^{\alpha_1} + \ldots + a_k t^{\alpha_k}]$ and a positive number N such that

$$|[\sum_{j,k=0,j+k\leq\ell}^{\ell} \left(\ln\left(\frac{t}{\theta_1}\right)\right)^j \left(\ln\left(\frac{t}{\theta_2}\right)\right)^k \Phi_{jk}(t)]| < A(t)$$

and

$$\exp[-\left(\frac{t}{\theta_1}\right)^{\beta_1} - \left(\frac{t}{\theta_2}\right)^{\beta_2}] \leq e^{-Nt}.$$

From these, we have

$$|g| < A(t)e^{-Nt}$$

and $A(t)e^{-Nt}$ is bounded for all $t > 0$. Therefore, the two parts of condition (ii) are satisfied. For proving the third part of condition (ii), we have the third order partial derivatives of f with respect to the parameters which are of the following form

$$\frac{\partial^3 f}{\partial v_1 \partial v_2 \partial v_3}; 2.0cm v_j \varepsilon \{\theta_1, \theta_2, \beta_1, \beta_2\}, 0.5cm for 0.1cm j = 1,2,3.$$

On similification, it can produce a form

$$[\sum_{j,k=0, j+k\le 3}^{3} \left(\ln\left(\frac{t}{\theta_1}\right)\right)^j \left(\ln\left(\frac{t}{\theta_2}\right)\right)^k \Phi_{jk}(t)] \times \exp[-\left(\frac{t}{\theta_1}\right)^{\beta_1} - \left(\frac{t}{\theta_2}\right)^{\beta_2}].$$

Again, corresponding to each $\frac{\partial^3 f}{\partial v_1 \partial v_2 \partial v_3}$, we can easily find a polynomial $A_1(t)$ such that

$$|\frac{\partial^3 f}{\partial v_1 \partial v_2 \partial v_3}| < A_1(t) \times \exp[-\left(\frac{t}{\theta_1}\right)^{\beta_1} - \left(\frac{t}{\theta_2}\right)^{\beta_2}].$$

Let us denote $A_1(t) \times \exp[-\left(\frac{t}{\theta_1}\right)^{\beta_1} - \left(\frac{t}{\theta_2}\right)^{\beta_2}]$ by $H(t)$. For $H(t)$, we can write

$$\int_0^\infty H(t)f(t)dt = \int_0^\infty A_1'(t)\exp[-2\left(\frac{t}{\theta_1}\right)^{\beta_1} - 2\left(\frac{t}{\theta_2}\right)^{\beta_2}]dt,$$

where $A_1'(t) = [b_1 t^{\gamma_1} + \ldots + b_m t^{\gamma_m}]$. Since $t \ge 0$ and $\theta_1 > 0, \theta_2 > 0$, hence replacing $\exp[-2\left(\frac{t}{\theta_2}\right)^{\beta_2}]$ by its maximum value, which is 1, we can write

$$\int_0^\infty H(t)f(t)dt < \int_0^\infty [b_1 t^{\gamma_1} + \ldots + b_m t^{\gamma_m}]\exp[-2\left(\frac{t}{\theta_1}\right)^{\beta_1}]dt. \tag{40}$$

Let us consider a integral of the following type

$$\int_0^\infty t^{\gamma_j}\exp[-2\left(\frac{t}{\theta_1}\right)^{\beta_1}]dt = \frac{\theta_1^{\gamma_j+1}}{\beta_1 2^{\frac{(\gamma_j+1)}{\beta_1}}}\Gamma\left(\frac{\gamma_j+1}{\beta_1}\right) = \delta_j(say),$$

where δ_j is a moment of generalized $gamma$ density and exists for all $\gamma_j > -1, \beta_1 > 0$ and $\theta_1 > 0$. Hence, each term in integral of right hand side of inequality (40) results in constant multiple of moment of generalized $gamma$ density, which exist. If we denote $\sum_{j=1}^{m} b_j \delta_j$ by C_M, we can write

$$\int_0^\infty H(t)f(t)dt < C_M < \infty.$$

With this, third part of condition(ii) is verified.

For verifying condition (iii), we need to prove the matrix J= $((J_{rs}(v)))$ positive definite. The matrix J can be written as

$$J = \int_{t=0}^\infty \begin{bmatrix} (\frac{\partial\ln(f)}{\partial\theta_1})^2 & (\frac{\partial\ln(f)}{\partial\theta_1})(\frac{\partial\ln(f)}{\partial\theta_2}) & (\frac{\partial\ln(f)}{\partial\theta_1})(\frac{\partial\ln(f)}{\partial\beta_1}) & (\frac{\partial\ln(f)}{\partial\theta_1})(\frac{\partial\ln(f)}{\partial\beta_2}) \\ (\frac{\partial\ln(f)}{\partial\theta_2})(\frac{\partial\ln(f)}{\partial\theta_1}) & (\frac{\partial\ln(f)}{\partial\theta_2})^2 & (\frac{\partial\ln(f)}{\partial\theta_2})(\frac{\partial\ln(f)}{\partial\beta_1}) & (\frac{\partial\ln(f)}{\partial\theta_2})(\frac{\partial\ln(f)}{\partial\beta_2}) \\ (\frac{\partial\ln(f)}{\partial\theta_1})(\frac{\partial\ln(f)}{\partial\beta_1}) & (\frac{\partial\ln(f)}{\partial\beta_1})(\frac{\partial\ln(f)}{\partial\theta_2}) & (\frac{\partial\ln(f)}{\partial\beta_1})^2 & (\frac{\partial\ln(f)}{\partial\beta_1})(\frac{\partial\ln(f)}{\partial\beta_2}) \\ (\frac{\partial\ln(f)}{\partial\beta_2})(\frac{\partial\ln(f)}{\partial\theta_1}) & (\frac{\partial\ln(f)}{\partial\beta_2})(\frac{\partial\ln(f)}{\partial\theta_2}) & (\frac{\partial\ln(f)}{\partial\beta_2})(\frac{\partial\ln(f)}{\partial\beta_1}) & (\frac{\partial\ln(f)}{\partial\beta_2})^2 \end{bmatrix} f dt \tag{41}$$

$$J = \int_{t=0}^\infty \left\{ \begin{bmatrix} \frac{\partial\ln(f)}{\partial\theta_1} \\ \frac{\partial\ln(f)}{\partial\theta_2} \\ \frac{\partial\ln(f)}{\partial\beta_1} \\ \frac{\partial\ln(f)}{\partial\beta_2} \end{bmatrix} \times \begin{bmatrix} \frac{\partial\ln(f)}{\partial\theta_1} & \frac{\partial\ln(f)}{\partial\theta_2} & \frac{\partial\ln(f)}{\partial\beta_1} & \frac{\partial\ln(f)}{\partial\beta_2} \end{bmatrix} \right\} f dt. \tag{42}$$

Let $\frac{\partial\ln(f)}{\partial v}$ denotes the vector of all first order partial derivatives of $\ln(f)$ with respect to parameters $v = (\theta_1, \theta_2, \beta_1, \beta_2)$ and can be written as

$$\frac{\partial \ln(f)}{\partial v} = \begin{bmatrix} \frac{\partial \ln(f)}{\partial \theta_1} \\ \frac{\partial \ln(f)}{\partial \theta_2} \\ \frac{\partial \ln(f)}{\partial \beta_1} \\ \frac{\partial \ln(f)}{\partial \beta_2} \end{bmatrix}.$$

Mathematically, any matrix A that can be written as $A = BB'$, where B may be square or rectangular, is at least positive semidefinite. Hence matrix in the integral of equation (41) is at least positive semidefinite. The matrix J is expected value of the matrix $(\frac{\partial \ln(f)}{\partial v})(\frac{\partial \ln(f)}{\partial v})'$. Thus J has a covariance structure and such a structure can be singular only when two or more elements in vector $\frac{\partial \ln(f)}{\partial v}$ are linearly related. But in our case, the elements are partial derivatives of same function with respect to different parameters, that is, $\frac{\partial \ln(f)}{\partial v_j}$; $j = 1,2,3,4$ and all v_js are independent of each other. Hence there is no linear relationship among the elements $\frac{\partial \ln(f)}{\partial v_j}$. Obviously, J is nonsingular or in other words J is positive definite. Again since all the first order partial derives of $\ln(f)$ are shown to exist and finite, hence determinant of matrix J is also finite.

Therefore, we can say that the density f offers unique and consistent ML estimates of the parameters.

Comparison of usage of crowdsourcing in traditional and agile software development methodologies on the basis of effectiveness

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Abstract

The authors here try to forecast the Effort in Person Months for developing Agile and the traditional way of software development including Prototyping. The comparison is made on the basis of considering both Agile and traditional software development methodologies in addition to the crowdsourcing paradigm applied to both approaches. DNA Matcher (DNAM) has been developed using both prototyping and Agile software development with crowd sourcing. For Agile development, first of all COCOMO II Model is applied on it utilizing the crowdsourcing technique. The authors have determined that Agile development proves to be considerably economical when both techniques use crowdsourcing. The case study used here is DNAM. First DNAM was developed using traditional prototyping methods. During its analysis, costing is done. This is done in accordance with the crowdsoursing used in parallel to the Prototyping method. The time and effort in Person Months (PM) was known. Then AGILE development methodology is used in the development of DNAM. Agile is used along with the crowdsourcing paradigm. As soon as the analysis phase is completed, Simple Build-up Approach forecasts the time and effort in terms of Number of Iterations and Person Months and we compare the results of Effort and Cost of both the techniques. The Agile method is found to be both, less in cost and effort, thereby increasing the Effectiveness and Efficiency of the progression of Software development.

Keywords: Crowdsourcing, Genomic Information Retrieval, Agile software development, COCOMO, Simple Build-up Method

1. Introduction

The DNAM System [1] comprises of 4 Modules that are refined though roles that came into existence though the Goals which are at first identified and for the most basic step in prototype design. The DNAM (Genomic Information Retrieval) System takes as input a DNA Pattern that has to be searched in the varied heterogeneous databases. Some even consisting of plain Text Files or Excel Sheets [3]. Here, the authors inspect the pertinence of Crowdsourcing and a couple of procedures from Open Innovation to the intelligent technique and major science in a non-advantage condition [4], discover a U-shaped association between the convenience time and winning in the two sorts of difficulties. Social capital lifts the probability of winning a gathering assessed challenge just if the social capital is enough high [5], report a preliminary give an account of crowdsourcing testing for informational endeavors. We introduce three business programming things as enlightening testing wanders, which are crowdsourced by our indicating candidly strong system. We call this "Semi Crowdsourcing Test" (QCT) in light of the fact that the contender workers are understudies, who have certain social relations. The examination occurs are encouraging and show to be favorable to both the understudies and industry in QCT wanders. [6], Revolves just around a legitimate perspective and on the frameworks available to these affiliations. The consequent estimations are pre choice of supporters, accessibility of partner responsibilities, collection of duties, and pay for responsibilities. By gathering the strategies of 46 crowdsourcing outlines, we perceive 19 unmistakable process creates. A resulting group examination exhibits general cases among these sorts and demonstrates an association with particular usages of crowdsourcing. [7], will demonstrate that the crowdsourcing model of research can make hurt individuals, controls the part into continued with help, and uses individuals as exploratory subjects. We assume that traditions relying upon this model require institutional review board (IRB) examination. [8], attempts to get an unrivaled appreciation of what crowdsourcing systems are and what general arrangement points are considered in the change of such structures. In this paper, the maker drove a ponder composing review in the space of crowdsourcing structures.

[10], portrayed the qualities of some conventional and agile techniques that are generally utilized as a part of software development. I have likewise talked about the qualities and shortcoming between the two contradicting procedures and furnished the difficulties related with actualizing agile procedures in the software industry. This episodic confirmation is rising with respect to the adequacy of agile methodology in specific situations; however there have not been much gathering and investigation of observational proof for agile products. Notwithstanding, to help my exposition I led a poll, requesting criticism from programming industry professionals to assess which approach has a superior achievement rate for various sizes of software development. As per our discoveries dexterous procedures can give great advantages to little scaled and medium scaled undertakings yet for substantial scaled activities customary strategies appear to be overwhelming.

UI Module deals with taking the DNA pattern (Consisting of four letters, A,C,T and G sequences) [3]. It forwards this to the module which by looking at the routing information forwards the segmented sub-patterns to the various “source modules”. The source agents construct a local ontology and the patterns are searched in those ontologies. Then the results are combined to form a merged ontology and the links to the data sources are sent back to the user.

OBJECTIVE

DNA SEQUENCING

A genome contains the complete set of DNA including all the genes. And this includes all the information required to build and uphold that organism[23]. A great amount of Genetic material is similar between organisms [13]. Understanding the relationship of an unknown gene or DNA sequence to known sequences is the key to assigning its function.
The main theme of interest for a Computer Science Researcher in the field of Genetics can be given as under:-
The Nucleotide Bases are used to identify the characteristics of all the species. The Nucleotide Bases [16] comprise of the following elements:

1) Thymine (T)
2) Cytosine (C)
3) Guanine (G)
4) Adenine (A)

It is agreed that we have a big, monolithic anthology of heterogeneous and distributed databases containing the Nucleotide Base Sequences of many species. But, the size and divergence of this raw information makes it very difficult for the Biologists to search for a "Pattern of some sequence" in these databases. The Biologists or Life Science Experts need to search for a specific sequence in the existing sequences stored in the databases. This pattern matching is done in order to find homologous patterns. This is helpful because if suppose there is a mutation in DNA of an organism and this mutated DNA is the reason for some problem eg. a type of cancer in that organism. Now if this mutated pattern matches with some other organism, and that organism got treated with some enzyme, the research can be directed to that hormone and can be used to remedy the cancer in the original organism. The DNAM project is taken as a case study for comparison of Agile with crowdsourcing and traditional prototyping also assisted by crowdsourcing.

CROWDSOURCING WITH TRADITIONAL PROTOTYPING

The work of Prototyping with crowd sourcing begins with identification of goals that need to be accomplished. The goal diagram is the best pictorial representation for what needs to be done at different levels of abstraction. The goals are broken down into sub tasks and this phenomenon is called task reduction resulting into Work breakdown Structure (WBS). The sub divided goals need someone to take the responsibility to put them into action. Once the Goal diagram is drawn we assign different goals to the identified Roles as shown in the figure 2.
Now the process of assigning Roles to various identified crowd sourcing worker needs to be executed.
The various crowd sourcing workers work on these segmented modules:

- Initiator
  This module deals with the starting up of the system eg: reading the configuration file etc.
- User Interface
  Deals with sub-modules like the way the end user will interact with the system.
- Wrapper
  Deals with modules for routing the DNA Pattern over the network.
- Source
  The sub-modules in Source module take care of platform transparency such that information from heterogeneous data sources can be extracted ad the DNA Pattern be searched.

This is shown in figure 3.

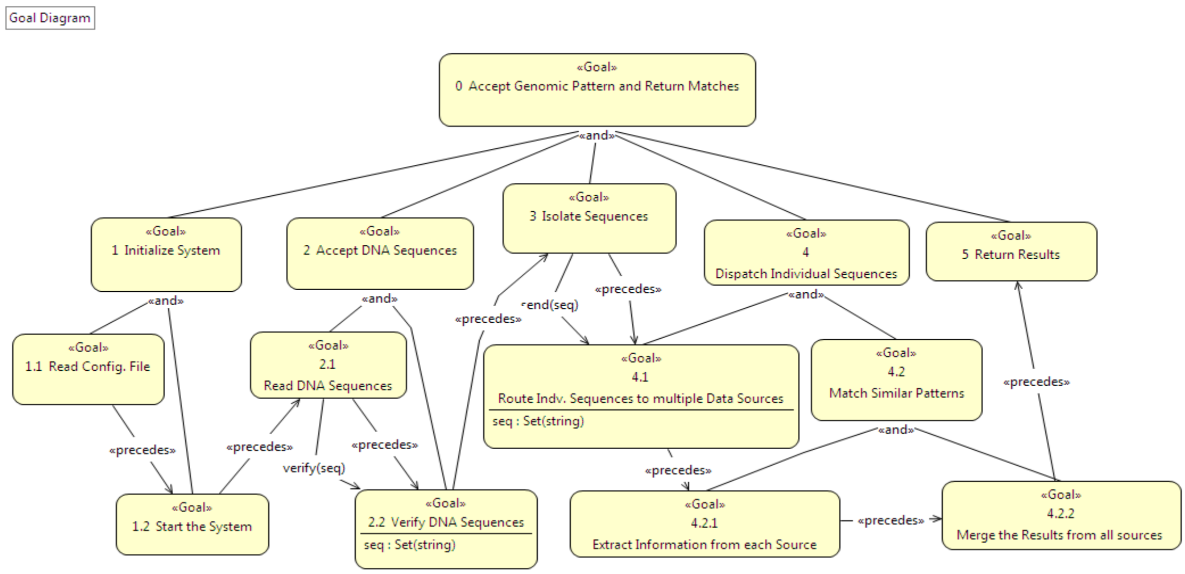

Fig. 1 DNAM Goal Diagram

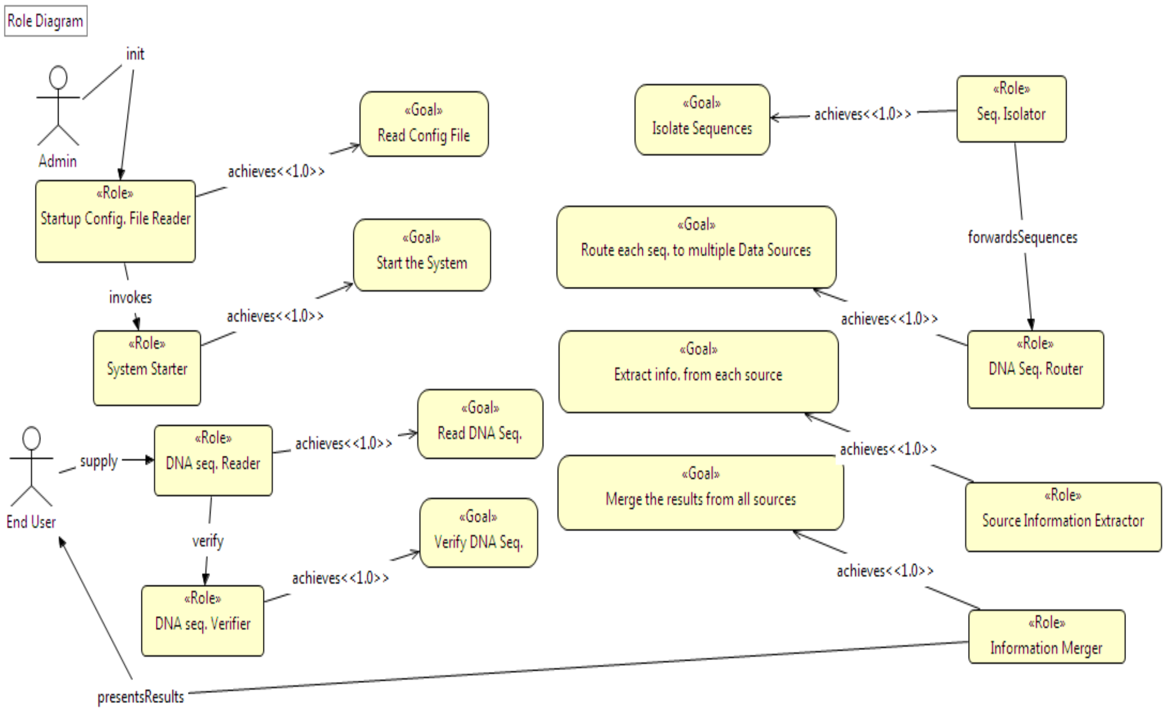

Fig. 2. DNAM Role Diagram

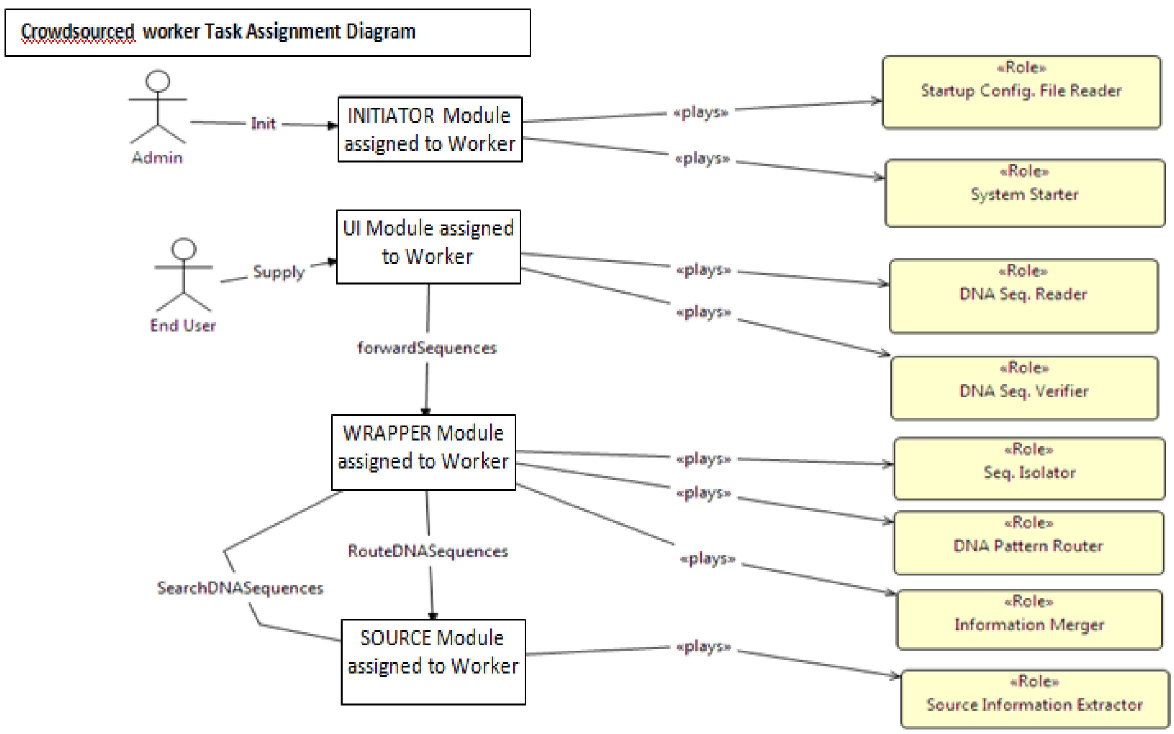

Fig. 3. DNAM CrowdSourced Worker Task Assignment Diagram

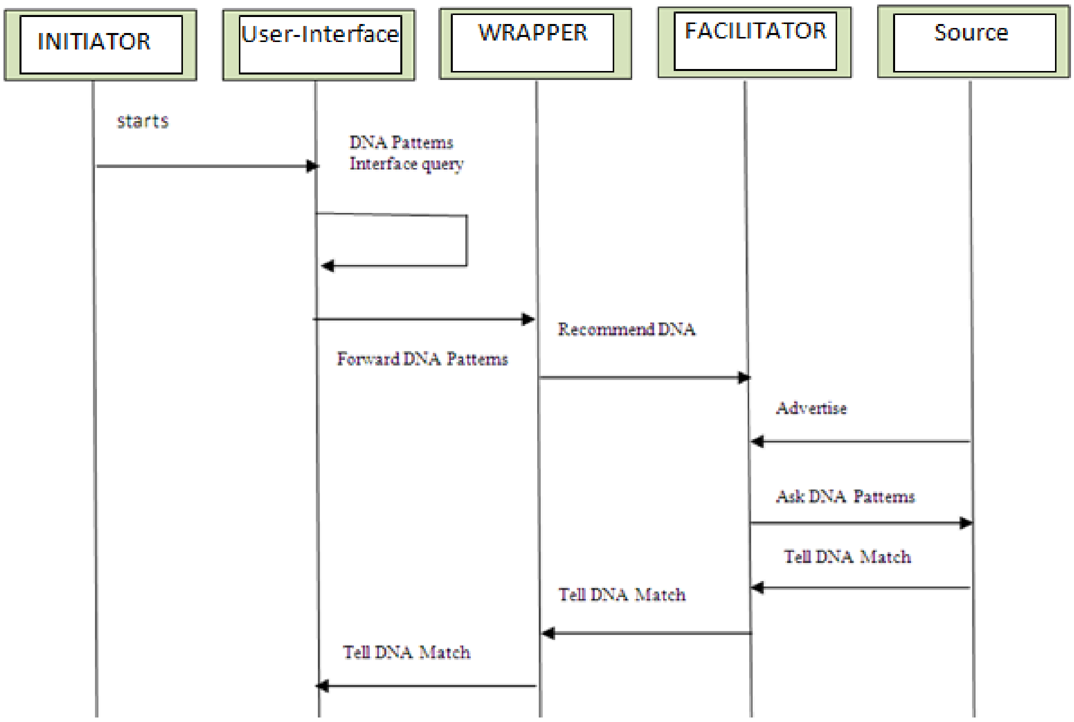

Fig. 4. Interactions and Messages Exchanged Between Modules

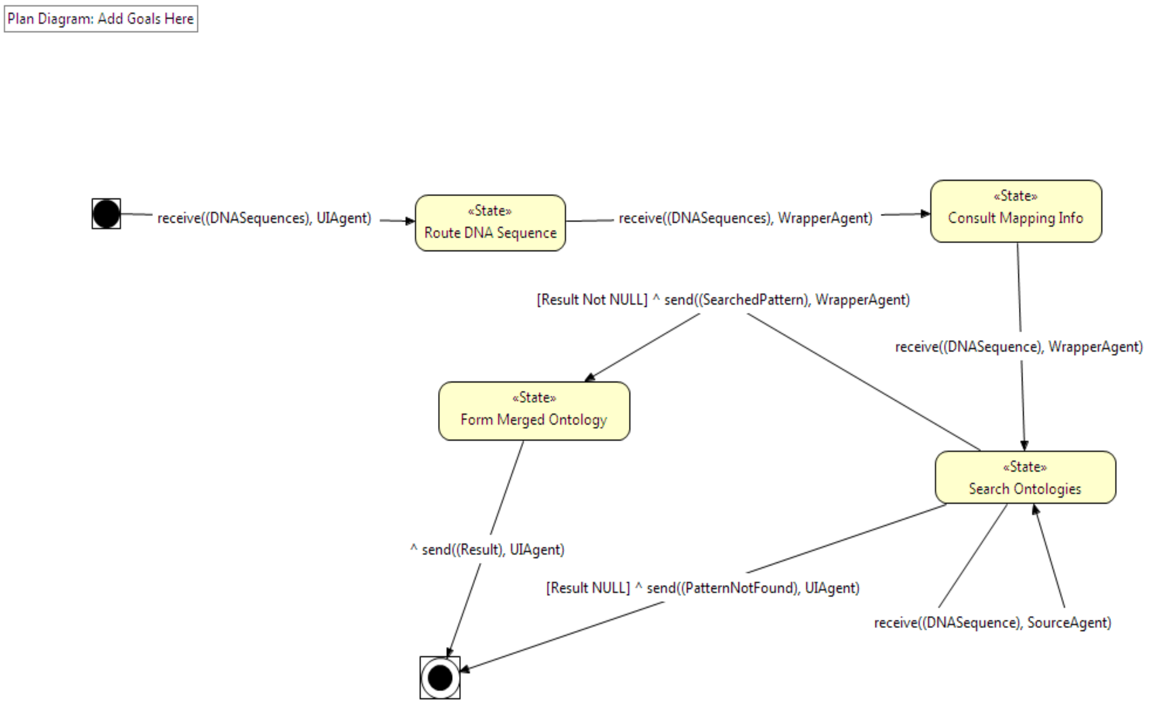

Fig. 5. DNAM State Diagram

2. COST ESTIMATION MODEL COCOMO II in Traditional (Prototyping) Software Development approaches combined with CrowdSourcing

