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Shukla Distribution And Its Application

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Abstract

In this paper a two-parameter lifetime distribution named, 'Shukla distribution' which includes several one parameter lifetime distributions including exponential, Shanker, Ishita, Pranav ,Rani and Ram Awadh as particular cases, has been proposed and investigated. Its moments have been obtained. The hazard rate function, mean residual life function and stochastic ordering of the distribution have been discussed. Maximum likelihood estimation has been explained for estimating the parameters of the distribution. Applications of the distribution have been explained through real life time data and its fit has been found satisfactory over well-known one parameter and two-parameter lifetime distributions.

Keywords: Lifetime distributions, Moments, Hazard rate function, Mean residual life function, Maximum likelihood estimation, Goodness of fit.

1. Introduction

In the new era of the world, it is important to study through the model for systematic approach and statistical approach. In this case approach of distribution theory is crucial to develop statistical model for knowing the occurrence of some event and their interest for some populations of individuals in almost every field of knowledge. The statistical modeling and their studies along with lifetime data has been drawn interest to researchers in engineering, biomedical science, insurance, finance, amongst others. Applications of lifetime distributions range from investigations into the endurance of manufactured items in engineering to research involving human diseases in biomedical sciences.

In the recent past years, a number of one parameter and two-parameter lifetime distributions for modeling lifetime data have been proposed by different statisticians. As we know that classical one parameter exponential distribution including other popular distribution such as Lindley, Akash, Shanker, Ishita, Pranav, Rani, Ram Awadh distributions are proposed and applied on life time data from various field. The two-parameter lifetime distributions popular in statistics are gamma, Weibull, Power Lindley, Quasi Lindley and Exponentiated exponential.

The probability density function (pdf) along with introducer (year) of exponential, Lindley, Akash, Shanker, Pranav and Ram Awadh distributions are presented in table 1.

Table 1: The pdf of exponential, Lindley, Shanker, Pranav, Rani and Ram Awadh distributions

Distributions	pdf	Introducer (Year)
Exponential	$f(x;\theta)=\theta e^{-\theta x}$	
Lindley	$f(x)=\frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}$	Lindley (1958)
Shanker	$f(x)=\frac{\theta^2}{\theta^2+1}(\theta+x)e^{-\theta x}$	Shanker (2015 a)
Akash	$f(x;\theta)=\frac{\theta^3}{\theta^2+2}(1+x^2)e^{-\theta x}$	Shanker (2015 b)
Ishita	$f(x;\theta)=\frac{\theta^3}{\theta^3+2}(\theta+x^2)e^{-\theta x}$	Shanker and Shukla (2017)
Pranav	$f(x;\theta)=\frac{\theta^4}{\theta^4+6}(\theta+x^3)e^{-\theta x}$	Shukla (2018)
Rani	$f(x;\theta)=\frac{\theta^5}{\theta^5+24}(\theta+x^4)e^{-\theta x}$	Shanker (2017)
Ram Awadh	$f(x;\theta)=\frac{\theta^6}{\theta^6+120}(\theta+x^5)e^{-\theta x}$	Shukla(2018)

Ghitany *et al* (2008) have discussed various statistical properties, estimation of parameter and application of Lindley distribution to model waiting time data in a bank and showed that Lindley distribution is a suitable model over exponential distribution. Shanker *et al* (2015) have detailed comparative and critical study on applications of exponential and Lindley distributions for modeling real lifetime datasets from biomedical science and engineering and showed that in majority of datasets exponential distribution shows satisfactory fit over Lindley distribution.

Recently, Shanker and Shukla (2019) proposed a two-parameter lifetime distribution named Rama-Kamlesh distribution (RKD) defined by its pdf and survival function as

$$f(x;\theta,\alpha)=\frac{\theta^{\alpha+1}}{\theta^{\alpha}+\Gamma(\alpha+1)}(1+x^{\alpha})e^{-\theta x};x>0,\theta>0,\alpha\geq 0 \tag{1.1}$$

$$S(x;\theta,\alpha)=\frac{\theta^{\alpha}(1+x^{\alpha})e^{-\theta x}+\alpha\Gamma(\alpha,\theta x)}{\theta^{\alpha}+\Gamma(\alpha+1)};x>0,\theta>0,\alpha\geq 0, \tag{1.2}$$

where $\Gamma(\alpha,\theta x)$ is the lower incomplete gamma function defined as

$$\Gamma(\alpha,z)=\int_0^z e^{-t}t^{\alpha-1}\,dt \tag{1.3}$$

It has been mentioned by Shanker and Shukla (2019) that RKD includes several one parameter lifetime distributions. Various interesting properties, estimation of parameters and application of the distribution have been given in Shanker and Shukla (2019).

