Time Dependent Analysis of an $M/M/2/N$ Queue With Catastrophes

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Abstract

We consider a Markovian queueing system with two identical servers subjected to catastrophes. When the system is not empty, catastrophes may occur and destroy all present customers in the system. Simultaneously the system is ready for new arrivals. The time dependent and the steady state solution are obtained explicitly. Further we have obtained some important performance measures of the studied queueing model.

Keywords: Markovian queueing system, catastrophes, limited capacity, Time Dependent Solution.

1 Introduction

During the last 40 years the attention of the queueing models has been focused on the effect of catastrophes, in particulars, birth and death models. The catastrophes arrive as negative customers to the system and their characteristic is to remove some or all of the regular customers in the system. The catastrophes may come either from outside the system or from another service station. For example, in computer networks, if a job infected with a virus, it transmits the virus to other processors and inactivities them [8]. Other interesting articles in this area include ([2],[6],[7]). In real life it is not necessary that a queueing system should have only one server. Practically they may have more than one server identical or non identical in their functioning .Krishna kumar et. al.[7] obtained the time dependent solution of two identical servers Markovian queueing system with catastrophes.Dharmaraja and kumar[3] consider a multi-server Markovian queueing system with heterogeneous servers and catastrophes.Jain and Bura [5]obtained the transient solution of an M/M/2/N queuing system with varying catastrophic intensity and restoration. We in this paper confine ourselves to a Markovian queueing system with two identical servers subjected to catastrophes.

Rest of the paper is organized as follows:In section 3, we describe the mathematical form of the model and obtained the time dependent solution of the model. In section 4, we obtain the time dependent performance measures of the system. Section 5 provides the steady state probabilities. In section 6, we obtain the expression for steady state mean and variance. Finally, the conclusion have been given in section 6.

2 Model description and analysis

We consider an $M/M/2/N$ queueing system with first come first out discipline that is subjected to catastrophes at the service station. Customers arrive in the system according to a Poisson stream with parameter λ.The service time distribution is independently identically exponential with parameter μ. When the system is not empty, catstrophes occur according to a Poisson process of rate ξ. Let $X(t)$ denote the number of customers in the system at time t.

Define $P_n(t) = P(X(t) = n); n = 0,1,2,\dots,N$ be the transient state probability that there are n customers in the system at time t, and $P(z,t) = \sum_{n=0}^{N} P_n(t)z^n$ be the probability generating function.

From the above assumption, the probability satisfies the following system of the differential- difference equations:

$$p_0'(t) = -\lambda p_0(t) + \mu p_1(t) + \xi[\textstyle\sum_{n=1}^{N} p_n(t)] \qquad ; n = 0 \tag{2.1}$$

$$p_1'(t) = -(\lambda + \mu + \xi)p_1(t) + \lambda p_0(t) + 2\mu p_2(t) \qquad ; n = 1 \tag{2.2}$$

$$p_n'(t) = -(\lambda + 2\mu + \xi)p_n(t) + \lambda p_{n-1}(t) + 2\mu p_{n+1}(t) \quad ; n = 2,3,\dots,(N-1) \tag{2.3}$$

$$p_N'(t) = -(2\mu + \xi)p_N(t) + \lambda p_{N-1}(t) \tag{2.4}$$

It is assumed that initially the system is empty i.e.

$$P_0(0) = 1 \qquad P_n(0) = 0 \ , n = 1,2,\dots,N \tag{2.5}$$

After Multiplying equations (2.1) to (2.4) by z^n for all $n \geq 0$, then summed on n from $n = 0$ to N and adding, we have

$$\sum_{n=0}^{N} p_n'(t)z^n = [\lambda z + \frac{2\mu}{z} - (\lambda + 2\mu + \xi)]P(z,t)$$
$$+2\mu(1 - \frac{1}{z})p_0(t) + \lambda z^N(1 - z)p_N(t) + \mu p_1(t)(z - 1) + \xi \tag{2.6}$$