There are many mathematical formulae used to forecast Effort required in developing a Software System in different methodologies like Waterfall, Spiral or Incremental and Agile Approaches. I have used COCOMO II Early Design Model [15] using 5 scale factors and 7 Effort Multipliers to project size (SLOC), effort (Person Months) and time (Months). The scale factors and Effort Multipliers are derived from the Analysis done in a traditional case study involving the same Project DNAM(Genomic Information Retrieval) accompanied by Crowdsourcing.

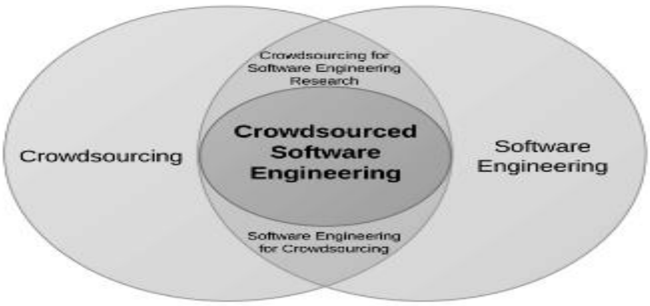

Fig. 6. Coagulation of Traditional Software Engineering and CrowdSourcing

The diagrams are refined from Highest Level of Abstraction to Lowest Level of Abstraction. High Abstraction deals with Goals and Plans and Lowest deal with the State Machines and coding and implementation.

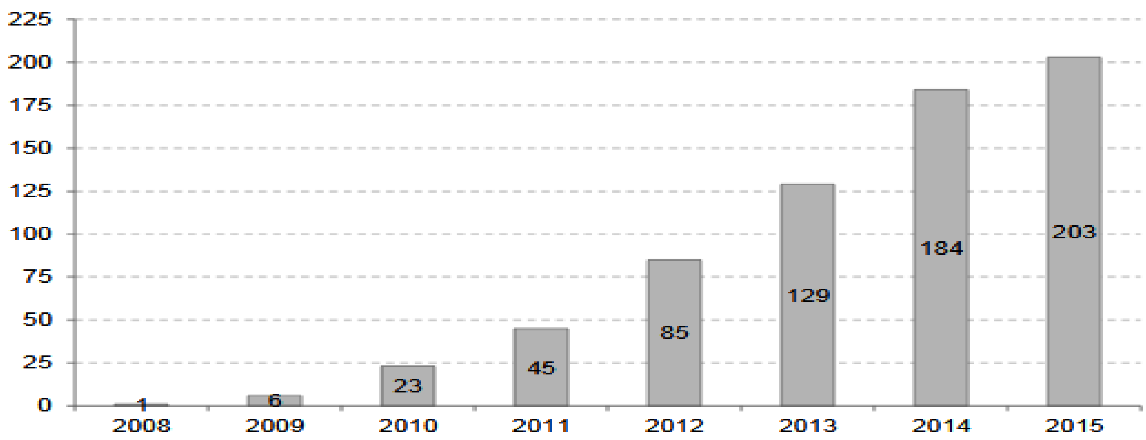

Fig. 7. Cumulative growth of Crowdsourced Software Engineering studies published before April 2015.

2. 1 Effort Estimation

The values for A, B, EMi and SFj in COCOMO II in Agile are standardized values taken from the effort of 161 projects in the model database. The formula for Effort can referred from Boehm (2000:13). In case of DNAM only 7 Effort multipliers are used to remove confusion.

2.1.1 DNAM through Prototyping (Traditional SDLC)

$$PM = A \text{ x Size}^{E} \prod_{i=1}^{17} EM_i$$

A = 2.94 (for COCOMO II.2000) (1)

$$E = B + 0.01 \sum_{j=1}^{5} SF_j$$

B = 0.91 (for COCOMO II.2000) (2)

The values and calculations provided in the charts shown under represent the analysis done on Agile Development techniques involving crowdsoursing.

Table 1
Details identified of Agents Through UML Design Toolkit in Prototype Model

Element	**DNAM (Prototype)**
Total No. of Interactions with other modules	6
Tot. No. of messages interchanged	9
Tot. No. of Roles	8
Tot. No. of Requirements	8
Tot. No. of Events	9
Total no. of state machines applied	4
Tot. No. of states in every machine.	4

Rules identified in DNAM
R1: Maximum Time Limit: 30ns
R2: ONLY A, C, T, or G characters can be used to represent a DNA Pattern.
R3: The search string cannot contain white spaces, hyphens or any special character or numbers.
R4: Max Length of a search pattern can be set but assumed to be 1024 chars.
R5: Break-up size of search pattern can be set but assumed to be 11 chars.
R6: Patterns not conforming to P1, P2, P3 and P4 will be immediately discarded.
R7: Total time from query submission to result display can be maximum 8.00 secs.
R8: Queries from Agile to crowds failing to meet R6 will be held for resubmission.

Table 2
Statistical Data About Functional Behaviour

Element	**DNAM (Prototype)**
Tot. No. of Roles R	8
Tot. No. of Events E	9
Tot. No. of State Machines S	4
Total number of types of entities (R+E+S)	21
Rules dedicated to management of entities	8

Table 3
Scale Factors Applied To The Projects(Both in Agile and Traditional S/W Development)

Scale Factors	**Range (DNAM Prototype)**	**Value (DNAM Prototype)**
Precedentness	Nominal	3.72
Development Flexibility	Low	6.24
Architecture/ Risk Resolution	Low	6.24
Team Cohesion	Low	2.19
Process Maturity	Low	6.24
		18.110

2. 1.2 Scale Factors

The exponential part of the application is governed by five scale factors (SF) that describe relative economies or diseconomies of scale. A project has economies of scale if the exponent is less than 1. if Exponent=1 then Economies and diseconomies of scale are in balance. If the exponent is more than one the project has diseconomies (Boehm 2000:30).

Table 4
Cost Drivers

Cost Drivers	**Range(Prototype)**	**Value(Prototype)**
Product Reliability and Complexity	High	1.10
Reusability	Nominal	1.00
Platform difficulty	Low	1.10
Personnel Capability	Nominal	1.00
Personnel Experience	Nominal	1.00
Facilities	Nominal	1.00
Required Development Schedule	Low	1.14 ΠVal=1.38

In addition to the scale factors there are other relevant factors that affect the efforts done by the developer, called cost drivers [15]. These are:

1.Product Reliability and Complexity (RCPX) is the combination of Software Reliability (RELY), Database Size (DATA), Software Complexity (CPLX) and Documentation (DOCU).

2.Reusability (RUSE)

3.Platform Difficulty (PDIF) is the combination of Time Constraint (TIME), Main Storage Constraint (STOR) and Platform Volatility (PVOL).

4.Personnel Capability (PERS) is Analyst Capability (ACAP), Programmer Capability (PCAP) and Personnel Continuity (PCON).

5.Personnel Experience (PREX) combines Analyst Experience (AEXP), Programmer Experience (PEXP) and Language and Tools Experience (LTEX).

6.Facilities (FCIL) combines Uses of Software Tools (TOOL) and Site Environment (SITE)

7.Required Development Schedule (SCED)

The values for scale and cost factors are taken from the COCOMO Manual http://csse.usc.edu/csse/research/COCOMOII/cocomo2000.0/CII_modelman2000.0.pdf

Table 5
Equivalence Of Each Elements Into Sloc

Element	**Element Count DNAM(Prototype)**	**SLOC / Element (Prototype)**	**SLOC(Prototype)**
Event	9	100	900
Rule	8	80	640
Goal	8	200	1600
Task	8	100	800
State Machines	4	150	600
TOTAL SLOC	--------------------	---------------------	4540
			4.5 KSLOC

- KSLOC decreases in Agile development due to the modern languages like PYTHON, RUBY, etc where we just need the required Tools (.dll files) that get easily integrated with our project.

Now Through Eq. 2:

$$PM = A \text{ x Size}^{E} \prod_{i=1}^{17} EM_i$$

A = 2.94 (for COCOMO II.2000)

$$E = B + 0.01 \sum_{j=1}^{5} SF_j$$

B = 0.91 (for COCOMO II.2000)

Eprototype = 0.91+ 0.01*(18.110) = 1.091

PMprototype = $2.94*(4.5)^{1.091*}(1.38)$**=20.94PM**

This means that a single croudsourced worker can complete the whole process in nearly 21 months.

If there are n workers the job can be accomplished in **21/n** months.

3 Analysis and Estimation of DNAM project using Agile with Crowdsourcing

3.1 Terminology

3.1.1 User Story

A "User Story" is a simple statement about what a user wants to do with a feature and the value the user will gain from that feature.

1. The User
2. What the user wants to do with a feature
3. Value gained by the user
4. Acceptance Criteria
5. Front and back of a card

3.1.2 Story Points

A story Point represents a value given to a user story that is used to measure the effort required to

implement the story.
It is a number that represents story size based on how hard a story is to implement.

3.1.3 Story Point Velocity

Velocity measures how much of something can be achieved over a fixed period of time; e.g. how many Story Points are completed during a Sprint.

3.1.4 Simple Buildup Approach for Agile Project Cost Estimation

Velocity(V)=Iteration Duration / Completed Total SP………........................1
Iterations needed = Total SP / Velocity ……………………………..2

In most cases a story point uses one of the following scales for sizing:
• 1,2,4,8,16
• X-Small, Small, Medium, Large, Extra-Large (known as "T-Shirt Sizing")
• Fibonacci sequence: 1,2,3,5,8,13,21

3.1.5 Crowdsourcing the Agile Software Development

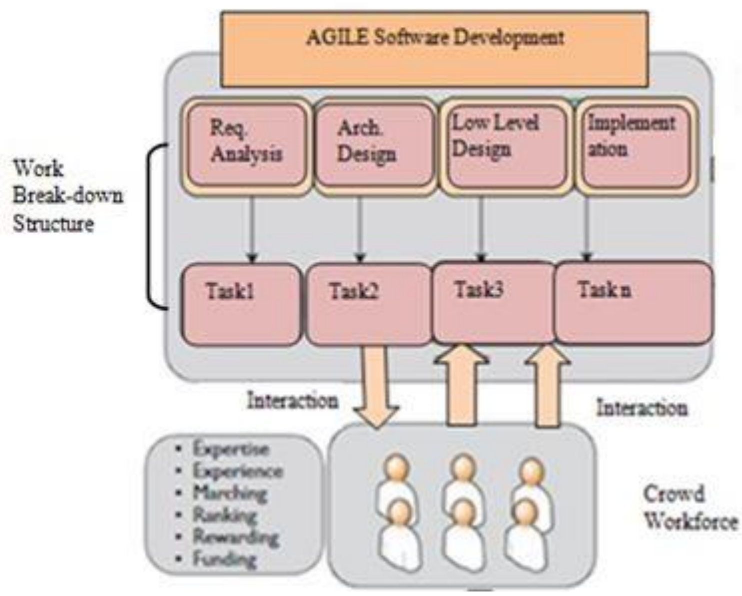

Fig. Agile with Crowdsourcing

The process of crowdsourcing Agile Software development begins with the construction of Work Break-down Structure, also called Task Reduction. Figure 2 shows this process. Now when the work is decomposed into tasks, Goals are identified by grouping similar tasks together. It is not always the case that multiple tasks unite to form a goal. A standalone task can result into a single goal as well.

3.2 Identifying Stories

Login SP=1 > Registered user can log into the system. > Unsuccessful Login should be prompted with an Error Message. > User should be allowed login attempts upto 3 times	Accept DNA pattern SP=1 > DNA Input must only contain 'A', 'C','T' or 'G' characters. > No whitespaces, special symbols or digits are permissible. > The input DNA Pattern's length must only be 11 characters.
Forward Pattern to Wrapper module SP= 2 > Route patterns to source modules. > Refer ontology > Merge results > Return matched results back to UI module	Receival of Pattern by Source Module SP=2 > Convert local database to XML > search local repository > Find matches > Pass matched DNA content to Wrapper Module
User Interface SP=1 > Accept matched DNA sources > Return matched DNA sources list to the end user > Ready to accept new DNA patterns as query	

The authors have developed first two User Stories.
By equation 1 and 2,

Velocity(V)=2/2=1;
Total Iterations needed=Total SP/V=7/1=7;
if 1 iteration is assumed to be 2 weeks, then there are total 14weeks=**3.11months** and team size is 4.

In case of Prototype Model used along with crowdsourcing, we got the total time needed for development=21/n
Since in Agile with crowdsourcing we took the team size to be equal to 4,
Time consumed by Prototype with Crowdsourcing=21/4= **5.25 months**

This represents a significant increase in efficiency of the Agile developers in crowdsourced projects. As regarding the quality of these types of methods a brief theoretical comparison is shown under:

Costs & Benefits attached with Crowdsourcing

- ❖ Concerns with regard to cost:

- Reduced overhead Cost (OC)
- Increased Development Cost (DC)
- Schedule
- The problem with scheduling traditional software development is project completion dates vary with different developers. The cool undertaking does sometimes incur greater cost and time. Let us say, costs through scheduling to be CS.

Concerns with regard to Quality

Practical Quality of the software is thought to be met, when the predetermined necessities are met. The prerequisites ought to be successfully assembled. In conventional software development, the dispersal of assignments amid the arranging stage is less demanding when contrasted with crowdsourcing software development. Likewise coordinating distinctive modules after their advancement is finished is moderately less complex. Quality confirmation in crowdsourcing software development can be extremely testing. It turns out to be exceptionally troublesome and wrong to expel the bugs in the later phase of the software lifecycle. A portion of the difficulties experienced while keeping up the nature of software are: partitioning expansive work into little assignments, sorting out and conveying successfully. Plainly the general population associated with crowdsourcing is not a piece of any comparable association subsequently there is an issue of contradiction.

4. Conclusion and Future Scope

This paper tries to apply Effort Forecasting in Genomic Information Retrieval (DNAM) taken as case study for comparing Traditional and Agile System currently developing under crowdsorucing. The authors predicted that Agile is better in reducing development time thereby increasing efficiency greater as compared to traditional SDLC Models. The estimation in prototyping is done using COCOMO II Model amalgamating crowdsoucing. The estimation of Agile with crowdsourcing is done using Simple Buildup Approach. In this paper the authors have tried to calculate the Effort required for a developer in manifesting the DNAM project thereby projecting the overall time taken in completion of the project by both software development methodologies.

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The Necessary Stability Conditions of a Tandem System With Feedback

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Abstract

In this paper, we consider Markovian model of a two-station tandem network with the following feedback admission control policy: the first station rejects new arrivals when the queue size in the second station exceeds a certain threshold N. We provide necessary stability conditions of this model. Each station operates as a multiserver queuieng system, and thus work in part generalizes the results from the paper [1] in which single-server stations have been considered. The analysis is based on the Burke's theorem and stochastic monotonicity of the Birth-Death process describing the number of customers in the second station.

Keywords: queuing system, ergodicity, input flow, feedback

I Introduction

We consider the following two-station queueing system with a feedback admission control policy. The input flow in this system is Poisson with the parameter λ. Station i has N_i servers, and the service time of each server in station i is exponentially distributed with parameter $\mu_i, i = 1,2$.

We consider a *feedback admission control* when the 1st station closes the admission gate provided the *queue size* (number of customers) in the 2nd station exceeds a fixed threshold $N \geq 1$. When the queue length of the 2nd station falls below the threshold, admission gate opens again. With this non-idling control policy, the system losses arrivals during the period when the gate is closed. We assume the FIFO service discipline at both stations. (In general, under the same conditions, stability of the system holds true for any work-conserving service discipline.) The detailed motivation of this model can be found in [1].

Our analysis is based on the dependencies between the rates of the flows, in particular, input rate and output rate from the first station, in stationary regime. Also the analysis is heavily based on the Burke's theorem stating the equality of the input and output rates in the stationary (non-overloaded) multiserver first station. Finally, we apply stochastic monotonicity of the Birth-Death (BD) process, describing the multiserver queuing system.

II Stability Conditions

In this section, we establish the necessary stability conditions of the basic model described shortly above.

First of all, we give more detailed description of the model. We consider the described above two-station tandem system with Piosson input with rate λ and feedback admission control, assuming that the first station operates as a queueing system $M|M|N_1$ with N_1 identical servers and infinite buffer. The second station is the system $M|M|N_2$, also with infinity capacity buffer. The service rate is μ_i at each server of station $i = 1,2$. Because all governing distributions are exponential, this feedback system is completely defined by the parameters $\lambda,\ \mu_i,\ N_i, N$.

The dynamics of this model can be described by a continuous-time discrete-valued Markov process $Z(t) =: (z_1(t), z_2(t)),\ t \geq 0$, where component $z_i(t)$ is the number of customers at station i at instant $t, i = 1,2$. Denote $y(t)$ the number of arrivals in the interval $(0, t],\ y(0) = 0$, in the Poisson input flow (with the intensity λ), and define $x(t)$, the *actual* number of arrivals to the 1st station in interval $(0, t],\ x(0) = 0$.

The following statement generalizes the necessary stability conditions found in [1] for the single-server stations.

Theorem 1. *Assume the Markov process Z is ergodic. If i) $N_1\mu_1 < N_2\mu_2$, then $\lambda < F_N(N_1\mu_1)$;*

ii) otherwise, if $N_1\mu_1 \geq N_2\mu_2$, that there are no other restrictions except $\lambda < \infty$.

Proof. Assume that the Markov process Z is in steady state, and denote $P_N = P(z_2(t) > N)$ the stationary probability that there are at least N customers in the 2nd station. The Poisson arrivals with the intensity λ enter the 1st station. Then, at an arrival instant a transition $y(t) \to y(t) + 1$ happens , and moreover, transition $x(t) \to x(t) + 1$ happens if and only if $z_2(t) \leq N$. Thus, the transition rate $x(t) \to x(t) + 1$ equals $\nu := \lambda P_N$.

Therefore, for each t and constant T, the number of customers entering the 1st station in interval $[t, t + T)$ does not depend on the number of customers arriving in interval $(0, t],\ t > 0$. Then it follows from [2], [3] that the rate of the arrivals entering the 1st station equals $\nu = \lambda P_N$ as well. Since the flow of arrivals entering the 1st station is Poisson with rate ν and the process Z is ergodic, then the process $z_1(t),\ t \geq 0$, turns out to be ergodic also. As a result , the process $z_1(t)$ is distributed as a BD process with the birth rate ν and the death rates $\mu_k = \min(k, N_1)\mu_1$ [§ 1.2][4]. It then follows from Karlin – McGregor criterion [6], we obtain the inequality $\nu < N_1\mu_1$. Because the stationary output from the 1st station is also Poisson process with the rate $\nu = \lambda P_N$, then we may notation $P_N = P_N(\nu)$ which is heavily used below.

Apply now a similar analysis to the 2nd station. Since the input to the 2nd station (output from the 1st station) is Poisson with rate ν, and the process Z is ergodic then the process $z_2(t),\ t \geq 0$, is ergodic also.

As above then the process $z_2(t)$ is distributed as a BD process with the birth rate ν and the death rates $\psi_k = \min(k, N_2)\mu_2$. Then, as above it follows from Karlin – McGregor criterion, that the inequality $\nu < N_2\mu_2$ holds. Thus, we obtain the following relations:

$$\nu = \lambda P_N,\ \nu < N_1\mu_1,\ \nu < N_2\mu_2. \tag{1}$$

Consider another BD process $z_2'(t),\ t \geq 0$, with the same death rates $\{\psi_k\}$ and a birth rate $\nu' > \nu$. Moreover, we assume the same initial state in both processes, that is $z_2(0) = z_2'(0)$. Then it follows from Theorem 4.2.1 in [8], that the following inequality holds:

$$\lim_{t\to\infty} P(z_2(t) > N) = P_N(\nu) \geq \lim_{t\to\infty} P(z_2't) > N) =: P_N(\nu'). \tag{2}$$

Because $\psi_j = \min(j, N_2)\mu_2,\ j \geq 1$, then it follows from [5] (Chapter 2, Section 3), that for each fixed $N > 0$ and for all $\nu,\ 0 < \nu < N_2\mu_2$, the function $P_N(\nu)$ has the following explicit expression

$$P_N(\nu) = 1 + \sum_{k=1}^{N} \nu^k / \prod_{j=1}^{k} \psi_j 1 + \sum_{k=1}^{\infty} \nu^k / \prod_{j=1}^{k} \psi_j,$$

and moreover, is monotonically decreasing (2) and continuous in ν. Because, under condition $\nu \geq N_2\mu_2$, the process $z_2(t)$ is not ergodic, then we obtain $P_N(\nu) = 0$ for $\nu \geq N_2\mu_2$. Therefore, for the fixed $N > 0$, the function

$$F_N(\nu) = \frac{\nu}{P_N(\nu)} \tag{3}$$

is continuous and monotonically increases in ν, as long as $0 < \nu < N_2\mu_2$, while we put $F_N(\nu) = \infty$ if $\nu \geq N_2\mu_2$. Then the equality

$$\nu = \lambda P_N(\nu) = F_N(\nu)P_N(\nu)$$

in (1) can be rewritten as $\nu = F^{-1}(\lambda)$, where F^{-1} is the inverse function to function F. Hence, by the monotonicity, we obtain from (1) that, for $N_1\mu_1 < N_2\mu_2$,

$$\lambda < F_N(N_1\mu_1). \tag{4}$$

Assume that $N_1\mu_1 \geq N_2\mu_2$. Take an arbitrary $\varepsilon \in (0, N_2\mu_2)$. Then, by the ergodicity of the Markov process $Z(t),\ t \geq 0$, the inequality $\nu < N_2\mu_2 - \varepsilon < N_1\mu_1$ follows, which in turn, is equivalent to the inequality $\nu < N_2\mu_2 - \varepsilon$. The latter inequality implies $\lambda < F_N(N_2\mu_2 - \varepsilon)$ by the monotonicity of function F_N. Because ε is arbitrary and

$$F_N(N_2\mu_2 - \varepsilon) \to F_N(N_2\mu_2) = \infty,\ \varepsilon \to 0,$$

then (4) becomes $\boldsymbol{\lambda} < \infty$, and the proof is completed.

