Events Dependence Used in Reliability Speaks More to Know. Modeling Competing Risks

Boyan Dimitrov

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Department of Mathematics, Kettering University,
1700 West University Avenue, Flint, Michigan, USA
email: bdimitro@kettering.edu

Abstract

In this article we show how some known to us measures of dependence between random events can be easily transferred into measures of local dependence between random variables. This enables everyone to see and visually evaluate the local dependence between uncertain units on every region of their particular values. We believe that the true value of the use of such dependences is in applications on non-numeric variables, as well as in finances and risk studies. We also trust that our approach may give a serious push into the microscopic analysis of the pictures of dependences offered in big data. Numeric and graphical examples should confirm the beauty, simplicity and the utility of this approach, especially in reliability models.

Keywords: local measures of dependence, local regression coefficienstins, local correlation, mapping the local dependence, big data tools, microscopic analysis of dependence in reliability models - graphic illustrations

1. Introduction

The big data files contain a number of simultaneous multi-dimensional observations. This fact offers plenty of opportunities for establishing possible dependences between observed variables. Most of these dependences will be of global nature. However, there exist (or can be created) techniques to take a microscopic look on more details into it. In this article we want to show the ideas of these microscopic looks.

The concepts of measuring dependence should start from the very roots of Probability Theory. Independence for random events is introduced simultaneously with conditional probability. Where independence does not hold, events are dependent. Further, the focus in text-books is on the independence. No text-books usually discus what to do if events are dependent. However, there are ways to go deeply in the analysis of dependence, to see some detailed pictures, and use it later in the studies of random variables. This question is discussed in our previous articles (Dimitrov 2010, 2015) and more (Esa-Dimitrov 2013, 2017). Some particular situations are analyzed in Dimitrov and Esa 2014 and Esa, Dimitrov 2017. Applications in study of politics are used in Esa, Dimitrov 2013. We refer to these articles for making a quick passage to the essentials.

First we notice here that the most informative measures of dependence between random events are the two *regression coefficients*. Their definition is given here:

Definition1. Regression coefficient $R_B(A)$ of the event A with respect to the event B is called the difference between the conditional probability for the event A given the event B, and the conditional probability for the event A given the complementary event $\overline{B}$, namely

$$R_B(A) = P(A|B) - P(A|\overline{B}).$$

This *measure of the dependence of the event A on the event B,* is *directed dependence*.

The regression coefficient $R_A(B)$ of the event B with respect to the event A is defined analogously.

From the many interesting properties of the regression coefficients we would like to point out here just few:

(R1) The equality to zero $R_B(A) = R_A(B) = 0$ takes place if and only if the two events are independent.

(R2) The regression coefficients $R_B(A)$ and $R_A(B)$ are numbers with equal signs and this is the sign of their connection $\delta(A,B) = P(A \cap B) - P(A)P(B)$. The relationships

$$R_B(A) = \frac{P(A \cap B) - P(A)P(B)}{P(B)[1 - P(B)]}, \text{ and } R_A(B) = \frac{P(A \cap B) - P(A)P(B)}{P(A)[1 - P(A)]}.$$

The numerical values of $R_B(A)$ and $R_A(B)$ may not always be equal. There exists an asymmetry in the dependence between random events, and this reflects the nature of real life.

(R3) The regression coefficients $R_B(A)$ and $R_A(B)$ are numbers between –1 and 1, i.e. they satisfy the inequalities

$$-1 \le R_B(A) \le 1; \qquad -1 \le R_A(B) \le 1.$$

(R4.1) The equality $R_B(A) = 1$ holds only when the random event A coincides with (or is equivalent to) the event B. Then it is also valid the equality $R_A(B) = 1$;

(R4.2) The equality $R_B(A) = -1$ holds only when the random event A coincides with (or is equivalent to) the event $\overline{B}$ - the complement of the event B. Then it is also valid $R_A(B) = -1$, and respectively $\overline{A} = B$.

We interpret the properties (r4) of the regression coefficients in the following way: As closer is the numerical value of $R_B(A)$ to 1, "as denser inside within each other are the events A and B, considered as sets of outcomes of the experiment". In a similar way we interpret also the negative values of the regression coefficient.

There is a symmetric measure of dependence between random events, and this is their coefficient of correlation.

Definition 2. Correlation coefficient between two events A and B we call the number

$$\rho_{A,B} = \pm\sqrt{R_B(A) \cdot R_A(B)},$$

where the sign, plus or minus, is the sign of the either of the two regression coefficients.

Remark. The correlation coefficient $\rho_{A,B}$ between the events A and B equals to the formal correlation coefficient ρ_{I_A,I_B} between the random variables I_A and I_B, the indicators of the two random events A and B.