The main aim of the present paper is to introduce two-parameter lifetime distribution named Shukla distribution (SD)' which includes many one parameter distributions including exponential distribution as particular case. Several other one parameter lifetime distributions can also be generated from SD. Its moments about origin and the variance have been obtained. The hazard rate function and stochastic ordering have been discussed. Maximum likelihood estimation has been discussed for estimating the parameters of the distribution. Applications of the distribution have been discussed with real lifetime dataset and the goodness of fit of the distribution has been compared with well known one parameter and two-parameter lifetime distributions.

2. Shukla Distribution

The pdf of Shukla distribution (SD) having parameters θ and α can be defined as

$$f(x;\theta,\alpha)=\frac{\theta^{\alpha+1}}{\theta^{\alpha+1}+\Gamma(\alpha+1)}(\theta+x^{\alpha})e^{-\theta x};x>0,\theta>0,\alpha\geq 0 \tag{2.1}$$

It can be easily verified that exponential, Shanker, Ishita, Pranav, Rani and Ram Awadh distributions are particular cases of SD for $\alpha=0,\alpha=1,\alpha=2,\alpha=3,$ $\alpha=4$ and $\alpha=5$ respectively. The pdf (2.1) can be shown as a convex combination of exponential (θ) and gamma (α,θ) distributions. We have

$$f(x;\theta,\alpha)=pg_1(x;\theta)+(1-p)g_2(x;\alpha,\theta),$$

where

$$p=\frac{\theta^{\alpha+1}}{\theta^{\alpha+1}+\Gamma(\alpha+1)},\ g_1(x;\theta)=\theta e^{-\theta x},\ g_2(x;\alpha,\theta)=\frac{\theta^{\alpha+1}}{\Gamma(\alpha+1)}e^{-\theta x}x^{\alpha+1-1}.$$

$$S(x;\theta,\alpha)=P(X>x)=\int_x^{\infty}f(t;\theta,\alpha)dt=\frac{\theta^{\alpha+1}}{\theta^{\alpha+1}+\Gamma(\alpha+1)}\int_x^{\infty}(\theta+t^{\alpha})\,e^{-\theta t}dt$$

$$=\frac{\theta^{\alpha+1}}{\theta^{\alpha+1}+\Gamma(\alpha+1)}\left[\theta\int_x^{\infty}e^{-\theta t}dt+\int_x^{\infty}e^{-\theta t}\,t^{\alpha}dt\right]$$

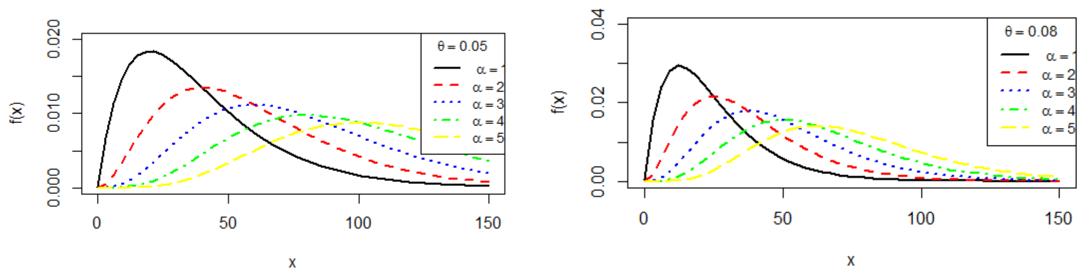
$$=\frac{\theta^{\alpha+1}}{\theta^{\alpha+1}+\Gamma(\alpha+1)}\left[\frac{e^{-\theta x}}{1}+\frac{e^{-\theta x}(\theta x)^{\alpha}+\alpha\Gamma(\alpha,\theta x)}{\theta^{\alpha+1}}\right]$$

$$=\frac{\theta^{\alpha}(\theta+x^{\alpha})e^{-\theta x}+\alpha\Gamma(\alpha,\theta x)}{\theta^{\alpha+1}+\Gamma(\alpha+1)},$$

Thus the corresponding cdf of SD can be obtained as

$$F(x;\theta,\alpha)=1-S(x;\theta,\alpha)=1-\frac{\theta^{\alpha}(\theta+x^{\alpha})e^{-\theta x}+\alpha\Gamma(\alpha,\theta x)}{\theta^{\alpha+1}+\Gamma(\alpha+1)};x>0,\theta>0,\alpha\geq 0 \tag{2.2}$$

Behaviors of pdf and survival function of SD for varying values of parameters θ and α have been shown in figures 1 and 2, respectively.