It is easily seen that the probability generating function $P(z,t)$ satisfies the following differential equation:

$$\frac{\partial}{\partial t}[P(z,t)] = [\lambda z + \frac{2\mu}{z} - (\lambda + 2\mu + \xi)]P(z,t)$$
$$+2\mu(1 - \frac{1}{z})p_0(t) + \lambda z^N(1 - z)p_N(t) + \mu p_1(t)(z - 1) + \xi \tag{2.7}$$

with the initial condition

$$P(Z,0) = 1 \tag{2.8}$$

The equation (2.7) can be considered as a first order differential equation in $P(z,t)$ and by finding the integrating factor and using the initial condition (2.8),the solution of the equation (2.7) is obtained as

$$P(z,t) = 2\mu(1 - \frac{1}{z}) \int_0^t P_0(t-u)e^{(\lambda z + \frac{2\mu}{z})u} e^{-(\lambda+2\mu+\xi)u} du$$
$$+\lambda z^N(1 - Z) \int_0^t P_N(t-u)e^{(\lambda z + \frac{2\mu}{z})u} e^{-(\lambda+2\mu+\xi)u} du$$
$$+\mu(Z - 1) \int_0^t P_1(t-u)e^{(\lambda z + \frac{2\mu}{z})u} e^{-(\lambda+2\mu+\xi)u} du$$
$$+\xi \int_0^t e^{(\lambda z + \frac{2\mu}{z})u} e^{-(\lambda+2\mu+\xi)u} du + e^{(\lambda z + \frac{2\mu}{z})t} e^{-(\lambda+2\mu+\xi)t} \tag{2.9}$$

Using the Bessel function identity, if $\alpha = 2\sqrt{\lambda 2\mu}$ and $\beta = \sqrt{\frac{\lambda}{2\mu}}$ then,

$$exp(\lambda z + \frac{2\mu}{z})t = \sum_{n=-\infty}^{\infty} I_n(\alpha t)(\beta z)^n$$

where $I_n(.)$ is the moddified Bessel function of order n. Substituting this equation in (2.9) and compairing the coefficient of z^n on either side, we have, for $n = 0,1,\dots,N$

$$P_n(t) = 2\mu\beta^n \int_0^t P_0(t-u)e^{-(\lambda+2\mu+\xi)u}[I_n(\alpha u) - \beta I_{n+1}(\alpha u)]du$$

$$+\lambda\beta^n \int_0^t P_N(t-u)e^{-(\lambda+2\mu+\xi)u}[\beta^{-N}I_{N-n}(\alpha u)-\beta^{-(N+1)}I_{(N+1)-n}(\alpha u)]du$$
$$+\mu\beta^n \int_0^t P_1(t-u)e^{-(\lambda+2\mu+\xi)u}[\beta^{-1}I_{n-1}(\alpha u)-I_n(\alpha u)]du$$
$$+\xi\beta^n \int_0^t e^{-(\lambda+2\mu+\xi)u}I_n(\alpha u)du+\beta^n e^{-(\lambda+2\mu+\xi)t}I_n(\alpha t) \quad (2.10)$$

where we have used $I_{-n}(.)=I_n(.)$

Here, we have obtained $P_n(t)$ for $n=1,\dots,N-1$. However, this expression depends upon $P_0(t)$ and $P_N(t)$. In order to determine, $P_0(t)$ and $P_N(t)$ we introduce the Laplace transform. In the sequel, for any function $f(.)$, let $f^*(s)$ denote its Laplace transform i.e. , $f^*(s)=\int_0^\infty e^{-st}f(t)dt$

