Transient Numerical Analysis of a Queueing Model with Correlated Reneging, Balking and Feedback

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Abstract

We consider a single server queuing model with correlated reneging, balking and feedback. The time-dependent behavior of the model is studied using Runge-Kutta method. Some measures of the performance like expected system size and expected waiting time are computed.

Keywords: Queuing model, Correlated reneging, Balking, Feedback, Numerical analysis

I Introduction

Queuing models with reneging have attracted the attention of many researchers for their real-life applications in industry and communication networks. The study of the reneging behavior of customers plays an important role in the design of queuing systems for various production and services systems. The pioneer work dealing with customers' impatience was initiated by Haight [8, 9], Ancker and Gafarian [1, 2], and Subba Rao [21, 22]. They have developed the basic queuing models with renging and balking. Since then, a number of researchers have worked on various queuing models with reneging and balking. Recenty, Kumar and Sharma [14] put forth a new concept of retention of reneging customers in queuing theory. They derived the steady-state solution and computed some performance measures, and also showed the effect of probability of customers' retention on expected system size. Kumar and Sharma [15] obtained the transient solution with the probability generating technique for the single server queuing model with retention of reneging customers. Kumar and Sharma [16] obtained the transient and steady-state probabilities for a two-heterogenous servers' Markovian queuing system with retention of reneging.

Mohan [17] was the first to introduce the concept of correlation in gambler's ruin problem. Conolly [4] considered a queuing system having services depending on inter-arrival times. Conolly and Hadidi [5] considered a model having arrival pattern impacting the service pattern. They examined the initial busy period, state and output processes. Murari [19] studied a queuing system with correlated arrivals and general service time distribution. Mohan and Murari [18] obtained the transient solution of a queuing model with correlated arrivals and variable service capacity. Cidon et al. [3] considered a queue in which service time is correlated to inter-arrival time. They studied this correlation in case of communication systems and showed the impact through numerical results by comparing with less reliable models. Patuwo et al. [20] worked on serial correlation in the arrivals. He studied the consequences of correlation on mean queuing performances. Kamoun [13] considered a single server queuing model with finite capacity and correlated arrival in which the packets are submitted to random interruptions. Drezner [7] performed the performance analysis of $M^c/G/1$ queues. Iravani and Luangkesorn [12] studied a model of parallel queues with correlated arrivals and bulk services. To get the performance measures they used the matrix geometric method. Hwang and Sohraby [11] considered a correlated queue of packets moving in transmission line with finite capacity. Numerical examples are illustrated to exhibit the importance of correlation on system performances. Hunter [10] studied the consequences of correlated arrivals on the steady-state queue length process for single server queuing model.

The concept of feedback in queuing theory is used to model the situations when the customers are not satisfied with their first service. A dissatisfied customer retries for service with certain probability. Takacs [24] studied a single server queuing model with feedback mechanism. Davignon and Disney [6] considered an $M/G/1$ queuing system where the served customer either joins the queue again with some probability or depart permanently. They studied the stationary queue length and departure process. Santhakumaran and Thangaraj [23] studied a single server queuing system with feedback and impatient customers.

The conventional reneging considered in the literature so far has the assumption that the reneging times happen to follow certain probability distribution and the reneging of the customers occur with certain rate. But, this assumption may not hold where the behavior of reneging customers can be bursty as this case may be possible in many practical scenarios. For example, consider a central system of an online shopping company where all the orders as well as the cancellation requests of orders are received. The arrival of orders is analogous to the arrival of customers, the dispatching of orders is analogous to the service of customers, and the orders cancelled before dispatching can be considered as reneging customers. A customer who visits a shopping site and does not find a satisfactory product may not place any order. This situation is similar to balking behavior of customers. The cancellation of orders could be abrupt or bursty at times because of the reasons like delay in delivery, some other online shopping companies start offering discounts, bad reviews about the products become viral etc. That is, if an order is cancelled at any time instant, then there is a probability that an order may or may not be cancelled at the next time instant. Similarly, if an order is not cancelled at any time instant, then there is a probability that an order may or may not be cancelled at the next time instant. This kind of reneging is referred to as correlated reneging, and is better than conventional reneging to capture the burntness. Sometimes it happens that the received product is below the expectations of the buyer and he feels unsatisfied, so he may put a request for re-order of the same product to get a new one. This situation resembles with the feedback in queuing theory and re-order of the same product can be considered as a feedback customer.

