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SPECIAL ISSUE:

QUEUING AND RELIABILITY
MODELS WITH THEIR
APPLICATIONS

San Diego

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Special Issue: Queuing and Reliability Models
with their applications

San Diego
2019

Editorial Board

e-Journal ***Reliability: Theory & Applications*** publishes papers, reviews, memoirs, and bibliographical materials on Reliability, Quality Control, Safety, Survivability and Maintenance.

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Priority is given to descriptions of case studies.
General requirements for presented papers

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Editorial. Special issue on Queuing and Reliability Models with their Applications

Guest Editors:

Dr. Pawan Kumar
(Professor), Department of
Statistics, University of Jammu, Jammu, India.
Email: pkk_skumar@yahoo.co.in

Dr. Rakesh Kumar
(Assistant Professor), School
of Mathematics, Shri Mata Vaishno Devi
University, Katra, Jammu and Kashmir, India.
Email: rakesh.kumar@smvdu.ac.in

This special issue is the outcome of the selected papers presented in the twenty-seventh international conference of Forum for Interdisciplinary Mathematics (FIM) in conjunction with third convention of IARS on Interdisciplinary Mathematics, Statistics and Computational Techniques (IMSCT 2018–FIM XXVII) which is held in the Department of Statistics, University of Jammu, Jammu & Kashmir, India during November 02-04, 2018.

In the paper, 'A Discrete Parametric Markov-Chain Model of a Two Unit Cold Standby System with Appearance and Disappearance of Repairman' by Rakesh Gupta and Parul Bhardwaj, they have performed the stochastic analysis of a two identical unit system model in which repairman does not always remain with the system and appears and disappears at random epochs. In this paper authors have taken discrete failure and repair time distribution, a rare consideration in reliability models. All the failure time distributions, repair time distributions and the appearance and disappearance time distributions of repairman have been considered geometric with different parameters. Graphs have been drawn for profit function and mean time to system failure to give more clarity about the behaviour of performance measures of the system model in respect to different parameters of various distributions considered in the study.

The paper, 'A Discrete Parametric Markov-Chain Model of a Two-Unit Cold Standby System with Repair Efficiency Depending on Environment' by Rakesh Gupta and Arti Tyagi deals with a two-unit cold standby system with repair efficiency depending on environmental conditions. The repairman at the time of need may be in poor or good physical condition. All the failure, repair and change of environment conditions time distributions are taken to be geometric with different parameters. Various measures of system effectiveness are obtained and behaviour of some of them is explained through graphs.

In the paper, 'A Two Identical Unit Cold Standby System Subject to Two Types of Failures' by Pradeep Chaudhary and Rashmi Tomar Authors have performed the stochastic analysis of a two identical unit system model with two types of failure viz. normal failure and failure due to chance causes which may occur randomly and are beyond human control. Several reliability measures of system performance have been obtained by taking failure time distribution as exponential and repair time distribution as general. Graphical study of some of the obtained reliability characteristics is also performed.

In the paper, 'Analysis of Reliability Measures of Two Identical Unit System with One Switching Device and Imperfect Coverage' by Akshita Sharma and Pawan Kumar a two identical unit system model with safe and unsafe failures, switching device and rebooting is investigated and analysed. The purpose of rebooting is to convert the unsafe failures in to safe failures and make system ready for repair. A repaired and standby unit is put into operation through a switching device. All the failure time distributions are considered to be exponentially distributed and repair time distribution as arbitrary. Regenerative point technique is used to perform the reliability analysis, besides finding the expressions for important reliability characteristics their behaviour w.r.t. failure and repair parameters has also been studied.

In the paper, 'Performance Measures of a Two Non-Identical Unit System Model With Repair and Replacement Policies' by Urvashi Gupta and Pawan Kumar, a two non-identical unit system model with repair and replacement policies has been developed for its stochastic analysis. Here authors have considered that a unit gives an indication of failure before it actually fails and the possible steps may be taken to prevent its failure. Also a failed unit needs some preparation time (which is a random variable) to start its repair. All the failure time distributions are taken as exponential and repair time distribution as general. Reliability characteristics useful to system manager have been found and their nature and pattern of variation for varying values of failure and repair parameters have been studied through graphs.

The paper, 'Assessment and Prediction of Reliability of an Automobile Component Using Warranty Claims Data' by Tahsina Aziz and M. Rezaul Karim, an analysis of warranty claims data of a component of an automobile is performed. The objectives of the analysis are to assess and predict the reliability of the component. To do this they present non-parametric and parametric analyses for the lifetime variable, age in month, based on warranty claims data. It also investigates on the variation of reliability of the component with respect to month of production and dominant failure modes. The work could be useful to the manufacturers for assessing and predicting reliability and warranty costs and for assuring customer satisfaction and product reputation.

The paper by Gulab Singh Bura and Shilpi Gupta entitled, 'Time Dependent Analysis of an M/M/2/N Queue with Catastrophes' considers a Markovian queueing system with two identical servers subjected to catastrophes. When the system is not empty, catastrophes may occur and destroy all present customers in the system. Simultaneously the system is ready for new arrivals. The time dependent and the steady-state solutions are obtained explicitly. Further, they have obtained some important performance measures of the queueing model.

In the paper, 'Cost-profit Analysis of Stochastic Heterogeneous Queue with Reverse Balking, Feedback and Retention of Impatient Customers' by Bhupender Kumar Som, the author has developed feedback queuing system with heterogeneous service, reverse balking, reneging and retention. The model is solved in steady-state recursively. Necessary measures of performance are drawn. Numerical interpretation of the model is presented. Cost-profit analysis of the system is performed by developing a cost model. Sensitivity analysis of the model is also presented arbitrarily.

Finally, in the paper, 'Transient Analysis of a Single-Server Queuing System with Correlated Inputs and Reneging' by Rakesh Kumar and Bhavneet Singh Soodan, the authors study a continuous-time single-server queuing system, wherein the arrivals at two consecutive transition marks are correlated. The service times and the reneging times are exponential distributed. The time-dependent behavior of the model is studied using Runge-Kutta method.

At last, the guest editors would like to acknowledge the contributions of all those professors who helped them to bring this special issue in current form. The guest editors would also like to thank the Editor-in-Chief, and also the Managing Editor, Prof. Alexander Bochkov of the journal Reliability: Theory and Applications for providing a platform for publishing the research papers in special issue.

A Discrete Parametric Markov-Chain Model Of A Two Unit Cold Standby System With Appearance And Disappearance Of Repairman

Rakesh Gupta, Parul Bhardwaj

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Department of Statistics,
Ch. Charan Singh University, Meerut
E-mails: smprgccsu@gmail.com,
parul.bhardwaaj@gmail.com

Abstract

The paper deals with the stochastic analysis of a two-identical unit cold standby system model assuming two modes- normal and total failure. A single repair facility appears in and disappears from the system randomly. The random variables denoting the failure time, repair time, time to appearance and time to disappearance of repairman are independent of discrete nature having geometric distributions with different parameters. The various measures of system effectiveness are obtained by using regenerative point technique.

Keywords: Transition probability, mean sojourn time, regenerative point, reliability, MTSF, availability of system, busy period of repairman.

1. Introduction

In reliability modeling repair maintenance is concerned with increasing system reliability, availability and net expected profit earned by the system with the implementation of major changes in the failed components of a unit. In order to achieve this goal, the failed components of a unit may be either repaired or replaced by new ones.

Numerous authors [1, 5, 6, 7, 8] have analyzed various system models considering different repair policies. Goyal and Murari [1] analyzed a two-identical unit standby system model considering two types of repairmen- regular and expert. The regular repairman is always available with the system whereas expert repairman can be made available from the outside instantaneously. Mokaddis et. al. [6] obtained the busy period analysis of a man-machine system model assuming different physical conditions of repairman. Pandey and Gupta [7] analyzed a two-unit standby redundant system model assuming that a delay occurs due to some administrative action in locating and getting repairman available to the system. Sharma et. al. [8] considered a two-unit parallel system assuming dependent failure rates and correlated working and rest time of repairman.

Gupta et. al. [5] investigated a two unit standby system with correlated failure and repair and random appearance and disappearance of repairman. They have also assumed that the failure time, repair time, time to appearance and time to disappearance of a repairman are continuous random variables. Few works by the authors [3, 4, 5] is carried out in the literature of reliability analyzing the system models taking geometric failure and repair time distributions.

The purpose of the present paper is to analyze a two-unit cold standby system model with appearance and disappearance of repairman under discrete parametric Markov-Chain i.e. **failure and repair times and appearance and disappearance times of repairman follow geometric distributions with different parameters.** The phenomena of discrete failure and repair time distributions may be observed in the following situation.

Let the continuous time period $(0,\infty)$ is divided as 0, 1, 2,…, n,… of equal distance on real line and the probability of failure of a unit during time (i, i+1); i = 0, 1, 2,….. is p, then the probability that the unit will fail during (t, t+1) i.e. after passing successfully t intervals of time is given by $p(1-p)^t$; t = 0, 1, 2,…. This is the p.m.f of geometric distribution. Similarly, if r denotes the probability that a failed unit is repaired during (i, i+1); i = 0, 1, 2,... then the probability that the unit will be repaired during (t, t+1) is given by $r(1-r)^t$; t = 0, 1, 2,…. On the same way, the random variables denoting appearance and disappearance of repairman may follow geometric distributions.

The following economic related measures of system effectiveness are obtained by using regenerative point technique-

i) Transition probabilities and mean sojourn times in various states.
ii) Reliability and mean time to system failure.
iii) Point-wise and steady-state availabilities of the system as well as expected up time of the system during interval (0, t).
iv) Expected busy period of the repairman during time interval (0, t).
v) Net expected profit earned by the system during a finite interval and in steady-state.

1. Model Description and Assumptions

i) The system comprises of two-identical units. Initially, one unit is operative and other is kept into cold standby.
ii) Each unit of the system has two modes- Normal (N) and Total failure (F).
iii) There is a single repair facility which appears in and disappears from the system randomly. Once the repairman starts the repair of a failed unit, he does not leave the system till all the units are repaired that failed during his stay in the system.
iv) All random variables denoting failure time, repair time, time to appearance and disappearance of repairman are independent of discrete nature and follow geometric distributions with different parameters.
v) The system failure occurs when both the units are in total failure mode.
vi) The repaired unit works as good as new.

2. Notations and States of the System

a) Notations:

pq^t : p.m.f. of failure time of an operating unit $(p+q=1)$.

rs^t : p.m.f. of repair time of a failed unit $(r+s=1)$.

ab^t	:	p.m.f. of disappearance of repairman from the system $(a+b=1)$.
cd^t	:	p.m.f. of appearance of repairman in the system $(c+d=1)$.
$q_{ij}(\cdot), Q_{ij}(\cdot)$	:	p.m.f. and c.d.f. of one step or direct transition time from state S_i to S_j.
p_{ij}	:	Steady state transition probability from state S_i to S_j. $p_{ij} = Q_{ij}(\infty)$
$Z_i(t)$	:	Probability that the system sojourns in state S_i at epochs 0, 1, 2,……,up to (t-1).
ψ_i	:	Mean sojourn time in state S_i.
$*, h$	:	Symbol and dummy variable used in geometric transform e. g. $GT\left[q_{ij}(t)\right] = q_{ij}^*(h) = \sum_{t=0}^{\infty} h^t q_{ij}(t)$
©	:	Symbol for ordinary convolution i.e. $A(t)©B(t) = \sum_{u=0}^{t} A(u)B(t-u)$

b) Symbols for the states of the system:

N_O / N_S : Unit in normal (N) mode and operative/standby.

F_r / F_w : Unit in total failure (F) mode and under repair/waiting for repair

A/ NA : Repairman is available/not available with the system.

TRANSITION DIAGRAM

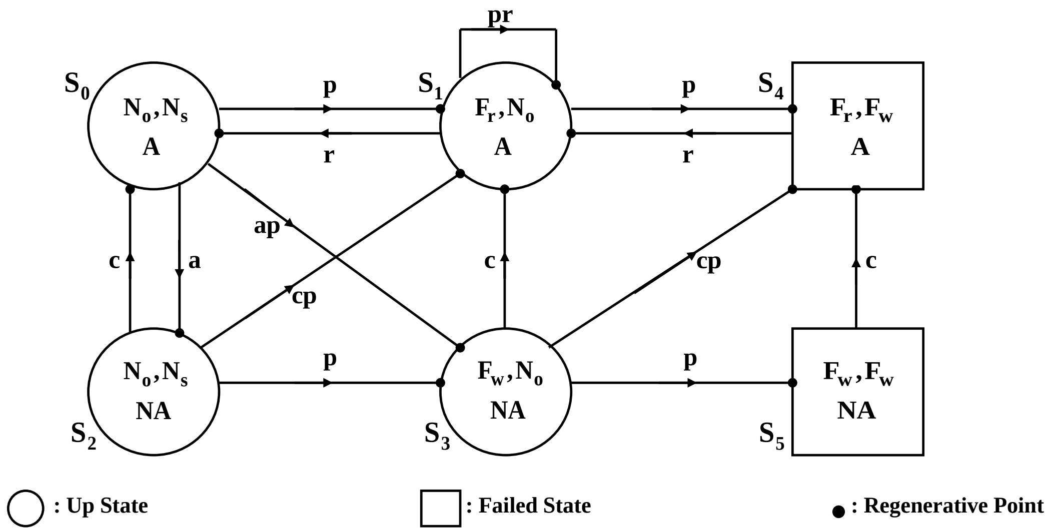

Fig.1

With the help of above symbols the possible states of the system along with failure and repair rates are shown in the transition diagram (Fig.1)

3. Transition Probabilities

Let $Q_{ij}(t)$ be the probability that the system transits from state S_i to S_j during time interval (0, t) i.e., if T_{ij} is the transition time from state S_i to S_j then $Q_{ij}(t)=P\left[T_{ij}\le t\right]$. In particular, $Q_{01}(t)$ is the probability that the operative unit fails at an epoch u between (0,t) and repairman does not disappear up to the epoch u.

$Q_{01}(t)=\sum_{u=0}^{t}$ P (Operative unit fails at epoch u) X P (Repairman does not disappear up to the epoch u) $=\sum_{u=0}^{t} pq^u b^{u+1}=\frac{bp}{(1-bq)}\left\{1-(bq)^{t+1}\right\}$

Similarly,

$$Q_{02}(t)=\frac{aq}{(1-bq)}\left\{1-(bq)^{t+1}\right\},\qquad Q_{03}(t)=\frac{ap}{(1-bq)}\left\{1-(bq)^{t+1}\right\}$$

$$Q_{10}(t)=\frac{qr}{(1-qs)}\left\{1-(qs)^{t+1}\right\},\qquad Q_{11}(t)=\frac{pr}{(1-qs)}\left\{1-(qs)^{t+1}\right\}$$

$$Q_{14}(t)=\frac{ps}{(1-qs)}\left\{1-(qs)^{t+1}\right\},\qquad Q_{20}(t)=\frac{cq}{(1-dq)}\left\{1-(dq)^{t+1}\right\}$$

$$Q_{21}(t)=\frac{cp}{(1-dq)}\left\{1-(dq)^{t+1}\right\},\qquad Q_{23}(t)=\frac{dp}{(1-dq)}\left\{1-(dq)^{t+1}\right\}$$

$$Q_{31}(t)=\frac{cq}{(1-dq)}\left\{1-(dq)^{t+1}\right\},\qquad Q_{34}(t)=\frac{cp}{(1-dq)}\left\{1-(dq)^{t+1}\right\}$$

$$Q_{35}(t)=\frac{dp}{(1-dq)}\left\{1-(dq)^{t+1}\right\},\qquad Q_{41}(t)=\left(1-s^{t+1}\right)$$

$$Q_{54}(t)=\left(1-d^{t+1}\right)\qquad (1-14)$$

The steady state transition probabilities from state S_i to S_j can be obtained from $(1-14)$ by taking $t\to\infty$, as follows:

$$p_{01}=\frac{bp}{(1-bq)},\quad p_{02}=\frac{aq}{(1-bq)},\quad p_{03}=\frac{ap}{(1-bq)},\quad p_{10}=\frac{qr}{(1-qs)}$$

$$p_{11}=\frac{pr}{(1-qs)},\quad p_{14}=\frac{ps}{(1-qs)},\quad p_{20}=\frac{cq}{(1-dq)},\quad p_{21}=\frac{cp}{(1-dq)}$$

$$p_{23}=\frac{dp}{(1-dq)},\quad p_{31}=\frac{cq}{(1-dq)},\quad p_{34}=\frac{cp}{(1-dq)},\quad p_{35}=\frac{dp}{(1-dq)}$$

$$p_{41}=1,\quad p_{54}=1$$

We observe that the following relations hold-

$$p_{01}+p_{02}+p_{03}=1,\quad p_{10}+p_{11}+p_{14}=1,\quad p_{20}+p_{21}+p_{23}=1$$

$$p_{31}+p_{34}+p_{35}=1,\quad p_{41}=p_{54}=1\qquad (15-19)$$

4. Mean Sojourn Times

Let ψ_i be the sojourn time in state S_i (i = 0, 1, 2, 3, 4, 5) then mean sojourn time in state S_i is given by

$$\psi_i = \sum_{t=1}^{\infty} P\left[T \geq t\right]$$

In particular,

$$\psi_0 = \frac{bq}{(1-bq)}, \qquad \psi_1 = \frac{qs}{(1-qs)}, \qquad \psi_2 = \psi_3 = \frac{dq}{(1-dq)} = \psi \text{, say}$$

$$\psi_4 = \frac{s}{r}, \qquad \psi_5 = \frac{d}{c} \tag{20-24}$$

The evaluation of steady-state transition probabilities and mean sojourn time play the vital role as the various measures of system effectiveness are obtained in these terms.

5. Methodology for Developing Equations

In order to obtain various interesting measures of system effectiveness we develop the recurrence relations for reliability, availability and busy period of repairman as follows-

a) Reliability of the system

Here we define $R_i(t)$ as the probability that the system does not fail up to t epochs 0, 1, 2,….,(t-1) when it is initially started from up state S_i. To determine it, we regard the failed states S_4 and S_5 as absorbing states. Now, the expressions for $R_i(t)$; i = 0, 1, 2, 3; we have the following set of convolution equations.

$$R_0(t) = b^t q^t + \sum_{u=0}^{t-1} q_{01}(u) R_1(t-1-u) + \sum_{u=0}^{t-1} q_{02}(u) R_2(t-1-u) + \sum_{u=0}^{t-1} q_{03}(u) R_3(t-1-u)$$
$$= Z_0(t) + q_{01}(t-1) © R_1(t-1) + q_{02}(t-1) © R_2(t-1) + q_{03}(t-1) © R_3(t-1)$$

Similarly,

$$R_1(t) = Z_1(t) + q_{10}(t-1) © R_0(t-1) + q_{11}(t-1) © R_1(t-1)$$
$$R_2(t) = Z_2(t) + q_{20}(t-1) © R_0(t-1) + q_{21}(t-1) © R_1(t-1) + q_{23}(t-1) © R_3(t-1)$$
$$R_3(t) = Z_3(t) + q_{31}(t-1) © R_1(t-1)$$
$$(25-28)$$

Where,

$$Z_1(t) = b^t q^t, \qquad Z_2(t) = Z_3(t) = d^t q^t = Z(t) \text{, say}$$

b) Availability of the system

Let $A_i(t)$ be the probability that the system is up at epoch (t-1), when it initially starts from state S_i. By using simple probabilistic arguments, as in case of reliability the following recurrence relations can be easily developed for $A_i(t)$; i = 0 to 5.

$$A_0(t) = Z_0(t) + q_{01}(t-1)©A_1(t-1) + q_{02}(t-1)©A_2(t-1) + q_{03}(t-1)©A_3(t-1)$$
$$A_1(t) = Z_1(t) + q_{10}(t-1)©A_0(t-1) + q_{11}(t-1)©A_1(t-1) + q_{14}(t-1)©A_4(t-1)$$
$$A_2(t) = Z(t) + q_{20}(t-1)©A_0(t-1) + q_{21}(t-1)©A_1(t-1) + q_{23}(t-1)©A_3(t-1)$$
$$A_3(t) = Z(t) + q_{31}(t-1)©A_1(t-1) + q_{34}(t-1)©A_4(t-1) + q_{35}(t-1)©A_5(t-1)$$
$$A_4(t) = q_{41}(t-1)©A_1(t-1)$$
$$A_5(t) = q_{54}(t-1)©A_4(t-1)$$

$$(29-34)$$

Where,

The values of $Z_i(t)$; i = 0, 1 and $Z(t)$ are same as given in section 6(a).

c) Busy period of repairman

Let $B_i(t)$ be the respective probability that the repairman is busy at epoch (t-1) in the repair of each unit, when system initially starts from S_i. Using simple probabilistic arguments as in case of reliability, the recurrence relations for $B_i(t)$; i = 0 to 5 can be easily developed as below-

$$B_0(t) = q_{01}(t-1)©B_1(t-1) + q_{02}(t-1)©B_2(t-1) + q_{03}(t-1)©B_3(t-1)$$
$$B_1(t) = Z_1(t) + q_{10}(t-1)©B_0(t-1) + q_{11}(t-1)©B_1(t-1) + q_{14}(t-1)©B_4(t-1)$$
$$B_2(t) = q_{20}(t-1)©B_0(t-1) + q_{21}(t-1)©B_1(t-1) + q_{23}(t-1)©B_3(t-1)$$
$$B_3(t) = q_{31}(t-1)©B_1(t-1) + q_{34}(t-1)©B_4(t-1) + q_{35}(t-1)©B_5(t-1)$$
$$B_4(t) = Z_4(t) + q_{41}(t-1)©B_1(t-1)$$
$$B_5(t) = q_{54}(t-1)©B_4(t-1)$$

$$(35-40)$$

Where,

$Z_1(t)$ has the same values as in section 6(a) and $Z_4(t) = s^t$.

6. Analysis of Characteristics

a) Reliability and MTSF

Taking geometric transforms of $(6.1-6.4)$ and simplifying the resulting set of algebraic equations for $R_0^*(h)$, we get

$$R_0^*(h) = \frac{N_1(h)}{D_1(h)} \qquad (41)$$

Where,

$$N_1(h) = (1 - hq_{11}^*)\left[Z_0^* + hq_{02}^* Z^* + (hq_{03}^* + h^2 q_{02}^* q_{23}^*)Z^*\right]$$

$$+\left[hq_{01}^{*}+hq_{02}^{*}\left(hq_{21}^{*}+h^{2}q_{23}^{*}q_{31}^{*}\right)+h^{2}q_{03}^{*}q_{31}^{*}\right]Z_{1}^{*}$$

$$D_{1}(h)=\left(1-hq_{11}^{*}\right)\left(1-h^{2}q_{02}^{*}q_{20}^{*}\right)-hq_{10}^{*}\left[hq_{01}^{*}+hq_{02}^{*}\left(hq_{21}^{*}+h^{2}q_{23}^{*}q_{31}^{*}\right)+h^{2}q_{03}^{*}q_{31}^{*}\right]$$

Collecting the coefficient of h^{t} from expression (7.1), we can get the reliability of the system $R_{0}(t)$. The MTSF is given by-

$$E(T)=\lim_{h\to 1}\sum_{t=1}^{\infty}h^{t}R(t)=\frac{N_{1}(1)}{D_{1}(1)}-1 \tag{42}$$

Where, on noting that $q_{ij}^{*}(1)=p_{ij}$, $Z_{i}^{*}(0)=\psi_{i}$, we have

$$N_{1}(1)=(1-p_{11})\left[\psi_{0}+\left\{p_{03}+p_{02}(1+p_{23})\right\}\psi\right]+\left[p_{01}+p_{02}(p_{21}+p_{23}p_{31})+p_{03}p_{31}\right]\psi_{1}$$

$$D_{1}(1)=(1-p_{11})(1-p_{02}p_{20})-p_{10}\left[p_{01}+p_{02}(p_{21}+p_{23}p_{31})+p_{03}p_{31}\right]$$

b) Availability Analysis. On taking geometric transform of $(6.5-6.10)$ and simplifying the resulting equations, we get

$$A_{0}^{*}(h)=\frac{N_{2}(h)}{D_{2}(h)} \tag{43}$$

Where,

$$N_{2}(h)=\left(1-hq_{11}^{*}-h^{2}q_{14}^{*}q_{41}^{*}\right)\left[Z_{0}^{*}+hq_{12}^{*}Z^{*}+\left(hq_{03}^{*}+h^{2}q_{02}^{*}q_{23}^{*}\right)Z^{*}\right]$$
$$+\left[\left(hq_{01}^{*}+h^{2}q_{02}^{*}q_{21}^{*}\right)+\left\{hq_{31}^{*}+hq_{41}^{*}\left(hq_{03}^{*}+h^{2}q_{02}^{*}q_{23}^{*}\right)\right\}\left(hq_{03}^{*}+h^{2}q_{02}^{*}q_{23}^{*}\right)\right]Z_{1}^{*}$$

$$D_{2}(h)=\left(1-hq_{11}^{*}-h^{2}q_{14}^{*}q_{41}^{*}\right)\left(1-h^{2}q_{02}^{*}q_{20}^{*}\right)-hq_{10}^{*}\left[\left(hq_{01}^{*}+h^{2}q_{02}^{*}q_{21}^{*}\right)\right.$$
$$\left.+\left\{hq_{31}^{*}+hq_{41}^{*}\left(hq_{34}^{*}+h^{2}q_{35}^{*}q_{54}^{*}\right)\right\}\left(hq_{03}^{*}+h^{2}q_{02}^{*}q_{23}^{*}\right)\right]$$

The steady state availability of the system is given by-

$$A_{0}=\lim_{t\to\infty}A_{0}(t)=\lim_{h\to 1}(1-h)\frac{N_{2}(h)}{D_{2}(h)}$$

As $D_{2}(h)$ at h=1 is zero, therefore by applying L. Hospital rule, we get

$$A_{0}=-\frac{N_{2}(1)}{D_{2}'(1)} \tag{44}$$

Where,

$$N_{2}(1)=p_{10}\left[\psi_{0}+p_{02}\psi+(p_{03}+p_{02}p_{23})\psi\right]+(1-p_{02}p_{20})\psi_{1}$$

and

$$D_{2}'(1)=p_{10}\left[\psi_{0}+p_{02}\psi+(p_{03}+p_{02}p_{23})\psi+p_{35}\psi_{5}+(1-p_{31})\psi_{4}\right]$$
$$+(1-p_{02}p_{20})(\psi_{1}+p_{14}\psi_{4})$$

Now the expected up time of the system up to epoch (t-1) is given by

$$\mu_{up}(t)=\sum_{x=0}^{t-1}A_{0}(x)$$

so that

$$\mu_{up}^{*}(h)=\frac{A_0^{*}(h)}{(1-h)} \tag{45}$$

c) Busy Period Analysis. On taking geometric transforms of $(6.11-6.16)$ and simplifying the resulting equations, we get

$$B_0^{*}(h)=\frac{N_3(h)}{D_2(h)} \tag{46}$$

Where,

$$\begin{aligned} N_3(h)=&\Big[\left(hq_{01}^{*}+h^2q_{02}^{*}q_{21}^{*}\right)+\left\{hq_{31}^{*}+hq_{41}^{*}\left(hq_{03}^{*}+h^2q_{02}^{*}q_{23}^{*}\right)\right\}\left(hq_{03}^{*}+h^2q_{02}^{*}q_{23}^{*}\right)\Big]Z_1^{*}\\ &+\Big[hq_{14}^{*}\left(hq_{01}^{*}+h^2q_{03}^{*}q_{31}^{*}\right)+\left(1-q_{11}^{*}\right)\left(hq_{34}^{*}+h^2q_{35}^{*}q_{54}^{*}\right)\left(hq_{03}^{*}+h^2q_{02}^{*}q_{23}^{*}\right)\\ &+h^2q_{02}^{*}q_{14}^{*}\left(hq_{21}^{*}+h^2q_{23}^{*}q_{31}^{*}\right)\Big]Z_4^{*}\end{aligned}$$

and $D_2(h)$ is same as in availability analysis.

In the long run, the respective probability that the repairman is busy in the repair of each unit is given by-

$$B_0=\lim_{t\to\infty}B_0(t)=\lim_{h\to 1}(1-h)\frac{N_3(h)}{D_2(h)}$$

But $D_2(h)$ at h=1 is zero, therefore by applying L. Hospital rule, we get

$$B_0=-\frac{N_3(1)}{D_2'(1)} \tag{47}$$

Where,

$$N_3(1)=(1-p_{02}p_{20})(\psi_1+p_{14}\psi_4)+p_{10}(1-p_{31})(p_{03}+p_{02}p_{23})$$

and $D_2'(1)$ is same as in availability analysis.

The expected busy periods of the repairman in the repair of both units up to epoch $(t-1)$ is given by-

$$\mu_b(t)=\sum_{x=0}^{t-1}B_0(x)$$

So that,

$$\mu_b^{*}(h)=\frac{B_0^{*}(h)}{(1-h)} \tag{48}$$

7. Profit Function Analysis

We are now in the position to obtain the net expected profit incurred up to epoch (t-1) by considering the characteristics obtained in earlier sections.
Let us consider,

K_0 = revenue per-unit time by the system when it is operative.

K_1 = cost per-unit time when repairman is busy in the repair of the failed units.