Substitute $n=0$, in equation (2.10) we get

$$P_0(t)=2\mu\int_0^t P_0(t-u)e^{-(\lambda+2\mu+\xi)u}[I_0(\alpha u)-\beta I_1(\alpha u)]du$$
$$+\lambda\int_0^t P_N(t-u)e^{-(\lambda+2\mu+\xi)u}[\beta^{-N}I_N(\alpha u)-\beta^{-(N+1)}I_{(N+1)}(\alpha u)]du$$
$$+\mu\int_0^t P_1(t-u)e^{-(\lambda+2\mu+\xi)u}[\beta^{-1}I_1(\alpha u)-I_0(\alpha u)]du$$
$$+\xi\int_0^t e^{-(\lambda+2\mu+\xi)u}I_0(\alpha u)du+e^{-(\lambda+2\mu+\xi)t}I_0(\alpha t) \quad (2.11)$$

Taking Laplace transform on both sides of equation (2.11) and solving for, $P_0^*(s)$ we obtain,

$$\left[\frac{\omega+\sqrt{\omega^2-\alpha^2}}{2}-2\mu\right]P_0^*(s)=\lambda P_N^*(s)\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{2\lambda}\right)^N$$
$$\left[1-\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{2\lambda}\right)\right]$$
$$+\mu P_1^*(s)\left[\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{2\lambda}\right)-1\right]+\frac{\xi}{s}+1$$

where $\omega=s+\lambda+2\mu+\xi$. After some algebra, the above equation can be expressed as

$$\left[1-\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{2\lambda}\right)\right]P_0^*(s)=\frac{\lambda}{2\mu}P_N^*(s)\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{2\lambda}\right)^{N+1}$$
$$\left[1-\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{2\lambda}\right)\right]$$
$$-\mu P_1^*(s)\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{2\lambda}\right)$$
$$\left[1-\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{2\lambda}\right)\right]\left(\frac{1}{2\mu}\right)$$
$$+\left(\frac{\xi}{s}+1\right)\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{2\lambda}\right)\left(\frac{1}{2\mu}\right) \quad (2.12)$$

By solving equation (2.12), we get,

$$P_0^*(s)=\frac{\lambda}{2\mu}P_N^*(s)\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{2\lambda}\right)^{N+1}-\frac{P_1^*(s)}{2}\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{2\lambda}\right)$$
$$+\left(\frac{\xi}{s}+1\right)\left(\frac{1}{2\mu}\right)\left[\frac{2\mu}{s+\xi}-\frac{\lambda}{s+\xi}\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{2\lambda}\right)\right] \quad (2.13)$$

On inversion, this equation yields an expression for $P_0(t)$ which depends upon $P_N(t)$.

$$P_0(t)=e^{-\xi t}+\left(\frac{2\mu}{\lambda}\right)^{\frac{N-1}{2}}\int_0^t P_N(t-u)e^{-(\lambda+2\mu+\xi)u}\left(\frac{N+1}{u}\right)I_{N+1}(\alpha u)du$$
$$\left(\frac{2\mu}{\lambda}\right)^{\frac{1}{2}}\int_0^t P_1(t-u)e^{-(\lambda+2\mu+\xi)u}\left(\frac{1}{2u}\right)I_1(\alpha u)du$$
$$+\xi\left[e^{-\xi t}-\sqrt{\frac{\lambda}{2\mu}}\int_0^t e^{-(\lambda+2\mu+\xi)u}e^{-\xi(t-u)}\frac{I_1(\alpha u)}{u}du\right]$$
$$-\sqrt{\frac{\lambda}{2\mu}}\int_0^t e^{-(\lambda+2\mu+\xi)u}e^{-\xi(t-u)}\frac{I_1(\alpha u)}{u}du \quad (2.14)$$

Substituting n=1 in equation (2.10), we get

$$P_1(t)=2\mu\beta\int_0^t P_0(t-u)e^{-(\lambda+2\mu+\xi)u}[I_1(\alpha u)-\beta I_2(\alpha u)]du$$
$$+\lambda\beta\int_0^t P_N(t-u)e^{-(\lambda+2\mu+\xi)u}[\beta^{-N}I_{N-1}(\alpha u)-\beta^{-(N+1)}I_N(\alpha u)]du$$

$$+\mu\beta \int_0^t P_1(t-u)e^{-(\lambda+2\mu+\xi)u}[\beta^{-1}I_0(\alpha u) - I_1(\alpha u)]du$$
$$+\xi\beta \int_0^t e^{-(\lambda+2\mu+\xi)u}I_1(\alpha u)du + \beta e^{-(\lambda+2\mu+\xi)t}I_1(\alpha t) \quad (2.15)$$