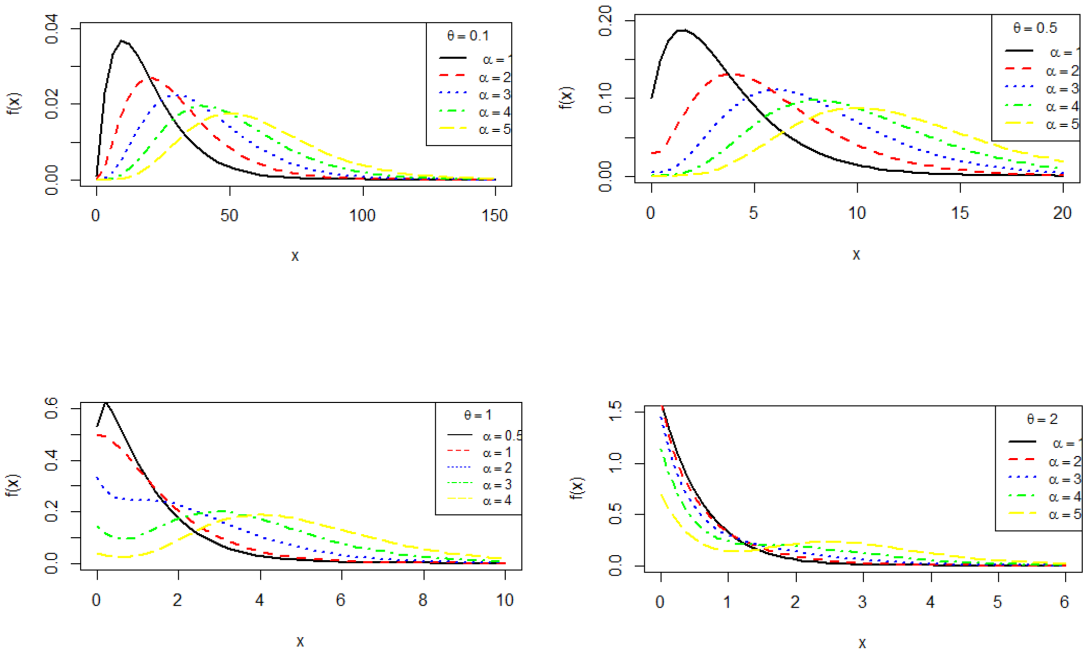


Fig.1: Behavior of the pdf of SD for varying values of parameters θ and α

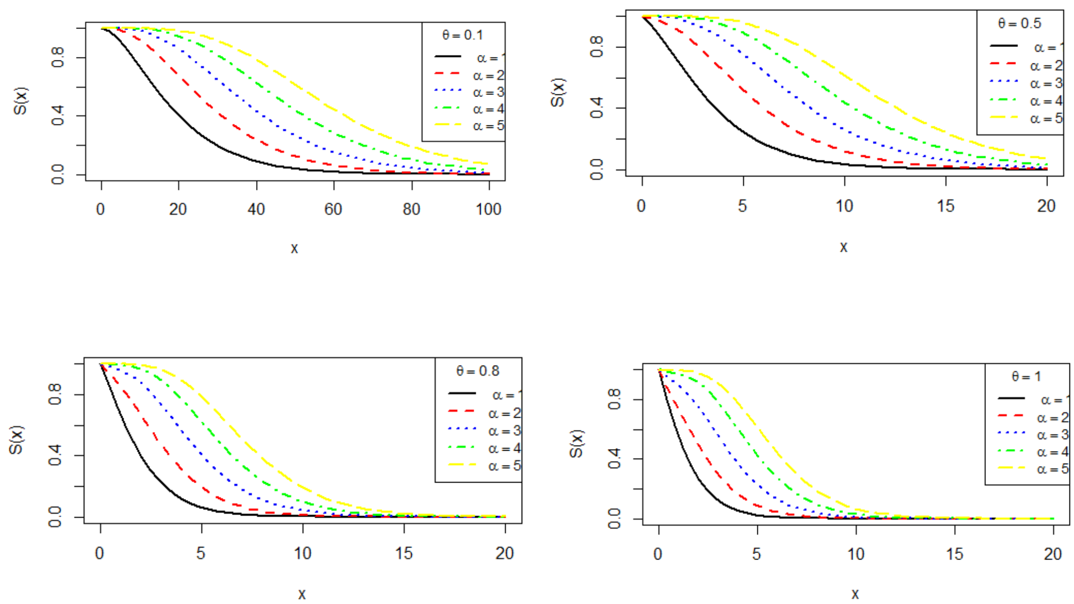


Fig.2: Behavior of the S(x) of SD for varying values of parameters θ and α

3. Moments

The rth moment about origin, μ_r' of Shukla distribution(SD) can be obtained as

$$\mu_r' = \frac{\theta^{\alpha+1}}{\theta^{\alpha+1}+\Gamma(\alpha+1)}\int_0^\infty x^r\,(\theta+x^\alpha)e^{-\theta x}dx$$
$$= \frac{\theta^{\alpha+1}\Gamma(r+1)+\Gamma(\alpha+r+1)}{\theta^r\{\theta^{\alpha+1}+\Gamma(\alpha+1)\}};r=1,2,3,\ldots$$

Thus the first four moments about origin of SD are obtained as

$$\mu_1' = \frac{\theta^{\alpha+1}+\Gamma(\alpha+2)}{\theta\{\theta^{\alpha+1}+\Gamma(\alpha+1)\}}$$

$$\mu_2' = \frac{2\theta^{\alpha+1}+\Gamma(\alpha+3)}{\theta^2\{\theta^{\alpha+1}+\Gamma(\alpha+1)\}}$$

$$\mu_3' = \frac{6\theta^{\alpha+1}+\Gamma(\alpha+4)}{\theta^3\{\theta^{\alpha+1}+\Gamma(\alpha+1)\}}$$

$$\mu_3' = \frac{24\theta^{\alpha+1}+\Gamma(\alpha+5)}{\theta^4\{\theta^{\alpha+1}+\Gamma(\alpha+1)\}}.$$

The variance of SD can be obtained as