Then, the net expected profit incurred up to epoch (t-1) is given by

$$P(t) = K_0\mu_{up}(t) - K_1\mu_b(t) \tag{49}$$

The expected profit per unit time in steady state is as follows-

$$P = \lim_{t\to\infty}\frac{P(t)}{t}$$

$$= K_0 \lim_{h\to1}(1-h)^2\frac{A_0^*(h)}{(1-h)} - K_1\lim_{h\to1}(1-h)^2\frac{B_0^*(h)}{(1-h)}$$

$$= K_0A_0 - K_1B_0 \tag{50}$$

8. Graphical Representation

The curves for MTSF and profit function have been drawn for different values of parameters p, a, c Fig. 2 depicts the variations in MTSF with respect to the rate of appearance of repairman (c) in the system for three different values of failure rate (p = 0.08, 0.10, 0.12) of an operative unit and two different values of rate of disappearance of repairman from the system (a = 0.5, 0.6). From these curves we observe that MTSF increases uniformly as the value of c increases. It also reveals that the MTSF decreases with the increase in p and decreases with the increase in a.

Similarly, Fig. 3 reveals the variations in profit (P) with respect to c for varying values of p and a, when the values of other parameters are kept fixed as r = 0.5, K_0=15 and K_1=80. From the curves we observe that profit decreases uniformly as the value of c increases. It also reveals that the profit decreases with the increase in p and decreases with the increase in a. From this figure it is clear from the dotted curves that the system is profitable only if the rate of appearance of repairman (c) in the system is less than 0.21, 0.41 and 0.70 respectively for p = 0.08, 0.10, and 0.12 for fixed value of a = 0.5. From smooth curves, we conclude that the system is profitable only if c is less than 0.19, 0.35 and 0.56 respectively for p = 0.08, 0.10, and 0.12 for fixed value of a = 0.

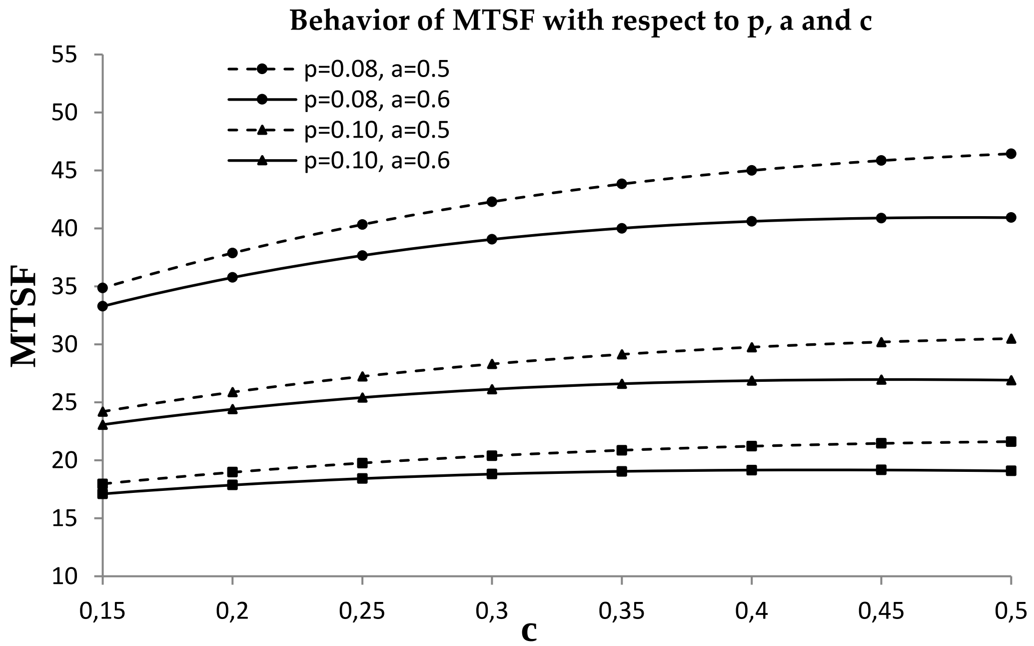

Fig. 2

Behavior of Profit (P) with respect to p, a and c

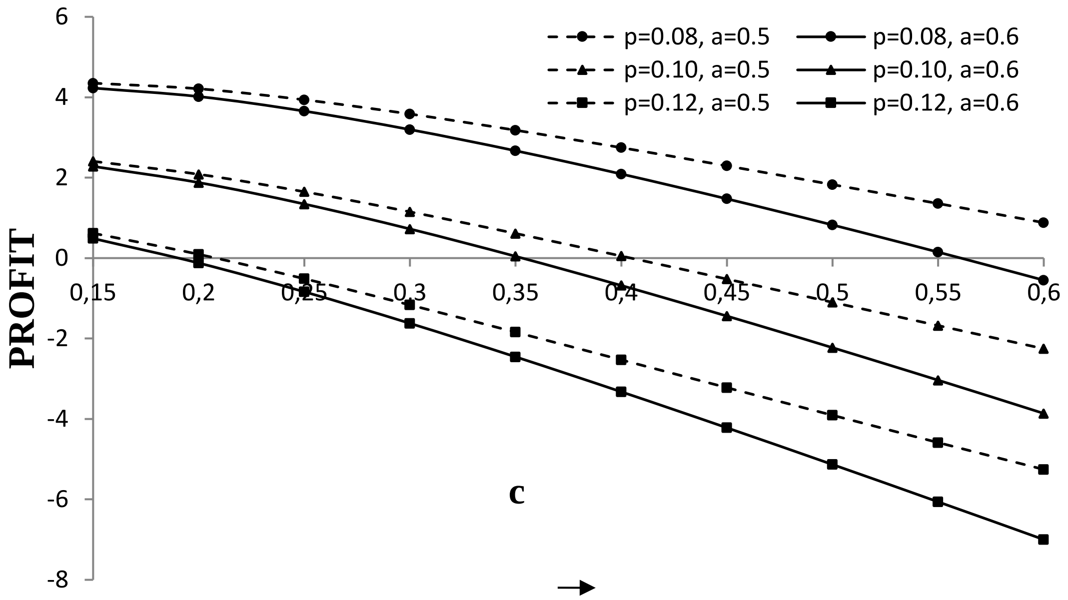

Fig. 3

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A Discrete Parametric Markov-Chain Model Of A Two-Unit Cold Standby System With Repair Efficiency Depending On Environment

Rakesh Gupta, Arti Tyagi

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Department of Statistics
Ch. Charan Singh University, Meerut-250004 (India)
Email ID: prgheadstats@yahoo.in

Abstract

The paper deals with stochastic analysis of a two identical unit cold standby system model with two modes of the units-normal (N) and total failure (F). A single repairman is always available with the system to repair a failed unit. Two physical conditions-good and poor of repairman depending upon the perfect and imperfect environment are considered. The system transits from perfect to imperfect environment and vice-versa after random periods of time. The failure and repair times of a unit are taken as independent random variables of discrete nature having geometric distributions with different parameters.

Keywords: Transition probability, Regenerative point, reliability, MTSF, availability of system, busy period of repairman.

1. Introdution

In order to fight with the competitive situations in modern business and industrial systems, the redundancy plays a vital role in the improvement of the various measures of system effectiveness. Numerous authors including [4,5,6,7] have studied two unit redundant systems with different sets of assumptions such as slow and imperfect switches, two types of repairmen, special warranty schemes, preparation time for repair etc. Various authors including [1,2] have analyzed the reparable system models by assuming that the repair rate of a failed unit is affected by the physical conditions of repairman. Goel et al [3] analyzed a two identical unit cold standby system model assuming the physical conditions of repairman and the repair time distributions are affected by good and poor physical conditions of repairman. They have not mentioned the cause of the change of physical conditions of repairman. All the above authors have assumed continuous distributions of various random variables such as failure time, repair time etc. Sometimes we may come across the situations when the physical conditions of repairman depend upon the changing in environmental conditions. For example- The repair efficiency of repairman is good in air-condition atmosphere as compared to non-air-condition atmosphere.

Keeping the above fact in view, the present paper deals with the analyses of a two identical unit standby system model assuming that the physical conditions of repairman depends upon the environmental conditions i.e. the perfect and imperfect environment. Here the parametric space of Markov-chain involved is taken of discrete nature and the random variables denoting failure times, repair time and time to change environments are assumed to follow geometric distributions with different parameters. Initially the repairman starts the repair of a failed unit in good physical condition. The following economic related measures of system effectiveness are obtained by using regenerative point technique-

vi) Transition probabilities and mean sojourn times in various states.
vii) Reliability and mean time to system failure.
viii) Point-wise and steady-state availabilities of the system as well as expected up time of the system during a finite interval of time.
ix) Expected busy period of the repairman in the perfect and imperfect environments during a finite interval of time.
x) Net expected profit incurred by the system during a finite interval of time and steady-state are obtained.

2. Model Description And Assumptions

i) The system comprises of two identical units with two modes of a unit- normal (N) total failure (F).
ii) Initially one unit is operative and other is kept into cold standby.
iii) A single repairman is always available with the system to repair a failed unit.
iv) Two physical conditions- good and poor of repairman depending upon the perfect and imperfect environments are considered. Initially the repairman starts the repair of a failed unit in good physical condition.
v) The system transits from perfect to imperfect environment and vice-versa after a random period of time.
vi) The repair rate of a failed unit in perfect environment is higher than the imperfect environment.
vii) A repaired unit works as good as new.
viii) Failure and repair times of the units follow independent geometric distributions with different parameters.

3. Notations And States Of The System

a) Notations :

pq^t : p.m.f. of failure time of a unit $(p+q=1)$.

rs^t : p.m.f. of repair time of a unit in perfect environment $(r+s=1)$.

$r's'^t$: p.m.f. of repair time of a unit in imperfect environment $(r'+s'=1)$.

ab^t : p.m.f. of time to change the environment from perfect to imperfect $(a+b=1)$.

cd^t : p.m.f. of time to change the environment from imperfect to perfect $(c+d=1)$

$q_{ij}(\cdot), Q_{ij}(\cdot)$: p.m.f. and c.d.f. of one step or direct transition time from state S_i to S_j.

p_{ij} : Steady-state transition probability from state S_i to S_j.

$$p_{ij} = Q_{ij}(\infty)$$

$Z_i(t)$: Probability that the system sojourn in state S_i at epochs 0, 1, 2 … (t-1).

ψ_i : Mean sojourn time in state S_i.

$*, h$: Symbol and dummy variable used in geometric transform e. g.

$$GT\left[q_{ij}(t)\right] = q_{ij}^{*}(h) = \sum_{t=0}^{\infty} h^t q_{ij}(t)$$

b) Symbols for the states of the systems:

N_0 / N_S : Unit in normal (N) mode and operative/standby.

F_r / F_w : Unit in total failure (F) mode in perfect environment and under repair/waiting for repair

$F_{r'} / F_{w'}$: Unit in total failure (F) mode in imperfect environment and under repair/ waiting for repair

G / P : System in perfect/ imperfect Environment.

With the help of above symbols the possible states S_0 to S_4 of the system are shown in **Fig.1,** where, S_0, S_1 and S_2 are the up states whereas S_3 and S_4 are the failed states

TRANSITION DIAGRAM

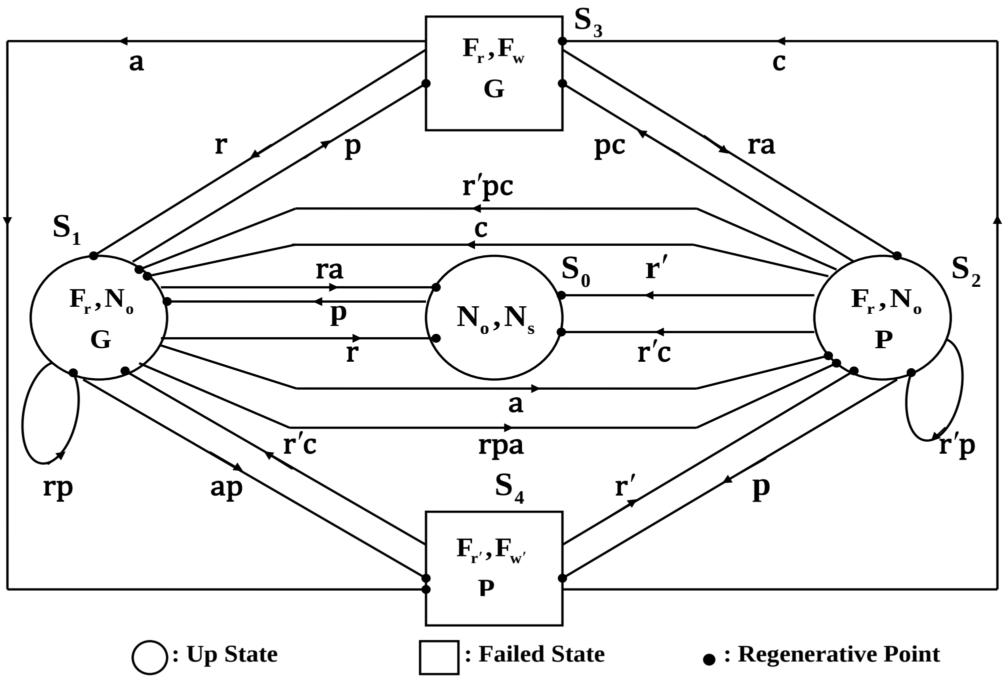

Fig.1

4. Transition Probabilities

Let $Q_{ij}(t)$ be the probability that the system transits from state S_i to S_j during time interval (0, t) i.e., if T_{ij} is the transition time from state S_i to S_j then

$$Q_{ij}(t) = P\left[T_{ij} \le t\right]$$

By using simple probabilistic arguments we have

$$Q_{01}(t)=1-q_1^{t+1},\quad Q_{10}(t)=rqA_1,\quad Q_{11}(t)=rbpA_1,$$
$$Q_{12}(t)=a(sq+rp)A_1$$
$$Q_{13}(t)=bpsA_1,\quad Q_{14}(t)=apsA_1,\quad Q_{20}(t)=r'qA_2,$$
$$Q_{21}(t)=c(qs'+pr')A_2$$
$$Q_{22}(t)=r'pdA_2,\quad Q_{23}(t)=s'cpA_2,\quad Q_{24}(t)=s'dpA_2,\quad Q_{31}(t)=rbA_3$$
$$Q_{32}(t)=raA_3,\quad Q_{34}(t)=saA_3,\quad Q_{41}(t)=r'cA_4,\quad Q_{42}(t)=r'dA_4$$
$$Q_{43}(t)=s'cA_4$$

(1-17)

Where, $A_1=\left[1-(sbq)^{t+1}\right]\big/(1-sbq)$, $\quad A_2=\left[1-(s'qd)^{t+1}\right]\big/(1-s'qd)$,

$A_3=\left[1-(sb)^{t+1}\right]\big/(1-sb)$, $\quad A_4=\left[1-(s'd)^{t+1}\right]\big/(1-s'd)$

The steady state transition probabilities from state S_i to S_j can be obtained from (1-17) by taking $t\to\infty$, as follows:

$$p_{01}=1,\ p_{10}=rqC,\ p_{11}=rbpC,\ p_{12}=a(rq+sq)C$$
$$p_{13}=sbpC,\ p_{14}=sapC$$

Where, $C=\dfrac{1}{1-sbq}$

Similarly, the values of other transition probabilities p_{20}, p_{21}, p_{22}, p_{23}, p_{24}, p_{31}, p_{32}, p_{34}, p_{41}, p_{42} and p_{43} can be evaluated.

We observe that the following relations hold-

$$p_{01}=1,\qquad p_{10}+p_{11}+p_{12}+p_{13}+p_{14}=1,$$
$$p_{20}+p_{21}+p_{22}+p_{23}+p_{24}=1$$
$$p_{31}+p_{32}+p_{34}=1,\qquad p_{41}+p_{42}+p_{43}=1 \tag{18-22}$$

5. Mean Sojourn Times

Let T_i be the sojourn time in state S_i (i=0, 1, 2, 3, 4) then mean sojourn time in state S_i is given by

$$\psi_i=\sum_{t=1}^{\infty}P\left[T\ge t\right]$$

In particular,

$$\psi_0=\sum_{t=1}^{\infty}P[\text{ The operating unit in state } S_0 \text{ doesn't fail up to epoch t-1}]$$
$$=\sum_{t=1}^{\infty}q^t=\frac{q}{p}$$

Similarly,

$$\psi_1=sbqC,\quad \psi_2=\frac{s'dq}{1-s'dq},\quad \psi_3=\frac{sb}{1-sb},\quad \psi_4=\frac{s'd}{1-s'd} \tag{23-27}$$

6. Methodology For Developing Equations

In order to obtain various interesting measures of system effectiveness we developed the recurrence relations for reliability, availability and busy period of repairman as follows

d) Reliability of the system- Here we define $R_i(t)$ as the probability that the system does not fail up to epochs 0, 1, 2,.., (t-1) when it is initially started from up state S_i. To determine it, we regard the failed state S_3 and S_4 as absorbing state. Now, the expression for $R_i(t)$; i=0, 1, 2 we have the following set of convolution equations.

$$R_0(t) = q^t + \sum_{u=0}^{t-1} q_{01}(u) R_1(t-1-u)$$
$$= Z_0(t) + q_{01}(t-1) © R_1(t-1)$$

Similarly,

$$R_1(t) = Z_1(t) + q_{10}(t-1) © R_0(t-1) + q_{11}(t-1) © R_1(t-1) + q_{12}(t-1) © R_2(t-1)$$
$$R_2(t) = Z_2(t) + q_{20}(t-1) © R_0(t-1) + q_{21}(t-1) © R_1(t-1) + q_{22}(t-1) © R_2(t-1)$$

(28-30)

Where,

$$Z_0(t) = q^t, \qquad Z_1(t) = b^t s^t q^t, \qquad Z_2(t) = d^t s'^t q^t$$

e) Availability of the system- Let $A_i(t)$ be the probability that the system is up at epoch (t-1), when it initially started from state S_i. By using simple probabilistic arguments as illustrated in case of reliability, the following recurrence relations can be easily developed for $A_i(t)$; i=0 to 4.

$$A_0(t) = Z_0(t) + q_{01}(t-1) © A_1(t-1)$$
$$A_1(t) = Z_1(t) + q_{10}(t-1) © A_0(t-1) + q_{11}(t-1) © A_1(t-1) + q_{12}(t-1) © A_2(t-1) + q_{13}(t-1) © A_3(t-1) + q_{14}(t-1) © A_4(t-1)$$
$$A_2(t) = Z_2(t) + q_{20}(t-1) © A_0(t-1) + q_{21}(t-1) © A_1(t-1) + q_{22}(t-1) © A_2(t-1) + q_{23}(t-1) © A_3(t-1) + q_{24}(t-1) © A_4(t-1)$$
$$A_3(t) = q_{31}(t-1) © A_1(t-1) + q_{32}(t-1) © A_2(t-1) + q_{34}(t-1) © A_4(t-1)$$
$$A_4(t) = q_{41}(t-1) © A_1(t-1) + q_{42}(t-1) © A_2(t-1) + q_{43}(t-1) © A_3(t-1)$$

(31-35)

Where,

The values of $Z_i(t)$; i=0 to 2 are same as given in section 8(a).

c) Busy period of repairman- Let $B_i^G(t)$ and $B_i^P(t)$ be the respective probabilities that the repairman is busy at epoch (t-1) in the repair of failed unit in good and poor conditions depending upon the perfect and imperfect environment are conditions when system initially starts from state S_i. Using simple probabilistic arguments as illustrated in case of availability analysis, the relations for $B_i^k(t)$; i=0 to 4 and k=G, P can be easily developed on replacing the following in expressions (31-35)-

i) A_i by B_i^k, $Z_0(t)$ by 0, $Z_1(t)$ by $(1-\delta)Z_1(t)$, $Z_2(t)$ by $\delta Z_2(t)$ and

ii) Considering one more contingency respectively $(1-\delta)Z_3(t)$ and $\delta Z_4(t)$ in expressions (34)

and (35). The resulting expressions may be denoted by (36-40).

Where, $\delta = 0$ and 1 respectively for k = G and P. Also, $Z_1(t)$ and $Z_2(t)$ are same as given in section 7(a). and $Z_3(t)$, $Z_4(t)$ are as follows-

$Z_3(t) = s^t b^t$, $Z_4 = s^t d^t$

7. Analysis Of Reliability And Mtsf

Taking geometric transforms of relations (28-30) and simplifying the resulting set of algebraic equations for $R_0^*(h)$ we get

$$R_0^*(h) = \frac{N_1(h)}{D_1(h)} \tag{41}$$

Where,

$$N_1(h) = \left[\left(1-hq_{11}^*\right)\left(1-hq_{22}^*\right) - h^2 q_{12}^* q_{21}^*\right] Z_0^* + \left[hq_{01}^*\left(1-hq_{22}^*\right)\right] Z_1^* + h^2 q_{01}^* q_{12}^* Z_2^*$$

$$D_1(h) = \left(1-hq_{11}^*\right)\left(1-hq_{22}^*\right) - h^2 q_{12}^* q_{21}^* - h^2 q_{01}^* q_{10}^*\left(1-hq_{22}^*\right) - h^3 q_{01}^* q_{12}^* q_{20}^*$$

Collecting the coefficient of h^t from expression (41), we can get the reliability of the system $R_0(t)$. The MTSF is given by-

$$E(T) = \lim_{h\to 1}\sum_{t=1}^{\infty} h^t R(t) = \frac{N_1(1)}{D_1(1)} - 1 \tag{42}$$

Where,

$$N_1(1) = \psi_0\left[(1-p_{11})(1-p_{22}) - p_{12}p_{21}\right] + \psi_1\left[p_{01}(1-p_{22})\right] + p_{01}p_{12}\psi_2$$

$$D_1(1) = (1-p_{11})(1-p_{22}) - p_{12}p_{21} - p_{01}(1-p_{22}) - p_{01}p_{12}p_{20}$$

8. Availability Analysis

On taking geometric transforms of relations (31-35) and simplifying the resulting equations we get-

$$A_0^*(h) = \frac{U_0^* Z_0^* + U_1^* Z_1^* + U_2^* Z_2^*}{U_0^* - hq_{10}^* U_1^* - hq_{20}^* U_2^*} = \frac{N_2(h)}{D_2(h)} \tag{43}$$

Where,

$$U_0^* = \left(1-hq_{11}^*\right)\left\{\left(1-hq_{22}^*\right)\left(1-h^2 q_{34}^* q_{43}^*\right) - hq_{23}^*\left(hq_{32}^* + h^2 q_{34}^* q_{42}^*\right) - hq_{24}^*\left(hq_{42}^* + h^2 q_{43}^* q_{32}^*\right)\right\}$$
$$-hq_{12}^*\left\{hq_{21}^*\left(1-h^2 q_{34}^* q_{43}^*\right) + hq_{23}^*\left(hq_{31}^* + h^2 q_{34}^* q_{41}^*\right) + hq_{24}^*\left(h^2 q_{43}^* q_{31}^* + hq_{41}^*\right)\right\}$$
$$-hq_{13}^*\left\{hq_{21}^*\left(hq_{32}^* + h^2 q_{34}^* q_{42}^*\right) + \left(1-hq_{22}^*\right)\left(hq_{32}^* + h^2 q_{34}^* q_{42}^*\right) + hq_{24}^*\left(h^2 q_{41}^* q_{32}^*\right.\right.$$

$$-h^2q_{31}^*q_{42}^*)\} - hq_{14}^*\{hq_{21}^*(h^2q_{43}^*q_{32}^* + hq_{42}^*) + (1-hq_{22}^*)(hq_{41}^* + h^2q_{43}^*q_{31}^*)$$
$$+hq_{22}^*(h^2q_{31}^*q_{42}^* - h^2q_{32}^*q_{41}^*)\}$$
$$U_1^* = \left[(1-hq_{22}^*)(1-h^2q_{34}^*q_{43}^*) - hq_{23}^*(hq_{32}^* + h^2q_{34}^*q_{42}^*) - hq_{24}^*(hq_{42}^* + h^2q_{43}^*q_{32}^*)\right]hq_{01}^*$$
$$U_2^* = \left[hq_{12}^*(1-h^2q_{34}^*q_{43}^*) + hq_{13}^*(hq_{32}^* + h^2q_{34}^*q_{42}^*) + hq_{14}^*(hq_{42}^* + h^2q_{43}^*q_{32}^*)\right]hq_{01}^*$$

The steady state availability of the system is given by-

$$A_0 = \lim_{t\to\infty} A_0(t) = \lim_{h\to1}(1-h)\frac{N_2(h)}{D_2(h)}$$

Now as $D_2(h)$ at h=1 is zero, therefore by applying L. hospital rule we get-

$$A_0 = -\frac{U_0\psi_0 + U_1\psi_1 + U_2\psi_2}{U_0\psi_0 + U_1\psi_1 + U_2\psi_2 + U_3\psi_3 + U_4\psi_4} = -\frac{N_2(1)}{D'_2(1)} \tag{44}$$

Where,

$$U_0 = p_{10}\{p_{21}(1-p_{34}p_{43}) + p_{23}(p_{31} + p_{34}p_{41}) + p_{24}(p_{41} + p_{43}p_{31})\} + p_{20}\{p_{12}(1-p_{34}p_{43})$$
$$+p_{13}(p_{32} + p_{34}p_{42}) + p_{14}(p_{42} + p_{43}p_{32})$$
$$U_1 = (1-p_{22})(1-p_{34}p_{43}) - p_{23}(p_{32} + p_{34}p_{42}) - p_{24}(p_{42} + p_{43}p_{32})$$
$$U_2 = p_{12}(1-p_{34}p_{43}) + p_{13}(p_{32} + p_{34}p_{42}) + p_{14}(p_{42} + p_{43}p_{32})$$
$$U_3 = p_{12}(p_{23} + p_{24}p_{43}) + p_{13}(1-p_{22} + p_{24}p_{42}) + p_{14}(p_{43}(1-p_{22}) - p_{42}p_{23})$$
$$U_4 = p_{12}(p_{24} + p_{23}p_{34}) + p_{13}((1-p_{22})p_{34} + p_{32}p_{24}) + p_{14}((1-p_{22}) - p_{23}p_{32})$$

Now, the expected up time of the system up to epoch t-1 (total t epochs) is given by-

$$\mu_{up}(t) = \sum_{x=0}^{t-1} A_0(x)$$

So that, $\mu_{up}^*(h) = A_0^*(h)/(1-h)$ (45)

9. Busy Period Analysis

On taking geometric transform of (36-40) and simplifying the resulting equations for k=G and P we get

$$B_0^{G*}(h) = \frac{U_1^*Z_1^* + U_3^*Z_3^*}{D_2(h)} \quad \text{and} \quad B_0^{P*}(h) = \frac{U_2^*Z_2^* + U_4^*Z_4^*}{D_2(h)} \tag{46-47}$$

Where,

$$U_3^* = \left[hq_{12}^*(hq_{23}^* + h^2q_{24}^*q_{43}^*) + hq_{13}^*(1-hq_{22}^* + h^2q_{24}^*q_{42}^*) + hq_{14}^*(hq_{43}^*(1-hq_{22}^*) - h^2q_{42}^*q_{23}^*)\right]hq_{01}^*$$
$$U_4^* = \left[hq_{12}^*(hq_{24}^* + h^2q_{23}^*q_{34}^*) + hq_{13}^*(hq_{34}^*(1-hq_{22}^*) + h^2q_{32}^*q_{24}^*) + hq_{14}^*(1-hq_{22}^* + h^2q_{24}^*q_{32}^*)\right]hq_{01}^*$$

and $D_2(h)$ is same as given in section 10.

In the long run the respective probabilities that the repairman is busy in the repair of failed unit in perfect and imperfect environment conditions are given by-

$$B_0^G = \lim_{t\to\infty} B_o^G(t) = \lim_{h\to 1}(1-h)\frac{N_3(h)}{D_2(h)}$$

$$B_0^P = \lim_{t\to\infty} B_o^P(t) = \lim_{h\to 1}(1-h)\frac{N_4(h)}{D_2(h)}$$

But $D_2(h)$ at h=1 is zero, therefore by applying L. Hospital rule, we get

$$B_0^G = -\frac{U_1\psi_1 + U_3\psi_3}{D_2'(1)} \qquad \text{and} \qquad B_0^P = -\frac{U_2\psi_2 + U_3\psi_3}{D_2'(1)}$$

(48-49)

and $D_2'(1)$ is same as given in section 10.

Now the expected busy period of the repairman is busy in the repair of failed unit in perfect and imperfect environment conditions up to epoch (t-1) are respectively given by-

$$\mu_b^G(t) = \sum_{x=0}^{t-1} B_0^G(x), \qquad \mu_b^P(t) = \sum_{x=0}^{t-1} B_0^P(x)$$

So that,

$$\mu_b^{G*}(h) = \frac{B_0^{G*}(h)}{(1-h)}, \qquad \mu_b^{P*}(h) = \frac{B_0^{P*}(h)}{(1-h)}$$

(50-51)

10. Profit Function Analysis

We are now in the position to obtain the net expected profit incurred up to epoch (t-1) by considering the characteristics obtained in earlier section.

Let us consider,

K_0 =revenue per-unit time by the system when it is operative.

K_1 =cost per-unit time when repairman is busy in the repairing failed unit in perfect environment condition.

K_2 =cost per-unit time when repairman is busy in the repairing failed unit in imperfect environment condition.