Taking Laplace transform on both sides of equation (2.15) and solving for, $P_1^*(s)$ we obtain,

$$P_1^*(s)\left[\sqrt{\omega^2-\alpha^2} - \mu\left\{1-\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{4\mu}\right)\right\}\right] = 2\mu\left\{1-\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{4\mu}\right)\right\}$$
$$\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{4\mu}\right)P_0^*(s)$$
$$+\lambda\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{2\lambda}\right)^{N-1}\left\{1-\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{2\lambda}\right)\right\}P_N^*(s) + \left(\frac{\xi}{s}+1\right)$$
$$\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{4\mu}\right) \quad (2.16)$$

Substituting n=N in equation (2.10), we get

$$P_N(t) = 2\mu\beta^N \int_0^t P_0(t-u)e^{-(\lambda+2\mu+\xi)u}[I_N(\alpha u) - \beta I_{N+1}(\alpha u)]du$$
$$+\lambda \int_0^t P_N(t-u)e^{-(\lambda+2\mu+\xi)u}[I_0(\alpha u) - \beta^{-1}I_1(\alpha u)]du$$
$$+\mu\beta^N \int_0^t P_1(t-u)e^{-(\lambda+2\mu+\xi)u}[\beta^{-1}I_{N-1}(\alpha u) - I_N(\alpha u)]du$$
$$+\xi\beta^N \int_0^t e^{-(\lambda+2\mu+\xi)u}I_N(\alpha u)du + \beta^N e^{-(\lambda+2\mu+\xi)t}I_N(\alpha t) \quad (2.17)$$

By taking Laplace transform and solving for $P_N^*(s)$, we obtain from equation (2.17),

$$\left(\frac{\omega+\sqrt{\omega^2-\alpha^2}}{2} - \lambda\right)P_N^*(s) = 2\mu\left\{1-\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{4\mu}\right)\right\}\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{4\mu}\right)^N$$
$$\left[\frac{\lambda}{2\mu}\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{2\lambda}\right)^{N+1} P_N^*(s) \quad (2.18)$$
$$-\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{2\lambda}\right) + \frac{1}{2\mu}\left(\frac{\xi}{s}+1\right)\right]$$

After some algebra, equation (2.18) can be expressed as

$$[1-f^*(s)]P_N^*(s) = g^*(s) \quad (2.19)$$

where

$$f^*(s) = \left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{4\mu}\right)^{N+1}\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{2\lambda}\right)^{N+1}$$
$$\left\{1-\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{4\mu}\right)\right\} + \left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{2\mu}\right) \quad (2.20)$$

$$g^*(s) = \frac{1}{\lambda}\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{4\mu}\right)^{N+1}\left(\frac{\xi}{s}+1\right)$$
$$\left[1+\left\{1-\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{4\mu}\right)\right\}\left\{\frac{2\mu}{s+\xi} - \frac{\lambda}{s+\xi}\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{2\lambda}\right)\right\}\right]$$
$$+\frac{\mu}{\lambda}\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{4\mu}\right)^{N+1}\left[\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{4\mu}\right)^{-1} - 1\right]P_1^*(s)$$
$$-\frac{\mu}{\lambda}\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{2\lambda}\right)\left[1-\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{4\mu}\right)\right]$$
$$\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{4\mu}\right)^{N+1} P_1^*(s) \quad (2.21)$$