III A Generalization

In the paper [1], also the following more general m-station system, $m \geq 2$, is considered: the external input (with rate λ) is rejected at the first station, if the number of customers $z_k(t)$ in each remaining station k exceeds a given threshold $N^{(k)}$. Moreover, the output from station k is the input to station $k+1, k = 1, \dots, m-1$. Denote $z_k(t)$ the number of customers at station k at instant t. In more detail, keeping other notation, consider an m - station exponential queueing system, in which station k has N_k (stochastically equivalent) servers with exponential service time with rate $\mu_k,\ k = 1, \dots, m$. It is assumed that a customer of the external Poisson input is rejected if the following inequalities hold true:

$$z_2(t) > N^{(2)}, \dots, z_m(t) > N^{(m)}.$$

The dynamics of this system is described by the following m-dimensional Markov process

$$Z = (z_1(t), \dots, z_m(t)),\ t \geq 0.$$

Theorem 2. *Assume the process Z is ergodic. If*

$$N_1\mu_1 < \min_{2\leq k\leq m} N_k\mu_k,$$

then $\lambda < F_N(N_1\mu_1)$. Otherwise, if

$$N_1\mu_1 \geq \min_{2\leq k\leq m} N_k\mu_k,$$

that only requirement is $\lambda < \infty$.

Proof. Denote ν the output rate of the (Poisson) flow of each station $1, \dots, m$. (This rate is the same for all stations by the ergodicity.) By the product-form theorem for stationary regime [9], the joint stationary distribution of the basic process satisfies

$$P(z_2(t) > N^{(2)}, \dots, z_m(t) > N^{(m)}) = \prod_{k=2}^{m} P(z_k(t) > N^{(k)}) =: P_{N^{(2)},\dots,N^{(m)}}(\nu). \tag{5}$$

The component processes $z_2(t), \dots, z_m(t)$ are the BD processes. Moreover, the process $z_k(t)$ has the birth rate ν and, if $z_k(t) = i$, the death rate $\mu_{k,i} = \min(i, N_k)\mu_k,\ k = 2, \dots, m$. It follows by Theorem 4.2.1 [8] and from analysis of the proof of Theorem 1 above, that the kth multiplier $P(z_k(t) > N^{(k)})$ in (5) (as function of ν) is continuous and decreases for all $\nu,\ 0 < \nu < N_k\mu_k,\ k = 2, \dots, m$. Thus, function $P_{N^{(2)},\dots,N^{(m)}}(\nu)$ is monotonically decreasing (and continuous) in ν as long as

$$0 < \nu < \min_{2\leq k\leq m} N_k\mu_k.$$

Because the process Z is ergodic, then the rate of the (Poisson) process entering the 1st station is $\nu = \lambda P_{N^{(2)},\dots,N^{(m)}}$. Furthermore, the output flows of all stations in the system are Poisson with the same rate ν. Now, repeating the arguments used in the proof of Theorem 1, we obtain the following relations

$$\nu = \lambda P_{N^{(2)},\dots,N^{(m)}}(\nu),\ \nu < N_1\mu_1, \dots, \nu < N_m\mu_m. \tag{6}$$

At that, the equality

$$\nu = \lambda P_{N^{(2)},\dots,N^{(m)}}(\nu) =: F_{N^{(2)},\dots,N^{(m)}}(\nu)$$

in (6) can be rewritten as

$$\nu = F^{-1}_{N^{(2)},\dots,N^{(m)}}(\lambda),$$

where $F^{-1}_{N^{(2)},\dots,N^{(m)}}$ is the inverse function to function $F_{N^{(2)},\dots,N^{(m)}}$. Now, by the monotonicity, we obtain from (6), for $N_1\mu_1 < \min_{2\le k\le m} N_k\mu_k$, the following inequality

$$\lambda < F_{N^{(2)},\dots,N^{(m)}}(N_1\mu_1). \qquad (7)$$

If $N_1\mu_1 \ge \min_{2\le k\le m} N_k\mu_k$, then again repeating arguments used in the proof of Theorem 1, we obtain finally the inequality $\lambda < \infty$, which completes the proof.

IV Conclusion

The necessary stability conditions of the Markovian model of a two-station tandem queueing network with a special type of feedback are found. Under this feedback, the input to the first station is rejected as long as the queue size in the second station exceeds a predefined fixed level. The analysis is based on the introduction of a function expressing the dependence between the rates of input and output at the first station. We apply stochastic monotonicity of the Birth-Death process describing the dynamics of the system, to obtain the necessary conditions in an explicit form. Analysis of the two-station system is then generalized to multi-station system.

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Reliability Analysis of a Maintenance Scheduling Model Under Failure Free Warranty Policy

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Abstract

This paper considers a maintenance scheduling model by using the concepts of failure free warranty policy. In this model, all the repairs during warranty are cost-free to the users, provided failures are not due to the negligence of the users. However, the users will have to repair the failed unit at their own expenses beyond warranty. During their formulation, the failure rate of the system is considered to be negative exponential distribution while the preventive maintenance (PM), repair and replacement time distributions are taken to be arbitrary with different probability density functions. Under these assumptions, using the supplementary variable technique, the various expressions which depict the behavior of the system such as reliability of the system, Mean Time to System Failure (MTSF), availability and profit function have been derived. Further, steady-state behavior of the system has also been derived. To substantiate the proposed approach, the effect of the parameters of the system has been analyzed through the system reliability, and expected profit through an illustrative example.

Keywords: Warranty, Reliability, Maintenance, Inspection, degradation, Mean Time to System Failure

1. Introduction

In the present era of industrial growth, the optimal efficiency and minimum hazards are more challengeable to maintain. To overcome these issues, reliability technology can play an important role. Reliability is measured as the ability of a system to perform its intended function, successfully, for a specified period, under predetermined conditions. This attribute has far-reaching consequences on the durability, availability, and life cycle cost of a product or system and is of great importance to the end user/engineer [8]. Typically, high-reliability targets or specifications are set for the system, and ways to achieve them are then examined, taking into account resource constraints. Apart from the limitations of resources, the targets set may be in dispute. For instance, high reliability generally means a high cost, weight, and volume. Also at the same time the unfortunate penalty of low availability and high maintenance cost need to be improved for their survival. To achieve this end, availability and reliability of equipment in the process must be maintained at the higher order. Thus, reliability and maintainability concepts are mainly applicable at the design stage of a machinery or plant layout, while the availability concept is mostly applicable after commissioning the plant or after a steady state of production is reached

[6, 7]. Modern technology has developed a tendency to design and manufacture equipment and systems of greater capital cost, sophistication, complexity, and capacity. In the literature, a variety of methods exists for failure analysis which includes reliability block diagrams, Markov Modeling, failure mode and effect analysis, Petri nets, fault tree analysis, and so forth [1, 4, 6, 7, 10, 11].

In a system where a certain amount of failures is allowed, the efficient repair and/or replacement of these failures is critical to the continued usefulness of the system. This repair and replacement of failures are called maintenance. Maintenance has a definite influence on operating costs, either through its own (maintenance) labor or through its effect of system downtime and efficiency. In reliability, maintainability can also be used to increase the probability that a system will continue to operate efficiently, given that it is allowed a certain amount of downtime of repairs. The purpose of maintainability is to return a failed or deteriorating system to a satisfactory operating state. To do this, there are two extreme maintenance policies that can be applied. The first is to unplanned (corrective) maintenance while the second one is planned (preventive) maintenance. In unplanned, corrective strategy, no maintenance action is carried out until the component or structure breaks down or when its cost of operation becomes creeping or wear-out failures. This is called corrective maintenance (CM) or emergency repair. A study on the effect of CM in maintenance policies was done by Samrout et al. [35]. However, to avoid failures at occasions that have high cost consequences preventive maintenance (PM) is normally chosen. The main function of planned maintenance is to restore equipment to the "as good as new" condition; periodical inspections must control equipment condition and both actions will ensure equipment availability. PM of the systems is necessary after a pre-specific period of time not only to maintain the operational power but may also reduce the failure rate. A study on the effect of PM on a single-unit system was done by Kapur et al. [19]. PM can increase the performance of two or more unit system model if it starts for an operative unit only when the other unit is in standby as discussed by Gupta [12].

Since the importance of PM during the reliability analysis, various authors have put forth the different approaches to enhance the reliability of the system. For instance, Mokaddis et al. [25] developed a three-unit standby redundant system with repair and PM. Yanagi and Sasaki [38] evaluated the availability of a parallel system with PM. Rander et al. [33] examined cost analysis of a two dissimilar cold standby system with preventive maintenance and replacement of standby. Hadidi and Rahim [13] analyzed the reliability for multiple units adopting sequential imperfect maintenance policies. Jin et al. [15] presented an option model for joint production preventive maintenance system. Garg et al. [9] presented different PM models to analyze the reliability of the system. Liao [22] evaluated optimum policy for a production system with major repair and preventive maintenance. Malik [24] studied reliability modeling of a computer system with preventive maintenance and priority subject to maximum operation and repair times while Kadyan [18] analyzed reliability and profit of a single-unit system with PM subject to maximum operation time. Apart from these, in the literature, numerous attempts have been made by the researchers to analyze the reliability of the system using different approaches [20, 18, 38, 32, 17, 17, 27].

From the above study, it is revealed that most of the above mentioned work considered that the unit works as new after PM and repair. Since, the working capability and efficiency of a unit after repair depend more or less on the quality of the unit and repair mechanism adopted, so in general, the assumption of considering the unit as good as new after repair is not always true. Moreover, continuous operation and ageing of the systems gradually reduce the performance of a system and repair action cannot bring the system to the good stage, but can make it operational [30]. Further, in such a situation, unit after its repair works with reduced capacity. To minimize the error in the study, Kumar et al. [20] and Malik [23] analyzed a single-unit system with degradation and maintenance by using degraded unit. Eryilmaz [5] studied a three

state reliability model in which degradation rates are random and statistically dependent. Apart from all these work, in our day-today life, there is always occur a situation where the repair of the failed degraded systems is neither possible nor economical to both manufacturer and the user due to wearout and other unforeseen conditions. In such cases, inspection can play an important role to see the feasibility of repair. On system reliability models, the concept of inspection with different maintenance policies has been discussed by various researchers such as Tuteja and Malik [36], Hariga [14], Leung [21], Zequeira and Berenguer [39], Nailwal and Singh [26], Pietruczuk and Wojciechowska [31], and Wang et al [37].
Since all the above system models are analyzed without considering the concept of failure free warranty policy. Under failure free warranty policy, the items are replaced/repaired free of cost to the users during warranty. Customers need assurance that the product they are buying will perform satisfactorily and warranty provides such assurance. Providing warranty to the system for a certain period of operation is one of the effective ways to ensure reliability of a sold product (or system) [29]. Also, it is an essential part of sale for commercial and industrial products. With these objectives, Kadyan and Niwas [16] and Niwas et al. [29] discussed reliability models of a single-unit system with warranty and different repair policies by using Supplementary Variable Technique [3]. Further, Niwas et al. [28] analyzed that replacement of failed degraded unit by new one is not economically beneficial. However, the unit or product can be restored to operate the required functions by repairing it rather than replacing the entire product.

Keeping these views in mind here we proposed a single-unit repairable system model with the concept of failure free Warranty Policy with PM and inspection for feasibility of repair of degraded unit by using Supplementary Variable Technique. In the proposed approach, to avoid unnecessary expenses on replacement of the entire product, inspection can be done to see the feasibility of repair of the degraded unit. On the other hand, if repair is feasible then the failed degraded unit will be repaired by the repairman otherwise it is replaced by a new one. Due to failure free warranty policy, users are secure about early failures of the products/system because all the repairs are cost free during warranty. Further, after expiration of failure free warranty policy, PM is conducted to improve the condition of the deteriorated product/system. The concepts of PM, degradation and inspection are conducted beyond warranty. So, these factors do not have any impact on failure free warranty policy but they have an impact on the overall performance of the system/product. And, for users prospective by using these concepts after expiration of warranty, can improve the overall performance of the product/system.

The remainder of this paper is organized as follows. Section 2 gives the description of the System containing the assumptions of the model, state-specifications and Notations related to the proposed system model, Section 3 presents the model analysis in which different system performance measures are computed such as steady-state behavior of the system, reliability and MTSF of the system, Section 4 shows the results and discussion with special case containing availability of the system and profit analysis of the user and provides a numerical result for these special cases. Finally, Section 5 presents concluding remarks.

2. Description of the System

2.1.Assumptions

(i) The system has a single unit.

(ii) There is a single repairman, who is always available with the system to do repair or replacement, PM and inspection of the failed unit.

(iii) The cost of repair during warranty is borne by the manufacturer provided failures are not due to the negligence of users such as cracked screen, accident, misuse, physical damage, damage due to liquid and unauthorized modifications etc.

(iv) Beyond warranty, the unit goes under PM and works as new after PM but works with some reduced capacity after its repair and so is called a degraded unit.
(v) Repairman inspects the failed degraded unit to see the feasibility of repair.
(vi) The distribution of failure time is taken as negative exponential while the PM, repair and replacement times are considered as arbitrary.

2.2. *State-Specification*

s_0 / s_1 : The unit is operative under warranty/ beyond warranty.

s_2 / s_4 : The unit is in failed state under warranty/ beyond warranty.

s_3 : The unit is under PM.

s_5 : The degraded unit is operative.

s_6 : The failed degraded unit is under inspection.

2.3. *Notations*

λ / λ_1	Constant failure rate of the new unit within/beyond warranty.
λ_2	Constant failure rate of the degraded unit beyond warranty.
λ_m	Transition rate with which a unit goes under PM for improvement.
α	Transition rate with which warranty of the system is completed.
p / q	Probability that repair is feasible/not feasible.
$\mu(x), S(x) / \mu_1(x), S_1(x)$	Repair rate of the unit and probability density function, for the elapsed repair time x within/ beyond warranty.
$\mu_2(y), S_2(y)$	PM rate of the unit and probability density function, for the elapsed PM time y .
$h(z), S_3(z)$	Inspection rate of the failed unit and probability density function, for the elapsed inspection time z .
$p_0(t) / p_1(t)$	Probability density that at time t, the system is within/ beyond warranty and in good state.
$p_i(x,t)$	Probability density that at time t, the system is in state $S_i, i = 2, 4$ and the system is under repair with elapsed repair time x .
$p_3(y,t)$	Probability density that at time t, the system is in state s_3 and the unit is under PM with elapsed PM time y .
$p_5(t)$	Probability density that at time t, the system is operable and in degraded state.
$p_6(z,t)$	Probability density that at time t, the system is in state s_6 and the failed degraded unit is under inspection with elapsed inspection time z.
$p(s)$	Laplace transform of function $p(t)$

$$S(x) = \mu(x)\, e^{\left[-\int_0^x \mu(x)dx\right]}$$

$$S_1(x) = \mu_1(x)\, e^{\left[-\int_0^x \mu_1(x)dx\right]}$$

$$S_2(y) = \mu_2(y)\, e^{\left[-\int_0^y \mu_2(y)dy\right]}$$

$$S_3(z) \quad = \quad h(z)\,e^{\left[-\int_0^z h(z)dz\right]}$$

3. Model Analysis

The system model consists of a single-unit in which there is a single repairman who always remains with the system and monitoring its performance. Initially, the unit is in operating within warranty and when it fails within warranty then it goes to repair with free of cost. On the other hand, if warranty is completed due to negligence of the users then system remain in working condition. In this case, the unit goes under PM and works as new after PM but with some reduced capacity after its repair and so is called a degraded unit. Degraded unit is inspected for feasibility of repair after its failure. It has been assumed the failure times of the system follow a negative exponential distribution while during the PM, replacement and/or repair time, its distribution is taken as arbitrary. The transition diagram of this system by considering all the states, namely up (i.e., good or working), failed and degrade states is shown in Fig. 1

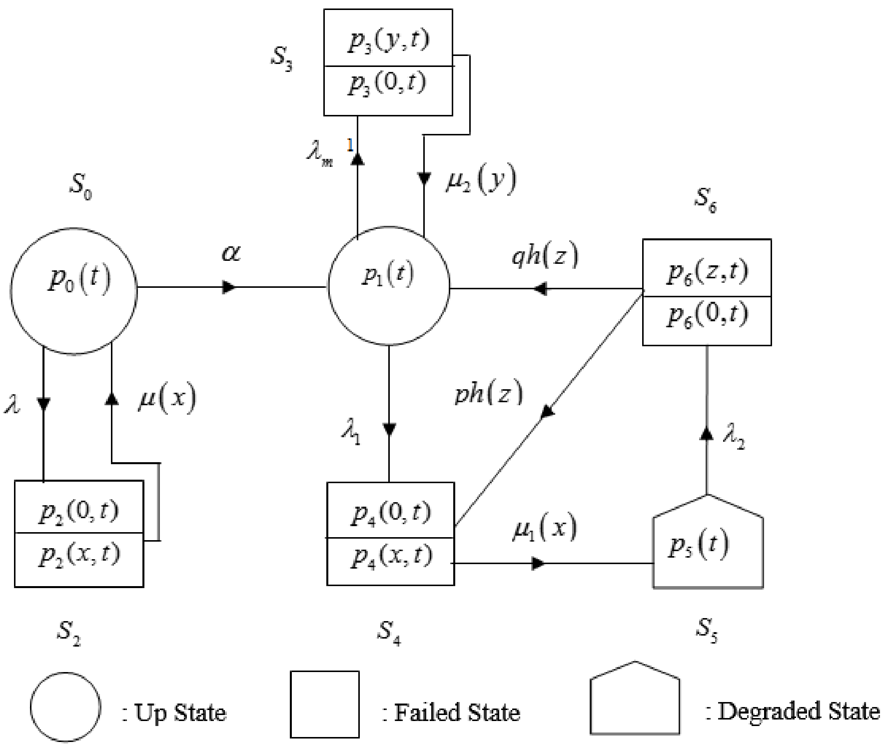

Figure 1: Transition diagram of the model

3.1. Formulation of mathematical model

Based on this diagram, we can formulate the difference-differential equations by using the probabilistic arguments of each state of the system and are summarized as follows [3], [29]:

$$\left[\frac{d}{dt}+\lambda+\alpha\right]p_0(t)=\int_0^\infty \mu(x)\,p_2(x,t)dx \tag{1}$$

$$\left[\frac{d}{dt}+\lambda_1+\lambda_m\right]p_1(t)=\alpha p_0(t)+q\int_0^\infty h(z)p_6(z,t)dz+\int_0^\infty \mu_2(y)p_3(y,t)dy \tag{2}$$

$$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+\mu(x)\right]p_2(x,t)=0 \tag{3}$$

$$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial y}+\mu_2(y)\right]p_3(y,t)=0 \tag{4}$$

$$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+\mu_1(x)\right]p_4(x,t)=0 \tag{5}$$

$$\left[\frac{d}{dt}+\lambda_2\right]p_5(t)=\int_0^\infty \mu_1(x)p_4(x,t)dx \tag{6}$$

$$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial z}+h(z)\right]p_6(z,t)=0 \tag{7}$$

Whereas, the boundary conditions [**Ошибка! Источник ссылки не найден.**] for the system are

$$p_2(0,t)=\lambda p_0(t), \tag{8}$$

$$p_3(0,t)=\lambda_m p_1(t) \tag{9}$$

$$p_4(0,t)=\lambda_1 p_1(t)+p\int_0^\infty h(z)p_6(z,t)dz, \tag{10}$$

$$p_6(0,t)=\lambda_2 p_5(t) \tag{11}$$

and the initial conditions are $p_i(0)=\begin{cases}1 & ; \quad i=0\\ 0 & ; \quad i\neq 0\end{cases}$ (12)

3.2.Solution of the equations

In order to solve the above formulated Eqs. (1) – (11), we use the Laplace transforms corresponding to initial condition given in Eq. (12) and get

$$\left[s+\lambda+\alpha\right]p_0(s)=1+\int_0^\infty \mu(x)p_2(x,s)dx \tag{13}$$

$$\left[s+\lambda_1+\lambda_m\right]p_1(s)=\alpha p_0(s)+q\int_0^\infty h(z)p_6(z,s)dz+\int_0^\infty \mu_2(y)p_3(y,s)dy \tag{14}$$

$$\left[\frac{\partial}{\partial x}+s+\mu(x)\right]p_2(x,s)=0 \tag{15}$$

$$\left[\frac{\partial}{\partial y}+s+\mu_2(y)\right]p_3(y,s)=0 \tag{16}$$

$$\left[\frac{\partial}{\partial x}+s+\mu_1(x)\right]p_4(x,s)=0 \tag{17}$$

$$\left[s+\lambda_2\right]p_5(t)=\int_0^\infty \mu_1(x)p_4(x,s)dx \tag{18}$$

$$\left[\frac{\partial}{\partial z}+s+h(z)\right]p_6(z,s)=0 \tag{19}$$

$$p_2(0,s)=\lambda p_0(s) \tag{20}$$

$$p_3(0,s)=\lambda_m p_1(s) \tag{21}$$

$$p_4(0,s) = \lambda_1 p_1(s) + p\int_0^\infty h(z) p_6(z,s) dz \tag{22}$$

$$p_6(0,s) = \lambda_2 p_5(s) \tag{23}$$

Thus, by integrating Eq. (15) and further using Eq. (20) we get

$$p_2(x,s) = p_2(0,s) e^{\left[-sx-\int_0^x \mu(x)dx\right]} \tag{24}$$

Similarly, by integrating Eqs. (16), (17) and (19) and using their corresponding Eqs. (21), (22) and (23) respectively, then we get

$$p_3(y,s) = p_3(0,s) e^{\left[-sy-\int_0^y \mu_2(y)dy\right]} \tag{25}$$

$$p_4(x,s) = p_4(0,s) e^{\left[-sx-\int_0^x \mu_1(x)dx\right]} \tag{26}$$

$$p_6(z,s) = p_6(0,s) e^{\left[-sz-\int_0^z h(z)dz\right]} \tag{27}$$

Further, by using Eq. (24), Eq. (13) yields

$$\left[s+\lambda+\alpha\right] p_0(s) = 1 + p_2(0,s)\int_0^\infty \mu(x) e^{\left[-sx-\int_0^x \mu(x)dx\right]} dx$$

$$= 1 + \lambda p_0(s) S(s) \tag{28}$$

$$\Rightarrow \quad p_0(s) = \frac{1}{T(s)} \tag{29}$$

$$\text{Where } T(s) = s + \alpha + \lambda\left(1 - S(s)\right) \tag{30}$$

Now, by using Eq. (27), the Eq. (22) yields

$$p_4(0,s) = \lambda_1 p_1(s) + \int_0^\infty p h(z) p_6(0,s) e^{\left[-sz-\int_0^z h(z)dz\right]}$$

$$p_4(0,s) = \lambda_1 p_1(s) + p\lambda_2 p_5(s) S_3(s) \tag{31}$$

Using Eq. (31), Eq. (26) yields

$$p_4(x,s) = \left(\lambda_1 p_1(s) + p\lambda_2 p_5(s) S_3(s)\right) e^{\left[-sx-\int_0^x \mu_1(x)dx\right]} \tag{32}$$