The correlation coefficient $\rho_{A,B}$ between two random events is symmetric, is located between the numbers $R_B(A)$ and $R_A(B)$.

The following statements hold:

ϱ1. $\rho_{A,B} = 0$ holds if and only if the two events A and B are independent. The use of the numerical values of the correlation coefficient is similar to the use of the two regression coefficients. As closer is $\rho_{A,B}$ located to the zero, as "closer" to the independence are the two events *A* and *B*.

For random variables similar statement is not true. The equality to zero of their mutual correlation coefficient does not mean independence

ϱ2. The correlation coefficient $\rho_{A,B}$ always is a number between –1 and +1, i.e.

$$-1 \leq \rho_{A,B} \leq 1.$$

ϱ2.1. The equality $\rho_{A,B} = 1$ holds if and only if the events *A* and *B* are equivalent, i.e. when A = *B*.

ϱ2.2. The equality $\rho_{A,B} = -1$ holds if and only if the events *A* and $\overline{B}$ are equivalent, i.e. when A = $\overline{B}$.

As closer is $\rho_{A,B}$ to the number 1, as "more dense one within the other" are the events A and *B*, and when $\rho_{A,B} = 1$, the two events coincide (are equivalent).

As closer is $\rho_{A,B}$ to the number -1, as "denser one within the other" are the events A and $\overline{B}$, and when $\rho_{A,B} = -1$, the two events coincide (are equivalent). Denser one within the other are then the events $\overline{A}$ and *B*.

2. The transfer rules

The above measures allow studying the behavior of interaction between any pair of numeric r.v.'s (*X,Y*) throughout the sample space, and better understanding and use of dependence.

Let the joint cumulative distribution function (c.d.f.) of the pair (*X,Y*) be $F(x,y)=P(X\leq x,\ Y\leq y)$, and marginals $F(x) = P(X \leq x)$, $G(y)=P(Y\leq y)$. Let introduce the events

$$A_x=\{x\leq X\leq x+\Delta_1 x\}; \qquad B_y=\{y\leq Y\leq y+\Delta_2 y\}, \text{ for any } x, y \in (-\infty, \infty).$$

Then the measures of dependence between events A_x and B_y turn into *a measure of local dependence between the pair of r.v.'s X and Y on the rectangle* $D=[x, x+\Delta_1 x]\times[y, y+\Delta_2 y]$. Naturally, they can be named and calculated as follows:

Regression coefficient of *X* with respect to *Y*, and of *Y* with respect to *X* on the rectangle $[x, x+\Delta_1 x]\times[y, y+\Delta_2 y]$. By the use of Definition 1 we get

$$R_Y((X,Y)\in D)=\frac{\Delta_D F(x,y)-[F(x+\Delta_1 x)-F(x)][G(y+\Delta_2 y)-G(y)]}{[F(x+\Delta_1 x)-F(x)]\{1-[F(x+\Delta_1 x)-F(x)]\}}.$$

Here $\Delta_D F(x,y)$ denotes the two dimensional finite difference for the function $F(x,y)$ on rectangle $D=[x, x+\Delta_1 x]\times[y, y+\Delta_2 y]$. Namely

$$\Delta_D F(x,y) = F(x+\Delta_1 x, y+\Delta_2 y)- F(x+\Delta_1 x, y)- F(x, y+\Delta_2 y)+ F(x, y).$$

In an analogous way is defined $\rho_X((X,Y)\in D)$. Just denominator in the above expression is changed respectively.

Correlation coefficient $\varrho_Y((X,Y)\in D)$ between the r.v.s *X* and *Y* on rectangle $D=[x, x+\Delta_1 x]\times[y, y+\Delta_2 y]$ can be presented in similar way by the use of Definition 2. We omit detailed expressions as something obvious.

It seems easier to find out the local dependence at a value (*X=i, Y=j*) for a pair of discretely distributed r.v. (*X,Y*). Regression coefficient of *X* with respect to *Y*, and of *Y* with respect to *X* at a value ($X=i$, $Y=j$) is determined by the rule

$$R_Y(X=i,Y=j)=\frac{P(X=i,Y=j)-P(X=i)P(Y=j)}{P(X=i)[1-P(X=i)]}=\frac{p(i,j)-p_{i.}p_{.j}}{p_{i.}(1-p_{i.})}.$$

Similarly, you can get that the local correlation coefficient between the values of the two r.v.'s (X,Y) is given by

$$\rho_{X,Y}(X=i,Y=j)=\frac{p(i,j)-p_{i.}p_{.j}}{\sqrt{p_{i.}(1-p_{i.})}\sqrt{p_{.j}(1-p_{.j})}}.$$

Using these rules one can see and visualize the local dependence between every pair of two r.v.'s with given joint distribution.