equation (2.21) can be written as

$$g^*(s) = \frac{1}{\lambda}\left(\frac{\xi}{s}+1\right)h^*(s) \quad (2.22)$$

where

$$h^*(s) = \left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{4\mu}\right)^{N+1}$$
$$\left[1+\left\{1-\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{4\mu}\right)\right\}\left\{\frac{2\mu}{s+\xi} - \frac{\lambda}{s+\xi}\left(\frac{2\mu}{\omega+\sqrt{\omega^2-\alpha^2}}\right)\right\}\right]$$

$$+\frac{\mu}{\lambda}\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{4\mu}\right)^{N+1} P_1^*(s)$$
$$\left[\left\{\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{4\mu}\right)^{-1}-1\right\}-\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{2\lambda}\right)\left\{1-\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{4\mu}\right)\right\}\right] \quad (2.23)$$

On inversion, the equation (2.20), (2.23) and (2.22) yield an expression for $f(t)$,$h(t)$ and $g(t)$ given by

$$f(t)=\sqrt{\frac{\lambda}{2\mu}}e^{-(\lambda+2\mu+\xi)t}\frac{I_1(\alpha t)}{t}+e^{-(\lambda+2\mu+\xi)t}(2N+2)\frac{I_{2N+2}(\alpha t)}{t}$$
$$-\sqrt{\frac{\lambda}{2\mu}}e^{-(\lambda+2\mu+\xi)t}(2N+3)\frac{I_{2N+3}(\alpha t)}{t} \quad (2.24)$$

$$h(t)=\left(\frac{\lambda}{2\mu}\right)^{\frac{(N+1)}{2}} e^{-(\lambda+2\mu+\xi)t}(N+1)\frac{I_{N+1}(\alpha t)}{t}+\left(\frac{\lambda}{2\mu}\right)^{\frac{(N+1)}{2}}$$
$$\left[\int_0^t e^{-(\lambda+2\mu+\xi)u}e^{-\xi(t-u)}\left\{2\mu(N+1)\frac{I_{N+1}(\alpha u)}{u}-\alpha(N+2)\frac{I_{N+2}(\alpha u)}{u}\right\}du\right]$$
$$+\left(\frac{\lambda}{2\mu}\right)^{\frac{(N+1)}{2}}\lambda\int_0^t e^{-(\lambda+2\mu+\xi)u}e^{-\xi(t-u)}(N+3)\frac{I_{N+3}(\alpha u)}{u}du$$
$$+\frac{\mu}{\lambda}\left(\frac{\lambda}{2\mu}\right)^{\frac{N}{2}}\int_0^t e^{-(\lambda+2\mu+\xi)u}N\frac{I_N(\alpha u)}{u}P_1(t-u)du$$
$$-\frac{\mu}{\lambda}\left(\frac{\lambda}{2\mu}\right)^{\frac{N+1}{2}}\int_0^t e^{-(\lambda+2\mu+\xi)u}(N+1)\frac{I_{N+1}(\alpha u)}{u}P_1(t-u)du$$
$$+2e^{-(\lambda+2\mu+\xi)t}\frac{I_2(\alpha t)}{t} \quad (2.25)$$

$$g(t)=\frac{1}{\lambda}(\xi+1)h(t) \quad (2.26)$$

Since $0\le f^*(s)<1$ so equation (2.19) can be written as

$$P_N^*(s)=g^*(s)\sum_{r=0}^{\infty}[f^*(s)]^r \quad (2.27)$$

On inversion, this equation yields an expression for $P_N(t)$ given by

$$P_N(t)=g(t)*\sum_{r=0}^{\infty}[f(t)]^{*r} \quad (2.28)$$

where $[f(t)]^{*r}$ is the r-fold convolution of $f(t)$ with itself. We note that

$$[f(t)]^{*0}=1$$

3 Performance measures

Mean

we know that

$$m(t)=E[X(t)]=\sum_{n=1}^{N} nP_n(t)$$
$$m(0)=\sum_{n=1}^{N} nP_n(0)=0$$
$$m'(t)=\sum_{n=1}^{N} nP_n'(t)$$