Then, the net expected profit incurred up to epoch (t-1) given by

$$P(t) = K_0\mu_{up}(t) - K_1\mu_b^G(t) - K_2\mu_b^P(t) \tag{52}$$

The expected profit per unit time in steady state is given by-

$$P = \lim_{t\to\infty}\frac{P(t)}{t} = \lim_{h\to 1}(1-h)^2 P^*(h)$$

$$= K_0 \lim_{h\to 1}(1-h)^2\frac{A_0^*(h)}{(1-h)} - K_1 \lim_{h\to 1}(1-h)^2\frac{B_0^{G*}(h)}{(1-h)} - K_2 \lim_{h\to 1}(1-h)^2\frac{B_0^{P*}(h)}{(1-h)}$$

$$= K_0A_0 - K_1B_0^G - K_2B_0^P \tag{53}$$

11. Graphical Representation

The curves for MTSF and profit function have been drawn for different values of parameters. Fig.2 depicts the variations in MTSF with respect to failure rate (p) of operative unit for different values of repair rate (r) of failed unit in perfect environmental condition and rate of the change of environment from perfect to imperfect (a) when repair rate of failed unit in imperfect environmental condition and rate of change of environment from imperfect to perfect are kept fixed as $r' = 0.07$ and $c = 0.1$.

The smooth curves shows the trends for three different values 0.2, 0.3 and 0.4 of r when 'a' is taken as 0.01 whereas dotted curves shows the trends for same three values of 'r' as above when 'a' is taken as 0.10. From these curves we observed that MTSF decreases uniformly as the values of 'p' and 'a' increase and increases with the increase in 'r'. From the curve of MTSF we also conclude that to achieve at least a specified value of expected life of the system say 3000 units, the failure rate p of a unit should not exceed 0.0124 and 0.0142 respectively for a = 0.10 and 0.01 when r is fixed as 0.4. Similarly when r = 0.3 and 0.2 one can find the upper bonds for a = 0.10 and 0.01.

Similarly, Fig. 3 reveals the variations in profit (P) with respect to p for varying values of r and a as in case of MTSF, when the values of other parameters are kept fixed as $r' = 0.07$, $c = 0.1$ $K_0 = 11$, $K_1 = 280$ and $K_2 = 400$. From this figure same trends in respect of p, r, a have been observed as in MTSF. Further it is also revealed by smooth curves that system is profitable only if p is less than 0.0153, 0.026, 0.041 respectively for r = 0.2, 0.3 and 0.4 for fixed $a = 0.01$. From dotted curves it is obvious that system is profitable only if p is less than 0.0105, 0.017 and 0.025 respectively for r = 0.2, 0.3 and 0.4 for fixed $a = 0.10$.

Thus the above graphical study reveals that the bonds of any parameter can be evaluated for fixed values of other parameters to get non-negative profit. Moreso, one can also obtain the upper bond of any parameter (in case the curve is of decreasing nature w.r.t. this paramter) to achieve at least any specific value of MTSF and the lower bond of any parameter (in case the curve is of increasing nature w.r.t this parameter) to achieve at least any particular value of MTSF.

This study will help the industrial manager to take decision to reduce the failure rate of a unit by incorporating its redundancy or to increase the repair rate of a failed unit by adopting various repair policies to get a specified value of expected life time and non-negative expected profit by the system.

Behavior of MTSF with respect to p, r and a

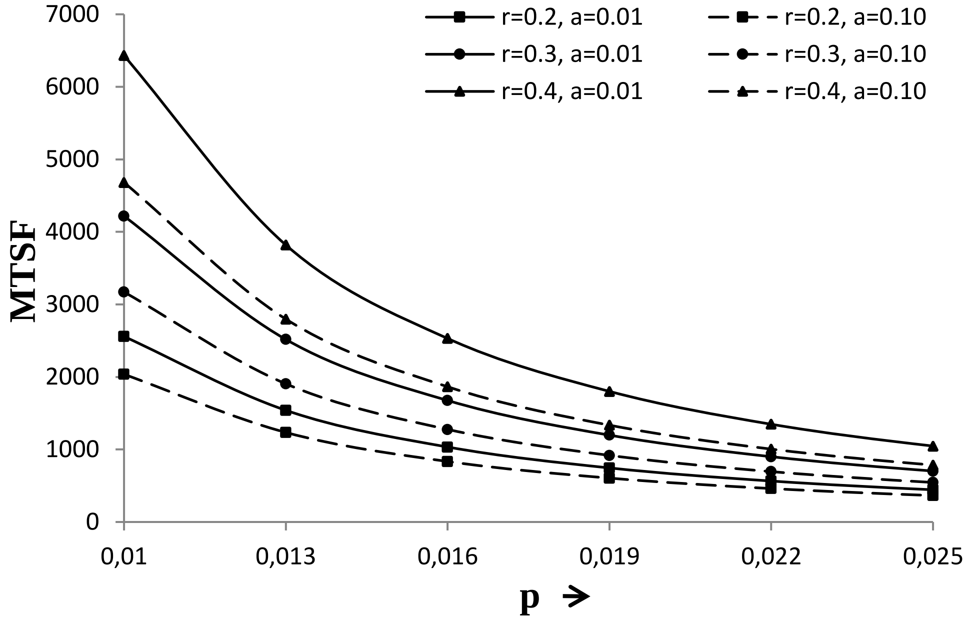

Fig. 2

Behavior of Profit (P) with respect to p, r and a

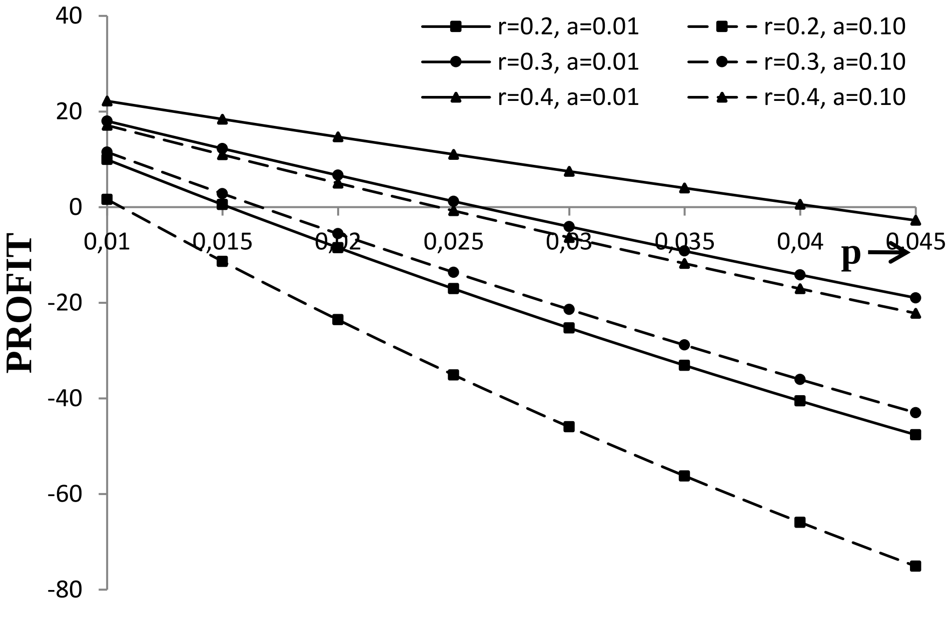

Fig. 3

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A Two Identical Unit Cold Standby System Subject To Two Types Of Failures

Pradeep Chaudhary*, Rashmi Tomar
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Department of Statistics
Ch. Charan Singh University, Meerut-250004(India)
*E-mail ID:pc25jan@gmail.com

Abstract

The paper deals with a system model composed of a two identical unit standby system in which initially one is operative and other is kept as cold standby. Each unit of the system has two possible modes – Normal (N) and Total Failure (F). An operating unit may fail either due to normal or due to chance causes. A single repairman is always available with the system to repair a unit failed due to any of the above causes. The system failure occurs when both the units are in total failure mode. The failure time distributions of a unit failed due to both the causes are taken as exponentials with different parameters whereas the repair time distributions of a failed unit in both types of failure are taken as general with different CDFs. Using regenerative point technique, the various important measures of system effectiveness have been obtained:

Keywords: Reliability, Mean time to system failure, availability, expected busy period of repairman, net expected profit.

1. Introduction

The two unit cold standby systems have been widely studied in the literature of reliability as they are frequently used in modern business and industries. It is obvious that the standby unit is switched to operate when the operating unit fails and the switching device which is used to put the standby unit into operation may be perfect or imperfect at the time of need. In past years various authors including [1,2,4,5,6,9,10,11] analyzed the two identical and non-identical units standby redundant system models with different sets of assumptions such as imperfect switching device, slow switching device, waiting time distribution of repairman, repair machine failure etc. They have analyzed the two identical and non-identical unit system models by taking the single failure mode of an operating unit i.e. due to normal (ageing effect).

In many realistic situations, the systems are subject to two types of failure .One occurs by a normal cause and the other due to chance cause such as (i) abnormal environmental condition i.e. temperature, pressure, vibration etc. (ii) defective design (iii) misunderstanding the process variables (iv) operator's negligence and mishandling of the system etc. Keeping this fact in view few authors [3,7,8] analyzed the system models assuming two failure modes of each unit.

The purpose of the present paper is to deal with a stochastic model of a two identical unit cold standby redundant system subject to two types of failure in each of the operating unit. By using regenerative point technique, the following important measures of system effectiveness are obtained.

i. Transient-state and steady-state transition probabilities.
ii. Mean sojourn time in various regenerative states.
iii. Reliability and mean time to system failure (MTSF).
iv. Point-wise and steady-state availabilities of the system as well as expected up time of the system during time interval (0, t).
v. The expected busy period of repairman in time interval (0, t).
vi. Net expected profit earned by the system in time interval (0, t) and in steady-state.

2. System Description and Assumptions

1. The system consists of two identical units. Initially, one unit is operative and other is kept as cold standby.
2. Each unit of the system has two possible modes: Normal (N) and Total Failure (F).
3. The switching device used to put the standby unit into operation is always perfect and instantaneous.
4. An operative unit may fail either due to normal cause i.e. due to ageing effect or due to chance cause.
5. The system failure occurs when both the units are in total failure mode.
6. A single repairman is always available at the system to repair a unit failed due to normal cause or due to chance cause.
7. The failure time distributions of the units to reach into the failure mode either due to normal or due to chance cause are taken as exponential whereas the repair time distributions of a unit failed due to both causes are taken as general with different CDF's.
8. A repaired unit always works as good as new.

3. Notations and States of the System

We define the following symbols for generating the various states of the system-

N_0, N_s	:	Unit is in N-mode and operative/standby
F_{1r}, F_{2r}	:	Unit is in failure mode due to normal cause/due to chance cause and under repair.
F_{1w}, F_{2w}	:	Unit is in failure mode due to normal cause/due to chance cause and waiting for repair.

Considering the above symbols in view of assumptions stated in section-2, the possible states of the system are shown in the transition diagram represented by **Fig. 1**. It is to be noted that the epochs of transitions into the state S_4 from S_1, S_3 from S_1, S_5 from S_2, S_6 from S_2 are non-regenerative, whereas all the other entrance epochs into the states of the system are regenerative. The states S_0, S_1 and S_2 are the up-states of the system and the states S_3, S_4, S_5 and S_6 are the failed states of the system.

TRANSITION DIAGRAM

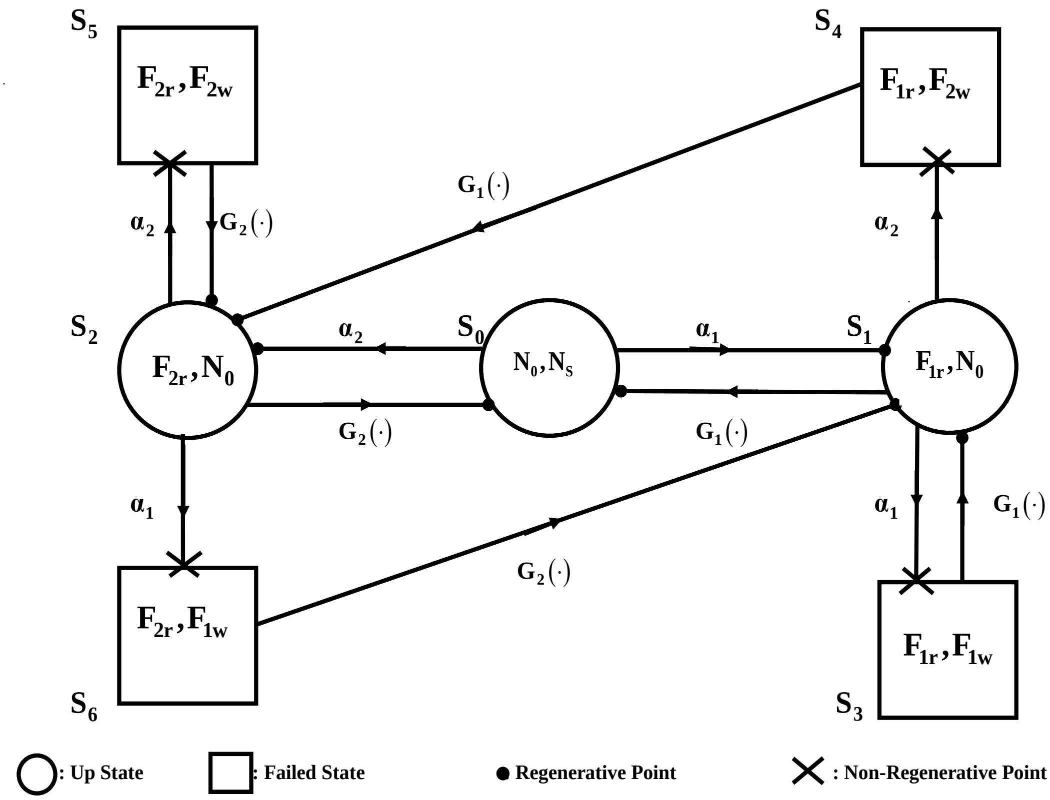

Fig. 1

The other notations used are defined as follows:

E : Set of regenerative states $\equiv \{S_0, S_1, S_2\}$

$\bar{E}$: Set of non-regenerative states $\equiv \{S_3, S_4, S_5, S_6\}$

α_1, α_2 : Constant failure rate of an operative unit due to normal cause/chance cause

$G_1(\cdot), G_2(\cdot)$: CDF of repair time of failed unit due to normal cause/chance cause.

$q_{ij}(\cdot)$: p.d.f of transition time from regenerative state S_i to S_j.

$q_{ij}^{(k)}(\cdot)$: p.d.f of transition time from regenerative state S_i to S_j via non-regenerative state S_k
.

$p_{ij}(\cdot)$: One[1] step steady-state transition probability from regenerative state S_i to $S_j = \int q_{ij}(u)du$.

$p_{ij}^{(k)}(\cdot)$: Two step steady-state transition probability from regenerative state S_i to S_j via non-regenerative state $S_k = \int q_{ij}^{(k)}(u)du$.

n_1, n_2 : Mean repair times of operative unit and standby unit

[1] The limits of integration are taken to be 0 to ∞ whenever they are not mentioned

$$=\int \bar{G}_1(t)dt \text{ and } \int \bar{G}_2(t)dt$$

⊔ : Symbol for Laplace Stieltjes Transform. i.e. $\tilde{Q}_{ij}(s)=\int e^{-st}dQ_{ij}(t)$

© : Symbol for ordinary convolution i.e. $A(t)©B(t)=\int_0^t A(u)©B(t\text{-}u)du$

$*$: Symbol for Laplace Transform. i.e. $q_{ij}^*(s)=\int e^{-st}q_{ij}(u)du$

4. Transition Probabilities and Sojourn Times

(a) The direct or one step steady-state transition probabilities are as follows

$$p_{01}=\int e^{\alpha_2 t}\alpha_1 e^{\alpha_1 t}dt=\frac{\alpha_1}{\alpha_1+\alpha_2}$$

$$p_{02}=\int e^{-\alpha_1 t}\alpha_2 e^{-\alpha_2 t}dt=\frac{\alpha_2}{\alpha_1+\alpha_2}$$

$$p_{10}=\int e^{-(\alpha_1+\alpha_2)t}dG_1(t)=\tilde{G}_1(\alpha_1+\alpha_2)$$

$$p_{20}=\int e^{-(\alpha_1+\alpha_2)t}dG_2(t)=\tilde{G}_2(\alpha_1+\alpha_2)$$

(b) The two step steady-state transition probabilities are given by

$$p_{11}^{(3)}=\int \alpha_1 e^{-\alpha_1 u}du\, e^{\alpha_2 u}\bar{G}_1(u)\int_u^\infty \frac{d\bar{G}_1(t)}{\bar{G}_1(u)}$$

$$=\alpha_1\int dG_1(t)\int_0^t e^{-(\alpha_1+\alpha_2)u}du$$

$$=\frac{\alpha_1}{\alpha_1+\alpha_2}\int\left[1-e^{-(\alpha_1+\alpha_2)t}\right]dG_1(t)$$

$$=\frac{\alpha_1}{\alpha_1+\alpha_2}\left[1-\tilde{G}_1(\alpha_1+\alpha_2)\right]$$

Similarly,

$$p_{12}^{(4)}=\frac{\alpha_2}{\alpha_1+\alpha_2}\left[1-\tilde{G}_1(\alpha_1+\alpha_2)\right]$$

$$p_{22}^{(5)}=\frac{\alpha_2}{\alpha_1+\alpha_2}\left[1-\tilde{G}_2(\alpha_1+\alpha_2)\right]$$

$$p_{21}^{(6)}=\frac{\alpha_1}{\alpha_1+\alpha_2}\left[1-\tilde{G}_2(\alpha_1+\alpha_2)\right]$$

We observe the following relationship

$$p_{01}+p_{02}=1, \qquad p_{10}+p_{11}^{(3)}+p_{12}^{(4)}=1, \qquad p_{20}+p_{22}^{(5)}+p_{21}^{(6)}=1$$

(1-3)

(a) The mean sojourn times in various states are as follows:

$$\psi_0 = \int e^{-(\alpha_1+\alpha_2)t} dt = \frac{1}{\alpha_1+\alpha_2}$$

Similarly,

$$\psi_1 = \int e^{-(\alpha_1+\alpha_2)t} \bar{G}_1(t) dt$$

$$\psi_2 = \int e^{-(\alpha_1+\alpha_2)t} \bar{G}_2(t) dt$$

5. Analysis of Characteristics

(a) RELIABILITY AND MTSF

Let $R_i(t)$ be the probability that the system is operative during $(0,t)$ given that at t=0 it starts from state $S_i \in E$. By simple probabilistic arguments, we have the following recurrence relations in $R_i(t)$; i = 0, 1, 2

$$R_0(t) = Z_0(t) + q_{01}(t)©R_1(t) + q_{02}(t)©R_2(t)$$

Similarly,

$$R_1(t) = Z_1(t) + q_{10}(t)©R_0(t)$$

$$R_2(t) = Z_2(t) + q_{20}(t)©R_0(t) \tag{4-6}$$

Where,

$$Z_0(t) = e^{-(\alpha_1+\alpha_2)t}, \quad Z_1(t) = e^{-(\alpha_1+\alpha_2)t}\bar{G}_1(t), \qquad Z_2(t) = e^{-(\alpha_1+\alpha_2)t}\bar{G}_2(t)$$

Taking Laplace Transforms of the relation (4-6) and solving the resulting set of algebraic equations for $R_0^*(s)$, we get

$$R_0^*(s) = \frac{Z_0^* + q_{01}^* Z_1^* + q_{02}^* Z_2^*}{1 - q_{01}^* q_{10}^* - q_{02}^* q_{20}^*} \tag{7}$$

We have omitted the argument's' from $q_{ij}^*(s)$ and $Z_i^*(s)$.

The expression of mean time to system failure is given by

$$E(T_0) = \lim_{s \to 0} R_0^*(s)$$

Observing that $q_{ij}^*(0) = p_{ij}$ and $Z_i^*(0) = \psi_i$, we get

$$E(T_0) = \frac{\psi_0 + p_{01}\psi_1 + p_{02}\psi_2}{1 - p_{01}p_{10} - p_{02}p_{20}} \tag{8}$$

b) AVAILABILITY ANALYSIS

Let $A_i(t)$ be the probability that the system is up at epoch t, when initially it starts operation from state $S_i \in E$. Using the regenerative point technique and the tools of Laplace transform, one can obtain the value of $A_0(t)$ in terms of its Laplace transforms i.e. $A_0^*(s)$ given as follows-

$$A_0^*(s) = \frac{N_1(s)}{D_1(s)} \tag{9}$$

Where,

$$N_1(s) = Z_0^*\left[\left(1-q_{11}^{(3)*}\right)\left(1-q_{22}^{(5)*}\right) - q_{12}^{(4)*}q_{21}^{(6)*}\right] + Z_1^*\left[q_{02}^*q_{21}^* + q_{01}^*\left(1-q_{22}^{(5)*}\right)\right]$$
$$+ Z_2^*\left[q_{01}^*q_{12}^{(4)*} + \left(1-q_{11}^{(3)*}\right)q_{02}^*\right]$$

$$D_1(s) = \left[\left(1-q_{11}^{(3)*}\right)\left(1-q_{22}^{(5)*}\right) - q_{12}^{(4)*}q_{21}^{(6)*}\right] - q_{01}^*\left[q_{20}^*q_{12}^{(4)*} + q_{10}^*\left(1-q_{22}^{(5)*}\right)\right]$$
$$- q_{02}^*\left[q_{10}^*q_{21}^{(6)*} + \left(1-q_{11}^{(3)*}\right)q_{20}^*\right]$$

The steady-state availability of the system is given by

$$A_0 = \lim_{s\to 0} sA_0^*(s) = \lim_{s\to 0} s\frac{N_1(s)}{D_1(s)} \tag{10}$$

We observe that

$$D_1(0) = 0$$

Therefore, by using L.Hospital's rule the steady state availability is given by

$$A_0 = \frac{N_1(0)}{D_1'(0)} \tag{11}$$

Where,

$$N_1(0) = \psi_0\left[p_{10}\left(1-p_{22}^{(5)}\right) - p_{20}p_{12}^{(4)}\right] + \psi_1\left[p_{01}\left(1-p_{22}^{(5)}\right) + p_{02}p_{20}\right]$$
$$+ \psi_2\left[p_{01}p_{12}^{(4)} + p_{02}\left(1-p_{11}^{(3)}\right)\right]$$

$$\begin{aligned} D_1'(0) = {} & \left[p_{10}\left(1-p_{22}^{(5)}\right) + p_{12}^{(4)}p_{20}\right]\psi_0 + \left(p_{01}\left(1-p_{22}^{(5)}\right) + p_{02}p_{21}^{(6)}\right)n_1 \\ & + \left(p_{01}p_{12}^{(4)} + p_{02}\left(1-p_{11}^{(3)}\right)\right)n_2 \end{aligned} \tag{12}$$

The expected up time of the system in interval (0, t) is given by

$$\mu_{up}(t) = \int_0^t A_0(u)\,du$$

so that,

$$\mu^*_{up}(s) = \frac{A_0^*(s)}{s} \tag{13}$$

(c) BUSY PERIOD ANALYSIS

Let $B_i^1(t)$ and $B_i^2(t)$ be the probability that the repairman is busy in the repair of a failed unit due to normal cause and due to chance shock at time t when system initially starts from state $S_i \in E$. Using the simple probabilistic arguments in regenerative point technique and the tools of Laplace transforms, one can obtain the value of $B_i^1(t)$ and $B_i^2(t)$ in terms of their Laplace transforms as follows-

$$B_0^{1*}(s) = \frac{N_2(s)}{D_1(s)} \quad \text{and} \quad B_0^{2*}(s) = \frac{N_3(s)}{D_1(s)} \tag{14-15}$$

where,

$$N_2(s) = \left[q_{01}^*\left(1 - q_{22}^{(5)*}\right) + q_{02}^* q_{21}^{(6)*} \right] \bar{G}_1^*$$

$$N_3(s) = \left[q_{02}^*\left(1 - q_{11}^{(3)*}\right) + q_{01}^* q_{12}^{(4)*} \right] \bar{G}_2^*$$

and $D_1(s)$ is already defined in section 5(b). $\bar{G}_1^*$ and $\bar{G}_2^*$ are the L.T. of $\bar{G}_1(t)$ and $\bar{G}_2(t)$.

In the long run, the probabilities that the repairman will be busy in repair of normal cause and chance causes are as follows-

$$B_0^1 = \frac{N_2(0)}{D_1'(0)} \quad \text{and} \quad B_0^2 = \frac{N_3(0)}{D_2'(0)} \tag{16-17}$$

where,

$$N_2(0) = \left[p_{01}\left(1 - p_{22}^{(5)}\right) + p_{02} p_{21}^{(6)} \right] n_1$$

$$N_3(0) = \left[p_{02}\left(1 - p_{11}^{(3)}\right) + p_{01} p_{12}^{(4)} \right] n_2$$

The value of $D_1'(0)$ is same as given in expression (12).

The expected busy period of the repairman in repair in repairing during (0, t) are given by

$$\mu_b^1(t) = \int_0^t B_0^1(u)\,du \quad \text{and} \quad \mu_b^2(t) = \int_0^t B_0^2(u)\,du$$

so that,

$$\mu_b^{1*}(s) = \frac{B_0^{1*}(s)}{s} \quad \text{and} \quad \mu_b^{2*}(s) = \frac{B_0^{2*}(s)}{s} \tag{18-19}$$

(d) PROFIT FUNCTION ANALYSIS

The net expected total profit incurred by the system in time interval (0, t) is given by
$P(t) =$ Expected total revenue in (0, t) - Expected cost of repair in (0, t)

$$= K_0\mu_{up}(t) - K_1\mu_b^1(t) - K_2\mu_b^2(t) \tag{20}$$

Where, K_0 is the revenue per- unit up time by the system during its operation. K_1 and K_2 are the amounts paid to the repairman per-unit of time when he is busy in repair of a unit failed due to normal cause and due to chance cause respectively.

The expected total profit incurred per unit time in steady-state is given by

$$P = K_0 A_0 - K_1 B_0^1 - K_2 B_0^2 \tag{21}$$

6. Particular Cases

Case 1: When the repair time of both the units also follow exponential distribution with p.d.fs as follows-

$$g_1(t) = \eta_1 e^{-\eta_1 t}, \qquad g_2(t) = \eta_2 e^{-\eta_2 t}$$

The Laplace Transform of above density functions are as given below.

$$g_1^*(s) = \tilde{G}_1(s) = \frac{\eta_1}{s+\eta_1}, \qquad g_2^*(s) = \tilde{G}_2(s) = \frac{\eta_2}{s+\eta_2}$$

Here $\tilde{G}_i(s)$ are the Laplace-Stieltjes Transforms of the c.d.fs $G_i(t)$ corresponding to the p.d.fs $g_i(t)$.

In view of above, the changed values of transition probabilities and mean sojourn times are given below-

$$p_{01} = \frac{\alpha_1}{\alpha_1+\alpha_2}, \qquad p_{02} = \frac{\alpha_2}{\alpha_1+\alpha_2}, \qquad p_{10} = \frac{\eta_1}{\alpha_1+\alpha_2+\eta_1}$$

$$p_{11}^{(3)} = \frac{\alpha_1}{\alpha_1+\alpha_2+\eta_1}, \qquad p_{12}^{(4)} = \frac{\alpha_2}{\alpha_1+\alpha_2+\eta_1}, \qquad p_{20} = \frac{\eta_2}{\alpha_1+\alpha_2+\eta_2}$$

$$p_{22}^{(5)} = \frac{\alpha_2}{\alpha_1+\alpha_2+\eta_2}, \qquad p_{21}^{(6)} = \frac{\alpha_1}{\alpha_1+\alpha_2+\eta_2}$$

$$\psi_0 = \frac{1}{\alpha_1+\alpha_2}, \qquad \psi_1 = \frac{1}{\alpha_1+\alpha_2+\eta_1}, \qquad \psi_1 = \frac{1}{\alpha_1+\alpha_2+\eta_2}$$

7. Graphical Study Of Behaviour

The curves for **MTSF** and **profit function** are drawn for the two particular cases: case 1 and case 2 in respect of different parameters. **In Case 1**, **when the repair time of unit-1 also follow exponential distribution**. We plot curves for **MTSF** and **profit function** in **Fig. 2** and **Fig. 3** w.r.t. $\boldsymbol{\alpha_1}$ for three different values of $\boldsymbol{\eta_1}$ and two different values of $\boldsymbol{\eta_2}$ while the other parameters are kept fixed as $\boldsymbol{\alpha_2 = 0.029}$. From the curves of **Fig. 2** we observe that MTSF increases uniformly as the value of η_1 and η_2 increase and it decreases with the increase in $\boldsymbol{\alpha_1}$. Further, we also observed from **Fig. 2** that the value of $\boldsymbol{\alpha_1}$ must be less than **0.012, 0.114** and **0.017** corresponding to $\boldsymbol{\eta_1 = 0.1}$, **0.2** and **0.3** to achieve at least **300** units of **MTSF** when $\boldsymbol{\eta_2 = 0.9}$ is fixed as. Similarly, we can find the upper bounds of $\boldsymbol{\alpha_1}$ corresponding to the values of $\boldsymbol{\eta_1}$ to achieve **300** units of **MTSF** when $\boldsymbol{\eta_2}$ is kept fixed as **0.4**.

Similarly, **Fig. 3** reveals the variations in profit w.r.t. $\boldsymbol{\alpha_1}$ for varying values of $\boldsymbol{\eta_1}$ and $\boldsymbol{\eta_2}$, when the values of other parameters are kept fixed as $\boldsymbol{\alpha_2 = 0.00009}$, $\mathbf{K_0 = 70}$, $\mathbf{K_1 = 40}$ and $\mathbf{K_2 = 500}$. Here also the same trend in respect of $\boldsymbol{\alpha_1}$, $\boldsymbol{\eta_1}$ and $\boldsymbol{\eta_2}$ are observed as in case of **MTSF.** From the figure it is clearly observed from the smooth curves, that the system is profitable if the value of parameter $\boldsymbol{\alpha_1}$ is less than **0.41, 0.56** and **0.86** respectively for $\boldsymbol{\eta_1 = 0.2}$, **0.3** and **0.5** for fixed value of $\boldsymbol{\eta_2 = 0.9}$. From dotted curves, we conclude that system is profitable if the value of parameter α_1 is less than **0.44, 0.60** and **0.90** respectively for $\eta_1 = 0.2$, 0.3 and 0.5 for fixed value of $\eta_2 = 0.05$.