On the other hand, by using Eq. (32), Eq. (18) yields

$$\left[s+\lambda_2\right] p_5(s) = \left(\lambda_1 p_1(s) + p\lambda_2 p_5(s) S_3(s)\right)\int_0^\infty \mu_1(x) e^{\left[-sx-\int_0^x \mu_1(x)dx\right]} dx$$

$$p_5(s) = A(s) p_1(s) \tag{33}$$

$$\text{Where, } A(s) = \frac{\lambda_1 S_1(s)}{\left(s + \lambda_2 - p\lambda_2 S_1(s) S_3(s)\right)} \tag{34}$$

Now, the Eq. (14) can be simplified by using Eqs. (25), (27) and (33) and get

$$\left[s+\lambda_1+\lambda_m\right] p_1(s) = \alpha p_0(s) + q p_6(0,s)\int_0^\infty h(z) e^{\left[-sz-\int_0^z h(z)dz\right]} + p_3(0,s)\int_0^\infty \mu_2(y) e^{\left[-sy-\int_0^y \mu_2(y)dy\right]}$$

$$p_1(s) = \frac{B(s)}{T(s)} \tag{35}$$

$$\text{Where, } B(s)=\frac{\alpha}{\left(s+\lambda_1+\lambda_m-\lambda_m S_2(s)-q\lambda_2 A(s)S_3(s)\right)} \tag{36}$$

Using Eq. (35) in Eq. (33), we get

$$p_5(s)=\frac{A(s)B(s)}{T(s)} \tag{37}$$

Now, the Laplace transform of the probability that the system is in the failed state is given by

$$p_2(s)=\int_0^\infty p_2(s,x)dx=\lambda p_0(s)\frac{\left(1-S(s)\right)}{s}$$

$$p_2(s)=\frac{\lambda C(s)}{T(s)} \tag{38}$$

$$\text{Where } \ C(s)=\frac{\left(1-S(s)\right)}{s} \tag{39}$$

Similarly $\quad p_3(s)=\int_0^\infty p_3(s,y)dy=\lambda_m p_1(s)\frac{\left(1-S_2(s)\right)}{s}$

$$p_3(s)=\frac{\lambda_m B(s)D(s)}{T(s)} \tag{40}$$

$$\text{Where } \ D(s)=\frac{\left(1-S_2(s)\right)}{s} \tag{41}$$

Similarly $\quad p_4(s)=\int_0^\infty p_4(s,x)dx=\left(\lambda_1 p_1(s)+p\lambda_2 p_5(s)S_3(s)\right)\frac{\left(1-S_1(s)\right)}{s}$

$$p_4(s)=\frac{\left(\lambda_1 B(s)+p\lambda_2 B(s)A(s)S_3(s)\right)E(s)}{T(s)} \tag{42}$$

$$\text{Where } \ E(s)=\frac{\left(1-S_1(s)\right)}{s} \tag{43}$$

Now, $p_6(s)=\int_0^\infty p_6(s,z)dz=\lambda_2 p_5(s)\frac{\left(1-S_3(s)\right)}{s}$

$$p_6(s)=\frac{\left(\lambda_2 A(s)B(s)F(s)\right)}{T(s)} \tag{44}$$

$$\text{Where, } F(s)=\frac{\left(1-S_3(s)\right)}{s} \tag{45}$$

It is worth noticing that

$$p_0(s)+p_1(s)+p_2(s)+p_3(s)+p_4(s)+p_5(s)+p_6(s)=\frac{1}{s} \tag{46}$$

3.3.Evaluation of Laplace transforms of up and down state probabilities

The Laplace transforms of the probabilities that the system is in up (i.e. good State) and down (i.e. failed State) at time *t* are as follows

$$Av(s) \text{ or } P_{up}(s)=p_0(s)+p_1(s)+p_5(s)$$

$$Av(s)=\frac{\left(1+A(s)+B(s)A(s)\right)}{T(s)} \tag{47}$$

$$P_{down}(s)=p_2(s)+p_3(s)+p_4(s)+p_6(s)$$

$$P_{down}(s) = \frac{\left(\lambda C(s) + \lambda_m B(s)D(s) + \left(\lambda_1 + p\lambda_2 S_3(s)A(s)\right)B(s)E(s) + \lambda_2 B(s)A(s)F(s)\right)}{T(s)} \tag{48}$$

3.4.Steady-State Behavior of the System

Using Abel's Lemma [26] i.e., $\lim_{s\to 0} s\left[F(s)\right] = \lim_{t\to\infty}\left[F(t)\right] = F$

in Eqs. (47) and (48), Provided the limit on the right hand side exists, the following time independent probabilities have been obtained.

$$Av = \frac{\lambda_1 + q\lambda_2}{\left(\lambda_1 + q\lambda_2 - q\lambda_2\lambda_m S_2^{'}(0) - \lambda_1\lambda_2 S_1^{'}(0) - \lambda_1\lambda_2 S_3^{'}(0)\right)} \tag{49}$$

$$p_{down} = \frac{-q\lambda_m\lambda_2 S_2^{'}(0) - \lambda_1\lambda_2 S_1^{'}(0) - \lambda_1\lambda_2 S_3^{'}(0)}{\left(\lambda_1 + q\lambda_2 - q\lambda_m\lambda_2 S_2^{'}(0) - \lambda_1\lambda_2 S_1^{'}(0) - \lambda_1\lambda_2 S_3^{'}(0)\right)} \tag{50}$$

3.5. Reliability of the system

Reliability, $R(t)$ is the probability that the system functions well in a specified period of time. Using the method similar to that in section 3.1, the differential–difference equations for reliability are [4]:

$$\left[\frac{d}{dt} + \lambda + \alpha\right] p_0(t) = 0 \tag{51}$$

$$\left[\frac{d}{dt} + \lambda_1 + \lambda_m\right] p_1(t) = \alpha\, p_0(t) \tag{52}$$

Taking Laplace transforms of Eqs. (51) and (52) and using Eq. (12) we get

$$\left[s + \lambda + \alpha\right] p_0(s) = 1 \tag{53}$$

$$\left[s + \lambda_1 + \lambda_m\right] p_1(s) = \alpha\, p_0(s) \tag{54}$$

Using the initial conditions, the solution can be written as

$$p_0(s) = \frac{1}{\left(s + \alpha + \lambda\right)} \tag{55}$$

$$p_1(s) = \frac{\alpha}{\left(s + \alpha + \lambda\right)\left(s + \lambda_1 + \lambda_m\right)} \tag{56}$$

$$R(s) = p_0(s) + p_1(s) = \frac{1}{\left(s + \alpha + \lambda\right)} + \frac{\alpha}{\left(s + \alpha + \lambda\right)\left(s + \lambda_1 + \lambda_m\right)}$$

Taking inverse Laplace transform, we get

$$R(t) = e^{-(\lambda+\alpha)t}\left[\frac{\left(\lambda - \lambda_m - \lambda_1\right)}{\left(\lambda - \lambda_m - \lambda_1 + \alpha\right)}\right] + \left[\frac{\alpha}{\left(\lambda - \lambda_m - \lambda_1 + \alpha\right)}\right] e^{-(\lambda_1 + \lambda_m)t} \tag{57}$$

Now, based on Eq. (57), the Mean Time to System Failure (MTSF) is defined as:

$$MTSF = \int_0^\infty R(t)dt$$

$$= \int_0^\infty \left\{ e^{-(\lambda+\alpha)t}\left(\frac{\left(\lambda - \lambda_m - \lambda_1\right)}{\left(\lambda - \lambda_m - \lambda_1 + \alpha\right)}\right) + \left(\frac{\alpha}{\left(\lambda - \lambda_m - \lambda_1 + \alpha\right)}\right) e^{-(\lambda_1 + \lambda_m)t} \right\} dt$$

$$=\left[\frac{(\lambda-\lambda_m-\lambda_1)}{(\lambda-\lambda_m-\lambda_1+\alpha)(\lambda+\alpha)}\right]+\left[\frac{\alpha}{(\lambda-\lambda_m-\lambda_1+\alpha)(\lambda_1+\lambda_m)}\right] \tag{58}$$

4. Results and discussions

In this section, firstly, we have deduced the expression of the availability and the cost-benefit analysis by taking a particular case of the distribution function of the component of the system.

4.1. Availability of the system $A_v(t)$

Assume that the repairs, PM and Inspection time follow negative exponential distribution i.e., $S(s)=\frac{\mu}{(s+\mu)}, S_1(s)=\frac{\mu_1}{(s+\mu_1)}$, $S_2(s)=\frac{\mu_2}{(s+\mu_2)}$ and $S_3(s)=\frac{h}{(s+h)}$ where μ, and μ_1 are constant repair rates, μ_2 is constant PM rate and h is constant inspection rate. Putting these values in Eqs. (28)-(37) we get

$$p_0(s)=\frac{1}{I(s)} \tag{59}$$

$$\text{Where } I(s)=\frac{\left(s^2+s(\lambda+\alpha+\mu)+\alpha\mu\right)}{(s+\mu)} \tag{60}$$

$$p_1(s)=\frac{J(s)}{I(s)} \tag{61}$$

Where

$$J(s)=\left[\frac{\alpha(s+h)(s+\mu_2)}{(s+\lambda_1+\lambda_m)(s+\mu_2)(s+h)-\lambda_m\mu_2(s+h)-qh\lambda_2K(s)(s+\mu_2)}\right] \tag{62}$$

$$p_5(s)=\frac{J(s)K(s)}{I(s)} \tag{63}$$

Where

$$K(s)=\left[\frac{\mu_1\lambda_1(s+h)}{(s+\mu_1)(s+\lambda_2)(s+h)-ph\lambda_2\mu_1}\right] \tag{64}$$

$$A_v(s) \text{ or } P_{up}(s)=p_0(s)+p_1(s)+p_5(s)$$

$$=\left[\frac{\left(s^5+b_4s^4+b_3s^3+b_2s^2+b_1s+b_0\right)(s+\mu)}{s\left(s^2+s(\lambda+\alpha+\mu)+\alpha\mu\right)\left(s^4+s^3a_3+s^2a_2+sa_1+a_0\right)}\right] \tag{65}$$

Where

$$b_4=(\lambda_1+\mu_1+\mu_2+\alpha+h+\lambda_2+\lambda_m),$$

$$b_3=\begin{pmatrix}\lambda_1\mu_1+\lambda_1\mu_2+h\mu_1+\mu_2\mu_1+h\mu_2+\lambda_2\mu_2+\lambda_1\lambda_2+\lambda_2\mu_1+\lambda_2h\\+\lambda_1h+\lambda_m\mu_1+\lambda_mh+\lambda_m\lambda_2+\lambda_2\alpha+\mu_1\alpha+\mu_2\alpha+h\alpha\end{pmatrix},$$

$$b_2=\begin{pmatrix}\lambda_1\mu_1h+\lambda_2\mu_1h+\lambda_2\mu_1\mu_2+\lambda_2\mu_2h+\mu_1\mu_2h+\lambda_1\lambda_2\mu_1+\lambda_1\lambda_2h+\lambda_1\mu_1\mu_2+\lambda_1\mu_2h+\lambda_1\lambda_2\mu_2\\+\lambda_m\mu_1h+\lambda_2\lambda_m\mu_1+\lambda_2\lambda_mh+\lambda_2\mu_1\alpha+\mu_1h\alpha+\lambda_2h\alpha+\mu_1\mu_2\alpha+\mu_2h\alpha+\lambda_2\mu_2\alpha+\alpha\lambda_1\mu_1\end{pmatrix},$$

$$b_1=\begin{pmatrix}\lambda_1\lambda_2\mu_1\mu_2+\lambda_1\lambda_2h\mu_2+\lambda_m\lambda_2\mu_1h-ph\lambda_2\lambda_m\mu_1-p\lambda_2\mu_1\mu_2h+\alpha\lambda_2\mu_1h\\+\alpha\lambda_2\mu_1\mu_2+\alpha\lambda_2\mu_2h+\alpha\lambda_1\mu_1h+\alpha\lambda_1\mu_1\mu_2+\alpha\mu_1\mu_2h-p\alpha\lambda_2\mu_1h\end{pmatrix}$$

$$\text{and } b_0 = \alpha q h \lambda_2 \mu_1 \mu_2 + \alpha h \lambda_1 \mu_1 \mu_2$$

$$\text{and } a_3 = \left(\mu_1 + \mu_2 + \lambda_1 + h + \lambda_2 + \lambda_m\right),$$

$$a_2 = \begin{pmatrix} \lambda_1\mu_1 + \lambda_1 h + \mu_1 h + \lambda_1\lambda_2 + \mu_1\lambda_2 + \mu_1\mu_2 + \mu_2 h \\ +\mu_2\lambda_2 + \lambda_1\mu_2 + \lambda_m\mu_1 + \lambda_m h + h\lambda_2 + \lambda_m\lambda_2 \end{pmatrix},$$

$$a_1 = \begin{pmatrix} \lambda_1\mu_1 h + \lambda_2\mu_1 h + \lambda_2\mu_1\mu_2 + \lambda_2\mu_2 h + \mu_1\mu_2 h + \lambda_1\lambda_2\mu_1 + \lambda_1\lambda_2 h \\ +\lambda_1\mu_1\mu_2 + \lambda_1\mu_2 h + \lambda_m\mu_1 h + \lambda_2\lambda_m\mu_1 + \lambda_2\lambda_m h + \lambda_2\lambda_1\mu_2 \end{pmatrix}$$

and

$$a_0 = \left(\lambda_1\lambda_2\mu_1\mu_2 + \lambda_2 h\mu_1\mu_2 + \lambda_1\lambda_2 h\mu_2 + \lambda_m\lambda_2\mu_1 h - ph\lambda_2\lambda_m\mu_1 - p\lambda_2\mu_1\mu_2 h\right)$$

Taking inverse Laplace transforms of Eq. (65) we get

$$\begin{aligned} A_v(t) = {} & \frac{\left(\alpha q\mu_2\lambda_2\mu_1 h + \alpha\mu_2\lambda_1\mu_1 h\right)\mu}{z_1 z_2 z_3 z_4 z_5 z_6} + \left\{\frac{\left(z_1^5 + b_4 z_1^4 + b_3 z_1^3 + b_2 z_1^2 + b_1 z_1 + b_0\right)(z_1+\mu)}{z_1(z_1-z_2)(z_1-z_3)(z_1-z_4)(z_1-z_5)(z_1-z_6)}\right\} e^{z_1 t} \\ & + \left\{\frac{\left(z_2^5 + b_4 z_2^4 + b_3 z_2^3 + b_2 z_2^2 + b_1 z_2 + b_0\right)(z_2+\mu)}{z_2(z_2-z_1)(z_2-z_3)(z_2-z_4)(z_2-z_5)(z_2-z_6)}\right\} e^{z_2 t} + \left\{\frac{\left(z_3^5 + b_4 z_3^4 + b_3 z_3^3 + b_2 z_3^2 + b_1 z_3 + b_0\right)(z_3+\mu)}{z_3(z_3-z_1)(z_3-z_2)(z_3-z_4)(z_3-z_5)(z_3-z_6)}\right\} e^{z_3 t} \\ & + \left\{\frac{\left(z_4^5 + b_4 z_4^4 + b_3 z_4^3 + b_2 z_4^2 + b_1 z_4 + b_0\right)(z_4+\mu)}{z_4(z_4-z_1)(z_4-z_2)(z_4-z_3)(z_4-z_5)(z_4-z_6)}\right\} e^{z_4 t} + \left\{\frac{\left(z_5^5 + b_4 z_5^4 + b_3 z_5^3 + b_2 z_5^2 + b_1 z_5 + b_0\right)(z_5+\mu)}{z_5(z_5-z_1)(z_5-z_2)(z_5-z_3)(z_5-z_4)(z_5-z_6)}\right\} e^{z_5 t} \\ & + \left\{\frac{\left(z_6^5 + b_4 z_6^4 + b_3 z_6^3 + b_2 z_6^2 + b_1 z_6 + b_0\right)(z_6+\mu)}{z_6(z_6-z_1)(z_6-z_2)(z_6-z_3)(z_6-z_5)(z_6-z_4)}\right\} e^{z_6 t} \end{aligned} \tag{66}$$

z_1 and z_2 are roots of the equation $s^2 + s\left(\lambda + \alpha + \mu\right) + \alpha\mu = 0$ and z_3, z_4, z_5 and z_6 are roots of the equation $s^4 + s^3 a_3 + s^2 a_2 + s a_1 + a_0 = 0$

4.2. *Profit analysis of the user*

Suppose that the warranty period of the system is $\left(0, w\right]$ includes the second state. Since the repairman is always available with the system, therefore beyond warranty period, it remains busy for time $\left(t-w\right)$ during the interval $\left(w,t\right]$. Let K_1 be the revenue per unit time and K_2 be the repair cost per unit time respectively, then the expected profit $H\left(t\right)$ during the interval $\left(0,t\right]$ is given by [29] $H(t) = K_1\int_0^t A_v(t)dt - K_2(t-w)$.

By using Eq. (66) and after solving, we get

$$H(t) = K_1 \left\{ \begin{aligned} & \frac{\left(\alpha q\mu_2\lambda_2\mu_1 h + \alpha\mu_2\lambda_1\mu_1 h\right)\mu t}{z_1 z_2 z_3 z_4 z_5 z_6} \\ & + \left\{\frac{\left(z_1^5 + b_4 z_1^4 + b_3 z_1^3 + b_2 z_1^2 + b_1 z_1 + b_0\right)(z_1+\mu)}{z_1^2(z_1-z_2)(z_1-z_3)(z_1-z_4)(z_1-z_5)(z_1-z_6)}\right\}\left(e^{z_1 t}-1\right) \\ & + \left\{\frac{\left(z_2^5 + b_4 z_2^4 + b_3 z_2^3 + b_2 z_2^2 + b_1 z_2 + b_0\right)(z_2+\mu)}{z_2^2(z_2-z_1)(z_2-z_3)(z_2-z_4)(z_2-z_5)(z_2-z_6)}\right\}\left(e^{z_2 t}-1\right) \\ & + \left\{\frac{\left(z_3^5 + b_4 z_3^4 + b_3 z_3^3 + b_2 z_3^2 + b_1 z_3 + b_0\right)(z_3+\mu)}{z_3^2(z_3-z_1)(z_3-z_2)(z_3-z_4)(z_3-z_5)(z_3-z_6)}\right\}\left(e^{z_3 t}-1\right) \\ & + \left\{\frac{\left(z_4^5 + b_4 z_4^4 + b_3 z_4^3 + b_2 z_4^2 + b_1 z_4 + b_0\right)(z_4+\mu)}{z_4^2(z_4-z_1)(z_4-z_2)(z_4-z_3)(z_4-z_5)(z_4-z_6)}\right\}\left(e^{z_4 t}-1\right) \\ & + \left\{\frac{\left(z_5^5 + b_4 z_5^4 + b_3 z_5^3 + b_2 z_5^2 + b_1 z_5 + b_0\right)(z_5+\mu)}{z_5^2(z_5-z_1)(z_5-z_2)(z_5-z_3)(z_5-z_4)(z_5-z_6)}\right\}\left(e^{z_5 t}-1\right) \\ & + \left\{\frac{\left(z_6^5 + b_4 z_6^4 + b_3 z_6^3 + b_2 z_6^2 + b_1 z_6 + b_0\right)(z_6+\mu)}{z_6^2(z_6-z_1)(z_6-z_2)(z_6-z_3)(z_6-z_5)(z_6-z_4)}\right\}\left(e^{z_6 t}-1\right) \end{aligned} \right\} - K_2(t-w) \tag{67}$$

To analyze the behavior of the system, we conducted an analysis where we vary the values of the parameter such as failure rates (λ and λ_1), transition rate (λ_m) and transition rate of completion of warranty (α). Based on it, the ranges of the system reliability and profit are computed and depicted in Tables 1 and 2 respectively. Further, to investigate the effect of individual component onto the system reliability, we vary the parameter λ from 0.01 to 0.03 and then further to 0.05 by fixing other parameters. By doing this, we compute that at time say 15 units, reliability of system decreased by 25.58% and further to 25.51% respectively. However, the complete variation of reliability with λ is summarized in Figure 2(a). On the other hand, if we increase the parameter λ_1 from 0.02 to 0.04 and further to 0.06 then reliability of the system decreases with the passage of time from 0.849405 to 0.84619 and then further to 0.843514 for the time 15units. The complete variation of this parameter is shown in Figure 2(b). However, Figures 2(c) and 2(d) respectively depicts the variation of the reliability with respect to the parameters α and λ_m .From these graphs, we conclude that the effect of α is more on to the system reliability than λ_m .