This ends our theoretical background of the local dependence structural study. Next we illustrate its application on qualitative and quantitative probability models.

3. Illustrations

3.1 Reliability systems

In this case let us consider the two traditional systems of independent components, the system in series and the system in parallel. We want to study how the regression coefficients of a component with respect to the system, and vise verse, regression coefficient of the system with respect to a component change in time during the work of the system. For simplicity consider system of just two components, since considering one component, everything else can be considered as a second component. Results of the studies are shown next.

3.1. A system in series. Assume both components have live times exponentially distributed with parameters λ_1 and λ_2 . Then the reliability function at any time instant t (this is the event B) equals r(t)= $e^{-(\lambda_1+\lambda_2)t}$, and the probability that component 1 functions (this is the event A) is $e^{-\lambda_1 t}$. The regression coefficient of the system with respect to component 1 is then

$$R_1(S)=\frac{r(t)-r(t)e^{-\lambda_1 t}}{e^{-\lambda_1 t}(1-e^{-\lambda_1 t})}=e^{-\lambda_2 t}.$$

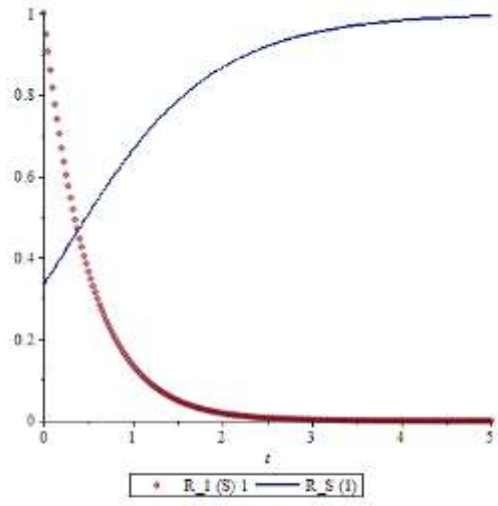
Analogously we evaluate the regression coefficient of the component 1 with respect to the system at time t. It is given by the relation

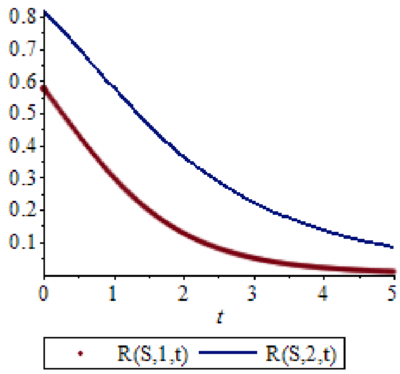
$$R_S(1)=\frac{r(t)-r(t)e^{-\lambda_1 t}}{r(t)[1-r(t)]}=\frac{1-e^{-\lambda_1 t}}{1-e^{-(\lambda_1+\lambda_2)t}}.$$

And the correlation coefficient between system reliability and the component reliability are changing during the time according to the relations

$$\rho_{s,1}(t)=\sqrt{\frac{e^{-\lambda_2 t}(1-e^{-\lambda_1 t})}{1-e^{-(\lambda_1+\lambda_2)t}}}\ ;\quad \rho_{s,2}(t)=\sqrt{\frac{e^{-\lambda_1 t}(1-e^{-\lambda_2 t})}{1-e^{-(\lambda_1+\lambda_2)t}}}.$$

Notice that all dependences are positive. Graphs of these functions of local dependence in time for $\lambda_1=1$ and $\lambda_2=2$ are shown on next figures.





We observe that the system reliability local correlation measures of dependence is decreasing to 0 for both components, but is higher with the weakest component 2, when the time increases. In the same time the regression coefficients between the system and the strongest component behave different: Local dependence $R_1(S)$ approaches 0 with the time (like system becomes independent on component 1with the growth of the time) when the local dependence $R_S(1)$ of strongest component 1 on the system reliability approaches 1 with the growth of the time.

3.2. System in parallel. Assume again both components have live times exponentially distributed with parameters λ_1 and λ_2. Then the reliability function at any time instant t (this is the event B) equals $r(t)=1-(1-e^{-\lambda_1 t})(1-e^{-\lambda_2 t})$, and the probability that component 1 functions (this is the event *A*) is $e^{-\lambda_1 t}$. Applying the rules we obtain:

The regression coefficient of the system with respect to component 1 is then

$$R_1(S)=\frac{r(t)-r(t)e^{-\lambda_1 t}}{e^{-\lambda_1 t}(1-e^{-\lambda_1 t})}=1-e^{-\lambda_2 t}.$$

Analogously we evaluate the regression coefficient of the component 1 with respect to the system at time t. It is given by the relation