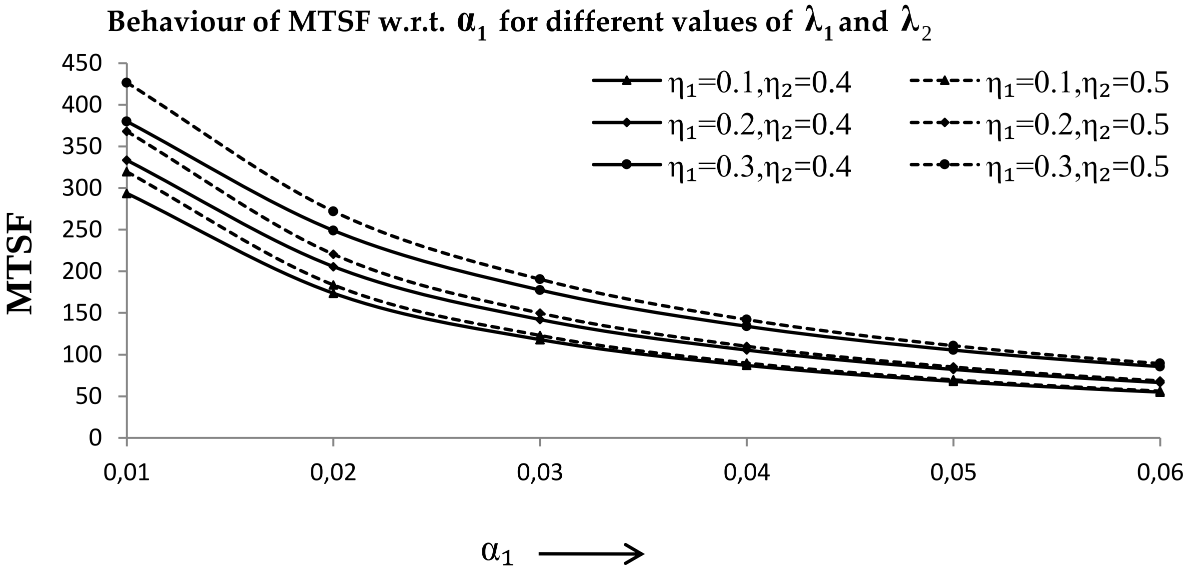

Fig. 2

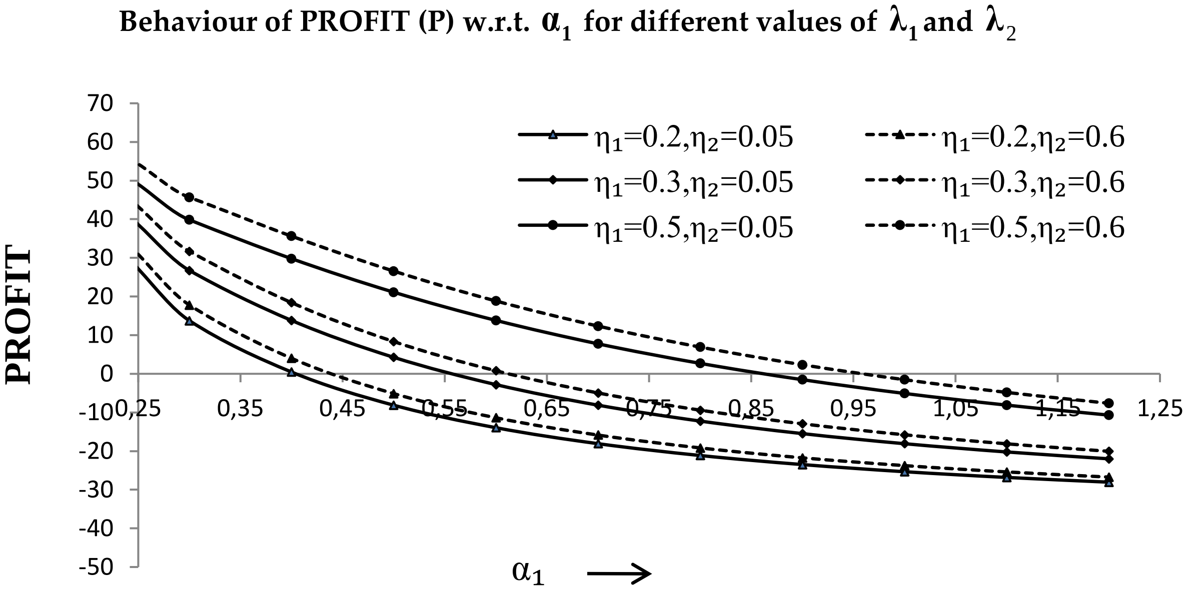

Fig. 3

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Analysis of Reliability Measures of Two Identical Unit System with One Switching Device and Imperfect Coverage

Akshita Sharma and Pawan Kumar

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Department of Statistics
University of Jammu, Jammu-180006, J&K, India
E-mail: akshita.sharma93@gmail.com

Abstract

The present paper deals with the reliability analysis of a two identical unit system model with safe and unsafe failures, switching device and rebooting. Initially one of the units is in operative state and other is kept in standby mode. A single repairman is always available with the system for repairing and rebooting the failed units. In case of unsafe failure, repair cannot be started immediately but first rebooting is done which transforms the unsafe failure to safe failure and thereafter repair is carried out as usual. Switching device is used to put the repaired and standby units to operation. The failure time distributions of both the units and switch are also assumed to be exponential while the repair time distributions are taken general in nature. Reboot delay time is assumed to be exponentially distributed. Using regenerative point techniques, various measures of system effectiveness such as transition probabilities, availability, busy period, expected numbers of repairs etc. have been obtained, to make the study more informative some of them have been studied graphically.

Key Words: reliability, availability, regenerative point technique, rebooting, coverage probability

1. Introduction

In the context of global competition and paced development, it has become the foremost concern to make the apt decisions in order to increase the reliability and profit margin of every institution. With the advent of complexity of machines and more advancements in industrial sectors, the focus on increasing reliability and profit margins of any firm is increasing day by day as it is the sole aim on which most of the firms/industries are flourishing. It has become an important point which has to be kept in mind that the designs and layout of complex equipments should be in such a way that it enhances the reliability of the system and try to minimize the loopholes which are responsible for its degradation. Hence, designing the reliable systems and determining their availability have become the relevant steps in almost every sector.

In many situations from daily life, we find that the breakdown of the units' results into machine failure which results into huge losses and one of the ways to increase reliability is to introduce standby units which increase its reliability. There also arises certain situations when reason for the failure of unit is not detected immediately which leads to the situation of imperfect coverage, which is further tackled by reboot. Depending upon the complexity the timings of reboot delay vary from system to system.. In the last few decades, elaborated and comprehensive research work regarding the reliability, availability, standby systems, imperfect coverage, reboot etc has been carried out. The concept of reboot is discussed by Trivedi [8] in his book 'Probability and Statistics with Reliability, Queueing and Computer Science Applications'. Several empirical studies are proposed by P.A. Keiller and D.R. Miller [3] to increase the reliability of system. The imperfect coverage models with various status and trends were given by Amari, et al. [1]. Hsu, et al. [4] have studied the machine repair problem with standby system, repair and reboot delay. The reliability measure of repairable system with standby switching failures and reboot delay is studied by Jyh-Bin, et al. [5]. The other important contributions are made by Amari, et al. [2], Wang and Chen [9], Ke and Liu [7]. Ke, et al. [6] has also done the analysis by considering detection, imperfect coverage and reboot as major factor.

The present paper here deals with the reliability analysis of a system model with two identical units and one switching device. The switching device is used to turn the unit from standby or repaired state to operative state and is assumed to be in good condition when the system initially starts. The failure in any of the identical unit or switch may result into safe/ unsafe failures. Unsafe failure is the situation when reason for any of the breakdown is not known which is cleared by reboot first. Reboot delay time, failure time of both the units and switch are assumed to be exponentially distributed while the repair time distributions are general in nature. Other measures of system effectiveness such as mean time to system failure, reliability, availability, expected number of repair have been evaluated using regenerative point techniques.

2. System Description and Assumptions

1. The system comprises of two identical units N_0 and N_s, one switch S is attached to it.
2. Initially one of the units is in operative state and other is kept in standby mode. Switching device is used to put the repaired and standby units to operation. In the initial phase, switching device is assumed to be in good condition.
3. The failures of units and switch in system might be safe and unsafe. Whenever any of the unit or switch of system results in safe failure, it may be immediately detected and located with coverage probability c and it will be repaired immediately if the repairman is available.
4. In case of unsafe failure, repair cannot be started immediately but first rebooting is done which transforms the unsafe failure to safe failure and thereafter repair is carried out as usual. Reboot delay times for units and switch are considered to be exponentially distributed random variables with different parameters.
5. A single repairman is always available with the system for repair and rebooting the failed units. Switch is always given preference over the failed units in the system for its repair.

6. The failure time distributions of both the units and switch are assumed to be exponential while the repair time distributions are taken general in nature.
7. Once a component is repaired it is as good as new.

3. Notations And Symbols

α : Failure rate of identical units
β : Failure rate of switching device
$H_1(.)$: Repair rate of failed unit
$H_2(.)$: Repair rate of switching device
c : Coverage probability
γ : Rebooting delay rate for unsafe failure for units
δ : Rebooting delay rate for unsafe failure for switching device

SYMBOLS FOR THE STATES OF THE SYSTEM

$N_0/N_s/N_g$: Unit is in operative / standby / good condition.
N_{usf}/S_{usf} : Unit/ switching device has undergone unsafe failure
N_r/N_{wr} : Unit is under repair / waiting for repair
S_r : Switch is under repair

With the help of the symbols defined above, the possible states of the system are:

$S_0 = [N_0, N_s, S_g]$ $S_1 = [N_r, N_0, S_g]$ $S_2 = [N_{usf}, N_g, S_g]$ $S_3 = [N_0, N_s, S_r]$
$S_4 = [N_g, N_s, S_{usf}]$ $S_5 = [N_g, N_g, S_r]$ $S_6 = [N_{usf}, N_g, S_r]$ $S_7 = [N_{wr}, N_g, S_r]$
$S_8 = [N_r, N_{usf}, S_g]$ $S_9 = [N_r, N_{wr}, S_g]$ $S_{10} = [N_r, N_g, S_{usf}]$

The transition diagram along with all transitions is shown in fig.1

TRANSITION DIAGRAM

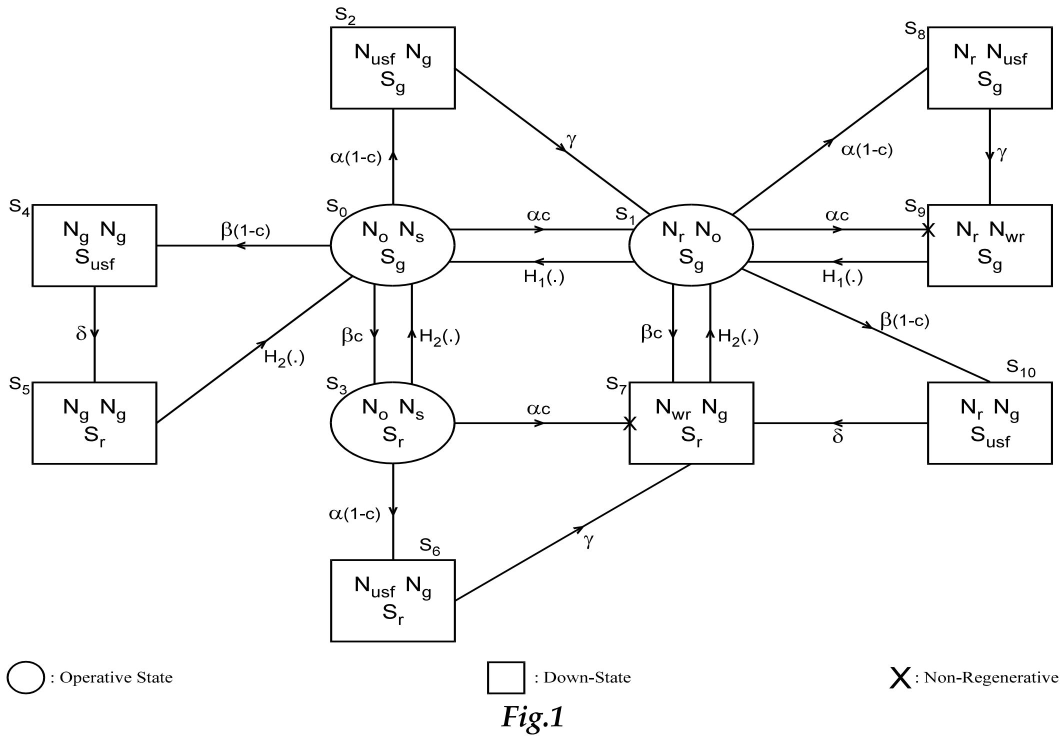

Fig.1

4. Transition Probabilities And Sojourn Times

Let X_n denotes the state visited at epoch T_{n^+} just after the transition at T_n, where T_1, T_2 .... represents the regenerative epochs. Then, Markov-Renewal process is constituted by $\{X_n, T_n\}$ with state space E representing set of regenerative states and

$$Q_{ij}(t) = P[X_{n+1} = j, T_{n+1} - T_n \leq t | X_n = i]$$

is the semi Markov kernel over E.

Then the transition probability matrix of the embedded Markov chain is

$$P = p_{ij} = Q_{ij}(\infty) = Q(\infty)$$

First we obtain the following direct steady-state transition probabilities:

$$p_{01} = \alpha c \int e^{-(\alpha+\beta)u}\, du = \frac{\alpha c}{(\alpha+\beta)}$$

Similarly,

$$p_{02} = \frac{\alpha(1-c)}{(\alpha+\beta)} \qquad p_{03} = \frac{\beta c}{(\alpha+\beta)}$$

$$p_{04} = \frac{\beta(1-c)}{(\alpha+\beta)} \qquad p_{10} = \widetilde{H}_1(\alpha+\beta)$$

$$p_{17} = \frac{\beta c}{(\alpha+\beta)}\left[1 - \widetilde{H}_1(\alpha+\beta)\right] \qquad p_{18} = \frac{\alpha(1-c)}{(\alpha+\beta)}\left[1 - \widetilde{H}_1(\alpha+\beta)\right]$$

$$p_{1,10} = \frac{\beta(1-c)}{(\alpha+\beta)}\left[1 - \widetilde{H}_1(\alpha+\beta)\right] \qquad p_{30} = \widetilde{H}_2(\alpha)$$

$$p_{36} = (1-c)\left[1 - \widetilde{H}_2(\alpha)\right]$$

$$p_{21} = p_{45} = p_{50} = p_{67} = p_{71} = p_{89} = p_{91} = p_{10,7} = 1$$

The indirect transition probability may be obtained as follows:

$$Q_{11}^{(9)}(t) = \alpha c \int_0^t e^{-(\alpha+\beta)u}\, \overline{H}_1(u) du \int_t^v \frac{dH_1(v)}{\overline{H}_1(u)}$$

$$= \alpha c \int_0^t dH_1(v) \int_0^v e^{-(\alpha+\beta)u}\, du$$

$$= \frac{\alpha c}{\alpha+\beta} \int_0^t (1 - e^{-(\alpha+\beta)v})\, dH_1(v)$$

By taking $t \to \infty$, we obtain the following indirect steady-state transition probability:

$$p_{11}^{(9)} = \frac{\alpha c}{\alpha+\beta} \int (1 - e^{-(\alpha+\beta)v}) dH_1(v) = \frac{\alpha c}{\alpha+\beta}\left[1 - \widetilde{H}_1(\alpha+\beta)\right]$$

Similarly,

$$p_{31}^{(7)} = c\left[1 - \widetilde{H}_2(\alpha)\right] \tag{1}$$

From these steady state probabilities obtained above, it can be easily verified that the following results holds good:

$$p_{01} + p_{02} + p_{03} + p_{04} = 1, \qquad p_{10} + p_{11}^{(9)} + p_{17} + p_{18} + p_{1,10} = 1$$

$$p_{30} + p_{36} + p_{31}^{(7)} = 1, \qquad p_{21} = p_{45} = p_{50} = p_{67} = p_{71} = p_{89} = p_{91} = p_{10,7} = 1 \tag{2}$$

Mean sojourn times

Mean sojourn time is defined as the expected time taken by the system in a state before making transition to any other state. Let Ψ_i be the mean sojourn time for state S_i, then to obtain mean sojourn time Ψ_i in state S_i, we observe that there is no transition from S_i to any other state as long as the system is in state S_i. If T_i denotes the sojourn time in state S_i then mean sojourn time Ψ_i in state S_i is:

$$\Psi_i = E[T_i] = \int P(T_i > t) dt$$

Hence, using it following expressions for mean sojourn time is obtained:

$$\Psi_0 = \frac{1}{(\alpha+\beta)} \qquad \Psi_1 = \frac{1}{(\alpha+\beta)}(1 - \widetilde{H}_1) \qquad \Psi_2 = \Psi_6 = \Psi_8 = \frac{1}{\gamma}$$

$$\Psi_3 = \frac{1}{\alpha}(1 - \widetilde{H}_2) \qquad \Psi_4 = \Psi_{10} = \frac{1}{\delta} \qquad \Psi_5 = \Psi_7 = \int \widetilde{H}_2(t) dt$$

$$\Psi_9 = \int \widetilde{H}_1(t)\, dt \tag{3}$$

5. Analysis Of Reliability And MTSF

Let T_i be the random variable denoting time to system failure when system starts up from state $S_i \in E_i$, then the reliability of the system is given by
$R_i(t) = P[T_i > t]$
To obtain $R_i(t)$, we consider failed states as absorbing states.
By referring to the state transition diagram, the recursive relations among $R_i(t)$ can be formulated on the basis of probabilistic arguments. Taking their Laplace Transform and solving the resultant set of equations for $R_0^*(s)$, we get

$$R_0^*(s) = N_1(s)/D_1(s) \tag{4}$$

where,
$N_1(s) = Z_0^* + q_{01}^* Z_1^* + q_{03}^* Z_3^*$
and
$D_1(s) = 1 - q_{01}^* q_{10}^* - q_{03}^* q_{30}^*$
Taking inverse Laplace Transform of **(4),** we get reliability of the system.
To get MTSF, we use the well known formula
$E(T_0) = \int R_0(t)dt = \lim_{s\to 0} R_0^*(s) = N_1(0)/D_1(0)$
where,
$N_1(0) = \Psi_0 + p_{01}\Psi_1 + p_{03}\Psi_3$
and
$D_1(0) = 1 - p_{01}p_{10} - p_{03}p_{30}$
Since, we have $q_{ij}^*(0) = p_{ij}$ *and* $\lim_{s\to 0} Z_i^*(s) = \int Z_i(t)dt = \Psi_i$

6. Availability Analysis

Define $A_i(t)$ as the probability that the system is available at time 't' given that initially started from state $S_i \in E_i$. Point wise availability is another measure of system effectiveness and is defined as the probability that the system will be able to work satisfactorily within tolerances at any instant of time. By using simple stochastic arguments, the recurrence relations among different point wise availabilities are obtained and taking the Laplace transforms and solving the resultant set of equations for $A_0^*(s)$, we have

$$A_0^*(s) = N_2(s)/D_2(s) \tag{5}$$

where,

$$N_2(s) = (1 - q_{11}^{(9)*} - q_{18}^* q_{89}^* q_{91}^* - q_{1,10}^* q_{10,7}^* q_{71}^* - q_{17}^* q_{71}^*)(Z_0^* + q_{03}^* Z_3^*) + Z_1^*(q_{01}^* + q_{03}^* q_{31}^{(7)*} + q_{03}^* q_{36}^* q_{67}^* q_{71}^* + q_{02}^* q_{21}^*) \tag{6}$$

and,

$$D_2(s) = (1 - q_{11}^{(9)*} - q_{18}^* q_{89}^* q_{91}^* - q_{1,10}^* q_{10,7}^* q_{71}^* - q_{17}^* q_{71}^*)(1 - q_{03}^* q_{30}^* - q_{04}^* q_{45}^* q_{50}^*) - q_{01}^* q_{10}^* - q_{10}^* q_{03}^* q_{31}^{(7)*} - q_{10}^* q_{03}^* q_{36}^* q_{67}^* q_{71}^* - q_{10}^* q_{02}^* q_{21}^* \tag{7}$$

The steady state availability is given by
$A_0 = \lim_{t\to\infty} A_0(t) = \lim_{s\to 0} s\, A_0^*(s) = N_2(0)/D_2(0)$
as we know that, $q_{ij}(t)$ is the pdf of the time of transition from state S_i to S_j and $q_{ij}(t)dt$ is the probability of transition from state S_i to S_j during the interval $(t, t + dt)$, thus
$q_{ij}^*(s)/_{s=0} = q_{ij}^*(0) = p_{ij}$
Also we know that
$\lim_{s\to 0} Z_i^*(s) = \int Z_i(t)dt = \Psi_i$

Therefore,

$$N_2(0) = (1 - p_{11}^{9} - p_{18}p_{89}p_{91} - p_{17}p_{71} - p_{1,10}p_{10,7}p_{71})(\Psi_0 + p_{03}\Psi_3) + \Psi_1(p_{01} + p_{03}p_{31}^{7} + p_{02}p_{21} + p_{03}p_{36}p_{67}p_{71}) \quad (8)$$

$$D_2(0) = (1 - p_{11}^{9} - p_{18}p_{89}p_{91} - p_{17}p_{71} - p_{1,10}p_{10,7}p_{71})(1 - p_{03}p_{30} - p_{04}p_{45}p_{50}) - p_{10}p_{01} - p_{10}p_{03}p_{31}^{7} - p_{10}p_{03}p_{36}p_{67}p_{71} - p_{10}p_{02}p_{21} \quad (9)$$

The steady state probability that the system will be up in the long run is given by

$$A_0 = \lim_{t\to\infty} A_0(t) = \lim_{s\to 0} s\, A_0^*(s)$$

$$\lim_{s\to 0} \frac{sN_2(s)}{D_2(s)} = \lim_{s\to 0} N_2(s) \lim_{s\to 0} \frac{s}{D_2(s)}$$

Since as $s \to 0, D_2(s)$ becomes zero.

Hence, on using L'Hospital's rule, A_0 becomes

$$A_0 = N_2(0)/D_2'(0) \quad (10)$$

where,

$$D_2'(0) = p_{10}(\Psi_0 + \Psi_2 p_{02} + \Psi_3 p_{03}) + p_{10}p_{04}(\Psi_4 + \Psi_5) + p_{10}p_{03}p_{36}(\Psi_6 + \Psi_7) + (1 - p_{03}p_{30} - p_{04})[\Psi_1 + p_{110}(\Psi_7 + \Psi_{10}) + \Psi_7 p_{17}] + (1 - p_{03}p_{30} - p_{04})p_{18}(\Psi_8 + \Psi_9) \quad (11)$$

Using **(8)** and **(11)** in **(10)**, we get the expression for A_0.

The expected up time of the system during (0, t] is given by

$$\mu_{up}(t) = \int_0^t A_0(u)\, du$$

So that,

$$\mu_{up}^*(s) = A_0^*(s)/s.$$

7. Busy Period Analysis

$B_i\,(t)$ is defined as the probability that the system having started from regenerative state $S_i \in E$ at that t=0, is under repair at time t due to failure of the unit. Now to determine these probabilities, we use simple probabilistic arguments and further taking the Laplace transform and solving the resultant set of equations for $B_0^*(s)$, we have

$$B_0^*(s) = N_3(s)/D_2(s) \quad (12)$$

where,

$$N_3(s) = (1 - q_{11}^{(9)*} - q_{18}^*q_{89}^*q_{91}^* - q_{1,10}^*q_{10,7}^*q_{71}^* - q_{17}^*q_{71}^*)[q_{02}^*Z_2^* + q_{03}^*Z_3^* + q_{04}^*(Z_4^* + q_{45}^*Z_5^*) + q_{03}^*q_{36}^*Z_6^*] + \left[q_{01}^* + q_{02}^*q_{21}^* + q_{03}^*(q_{31}^{(7)*} + q_{36}^*q_{67}^*q_{71}^*)\right]\left[Z_1^* + q_{18}^*(Z_8^* + q_{89}^*Z_9^*) + q_{1,10}^*Z_{10}^*\right] + Z_7^*\left\{q_{03}^*q_{36}^*q_{67}^*(1 - q_{11}^{(9)*} - q_{17}^*q_{71}^* - q_{18}^*q_{89}^*q_{91}^* - q_{1,10}^*q_{10,7}^*q_{71}^*) + (q_{1,10}^*q_{10,7}^* + q_{17}^*)\left[q_{01}^* + q_{02}^*q_{21}^* + q_{03}^*(q_{31}^{(7)*} + q_{36}^*q_{67}^*q_{71}^*)\right]\right\} \quad (13)$$

and, $D_2(s)$ is same as given by (7).

In the steady state, the probability that the repairman will be busy is given by

$$B_0 = \lim_{t\to\infty} B_0(t) = \lim_{s\to 0} s\, B_0^*(s) = N_3(0)/D_2'(0) \quad (14)$$

where,

$$N_3(0) = (1 - p_{11}^{(9)} - p_{17}p_{71} - p_{18}p_{89}p_{91} - p_{1,10}p_{10,7}p_{71})[\, p_{02}p_{04} + p_{03}\Psi_3 + p_{04}(p_{04} + p_{45}\Psi_5) + p_{03}p_{36}\Psi_6] + \left[p_{01} + p_{02}p_{21} + p_{03}(p_{31}^{(7)} + p_{36}p_{67}p_{71})\right]\left[\Psi_1 + p_{18}(\Psi_8 + p_{89}\Psi_9) + p_{1,10}\Psi_{10}\right] + \Psi_7\left\{p_{03}p_{36}p_{67}(1 - p_{11}^{(9)} - p_{17}p_{71} - p_{18}p_{89}p_{91} - p_{1,10}p_{10,7}p_{71}) + (p_{1,10}p_{10,7} + p_{17})\left[p_{01} + p_{02}p_{21} + p_{03}(p_{31}^{(7)} + p_{36}p_{67}p_{71})\right]\right\} \quad (15)$$

and $D_2'(0)$ is same as obtained in **(11)**.

The expected busy period of the repairman during $(0,\, t]$ is given by

$\mu_b(t) = \int_0^t B_0(u)\, du$

So that, $\mu_b^*(s) = B_0^*(s)/s$

8. Expected number of Repairs

$V_i(t)$ is defined as the expected number of repairs of the failed units during the time interval $(0, t]$ when the system initially starts from regenerative state S_i. Further, using the definition of $V_i(t)$ and by probabilistic reasoning the recurrence relations are easily obtained and taking their Laplace- Stieltjes transforms and solving the resultant set of equations for $\tilde{V}_0(s)$, we get

$$\tilde{V}_0(s) = N_4(s)/D_3(s) \tag{16}$$

where,

$$N_4(s) = \left(1 - \tilde{Q}_{11}^9 - \tilde{Q}_{17}\tilde{Q}_{71} - \tilde{Q}_{18}\tilde{Q}_{89}\tilde{Q}_{91} - \tilde{Q}_{1,10}\tilde{Q}_{10,7}\tilde{Q}_{71}\right)\left(\tilde{Q}_{03}\tilde{Q}_{30} + \tilde{Q}_{04}\tilde{Q}_{45}\tilde{Q}_{50}\right) + \tilde{Q}_{10}(\tilde{Q}_{01} + \tilde{Q}_{02}\tilde{Q}_{21} + \tilde{Q}_{03}\tilde{Q}_{31}^{(7)} + \tilde{Q}_{03}\tilde{Q}_{36}\tilde{Q}_{67}\tilde{Q}_{71}) \tag{17}$$

and $D_3(s)$ can be written on replacing q_{ij}^* and $q_{ij}^{(k)*}$ by $\tilde{Q}_{ij}$ and $\tilde{Q}_{ij}^{(k)}$ respectively in the equation (7) .

In the steady state, the expected number of repairs per unit time is given by

$$V_0 = \lim_{t\to\infty}[V_0(t)/t] = \lim_{s\to 0} s\, \tilde{V}_0(s) = N_4(0)/D_2'(0) \tag{18}$$

where,

$$N_4(0) = (1 - p_{11}^9 - p_{17}p_{71} - p_{18}p_{89}p_{91} - p_{1,10}p_{10,7}p_{71})(p_{03}p_{30} + p_{04}p_{45}p_{50}) + p_{10}(p_{01} + p_{02}p_{21} + p_{03}p_{31}^{(7)} + p_{03}p_{36}p_{67}p_{71}) \tag{19}$$

9. Profit Function Analysis

With the help of reliability characteristics obtained, the profit function $P(t)$ for the system can be obtained easily. Profit is excess of revenue over the cost, therefore, the expected total profits generated during (*0, t*] is:

$P(t) =$ Expected total revenue in (*0,t*]− Expected total expenditure in (*0,t*]

$$= K_0\mu_{up}(t) - K_1\mu_b(t) - K_2V_0 \tag{20}$$

where,

K_0 : Revenue per unit up time of the system.