$$\mu_2 = \mu_2' - (\mu_1)^2 = \frac{\{2\theta^{\alpha+1}+\Gamma(\alpha+3)\}\{\theta^{\alpha+1}+\Gamma(\alpha+1)\}-\{\theta^{\alpha+1}+\Gamma(\alpha+2)\}^2}{\theta^2\{\theta^{\alpha+1}+\Gamma(\alpha+1)\}^2}$$

Taking $r=1,2,3$ and 4, the first four moments about origin, μ_r' of SD can be obtained. It should be noted that the rth moment about origin, μ_r' of exponential, Shanker, Ishita, Pranav, Rani and Ram Awadh distribution can be obtained from the μ_r' of SD by taking $\alpha = 0,1,2,3,4,$ and 5.

4. Hazard Rate Function and Mean Residual Life Function

For a continuous random variable X having pdf $f(x)$ and cdf $F(x)$, the hazard rate function (also known as the failure rate function), $h(x)$, is defined as

$$h(x) = \lim_{\Delta x\to 0}\frac{P(X<x+\Delta x|X>x)}{\Delta x} = \frac{f(x)}{1-F(x)}.$$

Thus, hazard rate function, $h(x)$ of Shukla distribution can be expressed as $h(x) = h(x;\theta,\alpha) =$

$$\frac{f(x;\theta,\alpha)}{1-F(x;\theta,\alpha)} = \frac{\theta^{\alpha+1}(\theta+x^\alpha)e^{-\theta x}}{\theta^\alpha(\theta+x^\alpha)e^{-\theta x}+\alpha\Gamma(\alpha,\theta x)};x>0,\theta>0,\alpha\geq 0$$