From equation (3.2), (3.3) and (3.4),

$$m'(t)=(\lambda+2\mu+\xi)\sum_{n=1}^{N} nP_n(t)+\lambda NP_N(t)+\lambda\sum_{n=1}^{N} nP_{n-1}(t)$$
$$+2\mu\sum_{n=1}^{N-1} nP_{n+1}(t)+\mu P_1(t)$$

After some algebra, the above equation can be expressed as

$$m'(t)=-\xi m(t)+(\lambda-2\mu)+2\mu P_0(t)-\lambda P_N(t)+\mu P_1(t) \quad (3.1)$$

The above equation can be considered as a first order linear differential equation in $m(t)$. By finding the integrating factor and using the initial condition $m(0)=0$, the solution of the above equation is obtained as follows:

$$m(t)=\frac{(\lambda-2\mu)}{\xi}(1-e^{-\xi t})-\lambda\int_0^t P_N(u)e^{-\xi(t-u)}du$$
$$+2\mu\int_0^t P_0(u)e^{-\xi(t-u)}du+\mu\int_0^t P_1(u)e^{-\xi(t-u)}du \quad (3.2)$$

Variance

We know that

$$Var[X(t)] = E[X^2(t)] - [E\{X(t)\}]^2$$
$$Var[X(t)] = k(t) - [m(t)]^2 \tag{3.3}$$

where

$$k(t) = E[X^2(t)] = \sum_{n=1}^{N} n^2 P_n(t)$$

Also,

$$k(0) = \sum_{n=1}^{N} n^2 P_n(0) = 0$$

and

$$k'(t) = \sum_{n=1}^{N} n^2 P_n'(t)$$

From equation (3.2), (3.3) and (3.4),

$$k'(t) = -(\lambda + 2\mu + \xi)\sum_{n=1}^{N} n^2 P_n(t) + \lambda N^2 P_N(t) + \lambda \sum_{n=1}^{N} n^2 P_{n-1}(t) + 2\mu \sum_{n=1}^{N-1} n^2 P_{n+1}(t) + \mu P_1(t)$$

After some algebra, the above equation can be expressed as

$$k'(t) = -\xi k(t) + (\lambda + 2\mu) - 2\mu P_0(t) - \lambda(2N+1)P_N(t) + 2(\lambda - 2\mu)m(t) + \mu P_1(t) \tag{3.4}$$

The above equation can be considered as a first order linear differential equation in $k(t)$. By finding the integrating factor and using the initial condition $k(0) = 0$, the solution of the above equation is obtained as follows:

$$k(t) = \frac{(\lambda+2\mu)}{\xi}(1 - e^{-\xi t}) - \lambda(2N+1)\int_0^t P_N(u)e^{-\xi(t-u)}du - 2\mu \int_0^t P_0(u)e^{-\xi(t-u)}du + 2(\lambda - \mu)\int_0^t m(u)e^{-\xi(t-u)}du + \mu \int_0^t P_1(u)e^{-\xi(t-u)}du + C \tag{3.5}$$

Substituting the above equation in equation (3.3), we get

$$Var[X(t)] = \frac{(\lambda+2\mu)}{\xi}(1 - e^{-\xi t}) - \lambda(2N+1)\int_0^t P_N(u)e^{-\xi(t-u)}du - 2\mu \int_0^t P_0(u)e^{-\xi(t-u)}du + 2(\lambda - \mu)\int_0^t m(u)e^{-\xi(t-u)}du + \mu \int_0^t P_1(u)e^{-\xi(t-u)}du - \{m(t)\}^2$$

4 Steady state probabilities

In this section, we shall discuss the structure of the steady state probabilities.