The literature survey shows that no work has appeared on correlated reneging till date. Moreover, because of the usefulness of the concept of correlated reneging as discussed in the previous paragraph we develop a single server queuing model with correlated reneging, balking, and feedback. We perform the transient numerical analysis of the queuing model. Rest of the paper is as follows: In section 2, the stochastic queuing model is described. In section 3, the mathematical model is presented. Section 4 deals with the transient analysis of the model. The sensitivity analysis of the model is presented in section 5. Finally, the paper is concluded in section 6.

II Queuing Model Description

The queueing model considered is based on the following assumptions: The customers arrive at a service facility one by one in accordance with Poisson process with parameter λ. There is a single queue and a single server. The service-times are independently, identically and exponentially distributed with parameter μ. On arrival, an incoming customer may decide not to join the queue (i.e. balk) with certain probability (say, $1-\beta$). This means that the arrival customer may join the queue with probability β. After being served, a customer either leaves the system with probability q or rejoins the queue as a feedback customer with complementary probability p=(1-q). The capacity of the system is finite (say, N). After joining the queue and waiting for sometime, a customer may get impatient and leave the queue(renege) without getting the service. The reneging of the customers can take place only at the transition marks $t_0, t_1, t_2, \dots$ where $\theta_r = t_r - t_{r-1}, r = 1,2,3\dots$, are random variables with $P[\theta_r \leq x] = 1 - exp(-\xi x); \xi > 0, r = 1,2,3, \dots$ That is, the distribution of inter-transition marks is negative exponential with parameter ξ. The reneging at two consecutive transition marks is governed by the following transition probability matrix:

$$\begin{array}{ccc} & & \text{to } t_r \\ & & \begin{array}{cc} 0 & 1 \end{array} \\ \text{from } t_{r-1} & \begin{array}{c} 0 \\ 1 \end{array} & \left\| \begin{array}{cc} p_{00} & p_{01} \\ p_{10} & p_{11} \end{array} \right\| \end{array} \text{ where } p_{00} + p_{01} = 1 \text{ and } p_{10} + p_{11} = 1$$

0 refers to no reneging and 1 refers to the occurrence of reneging.

Thus, the reneging at two consecutive transition marks is correlated.

III Mathematical Model

Defining the probabilities:

$Q_{0,0}(t)$ = Probability that at time t the queue is empty, the server is idle, and a customer has not reneged at the previous transition mark.

$Q_{0,1}(t)$ = Probability that at time t the queue is empty, the server is idle, and a customer has reneged at the previous transition mark.

$P_{0,0}(t)$ = Probability that at time t the queue is empty, the server is not idle, and a customer has not reneged at the previous transition mark.

$P_{0,1}(t)$ = Probability that at time t the queue is empty, the server is not idle, and a customer has reneged at the previous transition mark.

$P_{n,0}(t)$ = Probability that at time t the queue length is n , the server is not idle, and a customer has not reneged at the previous transition mark.

$P_{n,1}(t)$ = Probability that at time t the queue length is n, the server is not idle, and a customer has reneged at the previous transition mark.

$P_{N,0}(t)$ = Probability that at time t the queue length is N, the server is not idle, and a customer has not reneged at the previous transition mark.

$P_{N,1}(t)$ = Probability that at time t the queue length is N, the server is not idle, and a customer has reneged at the previous transition mark.

The differential equations of the model are:

$$\frac{d}{dt}Q_{0,0}(t) = -\lambda Q_{0,0}(t) + \mu q P_{0,0}(t) \quad (1)$$

$$\frac{d}{dt}P_{0,0}(t) = -(\lambda + \mu q)P_{0,0}(t) + \mu q P_{1,0} + \lambda Q_{0,0}(t) \quad (2)$$

$$\frac{d}{dt}P_{1,0}(t) = -(\lambda\beta + \mu q + n\xi)P_{1,0}(t) + \mu q P_{2,0}(t) + \lambda P_{0,0}(t)$$
$$+\xi[p_{00}P_{1,0}(t) + p_{10}P_{1,1}(t)] \quad (3)$$