Table-1: Effect of failure rates (λ and λ_1), transition rate (λ_m) and transition rate of completion of warranty (α) on Reliability of the system $\left(R(t)\right)$

Time (t)	λ_1=0.02, α=0.003, λ_m=0.04	λ_1=0.02, α=0.003, λ_m=0.04	λ=0.01, α=0.003, λ_m=0.04	λ=0.01, λ_1=0.02, λ_m=0.04	λ=0.01, λ_1=0.02, α=0.003
	$R(t)$ (for λ =0.01)	$R(t)$ (for λ =0.03)	$R(t)$ (for λ_1 =0.04)	$R(t)$ (for α =0.005)	$R(t)$ (for λ_m =0.06)
10	0.899114	0.7378251	0.897294	0.895363	0.897294
11	0.889088	0.7154459	0.886992	0.884676	0.886992
12	0.8791	0.6937016	0.876723	0.873994	0.876723
13	0.869154	0.672577	0.866496	0.863327	0.866496
14	0.859254	0.652057	0.856317	0.852681	0.856317
15	0.849405	0.6321266	0.846192	0.842066	0.846192
16	0.83961	0.6127712	0.836125	0.831487	0.836125
17	0.829873	0.5939763	0.826122	0.820952	0.826122

Table 2: Effect of repair cost (K_2), PM rate (μ_2), transition rate of completion of warranty (α), inspection rate (h) and failure rate of degraded unit (λ_2) on expected profit ($H(t)$)

Time (t)	λ=0.01, μ=0.2, λ_2=0.04, λ_m=0.04, α=0.003, λ_1=0.02, p=0.6, μ_1=0.1 q=0.4 μ_2=0.4, h=0.5, w=3, K_1=500	λ=0.01, μ=0.2, λ_1=0.02, λ_2=0.04, λ_m=0.04, α=0.003, p=0.6, μ_1=0.1 q=0.4 μ_2=0.4, h=0.5, w=3, K_1=500	λ=0.01, λ_1=0.02, λ_2=0.04, λ_m=0.04, K_2=150, α=0.003, p=0.6, μ_1=0.1 q=0.4, μ=0.2, h=0.5, w=3, K_1=500	λ=0.01, λ_1=0.02, λ_2=0.04, λ_m=0.04, K_2=150, p=0.6, μ_1=0.1, μ=0.2 q=0.4, μ_2=0.4 h=0.5, w=3, K_1=500	λ=0.01, λ_1=0.02, λ_2=0.04, λ_m=0.04, μ_1=0.1, μ=0.2, K_2=150, α=0.003, q=0.4, μ_2=0.4 p=0.6, w=3, K_1=500	λ=0.01, λ_1=0.02, μ=0.2, λ_m=0.04, μ_1=0.1, h=0.5, K_2=150, α=0.003, q=0.4, μ_2=0.4 p=0.6, w=3, K_1=500
	$H(t)$	$H(t)$	$H(t)$	$H(t)$ (For α=0.006)	$H(t)$	$H(t)$

	(For K_2 =150)	(For K_2 =100)	(For μ_2 =0.45)		(For h =0.7)	(For λ_2 =0.02)
10	3984.972	4334.972	4211.116	3803.535	3801.078	3846.46
11	4293.353	4693.353	4563.899	4119.926	4101.091	4157.451
12	4593.596	5043.596	4908.402	4430.105	4393.336	4462.041
13	4885.589	5385.589	5244.42	4733.752	4677.603	4760.034
14	5169.253	5719.253	5571.809	5030.638	4953.735	5051.281
15	5444.534	6044.534	5890.476	5320.608	5221.625	5335.673
16	5711.41	6361.41	6200.37	5603.571	5481.207	5613.137
17	5969.883	6669.883	6501.481	5879.487	5732.451	5883.631

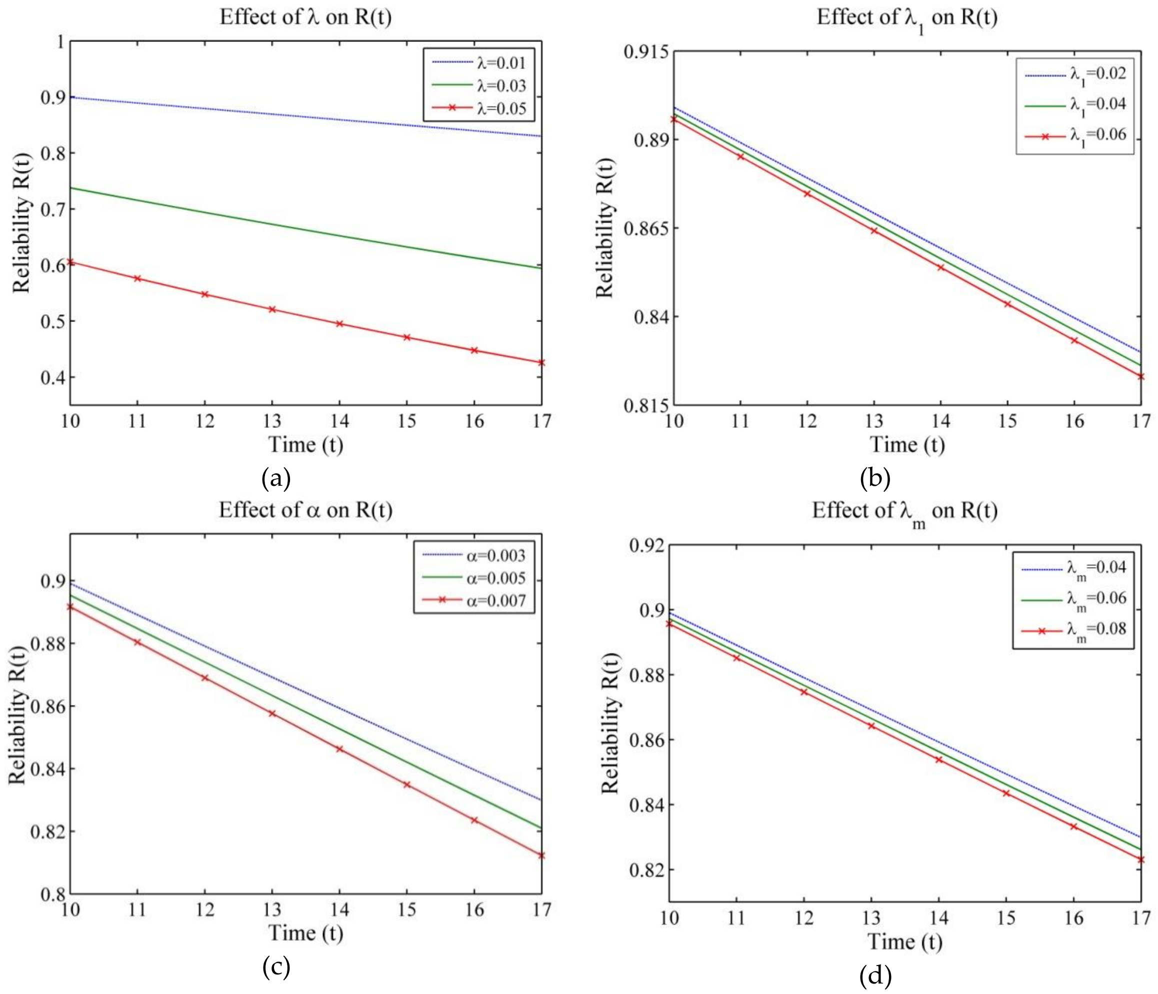

Figure 2: **Effect of the parameters $\lambda, \lambda_1, \alpha, \lambda_m$ on to the system reliability $R(t)$**

On the other hand, if we analyze the effect of the various parameters on to the expected profit $H(t)$ during the interval $(0,t]$ as given in Eq. (67). Firstly, we fix the different parameters as λ =0.01, λ_1=0.02, λ_2 =0.03, λ_m =0.04, α =0.003, μ =0.2, p =0.7, μ_1 =0.1, q =0.3 μ_2 =0.3, h =0.4, w =3, K_1=500. Now, the effect of the parameters K_2, μ_2, α and h is analyzed on $H(t)$ are analyzed with the passage of the time. For it, firstly if we decrease the repair cost K_2 from 150 to 100 and then to 50, then the expected profit at a time 15 units is increased from 5444.534 to 6044.534 and to 6644.534. The complete variation of the profit is summarized in Figure 3(a). On the other hand, the expected profit increases by 8.19% and further to 8.03% when value of μ_2 changes from 0.4 to 0.45 then further to 0.5. The variation corresponding to this is given in Figure 3(b). Finally, in Figures 3(c) and 3(d) respectively give the variation of the profit values with α and h. From these graphs, it is interpreted that the expected profit increases if i decrease the values of α and h and conclude that the effect of h is more on to expected profit $H(t)$ than α. Also, Table 2 depicts that whenever the failure rate of degraded unit (λ_2) changes from 0.04 to 0.02 then expected profit $H(t)$ decreases from 5444.534 to 5335.763 corresponding to time 15 units. Similar observations have been found for different time periods in it.

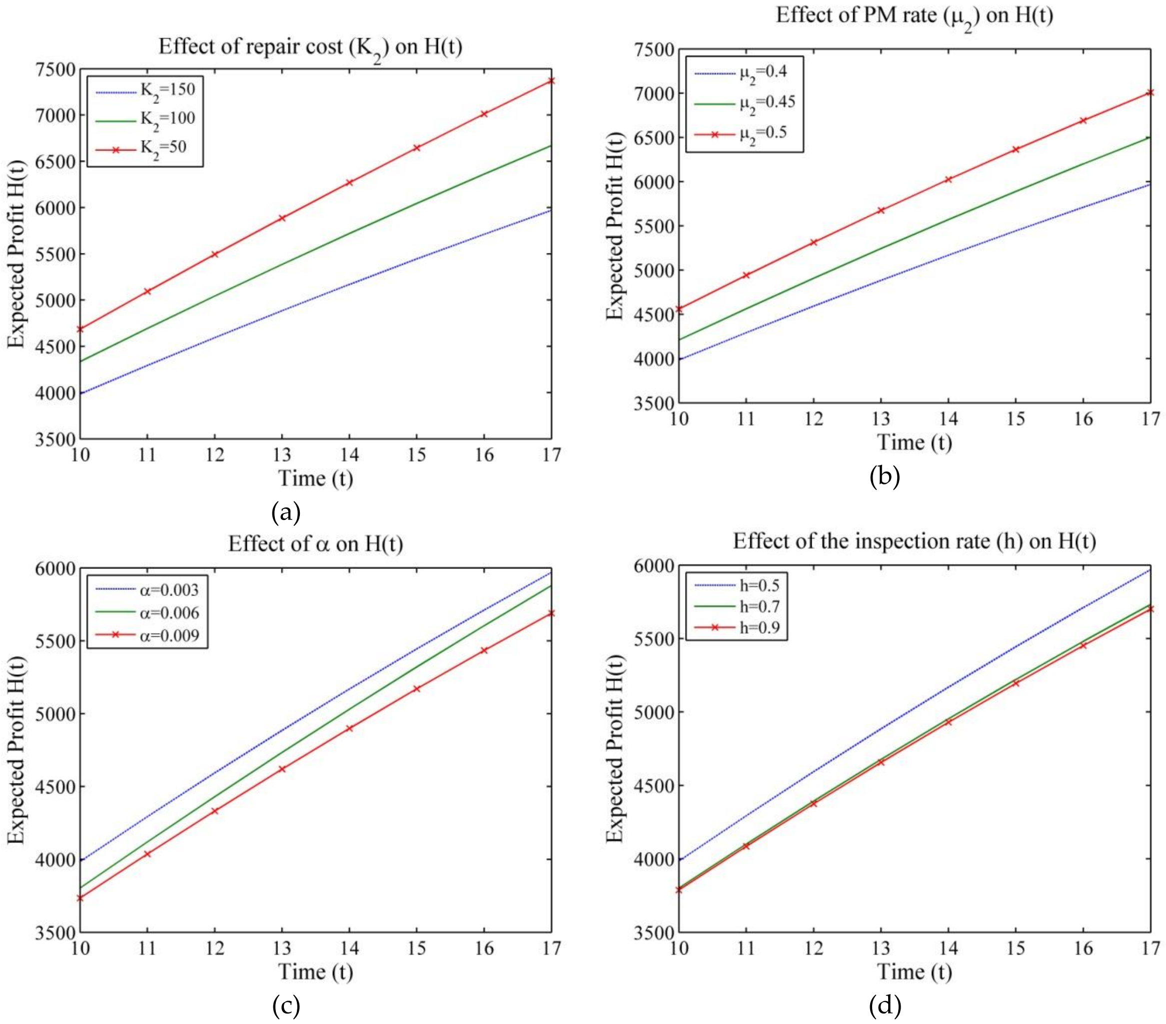

Figure 3. Effect of the parameters K_2, μ_2, α, and h on expected profit $H(t)$

As compared to the existing model proposed by Kadyan and Ramniwas [16], when we set parameters $\lambda_m, \mu_2, \lambda_2, p, q, h$ are all zero i.e., when beyond warranty, the system does not go under PM, there is no inspection of failure unit and the unit works as like a new unit after its repair, then the proposed model reduced to Kadyan and Niwas [16]. Additionally, it is observed that when parameters $\lambda_m = \mu_2 = \lambda_2 = 0$ i.e., the unit neither maintained nor works with reduced capacity, then the current model reduces to Niwas et al. [29] model. Thus, it is clearly seen that the proposed model is an extension of these existing model. Therefore, the study reveals that after getting PM beyond warranty, a system in which unit works with reduced capacity after its repair will be economically beneficial; if failed degraded unit is inspected for feasibility of repair. So, our studying model is more reasonable and advance than the existing models.

5. Conclusion

In the present paper, we have proposed an approach for analyzing a maintenance scheduling model using failure free warranty policy. In it, all the repairs during warranty are cost-free to the users, provided failures are not due to the negligence of users. For improving the performance of system PM is conducting beyond warranty and the unit works as new after PM but becomes degraded after its repair. Degraded unit is inspected by the repairman for feasibility of repair after its failure. Further, the effect of the various parameters on to system reliability and expected profit have been analyzed and found that by varying μ_2, h and α, expected profit is increased. Based on it, the system analyst may focus on μ_2, h and α parameters so as to increase the performance and productivity of the system. In future work, we shall extend our work to different approaches such as reliability-cost optimization model, fuzzy reliability, and geometric process for two or more unit system models using Weibull- Gnedenko distribution [2, 34].

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A Pareto II Model With Inliers at Zero and One Based on TYPE-II Censored Samples

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Abstract

Inliers (instantaneous or early failures) are natural occurrences of a life test, where some of the items fail immediately or within a short time of the life test due to mechanical failure, inferior quality or faulty construction of items and components. The inconsistency of such life data is modeled using a nonstandard mixture of distributions; where degeneracy can happen at discrete points at zero and one. In this paper, parameters estimation based on Type-II censored sample from a Pareto type II distribution with a discrete mass at zero and one is study. The Maximum Likelihood Estimators (MLE) are developed for estimating the unknown parameters. The Fisher information matrix, as well as the asymptotic variance-covariance matrix of the MLEs, are derived. Uniformly Minimum Variance Unbiased Estimate (UMVUE) of model parameters as well as UMVUE of the density function, reliability function, and some other parametric function are obtained along with the standard error of estimators. The model is implemented on various real data sets and compared with Weibull inliers model.

Keywords: early failures; failure time distribution; infant mortality rate; inliers; instantaneous failures; type-II censored sample.

I. Introduction

There are a plethora of examples of phenomena concerning nature, life and human activities where the real data do not conform to the standard distributions. In such cases, we either use mixtures of standard distributions of similar types or non-standard mixtures of degenerate distribution and a standard distribution, which may be again a discrete or continuous one. Since *inliers* are inconsistent observations, which are generally the results of instantaneous and early failures, modeling with inliers involve non-standard mixtures of distributions. In the former case, the random variable will have a discrete probability mass at the origin (that is life will be zero) and some positive lifetimes, and in the latter case, the failure times may be smaller in relation to other lifetimes. These occurrences may be due to mechanical failure, inferior quality or faulty construction or defective parts of items and components. Such failures usually discard the assumption of a single mode distribution and hence the usual method of modeling and inference procedures may not be accurate in practice. [2] was the first to discuss the inference problem of instantaneous failures in life testing. The author has provided the efficient estimation of parametric functions under various probability models. [13] have introduced the term *inliers* in connection with the estimation of (p, θ) of early failure model with modified failure time distribution (FTD) being an exponential distribution with mean θ assuming p known. Later on, many authors have studied these kinds of models (see [18], [13] and [15]).

There are many practical contexts, where inliers can be natural occurrences of the specific situations involved and degeneracy can happen at two discrete points and a positive distribution for the remaining lifetimes. Some of the situations are as follows:

1. The size of tumor lesions is of interest to treat *Hematologic malignancy* patients. The measurement effect is zero who have lesions absent (or due to disappearance of tumor during treatment), though who have lesions present at baseline that are measurable but do not meet the definitions of measurable disease may be considered as measurement 1, otherwise lesions can be accurately measured as longest diameter to be recorded in at least one dimension by chest x-ray, with CT scan or with calipers by clinical examination. Similarly, in studies like Bone lesions, leptomeningeal disease, ascites, pleural/pericardial effusions, lymphangitis cutis/pulmonitis, inflammatory breast disease, and abdominal masses, either the effect is absent or present but not followed by CT or MRI, are considered as non-measurable otherwise accurately measurable on a continuous scale.
2. In the mass production of technological components of hardware, intended to function over a period of time, some components may fail on installation and therefore have zero life lengths, some component that does not fail on installation but fails with negligible life (may be coded as one for simplicity), and others that will have a life length a positive random variable whose distribution may take different forms.
3. In a clinical trial laboratory, a particular drug is designed and given to certain species of hens so that the new chicks have a weight greater than usual. The possible weight of chicks may be modeled as a continuous distribution, with discrete mass at ‘zero’ and ‘one’, where zero measures those chicks having no gain of weight, and one measures those chicks with negligible gain of weight than usual, and the remaining chicks having weight gain in some continuous measurement.
4. The rainfall measurement at a place recorded during a season is modeled as a continuous distribution, with a discrete mass at ’zero’ where zero measures those days having no rainfall, and at ‘one’, one measures those days with no rain but humid and cloudy conditions, and a continuous variable having some positive amount of rain.
5. In the studies of genetic birth defects, children can be characterized by three variables: first, a discrete variable to indicate whether a child is affected and born dead; second, a child is affected and has a neonatal death; and third, a continuous variable measuring the survival time of affected children born alive. We may consider this as a nonstandard mixture of the mass point at “zero” (for children born dead), at “one” (for children born and neonatal death), and a nontrivial continuous distribution for other surviving children.

Similarly, one can contemplate many such examples in practical situations involving degeneracy at two or more points and positive configurations of observations. Authors [16] and [17], have modeled the above situation using exponential distribution and Weibull distribution respectively. In this article, we model the inliers situation using the type-II censored lifetime data from a Pareto II distribution. As per the scheme, if n units are placed on the test and the experiment is terminated after a prefixed number of failures say, $c < n$, then the observed failure times are $X_{(1)}, X_{(2)}, \dots, X_{(c)}$ where $X_{(c)} < X_{(n)}$. The remaining $n - c$ items are regarded as censored data. The family of the Pareto distribution is well known in the literature for its capability in modeling the heavy-tailed distributions. The Pareto Type II distribution (also called Lomax distribution with location parameter zero) has the probability distribution function (pdf)

$$f(x, \underline{\alpha}) = \frac{\theta\beta^{\theta}}{(x+\beta)^{(1+\theta)}}, \; x > 0, \beta > 0, \theta > 0 \tag{1}$$

where $\underline{\alpha} = (\beta, \theta)$, $\beta > 0$ is a scale parameter and $\theta > 0$ is a shape parameter. The Pareto distribution has been used in connection with studies of income, property values, insurance risk, migration, size of cities and firms, word frequencies, business mortality, service time in queuing systems, etc. The paper by [1] contains a detailed list of important areas where heavy-tailed distributions are found applicable. There are also recent applications of the Pareto distribution in data sets on earthquakes, forest fire areas, fault lengths on Earth and Venus, and on oil and gas fields sizes, see [22] for details.

The presentation of the paper is as follows: The model description is given in Section II. In Section III, we derive the MLE of the unknown parameters along with the interval estimation of parameters. The UMVU estimation of model parameters and various parametric functions are given in Section IV. For illustration, we consider four real datasets for implementing the proposed model in Section V.

II. Model description

If 0 and 1 are natural occurrence of a life test as described above with other positive observations, then the distribution function of such a inliers model can be written as:

$$H(x; p_1, p_2, \underline{\alpha}) = \begin{cases} 0, & x < 0 \\ p_1, & 0 \le x < 1 \\ p_1 + p_2, & x = 1 \\ p_1 + p_2 + (1 - p_1 - p_2)\frac{F(x;\underline{\alpha}) - F(1;\underline{\alpha})}{1 - F(1;\underline{\alpha})}, & x \ge 1 \end{cases} \tag{2}$$

The fact is that the probability measure generated by $H(.)$ is composed of three measures, say μ_1, μ_2, and μ_3, where μ_3 is absolutely continuous with respect to the Lebesgue measure on R and μ_1 and μ_2 are singular with respect to the Lebesgue measure on R. The corresponding likelihood function of the model is

$$h(x; p_1, p_2, \underline{\alpha}) = \begin{cases} p_1, & x = 0 \\ p_2, & x = 1 \\ (1 - p_1 - p_2)\frac{f(x;\underline{\alpha})}{1 - F(1;\underline{\alpha})}, & x > 1 \end{cases} \tag{3}$$

where p_1 and p_2 are the proportion of 0 and 1 observations respectively. For β =1, the Pareto Type II inliers distribution has the likelihood function

$$h(x; p_1, p_2, \theta) = \begin{cases} p_1, & x = 0 \\ p_2, & x = 1 \\ (1 - p_1 - p_2)\frac{\theta}{(1+x)}\left(\frac{2}{(1+x)}\right)^{\theta}, & x > 1 \end{cases} \tag{4}$$

The parameter estimates are obtained in the next section.

III. The Maximum Likelihood Estimation of $\underline{\boldsymbol{\theta}} = (\boldsymbol{p_1}, \boldsymbol{p_2}, \boldsymbol{\theta})$

Suppose n items placed on life test, where r_1 items have life zero where as r_2 items have life 1 and remaining $n - r_1 - r_2$ items have life greater than 1, is denoted by $X_1, X_2, \dots, X_{n-r_1-r_2}$. By applying the technique of 'Type-II censored sample', the experiment terminates after prefixed number of failures $n - r_1 - r_2 - c$ out of $n - r_1 - r_2$ items, where, $n - r_1 - r_2 - c < n - r_1 - r_2$. Clearly, if $n - r_1 - r_2 - c = n - r_1 - r_2$, then the experiment is not terminated and all $n - r_1 - r_2$ lifetimes are observed. Let $n - r_1 - r_2 - c^* = \min(n - r_1 - r_2 - c,\ n - r_1 - r_2)$ and $X_{(1)}, X_{(2)}, \dots, X_{(n-r_1-r_2-c^*)}$ denote ordered observed failure time of these $n - r_1 - r_2 - c^*$ items from $h \in \mathcal{H}$ as given in (4). Then the likelihood equation can be written as

$$L(\underline{x}; \underline{\theta}) = \prod_{i=1}^{n} h(x_i; \underline{\theta})$$

If we define

$$I_1(x) = \begin{cases} 1, & x = 0 \\ 0, & otherwise \end{cases}$$

and

$$I_2(x) = \begin{cases} 1, & x = 1 \\ 0, & otherwise \end{cases}$$

Then the likelihood equation can be written as

$$= {p_1}^{r_1}{p_2}^{r_2}(1-p_1-p_2)^{(n-r_1-r_2)} \frac{(n-r_1-r_2)!}{c^*!} \theta^{n-r_1-r_2-c^*} \prod_{i=1}^{n-r_1-r_2-c^*} \frac{1}{1+x_{(i)}}$$

$$e^{-\theta\left\{\sum_{i=1}^{n-r_1-r_2-c^*}\left[\log\left(\frac{1+x_{(i)}}{2}\right)\right]+c^*\left[\log\left(\frac{1+x_{(n-r_1-r_2-c^*)}}{2}\right)\right]\right\}} \tag{5}$$

where $r_1 = \sum_{i=1}^{n} I_1(x_{(i)})$ and $r_2 = \sum_{i=1}^{n} I_2(x_{(i)})$, denotes the number of zero and one observations respectively. We now investigate the following four possible cases of likelihood estimates:

Case (*i*). $r_2 = 0$, that is $r_1 = n$. The likelihood function simply reduces to $L(\underline{x};\underline{\theta}) = p_1^n$. Obviously, this is maximum when $p_1 = 1$. This corresponds to the maximum likelihood estimator $\hat{p}_1 = \frac{r_1}{n}$. Since $L(\underline{x};\underline{\theta}) = p_1^n$ is free from the other parameters, the maximum likelihood estimator of other parameters do not exist.
Case (*ii*). $r_1 = 0$, that is $r_2 = n$. The likelihood function simply reduces to $L(\underline{x};\underline{\theta}) = p_2^n$. Obviously, this is maximum when $p_2 = 1$. This corresponds to the maximum likelihood estimator $\hat{p}_2 = \frac{r_2}{n}$. Since $L(\underline{x};\underline{\theta}) = p_2^n$ is free from the other parameters, the maximum likelihood estimator of other parameters do not exist.
Case (*iii*). $r_1 < n$, $r_2 < n$ but $r_1 + r_2 = n$. The likelihood function simply reduces to $L(\underline{x};\underline{\theta}) = p_1^{r_1}p_2^{r_2}$. Here $p_1 + p_2 < n$. Then the likelihood function $L(\underline{x};\underline{\theta}) < \left(\frac{x_1}{n}\right)^{r_1}\left(\frac{x_2}{n}\right)^{r_2}$ So $\hat{p}_1 = \frac{r_1}{n}$ and $\hat{p}_2 = \frac{r_2}{n}$. The maximum likelihood of other parameters do not exist.
Case (*iv*). $r_1 + r_2 < n$. The log-likelihood function is given by

$$\begin{aligned}\log L(\underline{x};\underline{\theta}) &= r_1 \log p_1 + r_2 \log p_2 + (n-r_1-r_2)\log(1-p_1-p_2) + \log(n-r_1-r_2)! \\ &\quad -\log c^*! + (n-r_1-r_2-c^*)\log\theta - \sum_{i=1}^{n-r_1-r_2-c^*} \log(1+x_{(i)}) \\ &\quad -\theta\left\{\sum_{i=1}^{n-r_1-r_2-c^*}\left[\log\left(\frac{1+x_{(i)}}{2}\right)\right] + c^*\left[\log\left(\frac{1+x_{(n-r_1-r_2-c^*)}}{2}\right)\right]\right\}\end{aligned} \tag{6}$$

The maximum likelihood estimator of parameter $\underline{\theta} = (p_1, p_2, \theta)$ is obtained by solving the following likelihood equations:

$$\frac{\partial \log L(\underline{x};\underline{\theta})}{\partial p_1} = \frac{r_1}{p_1} - \frac{n-r_1-r_2}{1-p_1-p_2} = 0 \tag{7}$$

$$\frac{\partial \log L(\underline{x};\underline{\theta})}{\partial p_2} = \frac{r_2}{p_2} - \frac{n-r_1-r_2}{1-p_1-p_2} = 0 \tag{8}$$

and

$$\frac{\partial \log L(\underline{x};\underline{\theta})}{\partial \theta} = \frac{n-r_1-r_2-c^*}{\theta} - \left\{\sum_{i=1}^{n-r_1-r_2-c^*}\left[\log\left(\frac{1+x_{(i)}}{2}\right)\right] + c^*\left[\log\left(\frac{1+x_{(n-r_1-r_2-c^*)}}{2}\right)\right]\right\} = 0 \tag{9}$$

Solving (7) and (8) simultaneously, we get

$$\hat{p}_1 = \frac{r_1}{n} \tag{10}$$

$$\hat{p}_2 = \frac{r_2}{n} \tag{11}$$

From (9), the estimate of θ is

$$\hat{\theta} = \frac{n-r_1-r_2-c^*}{\sum_{i=1}^{n-r_1-r_2-c^*}[\log(1+x_{(i)})-\log 2] + c^*\left[\log\left(1+x_{(n-r_1-r_2-c^*)}\right)-\log 2\right]} \tag{12}$$

The approximate $(1-\alpha)\%$ confidence interval for $p_1,\ p_2$ and θ are respectively given by

$\hat{p}_1 \pm z\alpha/_2 \sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n}}$, $\hat{p}_2 \pm z\alpha/_2 \sqrt{\frac{\hat{p}_2(1-\hat{p}_2)}{n}}$ and $\hat{\theta} \pm z\alpha/_2 \sqrt{\frac{\hat{\theta}^2}{(n-c^*)\,\hat{p}^*}}$ where, $\hat{p}^* = 1-\hat{p}_1-\hat{p}_2$.