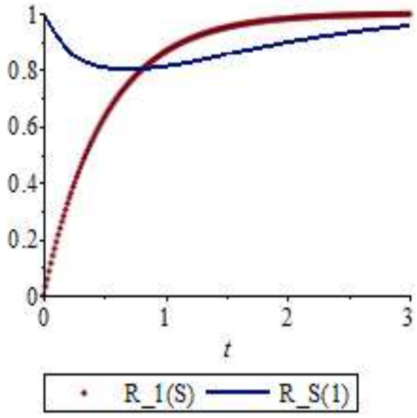
$$R_S(1)=\frac{r(t)-r(t)e^{-\lambda_1 t}}{r(t)[1-r(t)]}=\frac{e^{-\lambda_1 t}}{1-(1-e^{-\lambda_1 t})(1-e^{-\lambda_2 t})}.$$

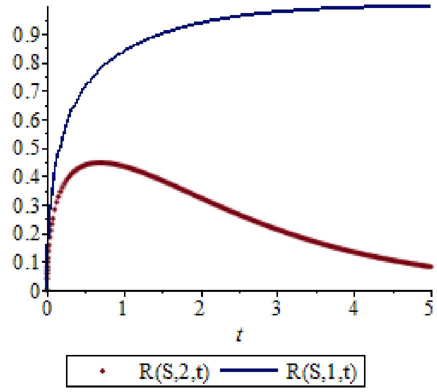
And the correlation coefficient between system reliability and the component reliability are changing during the time according to the relations

$$\rho_{s,1}(t)=\sqrt{\frac{e^{-\lambda_1 t}(1-e^{-\lambda_2 t})}{1-(1-e^{-\lambda_1 t})(1-e^{-\lambda_2 t})}};\quad \rho_{s,2}(t)=\sqrt{\frac{e^{-\lambda_2 t}(1-e^{-\lambda_1 t})}{1-(1-e^{-\lambda_1 t})(1-e^{-\lambda_2 t})}}.$$

Notice that all dependences are positive. Graphs of these functions of local dependence in time for λ_1=1 and λ_2=2 are shown on next figures. First one represents the two regression coefficients superimposed on the same graph, and he second represents the two correlation coefficients.

We see that the system reliability local correlation measure of dependence is approaching 1 with the strongest component 1, and approaches 0 with the weakest component when the time increases. In the same time the both regression coefficient between the system and the strongest component approach 1 with the growth of the time.





3.3. Categorical variables

The most interesting and valuable applications in the Big Data analysis we see in the analysis of local dependences between non-numeric vs non numeric variables, as well as between non-numeric vs numeric variables. Since analysis in this kind of studies is (according to us obviously) too similar, we recommend here as an example of local dependence between categories of two non-numeric random variables. It is just an illustration of the proposed measures of dependence between random events. We analyze here an example from the book of Alan Agresti, (2006). You can see this illustration in the work of Dimitrov, 2010.

3.4. A challenging idea in modeling dependent variables

Modeling dependence in multivariate distributions always has been and still is a hot topic in applied probability, statistics and risk studies. One of the most popular approach in modeling dependence is known as Farlie–Gumbel–Morgenstern dependence model. It is using a construction of bivariate distributions as a mixture of two or more marginal distributions. The main disadvantage of this approach is that it produces multivariate distributions with limited magnitude of the correlation coefficient ρ_{XY}. Original construction gives ρ_{XY} within [-.32, .32]. Some generalizations lately (Bekrizadeh et al, 2012) expanded this range to [-.5, +.43]. Other approaches based on copula constructions (Joe, 1997, Nelsen 2006) offer constructions for dependent multivariate distributions with desired marginal. In most of these constructions is used mostly analytical instrumentation where one can get the goal, but loses the meaning.

In this subsection we offer a construction which is based on the dependence between the two components of the random vector due to the presence of a common random component in each. In our opinion, such models are of interest in reliability and risk modeling where competitive risks are presented and have realistic meaning. And each risk is presented by a r.v. kind of independent on the others. We illustrate this approach on a very particular bivariate dependence, where components are indicator variables.

Let U, V and W be independent one dimensional r.v. Consider the following constructions:

A) $X = \min(U, W)$; $Y = \min(V, W)$, and the pair (X, Y);

B) $X = \max(U, W)$; $Y = \max(V, W)$, and the pair (X, Y);

C) $X = \min(U, W)$; $Y = \max(V, W)$, and the pair (X, Y);

D) $X = U + W$; $Y = V + W$, and the pair (X, Y).