$$\frac{d}{dt}P_{n,0}(t) = -(\lambda\beta + \mu q + n\xi)P_{n,0}(t) + \mu q P_{n+1,0}(t) + \lambda\beta P_{n-1,0}(t)$$
$$+n\xi[p_{00}P_{n,0}(t) + p_{10}P_{n,1}(t)], 1 < n < N \quad (4)$$

$$\frac{d}{dt}P_{N,0}(t) = -(\mu q + N\xi)P_{N,0}(t) + \lambda\beta P_{N-1,0}(t) + N\xi[p_{00}P_{N,0}(t)$$
$$+p_{10}P_{N,1}(t)] \quad (5)$$

$$\frac{d}{dt}Q_{0,1}(t) = -\lambda Q_{0,1}(t) + \mu q P_{0,1}(t) \quad (6)$$

$$\frac{d}{dt}P_{0,1}(t) = -(\lambda + \mu q)P_{0,1}(t) + \mu q P_{1,1} + \lambda Q_{0,1}(t) + \xi[p_{11}P_{1,1}(t)$$

$$+p_{01}P_{1,0}(t)] \tag{7}$$

$$\frac{d}{dt}P_{1,1}(t) = -(\lambda\beta + \mu q + n\xi)P_{1,1}(t) + \mu q P_{2,1}(t) + \lambda P_{0,1}(t)$$
$$+2\xi[p_{01}P_{2,0}(t) + p_{11}P_{2,1}(t)] \tag{8}$$

$$\frac{d}{dt}P_{n,1}(t) = -(\lambda\beta + \mu q + n\xi)P_{n,1}(t) + \mu q P_{n+1,1}(t) + \lambda\beta P_{n-1,1}(t)$$
$$+(n+1)\xi[p_{01}P_{n+1,0}(t) + p_{11}P_{n+1,1}(t)], 1 < n < N \tag{9}$$

$$\frac{d}{dt}P_{N,1}(t) = -(\mu q + N\xi)P_{N,1}(t) + \lambda\beta P_{N-1,1}(t) \tag{10}$$

IV Transient Analysis of the Model

In this section we perform the transient analysis of the model. We use the Runge-Kutta method of fourth order to obtain the transient solution as it is quite difficult to obtain analytical solution explicitly. The $''ode45''$ function of MATLAB software is used to compute the transient numerical results.

4.1 Performance measures

We study the following performance measures:

1. Expected system Size ($L_s(t)$):
$$L_s(t) = \sum_{n=0}^{N} (n+1)[P_{n,0}(t) + P_{n,1}(t)]$$

2. Expected waiting time in the system ($W_s(t)$):
$$W_s(t) = \frac{L_s(t)}{\mu(1 - Q_{0,0}(t) - Q_{0,1}(t))}$$

Where $L_s(t)$ is mean system size at time t.

Now, we illustrate the transient behaviour of the model with the help of a numerical example. We take $\lambda = 2.3, \mu = 2.9, \xi = 0.3, \beta = 0.8, q = 0.7, p_{00} = 0.8, p_{01} = 0.2, p_{10} = 0.7, p_{11} = 0.3, N = 6$. In figures 1 and 2 the system size probabilities are plotted against time. We can observe that all the probabilities increase to a certain extent and after sometime they become stationary. However, the probability $P_{0,0}(t)$ has highest value in the beginning and it decreases to a certain extent and after sometime it becomes stationary. This behaviour of $P_{0,0}(t)$ is due to initial condition, that is, $P_{0,0}(0) = 1$. In figure 3, the variation in expected system size is plotted against time. The expected system size gradually increases from the initial state and achieves a constant value after some time. In figure 4, the variation in expected system size is plotted against time. The expected system size gradually increases from the initial state and achieves a constant value after some time.