IV. Unbiased estimation

Many authors have studied the problem of minimum variance unbiased estimation for different classes of distributions. [23], [12] and [5] have studied the estimation problem for power series distribution, [20] has studied the same for generalized power series distribution, [7] and [5] have studied for modified power series distribution. [19] has studied the UMVUE of parameters for the multivariate modified power series distribution. All these studies include discrete distributions only. [9] has studied the problem of MVU estimation in one parameter exponential family of distributions which includes power series distribution, modified power series distribution and univariate continuous distributions. Further, a characterization property of power series distribution using one and two moments was given by [14]. [8] extended this for the one-parameter exponential family of distribution which includes all earlier cases. [10] have further studied MVU estimation in the multi-parameter exponential family of distributions. Here, we propose the distributional properties of complete sufficient statistic and study UMVU estimation for various parametric functions of the model.

The model in (4) can be expressed as

$$h(x;\underline{\theta}) = \frac{\left(\frac{1}{(1+x)}\right)^{(1-I_1(x)-I_2(x))}\left(\frac{p_1}{\theta(1-p_1-p_2)}\right)^{I_1(x)}\left(\frac{p_2}{\theta(1-p_1-p_2)}\right)^{I_2(x)}\left(e^{-\theta}\right)^{\left\{\left[\log\left(\frac{1+x}{2}\right)\right](1-I_1(x)-I_2(x))\right\}}}{\left(\frac{1}{\theta(1-p_1-p_2)}\right)} = (a(x))^{(1-C_1(x)-C_2(x))}\frac{\prod_{i=1}^{3}\left(h_i(\underline{\theta})\right)^{C_i(x)}}{g(\underline{\theta})} \tag{13}$$

where, $a(X) = \frac{1}{(1+X)}$; $h_1(\underline{\theta}) = \frac{p_1}{\theta(1-p_1-p_2)}$; $h_2(\underline{\theta}) = \frac{p_2}{\theta(1-p_1-p_2)}$; $h_3(\underline{\theta}) = e^{-\theta}$; $g(\underline{\theta}) = \frac{1}{\theta(1-p_1-p_2)}$; $C_1(X) = I_1(X)$; $C_2(X) = I_2(X)$ and $C_3(X) = \left[\log\left(\frac{1+X}{2}\right)\right]\left(1-I_1(X)-I_2(X)\right)$. Also $a(X)>0$, $C_i(X), i=1,2$ and 3 are nontrivial real- valued statistics, $g(\underline{\theta})$ and $h_i(\underline{\theta})$ are at least twice differentiable functions of θ_i, i=1,2 and 3. Here $g(\underline{\theta}) = \int_{x>1}(a(x))^{(1-C_1(x)-C_2(x))}\prod_{i=1}^{3}\left(h_i(\underline{\theta})\right)^{C_i(x)}dx$. The density in (13) so obtained is defined with respect to a measure $\mu(x)$ which is the sum of Lebesgue measure over $(1,\infty)$ a well-known form of a three parameter exponential family with natural parameters $(\eta_1,\eta_2,\eta_3) = \left(\log\left(\frac{p_1}{\theta(1-p_1-p_2)}\right), \log\left(\frac{p_2}{\theta(1-p_1-p_2)}\right), \log\left(e^{-\theta}\right)\right)$ generated by underlying indexing parameters $\underline{\theta} = (p_1,p_2,\theta)$. Hence $C(X) = \left(C_1(X),C_2(X),C_3(X)\right) = \left(I_1(X),\ I_2(X), \left[\log\left(\frac{1+X}{2}\right)\right]\left(1-I_1(X)-I_2(X)\right)\right)$ is jointly complete sufficient for $\underline{\theta} = (p_1,p_2,\theta)$. The distributional properties of $C(X) = \left(C_1(X),C_2(X),C_3(X)\right)$ are presented in appendix A. We now propose some uniformly minimum variance unbiased estimators for parameters and some parametric function of the model (13) in various subsections below.

I. Uniformly Minimum Variance Unbiased Estimation of parameters

For the Type-II censored sample discussed in the previous section, consider the following transformation

$$Y_1 = (n-r_1-r_2)\left(\left[\log\left(\frac{1+x_{(i)}}{2}\right)\right]\right),$$

and

$$Y_i = (n - r_1 - r_2 - i + 1)\left\{\left[\log\left(\frac{1+x_{(i)}}{2}\right)\right] - \left[\log\left(\frac{1+x_{(i-1)}}{2}\right)\right]\right\};$$
$$i = 2,3,\ldots, n - r_1 - r_2 - c^* \tag{14}$$

It can be seen that

$$\sum_{i=1}^{n-r_1-r_2-c^*} Y_i = \sum_{i=1}^{n-r_1-r_2-c^*}\left[\log\left(\frac{1+x_{(i)}}{2}\right)\right] + c^*\left[\log\left(\frac{1+x_{(n-r_1-r_2-c^*)}}{2}\right)\right]$$

and

$$|J| = \frac{c^*!\prod_{i=1}^{n-r_1-r_2-c^*}(1+x_{(i)})}{(n-r_1-r_2)!} \tag{15}$$

Using (14) and (15),

$$h\left(\underline{y};\,\underline{\theta}\right) = {p_1}^{r_1}{p_2}^{r_2}\,(1-p_1-p_2)^{(n-r_1-r_2)}\,\theta^{(n-r_1-r_2-c^*)}\;e^{-\theta\sum_{i=1}^{n-r_1-r_2-c^*} y_i} \tag{16}$$

$$= \frac{\left(\frac{p_1}{\theta(1-p_1-p_2)}\right)^{z_1}\left(\frac{p_2}{\theta(1-p_1-p_2)}\right)^{z_2}\left(e^{-\theta}\right)^{z_3}(1-p_1-p_2)^{c^*}}{\left(\frac{1}{\theta(1-p_1-p_2)}\right)^{n-c^*}}$$

where

$$Z_1 = \sum_{i=1}^{n} C_1(X_i) = \sum_{i=1}^{n-c^*} I_1(Y_i) = r_1$$
$$Z_2 = \sum_{i=1}^{n} C_2(X_i) = \sum_{i=1}^{n-c^*} I_2(Y_i) = r_2$$

and

$$Z_3 = \sum_{i=1}^{n} C_3(X_i) = \sum_{i=1}^{n-r_1-r_2-c^*} Y_i$$

Hence by Neyman Factorization theorem $Z = (Z_1, Z_2, Z_3)$ is jointly sufficient for $\underline{\theta} = (p_1, p_2, \theta)$. Also,

$$h\left(\underline{y};\underline{\theta}\right) = \frac{n!}{r_1!\;r_2!\;(n-r_1-r_2)!}\,{p_1}^{r_1}{p_2}^{r_2}(1-p_1-p_2)^{\,(n-r_1-r_2)}\,\frac{\theta^{(n-r_1-r_2-c^*)}}{\left(\frac{n!}{r_1!\;r_2!\;(n-r_1-r_2)!}\right)}\;e^{-\theta\sum_{i=1}^{n-r_1-r_2-c^*} y_i}$$

$$= \mathrm{P}(Z_1 = r_1, Z_2 = r_2)\;\;h(\underline{y};\theta|Z_1 = r_1, Z_2 = r_2)$$

Here distribution of (Z_1, Z_2) is trinomial and is a complete family of distribution and

$$h\left(\underline{y};\theta\middle|Z_1 = r_1, Z_2 = r_2\right) = \frac{\theta^{(n-r_1-r_2-c^*)}\;e^{-\theta\sum_{i=1}^{n-r_1-r_2-c^*} y_i}}{\left(\frac{n!}{r_1!\;r_2!\;(n-r_1-r_2)!}\right)}$$

which belongs to the one-parameter exponential family. Hence $Z_3|Z_1, Z_2$ is complete sufficient for θ and also a member of the exponential family. The distribution of $Z_3|Z_1, Z_2$ is Gamma with parameter $(n - r_1 - r_2 - c^*, \theta)$ with pdf

$$h(z_3;\theta|n - r_1 - r_2 - c^*) = \frac{{z_3}^{(n-r_1-r_2-c^*-1)}\;\theta^{n-r_1-r_2-c^*}\;e^{-\theta\,z_3}}{\Gamma n-r_1-r_2-c^*}, z_3 > 0;\;\theta > 0$$

which depends only on θ and is also a complete family of distribution. Therefore, using result of [11] $Z = (Z_1, Z_2, Z_3)$ is complete sufficient for $\underline{\theta} = (p_1, p_2, \theta)$. The Joint distribution of $Z = (Z_1, Z_2, Z_3)$ is

$$h_Z(z;\underline{\theta}) = \frac{n!}{r_1!\ r_2!\ (n-r_1-r_2)!} {p_1}^{r_1} {p_2}^{r_2} (1-p_1-p_2)^{(n-r_1-r_2)} \frac{{z_3}^{(n-r_1-r_2-c^*-1)}}{\Gamma n-r_1-r_2-c^*} \theta^{n-r_1-r_2-c^*} e^{-\theta z_3},$$

$$0 \le r_1, r_2 \le n-c^*;\ z_3 > 0;\ 0 \le p_1, p_2 \le 1;\ \theta > 0$$

$$= B(z_1, z_2, z_3, c^*, n)\ \frac{\prod_{i=1}^{3}(h_i(\underline{\theta}))^{z_i}}{g(\underline{\theta})^{n-c^*}} (1-p_1-p_2)^{c^*}$$

where

$$B(z_1, z_2, z_3, c^*, n) = \begin{cases} \frac{n!}{r_1!\ r_2!\ (n-r_1-r_2)!} \frac{{z_3}^{(n-r_1-r_2-c^*-1)}}{\Gamma n-r_1-r_2-c^*},\ z_3 > 0;\ r_1+r_2-1 < n-c^* \\ 1, \qquad z_3 = 0;\ r_1 = 0\ or\ r_2 = 0 \end{cases} \tag{17}$$

$z_i \in T(n-c^*) \subseteq \mathbb{R}, \underline{\theta} \in \Omega$. Here $z = (z_1, z_2, z_3, c^*, n)$ and $B(z_1, z_2, z_3, c^*, n)$ are such that

$$\frac{g(\theta)^{n-c^*}}{(1-p_1-p_2)^{c^*}} = \int_{z_1 \in T(n-c^*)} \int_{z_2 \in T(n-c^*)} \int_{z_3 \in T(n-c^*)} B(z_1, z_2, z_3, c^*, n) \prod_{i=1}^{3} (h_i(\underline{\theta}))^{z_i}\, dz_1\, dz_2\, dz_3$$

Since $(C_1(x)) = p_1$, $E(C_2(x)) = p_2$ and $E(C_3(x)) = \frac{(1-p_1-p_2)}{\theta}$ (see Appendix A for details). Hence,

$$E(Z_1) = E\left(\sum_{j=1}^{n} C_1(x_j)\right) = \sum_{j=1}^{n-c^*} E\left(I_1(y_j)\right) = (n-c^*)\, p_1,$$

$$E(Z_2) = E\left(\sum_{j=1}^{n} C_2(x_j)\right) = \sum_{j=1}^{n-c^*} E\left(I_2(y_j)\right) = (n-c^*)\, p_2,$$

and

$$E(Z_3) = E\left(\sum_{j=1}^{n} C_3(x_j)\right) = \sum_{i=1}^{n-r_1-r_2-c^*} E(Y_i) = (n-c^*) \frac{(1-p_1-p_2)}{\theta},$$

which in turn give UMVUE's of p_1, p_2 and θ as

$$\hat{p}_1 = \frac{Z_1}{n-c^*} = \frac{r_1}{n-c^*} \tag{18}$$

$$\hat{p}_2 = \frac{Z_2}{n-c^*} = \frac{r_2}{n-c^*} \tag{19}$$

and

$$\hat{\theta} = \frac{(n-c^*)(1-\hat{p}_1-\hat{p}_2)}{Z_3} \tag{20}$$

For variance computation, see Appendix A. Note that, the likelihood estimate and minimum variance unbiased estimate of the parameters coincides everywhere when c^*=0.

II. Uniformly Minimum Variance Unbiased Estimation of parametric functions

Let $X_1, X_2, \dots, X_{n-c^*}$ be Type-II censored random sample from (13), then there exists an UMVUE of $\Phi(\underline{\theta})$ if and only if $\Phi(\underline{\theta})[g(\underline{\theta})]^{n-c^*}$ can be expressed in the form

$$\frac{\Phi(\underline{\theta})[g(\underline{\theta})]^{n-c^*}}{(1-p_1-p_2)^{c^*}} = \int_{z_1 \in T(n-c^*)} \int_{z_2 \in T(n-c^*)} \int_{z_3 \in T(n-c^*)} \alpha(z_1, z_2, z_3, c^*, n) \prod_{i=1}^{3} (h_i(\underline{\theta}))^{z_i}\, dz_1\, dz_2\, dz_3$$

Thus, the UMVUE of a function $\Phi(\underline{\theta})$ of $\underline{\theta}$ in $h(x;\underline{\theta})$ is given by

$$\psi(Z_1, Z_2, Z_3, c^*, n) = \frac{\alpha(Z_1, Z_2, Z_3, c^*, n)}{B(Z_1, Z_2, Z_3, c^*, n)}, \qquad B(Z_1, Z_2, Z_3, c^*, n) \neq 0$$

The following results are now obvious.

Result 1 The UMVUE of $\prod_{i=1}^{3}(h_i(\underline{\theta}))^{k_i} = \left(\frac{1}{\theta(1-p_1-p_2)}\right)^{k_1+k_2} {p_1}^{k_1}\, {p_2}^{k_2} e^{-\theta k_3}$ is given by

$$H_{k_1,k_2,k_3}(z_1, z_2, z_3, c^*, n) = \frac{B(z_1-k_1,\ z_2-k_2,\ z_3-k_3,\ c^*, n)}{B(z_1, z_2, z_3, c^*, n)}$$

$$= \frac{(r_1)_{k_1}\, (r_2)_{k_2}\, \left(1-\frac{k_3}{z_3}\right)^{(n-r_1-r_2-c^*-1)} (z_3-k_3)^{k_1+k_2}}{[n-r_1-r_2+1]_{k_1+k_2} [n-r_1-r_2-c^*]_{k_1+k_2}},$$

where $k_1 \le r_1;\ k_2 \le r_2;\ k_3 \le z_3;\ k_1 + k_2 \le n - r_1 - r_2 - c^*;\ r_1 + r_2 - 1 < n - c^*$, and $(r)_k = \frac{r!}{(r-k)!}$, $[r]_k = \frac{\Gamma r+k}{\Gamma r}$.

Corollary 1 If $k_1 \ne 0,\ k_2 = 0$ and $k_3 = 0$, then UMVUE of $\left(h_1(\underline{\theta})\right)^{k_1} = \left(\frac{p_1}{\theta(1-p_1-p_2)}\right)^{k_1}$ is given by

$$H_{k_1}(z_1, z_2, z_3, c^*, n) = \frac{B(z_1-k_1, z_2, z_3, c^*, n)}{B(z_1, z_2, z_3, c^*, n)} = \frac{(r_1)_{k_1}\ {z_3}^{k_1}}{[n-r_1-r_2+1]_{k_1}[n-r_1-r_2-c^*]_{k_1}},$$
$$k_1 \le r_1;\ k_1 \le n - r_1 - r_2 - c^*;\ r_1 + r_2 - 1 < n - c^*$$

Corollary 2 If $k_1 = 0, k_2 \ne 0$ and $k_3 = 0$, then UMVUE of $\left(h_2(\underline{\theta})\right)^{k_2} = \left(\frac{p_2}{\theta(1-p_1-p_2)}\right)^{k_2}$ is given by

$$H_{k_2}(z_1, z_2, z_3, c^*, n) = \frac{B(z_1, z_2-k_2, z_3, c^*, n)}{B(z_1, z_2, z_3, c^*, n)}$$
$$= \frac{(r_2)_{k_2}\ {z_3}^{k_2}}{[n-r_1-r_2+1]_{k_2}[n-r_1-r_2-c^*]_{k_2}},\ k_2 \le r_2;\ k_2 \le n - r_1 - r_2 - c^*;\ r_1 + r_2 - 1 < n - c^*$$

Corollary 3 If $k_1 = 0, k_2 = 0$ and $k_3 \ne 0$, then UMVUE of $\left(h_3(\underline{\theta})\right)^{k_3} = e^{-\theta k_3}$ is given by

$$H_{k_3}(z_1, z_2, z_3, c^*, n) = \frac{B(z_1, z_2, z_3 - k_3,\ c^*, n)}{B(z_1, z_2, z_3, c^*, n)} = \left(1 - \frac{k_3}{z_3}\right)^{n-r_1-r_2-c^*-1},\ k_3 \le z_3;\ r_1 + r_2 - 1 < n - c^*$$

Result 2 The UMVUE of the variance of $H_{k_1,k_2,k_3}(Z_1, Z_2, Z_3, c^*, n)$, is given by

$$\widehat{var}\left[H_{k_1,k_2,k_3}(z_1, z_2, z_3, c^*, n)\right] = H^2_{k_1,k_2,k_3}(z_1, z_2, z_3, c^*, n) - H_{2k_1,2k_2,2k_3}(z_1, z_2, z_3, c^*, n)$$
$$= \left[\frac{(r_1)_{k_1}(r_2)_{k_2}\left(1-\frac{k_3}{z_3}\right)^{(n-r_1-r_2-c^*-1)}(z_3-k_3)^{k_1+k_2}}{[n-r_1-r_2+1]_{k_1+k_2}[n-r_1-r_2-c^*]_{k_1+k_2}}\right]^2 - \frac{(r_1)_{2k_1}(r_2)_{2k_2}\left(1-\frac{2k_3}{z_3}\right)^{(n-r_1-r_2-c^*-1)}(z_3-2k_3)^{2(k_1+k_2)}}{[n-r_1-r_2+1]_{2(k_1+k_2)}[n-r_1-r_2-c^*]_{2(k_1+k_2)}},$$

where $2k_1 \le r_1;\ 2k_2 \le r_2;\ 2k_3 \le z_3;\ 2(k_1 + k_2) \le n - r_1 - r_2 - c^*;\ r_1 + r_2 - 1 < n - c^*$.

Corollary 4 The UMVUE of the variance of $H_{k_1}(Z_1, Z_2, Z_3, c^*, n)$, is given by

$$\widehat{var}\left[H_{k_1}(z_1, z_2, z_3, c^*, n)\right] = H^2_{k_1}(z_1, z_2, z_3, c^*, n) - H_{2k_1}(z_1, z_2, z_3, c^*, n)$$
$$= \left[\frac{(r_1)_{k_1}\ {z_3}^{k_1}}{[n-r_1-r_2+1]_{k_1}[n-r_1-r_2-c^*]_{k_1}}\right]^2 - \frac{(r_1)_{2k_1}\ {z_3}^{2k_1}}{[n-r_1-r_2+1]_{2k_1}[n-r_1-r_2-c^*]_{2k_1}},$$
$$2k_1 \le r_1;\ 2k_1 \le n - r_1 - r_2 - c^*;\ r_1 + r_2 - 1 < n - c^*$$

Corollary 5 The UMVUE of the variance of $H_{k_2}(Z_1, Z_2, Z_3, c^*, n)$, is given by

$$\widehat{var}\left[H_{k_2}(z_1, z_2, z_3, c^*, n)\right] = H^2_{k_2}(z_1, z_2, z_3, c^*, n) - H_{2k_2}(z_1, z_2, z_3, c^*, n)$$
$$= \left[\frac{(r_2)_{k_2}\ {z_3}^{k_2}}{[n-r_1-r_2+1]_{k_2}\ [n-r_1-r_2-c^*]_{k_2}}\right]^2 - \frac{(r_2)_{2k_2}\ {z_3}^{2k_2}}{[n-r_1-r_2+1]_{2k_2}\ [n-r_1-r_2-c^*]_{2k_2}},$$
$$2k_2 \le r_2;\ 2k_2 \le n - r_1 - r_2 - c^*;\ r_1 + r_2 - 1 < n - c^*$$

Corollary 6 The UMVUE of the variance of $H_{k_3}(Z_1, Z_2, Z_3, c^*, n)$, is given by

$$\widehat{var}\left[H_{k_3}(z_1, z_2, z_3, c^*, n)\right] = H^2_{k_3}(z_1, z_2, z_3, c^*, n) - H_{2k_3}(z_1, z_2, z_3, c^*, n)$$
$$= \left(1 - \frac{k_3}{z_3}\right)^{2(n-r_1-r_2-c^*-1)} - \left(1 - \frac{2k_3}{z_3}\right)^{n-r_1-r_2-c^*-1},$$
$$2k_3 \le z_3;\ r_1 + r_2 - 1 < n - c^*$$

Result 3 The UMVUE of $\left[g(\underline{\theta})\right]^k = \left(\frac{1}{\theta(1-p_1-p_2)}\right)^k, k \ne 0$ as per the model given in (13) is

$$G_k(z_1, z_2, z_3, c^*, n) = \frac{B(z_1, z_2, z_3, c^*, n+k)}{B(z_1, z_2, z_3, c^*, n)}$$
$$= \frac{[n+1]_k \quad {z_3}^k}{[n-r_1-r_2+1]_k \quad [n-r_1-r_2-c^*]_k}, \; k \leq n-r_1 - r_2 - c^*; \; r_1 + r_2 - 1 < n - c^*$$

Result 4 The UMVUE of the variance of $G_k(Z_1, Z_2, Z_3, c^*, n)$ is given by

$$\widehat{var}[G_k(z_1, z_2, z_3, n)] = G_k^2(z_1, z_2, z_3, c^*, n) - G_{2k}(z_1, z_2, z_3, c^*, n)$$
$$= \left[\frac{[n+1]_k \quad {z_3}^k}{[n-r_1-r_2+1]_k \quad [n-r_1-r_2-c^*]_k}\right]^2 - \frac{[n+1]_{2k} \quad {z_3}^{2k}}{[n-r_1-r_2+1]_{2k} \quad [n-r_1-r_2-c^*]_{2k}},$$
$$2k \leq n-r_1 - r_2 - c^*; \; r_1 + r_2 - 1 < n - c^*$$

Result 5 For fixed x, the UMVUE of the density given in (13) is

$$\phi_x(z_1, z_2, z_3, c^*, n) = a(x) \frac{B(z_1 - C_1(x), \, z_2 - C_2(x), \, z_3 - C_3(x), c^*, n-1)}{B(z_1, z_2, z_3, c^*, n)}$$
$$= \left(\frac{1}{1+x}\right) \frac{(r_1)_{I_1(x)} \, (r_2)_{I_2(x)} \, (n-r_1-r_2)_{(1-I_1(x)-I_2(x))} \, (n-r_1-r_2-c^*-1)_{(1-I_1(x)-I_2(x))}}{n\left[\, z_3 - \left[\log\left(\frac{1+x}{2}\right)\right](1-I_1(x)-I_2(x))\right]^{(1-I_1(x)-I_2(x))}}$$
$$\left(1 - \frac{\left[\log\left(\frac{1+x}{2}\right)\right](1-I_1(x)-I_2(x))}{z_3}\right)^{(n-r_1-r_2-c^*-1)}, z_3 > \left[\log\left(\frac{1+x}{2}\right)\right]; \; r_1 + r_2 - 1 < n - c^*$$