Other algebraic operations also may be used in similar constructions. Obviously, the components of each pair are dependent due to the presence of one and the same component W in both. The good thing here is that we see the interaction between *X*, and *Y*. And also, here one may use any distributions or the original risks *U*, *V* and *W*. Our goals here are to find the correlation coefficients in each of the above 4 constructions, and also to investigate the local correlation structure between *X*, and *Y* in the light of the proposed measures for the strength of local dependence explored recently (Dimitrov 2010, 2015). Actually, we will use the measure $\rho_{A,B}$ defined in Definition 2.We start on the grass roots, considering the examples when *U*, *V* and *W* have the simplest Bernoulli distribution i=1,2,3, or have the Uniform distributions on [0, 1].

U, V, W	0	1
$f_i(.)$	$q_i = 1 - p_i$	p_i

Everyone knows that the expected values and standard deviation of a Bernoulli distributed r.v. are E(*U*) = *p*, and $\sigma_U = \sqrt{pq}$.

3.4.1 Minimum-Minimum competing risks

Elementary combinatorial considerations will convince you, that the joint distribution of the random vector (*X*, *Y*) is presented by the table

Table 1

X \ Y	0	1	$f_X(.)$
0	$1 – p_3(p_1+p_2-p_1p_2)$	$q_1p_2p_3$	$1 - p_1p_3$
1	$p_1q_2p_3$	$p_1p_2p_3$	p_1p_3
$f_Y(.)$	$1 – p_2p_3$	p_2p_3	1

On the margins are the marginal distributions of the components *X*, and *Y*, and each of it is also a Bernoulli distributed r.v. This fact will simplify your calculation of the correlation coefficient ρ_{XY}, using the short cut rule

$$\rho_{XY} = [E(XY) – E(X)E(Y)]/ [\sigma_X \sigma_Y],$$

and the results above about Bernoulli distributed r.v. After several algebraic manipulations we arrive to the expression

$$\rho_{XY} = q_3\sqrt{\frac{p_1 p_2}{(1-p_1 p_3)(1-p_2 p_3)}} .$$

A brief analysis of this expression shows, that this correlation coefficient can take any value between 0 (when q_3 is close to 0, and p_1, p_2 are small), and 1 (when q_3 is close to 1, and so are p_1, p_2). Hence, pending on probabilities p_1, p_2 and p_3, any correlation between *X* and *Y* is feasible.

In particular if *U*, *V* and *W* are equally distributed, then

$$\rho_{XY} = p/(1+p).$$

But this time the correlation coefficient may take values only between 0 and 0.5. Of course, you may get negative correlations of same size if change second component *Y* by its negative –*Y*.

Local dependence magnitudes.

Now let see the strength of dependence of the event {*Y*=0} with respect to the event {*X*=0}. It means that we may predict the event {*Y*=0} if we know that it occurred {*X*=0}, by making use of relations above. First determine the local regression coefficient using Definition 1 and data in Table 1. We get

$$R_{X=0}(Y=0) = \frac{p_2 q_3}{1-p_1 p_3}.$$

Further considerations show, that if we know individual parameters of variables *U,V* and *W*, and know that event (*X*=0) occurred, then our prediction of event (*Y*=0) will be given by the posterior probability

$$P(Y=0 \mid X=0) = 1 - q_1 p_2 p_3/[1 - p_1 p_3],$$

Due to the positive dependence, the prediction probability increases with the information of the known value of component *X*.

Not going into detailed explanations, we get

$$R_{X=0}(Y=1) = -\frac{p_2 q_3}{1-p_1 p_3}; \quad P(Y=1 \mid X=0) = \mathbf{q_1 p_2 p_3/(1-p_1 p_3)}$$

As we expect, we have $P(Y=0 \mid X=0) + P(Y=1 \mid X=0) = 1$.

Let see the local strength of dependence of the event {*Y*=0} with respect to the event {*X*=1}. It means that we may predict the event {*Y*=0} if we know that it occurred {*X*=1}, by making use of Definition 2. First determine the local regression coefficient using Definition 1 and data in Table 1. We get

$$Rr_{X=1}(Y=0) = \frac{-p_2 q_3}{1-p_1 p_3}.$$

This negative regression coefficient indicates that chances of the event {*Y*=0} to happen decrease if it is known that event {*X*=1} occurred. And the equivalent to the Bayes posterior probability rule now is valid

$$P(Y=0 \mid X=1) = 1 - p_2 p_3 - p_2 q_3 = 1 - p_2.$$

Similarly, we determine $R_{X=1}(Y=1)$, and respective posterior probabilities:

$$Rr_{X=1}(Y=1) = \frac{p_2 q_3}{1-p_1 p_3}; \quad P(Y=1 \mid X=1) = p_2 p_3 + p_2 q_3 = p_2.$$

Compare all four results, we observe complete symmetry in regards of the local dependence strengths: Positive regression coefficients for same results in *Y* as in *X*, and negative (same magnitudes) for opposite result. So to say, *the two risks support each other in the sense that they act in same direction.*