The mean residual life function, $m(x)$ of Shukla distribution can be obtained as

$$m(x;\theta,\alpha) = \frac{1}{S(x;\theta,\alpha)}\int_x^\infty tf(t;\theta,\alpha)\,dt - x$$
$$= \frac{\theta^{\alpha+1}+\Gamma(\alpha+1)}{\theta^\alpha(\theta+x^\alpha)e^{-\theta x}+\alpha\Gamma(\alpha,\theta x)}\int_x^\infty t\frac{\theta^{\alpha+1}}{\theta^{\alpha+1}+\Gamma(\alpha+1)}(\theta+t^\alpha)e^{-\theta t}\,dt - x$$

$$= \frac{\theta^{\alpha+1}}{\theta^\alpha(\theta+x^\alpha)e^{-\theta x}+\alpha\Gamma(\alpha,\theta x)}\left[\theta\int_x^\infty e^{-\theta t}tdt + \int_x^\infty e^{-\theta t}t^{\alpha+1}\,dt\right] - x$$

$$= \frac{\theta^{\alpha+1}}{\theta^{\alpha}(\theta+x^{\alpha})e^{-\theta x}+\alpha\Gamma(\alpha,\theta x)}\left[\frac{e^{-\theta x}(\theta x+1)}{\theta}+\frac{e^{-\theta x}(\theta x)^{\alpha}(\theta x+\alpha+1)+\alpha(\alpha+1)\Gamma(\alpha,\theta x)}{\theta^{\alpha+2}}\right]-x$$

$$=\frac{e^{-\theta x}\{\theta^{\alpha+1}+(\alpha+1)(\theta x)^{\alpha}\}+\alpha(\alpha+1-\theta x)\Gamma(\alpha,\theta x)}{\theta\{\theta^{\alpha}(\theta+x^{\alpha})e^{-\theta x}+\alpha\Gamma(\alpha,\theta x)\}}.$$

Note that $h(0)=\frac{\theta^{\alpha+2}}{\theta^{\alpha+1}+\Gamma(\alpha+1)}=f(0)$ and $m(0)=\frac{\theta^{\alpha+1}+\Gamma(\alpha+2)}{\theta\{\theta^{\alpha+1}+\Gamma(\alpha+1)\}}=\mu_1{'}$. The behaviors of $h(x)$ and $m(x)$ of SD for varying values of parameters θ and α have been shown in figures 3 and 4 respectively.