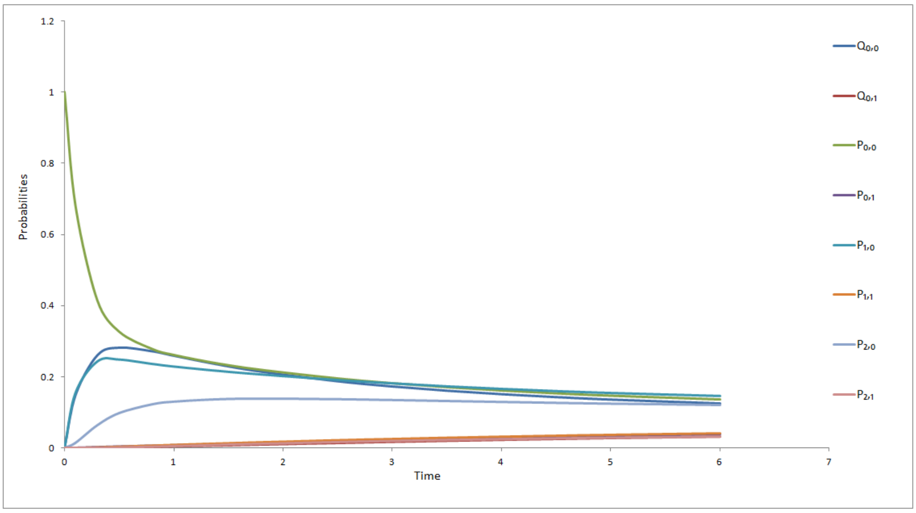

Figure 1: Probabilities vs Time

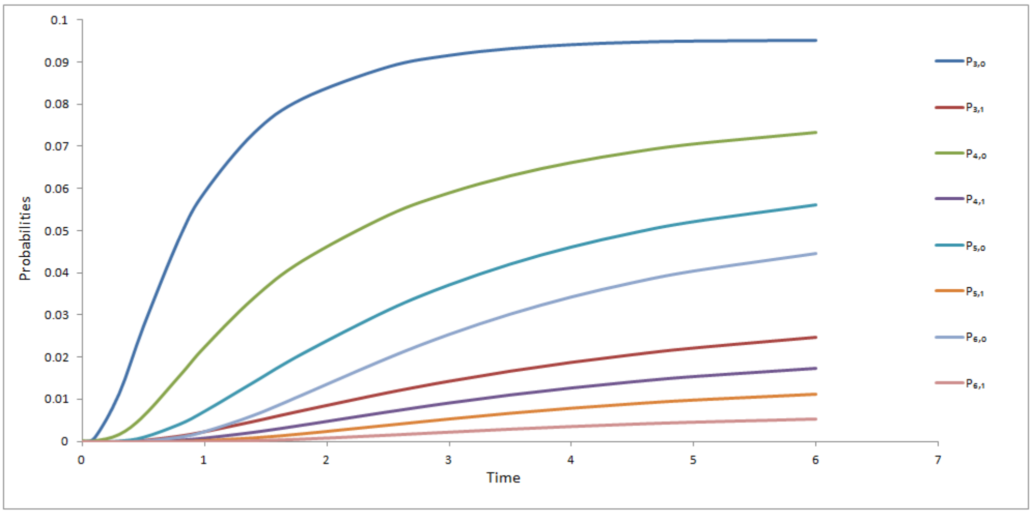

Figure 2: Probabilities vs Time

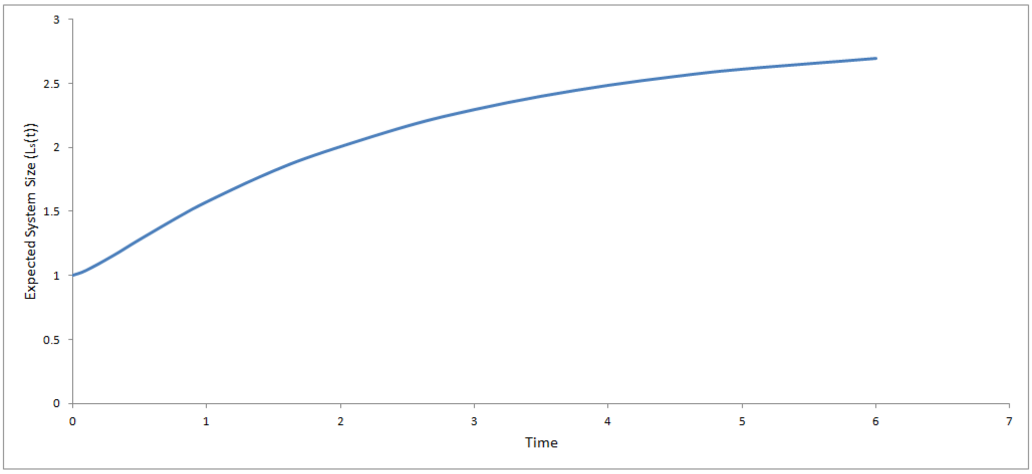

Figure 3: Expected system size vs Time

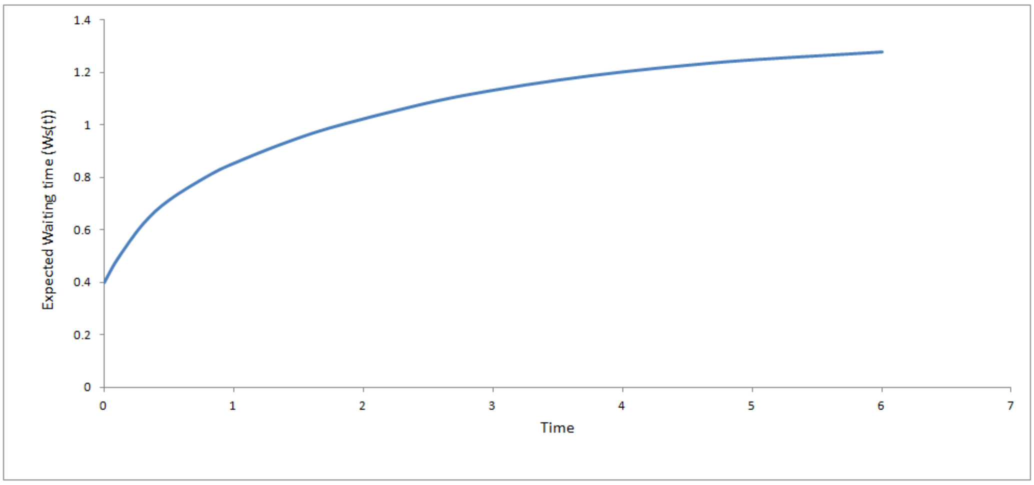

Figure 4: Expected waiting time vs Time

V Sensitivity analysis of the model

In this section, we study the variation in performance measures with respect to the change in system parameters. In table 1, the variation in expected system size and in expected waiting time with respect to mean arrival rate is presented. One can see that the performance measures decreases with the increase in mean arrival rate. The variation in performance measures with respect to mean service rate is shown in table 2. With the increase in the mean service rate the expected system size increases. Similar is the case with expected waiting time. In table 3, the variation in performance measure with respect to the probability p_{00} is studied. An increase in p_{00} leads to the increase in performance measures $L_s(t)$ and $W_s(t)$. Since $p_{01} = 1 - p_{00}$, the variation in performance measures is reverse for p_{01}. Table 4 deals with the changes in $L_s(t)$ and $W_s(t)$ with respect to change in the probability p_{10}. One can observe that the increase in p_{10} increases $L_s(t)$ and $W_s(t)$. The variations are in reverse order for probability $p_{11}(= 1 - p_{10})$. The variations in performance measures with respect to the change in feedback probability are presented in table 5. One can see that with the increase in feedback probability the measures of performance $L_s(t)$ and $W_s(t)$ show increasing trend. The increase in feedback probability means more number of feedback customers join the queue and thus increase the system size and hence the waiting time in the system also increases. The variations in performance measures with respect to the change in balking probability are presented in table 6. One can see that with the increase in balking probability the measures of performance $L_s(t)$ and $W_s(t)$ show decreasing trend. The increase in balking probability means more number of customers do not join the queue and thus decreases the system size and hence the waiting time in the system also decreases. The numerical results discussed in tables 1-6 describe the functioning of our model. font=normalsize,sf

Table 1: Variation in performance measures w.r.t. mean arrival rate
Here, $\mu = 3.3, \xi = 0.8, \beta = 0.8, q = 0.7, p_{00} = 0.8, p_{01} = 0.2, p_{10} = 0.7, p_{11} = 0.3, N = 6, t = 4.$