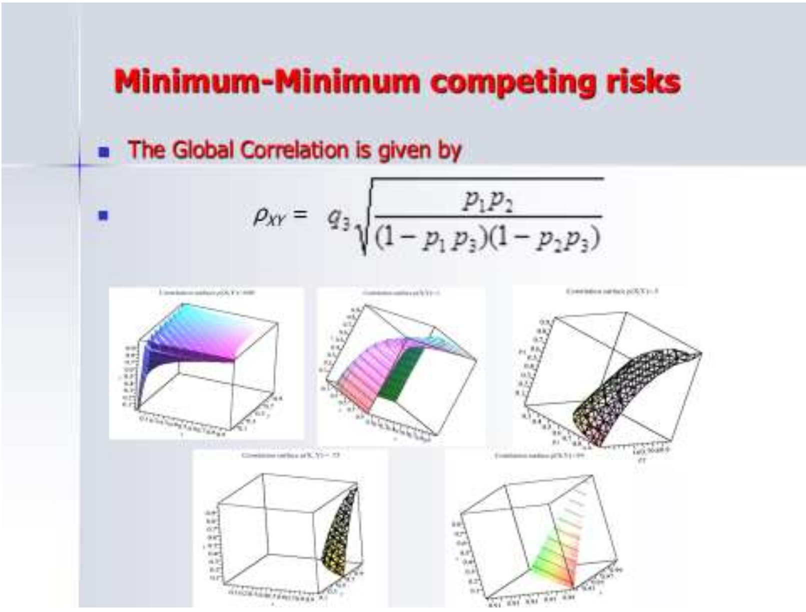
Since symmetric constructions, the regression coefficients $R_{Y=j}(X=i)$, for i , j = 0, 1 in relationships above can be found by same expressions, when keeping p_3 and q_3 as is, and changing indices p_1 and q_1 to p_2 and q_2, and vice versa. We skip details, but give a numeric example for (A): p_1=.3, p_2=.6 and p_3=.9 with calculated correlation coefficient (as measures of global dependence), the regression coefficients (as measures of local dependence) and the posterior probabilities for each variable. For comparison, we also give the same characteristics calculated for the combination of numeric parameters (B): p_1=.3, p_2=.6 and p_3=.1. We have

$$\rho_{XY}(p_1=.3, p_2=.6, p_3=.9) = 0.0732143;$$

Table 2. Joint distribution (X,Y)

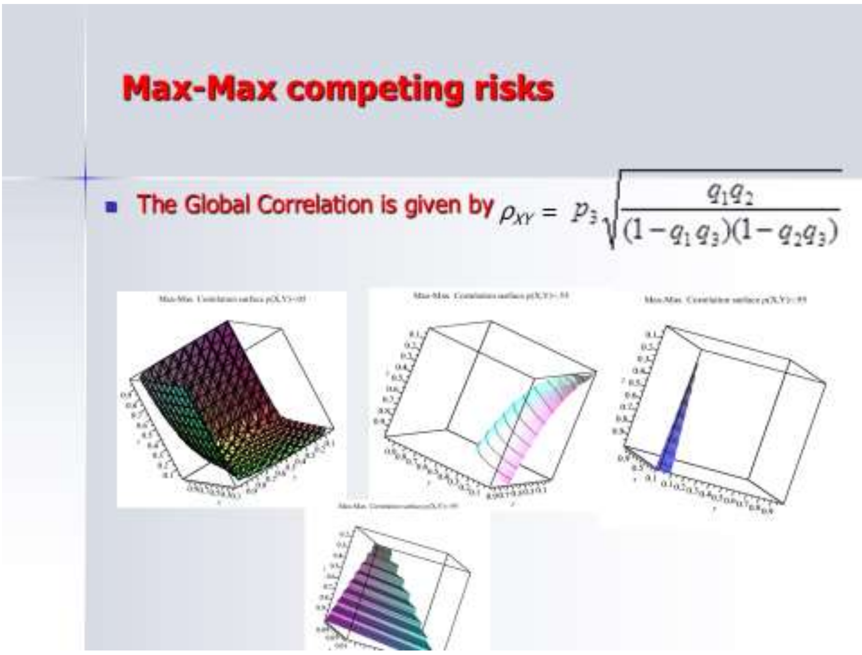
X \ Y	0	1	$f_X(.)$
0	.352	.378	.73
1	.108	.162	.27
$f_Y(.)$	.46	.54	1

The next graphs show surfaces within the cube {0,1]x[0,1]x[0,1], where combination of values p_1, p_2, and p_3 produce correlation coefficient of equal values.



In the next illustrations we will not give detailed numerical analysis, and will show jist the summary graphs similar to this one.