K_1 : Cost per unit time for which repair man is busy in repairing the failed unit.

K_2 : Cost of repair per unit.

In steady state, the expected total profits per unit time, is

$P = \lim_{t\to\infty}[P(t)/t] = \lim_{s\to 0} s^2P^*(s)$

Therefore, we have

$$P = K_0A_0 - K_1B_0 - K_2V_0 \tag{21}$$

10. Graphical Study Of The System Model

Graphical study of the system model gives a more perceived picture of system behaviour. So, for more concrete study, we plot MTSF and Profit function with respect to α, failure rate of identical unit for different values of h_1, repair rate of identical unit.

Fig. 2 represents the variations in MTSF with respect to α for different values of $h_1$as 0.05, 0.40, 0.80 by keeping all the other parameters fixed at $\beta = 0.03$, $h = 0.20$, $\gamma = 0.35$, $\delta = 0.45$. The coverage probability c for the system is set at 0.70. It can be clearly seen from the graph of MTSF that it decreases continuously with increase in failure rate α and by increasing repair rate h_1, the value for MTSF also increases, thereby concluding that repair facility increases the lifetime of the system.

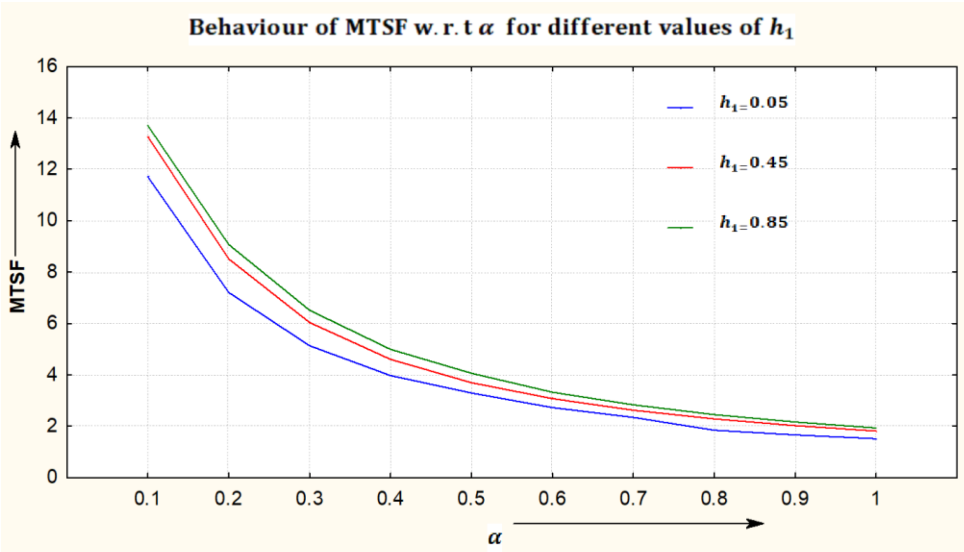

Fig. 2

Fig. 3 represents the change in Profit function P with respect to α for different values of h_1 as 0.05, 0.40, 0.80 by keeping all the other parameters fixed at $\beta = 0.03$, $h = 0.20$, $\gamma = 0.35$, $\delta = 0.45$. The coverage probability c for the system is set at 0.70. Clearly, it is observed that profit function decreases with increase in failure rate α but increases with increase in repair rate h_1. Hence, repairing the system from time to time will result in better performance of the system.

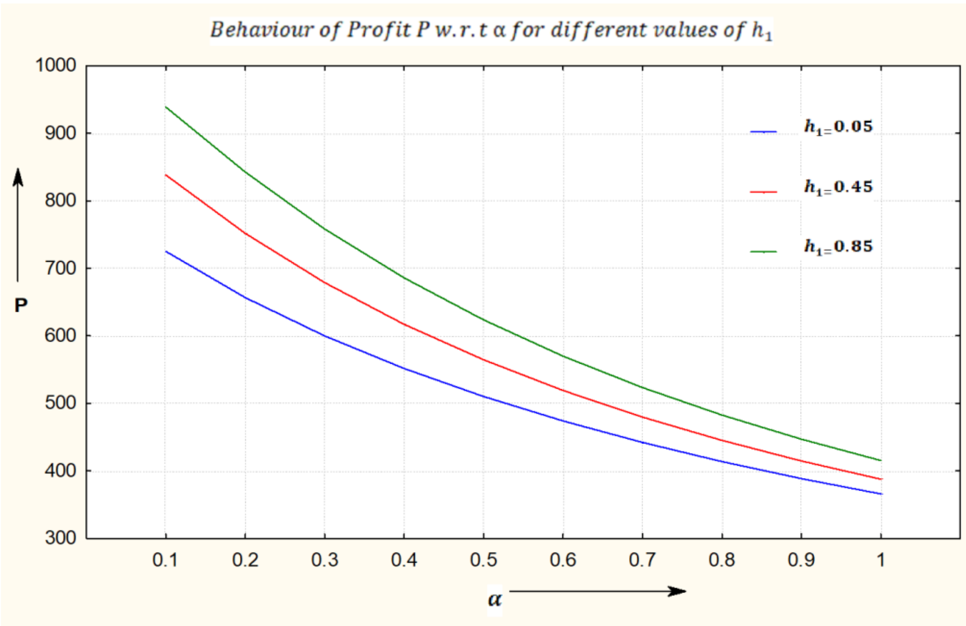

Fig. 3

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Performance Measures Of A Two Non-Identical Unit System Model With Repair And Replacement Policies

Urvashi Gupta, Pawan Kumar

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Department of Statistics
University of Jammu Jammu-180006
Email: urvigupta2791@gmail.con

Abstract

The present paper deals with two non-identical units A and B, both are in operative mode. If the unit A fails then it is taken up for preparation of repair before entering into repair mode and the unit-B gives a signal for repair before going into failure mode. If the unit gets repaired then it becomes operative otherwise it is replaced by the new one. A single repairman is always available with the system to repair the failed units and the priority in repair is always given to the unit-A. The failure time distributions of both the units are taken as exponential and the repair time distributions are taken as general. Using regenerative point technique the various characteristics of the system effectiveness have been obtained such as Transition Probabilities and Mean Sojourn times, Mean time to system failure (MTSF), Availability of the system, Busy Period of repairman, Expected number of Replacement, Expected profit incurred.

Keyword: Preparation, Signal, Repair and Replacement.

1.1 INTRODUCTION

Several researchers have considered and studied numerous reliability system models having identical units. In view of their growing use in modern technology the study of reliability characteristics of different stochastic models have attracted the attention of the researchers in the field of reliability theory and system engineering. To help system designers and operational managers, various researchers including [1,2,3] in the field of reliability theory have analysed two unit system models with two types of repairs, replacement policy, signal concept etc. They obtained various economic measures of system effectiveness by using regenerative point technique. The common assumption which is taken in most of these models is that a single repairman is always available with the system to repair the failed unit and after the repair the unit becomes as good as new. But in many practical situations, it is not possible that a single repairman perform the whole process of repair particularly in case of complicated unit/machine. Goel [1] analyzed that the multi standby, multi failure mode system model with repair and replacement policy and there are various authors who have carried out study on repair and replacement policies.

In the present paper, we study a two non-identical units system model. The units are named as A and B and are taken to be in operative mode. If the unit A fails then goes for preparation for repair before entering into repair. Unit-B while in operation gives a signal for its repair before going in to failure mode and if it gets repaired it starts its functioning in usual manner otherwise it is replaced by the new one. A single repairman is always available with the system to repair the failed units and the priority in repair is always given to the unit-A The failure time distributions of both the units are taken as exponential and the repair time distribution is taken as general. All random variable are statistically independent.

Using semi- Markov process and regenerative point technique the expressions for the following important performance measures of the system have been derived in steady state -

1. Transition Probabilities and mean Sojourn times.
2. Mean time to system failure (MTSF).
3. Availability of the system.
4. Busy period of repairman.
5. Expected number of replacement of the unit.
6. Net expected profit earned by the system during the interval (0,t) and in steady state.

1.2 MODEL DESCRIPTION AND ASSUMPTIONS

1. The system comprises of two non-identical units A and B initially both are in operative mode.

2. Upon the failure of unit A, it will go for preparation for repair before taken up for repair.

3. Unit-B while in operation gives a signal for its repair before going in to failure mode and if it is not repaired in a stipulated time it is replaced by the new one.

4. A single repairman is always available with the system to repair and replace the failed units and the priority in repair is always given to the unit A over unit B

5. The failures of the units are independent and the failure time distributions of the units are taken as Exponential.

6. The repair time distributions of the units are taken as general.

1.3 NOTATIONS AND STATES OF THE SYSTEM

We define the following symbols for generating the various states of the system.

A_{10}, B_{20} : Unit A and unit B are in operative mode.
$A_{1r}/\ A_{1P}$: Unit A under repair/ preparation for repair.
$B_{2sr}/\ B_{2srw}$: Unit B in operative mode and gives signal for repair/waiting of signal for repair.
$B_{2r}/\ B_{2rw}$: Unit B under repair/waiting for repair.
B_{2R}/B_{2Rw} : Unit B under replacement/waiting for replacement

b) NOTATIONS:

E : Set of regenerative states $= \{S_0, S_1, S_2, S_3, S_4, S_5, S_6, S_8\}$
$\overline{E}$: Set of non − regenerative states $= \{S_7, S_9, S_{10}, S_{11}\}$
α_1 : Failure rate of unit − A
α_2 : Repair rate of unit − B
β_1 : Parameter for signal
β_2 : Repair rate of unit − A
β_3 : Repalcement rate of unit − B
H_1 : cdf of repair time of unit − B
H_2 : cdf of repair time of unit − A
G_1 : cdf of replacement time of unit − B

TRANSITION DIAGRAM

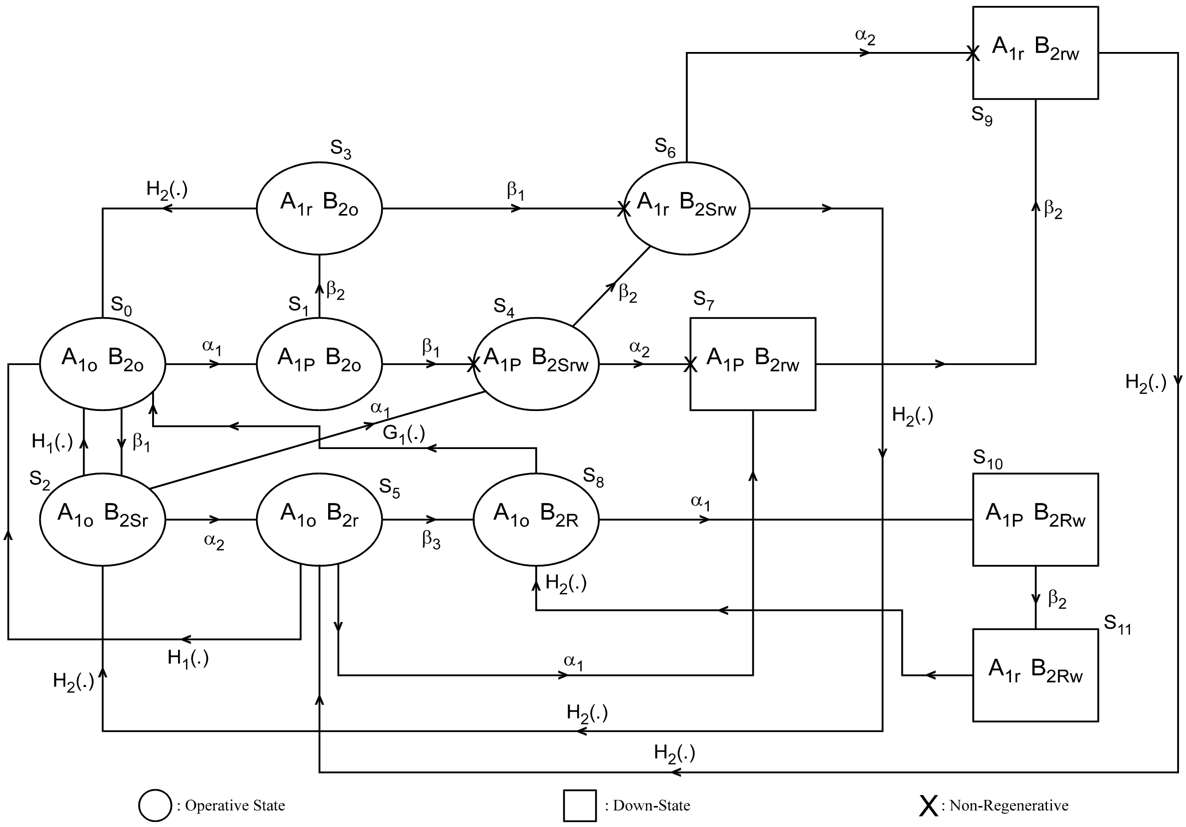

Fig.1.1

1.4 TRANSITION PROBABLITIES

Let X_n denotes the state visited at epoch T_{n^+} just after the transition at T_n, where T_1, T_2 .... represents the regenerative epochs, then$\{X_n, T_n\}$ constitute a Markov-Renewal process with state space E and

$Q_{ij}(t) = P[X_{n+1} = j, T_{n+1} - T_n \leq t | X_n = i]$

is the semi Markov kernel over E.

Then the transition probability matrix of the embedded Markov chain is

$P = p_{ij} = Q_{ij}(\infty) = Q(\infty)$

We obtain the following direct steady-state transition probabilities:

$$p_{01} = \alpha_1 \int e^{-(\alpha_1+\beta_1)u} du = \frac{\alpha_1}{(\alpha_1+\beta_1)}$$

Similarly,

$$p_{02} = \frac{\beta_1}{(\alpha_1+\beta_1)}, \quad p_{13} = \frac{\beta_2}{(\beta_2+\beta_1)}, \quad p_{20} = H_1^*(\alpha_2+\alpha_1)$$

$$p_{24} = \frac{\alpha_1}{(\alpha_2+\alpha_1)}[1 - H_1^*(\alpha_2+\alpha_1)], \quad p_{25} = \frac{\alpha_2}{(\alpha_2+\alpha_1)}[1 - H_1^*(\alpha_2+\alpha_1)]$$

$$p_{30} = H_2^*(\beta_1), \quad p_{46} = \frac{\beta_2}{\alpha_2+\beta_2}, \quad p_{50} = H_1^*(\alpha_1+\beta_3)$$

$$p_{57} = \frac{\alpha_1}{\alpha_1+\beta_3}[1 - H_1^*(\alpha_1+\beta_3)], \quad p_{58} = \frac{\beta_3}{\beta_3+\alpha_1}[1 - H_1^*(\alpha_1+\beta_3)]$$

$$p_{62} = H_2^*(\alpha_2), \quad p_{80} = G_1^*(\alpha_1), \quad p_{8,10} = 1 - G_1^*(\alpha_1)$$

$$p_{79} = p_{95} = p_{10,11} = p_{11,8} = 1$$

The indirect transition probability may be obtained as follows:

$$p_{16}^{(4)} = \frac{\beta_1\beta_2}{(\beta_1-\alpha_2)} \int e^{-(\beta_2+\alpha_2)v} - e^{-(\beta_2+\beta_1)v} du$$

$$= \frac{\beta_1\beta_2}{(\beta_2+\alpha_2)(\beta_2+\beta_1)}$$

Similarly,

$$p_{19}^{(4,7)} = 1 + \frac{\beta_2\alpha_2}{(\beta_1-\alpha_2)(\beta_2+\beta_1)} - \frac{\beta_1\beta_2}{(\beta_2-\alpha_2)(\beta_2+\alpha_2)}$$

$$p_{35}^{(6,9)} = 1 - \frac{\beta_1 H_2^*(\alpha_2)}{(\beta_1-\alpha_2)} + \frac{\alpha_2 H_2^*(\beta_1)}{(\beta_1-\alpha_2)}, \quad p_{32}^{(6)} = \frac{\beta_1}{(\beta_1-\alpha_2)}[H_2^*(\alpha_2) - H_2^*(\beta_1)]$$

$$p_{49}^{(7)} = \frac{\alpha_2}{\alpha_2+\beta_2} \quad p_{65}^{(9)} = 1 - H_2^*(\alpha_2)$$

It can be easily verified that

$$p_{01} + p_{02} = 1, \quad p_{13} + p_{19}^{(4,7)} + p_{16}^{(4)} = 1, \quad p_{20} + p_{24} + p_{25} = 1$$

$$p_{30} + p_{35}^{(6,9)} + p_{32}^{(6)} = 1, \quad p_{46} + p_{49}^{(7)} = 1, \quad p_{50} + p_{57} + p_{58} = 1$$

$$p_{62} + p_{65}^{(9)} = 1, \quad p_{80} + p_{8,10} = 1, \quad p_{79} = p_{95} = p_{10,11} = p_{11,8} = 1$$

A) MEAN SOJOURN TIMES

The mean sojourn time in state S_i denoted by μ_i is defined as the expected time taken by the system in state S_i before transiting to any other state. To obtain mean sojourn time μ_i, in state S_i, we observe that as long as the system is in state S_i, there is no transition from S_i to any other state. If T_i denotes the sojourn time in state S_i then mean sojourn time μ_i in state S_i is:

$$\mu_i = E[T_i] = \int P(T_i > t) dt$$

Therefore,

$$\mu_0 = \frac{1}{\alpha_1+\beta_1}, \quad \mu_1 = \frac{1}{\beta_1+\beta_2}, \quad \mu_2 = \frac{1}{\alpha_1} - H_1^*(\alpha_1)$$

$$\mu_3 = \frac{1}{\beta_1} - H_2^*(\beta_1), \quad \mu_4 = \frac{1}{\alpha_2+\beta_2}, \quad \mu_5 = \frac{1}{\beta_3+\alpha_1} - H_1^*(\beta_3+\alpha_1)$$

$$\mu_6 = \frac{1}{\alpha_2} - H_2^*(\alpha_2), \quad \mu_8 = \frac{1}{\alpha_1} - G_1^*(\alpha_1) \quad \mu_7 = \mu_{10} = \frac{1}{\beta_2}$$

$$\mu_9 = \mu_{11} = \int \overline{H}_2(t) dt$$

1.5 ANALYSIS OF RELIABILITY

Let T_i be the random variable denoting time to system failure when system starts up from state $S_i \in E_i$, then the reliability of the system is given by

$$R_i(t) = P[T_i > t]$$

To obtain $R_i(t)$, we consider failed states as absorbing states.

The recursive relations among $R_i(t)$ can be developed on the basis of probabilistic arguments. Taking their Laplace Transform and solving the resultant set of equations for $R_0^*(s)$, we get

$$R_0^*(s) = \frac{N_1(s)}{D_1(s)} \tag{1.5.1}$$

Where

$$\begin{aligned} N_1(s) = {} & [(1 - q_{24}^* q_{46}^* q_{62}^*)(Z_0^* + q_{01}^* Z_1^* + q_{01}^* q_{13}^* Z_3^*)] \\ & + \left[q_{01}^* \left(q_{13}^* q_{32}^{(6)*} + q_{16}^{(4)*} q_{62}^*\right) + q_{02}^*\right] (Z_2^* + q_{24}^* Z_4^* + q_{25}^* Z_5^* + q_{25}^* q_{58}^* Z_8^*) \\ & + \left[q_{01}^* q_{16}^{(4)*} + q_{24}^* q_{46}^* \left(q_{01}^* q_{13}^* q_{32}^{(6)*} + q_{02}^*\right)\right] Z_6^* \end{aligned}$$

$$\begin{aligned} D_1(s) = {} & [(1 - q_{24}^* q_{46}^* q_{62}^*)(1 - q_{01}^* q_{13}^* q_{30}^*)] \\ & - \left[\left(q_{13}^* q_{32}^{(6)*} + q_{16}^{(4)*} q_{62}^*\right) q_{01}^* - q_{02}^*\right] (q_{20}^* + q_{50}^* + q_{25}^* q_{58}^* q_{80}^*) \end{aligned}$$

Taking the Inverse Laplace Transform of **(1.5.1)**, one gets the reliability of the system.
To get MTSF, we use the well known formula

$$E(T_0) = \int R_0(t)dt = \lim_{s \to 0} R_0^*(s) = N_1(0)/D_1(0)$$

where,

$$\begin{aligned} N_1(0) = {} & [(1 - p_{24} p_{46} p_{62})(\mu_0 + \mu_1 p_{01} + \mu_3 p_{01} p_{13})] \\ & + \left[p_{01} \left(p_{13} p_{32}^{(6)} + p_{16}^{(4)} p_{62}\right) + p_{02}\right] (\mu_2 + p_{24}\mu_4 + p_{25}\mu_5 + p_{25} p_{58} \mu_8) \\ & + \left[p_{01} p_{16}^{(4)} + p_{24} p_{46} \left(p_{01} p_{13} p_{32}^{(6)} + p_{02}\right)\right] \mu_6 \end{aligned}$$

$$\begin{aligned} D_1(0) = {} & [(1 - p_{24} p_{46} p_{62})(1 - p_{01} p_{13} p_{30})] \\ & - \left[\left(p_{13} p_{32}^{(6)} + p_{16}^{(4)} p_{62}\right) p_{01} - p_{02}\right] (p_{20} + p_{50} + p_{25} p_{58} p_{80}) \end{aligned}$$

Since, we have $q_{ij}^*(0) = p_{ij}$ and $\lim_{s \to 0} Z_i^*(s) = \int Z_i(t)dt = \mu_i$

1.6 AVAILABILITY ANALYSIS

Let $A_i(t)$ be the probability that the system is available at epoch t, when it initially starts from $S_i \in E$. Using the regenerative point technique and the tools of L.T., one can obtain the value of above probabilities in terms of their L.T. i.e$A_i^{n*}(s)$.Solving the resultant set of equations and simplifying for $A_0^*(s)$, we have

$$A_0^{n*}(s) = N_2(s)/D_2(s) \tag{1.6.1}$$

$$\begin{aligned} N_2(s) = {} & q_{80}(1 - q_{57})(1 - q_{24} q_{46} q_{62})[Z_0 + q_{01} Z_1] + q_{01}[q_{13} q_{80}(1 - q_{57})(1 - q_{24} q_{46} q_{62}) Z_3] \\ & + q_{80}(1 - q_{57})\big(Z_2 + q_{24}(Z_4 + q_{46} Z_6)\big) \left[q_{01} q_{13} q_{32}^{(6)} + q_{02}\right] \\ & + \left\{(q_{80} Z_5 + q_{58} Z_8) \left[\left(q_{65}^{(9)} q_{46} + q_{4,9}^{(7)}\right) q_{24} + q_{25}\right]\right\} \left(q_{01} q_{32}^{(6)} + q_{02}\right) \\ & + q_{01}(q_{80} Z_5 + q_{58} Z_8)(1 - q_{24} q_{46} q_{62}) \left[q_{35}^{(6,9)} + q_{19}^{(4,7)}\right] \\ & + q_{01} q_{16}^{(4)} \Big[q_{80}(1 - q_{57}) \big[(Z_6 + q_{62} Z_2 + q_{24} q_{62} Z_4) \\ & + (q_{80} Z_5 + q_{58} Z_8) \left\{q_{6,5}^{(9)} + q_{62} q_{25} + q_{25} q_{4,9}^{(7)}\right\}\big]\Big] \end{aligned} \tag{1.6.2}$$

and

$$D_2(s) = q_{80}(1-q_{57})(1-q_{24}q_{46}q_{62}) - \left\{q_{01}\left[q_{13}q_{80}(1-q_{57})(1-q_{24}q_{46}q_{62})\left(1-q_{32}^{(6)}\right)\right]\right\} - q_{01}q_{13}q_{80}q_{32}^{(6)}q_{80}(1-q_{57})(1-q_{24}q_{46}q_{62}) - q_{01}q_{80}q_{19}^{(4,7)}(1-q_{57})(1-q_{24}q_{46}q_{62}) - q_{01}q_{80}q_{16}^{(4)}(1-q_{57})(1-q_{24}q_{46}q_{62}) - q_{02}q_{80}(1-q_{57})(1-q_{24}q_{46}q_{62}) \quad (1.6.3)$$

The steady state availability is given by

$A_0 = \lim_{t\to\infty} A_0(t) = \lim_{s\to 0} sA_0^*(s) = \frac{N_2(0)}{D_2(0)}$

As we know that, $q_{ij}(t)$ is the pdf of the time of transition from state S_i to S_j and $q_{ij}(t)dt$ is the probability of transition from state S_i to S_j during the interval $(t, t+dt)$, thus

$\lim_{s\to 0} Z_i^*(s) = \int Z_i(t)dt = \mu_i$ and $q_{ij}^*(s) = q_{ij}^*(0) = p_{ij}$, we get

Therefore,

$$N_2(0) = p_{80}(1-p_{57})(1-p_{24}p_{46}p_{62})[\mu_0 + p_{01}\mu_1] + p_{01}[p_{13}p_{80}(1-p_{57})(1-p_{24}p_{46}p_{62})\mu_3] + p_{80}(1-p_{57})(\mu_2 + p_{24}(\mu_4 + p_{46}\mu_6))\left[p_{01}p_{13}p_{32}^{(6)} + p_{02}\right] + \left\{(p_{80}\mu_5 + p_{58}\mu_8)\left[\left(p_{65}^{(9)}p_{46} + p_{4,9}^{(7)}\right)p_{24} + p_{25}\right]\right\}\left(p_{01}p_{32}^{(6)} + p_{02}\right) + p_{01}(p_{80}\mu_5 + p_{58}\mu_8)(1-p_{24}p_{46}p_{62})\left[p_{35}^{(6,9)} + p_{19}^{(4,7)}\right] + p_{01}p_{16}^{(4)}\left[p_{80}(1-p_{57})\left[(\mu_6 + p_{62}\mu_2 + p_{24}p_{62}\mu_4) + (p_{80}\mu_5 + p_{58}\mu_8)\left\{p_{6,5}^{(9)} + p_{62}p_{25} + p_{25}p_{4,9}^{(7)}\right\}\right]\right] \quad (1.6.4)$$

$$D_2(0) = p_{80}(1-p_{57})(1-p_{24}p_{46}p_{62}) - \left\{p_{01}\left[p_{13}p_{80}(1-p_{57})(1-p_{24}p_{46}p_{62})\left(1-p_{32}^{(6)}\right)\right]\right\} - p_{01}p_{13}p_{80}p_{32}^{(6)}p_{80}(1-p_{57}) - p_{01}p_{80}p_{19}^{(4,7)}(1-p_{57})(1-p_{24}p_{46}p_{62}) - p_{01}p_{80}p_{16}^{(4)}(1-p_{57})(1-p_{24}p_{46}p_{62}) - p_{02}p_{80}(1-p_{57})(1-p_{24}p_{46}p_{62})$$

The steady state probability that the system will be up in the long run is given by

$A_0 = \lim_{t\to\infty} A_0(t) = \lim_{s\to 0} sA_0^*(s) \lim_{s\to 0}\frac{sN_2(s)}{D_2(s)} = \lim_{s\to 0} N_2(s) \lim_{s\to 0}\frac{s}{D_2(s)}$

as $s\to 0$, $D_2(s)$ becomes zero.

Therefore, by L' Hospital's rule, A_0 becomes

$$A_0 = N_2(0)/D_2'(0) \quad (1.6.5)$$

where,

$$D_2'(0) = \mu_0\{p_{80}(1-p_{57})(1-p_{24}p_{46}p_{62})\} + \mu_1\{p_{01}p_{80}(1-p_{57})(1-p_{24}p_{46}p_{62})\} + \mu_2\left\{p_{80}(1-p_{57})p_{01}\left[p_{13}p_{32}^{(6)} + p_{16}^{(4)}p_{62}\right] + p_{80}(1-p_{57})p_{02}\right\} + \mu_3\{p_{01}p_{80}p_{13}(1-p_{57})(1-p_{24}p_{46}p_{62})\} + \mu_4\left\{p_{80}(1-p_{57})p_{24}\left[p_{01}p_{16}^{(4)}p_{62} + p_{01}p_{13}p_{32}^{(6)} + p_{02}\right]\right\} + (\mu_5 + \mu_7 + \mu_8 + \mu_9 + \mu_{10} + \mu_{11})(1) + \mu_6\left\{p_{02}p_{80}(1-p_{57})p_{24}p_{46} + p_{01}p_{80}(1-p_{57})p_{24}p_{46}p_{32}^{(6)}\right\} \quad (1.6.6)$$

Using the results **(1.6.4)** and **(1.6.6)** in **(1.6.5)**, we get the expressions for A_0.