S. No.	Mean arrival rate (λ)	Expected system size ($L_s(t)$)	Expected waiting time ($W_s(t)$)
1	1.3	0.8674	0.5182
2	1.5	1.0563	0.5627
3	1.7	1.259	0.6102
4	1.9	1.4743	0.6607
5	2.1	1.7005	0.7139
6	2.3	1.9358	0.7694
7	2.5	2.1777	0.8268

Table 2: Variation in performance measures w.r.t. mean service rate
Here, $\lambda = 3.3, \xi = 0.8, \beta = 0.8, q = 0.7, p_{00} = 0.8, p_{01} = 0.2, p_{10} = 0.7, p_{11} = 0.3, N = 6, t = 4.$

S. No.	Mean service rate (μ)	Expected system size ($L_s(t)$)	Expected waiting time ($W_s(t)$)
1	3.1	2.3282	0.9161
2	3.3	2.1777	0.8268
3	3.5	2.0384	0.7502
4	3.7	1.9098	0.6839
5	3.9	1.7914	0.6265
6	4.1	1.6825	0.5762
7	4.3	1.5824	0.5321

Table 3: Variation in performance measures w.r.t. probability (p_{00})
Here, $\lambda = 2.5, \mu = 3.3, \xi = 0.9, \beta = 0.8, q = 0.7, p_{10} = 0.7, p_{11} = 0.3, N = 6, t = 4$.

S. No.	Probability (p_{00})	Expected system size $(L_s(t))$	Expected waiting time $(W_s(t))$
1	0.1	1.7244	0.6893
2	0.2	1.7549	0.6985
3	0.3	1.7917	0.7097
4	0.4	1.8367	0.7233
5	0.5	1.8925	0.7403
6	0.6	1.9633	0.7618
7	0.7	2.0551	0.7897
8	0.8	2.1777	0.8268
9	0.9	2.3465	0.8777

Table 4: Variation in performance measures w.r.t. probability (p_{10})
Here, $\lambda = 2.5, \mu = 3.3, \xi = 0.8, \beta = 0.8, q = 0.7, p_{00} = 0.8, p_{01} = 0.2, N = 6, t = 4$.

S. No.	Probability (p_{10})	Expected system size $(L_s(t))$	Expected waiting time $(W_s(t))$
1	0.1	1.9361	0.7541
2	0.2	1.9899	0.7705
3	0.3	2.0371	0.7848
4	0.4	2.0788	0.7973
5	0.5	2.1157	0.8083
6	0.6	2.1484	0.8181
7	0.7	2.1777	0.8268
8	0.8	2.2038	0.8346
9	0.9	2.2273	0.8416

Table 5: Variation in performance measures w.r.t. probability of feedback p
Here, $\mu = 3.3, \xi = 0.8, \beta = 0.8, q = 0.7, p_{00} = 0.8, p_{01} = 0.2, p_{10} = 0.7, p_{11} = 0.3, N = 6, t = 4$.

S. No.	Probability of feedback (p)	Expected system size $(L_s(t))$	Expected waiting time $(W_s(t))$
1	0.1	1.6101	0.6966
2	0.2	1.8664	0.7572
3	0.3	2.1777	0.8268
4	0.4	2.5511	0.9107
5	0.5	2.9909	1.0112
6	0.6	3.4952	1.1303
7	0.7	4.0537	1.2687
8	0.8	4.6453	1.4246
9	0.9	5.2388	1.5914

Table 6: Variation in performance measures w.r.t. probability of balking $1-\beta$
Here, $\lambda = 3.3, \xi = 0.8, \beta = 0.8, q = 0.7, p_{00} = 0.8, p_{01} = 0.2, p_{10} = 0.7, p_{11} = 0.3, N = 6, t = 4.$

S. No.	Probability of balking $(1-\beta)$	Expected system size $(L_s(t))$	Expected waiting time $(W_s(t))$
1	0.1	2.4152	0.8890
2	0.2	2.1777	0.8268
3	0.3	1.9561	0.7581
4	0.4	1.7547	0.6943
5	0.5	1.5764	0.6369
6	0.6	1.4228	0.5866
7	0.7	1.2935	0.5439
8	0.8	1.1868	0.5082
9	0.9	1.0099	0.4791

VI Conclusion

In this paper we have performed the transient numerical analysis of a single server queuing model with correlated reneging, balking and feedback. Sensitivity analysis has also been performed.

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