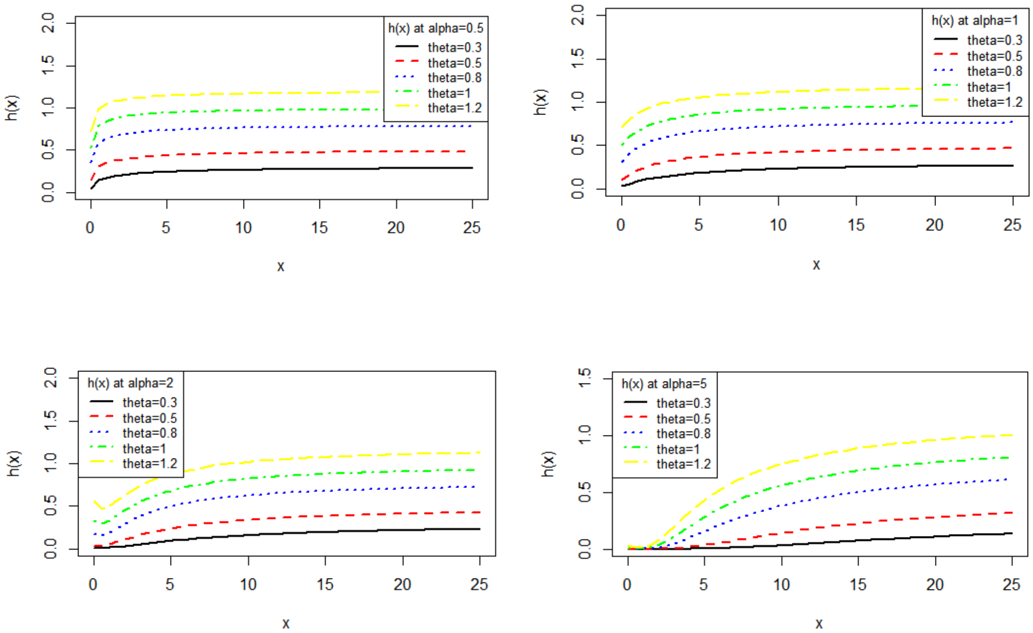


Fig.3: Behavior of $h(x)$ of SD for varying values of parameters θ and α

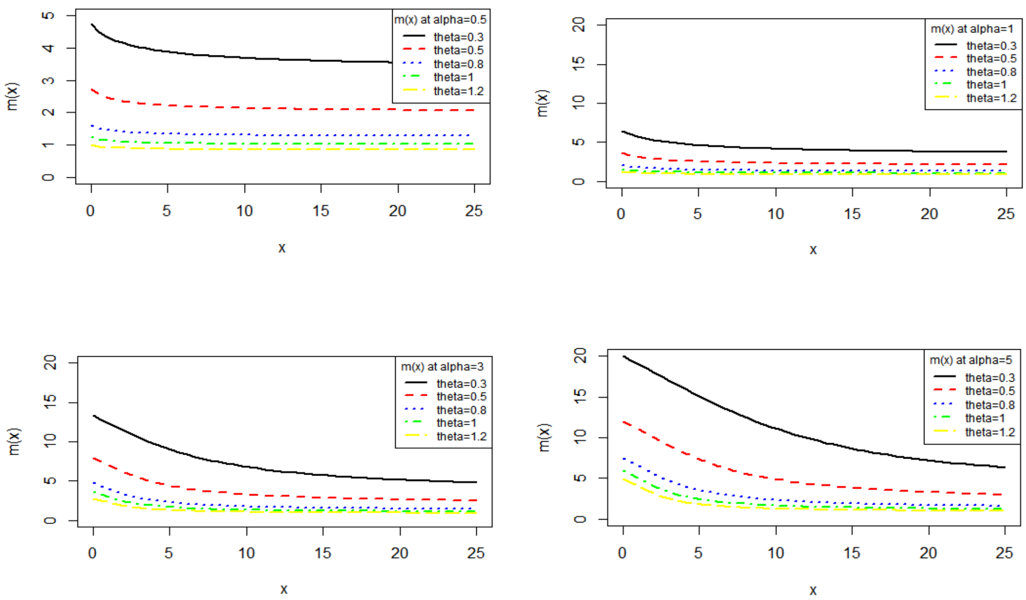


Fig. 4: Behavior of $m(x)$ of SD for varying values of parameters θ and α

5. Stochastic Ordering

Stochastic ordering of positive continuous random variables is an important tool for judging their comparative behavior. A random variable X is said to be smaller than a random variable Y in the

(i) stochastic order $(X \leq_{st} Y)$if $F_X(x) \geq F_Y(x)$for all x

(ii) hazard rate order $(X \leq_{hr} Y)$if $h_X(x) \geq h_Y(x)$ for all x

(iii) mean residual life order $(X \leq_{mrl} Y)$if $m_X(x) \leq m_Y(x)$for all x

(iv) likelihood ratio order $(X \leq_{lr} Y)$if $\frac{f_X(x)}{f_Y(x)}$ decreases in x.

The following results due to Shaked and Shanthikumar (1994) are well known for establishing stochastic ordering of distributions

$$X \leq_{lr} Y \Rightarrow X \leq_{hr} Y \Rightarrow X \leq_{mrl} Y$$
$$\Downarrow$$
$$X \leq_{st} Y$$

KRD is ordered with respect to the strongest 'likelihood ratio' ordering as shown in the following theorem:

Theorem: Let X ~ RKD(θ_1, α_1) and Y Y~ RKD$(\theta_2, \alpha_2)(\theta_2, \alpha_2)$. If $\alpha_1 \leq \alpha_2$ and $\theta_1 > \theta_2$, then $X \leq_{lr} Y$and hence$X \leq_{hr} Y$, $X \leq_{mrl} Y$and$X \leq_{st} Y$.

Proof: We have

$$\frac{f_X(x;\theta_1,\alpha_1)}{f_Y(x;\theta_2,\alpha_2)} = \frac{{\theta_1}^{\alpha_1+1}\left({\theta_2}^{\alpha_2+1}+\Gamma(\alpha_2+1)\right)}{{\theta_2}^{\alpha_2+1}\left({\theta_1}^{\alpha_1+1}+\Gamma(\alpha_1+1)\right)}\left(\frac{\theta_1+x^{\alpha_1}}{\theta_2+x^{\alpha_2}}\right)e^{-(\theta_1-\theta_2)x};\ x > 0$$

Now

$$ln\frac{f_X(x;\theta_1,\alpha_1)}{f_Y(x;\theta_2,\alpha_2)} = ln\left[\frac{{\theta_1}^{\alpha_1+1}\left({\theta_2}^{\alpha_2+1}+\Gamma(\alpha_2+1)\right)}{{\theta_2}^{\alpha_2+1}\left({\theta_1}^{\alpha_1+1}+\Gamma(\alpha_1+1)\right)}\right] + ln\left(\frac{\theta_1+x^{\alpha_1}}{\theta_2+x^{\alpha_2}}\right) - (\theta_1-\theta_2)x$$

.