Result 6 The UMVUE of the variance of $\phi_x(Z_1, Z_2, Z_3, c^*, n)$ is given by

$$\widehat{var}[\phi_x(z_1, z_2, z_3, c^*, n)] = \phi_x^2(z_1, z_2, z_3, c^*, n)$$
$$-\phi_x(z_1, z_2, z_3, c^*, n) \, \phi_x(z_1 - C_1(x), \, z_2 - C_2(x), \, z_3 - C_3(x), c^*, n-1)$$
$$= \phi_x^2(z_1, z_2, z_3, c^*, n) - \left(\frac{1}{1+x}\right)^2 \frac{(r_1)_{2I_1(x)} \, (r_2)_{2I_2(x)} \, (n-r_1-r_2)_{2(1-I_1(x)-I_2(x))} \, (n-r_1-r_2-c^*-1)_{2(1-I_1(x)-I_2(x))}}{n(n-1)\left[\, z_3 - 2\left[\log\left(\frac{1+x}{2}\right)\right](1-I_1(x)-I_2(x)\right]^{2(1-I_1(x)-I_2(x))}}$$
$$\left(1 - \frac{2\left[\log\left(\frac{1+x}{2}\right)\right](1-I_1(x)-I_2(x))}{z_3}\right)^{(n-r_1-r_2-c^*-1)}, z_3 > 2\left[\log\left(\frac{1+x}{2}\right)\right]; \; r_1 + r_2 - 1 < n - c^*$$

Result 7 For a fixed $z = (z_1, z_2, z_3, c^*, n)$, the UMVUE of the survival function $S(x) = p(X > x)$, $x \geq 0$ is obtained as

$$\hat{S}(x) = \left(\frac{(r_1)_{I_1(x)} (r_2)_{I_2(x)} (n-r_1-r_2)_{(1-I_1(x)-I_2(x))} (n-r_1-r_2-c^*-1)_{(1-I_1(x)-I_2(x))}}{n\,[(n-r_1-r_2-c^*) - (1-I_1(x)-I_2(x))]}\right)$$
$$\left(Z_3 - \left[\log\left(\frac{1+x}{2}\right)\right]\left(1 - I_1(x) - I_2(x)\right)\right)^{(I_1(x)+I_2(x))} \left(1 - \frac{\left[\log\left(\frac{1+x}{2}\right)\right](1-I_1(x)-I_2(x))}{Z_3}\right)^{(n-r_1-r_2-c^*-1)},$$
$$Z_3 > \left[\log\left(\frac{1+x}{2}\right)\right]; \; r_1 + r_2 - 1 < n - c^*$$

Result 8 For the fixed $z = (z_1, z_2, z_3, c^*, n)$, the UMVUE of the $var(\hat{S}(x))$, is obtained as

$$\widehat{var}(\hat{S}(x)) = [\hat{S}(x)]^2 - \frac{1}{n(n-1)}\left(1 - \frac{2\left[\log\left(\frac{1+x}{2}\right)\right](1-I_1(x)-I_2(x))}{z_3}\right)^{(n-r1-r2-c^*-1)}$$
$$\left(\frac{(r_1)_{2I_1(x)} (r_2)_{2I_2(x)} (n-r_1-r_2)_{2(1-I_1(x)-I_2(x))} \, (n-r_1-r_2-c-1^*)_{2(1-I_1(x)-I_2(x))}}{[(n-r_1-r_2-c^*)-2(1-I_1(x)-I_2(x))][(n-r_1-r_2-c^*+1)-2(1-I_1(x)-I_2(x))]}\right)$$
$$\left(\, z_3 - 2\left[\log\left(\frac{1+x}{2}\right)\right](1 - I_1(x) - I_2(x)\right)^{2(I_1(x)+I_2(x))},$$
$$Z_3 > 2\left[\log\frac{(1+x)}{2}\right]; \; r_1 + r_2 - 1 < n - c^*$$

III. Real data illustration

In this section, we have considered four inliers prone data set to illustrate our proposed work. The motivation behind considering a different variety of data sets is to show the flexibility of the proposed model in different situations. The detailed description regarding the data sets is given below:

Dataset 1: The data in Table 1 shows the loss ratios (yearly data) for earthquake insurance in

California from 1971 through 1993. The data are taken from [6] and also used by [4] for their study. Note that, for four years there was no loss for earthquake insurance and the information where loss of less than 1 billion dollars per year is considered as 1, for simplicity. The analysis of this data is carried out at the end of this section.

Table 1. *California earthquake insurance data*

Year	1971	1972	1973	1974	1975	1976	1977	1978	1979	1980	1981	1982
Loss ratios	17.4	0.0	0.6	3.4	0.0	0.0	0.7	1.5	2.2	9.2	0.9	0.0
Year	1983	1984	1985	1986	1987	1988	1989	1990	1991	1992	1993	
Loss ratios	2.9	5.0	1.3	9.3	22.8	11.5	129.8	47.0	17.2	12.8	3.2	

Dataset 2: The National Family Health Survey (NFHS) is a large-scale, multi-round survey conducted in a representative sample of households throughout India. The First National Family Health Survey (NFHS-1) was conducted in 1992-93, the Second National Family Health Survey (NFHS-2) was conducted in 1998-99 and the Third National Family Health Survey (NFHS-3) was carried out in 2005-06. The survey is based on a sample of households that is representative at the national and state levels. The NFHS-3 fieldwork, conducted by 18 research organizations between December 2005 and August 2006, interviewed women at age 15-49. We consider the data on child's age at death from the woman's questionnaire of NFHS-3. For comprehensive data, one may visit [24]. For Gujarat state, there are 15 stillbirths (the death of a baby before or during the birth after 28 weeks of gestation) considered as observation 0, 37 neonatal deaths (the death of a baby within the first 28 days of life) considered as observation 1 and other observations of age at death in days as: 30, 30, 30, 31, 31, 60, 62, 62, 62, 90, 90, 90, 92, 93, 150, 182, 213, 242, 272, 273, 300, 303, 333, 334, 335, 356, 360, 365, 366, 450, 730, 731, 732, 732 and 1462. This is a perfect data for inliers model with two discrete point at zero and one. Authors of this paper had already modeled this data using exponential and Weibull distribution. The analysis based on Pareto Type II distribution is presented below.

Dataset 3: [23] have analyzed and quantified forest burnt area in India using AWiFS data for the year 2014. The burnt area map from AWiFS data involves Forest type map of 2013 at 56 m resolution prepared as part of the national carbon project. India has a geographical area of about 3,287,263 sq. km. It comprises 29 states and 7 union territories. The country has 21% of the geographical area under forest cover. Forest fires occur in India mainly between January and June. They are more frequent between February and May in different biogeographic zones of India. State/Union Territory-wise analysis of the percentage of forest burnt area (area in sq. km) is available in [23], page 1531. We consider State/Union Territory burnt area from February to May 2014. There are six State/Union Territory (Delhi, Andaman and Nicobar, Chandigarh, Daman and Diu, Lakshadweep and Pondicherry) having burnt area zero, five State/Union Territory (Goa(0.04) , Jammu and Kashmir (0.11), Dadra and Nagar Haveli (0.23), Punjab (0.85) and Himachal Pradesh (0.91)) having percentage burnt area less than 1 sq. Km. conveniently considered here as observation 1, and the remaining 25 State/Union Territory burnt area in sq. Km. are: 6611.86, 102.70, 941.11, 1773.22, 4606.69, 487.81, 1.84, 2587.40, 1920.35, 82.01, 3342.66, 5066.66, 1974.23, 457.50, 421.03, 975.79, 8186.46, 364.17, 2.50, 4275.64, 2955.23, 739.00, 459.07, 42.01 and 386.37. The analysis is reported below.

Dataset 4: This data is about the amount of snowfall in all 50 states of US. According to the National Climatic Data Center, the data were populated considering the average snowfall for almost three decades from 1981 to 2010, available at [25]. The average amount snowfall per year (in inches) for 50 states of US are: 5.2, 0.5, 1.6, 74.5, 0.3, 0.0, 19.1, 40.5, 20.2, 0.0, 0.7, 0.0, 19.2, 24.6, 25.9, 34.9, 14.7, 12.5, 0.0, 61.8, 20.2, 43.8, 51.1, 54, 0.9, 17, 38.1, 25.9, 21.8, 60.8, 16.5, 9.6, 123.8, 7.6, 51.2, 27.5, 7.8,3, 28.2, 33.8, 43.9, 6.3, 1.5, 56.2, 81.2, 10.3, 5.0, 62.0, 50.9 and 91.4. It is observed that there are three decades having an average amount of snowfall zero and for four states having decades average amount of snowfall less than 1 inches (coded as observation1).

For all the data sets above we have calculated parameter estimates, goodness-of-fit criteria values, goodness-of-fit statistics and corresponding p-values (see Table 2 for details) for positive observations only. It may be noted from the table that for all the considered data sets, the Pareto Distribution fits well (see p-values).

Table 2. *The parameter estimates, goodness-of-fit criteria and corresponding* p*-value for various datasets (Pareto distribution).*

Data	MLE (SE)	AIC	BIC	K-S (p-value)	CVM (p-value)	AD (p-value)
Earthquake insurance	$\hat{\beta}$= 19.5743 (19.2742) $\hat{\theta}$= 2.0113 (1.4153)	124.7323	126.2778	0.1213 (0.9498)	0.0362 (0.9563)	0.2901 (0.9448)
NFHS-3	$\hat{\beta}$=18557.4806 (34321.4861) $\hat{\theta}$= 65.5015 (119.8512)	470.3576	473.4683	0.1210 (0.6848)	0.0898 (0.6400)	0.6150 (0.6327)
Forest burnt area	$\hat{\beta}$=3418.3510 (4828.3362) $\hat{\theta}$= 2.6249 (2.7363)	431.6623	434.1000	0.1446 (0.6214)	0.0984 (0.5964)	1.0663 (0.3236)
Snow fall	$\hat{\beta}$=2907.8650 (8293.9850) $\hat{\theta}$= 87.5320 (247.1416)	383.029	386.5043	0.1049 (0.7447)	0.0933 (0.6208)	0.5532 (0.6922)

(* Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Kolmogorov-Smirnov (K-S) Statistic, Cramer-Von Mises (CVM) statistics, Anderson-Darling (AD) statistic).

The plot of pdf, $h(x)$ and survival function, $S(x)$ for all four datasets under study, is displayed in Figure 1 and Figure 2 respectively for varying censoring schemes under Pareto II and the Weibull distribution. For the data sets under study, the summary of the various estimates of parameters and parametric functions along with their standard error (shown in bracket) and 95 % confidence interval considering censoring schemes at value c^* is given in Table 3. Whereas Table 4 shows, the UMVU estimate of pdf and survival function with Pareto II and the Weibull distribution for varying censoring schemes. It is observed that Pareto distribution has a heavier tail than Weibull.

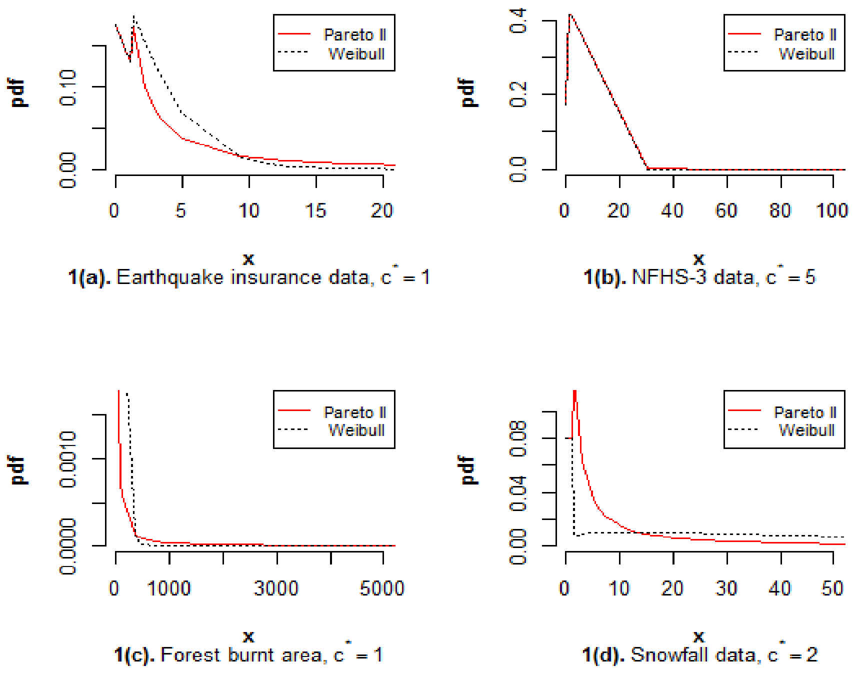

Fig. 1 *Density plot to various data sets censored at value* c^*

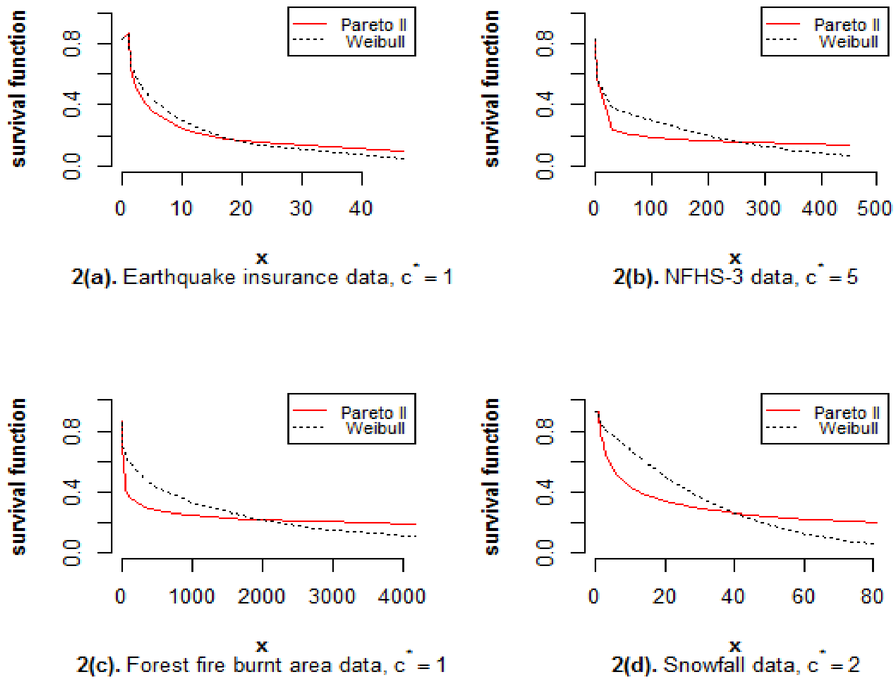

Fig. 2 *Survival function plot to various data sets censored at value* c^*

Table 3. *Summary of estimates of parameters/parametric functions of Pareto II distribution censored at c^*.*

Parameter/Parametric function	Earthquake insurance data	NFHS-3 data	Forest fire burnt area data	Snowfall data
	c^*=1	c^*=5	c^*=1	c^*=2
MLE (SE) of p_1	0.17391 (0.07904)	0.17241 (0.04050)	0.16667 (0.06212)	0.08000 (0.03837)
MLE (SE) of p_2	0.13043 (0.07022)	0.42529 (0.05300)	0.13889 (0.05764)	0.08000 (0.03837)
MLE (SE) of θ	0.61420 (0.15857)	0.19539 (0.03402)	0.16667 (0.03380)	0.38550 (0.06095)
95% CI of p_1	(0.01901, 0.32882)	(0.09304, 0.25179)	(0.04493, 0.28841)	(0.00480, 0.15520)
95% CI of p_2	(0.00000, 0.26807)	(0.32140, 0.52917)	(0.02592, 0.25186)	(0.00480, 0.15520)
95% CI of θ	(0.30648, 0.92191)	(0.12871, 0.26206)	(0.10041, 0.23292)	(0.26651, 0.50449)
UMVUE (SE) of p_1	0.18182 (0.08223)	0.18293 (0.04269)	0.17143 (0.06370)	0.08333 (0.03989)
UMVUE (SE) of p_2	0.13636 (0.07317)	0.45122 (0.05495)	0.14286 (0.05915)	0.08333 (0.03989)
UMVUE (SE) of θ	0.61420 (0.13812)	0.19539 (0.02791)	0.16667 (0.02967)	0.38550 (0.05643)
95% CI of UMVUE p_1	(0.02065, 0.34299)	(0.09925, 0.26660)	(0.04657, 0.29629)	(0.00515, 0.16152)
95% CI of UMVUE of p_2	(0.00000, 0.27976)	(0.34351, 0.55892)	(0.02693, 0.25879)	(0.00515, 0.16152)
95% CI of UMVUE θ	(0.34348 0.88492)	(0.14069, 0.25008)	(0.10850, 0.22483)	(0.27489, 0.49610)
$\prod_{i=1}^{3}\left(h_i(\underline{\theta})\right)^{k_i} = \left(\frac{\theta}{1-p_1-p_2}\right)^2 p_1 p_2 e^{-\frac{1}{\theta}}$ $k_1 = 1, k_2 = 1, k_3 = 1$	0.04992 (0.04299)	8.62541 (4.73007)	1.240801 (0.89610)	3.07667 (0.02775)
$h_1(\underline{\theta}) = \frac{\theta\, p_1}{1-p_1-p_2}, k_1 = 1, k_2 = 0, k_3 = 0$	0.38309 (0.22204)	2.13250 (0.74236)	1.38463 (0.66351)	0.24131 (0.12880)
$h_2(\underline{\theta}) = \frac{\theta\, p_2}{1-p_1-p_2}, k_1 = 0, k_2 = 1, k_3 = 0$	0.28732 (0.18391)	5.26018 (1.52329)	1.15386 (0.58886)	0.24131 (0.12880)
$h_3(\underline{\theta}) = e^{-\frac{1}{\theta}}, k_1 = 0, k_2 = 0, k_3 = 1$	0.55693 (0.08844)	0.82738 (0.00109)	0.85191 (0.02856)	0.68545 (0.04162)
$g(\underline{\theta}) = \frac{\theta}{1-p_1-p_2}, k = 1$	2.29856 (0.64068)	12.51070 (2.73276)	8.53853 (1.92013)	3.07667 (0.51361)

Table 4. *Summary of estimates of pdf and reliability function of the various data sets censored at c^*.*

Function	Earthquake insurance data		NFHS-3 data		Forest fire burnt area data		Snowfall data	
	c^*=1		c^*=5		c^*=1		c^*=2	
	Pareto-II	Weibull	Pareto-II	Weibull	Pareto-II	Weibull	Pareto-II	Weibull
pdf	ϕ_{10} = 0.01415 (0.00185)	ϕ_{10} = 0.02091 (0.00295)	ϕ_{100} = 0.00030 (5.043e-05)	ϕ_{100} = 0.0011 (1.968e-04)	ϕ_{650} = 6.912e-05 (7.524e-06)	ϕ_{650} = 0.00022 (3.235e-05)	ϕ_{25} = 0.00469 (0.00028)	ϕ_{25} = 0.01383 (0.00114)
	ϕ_{15} = 0.00784 (0.00112)	ϕ_{15} = 0.01317 (0.00113)	ϕ_{500} = 5.473e-05 (7.032e-06)	ϕ_{500} = 0.0003 (6.096e-05)	ϕ_{1350} = 2.966e-05 (3.199e-06)	ϕ_{1350} = 0.00012 (8.101e-06)	ϕ_{50} = 0.00186 (0.00013)	ϕ_{50} = 0.00676 (0.00068)
	ϕ_{40} = 0.00175 (0.00046)	ϕ_{40} = 0.00273 (0.00046)	ϕ_{1000} = 2.387e-05 (3.185e-06)	ϕ_{1000} = 1.851e-05 (1.282e-05)	ϕ_{2500} = 1.451e-05 (1.624e-06)	ϕ_{2500} = 6.421e-05 (3.185e-06)	ϕ_{100} = 0.00072 (7.193e-05)	ϕ_{100} = 0.00108 (0.00044)
Survival function	$\hat{S}_{10}$ = 0.25585 (0.07038)	$\hat{S}_{10}$ = 0.17398 (0.05400)	$\hat{S}_{100}$ = 0.18996 (0.03638)	$\hat{S}_{100}$ = 0.29777 (0.04249)	$\hat{S}_{650}$ = 0.27042 (0.07114)	$\hat{S}_{650}$ = 0.40038 (0.06434)	$\hat{S}_{25}$ = 0.31647 (0.05347)	$\hat{S}_{25}$ = 0.43451 (0.05291)
	$\hat{S}_{15}$ = 20013 (0.07279)	$\hat{S}_{15}$ = 0.00183 (0.04218)	$\hat{S}_{500}$ = 0.13903 (0.03296)	$\hat{S}_{500}$ = 0.05063 (0.02039)	$\hat{S}_{1350}$ = 0.23941 (0.06868)	$\hat{S}_{1350}$ = 0.05022 (0.06206)	$\hat{S}_{50}$ = 0.24389 (0.05082)	$\hat{S}_{50}$ = 0.18368 (0.04572)
	$\hat{S}_{40}$ = 0.10957 (0.05592)	$\hat{S}_{40}$ = 2.139e-06 (0.00953)	$\hat{S}_{1000}$ = 0.12136 (0.03136)	$\hat{S}_{1000}$ = 0.00293 (0.02039)	$\hat{S}_{2500}$ = 0.21592 (0.05811)	$\hat{S}_{2500}$ = 0.01212 (0.05510)	$\hat{S}_{100}$ = 0.18692 (0.04659)	$\hat{S}_{100}$ = 0.02462 (0.01331)

Appendix A. Distributional properties of $C(X)$

Since the moments of $C(X) = \big(C_1(X), C_2(X), C_3(X)\big)$ are functions of $\underline{\theta} = (p_1, p_2, \theta)$, and β assumed known, they are MVUE's of these functions. Hence, in order to find the moments, differentiating $g(\underline{\theta})$ partially with respect to p_1, p_2 and θ under the regularity conditions, we get

$$\underline{G} = A\,\underline{\mu}\,,\ |A| \neq 0 \tag{i}$$

where

$$\underline{G} = \begin{bmatrix} \dfrac{\partial \log g(\underline{\theta})}{\partial p_1} \\ \dfrac{\partial \log g(\underline{\theta})}{\partial p_2} \\ \dfrac{\partial \log g(\underline{\theta})}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \dfrac{1}{1-p_1-p_2} \\ \dfrac{1}{1-p_1-p_2} \\ -\dfrac{1}{\theta} \end{bmatrix}$$

$$\underline{\mu} = \begin{bmatrix} E\big(C_1(x)\big) \\ E\big(C_2(x)\big) \\ E\big(C_3(x)\big) \end{bmatrix} = \begin{bmatrix} E\big(I_1(x)\big) \\ E\big(I_2(x)\big) \\ E\Big([\log(1+x) - \log 2]\big(1 - I_1(x) - I_2(x)\big)\Big) \end{bmatrix}$$

and

$$A = \begin{bmatrix} \frac{\partial \log\, h_1(\underline{\theta})}{\partial p_1} & \frac{\partial \log h_2(\underline{\theta})}{\partial p_1} & \frac{\partial \log h_3(\underline{\theta})}{\partial p_1} \\ \frac{\partial \log\, h_1(\underline{\theta})}{\partial p_2} & \frac{\partial \log h_2(\underline{\theta})}{\partial p_2} & \frac{\partial \log h_3(\underline{\theta})}{\partial p_2} \\ \frac{\partial \log\, h_1(\underline{\theta})}{\partial \theta} & \frac{\partial \log h_2(\underline{\theta})}{\partial \theta} & \frac{\partial \log h_3(\underline{\theta})}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \frac{1}{p_1} + \frac{1}{1-p_1-p_2} & \frac{1}{1-p_1-p_2} & 0 \\ \frac{1}{1-p_1-p_2} & \frac{1}{p_2} + \frac{1}{1-p_1-p_2} & 0 \\ -\frac{1}{\theta} & -\frac{1}{\theta} & -1 \end{bmatrix}$$