The expected up (operative) time of the system during (0, t] is given by

$$\mu_{up}(t) = \int_0^t A_0(u)du$$

So that,

$$\mu_{up}^*(s) = \frac{A_0^*(s)}{s}$$

1.7 BUSY PERIOD OF REPAIRMAN

Let $B_i(t)$ be the probability that the repairman is busy in the repair of failed unit at epoch t, when the system initially starts operation from state $S_i \in E$. Developing the recursive relations among $B_i(t)$'s and solving the resultant set of equations and simplifying for $B_0^*(s)$, we have

$$B_0^*(s) = N_3(s)/D_2(s) \tag{1.7.1}$$

where

$$\begin{aligned}N_3(s) = \Big[&q_{01}^*(1 - q_{24}^* q_{46}^* q_{62}^*)\left[q_{13}^* q_{35}^{(6,9)*} + q_{65}^{(9)*} + q_{19}^{(4,7)*}\right] \\ &+ q_{01}^*(1 - q_{20}^* - q_{24}^* q_{46}^* q_{62}^*)\left(q_{32}^{(6)*} + q_{16}^{(4)*} q_{62}^*\right) \\ &+ q_{02}^*(1 - q_{20}^* - q_{24}^* q_{46}^* q_{62}^*)\Big]\left\{q_{80}^* M_5^* + q_{8,10}^* q_{58}^* M_{11}^* + q_{80}^* q_{57}^* q_{79}^* M_9^*\right\} \\ &+ \left\{q_{01}^* q_{80}^*(1 - q_{57}^*)(1 - q_{24}^* q_{46}^* q_{62}^*)\left[q_{13}^* M_3^* + q_{16}^{(4)*} M_6^*\right] + q_{01}^* q_{16}^{(4)*} M_9^*\right\} \\ &+ q_{80}^*(1 - q_{57}^*)\left[M_2^* + q_{24}^*\left(q_{46}^* M_6^* + q_{49}^{(7)*} M_9^*\right)\right]\left\{q_{01}^*\left(q_{13}^* q_{32}^{(6)*} + q_{16}^{(4)*} q_{62}^*\right) \right. \\ &\left. + q_{02}^*\right\}\end{aligned}$$

In the long run, the expected fraction of time for which the expert server is busy in the repair of failed unit is given by

$$B_0 = \lim_{t\to\infty} B_0(t) = \lim_{s\to 0} B_0^*(s) = \frac{N_3(0)}{D_2'(0)} = \frac{N_3}{D_2} \tag{1.7.2}$$

$$\begin{aligned}N_3(0) = \Big[&p_{01}(1 - p_{24}p_{46}p_{62})\left[p_{13}p_{35}^{(6,9)} + p_{65}^{(9)} + p_{19}^{(4,7)}\right] + p_{01}(1 - p_{20} - p_{24}p_{46}p_{62})\left(p_{32}^{(6)} + \right. \\ &\left. p_{16}^{(4)} p_{62}\right) + p_{02}(1 - p_{20} - p_{24}p_{46}p_{62})\Big]\left\{p_{80}\mu_5 + p_{8,10}p_{58}\mu_{11} + p_{80}p_{57}p_{79}\mu_9\right\} + \\ &\left\{p_{01}p_{80}(1 - p_{57})(1 - p_{24}p_{46}p_{62})\left[p_{13}\mu_3 + p_{16}^{(4)}\mu_6\right] + p_{01}p_{19}^{(4,7)}\mu_9\right\} + p_{80}(1 - p_{57})\left[\mu_2 + \right. \\ &\left. p_{24}\left(p_{46}\mu_6 + p_{49}^{(7)}\mu_9\right)\right]\left\{p_{01}\left(p_{13}p_{32}^{(6)} + p_{16}^{(4)}p_{62}\right) + p_{02}\right\}\end{aligned} \tag{1.7.3}$$

and $D_2(s)$ is same as given by **(1.6.6)**.

Thus using **(1.7.3)** and **(1.6.6)** in **(1.7.2)**, we get the expression for B_0.

The expected busy period of repairman during the time interval (0,t] is given by

$$\mu_b(t) = \int_0^t B_0(u)\,du$$

So that

$$\mu_b^*(s) = \frac{B_0^*(s)}{s}$$

1.8 EXPECTED NUMBER OF REPLACEMENTS

Let $V_i^{rp}(t)$ be the expected number of replacements by the server in (0,t] given that the system entered the regenerative state S_i at t=0. Framing the relations among $V_i^{rp}(t)$, taking L.S.T and solving for $\widetilde{V}_0^{r}(s)$, we get

$$\widetilde{V}_0^{rp}(s) = \frac{N_4^{rp}(s)}{D_2(s)} \tag{1.8.1}$$

where,

$$\begin{aligned}N_4^{rp}(s) = \widetilde{Q}_{01}\Big[&\widetilde{Q}_{12}\widetilde{Q}_{21} + \widetilde{Q}_{25}\widetilde{Q}_{21}(\widetilde{Q}_{78}\widetilde{Q}_{89} + \widetilde{Q}_{79})\widetilde{Q}_{57}\widetilde{Q}_{90} + \widetilde{Q}_{13}\widetilde{Q}_{34}\left(\widetilde{Q}_{46}\widetilde{Q}_{69}^{(8)} + \widetilde{Q}_{48}^{(7)}\widetilde{Q}_{89} + \widetilde{Q}_{49}^{(7)}\right) \\ &+ \widetilde{Q}_{13}\widetilde{Q}_{37}^{(5)}(\widetilde{Q}_{78}\widetilde{Q}_{89} + \widetilde{Q}_{79})\widetilde{Q}_{90}\Big]\end{aligned}$$

and $D_2(s)$ can be obtained on replacing q_{ij}'s by Q_{ij}'s in **1.6.6**

In steady-state per-unit of time expected number of replacement by server is given

$$V_0^{rp} = \lim_{t\to\infty}\frac{V_0^{rp}(t)}{t} = \lim_{s\to 0}\widetilde{V}_0^{rp}(s) = \frac{N_4^{rp}(0)}{D_2'(0)} = \frac{N_4^{rp}}{D_2} \tag{1.8.2}$$

Where

$$N_4^{rp} = p_{01}\Big[p_{12}p_{21} + p_{12}p_{25}(p_{78}p_{89} + p_{79})p_{57}p_{90} + p_{13}p_{34}\left(p_{46}p_{69}^{(8)} + p_{89}p_{48}^{(7)} + p_{49}^{(7)}\right) +$$

$$p_{13}p_{37}^{(5)}(p_{78}p_{89}+p_{79})p_{90}\Big] \tag{1.8.3}$$

Thus using **(1.8.3)** and **(1.6.6)** in **(1.8.2),** we get the expression for V_0^{rp}.

1.9 PROFIT FUNCTION ANALYSIS

The net profit incurred during (0,t) is given by

$$P(t) = \text{Expected total revenue in } (0,t] - \text{Expected total expenditure in } (0,t]$$
$$= K_0\mu_{up}(t) - K_1\mu_b^r(t) - K_2\mu_n^{rp}(t)$$

Where K_0 is the revenue per unit up time by the system, and K_1 repair cost per unit of time in repairing the failed unit by repairman and K_2 is per unit replacement cost of the failed unit.

Also,

$$\mu_{up}(t) = \int_0^t A_0(u)\,du$$

So that, $\mu_{up}^*(t) = \frac{A_0^*(s)}{s}$

In the similar way $\mu_b^r(t), \mu_n^{rp}(t)$can be defined.

Now the expected profit per unit of time in steady state is given by

$$P = \lim_{t\to\infty}\frac{P(t)}{t} = \lim_{s\to 0} s^2P^*(s)$$
$$= K_0A_0 - K_1B_0 - K_2V_1$$

1.10 CONCLUSION

To study the behavior of MTSF and profit function through graphs w.r.t *v*arious parameters, curves are plotted for these characteristics w.r.t failure parameter α_1 in Fig.2.1 and Fig.2.2 respectively for three different values of repair rate $\beta_2 = (0.20, 0.50, 0.60)$ whereas other parameters are kept fixed as $\alpha_2 = 0.03, \beta_1 = 0.25, \beta_3 = 0.20, h_1 = 0.30, h_2 = 0.02, g_1 = 0.03$.

Fig.2.1 represents variation in MTSF for varying values of failure parameter α_1 for three different values of repair rate β_2 . The graph shows decrease in MTSF with the increase in failure rate and an increase with the increase in repair rate. The curves also indicates that for the same value of failure rate, MTSF is higher for higher values of repair rate .So we conclude that the repair facility has a better understanding of failure phenomenon resulting in longer lifetime of the system.

Fig.2.2 represents the variation pattern in profit function w.r.t. varying values of failure parameter α_1 for three different values of repair rate β_2, it is observed from graph that profit decreases with the increase in failure rate α_1 and increases with increase in repair rate β_2 irrespective of other parameters. The curve also indicates that for the same value of repair rate, profit is lower for higher values of failure rate and decrease in both MTSF and profit function is almost exponential.

Hence, it can be concluded that the expected life of the system can be increased by decreasing failure rate and increasing repair rate of the unit which in turn will improve the reliability and hence the effectiveness of the system.

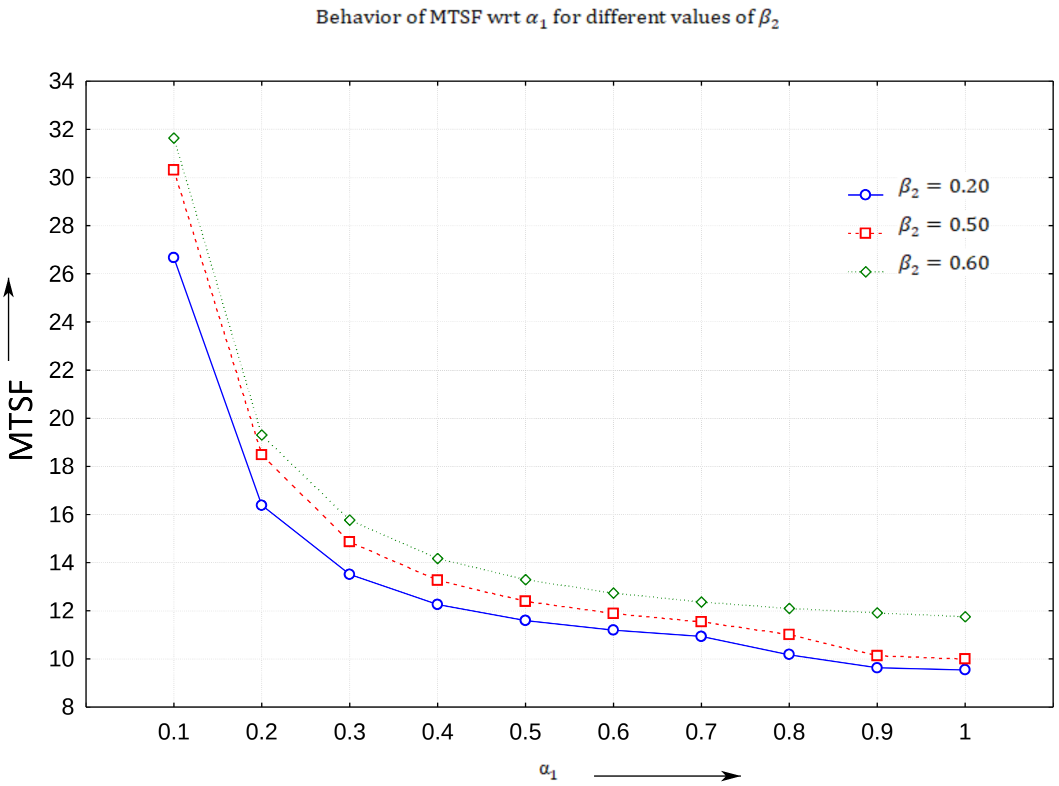
Behavior of MTSF wrt α_1 for different values of β_2
MTSF
α_1
$\beta_2 = 0.20$
$\beta_2 = 0.50$
$\beta_2 = 0.60$
8
10
12
14
16
18
20
22
24
26
28
30
32
34
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

FIG 2.1

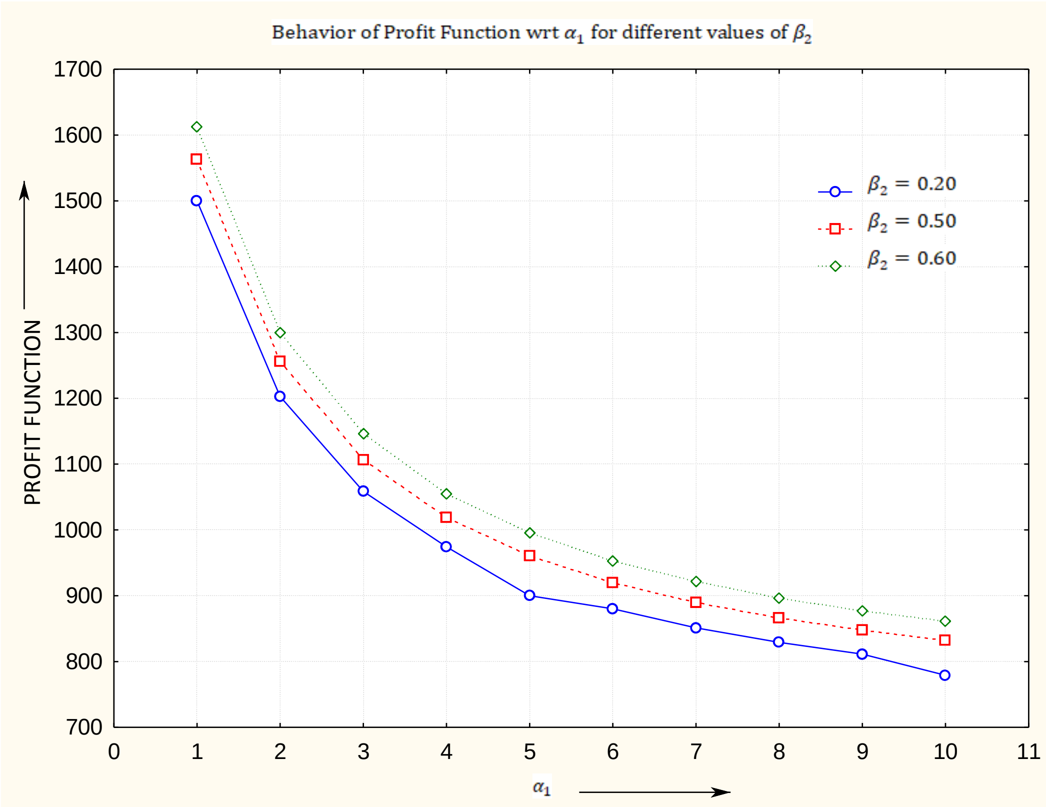
Behavior of Profit Function wrt α_1 for different values of β_2
PROFIT FUNCTION
α_1
$\beta_2 = 0.20$
$\beta_2 = 0.50$
$\beta_2 = 0.60$
700
800
900
1000
1100
1200
1300
1400
1500
1600
1700
0
1
2
3
4
5
6
7
8
9
10
11

FIG 2.2

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Assessment And Prediction Of Reliability Of An Automobile Component Using Warranty Claims Data

Tahsina Aziz, M. Rezaul Karim

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Department of Statistics, University of Rajshahi, Rajshahi-6205, Bangladesh. Email: taziz_stat@yahoo.com
Department of Statistics, University of Rajshahi, Rajshahi-6205, Bangladesh. Email: mrezakarim@yahoo.com; mrkarim@ru.ac.bd

Abstract

This paper presents an analysis of warranty claims data of a component of an automobile. The objectives of the analysis are to assess and predict the reliability of the component. To do these the paper present nonparametric and parametric analyses for the lifetime variable, age in month, based on warranty claims data. It also investigates on the variation of reliability of the component with respect to month of production and dominant failure modes. The paper will be useful to the manufacturer for assessing and predicting reliability and warranty costs and for assuring customer satisfaction and product reputation.

Keywords. Automobile component; Failure mode; Maximum likelihood estimate; Reliability; Warranty claims data; Warranty claim rate.

1. Introduction

The complexity of products has been increasing with technological advances. Over the last few decades there has been a heightened interest in improving quality, productivity and reliability of manufactured products. Rapid advances in technology and constantly increasing demands of customers for sophisticated products have put new pressure on manufacturers to produce highly reliable products. As a result, a product must be viewed as a system consisting of many elements and capable of decomposition into a hierarchy of levels, with the system at the top level and parts at the lowest level. Blischke, Karim and Murthy (2011) mentioned that there are many ways of describing this hierarchy. The modern automobile is a complex system consisting of over 15,000 components (Blischke et al., 2011). In this paper the warranty claims data of a component of automobile which belongs to the electrical sub-system, manufactured and sold in Asia, is considered. We analyze the warranty claims data of the component to investigate questions of interest to the manufacturers regarding reliability assessment and prediction.

As there are many aspects to warranty, a number of procedures have been developed for analyzing product warranty data, and the literature on this topic is very large. Detailed discussion on various aspects of warranty and reviews of subsequent recent literature on warranty analysis can be found in Thomas and Rao (1999), Murthy and Djamaludin (2002), Karim and Suzuki (2005), Blischke et al. (2011), Wu (2012) and Wang and Xie (2017). Many factors contribute to product failures that result in warranty claims. One of the most important factors is the age of the product. Age is calculated by the service time measured in terms of calendar time since the product was sold or entered service. The age-based (or age-specific) analysis of product failure data has engendered considerable interest in the literature (Kalbfleisch et al., 1991; Kalbfleisch and Lawless, 1996; Lawless, 1998; Karim et al., 2001; Suzuki et al., 2001) and a number of approaches have been developed with regard to addressing age-based analysis of warranty claims data.

Recently, Blischke et al. (2011) discussed the age-based analysis of an automobile component failure data in a case study. Here first we find the non-parametric estimates of cumulative density function $F(t)$ and reliability function $R(t)$ of lifetime random variable T measured by age in month. Next we apply the parametric approach to select the suitable lifetime models for the component and of different failure modes assuming that the number of failures at age t depends on the age of the product and is independent on other factors. The age-based warranty claim rates for different month of production are estimated for checking the quality variation problems with respect to production period. We also determine the dominant failure modes for the component and investigate how the reliability improves by successively removing the dominant failure modes. We consider a month as the unit for age without loss of generality. If necessary, the unit 'month' can be easily substituted with 'week', 'day' and so on.

The outline of the paper is as follows. Section 2 describes the warranty claims data set. Section 3 discusses the nonparametric approach of data analysis. Section 4 presents the parametric approach to analysis the warranty claims data. Finally, Section 5 concludes the paper.

2. Description of Data

This paper analyses a set of failure data of an automobile component manufactured and sold in Asia. The failure data are the warranty claims data of the component produced over 12 month period of a year and sold over a 26 month period. For reasons of commercial sensitivity we cannot disclose the names of the component and manufacturing company and call simply the "component". The component is non-repairable and the automobiles on which it is used are sold with a non-renewing free-replacement warranty (FRW) with 18 months (age limit) warranty period.[2] The data are collected during 26 months observation period. There are total 4746 failed observations and 64567 censored observations. For each claim, the available data relating to component consisted of the following:

- Serial number of claim
- Month of production
- Date of sale
- Date of failure
- Age of the component

[2] Generally in case of automobile components, the offered warranty is two-dimensional, where the warranty is characterized by a region in a two-dimensional plane, usually with one axis representing age and the other representing usage, whichever occurs first. However, the warranty of this component is one-dimensional, which is characterized by a single variable, age.

- Odometer readings (mileage in kilometers)
- Failure modes
- Used region

The manufacturer has identified 8 different failure modes for the component denoted by FM01, FM02, FM03, FM04, FM05, FM06, FM07 and FM08. Additional these, the database consists of the supplementary data: Production amount (monthly, for 12 months) and Sales amount (monthly, for 26 months).

3. Nonparametric Analysis

The nonparametric approach allows the user to analyze data without assuming an underlying distribution, that is, it does not require that the form of the sampled population be known. Blischke et al. (2011) recommended that any set of warranty data first be subjected to a nonparametric analysis before moving on to parametric analyses assuming a specific underlying probability distribution. Here we look at the nonparametric approach to inferences regarding quantities such as the cumulative density function (cdf) $F(t)$, reliability function $R(t)$, as well as warranty claim rates (WCR) of the component.

Kaplan and Meier (1958) derived the nonparametric estimator of the survival function for censored data which is known as the *product-limit* (PL) *estimator*. This estimator is also widely known as the *Kaplan-Meier* (KM) *estimator* of the survival function. We find the age-based Kaplan-Meier estimator of the survival function $S(t)$ or reliability function $R(t)$. Suppose that there are observation on n individuals and that there are k ($k \le n$) distinct times (say, age in month) $t_1 < t_2 < \ldots < t_k$ at which failures occur. Let d_i denote the number of units that failed at t_i and r_i represent the number of units that are right-censored at t_i, $i = 1, 2, \ldots, k$. Then the size of the *risk set* (number of units that are alive) at the beginning of time t_i is

$$n_i = n - \sum_{j=0}^{i-1} d_j - \sum_{j=0}^{i-1} r_j, \quad i = 1, 2, \ldots, k \tag{1}$$

where $d_0=0$ and $r_0=0$. Then, the estimator of the conditional probability that a unit fails in the time interval from $t_i - \delta t$ to t for small δt, given that the unit enters this interval, is the sampling proportion failing $\hat{p}_i = d_i / n_i, i = 1, 2, \ldots, k$ and the estimator of the corresponding survival probability is $1 - \hat{p}_i = (n_i - d_i)/n_i, i = 1, 2, \ldots, k$. Under this condition, the Kaplan-Meier estimator of the survival function $S(t)$ is given by

$$S(t) = P(T > t) = \prod_{j:t_j < t} (1 - \hat{p}_j) = \prod_{j:t_j < t} \frac{n_j - d_j}{n_j}, t \ge 0. \tag{2}$$

The nonparametric estimator of $F(t)$ is obtained using the Kaplan-Meier estimator as

$$\hat{F}(t) = 1 - \hat{S}(t), t \ge 0 \tag{3}$$

Meeker and Escobar (1998) discussed estimation methods for the variance and point-wise normal-approximation confidence intervals for $F(t)$. By using the logit transformation, they showed that two-sided approximate 100(1-α)% confidence intervals for $F(t)$ can be calculated as

$$\left[\frac{\hat{F}(t)}{\hat{F}(t)+(1-\hat{F}(t))\times w}, \frac{\hat{F}(t)}{\hat{F}(t)+(1-\hat{F}(t))/w}\right] \tag{4}$$

where $w=\exp\left\{z_{(1-\alpha/2)}\hat{se}_{\hat{F}(t)}/[\hat{F}(t)(1-\hat{F}(t))]\right\}$ and $\hat{se}_{\hat{F}(t)}=\sqrt{V(\hat{F}(t))}=\hat{S}(t)\sqrt{\sum_{j=1}^{t}\frac{\hat{p}_j}{n_j(1-\hat{p}_j)}}$.

The nonparametric estimates of reliability function $R(t)$ and cumulative density function $F(t)$ with their 95% confidence intervals are plotted in Figure 1. Minitab software and the R-function survfit(Surv()) under the library survival can be applied to estimate these functions. Figure 1 indicates that about 96% of the component is estimated to survive until 12 months. The value of the cdf at age 18 months is $F(t=18)=0.068$, indicating a claims rate of 6.8% within the warranty period. We are 95% confident that the probability of failing of the component within the warranty period of 18 months is between 6.6% and 7.0%.

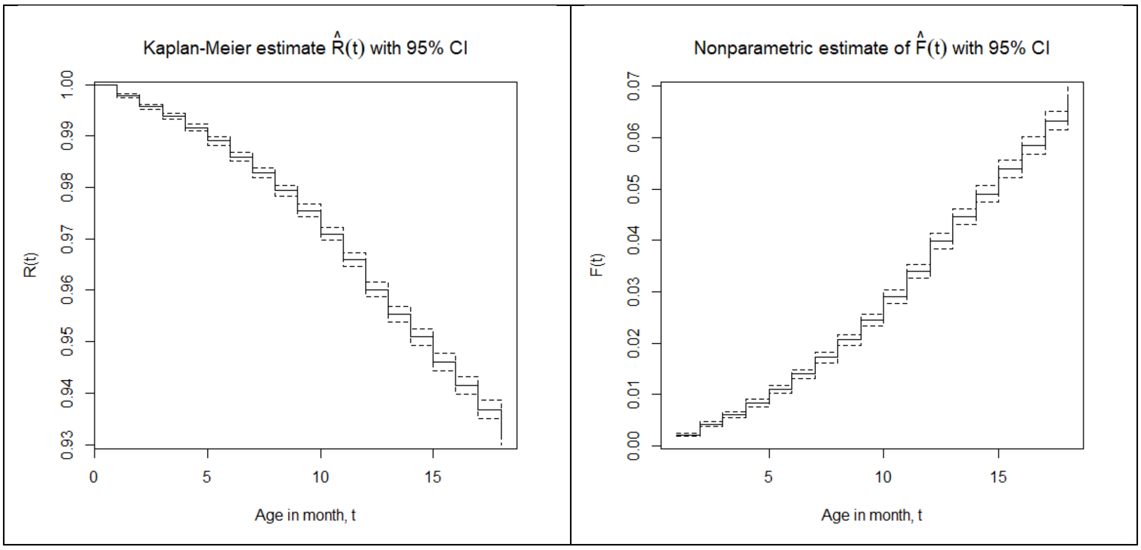

Figure 1: Nonparametric estimates of $R(t)$ (left side) and $F(t)$ (right side) with 95% confidence intervals

The Pareto chart of different failure modes, given in Figure 2, indicates that the three failure modes FM02, FM01, and FM03 account for 83.5% of the total claims. Failure modes from FM04 to FM08 have considerably lower frequencies. Based on Figure 2, we may conclude that efforts should be concentrated to eliminate or reduce the risks associated with the failure modes FM02, FM01 and FM03 in order to improve the reliability of the product and thereby decrease warranty claims and costs.

The summary statistics for the variables Age and Usage for important failure modes are given in Table 1. These summary statistics are the conditional estimates in the sense that they are estimated based on the items that failed during the warranty period and led to claims. This means the summary statistics for the variables Age and Usage given that the Age is less than or equal to 18 months.

Table 1 indicates that the conditional average and median lifetimes with respect to Age and Usage are smaller for failure mode FM02 among the three failure modes. In case of

Usage variable for three failure modes, the mean exceeds the median, indicating skewness to the right. On the other hand, Age variable shows negative skewness.

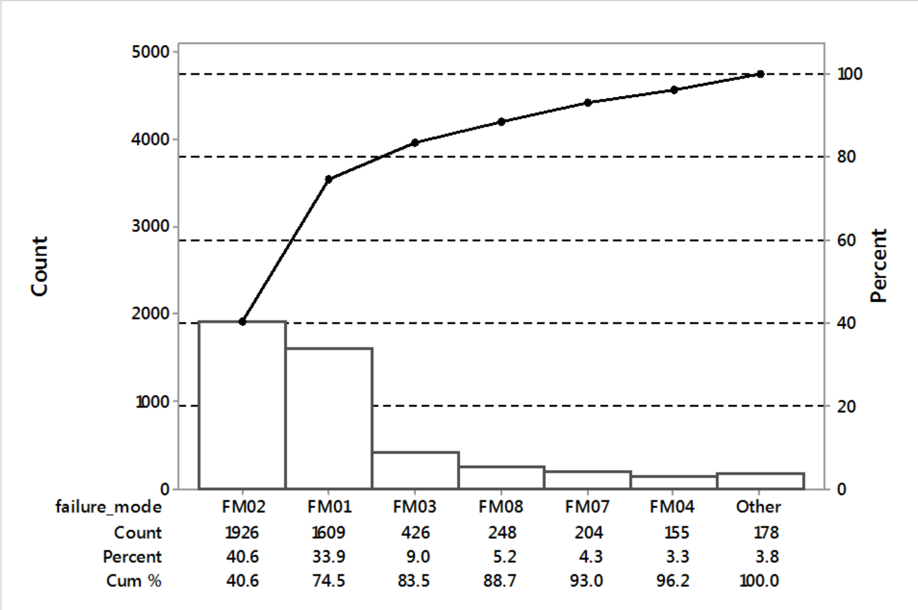

Figure 2: Pareto chart of failure modes

Table 1: Summary statistics for the variables Age and Usage for important failure modes

Failure Mode	Age (in month)							
	Count	Mean	StDev	Q1	Median	Q3	Skewness	Kurtosis
FM01	1609	11.6650	4.4155	9	12	15	-0.5080	-0.5107
FM02	1926	10.1277	4.7288	7	10	14	-0.1509	-0.9037
FM03	426	13.0493	4.4547	11	14	17	-0.9541	0.1410
Failure Mode	Usage (in km)							
	Count	Mean	StDev	Q1	Median	Q3	Skewness	Kurtosis
FM01	1609	27221.76	15053.99	16067.50	25819.00	36479.00	0.6809	0.7755
FM02	1926	24785.99	15905.73	12735.25	22593.50	34533.50	0.7978	0.6639
FM03	426	30018.70	16246.41	18825.25	26984.00	40408.50	0.8214	0.7578

Figure 3 shows the interval plots of Usage (km) for eight failure modes (FM01, FM02, …, FM08) under four used regions (geographic areas of a country with different environments, denoted by R1, R2, R3 & R4). Interval plot can be used to assess and compare both a measure of central tendency and variability of the data. The confidence intervals allow to assess the differences between group means in relation to within-group variance.

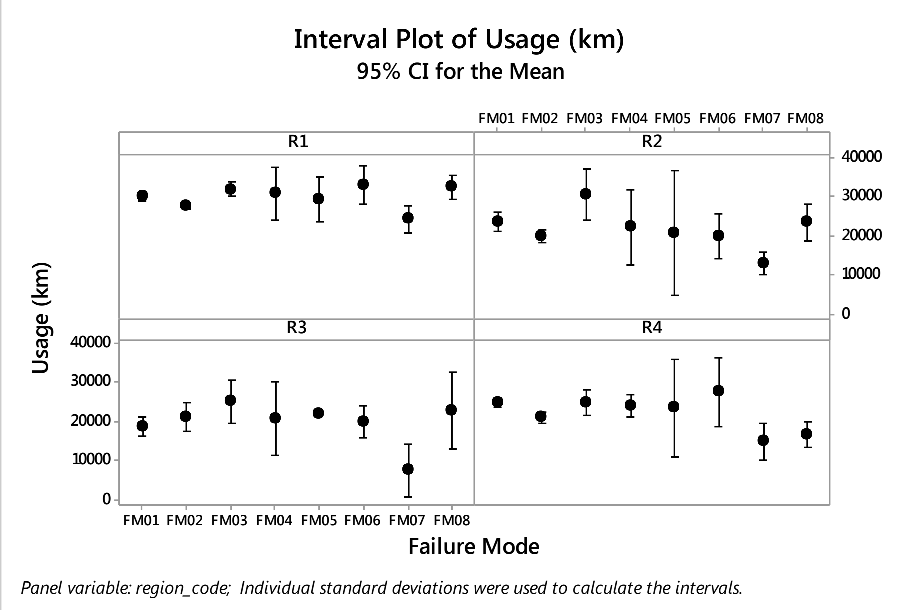

Figure 3: Interval plot of Usage (km) for various failure modes under four used regions

The interval plots show the means of usage for failure mode FM07 are shorter in all used regions. The means of usage for almost all failure modes are longer for used region R1 than other regions. The intervals of means for various failure modes are not all overlap, this indicates that some of the means are different. This indicates the variation of average lifetimes for different failure modes with respect to used regions.