This gives $$\frac{d}{dx}ln\frac{f_X(x;\theta_1,\alpha_1)}{f_Y(x;\theta_2,\alpha_2)} = \frac{\alpha_1\theta_2x^{\alpha_1-1}-\theta_1\alpha_2x^{\alpha_2-1}+(\alpha_1-\alpha_2)x^{\alpha_1+\alpha_2-1}}{(1+x^{\alpha_1})(1+x^{\alpha_2})} - (\theta_1-\theta_2)$$

Thus, for$\alpha_1 \leq \alpha_2$ and $\theta_1 > \theta_2$,$\frac{d}{dx}ln\frac{f_X(x;\theta_1,\alpha_1)}{f_Y(x;\theta_2,\alpha_2)} < 0$. This means that $X \leq_{lr} Y$and hence$X \leq_{hr} Y$, $X \leq_{mrl} Y$and$X \leq_{st} Y$. This shows flexibility of SD over one parameter exponential, Shanker, Ishita, Pranav, Rani and Ram Awadh distributions.

6. Maximum Likelihood Estimation

Let $(x_1, x_2, x_3, \ldots, x_n)$ be a random sample from SD (2.1). The likelihood function, L of (2.1) can be expressed as

$$L = \left(\frac{\theta^{\alpha+1}}{\theta^{\alpha+1}+\Gamma(\alpha+1)}\right)^n \prod_{i=1}^{n}(\theta + {x_i}^{\alpha})\, e^{-n\theta\bar{x}}$$

The natural log likelihood function is thus obtained as

$$ln\, L = n\, ln\left(\frac{\theta^{\alpha+1}}{\theta^{\alpha+1}+\Gamma(\alpha+1)}\right) + \sum_{i=1}^{n} ln(\theta + {x_i}^{\alpha}) - n\theta\bar{x}$$

$$= n\left[(\alpha+1)\, ln\, \theta - ln\left(\theta^{\alpha+1}+\Gamma(\alpha+1)\right)\right] + \sum_{i=1}^{n} ln(\theta + {x_i}^{\alpha}) - n\theta\bar{x}.$$

The maximum likelihood estimates (MLEs) $\left(\hat{\theta}\,, \hat{\alpha}\right)$ of parameters $\left(\theta,\, \alpha\right)$ of SD are the solution of the following nonlinear log likelihood equations

$$\frac{\partial\, ln\, L}{\partial\theta} = \frac{n(\alpha+1)}{\theta} - \frac{n(\alpha+1)\theta^{\alpha}}{\theta^{\alpha+1}+\Gamma(\alpha+1)} + \sum_{i=1}^{n}\frac{1}{\theta+{x_i}^{\alpha}} - n\bar{x} = 0$$

$$\frac{\partial \ln L}{\partial \alpha} = n \ln \theta - \frac{n[\theta^{\alpha+1} \ln \theta + \psi(\alpha+1)]}{\theta^{\alpha+1} + \Gamma(\alpha+1)} + \sum_{i=1}^{n} \frac{x_i^{\alpha} \ln(x_i)}{\theta + x_i^{\alpha}} = 0$$

where $\overline{x}$ is the sample mean and $\psi(\alpha+1) = \frac{d}{d\alpha} \ln \Gamma(\alpha+1)$ is the digamma function. These two natural log likelihood equations do not seem to be solved directly, because they cannot be expressed in closed forms. The (MLE's) $(\hat{\theta}, \hat{\alpha})$ of (θ, α) can be computed directly by solving the natural log likelihood equation using Newton-Raphson iteration available in R-software till sufficiently close values of $\hat{\theta}$ and $\hat{\alpha}$ are obtained.