Equation (i) gives

$$E\big(C_i(x)\big) = \frac{|A_i|}{|A|}\,, i = 1{,}2 \text{ and } 3$$

where A_i is obtained by replacing i[th] column of A by the elements of $\underline{G}$. Hence,

$$\underline{\mu} = \begin{bmatrix} E\big(C_1(x)\big) \\ E\big(C_2(x)\big) \\ E\big(C_3(x)\big) \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \\ \frac{(1-p_1-p_2)}{\theta} \end{bmatrix} \tag{ii}$$

Now joint moments of $C_1^{k_1}(\boldsymbol{x})$, $C_2^{k_2}(x)$ and $C_3^{k_3}(x)$ are given as

$$E\left(C_1^{k_1}(\boldsymbol{x})\, C_2^{k_2}(\boldsymbol{x})\, C_3^{k_3}(\boldsymbol{x})\right) = \int_x C_1^{k_1}(\boldsymbol{x})\, C_2^{k_2}(\boldsymbol{x})\, C_3^{k_3}(\boldsymbol{x})\, a(x) \frac{\prod_{i=1}^{3}\big(h_i(\underline{\theta})\big)^{C_i(x)}}{g(\underline{\theta})}\, dx$$

which on differentiating with respect to p_1, p_2 and θ and using (iv), gives a system of three non-homogeneous equations

$\underline{G_1} = A\,\underline{V}\,,\ |A| \neq 0$
(iii)

where

$$\underline{G}_1 = \begin{bmatrix} \dfrac{\partial \log E\left(C_1^{k_1}(x)\, C_2^{k_2}(x)\, C_3^{k_3}(x)\right)}{\partial p_1} \\ \dfrac{\partial \log E\left(C_1^{k_1}(x)\, C_2^{k_2}(x)\, C_3^{k_3}(x)\right)}{\partial p_2} \\ \dfrac{\partial \log E\left(C_1^{k_1}(x)\, C_2^{k_2}(x)\, C_3^{k_3}(x)\right)}{\partial \theta} \end{bmatrix}$$

$$\underline{V} = \begin{bmatrix} E\left(C_1^{k_1+1}(x)\, C_2^{k_2}(x)\, C_3^{k_3}(x)\right) - E(C_1(x))E\left(C_1^{k_1}(x)\, C_2^{k_2}(x)\, C_3^{k_3}(x)\right) \\ E\left(C_1^{k_1}(x)\, C_2^{k_2+1}(x)\, C_3^{k_3}(x)\right) - E(C_2(x))E\left(C_1^{k_1}(x)\, C_2^{k_2}(x)\, C_3^{k_3}(x)\right) \\ E\left(C_1^{k_1}(x)\, C_2^{k_2}(x)\, C_3^{k_3+1}(x)\right) - E(C_3(x))E\left(C_1^{k_1}(x)\, C_2^{k_2}(x)\, C_3^{k_3}(x)\right) \end{bmatrix} = \begin{bmatrix} \sigma_{1(1,2,3)} \\ \sigma_{2(1,2,3)} \\ \sigma_{3(1,2,3)} \end{bmatrix}, \text{(say)}.$$

Using Cramer's rule for the solution of a system of linear equations (iii) gives

$$\sigma_{i(1,2,3)} = \frac{|A_i|}{|A|}, i = 1,2 \text{ and } 3$$

where A_i is obtained by replacing i[th] column of A by the elements of $\underline{G}_1$. For $k_i = 1$ and $k_j = 0\ \forall\ i \neq j = 1,2$ *and* 3, we get covariance between $C_i(x)$ and $C_j(x)$ as

$$\sigma_{i(1,2,3)} = \frac{|A_i|_{(k_i=1;k_j=0),i\neq j}}{|A|}.$$

Thus, we have the variance-covariance matrix V as

$$\mathrm{V} = \left[\sigma_{ij}\right]_{3\times 3} = \frac{\left(|A_i|_{(k_i=1;k_j=0),i\neq j}\right)}{|A|}$$

If A_{ij} is the cofactor of the element a_{ij} of A, then

$$|A_i|_{(k_i=1;k_j=0),i\neq j=1,2,3} = A_{1i}\frac{\partial}{\partial p_1}E(C_i(x)) + A_{2i}\frac{\partial}{\partial p_2}E(C_i(x)) + A_{3i}\frac{\partial}{\partial \theta}E(C_i(x))$$

and hence

$$\mathrm{V} = \begin{bmatrix} p_1(1-p_1) & -p_1p_2 & -\theta\, p_1(1-p_1-p_2) \\ -p_1p_2 & p_2(1-p_2) & -\theta\, p_2(1-p_1-p_2) \\ -\frac{p_1(1-p_1-p_2)}{\theta} & -\frac{p_2(1-p_1-p_2)}{\theta} & \frac{[1-(p_1+p_2)^2]}{\theta^2} \end{bmatrix} \qquad \text{(iv)}$$

where $|A| = \frac{1}{p_1\, p_2\, (1-p_1-p_2)}$.

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Reliability analysis of a multi state system with common cause failures using Markov Regenerative Process

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Abstract

In this paper the dynamic reliability behaviour in terms of common cause failures is studied and a state space model has been formed for the evaluation of performance measures of multi state system. The concept of renewal is employed and the Markov Regenerative Process has been used for assessment of availability of the system. Using proposed technique we obtain the transition kernel and formulas for the steady state probabilities of the system. A numerical example is proposed to demonstrate the real possibility of the proposed technique.

Keywords: Multi state system, Common cause failures, Markov Regenerative Process, Availability

I Introduction

Failures of multiple components of a system due to a common cause is called Common Cause Failures (CCF). CCF is the one of the most important issues in evaluation of system reliability. When compared to random failures, which affect individual components, the frequency of CCF has relatively low expectancy. According to Rausand and Hoyland [11] common cause failures is a dependent failure in which two or more component fault states exist simultaneously or within short time interval and are direct result of a shared cause. Beta(β) factor model is the most commonly used model for common cause failures of the multi state system [3]. The β factor model describes the correlation between the independent random component failures and common cause failures in a redundant multi state system. A set of powerful techniques that proved for the solution of non-Markovian models is based on the ideas grouped under the Markov renewal theory. The application of Markov renewal theory for finding reliability and availability of stochastic systems is discussed in [6]. Semi-Markov process is the most widely used and adopted non-Markovian model for evaluating reliability and availability of multi state system. A good reference on the semi-Markov process (SMP) is [8] which discusses the the theory of SMP very clearly, also gives examples which helps to understanding the theory and how to apply the model in many real life situations. The stationary character of Markov regeneratve process (MRGP) has been studied in [10]. Most of the theoretical foundations of Markov regeneratve process (MRGP) were discussed in [2] in which it is named as semi regenerative process. One of the first paper which consider semi-regenerative processes is in Russian (refer [13]). For a concise review on Semi-regenerative, decomposable Semi-regenerative Processes and their applications one may refer to [12]. The transient and steady state analysis of stochastic petri nets are discussed analytically and numerically in [1]. MRGPs have been used to evaluating reliability and availability of the system. Some examples concerning reliability and availability of power plants and fault tree systems can be found in [4,5,9,16]. Many other examples and applications of MRGP in the dependability context has been solved using SHARPE software [14] as demonstrated in [17]. Semi Markov, Markov regenerative models and Phase type expansion with a number of solved examples were discussed in [15]. The system-level reliability of a heterogeneous double redundant renewable system under Marshall-Olkin failure model in the case when repair times of its components have a general continuous distribution is studied in [7]. The mathematical model proposed therein allows to obtain the explicit expression in terms of Laplace transform for the system reliability function.

II Markov Regenerative Process

Consider a stochastic process $\{Z(t), t \geq 0\}$ with state space Ω. Suppose every time a certain phenomena occurs, the future of the process Z after that time becomes a probabilistic replica of the future after time zero. Such time which is usually random is called regeneration time of Z. Such process is named as regeneration process. In a Markov Regenerative Process (MRGP) the stochastic evolution between two successive regeneration points depends only on the state of regeneration not on the evolution before regeneration.

Following [1] a stochastic process $\{Z(t), t \geq 0\}$ on Ω is called an MRGP if there exist a Markov renewal sequence $\{(Y_n, S_n), n \geq 0\}$ of random variable such that all conditional finite dimensional distribution of $\{Z(S_n + t), t \geq 0\}$ given $\{Z(u), 0 \leq u \leq S_n, Y_n = i\}$ $i \in \Omega$ are the same as those of $\{Z(t), t \geq 0\}$ given $Y_0 = i$.

From the above definition we obtain embedded Markov chain (EMC) in $\{Z(t), t \geq 0\}$. Global kernel $K(t)$ gives a description of the evolution of process from the Markovian regenerative moment without describing the happenings between regenerative moments.

$$K(t) = K_{ij}(t) = Pr\{Y_1 = j, S_1 \leq t / Y_0 = i\} \forall i, j \in \Omega$$

An MRGP can change states between two consecutive Markov renewal moments. $E(t)$ is the local kernel which explains the state probabilities of the process during the interval between successive Markov regenerative moments.

$$E(t) = E_{ij}(t) = Pr\{Z(t) = j, S_1 > t / Y_0 = i\} \forall i, j \in \Omega$$

The matrix of conditional transition probabilities are given by

$$V_{ij}(t) = Pr\{Z(t) = j / Z_0 = i\} \forall i, j \in \Omega$$

In many real life problems involving Markov Renewal Process our primary aim to compute $V_{ij}(t)$ effectively and hence several performance measures of interest like Availability, Reliability based on $V_{ij}(t)$

The conditional transition probabilities $V_{ij}(t)$ at any instant t can be computed as

$$V_{ij}(t) = Pr\{Z(t) = j, S_1 / Z_0 = i\} + \sum_{k \in \Omega'} \int_0^t dK(u) V_{kj}(t-u) \forall i, j \in \Omega$$

A Markov renewal equation is defined by this set of integral equations. Equation can be expressed in Matrix form as

$$V(t) = E(t) + K(t).V(t)$$

Laplace-Steiltjes transform $K(s)$ and $E(s)$ of $K(t)$ and $E(t)$ respectively can obtained as

$$K(s) = \int_0^\infty e^{-st} dK(t)$$
$$E(s) = \int_0^\infty e^{-st} dE(t)$$

Then

$$V(s) = E(s) + K(s)V(s) = [I - K(s)]^{-1} E(s)$$

$V(t)$ can be obtained by taking inverse laplace transform of $V(s)$

$$P(t)_{1\times\Omega} = P(0)_{1\times\Omega} \times V(t)_{\Omega\times\Omega}$$

For the purpose of the steady state analysis of an MRGP the following two matrices $\alpha =$

$[\alpha_{ij}]$ and $\phi = [\phi_{ij}]$ should be calculated. α_{ij} is the Mean time the process from state i spends in state j. $\phi = [\phi_{ij}]$ is the one step transition probability matrix of the embedded Markov chain.

The two matrices are defined as

$$\alpha = \int_{t=0}^{\infty} E(t)dt = \lim_{s\to 0}\frac{1}{s}E(s) \quad (1)$$

$$\phi = \lim_{t\to\infty} K(t) = \lim_{s\to 0} K(s) \quad (2)$$

To obtain the steady-state probabilities of the MRGP, at first we have to solve the steady-state probabilities of the embedded discrete time Markov chain by solving

$$\nu = \nu.\phi$$

$$\nu.e = 1$$

where e is a column vector with its elements equal to 1 and ν is a row vector. Steady state probability vector is

$\nu = [\nu_1, \nu_2, \dots \nu_k]$ where $k \in \Omega$

The steady state probability $\pi = [\pi_1, \pi_2, \dots \pi_k]$ of the MRGP is given by

$$\pi = \frac{\nu\alpha}{\nu\alpha e} \quad (3)$$

Steady state Availability of system

Let $\Omega = \{0,1,\dots,k\}$ be the set of all possible states of a system. Let Ω' denote the subset of states in which the system is functioning and let $F = \Omega - \Omega'$ denote the states in which the system is failed. The long term availability of the system is the mean proportion of time when the system is functioning. Steady state system availability can be obtained by

$$A_\infty = \sum_{j\in\Omega'} \pi_j \quad (4)$$

III Parallel System with Single Repair Facility and CCF

Consider a system which consists of two components named A and B. A single repairman is assigned for the system with the First Come First Served (FCFS) scheduling policy for repair. When the components A or B fails the repairman begins to repair if he is not busy. When one component is already under repair and the other component fails then the second component has to wait for repair till the repairman is free. The lifetime of components A and B are exponentially distributed with the rates λ_A and λ_B respectively. The distribution function of the repair times of components A and B are $G_A(t)$ and $G_B(t)$ respectively. Let $\mu_A(t)$ and $\mu_B(t)$ be the respective repair rates of components A and B. Also in this case common cause failure involving both components A and B can occur with probability β. Define the stochastic process $Z = \{Z(t); t \geq 0\}$ to represent the system state at any instant t. $Z(t) \in \{1,2,3,4,5\}$

System is in state

1, if both components are working at time t

2, if component A is under repair while component B is working at time t

3, if component B is under repair while component A is working at time t

4, if component A is under repair while component B is waiting for repair at time t or due to common cause failure in which the repairman randomly selects component A is the first to be repaired

5, if component B is under repair while component A is waiting for repair at time t or due to common cause failure in which the repairman randomly selects component B is the first to be repaired

We can define that all state transitions correspond to Markov renewal moments $S = \{S_n; n \in N\}$ and the embedded Markov chain $Y_n; n \in N$ such that Y_n is the state of the system at time

S_{n^+}(i.e,$Y_n = Z(S_{n^+})$)

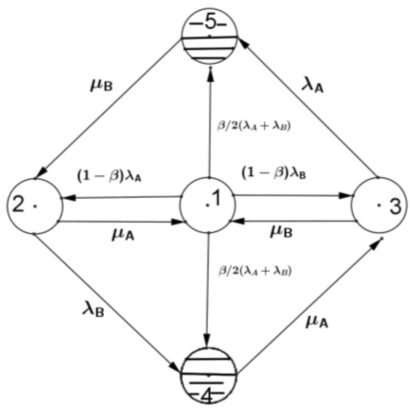

Figure 1: State transition diagram

Analysis of the above reliability transition diagram shows that Z is an MRGP with an embedded markov chain (EMC) defined by the states 1, 2 and 3. We can observe the transition to states 4 and 5 do not belong to the EMC since they are non-renewal moments. System is in state 1 if both A and B are up states and the repairman is free. Component A can fail at rate λ_A and reach state 2. The component A is repaired with cdf $G_A(t)$ to bring the system back to state 1. If component B fell down during repair time of component A , the system jumps to state 4. When the component B is down the system reaches the state 3 and when B is repaired with repair time cdf $G_B(t)$ to back the system state 1. But the component A fail jumping the state 3 to state 5. To find the distribution of Z for MRGP we have to construct kernel matrices [global kernel matrix $K(t)$and local kernel matrix $E(t)$]. R_A, R_B be the time to repair and L_A and L_B be the times to failure of A and B respectively.

$$K(t) = \begin{pmatrix} 0 & k_{12}(t) & k_{13}(t) \\ k_{21}(t) & 0 & k_{23}(t) \\ k_{31}(t) & k_{32}(t) & 0 \end{pmatrix},$$

$K_{12}(t)$ = Pr{If A fails before B or common cause failures occur and repairman chose to repair A first and completed the repair action}

$$= Pr\{Z(S_1) = 2, S_1 \leq t/Z_0 = 1\} = Pr\{(L_A \leq t \cap L_B > L_A) \cup (R_A \leq t \cap (L_A = L_B) \leq R_A)\}$$

$$= (1-\beta)\lambda_A \int_0^t e^{-(\lambda_A+\lambda_B)u} du + \frac{\beta}{2}(\lambda_A + \lambda_B) \int_0^t e^{-(\lambda_A+\lambda_B)u} G_A(t-u) du$$

$$K_{13}(t) = (1-\beta)\lambda_B \int_0^t e^{-(\lambda_A+\lambda_B)u} du + \frac{\beta}{2}(\lambda_A + \lambda_B) \int_0^t e^{-(\lambda_A+\lambda_B)u} G_B(t-u) du$$

$K_{21}(t)$ = Pr{Repair A is finished up to time t and B has not failed during repair A}

$$= Pr\{Z(S_1) = 1, S_1 \leq t/Z_0 = 2\} == Pr\{R_A \leq t \cap L_B > R_A\} = \int_0^t e^{-\lambda_B u} dG_A(u)$$

$K_{23}(t)$ = Pr{Repair A is not finished up to time t and B failed during the repair A}

$$= Pr\{Z(S_1) = 3, S_1 \leq t/Z_0 = 2\} = \int_0^t (1 - e^{-\lambda_B u}) dG_A(u)$$

$$K_{31}(t) = Pr\{Z(S_1) = 3, S_1 \leq t/Z_0 = 3\} = \int_0^t e^{-\lambda_A u} dG_B(u)$$

$$K_{32}(t) = Pr\{Z(S_1) = 2, S_1 \le t/Z_0 = 3\} = \int_0^t (1 - e^{-\lambda_A u}) dG_B(u)$$

$$E(t) = \begin{pmatrix} E_{11}(t) & 0 & 0 & E_{14}(t) & E_{15}(t) \\ 0 & E_{22}(t) & 0 & E_{24}(t) & 0 \\ 0 & 0 & E_{33}(t) & 0 & E_{35}(t) \end{pmatrix},$$

$E_{11}(t)$ = Pr{Remaining state 1 until time t}
$= Pr\{Z(t) = 1, S_1 > t/Z_0 = 1\} = (1-\beta)e^{-(\lambda_A+\lambda_B)t}$
$E_{22}(t)$=Pr{repair A is not finished up to time t and B has not failed}
$= Pr\{Z(t) = 2, S_1 > t/Z_0 = 2\} = (1 - G_A(t))e^{-\lambda_B t}$

$$E_{33}(t) = (1 - G_B(t)e^{-\lambda_A t}$$
$$E_{14}(t) = \frac{\beta}{2} e^{-(\lambda_A+\lambda_B)t}$$
$$E_{15}(t) = \frac{\beta}{2} e^{-(\lambda_A+\lambda_B)t}$$

$E_{24}(t)$ =Pr{repair A is not finished up to time t and B has not failed}
$= (1 - G_A(t))(1 - e^{-\lambda_B t})$

$$E_{35}(t) = (1 - G_B(t))(1 - e^{-\lambda_A t})$$

Laplace-Steiltjes transform of Global Kernel Matrix is

$$K(s) = \begin{pmatrix} 0 & \frac{(1-\beta)\lambda_A}{s+\lambda_A+\lambda_B} + \frac{\beta(\lambda_A+\lambda_B)G_A(s)}{2(s+\lambda_A+\lambda_B)} & \frac{(1-\beta)\lambda_B}{s+\lambda_A+\lambda_B} + \frac{\beta(\lambda_A+\lambda_B)G_A(s)}{2(s+\lambda_A+\lambda_B)} \\ G_A(s+\lambda_B) & 0 & G_A(s) - G_A(s+\lambda_B) \\ G_B(s+\lambda_A) & G_B(s) - G_B(s+\lambda_A) & 0 \end{pmatrix},$$

Laplace-Steiltjes transform of Local Kernel Matrix is

$$E(s) = \begin{pmatrix} \frac{(1-\beta)s}{s+\lambda_A+\lambda_B} & 0 & 0 & \frac{\beta s}{2(s+\lambda_A+\lambda_B)} & \frac{\beta s}{2(s+\lambda_A+\lambda_B)} \\ 0 & \frac{s}{s+\lambda_B}(1 - G_A(s+\lambda_B)) & 0 & \frac{\lambda_B}{s+\lambda_B} - G_A(s) + \frac{s}{s+\lambda_B} G_A(s+\lambda_B) & 0 \\ 0 & 0 & \frac{s}{s+\lambda_A}(1 - G_B(s+\lambda_A)) & 0 & \frac{\lambda_A}{s+\lambda_A} - G_B(s) + \frac{s}{s+\lambda_A} G_B(s+\lambda_A) \end{pmatrix},$$

IV Numerical Illustration

Consider a numerical example wherein the components have deterministic repair-times with distribution functions,

$$G_A(t) = u(t - \mu_A), \mu_A > 0$$
$$G_B(t) = u(t - \mu_B), \mu_B > 0$$

where u(t) is the unit step function. The units are hours for repair-time (parameters μ_A and μ_B) and $hour^{-1}$ for the failure rates (parameters λ_A and λ_B). The values of parameters of the system are given below.

Component	λ	μ
A	0.01	5
B	0.01	5

$$K(s) = \begin{pmatrix} 0 & \frac{(1-\beta)0.01}{s+0.02} + \frac{\beta 0.02 e^{-5s}}{2(s+0.02)} & \frac{(1-\beta)0.01}{s+0.02} + \frac{\beta 0.02 e^{-5s}}{2(s+0.02)} \\ e^{-5(s+0.01)} & 0 & e^{-5s} - e^{-5(s+0.01)} \\ e^{-5(s+0.01)} & e^{-5s} - e^{-5(s+0.01)} & 0 \end{pmatrix}$$

$$E(s) = \begin{pmatrix} \frac{(1-\beta)s}{s+0.02} & 0 & 0 & \frac{\beta s}{2(s+0.02)} & \frac{\beta s}{2(s+0.02)} \\ 0 & \frac{s}{s+0.01}(1-e^{-5(s+0.01)}) & 0 & \frac{0.01}{s+0.01} - e^{-5s} + \frac{s}{s+0.01} e^{-5(s+0.01)} & 0 \\ 0 & 0 & \frac{s}{s+0.01}(1-e^{-5(s+0.01)}) & 0 & \frac{0.01}{s+0.01} - e^{-5s} + \frac{s}{s+0.01} e^{-5(s+0.01)} \end{pmatrix}$$

The Steady state probability vector is

$$[\pi_1, \pi_2, \pi_3, \pi_4, \pi_5] = [0.892074(1-\beta), 0.045738, 0.045738, 0.446037\beta + 0.008225, 0.446037\beta + 0.008225]$$

Steady state Availability

$$A_\infty = \pi_1 + \pi_2 + \pi_3 = 0.98355(1-\beta) \quad (5)$$

Impact of the common cause failures on the system is evaluated for the corresponding model. The MRGP steady state availability can be calculated for varying common cause failure probability β value. By analyzing the MRGP for the above numerical values, the graph depicted in Fig. 2 is obtained.

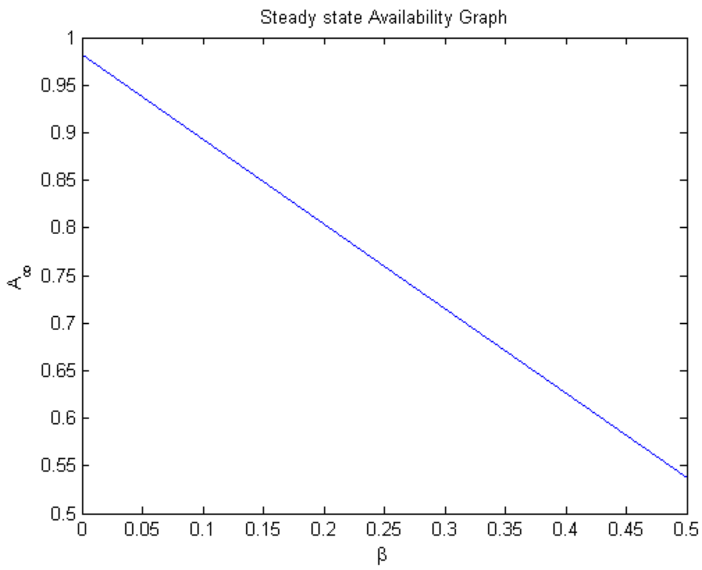

Figure 2: Steady state availability of the system for varying β

The graph reveals how the steady state availability (A_∞) of the system varies by changing the common cause failure probability β from 0 to 0.5. On viewing the graph we can observe a clear linear trend of the A_∞ with respect to β.

V Conclusion

In this paper analytical techniques based on MRGP are explored for modeling and evaluation of availability of multi state system. A parallel system of two components with common cause failures were elaborated to show the applicability of MRGP in the evaluation of performance measures with numerical example. Since MRGP can overcome limitations of SMP to some extent, one can solve a wide range of problems in system reliability on similar lines.

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