The component considered here was produced during one year from month January to month December. The information on the month of production for the failed items are given in the database. The following Figures 4 and 5 can be used to investigate whether there is any variation in quality with respect to the month of production. Figures 4 and 5 indicate that the items produced in month September have smallest mean lifetimes for both Age and Usage. The intervals (except the production month September) all overlap, so we cannot conclude that any of the means (except September) are different. This preliminary graphical investigation indicates that there might be some quality related problems for the items produced in month September.

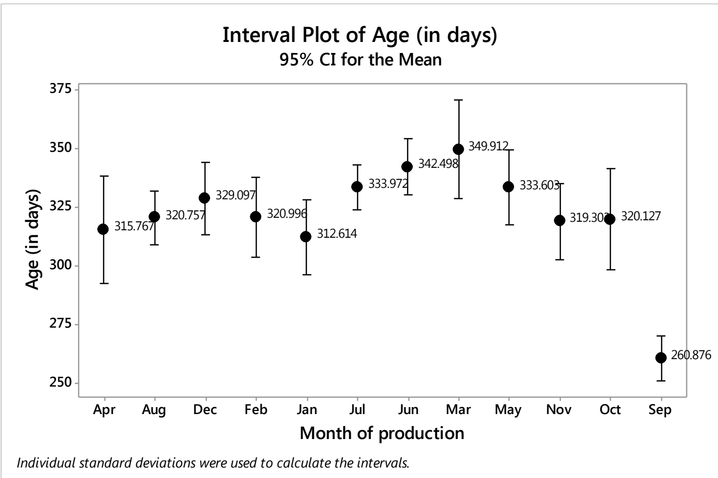

Figure 4: Interval plot of Age (in days) based on month of production

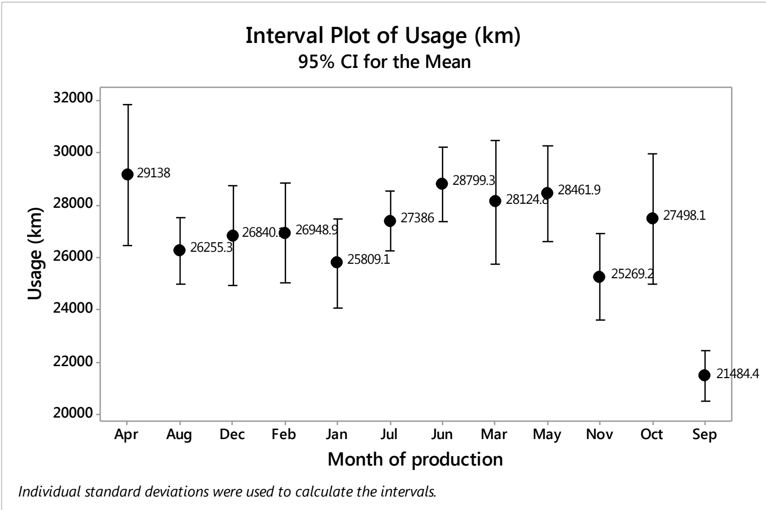

Figure 5: Interval plot of Usage (km) based on month of production

Figure 6 makes a comparison of nonparametric estimates of reliability functions for the main three failure modes FM01, FM02 and FM03. The figure indicates that $R(t)$ for FM02 less than the $R(t)$ for FM01 less than the $R(t)$ for FM03, for all t = 1,2,…, 18. For example, $R_{FM02}(t{=}12){=}0.9814$ < $R_{FM01}(t{=}12){=}0.9875$ < $R_{FM03}(t{=}12){=}0.9976$. Therefore, to improve the overall reliability of the component, efforts should be concentrated to eliminate or reduce the failure modes FM02 first, then FM01 and then FM03.

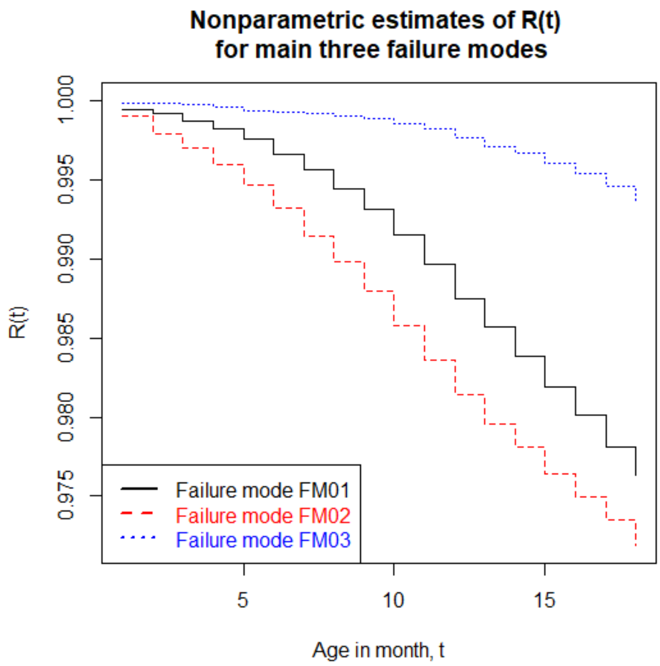

Figure 6: Comparison of nonparametric estimates of $R(t)$
for main three failure modes

The warranty claims data of the component considered here were manufactured over a 12 month period (Jan., Feb., .., Dec.) of a particular year. Monthly production amounts are given as supplementary data. The production month-wise monthly failure counts can be estimated from the warranty database. Due to variations in materials and/or production, the quality of components can vary from batch to batch or month of production (MOP) to MOP. We estimate the age-based warranty claim rates (WCR) for various MOP to provide a basis for checking quality variation problems with respect to production period. We define the WCR for MOP=*i* and Age=*t* as follows

$$WCR(i,t) = \frac{r_{it}}{M_i}, i = 1,2,...,12; t = 1,2,...,18 \tag{5}$$

where r_{it} represents the count of claims at age *t* occurred from month of production *i* and M_i is the total number of items produced in month *i*, *i*=1, 2, …, 12; *t*=1,2,…, 18. More detail on the estimation of WCR can be found in Blischke et al. (2011). The estimates of *WCR* (*i*, *t*) are shown in Figure 7.

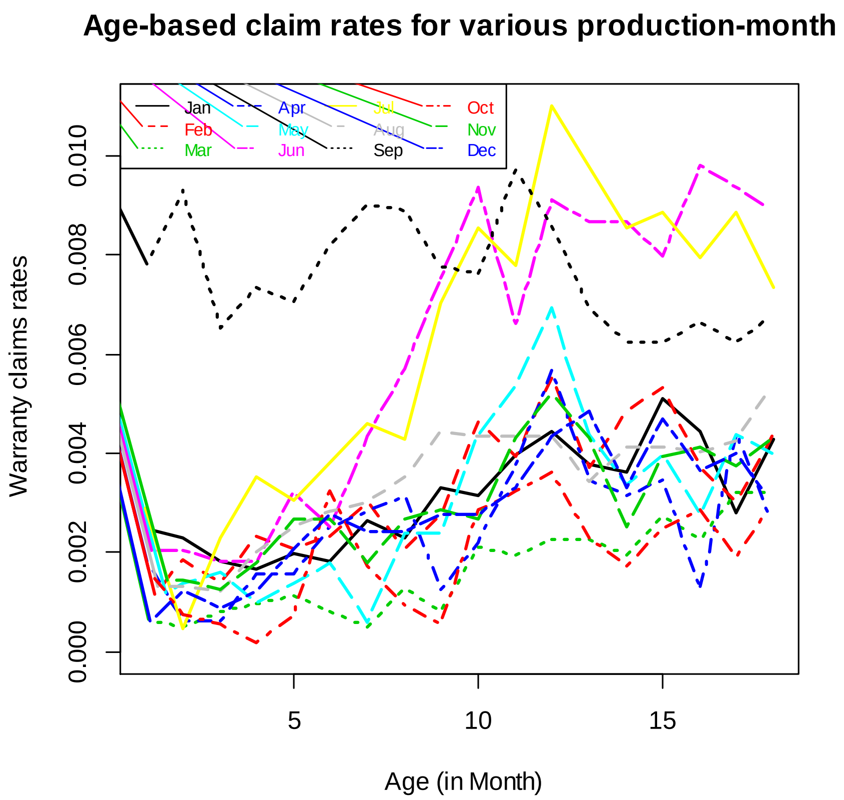

Figure 7: Age-based warranty claim rates for different months of production (Jan. to Dec.)

Figure 7 indicates that the warranty claim rates are very high for the three months of production September, July and June compared with other months of production. For the MOP September, the WCRs are approximately constant with respect to age. The WCRs for the MOP July and June are seems to be increasing with age. The quality of the items produced in the remaining MOP (January to May, August and October to December) is the best in the sense that the claim rates are low and age-wise approximately similar. This suggests that there were some problems in materials and/or production process in the MOP September, July and June.

To make the differences among WCRs clear in Figure 7, we separate the MOP in two groups: Group 1 contains three MOP September, July and June and Group 2 contains the remaining nine MOP (January to May, August and October to December). Then we estimate the age-based average WCRs for Group 1 and Group 2. For example, for Group 1, the average WCR at age t equals to $\{WCR(9,t)+WCR(7,t)+WCR(6,t)\}/3$, t=1,2,…, 18. Similarly, it can be estimated for Group 2 by averaging on nine MOP. The age-based average warranty claim rates for two groups are shown in Figure 8 which clearly indicates that the average warranty claim rates for Group 1 are higher than that of Group 2.

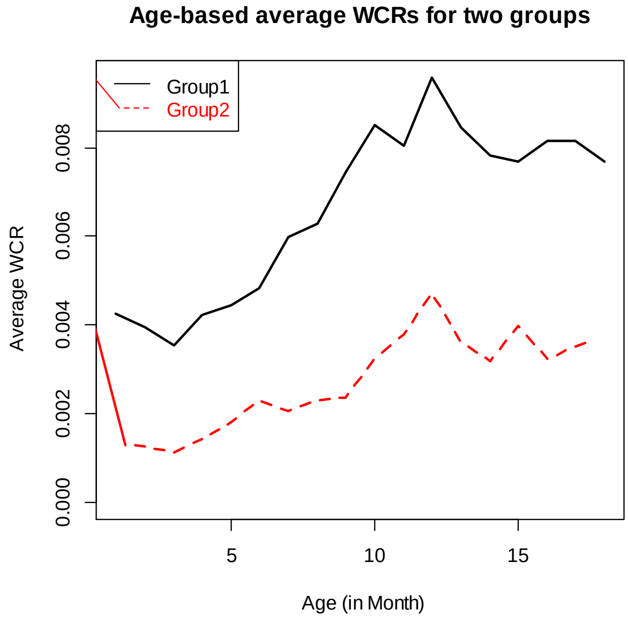

Figure 8: Age-based average warranty claim rates for two groups

4. Parametric Analysis

This section presents the parametric approach to analysis the warranty claims data set discussed in Section 2. The parametric approach to data analysis is concerned with the construction, estimation, and interpretation of mathematical models as applied to empirical data. This involves the tasks model selection, estimation of model parameters and validation of the model. Once these tasks are completed, the model may be used for prediction and other inferences.

To apply the parametric approach, we arrange the data in a concentrated form. Let t_i be the observed failure/censored lifetimes for the random variable T measured in month, m_i denote the number of units (frequency) that failed/censored at t_i and δ_i represent the failure-censoring indicator for t_i (taking on value 1 for failed items and 0 for censored), i = 1, 2, …, k (for the data set k=18). We assume a parametric model $f(t;\theta)$, with corresponding survival or reliability function $R(t;\theta)$, for the failure time variable T, where θ is a vector of model parameters. Under this scenario of data, the likelihood function can be written as

$$L(\theta) = \prod_{i=1}^{k} f(t_i;\theta)^{\delta_i m_i} R(t_i;\theta)^{(1-\delta_i)m_i} \tag{6}$$

The log likelihood becomes

$$\log L(\theta) = \sum_{i=1}^{k} \left[\delta_i m_i \log\{f(t_i;\theta)\} + (1-\delta_i) m_i \log\{R(t_i;\theta)\}\right] \tag{7}$$

We assume the eleven popular distributions, given in Appendix A (Table A.1), in the likelihood function (6) or log-likelihood function (7) and obtain the maximum likelihood estimator of θ by maximizing any of these likelihood functions. The log-likelihood function (7) is evaluated for the variable Age in month, T, and maximize to obtain the MLEs of the parameters assuming eleven distributions: (i) Smallest extreme value, (ii) Two-parameter Weibull, (iii) One-parameter exponential, (iv) Two-parameter exponential, (v) Normal, (vi) Two-parameter lognormal, (vii) Logistic, (viii) Loglogistic, (ix) Three-parameter Weibull, (x) Three-parameter lognormal and (xi) Three-parameter Loglogistic. We use the Minitab software to do this task.[3] The adjusted Anderson-Darling (AD) test statistic is used to select the best fitted distribution among the eleven distributions.[4] Figure 9 shows the Minitab output of distribution ID plots for the four distributions (Weibull, lognormal, loglogistic and 3-parameters lognormal) which give the smaller AD values among eleven distributions.

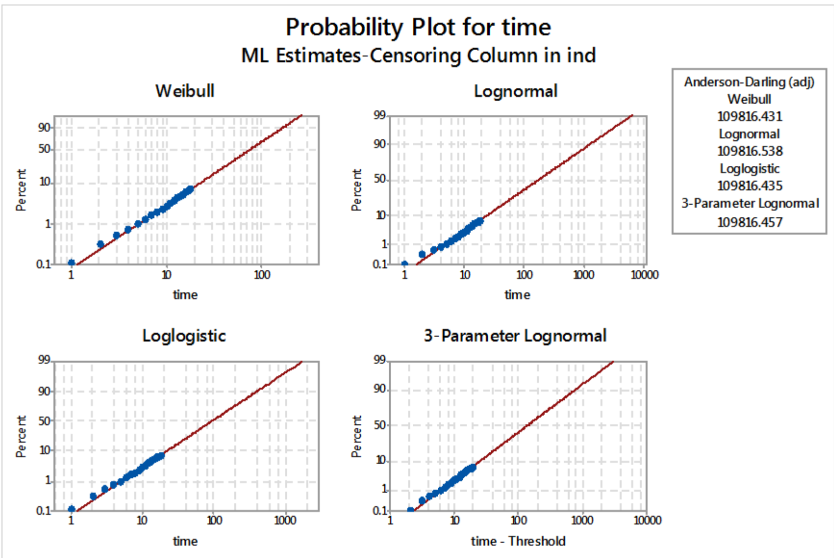

Figure 9: Four distributions probability plots of Age in month

In Figure 9, the overall appearance of the plots are not much changed, and the values of the AD statistic are approximately equal. However, the Weibull distribution shows the smallest AD statistic and so this distribution can be considered as the best distribution for the data among eleven distributions.

[3] The R functions mle(), optimize(), optim() or nlm() can also be used to do this task.

[4] Minitab (version 17) software creates probability plots and estimates adjusted Anderson-Darling (AD) test statistic for these eleven distributions.

The Weibull distribution overview plot shown in Figure 10, where the maximum likelihood estimates of the parameters are scale parameter $\hat{\eta}=99.0176$ and shape parameter $\hat{\beta}=1.5553$. The maximum likelihood estimates of mean age and median age are respectively 89.0239 months and 78.2292 months. As the estimate of the shape parameter of Weibull distribution is greater than one, the hazard function in Figure 10 indicates an increasing failure rate (IFR) with respect to age.

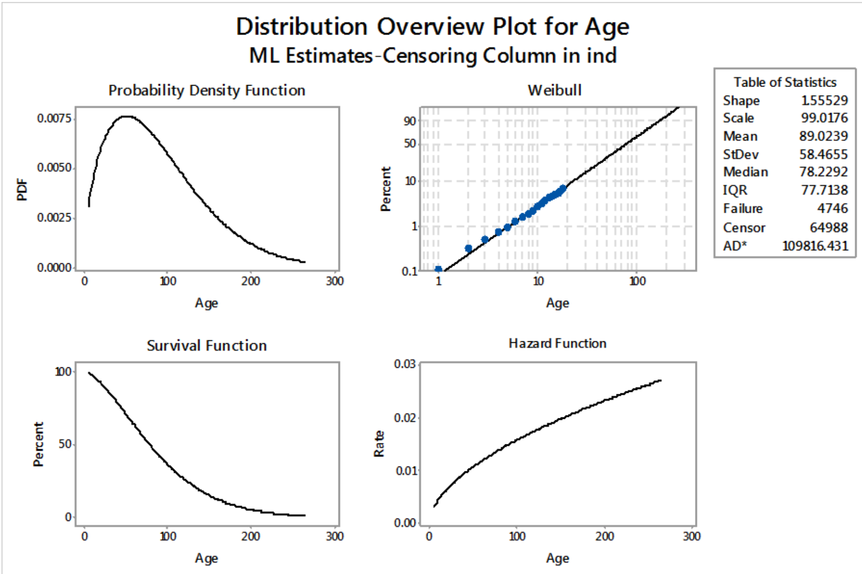

Figure 10: Weibull distribution overview plot for Age in month

The fitted Weibull cumulative density function, $\hat{F}(t;\eta,\beta)$, can be utilized to predict warranty cost of the component for a given warranty period. Let c_s denotes the average warranty cost (the cost incurred by the seller for servicing a claim which can be estimated from the warranty-service database) under a one-dimensional warranty with only first failure coverage. Then an estimate of the expected average cost per unit to the manufacturer for servicing a warranty up to t_w, denoted by $\hat{C}(t_w)$, is c_s times the proportion of units expected to fail within t_w (Karim and Suzuki, 2008), that is, $\hat{C}(t_w)=c_s\hat{F}(t_w;\eta,\beta),\ t_w>0$.

4.1. Analysis by Individual Failure Mode

If the manufacturer wants to improve the overall reliability of the component, it is important to find the suitable parametric distributions for each failure modes separately. Comparing the reliability functions of each failure modes, the manufacturer can redesign the component, if necessary, to optimize the overall reliability. This can be done by analyzing the competing risk models. In the competing risk setup, when we look at a single failure mode, all of the remaining items, including those that failed by another mode and common censored items, are right-censored. The distributions for individual failure modes are selected based on the minimum adjusted AD values and probability plots from a set of 11 distributions. It is found that the 3-parameter lognormal distribution can be selected as the best distribution for each failure modes. The maximum likelihood estimates of parameters of 3-parameter lognormal distribution for different failure modes are summarized in Table 2.

Figure 11 plots the individual reliability functions for eight different failure modes. It indicates that the reliability of failure modes FM02 and FM01 are very low compared with other failure modes.

Table 2: MLEs of the parameters of 3-parameter lognormal distribution

Failure Modes	Maximum likelihood estimates (MLEs)		
	Location ($\hat{\mu}$)	Scale ($\hat{\sigma}$)	Threshold ($\hat{\tau}$)
FM01	5.3630	1.1930	-2.0349
FM02	6.7479	2.0279	0.2808
FM03	4.7649	0.2886	-39.1793
FM04	6.6998	1.2714	-4.1681
FM05	7.9435	1.5531	-2.1135
FM06	7.8058	1.6704	-1.6149
FM07	22.1312	7.0101	0.9993
FM08	9.2202	2.3635	0.1461

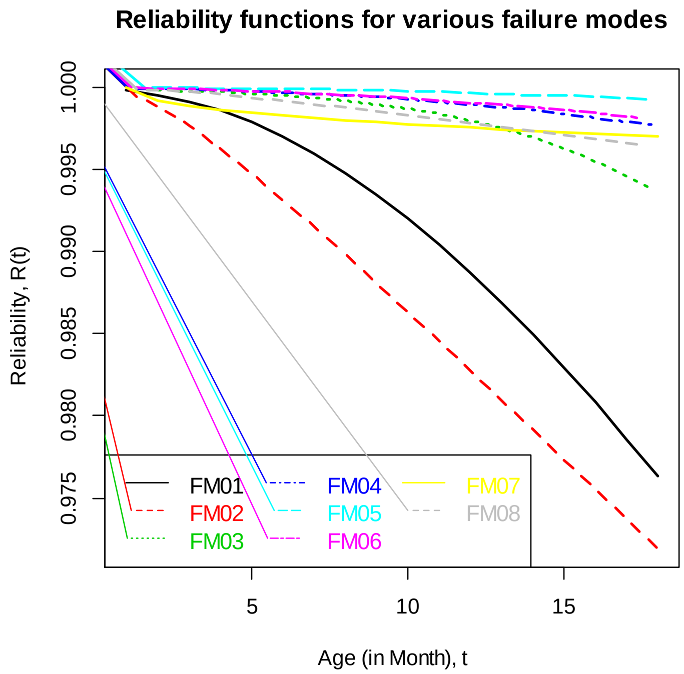

Figure 11: Reliability functions for different failure modes

Therefore, to increase the overall reliability of the component, effort should be concentrated on failure modes FM02 and FM01. Elimination of these or reducing the risks associated with them would significantly increase reliability and decrease warranty claims and costs. This investigation is important not only for assessing reliability and warranty costs, but also for assuring customer satisfaction and product reputation.

4.2. Elimination of Dominant Failure Mode

In this section, we look at modeling through elimination of the main failure modes one at a time. This enables us to investigate how the reliability of the component improves by successively removing failure modes. If $\hat{R}_{\text{FM}k}(t)$ be the estimated reliability function associated with the k^{th} failure mode, under competing risk setup, the estimate of overall reliability of the component at age t, $\hat{R}(t)$, can be expressed as

$$\hat{R}(t) = \prod_{k=01}^{K} \hat{R}_{\text{FM}k}(t),\ t = 1, 2, \ldots, 18 \tag{8}$$

where K is the number of failure modes and here K=08. FM01 eliminated means the first term of the right side of (8) equals 1, and so on for other failure modes. For example, the reliability of the component after eliminating failure mode FM*z*, let us denote by $\hat{R}_{[\text{-FMz}]}(t)$, z=01, 02, …, 08, can be estimated as

$$\hat{R}_{[-\text{FMz}]}(t) = \prod_{k=01}^{K} \hat{R}_{\text{FM}k}(t) / \hat{R}_{\text{FMz}}(t),\ t = 1, 2, \ldots, 18 \tag{9}$$

Figure 12 shows a comparison of reliability functions after eliminating failure modes FM01 or FM02. In this figure, "2-parameter-Weibull" means the estimated reliability function based on 2-parameter Weibull distribution fitted in Section 5, "Comp-risk All FM included" means the estimated reliability function based on competing risk model (8), "FM01 Eliminated" and "FM02 Eliminated" mean the estimated reliability functions by eliminating failure modes respectively FM01 and FM02 by (9).

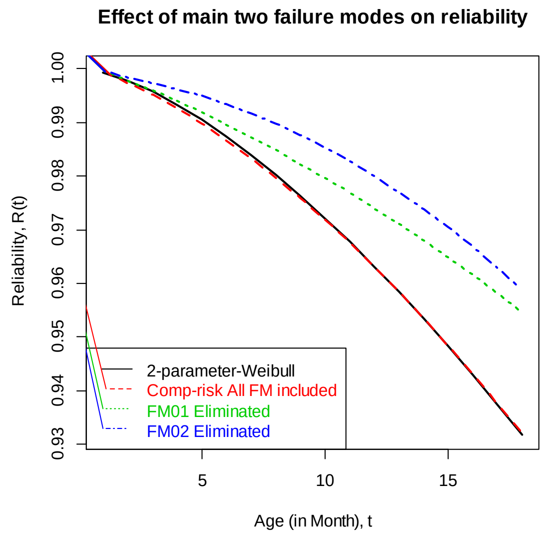

Figure 12: Comparison of reliability functions after eliminating failure modes FM01 or FM02

The overall reliability of the component estimated by 2-parameter Weibull distribution and by competing risk model are almost equal. The reliability of the component improves vastly after eliminating failure modes FM02 or FM01. For example, at age 18 months, the component reliability is 0.9319. This reliability improves to 0.9546 if failure mode FM01 eliminated and to 0.9589 if failure mode FM02 eliminated. The analysis suggests that if we design out failure mode FM02 and/or FM01, the reliability of the component improves vastly. This investigation is important in effective maintenance management (Murthy, et al., 2015) and managerial implications for cost-benefit analysis, including improvement in reliability, reduction in warranty cost, and forecasting claims rates and costs.

5. Conclusions

In this paper, we have attempted to analyze warranty claims data on a component of an automobile. Nonparametric and parametric analyses were employed for analyzing the warranty claims data. Some findings and recommendations are as follows:

- For this component, the warranty claim rates are significantly very high for the three months of production June, July and September (called Group1) compared with other months of production (called Group2). The claim rates for Group1 is approximately 2.5 times higher than that of Group2.
- The component has two dominating failure modes (denoted by FM02 and FM01) which vastly contribute in decreasing the reliability of the component. The overall 18-month component reliability is 0.9319. That is, 93.19% of the components survive past 18 months. If the failure modes FM01 or FM02 can be eliminated, 95.46% or 95.89% of the component will survive at the age of 18 months. To improve the overall reliability, we may need to improve both the failure modes FM01 and FM02. This analysis would be useful to the manufacturer if they decide to eliminate the dominant failure modes and to address the problem whether it is due to manufacturing or design.
- The paper presents age-based analysis of warranty claims data. The limitation of the paper is that it does not considered usage-based analysis. Future research on applications of usage-based modeling (e.g., Rai and Singh, 2005; Jiang and Jardine, 2006; Manna et al., 2007; Dai et al., 2017; He et al., 2018) and bivariate modeling (e.g., Moskowitz and Chun, 1994; Murthy et al., 1995; Blischke and Murthy, 1996; Kim and Rao, 2000; Pal and Murthy, 2003; Baik et al., 2004; Manna et al., 2008; Gupta et al., 2017) would enrich the analysis of the data.

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Time Dependent Analysis of an $M/M/2/N$ Queue With Catastrophes

Gulab Singh Bura and Shilpi Gupta
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Department of Mathematics & Statistics
Banasthali Vidyapith, Rajasthan (India).
E-mail: burastats@gmail.com

Abstract

We consider a Markovian queueing system with two identical servers subjected to catastrophes. When the system is not empty, catastrophes may occur and destroy all present customers in the system. Simultaneously the system is ready for new arrivals. The time dependent and the steady state solution are obtained explicitly. Further we have obtained some important performance measures of the studied queueing model.

Keywords: Markovian queueing system, catastrophes, limited capacity, Time Dependent Solution.

1 Introduction

During the last 40 years the attention of the queueing models has been focused on the effect of catastrophes, in particulars, birth and death models. The catastrophes arrive as negative customers to the system and their characteristic is to remove some or all of the regular customers in the system. The catastrophes may come either from outside the system or from another service station. For example, in computer networks, if a job infected with a virus, it transmits the virus to other processors and inactivities them [8]. Other interesting articles in this area include ([2],[6],[7]). In real life it is not necessary that a queueing system should have only one server. Practically they may have more than one server identical or non identical in their functioning .Krishna kumar et. al.[7] obtained the time dependent solution of two identical servers Markovian queueing system with catastrophes.Dharmaraja and kumar[3] consider a multi-server Markovian queueing system with heterogeneous servers and catastrophes.Jain and Bura [5]obtained the transient solution of an M/M/2/N queuing system with varying catastrophic intensity and restoration. We in this paper confine ourselves to a Markovian queueing system with two identical servers subjected to catastrophes.

Rest of the paper is organized as follows:In section 3, we describe the mathematical form of the model and obtained the time dependent solution of the model. In section 4, we obtain the time dependent performance measures of the system. Section 5 provides the steady state probabilities. In section 6, we obtain the expression for steady state mean and variance. Finally, the conclusion have been given in section 6.

2 Model description and analysis

We consider an $M/M/2/N$ queueing system with first come first out discipline that is subjected to catastrophes at the service station. Customers arrive in the system according to a Poisson stream with parameter λ.The service time distribution is independently identically exponential with parameter μ. When the system is not empty, catstrophes occur according to a Poisson process of rate ξ. Let $X(t)$ denote the number of customers in the system at time t.

Define $P_n(t) = P(X(t) = n); n = 0,1,2,\dots,N$ be the transient state probability that there are n customers in the system at time t, and $P(z,t) = \sum_{n=0}^{N} P_n(t)z^n$ be the probability generating function.