Theorem-

For $\xi > 0$, the steady state distribution $\{P_n : n \geq 0\}$ of the $M/M/2/N$ queue with catastrophe corresponds to

$$P_0 = \rho\rho_1 P_N + (1-\rho) - \frac{P_1}{2}\rho_1 \tag{4.1}$$

$$P_n = 2\sigma\mu\rho^{n+1}(1-\rho)\rho_1^N P_N + \sigma\lambda\rho_1^{N-n}(1-\rho_1)P_N + (1-\rho)\rho^n + \mu\sigma(1-\rho)\rho^n\left(\frac{\sqrt{\omega^2-\alpha^2}}{\lambda}\right)P_1 \tag{4.2}$$

$$P_N = \frac{[\{\xi + 2\mu(1-\rho)^2\} + \mu\{(\rho^{-1}-1) - \rho_1(1-\rho)\}P_1]\rho^{N+1}}{\lambda[1 - \rho - \rho_1^{N+1}\rho^{N+1}(1-\rho)]} \tag{4.3}$$

where

$$\rho = \frac{(\lambda+2\mu+\xi) - \sqrt{(\lambda+2\mu+\xi)^2 - 8\lambda\mu}}{4\mu} \tag{4.4}$$

$$\rho_1 = \frac{(\lambda+2\mu+\xi) - \sqrt{(\lambda+2\mu+\xi)^2 - 8\lambda\mu}}{2\lambda} \tag{4.5}$$

$$\sigma = \frac{1}{\sqrt{(\lambda+2\mu+\xi)^2-8\lambda\mu}} \tag{4.6}$$

Proof-

We have from equation (3.13),

$$P_0^*(s) = \frac{\lambda}{2\mu} P_N^*(s)\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{2\lambda}\right)^{N+1} - \left(\frac{\omega-\sqrt{\omega^2-\alpha 2}}{2\lambda}\right)\frac{P_1^*(s)}{2}$$
$$+\left(\frac{\xi}{s}+1\right)\left(\frac{1}{2\mu}\right)\left\{\frac{2\mu}{s+\xi} - \frac{\lambda}{s+\xi}\left(\frac{\omega-\sqrt{\omega^2-\alpha 2}}{2\lambda}\right)\right\} \tag{4.7}$$

Multiplying equation (4.7) by s on both sides and taking limit as $s \to 0$, we get

$$\lim_{s\to 0} sP_0^*(s) = \frac{\lambda}{2\mu}\rho_1^{N+1}P_N - \left(\frac{1}{2}\right)\lim_{s\to 0} sP_1^*(s)\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{2\lambda}\right)$$
$$+\left(\frac{1}{2\mu}\right)\lim_{s\to 0} s\left(\frac{\xi}{s}+1\right)\left\{\frac{2\mu}{s+\xi} - \frac{\lambda}{s+\xi}\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{2\lambda}\right)\right\}$$

Using the property

$$\lim_{s\to 0} sP_0^*(s) = P_0$$

After some algebra, the above expression becomes

$$P_0 = \rho\rho_1 P_N + (1-\rho) - \frac{P_1}{2}\rho_1 \tag{4.8}$$

By taking Laplace transform of the equation (3.10), for $n = 1,2,\ldots,N-1$, we get,

$$P_n^*(s) = 2\mu P_0^*(s)\left(\frac{1}{\sqrt{\omega^2-\alpha^2}}\right)\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{4\mu}\right)^n\left[1-\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{4\mu}\right)\right]$$
$$+\lambda P_N^*(s)\left(\frac{1}{\sqrt{\omega^2-\alpha^2}}\right)\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{2\lambda}\right)^{N-n}\left[1-\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{2\lambda}\right)\right]$$
$$+\frac{\mu}{\sqrt{\omega^2-\alpha^2}}\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{4\mu}\right)^{n-1}\left[1-\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{4\mu}\right)\right]P_1^*(s)$$
$$+\left(\frac{\xi}{s}+1\right)\left(\frac{1}{\sqrt{\omega^2-\alpha^2}}\right)\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{4\mu}\right)^n \tag{4.9}$$