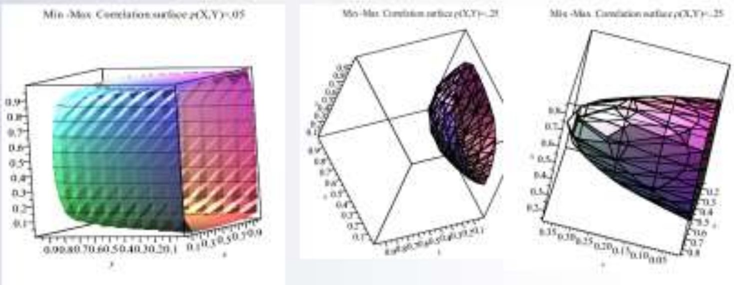
3.4.2 Maximin-Maximum competing risks



3.4.3 Minimum-Maximum competing risks

Min-Max competing risks

The Global Correlation is given by $\rho_{XY} = \sqrt{\frac{p_1 p_3}{1 - p_1 p_3} \frac{q_1 q\, q_3}{1 - q_1 q\, q_3}}$



- For this model maximum correlation that can be reached is 1/3.

3.4.4 Sums of competing risks

Sums of competing risks.

- Here we consider the **configuration** ***X = U + W; Y = V + W***, **and the pair (*X, Y*).**

Table 2.4. Joint distribution of (*X, Y*)

Y / X	0	1	2	$f_X(.)$
0	$q_1q_2q_3$	$q_1p_2q_3$	0	q_1q_3
1	$p_1q_2q_3$	$p_1p_2q_3 + q_1q_2p_3$	$q_1p_2p_3$	$1 - q_1q_3 - p_1p_3$
2	0	$p_1q_2p_3$	$p_1p_2p_3$	p_1p_3
$f_Y(.)$	q_2q_3	$1 - q_2q_3 - p_2p_3$	p_2p_3	1

Global Correlation coefficient $\rho_{XY} = \frac{p_3 q_3}{\sqrt{(p_1q_1 + p_3q_3)(p_2q_2 + p_3q_3)}}$

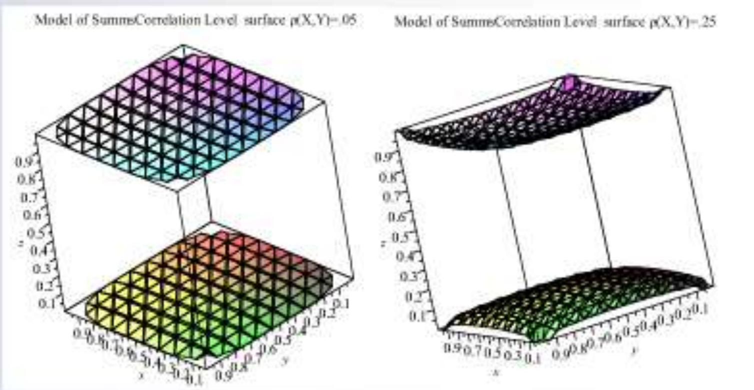
Sums of competing risks.

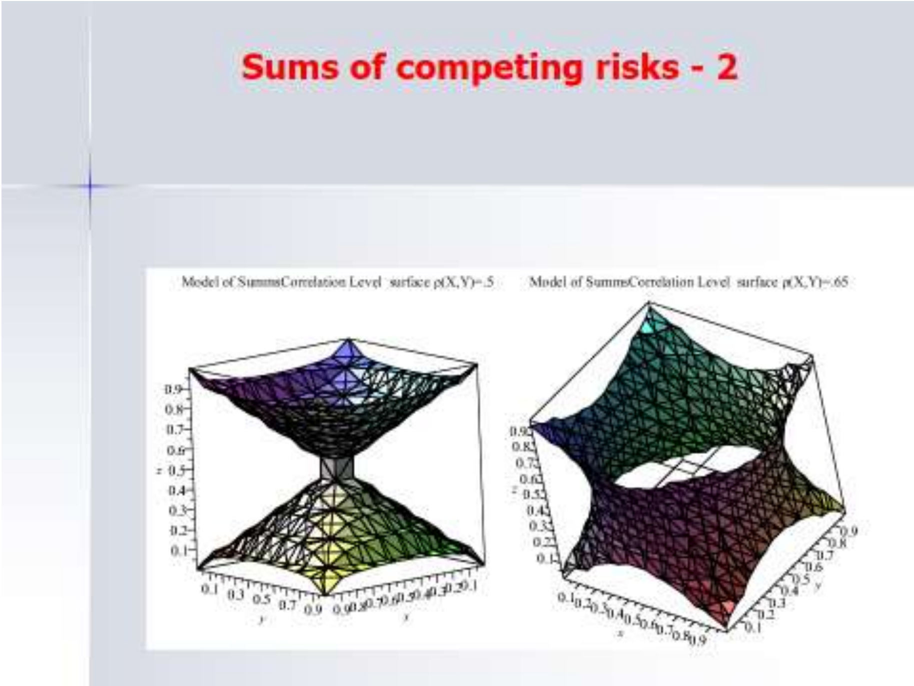
- **Local measures of dependence.**
- These are calculated by use of the rules
- $R_{Y=j}(X=i) = \dfrac{P\{X=i, Y=j) - P\{X=i\}P\{Y=j\}}{P\{X=i\}[1-P\{X=i\}]}$
- In this way we find:

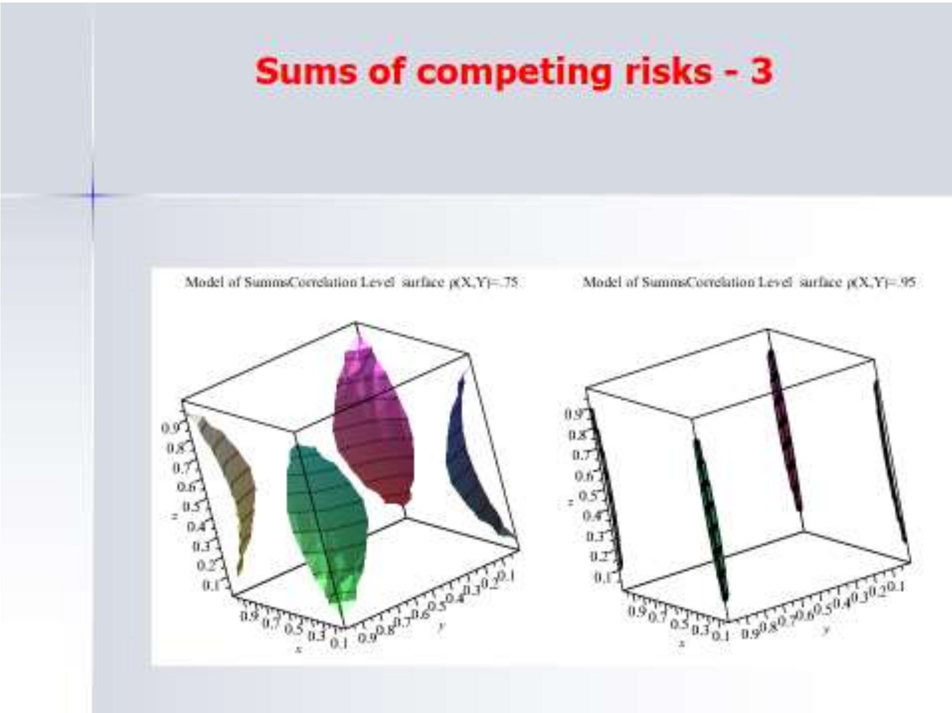
$$R_{Y=0}(X=0) = \frac{q_2 p_3}{1-q_1 q_3} \quad R_{Y=0}(X=1) = \frac{q_2 q_3 (p_1 - q_1)}{(1-q_1 q_3 - p_1 p_3)(q_1 q_3 + p_1 p_3)}$$

$$R_{Y=0}(X=2) = -\frac{q_2 q_3}{1-p_1 p_3}$$

Sums of competing risks -1




Sums of competing risks - 2
Model of SummsCorrelation Level surface ρ(X,Y)=.5
Model of SummsCorrelation Level surface ρ(X,Y)=.65


Sums of competing risks - 3
Model of SummsCorrelation Level surface ρ(X,Y)=.75
Model of SummsCorrelation Level surface ρ(X,Y)=.95

Conclusions

We extend our previous study of local dependence between random events to measures of local dependences between random variables. This turns into a study of the local dependence at a rectangle where interval values of the random variables meet. These local dependences are universally valid and can be continued for higher dimensions. As illustrations, we consider local dependences in reliability systems. The numerical illustrations can be graphically visualized, and show that local dependence is essentially different on different areas in the field. Graphics offer much more comments and further thoughts. Our expectations are that the analysis of Big Data sets will be enriched with the inclusion of our approach into its system tools. An excellent example of this approach can be seen in Dimitrov and Esa (2018).

We also discussed four models for constructing of dependence between two random variables (X,Y) build on 3 independent Bernoulli distributed r.v.'s Um V and W with different parameters.

These models are producing Correlation coefficients in different ranges. These ranges are shown on Correlation Level surfaces in the space of probabilities for success in the used Bernoulli variables in the models. Local dependences between values of X and Y are studied via the correlation coefficient' magnitudes.

Their numerical values serve are presented for particular combinations of parameters, and graphs of some level surfaces are shown.

We are sure that using other particular distributions of the components, different from the Bernoulli ones, may lead to more inetersting and useful results.

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