7. Data Analysis

The applications of SD have been discussed with the following dataset relating to engineering from Fuller et al (1994).This data set is the strength data of glass of the aircraft window reported by Fuller *et al* (1994):

18.83	20.8	21.657	23.03	23.23	24.05	24.321	25.5	25.52	25.8	26.69	26.77
26.78	27.05	27.67	29.9	31.11	33.2	33.73	33.76	33.89	34.76	35.75	35.91
36.98	37.08	37.09	39.58	44.045	45.29	45.381					

For the above dataset, SD has been fitted along with two- parameter distributions including Power Lindley distribution (PLD) proposed by Ghitany et al (2013), Weibull distribution suggested by Weibull distribution(1951), gamma distribution, Quasi Lindley distribution introduced by Shanker and Mishra (2013) and generalized exponential distribution proposed by Gupta and Kundu (1999), RKD and one parameter lifetime distributions including exponential, Lindley, Shanker, Akash, Ishita, Pranav, Rani and Ram Awadh . The ML estimates, value of $-2\log L$, Akaike Information criteria (AIC), K-S statistics and p-value of the fitted distributions are presented in tables 2 and 3. The AIC and K-S Statistics are computed using the following formulae: $AIC = -2 \ln L + 2k$ and K-S $= \sup_x |F_n(x) - F_0(x)|$, where k = the number of parameters, n = the sample size , $F_n(x)$is the empirical (sample) cumulative distribution function, and $F_0(x)$ is the theoretical cumulative distribution function. The best distribution is the distribution corresponding to lower values of $-2\log L$, AIC, and K-S statistics and higher p-value

Table 2: MLE's, Standard Errors, - 2ln L, AIC, K-S and p-values of the fitted distributions for dataset 1

Distributions	ML Estimates	$-2\log L$	AIC	BIC	K-S	p-value
SD	$\hat{\theta} = 0.6144$	**208.23**	**212.23**	**216.05**	**0.134**	**0.580**
	$\hat{\alpha} = 17.9299$					
PLD	$\hat{\theta} = 0.00243$	220.14	224.14	226.13	0.198	0.152
	$\hat{\alpha} = 1.9439$					
RKD	$\hat{\theta} = 0.61361$	208.23	212.23	216.05	0.134	0.580
	$\hat{\alpha} = 17.9060$					
Gamma	$\hat{\theta} = 0.61482$	208.22	212.22	216.05	0.134	0.578
	$\hat{\alpha} = 18.9433$					
Weibull	$\hat{\theta} = 0.00203$	241.61	245.61	247.61	0.353	0.000
	$\hat{\alpha} = 1.80566$					

QLD	$\hat{\theta}=0.03416$	274.45	278.45	281.32	0.458	0.000
	$\hat{\alpha}=18.9393$					
GED	$\hat{\theta}=0.16531$	208.27	212.27	215.13	0.135	0.581
	$\hat{\alpha}=92.0017$					
Exponential	$\hat{\theta}=0.0325$	274.53	276.53	277.96	0.459	0.000
Lindley	$\hat{\theta}=0.0629$	253.99	255.99	257.42	0.333	0.000
Akash	$\hat{\theta}=0.0970$	240.68	242.68	244.11	0.296	0.006
Shanker	$\hat{\theta}=0.06471$	252.35	254.35	255.78	0.357	0.000
Ishita	$\hat{\theta}=0.09732$	240.48	242.48	243.48	0.297	0.006
Pranav	$\hat{\theta}=0.1298$	232.77	234.77	235.77	0.253	0.030
Rani	$\hat{\theta}=0.1623$	277.25	229.25	230.24	0.220	0.080
Ram Awadh	$\hat{\theta}=0.19471$	223.07	225.07	226.07	0.197	0.155

It is obvious from the goodness of fit given in tables 2 that SD competes well with considered one-parameter and two-parameter lifetime distributions. Therefore, SD can be considered an important two-parameter lifetime distribution as.

8. Conclusions

In this paper a two-parameter lifetime distribution named, 'Shukla distribution (SD)' which includes one parameter lifetime distributions including exponential, Shanker, Ishita, Pranav, Rani and Ram Awadh as particular cases, has been proposed and studied. Its moments have been obtained. The hazard rate function, mean residual life function and stochastic ordering have been discussed. The estimation of its parameters using maximum likelihood estimation has been discussed. Goodness of fit has been presented with a real lifetime dataset and fit found quite satisfactory over all well- known considered lifetime distributions.

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