Multiplying the above equation by s on both sides and taking limit as $s \to 0$, we get

$$\lim_{s\to 0} sP_n^*(s) = 2\sigma\mu\rho^n(1-\rho)P_0 + \sigma\lambda\rho_1^{N-n}(1-\rho_1)P_N$$
$$+\mu\sigma\rho^{n-1}(1-\rho)P_1 + \sigma\xi\rho^n \tag{4.10}$$

Substituting equation (4.8) in the above equation, and solving, we get

$$P_n = 2\sigma\mu\rho^{n+1}\rho_1^N(1-\rho)P_N + \sigma\lambda\rho_1^{N-n}(1-\rho_1)P_N$$
$$+\mu\sigma(1-\rho)\rho^n\left(\frac{\sqrt{\omega^2-\alpha^2}}{\lambda}\right)P_1 + (1-\rho)\rho^n \qquad n = 1,2,\ldots,N-1 \tag{4.11}$$

Multiplying the equation (3.21) by s on both sides and taking limit as $s \to 0$, after some algebra, we get

$$\lim_{s\to 0} sg^*(s) = \frac{1}{\lambda}[\xi + 2\mu(1-\rho)^2]\rho^{N+1} + \frac{\mu}{\lambda}\rho^{N+1}[(\rho^{-1}-1) - \rho_1(1-\rho)]P_1 \tag{4.12}$$

Now taking limit as $s \to 0$ in the equation (3.20), we get

$$\lim_{s\to 0} f^*(s) = \rho[1 + \rho_1^{N+1}\rho^N(1-\rho)] \tag{4.13}$$

Multiplying the equation (3.19) by s on both sides and taking limit as $s \to 0$, we get

$$\lim_{s\to 0} sP_N^*(s) = \lim_{s\to 0}\frac{sg^*(s)}{1-f^*(s)} \tag{4.14}$$

Substituting equation (4.12) and (4.13) in the above equation

$$P_N = \frac{[\{\xi+2\mu(1-\rho)^2\}+\mu\{(\rho^{-1}-1)-\rho_1(1-\rho)\}P_1]\rho^{N+1}}{\lambda[1-\rho-\rho_1^{N+1}\rho^{N+1}(1-\rho)]} \tag{4.15}$$

Multiplying the equation (3.16) by s on both sides and taking limit as $s \to 0$, we get

$$\lim_{s\to 0} sP_1^*(s)\left[\sqrt{\omega^2-\alpha^2} - \mu\left\{1-\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{4\mu}\right)\right\}\right] =$$
$$2\mu\lim_{s\to 0} s\left\{1-\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{4\mu}\right)\right\}$$

$$\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{4\mu}\right)P_0^*(s)$$
$$+\lambda\lim_{s\to0} s\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{2\lambda}\right)^{N-1}\left\{1-\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{2\lambda}\right)\right\}P_N^*(s)$$
$$+\lim_{s\to0} s\left(\frac{\xi}{s}+1\right)\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{4\mu}\right) \quad (4.16)$$

After some algebra, the above expression becomes

$$P_1\left\{\frac{1}{\sigma}-\mu(1-\rho)\right\} = 2\mu(1-\rho)\rho P_0 + \lambda(\rho_1^{N-1}-\rho_1^N)P_N + \xi\rho$$

5 Steady state mean and variance

The corresponding values of the steady state mean and variance of the system length are obtained by taking limit as $t \to \infty$ in equation (4.2) and (4.3). These values are given by

$$m = E(X) = \frac{1}{\xi}[(\lambda-2\mu)+2\mu P_0-\lambda P_N+\mu P_1]$$
$$Var(X) = \frac{1}{\xi}[(\lambda+2\mu)+2(\lambda-\mu)m-2\mu P_0-\lambda(2N+1)P_N+\mu P_1]-m^2$$

6 Conclusion

In the present paper, we have discussed the $M/M/2/N$ queueing system subject to catastrophes. The transient as well as the steady state probabilities of the models have been determined analytically. Further, we have also obtained the performance measures of the system.

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