From the above assumption, the probability satisfies the following system of the differential- difference equations:

$$p_0'(t) = -\lambda p_0(t) + \mu p_1(t) + \xi[\textstyle\sum_{n=1}^{N} p_n(t)] \qquad ; n = 0 \tag{2.1}$$

$$p_1'(t) = -(\lambda + \mu + \xi)p_1(t) + \lambda p_0(t) + 2\mu p_2(t) \qquad ; n = 1 \tag{2.2}$$

$$p_n'(t) = -(\lambda + 2\mu + \xi)p_n(t) + \lambda p_{n-1}(t) + 2\mu p_{n+1}(t) \quad ; n = 2,3,\dots,(N-1) \tag{2.3}$$

$$p_N'(t) = -(2\mu + \xi)p_N(t) + \lambda p_{N-1}(t) \tag{2.4}$$

It is assumed that initially the system is empty i.e.

$$P_0(0) = 1 \qquad P_n(0) = 0 \ , n = 1,2,\dots,N \tag{2.5}$$

After Multiplying equations (2.1) to (2.4) by z^n for all $n \geq 0$, then summed on n from $n = 0$ to N and adding, we have

$$\sum_{n=0}^{N} p_n'(t)z^n = [\lambda z + \frac{2\mu}{z} - (\lambda + 2\mu + \xi)]P(z,t)$$
$$+2\mu(1 - \frac{1}{z})p_0(t) + \lambda z^N(1 - z)p_N(t) + \mu p_1(t)(z-1) + \xi \tag{2.6}$$

It is easily seen that the probability generating function $P(z,t)$ satisfies the following differential equation:

$$\frac{\partial}{\partial t}[P(z,t)] = [\lambda z + \frac{2\mu}{z} - (\lambda + 2\mu + \xi)]P(z,t)$$
$$+2\mu(1 - \frac{1}{z})p_0(t) + \lambda z^N(1 - z)p_N(t) + \mu p_1(t)(z-1) + \xi \tag{2.7}$$

with the initial condition

$$P(Z,0) = 1 \tag{2.8}$$

The equation (2.7) can be considered as a first order differential equation in $P(z,t)$ and by finding the integrating factor and using the initial condition (2.8),the solution of the equation (2.7) is obtained as

$$P(z,t) = 2\mu(1 - \frac{1}{z}) \int_0^t P_0(t-u)e^{(\lambda z + \frac{2\mu}{z})u}e^{-(\lambda+2\mu+\xi)u}du$$
$$+\lambda z^N(1 - Z) \int_0^t P_N(t-u)e^{(\lambda z + \frac{2\mu}{z})u}e^{-(\lambda+2\mu+\xi)u}du$$
$$+\mu(Z-1) \int_0^t P_1(t-u)e^{(\lambda z + \frac{2\mu}{z})u}e^{-(\lambda+2\mu+\xi)u}du$$
$$+\xi \int_0^t e^{(\lambda z + \frac{2\mu}{z})u}e^{-(\lambda+2\mu+\xi)u}du + e^{(\lambda z + \frac{2\mu}{z})t}e^{-(\lambda+2\mu+\xi)t} \tag{2.9}$$

Using the Bessel function identity, if $\alpha = 2\sqrt{\lambda 2\mu}$ and $\beta = \sqrt{\frac{\lambda}{2\mu}}$ then,

$$exp(\lambda z + \frac{2\mu}{z})t = \sum_{n=-\infty}^{\infty} I_n(\alpha t)(\beta z)^n$$

where $I_n(.)$ is the moddified Bessel function of order n. Substituting this equation in (2.9) and compairing the coefficient of z^n on either side, we have, for $n = 0,1,\dots,N$

$$P_n(t) = 2\mu\beta^n \int_0^t P_0(t-u)e^{-(\lambda+2\mu+\xi)u}[I_n(\alpha u) - \beta I_{n+1}(\alpha u)]du$$

$$+\lambda\beta^n \int_0^t P_N(t-u)e^{-(\lambda+2\mu+\xi)u}[\beta^{-N}I_{N-n}(\alpha u)-\beta^{-(N+1)}I_{(N+1)-n}(\alpha u)]du$$
$$+\mu\beta^n \int_0^t P_1(t-u)e^{-(\lambda+2\mu+\xi)u}[\beta^{-1}I_{n-1}(\alpha u)-I_n(\alpha u)]du$$
$$+\xi\beta^n \int_0^t e^{-(\lambda+2\mu+\xi)u}I_n(\alpha u)du+\beta^n e^{-(\lambda+2\mu+\xi)t}I_n(\alpha t) \quad (2.10)$$

where we have used $I_{-n}(.)=I_n(.)$

Here, we have obtained $P_n(t)$ for $n=1,\dots,N-1$. However, this expression depends upon $P_0(t)$ and $P_N(t)$. In order to determine, $P_0(t)$ and $P_N(t)$ we introduce the Laplace transform. In the sequel, for any function $f(.)$, let $f^*(s)$ denote its Laplace transform i.e. , $f^*(s)=\int_0^\infty e^{-st}f(t)dt$

Substitute $n=0$, in equation (2.10) we get

$$P_0(t)=2\mu\int_0^t P_0(t-u)e^{-(\lambda+2\mu+\xi)u}[I_0(\alpha u)-\beta I_1(\alpha u)]du$$
$$+\lambda\int_0^t P_N(t-u)e^{-(\lambda+2\mu+\xi)u}[\beta^{-N}I_N(\alpha u)-\beta^{-(N+1)}I_{(N+1)}(\alpha u)]du$$
$$+\mu\int_0^t P_1(t-u)e^{-(\lambda+2\mu+\xi)u}[\beta^{-1}I_1(\alpha u)-I_0(\alpha u)]du$$
$$+\xi\int_0^t e^{-(\lambda+2\mu+\xi)u}I_0(\alpha u)du+e^{-(\lambda+2\mu+\xi)t}I_0(\alpha t) \quad (2.11)$$

Taking Laplace transform on both sides of equation (2.11) and solving for, $P_0^*(s)$ we obtain,

$$\left[\frac{\omega+\sqrt{\omega^2-\alpha^2}}{2}-2\mu\right]P_0^*(s)=\lambda P_N^*(s)\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{2\lambda}\right)^N$$
$$\left[1-\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{2\lambda}\right)\right]$$
$$+\mu P_1^*(s)\left[\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{2\lambda}\right)-1\right]+\frac{\xi}{s}+1$$

where $\omega=s+\lambda+2\mu+\xi$. After some algebra, the above equation can be expressed as

$$\left[1-\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{2\lambda}\right)\right]P_0^*(s)=\frac{\lambda}{2\mu}P_N^*(s)\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{2\lambda}\right)^{N+1}$$
$$\left[1-\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{2\lambda}\right)\right]$$
$$-\mu P_1^*(s)\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{2\lambda}\right)$$
$$\left[1-\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{2\lambda}\right)\right]\left(\frac{1}{2\mu}\right)$$
$$+\left(\frac{\xi}{s}+1\right)\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{2\lambda}\right)\left(\frac{1}{2\mu}\right) \quad (2.12)$$

By solving equation (2.12), we get,

$$P_0^*(s)=\frac{\lambda}{2\mu}P_N^*(s)\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{2\lambda}\right)^{N+1}-\frac{P_1^*(s)}{2}\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{2\lambda}\right)$$
$$+\left(\frac{\xi}{s}+1\right)\left(\frac{1}{2\mu}\right)\left[\frac{2\mu}{s+\xi}-\frac{\lambda}{s+\xi}\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{2\lambda}\right)\right] \quad (2.13)$$

On inversion, this equation yields an expression for $P_0(t)$ which depends upon $P_N(t)$.

$$P_0(t)=e^{-\xi t}+\left(\frac{2\mu}{\lambda}\right)^{\frac{N-1}{2}}\int_0^t P_N(t-u)e^{-(\lambda+2\mu+\xi)u}\left(\frac{N+1}{u}\right)I_{N+1}(\alpha u)du$$
$$\left(\frac{2\mu}{\lambda}\right)^{\frac{1}{2}}\int_0^t P_1(t-u)e^{-(\lambda+2\mu+\xi)u}\left(\frac{1}{2u}\right)I_1(\alpha u)du$$
$$+\xi\left[e^{-\xi t}-\sqrt{\frac{\lambda}{2\mu}}\int_0^t e^{-(\lambda+2\mu+\xi)u}e^{-\xi(t-u)}\frac{I_1(\alpha u)}{u}du\right]$$
$$-\sqrt{\frac{\lambda}{2\mu}}\int_0^t e^{-(\lambda+2\mu+\xi)u}e^{-\xi(t-u)}\frac{I_1(\alpha u)}{u}du \quad (2.14)$$

Substituting n=1 in equation (2.10), we get

$$P_1(t)=2\mu\beta\int_0^t P_0(t-u)e^{-(\lambda+2\mu+\xi)u}[I_1(\alpha u)-\beta I_2(\alpha u)]du$$
$$+\lambda\beta\int_0^t P_N(t-u)e^{-(\lambda+2\mu+\xi)u}[\beta^{-N}I_{N-1}(\alpha u)-\beta^{-(N+1)}I_N(\alpha u)]du$$

$$+\mu\beta \int_0^t P_1(t-u)e^{-(\lambda+2\mu+\xi)u}[\beta^{-1}I_0(\alpha u) - I_1(\alpha u)]du$$
$$+\xi\beta \int_0^t e^{-(\lambda+2\mu+\xi)u}I_1(\alpha u)du + \beta e^{-(\lambda+2\mu+\xi)t}I_1(\alpha t) \tag{2.15}$$

Taking Laplace transform on both sides of equation (2.15) and solving for, $P_1^*(s)$ we obtain,

$$P_1^*(s)\left[\sqrt{\omega^2-\alpha^2} - \mu\left\{1-\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{4\mu}\right)\right\}\right] = 2\mu\left\{1-\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{4\mu}\right)\right\}$$
$$\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{4\mu}\right)P_0^*(s)$$
$$+\lambda\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{2\lambda}\right)^{N-1}\left\{1-\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{2\lambda}\right)\right\}P_N^*(s) + \left(\frac{\xi}{s}+1\right)$$
$$\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{4\mu}\right) \tag{2.16}$$

Substituting n=N in equation (2.10), we get

$$P_N(t) = 2\mu\beta^N \int_0^t P_0(t-u)e^{-(\lambda+2\mu+\xi)u}[I_N(\alpha u) - \beta I_{N+1}(\alpha u)]du$$
$$+\lambda \int_0^t P_N(t-u)e^{-(\lambda+2\mu+\xi)u}[I_0(\alpha u) - \beta^{-1}I_1(\alpha u)]du$$
$$+\mu\beta^N \int_0^t P_1(t-u)e^{-(\lambda+2\mu+\xi)u}[\beta^{-1}I_{N-1}(\alpha u) - I_N(\alpha u)]du$$
$$+\xi\beta^N \int_0^t e^{-(\lambda+2\mu+\xi)u}I_N(\alpha u)du + \beta^N e^{-(\lambda+2\mu+\xi)t}I_N(\alpha t) \tag{2.17}$$

By taking Laplace transform and solving for $P_N^*(s)$, we obtain from equation (2.17),

$$\left(\frac{\omega+\sqrt{\omega^2-\alpha^2}}{2}-\lambda\right)P_N^*(s) = 2\mu\left\{1-\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{4\mu}\right)\right\}\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{4\mu}\right)^N$$
$$\left[\frac{\lambda}{2\mu}\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{2\lambda}\right)^{N+1} P_N^*(s)\right. \tag{2.18}$$
$$\left.-\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{2\lambda}\right)+\frac{1}{2\mu}\left(\frac{\xi}{s}+1\right)\right]$$

After some algebra, equation (2.18) can be expressed as

$$[1-f^*(s)]P_N^*(s) = g^*(s) \tag{2.19}$$

where

$$f^*(s) = \left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{4\mu}\right)^{N+1}\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{2\lambda}\right)^{N+1}$$
$$\left\{1-\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{4\mu}\right)\right\}+\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{2\mu}\right) \tag{2.20}$$

$$g^*(s) = \frac{1}{\lambda}\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{4\mu}\right)^{N+1}\left(\frac{\xi}{s}+1\right)$$
$$\left[1+\left\{1-\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{4\mu}\right)\right\}\left\{\frac{2\mu}{s+\xi}-\frac{\lambda}{s+\xi}\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{2\lambda}\right)\right\}\right]$$
$$+\frac{\mu}{\lambda}\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{4\mu}\right)^{N+1}\left[\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{4\mu}\right)^{-1}-1\right]P_1^*(s)$$
$$-\frac{\mu}{\lambda}\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{2\lambda}\right)\left[1-\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{4\mu}\right)\right]$$
$$\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{4\mu}\right)^{N+1} P_1^*(s) \tag{2.21}$$

equation (2.21) can be written as

$$g^*(s) = \frac{1}{\lambda}\left(\frac{\xi}{s}+1\right)h^*(s) \tag{2.22}$$

where

$$h^*(s) = \left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{4\mu}\right)^{N+1}$$
$$\left[1+\left\{1-\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{4\mu}\right)\right\}\left\{\frac{2\mu}{s+\xi}-\frac{\lambda}{s+\xi}\left(\frac{2\mu}{\omega+\sqrt{\omega^2-\alpha^2}}\right)\right\}\right]$$

$$+\frac{\mu}{\lambda}\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{4\mu}\right)^{N+1} P_1^*(s)$$
$$\left[\left\{\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{4\mu}\right)^{-1}-1\right\}-\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{2\lambda}\right)\left\{1-\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{4\mu}\right)\right\}\right] \tag{2.23}$$

On inversion, the equation (2.20), (2.23) and (2.22) yield an expression for $f(t)$,$h(t)$ and $g(t)$ given by

$$f(t)=\sqrt{\frac{\lambda}{2\mu}}e^{-(\lambda+2\mu+\xi)t}\frac{I_1(\alpha t)}{t}+e^{-(\lambda+2\mu+\xi)t}(2N+2)\frac{I_{2N+2}(\alpha t)}{t}$$
$$-\sqrt{\frac{\lambda}{2\mu}}e^{-(\lambda+2\mu+\xi)t}(2N+3)\frac{I_{2N+3}(\alpha t)}{t} \tag{2.24}$$

$$h(t)=\left(\frac{\lambda}{2\mu}\right)^{\frac{(N+1)}{2}}e^{-(\lambda+2\mu+\xi)t}(N+1)\frac{I_{N+1}(\alpha t)}{t}+\left(\frac{\lambda}{2\mu}\right)^{\frac{(N+1)}{2}}$$
$$\left[\int_0^t e^{-(\lambda+2\mu+\xi)u}e^{-\xi(t-u)}\left\{2\mu(N+1)\frac{I_{N+1}(\alpha u)}{u}-\alpha(N+2)\frac{I_{N+2}(\alpha u)}{u}\right\}du\right]$$
$$+\left(\frac{\lambda}{2\mu}\right)^{\frac{(N+1)}{2}}\lambda\int_0^t e^{-(\lambda+2\mu+\xi)u}e^{-\xi(t-u)}(N+3)\frac{I_{N+3}(\alpha u)}{u}du$$
$$+\frac{\mu}{\lambda}\left(\frac{\lambda}{2\mu}\right)^{\frac{N}{2}}\int_0^t e^{-(\lambda+2\mu+\xi)u}N\frac{I_N(\alpha u)}{u}P_1(t-u)du$$
$$-\frac{\mu}{\lambda}\left(\frac{\lambda}{2\mu}\right)^{\frac{N+1}{2}}\int_0^t e^{-(\lambda+2\mu+\xi)u}(N+1)\frac{I_{N+1}(\alpha u)}{u}P_1(t-u)du$$
$$+2e^{-(\lambda+2\mu+\xi)t}\frac{I_2(\alpha t)}{t} \tag{2.25}$$

$$g(t)=\frac{1}{\lambda}(\xi+1)h(t) \tag{2.26}$$

Since $0\le f^*(s)<1$ so equation (2.19) can be written as

$$P_N^*(s)=g^*(s)\sum_{r=0}^{\infty}[f^*(s)]^r \tag{2.27}$$

On inversion, this equation yields an expression for $P_N(t)$ given by

$$P_N(t)=g(t)*\sum_{r=0}^{\infty}[f(t)]^{*r} \tag{2.28}$$

where $[f(t)]^{*r}$ is the r-fold convolution of $f(t)$ with itself. We note that

$[f(t)]^{*0}=1$

3 Performance measures

Mean

we know that

$$m(t)=E[X(t)]=\sum_{n=1}^{N}nP_n(t)$$
$$m(0)=\sum_{n=1}^{N}nP_n(0)=0$$
$$m'(t)=\sum_{n=1}^{N}nP_n'(t)$$

From equation (3.2), (3.3) and (3.4),

$$m'(t)=(\lambda+2\mu+\xi)\sum_{n=1}^{N}nP_n(t)+\lambda NP_N(t)+\lambda\sum_{n=1}^{N}nP_{n-1}(t)$$
$$+2\mu\sum_{n=1}^{N-1}nP_{n+1}(t)+\mu P_1(t)$$

After some algebra, the above equation can be expressed as

$$m'(t)=-\xi m(t)+(\lambda-2\mu)+2\mu P_0(t)-\lambda P_N(t)+\mu P_1(t) \tag{3.1}$$

The above equation can be considered as a first order linear differential equation in $m(t)$. By finding the integrating factor and using the initial condition $m(0)=0$, the solution of the above equation is obtained as follows:

$$m(t)=\frac{(\lambda-2\mu)}{\xi}(1-e^{-\xi t})-\lambda\int_0^t P_N(u)e^{-\xi(t-u)}du$$
$$+2\mu\int_0^t P_0(u)e^{-\xi(t-u)}du+\mu\int_0^t P_1(u)e^{-\xi(t-u)}du \tag{3.2}$$

Variance

We know that

$$Var[X(t)] = E[X^2(t)] - [E\{X(t)\}]^2$$
$$Var[X(t)] = k(t) - [m(t)]^2 \tag{3.3}$$

where

$$k(t) = E[X^2(t)] = \sum_{n=1}^{N} n^2 P_n(t)$$

Also,

$$k(0) = \sum_{n=1}^{N} n^2 P_n(0) = 0$$

and

$$k'(t) = \sum_{n=1}^{N} n^2 P_n'(t)$$

From equation (3.2), (3.3) and (3.4),

$$k'(t) = -(\lambda + 2\mu + \xi)\sum_{n=1}^{N} n^2 P_n(t) + \lambda N^2 P_N(t) + \lambda \sum_{n=1}^{N} n^2 P_{n-1}(t) +$$
$$2\mu \sum_{n=1}^{N-1} n^2 P_{n+1}(t) + \mu P_1(t)$$

After some algebra, the above equation can be expressed as

$$k'(t) = -\xi k(t) + (\lambda + 2\mu) - 2\mu P_0(t) - \lambda(2N+1)P_N(t)$$
$$+2(\lambda - 2\mu)m(t) + \mu P_1(t) \tag{3.4}$$

The above equation can be considered as a first order linear differential equation in $k(t)$. By finding the integrating factor and using the initial condition $k(0) = 0$, the solution of the above equation is obtained as follows:

$$k(t) = \frac{(\lambda+2\mu)}{\xi}(1 - e^{-\xi t}) - \lambda(2N+1)\int_0^t P_N(u)e^{-\xi(t-u)}du$$
$$-2\mu \int_0^t P_0(u)e^{-\xi(t-u)}du + 2(\lambda - \mu)\int_0^t m(u)e^{-\xi(t-u)}du$$
$$+\mu \int_0^t P_1(u)e^{-\xi(t-u)}du + C \tag{3.5}$$

Substituting the above equation in equation (3.3), we get

$$Var[X(t)] = \frac{(\lambda+2\mu)}{\xi}(1 - e^{-\xi t}) - \lambda(2N+1)\int_0^t P_N(u)e^{-\xi(t-u)}du$$
$$-2\mu \int_0^t P_0(u)e^{-\xi(t-u)}du + 2(\lambda - \mu)\int_0^t m(u)e^{-\xi(t-u)}du$$
$$+\mu \int_0^t P_1(u)e^{-\xi(t-u)}du - \{m(t)\}^2$$

4 Steady state probabilities

In this section, we shall discuss the structure of the steady state probabilities.

Theorem-

For $\xi > 0$, the steady state distribution $\{P_n : n \geq 0\}$ of the $M/M/2/N$ queue with catastrophe corresponds to

$$P_0 = \rho\rho_1 P_N + (1-\rho) - \frac{P_1}{2}\rho_1 \tag{4.1}$$
$$P_n = 2\sigma\mu\rho^{n+1}(1-\rho)\rho_1^N P_N + \sigma\lambda\rho_1^{N-n}(1-\rho_1)P_N + (1-\rho)\rho^n$$
$$+\mu\sigma(1-\rho)\rho^n\left(\frac{\sqrt{\omega^2-\alpha^2}}{\lambda}\right)P_1 \tag{4.2}$$
$$P_N = \frac{[\{\xi + 2\mu(1-\rho)^2\} + \mu\{(\rho^{-1}-1) - \rho_1(1-\rho)\}P_1]\rho^{N+1}}{\lambda[1-\rho-\rho_1^{N+1}\rho^{N+1}(1-\rho)]} \tag{4.3}$$

where

$$\rho = \frac{(\lambda+2\mu+\xi) - \sqrt{(\lambda+2\mu+\xi)^2 - 8\lambda\mu}}{4\mu} \tag{4.4}$$
$$\rho_1 = \frac{(\lambda+2\mu+\xi) - \sqrt{(\lambda+2\mu+\xi)^2 - 8\lambda\mu}}{2\lambda} \tag{4.5}$$

$$\sigma = \frac{1}{\sqrt{(\lambda+2\mu+\xi)^2-8\lambda\mu}} \tag{4.6}$$

Proof-

We have from equation (3.13),

$$P_0^*(s) = \frac{\lambda}{2\mu} P_N^*(s)\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{2\lambda}\right)^{N+1} - \left(\frac{\omega-\sqrt{\omega^2-\alpha 2}}{2\lambda}\right)\frac{P_1^*(s)}{2}$$
$$+\left(\frac{\xi}{s}+1\right)\left(\frac{1}{2\mu}\right)\left\{\frac{2\mu}{s+\xi} - \frac{\lambda}{s+\xi}\left(\frac{\omega-\sqrt{\omega^2-\alpha 2}}{2\lambda}\right)\right\} \tag{4.7}$$

Multiplying equation (4.7) by s on both sides and taking limit as $s \to 0$, we get

$$\lim_{s\to 0} sP_0^*(s) = \frac{\lambda}{2\mu}\rho_1^{N+1}P_N - \left(\frac{1}{2}\right)\lim_{s\to 0} sP_1^*(s)\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{2\lambda}\right)$$
$$+\left(\frac{1}{2\mu}\right)\lim_{s\to 0} s\left(\frac{\xi}{s}+1\right)\left\{\frac{2\mu}{s+\xi} - \frac{\lambda}{s+\xi}\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{2\lambda}\right)\right\}$$

Using the property

$$\lim_{s\to 0} sP_0^*(s) = P_0$$

After some algebra, the above expression becomes

$$P_0 = \rho\rho_1 P_N + (1-\rho) - \frac{P_1}{2}\rho_1 \tag{4.8}$$

By taking Laplace transform of the equation (3.10), for $n = 1,2,\ldots,N-1$, we get,

$$P_n^*(s) = 2\mu P_0^*(s)\left(\frac{1}{\sqrt{\omega^2-\alpha^2}}\right)\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{4\mu}\right)^n\left[1-\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{4\mu}\right)\right]$$
$$+\lambda P_N^*(s)\left(\frac{1}{\sqrt{\omega^2-\alpha^2}}\right)\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{2\lambda}\right)^{N-n}\left[1-\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{2\lambda}\right)\right]$$
$$+\frac{\mu}{\sqrt{\omega^2-\alpha^2}}\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{4\mu}\right)^{n-1}\left[1-\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{4\mu}\right)\right]P_1^*(s)$$
$$+\left(\frac{\xi}{s}+1\right)\left(\frac{1}{\sqrt{\omega^2-\alpha^2}}\right)\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{4\mu}\right)^n \tag{4.9}$$

Multiplying the above equation by s on both sides and taking limit as $s \to 0$, we get

$$\lim_{s\to 0} sP_n^*(s) = 2\sigma\mu\rho^n(1-\rho)P_0 + \sigma\lambda\rho_1^{N-n}(1-\rho_1)P_N$$
$$+\mu\sigma\rho^{n-1}(1-\rho)P_1 + \sigma\xi\rho^n \tag{4.10}$$

Substituting equation (4.8) in the above equation, and solving, we get

$$P_n = 2\sigma\mu\rho^{n+1}\rho_1^N(1-\rho)P_N + \sigma\lambda\rho_1^{N-n}(1-\rho_1)P_N$$
$$+\mu\sigma(1-\rho)\rho^n\left(\frac{\sqrt{\omega^2-\alpha^2}}{\lambda}\right)P_1 + (1-\rho)\rho^n \qquad n = 1,2,\ldots,N-1 \tag{4.11}$$

Multiplying the equation (3.21) by s on both sides and taking limit as $s \to 0$, after some algebra, we get

$$\lim_{s\to 0} sg^*(s) = \frac{1}{\lambda}[\xi + 2\mu(1-\rho)^2]\rho^{N+1} + \frac{\mu}{\lambda}\rho^{N+1}[(\rho^{-1}-1) - \rho_1(1-\rho)]P_1 \tag{4.12}$$

Now taking limit as $s \to 0$ in the equation (3.20), we get

$$\lim_{s\to 0} f^*(s) = \rho[1 + \rho_1^{N+1}\rho^N(1-\rho)] \tag{4.13}$$

Multiplying the equation (3.19) by s on both sides and taking limit as $s \to 0$, we get

$$\lim_{s\to 0} sP_N^*(s) = \lim_{s\to 0}\frac{sg^*(s)}{1-f^*(s)} \tag{4.14}$$

Substituting equation (4.12) and (4.13) in the above equation

$$P_N = \frac{[\{\xi+2\mu(1-\rho)^2\} + \mu\{(\rho^{-1}-1)-\rho_1(1-\rho)\}P_1]\rho^{N+1}}{\lambda[1-\rho-\rho_1^{N+1}\rho^{N+1}(1-\rho)]} \tag{4.15}$$

Multiplying the equation (3.16) by s on both sides and taking limit as $s \to 0$, we get

$$\lim_{s\to 0} sP_1^*(s)\left[\sqrt{\omega^2-\alpha^2} - \mu\left\{1-\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{4\mu}\right)\right\}\right] =$$
$$2\mu\lim_{s\to 0} s\left\{1-\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{4\mu}\right)\right\}$$

$$\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{4\mu}\right)P_0^*(s)$$
$$+\lambda\lim_{s\to 0} s\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{2\lambda}\right)^{N-1}\left\{1-\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{2\lambda}\right)\right\}P_N^*(s)$$
$$+\lim_{s\to 0} s\left(\frac{\xi}{s}+1\right)\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{4\mu}\right) \tag{4.16}$$

After some algebra, the above expression becomes

$$P_1\left\{\frac{1}{\sigma}-\mu(1-\rho)\right\}=2\mu(1-\rho)\rho P_0+\lambda(\rho_1^{N-1}-\rho_1^N)P_N+\xi\rho$$

5 Steady state mean and variance

The corresponding values of the steady state mean and variance of the system length are obtained by taking limit as $t\to\infty$ in equation (4.2) and (4.3). These values are given by

$$m=E(X)=\frac{1}{\xi}[(\lambda-2\mu)+2\mu P_0-\lambda P_N+\mu P_1]$$
$$Var(X)=\frac{1}{\xi}[(\lambda+2\mu)+2(\lambda-\mu)m-2\mu P_0-\lambda(2N+1)P_N+\mu P_1]-m^2$$

6 Conclusion

In the present paper, we have discussed the $M/M/2/N$ queueing system subject to catastrophes. The transient as well as the steady state probabilities of the models have been determined analytically. Further, we have also obtained the performance measures of the system.

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Cost-profit Anslysis of Stochastic Heterogeneous Queue with Reverse Balking, Feedback and Retention of Impatient Customers

Bhupender Kumar Som
•
Department of Management, Jagan Institute of Management Studies
Institutional Area, Sector -5, Rohini, New Delhi, India 110085

Abstract

Consider a system operating as an M/M/2/N queue. Increasing system size influences newly arriving customers to join the system (reverse balking). As the system size increases, the customers' waiting in the queue become impatient. After a threshold value of time, the waiting customer abandons the queue (reneging). These reneging customers can be retained with some probability (retention). Few customers depart dissatisfied with the service and rejoin the system as feedback customers. In this paper a feedback queuing system with heterogeneous service, reverse balking, reneging and retention is developed. The model is solved in steady-state recursively. Necessary measures of performance are drawn. Numerical interpretation of the model is presented. Cost-profit analysis of the system is performed by developing a cost model. Sensitivity analysis of the model is also presented arbitrarily.

Keyword – reverse balking, heterogeneous service, queuing theory, customer impatience

1. INTRODUCTION

Balking and reneging (impatience) are fundamental concepts in queuing literature introduced by Anker & Gafarian (1963a, 1963b). Further Haight (1957, 1959) and Bareer (1957) studied notion of customer reneging and balking in various ways. They state that an arriving customer shows least interest in joining a system which is already crowded. This behavior is termed as *Balking*. Since then researchers applied balking at various places and a number of research papers appeared on balking. Singh (1970) studied a two-server Markovian queues with Balking. He compared two heterogeneous servers with homogeneous servers. He also obtained the efficiency of heterogeneous system functioning under balking, Hassin, (1986) applied balking in customer information in markets with random product quality; they consider a revenue server suppressing the information using balking function. Falin (1995) approximated multi-server queues with balking/retrial discipline. They studied congestion in communication with balking discipline. Kumar (2006) further studied multi-server feedback retrial queues with balking and control retrial rate. They analyzed system as quasi-birth-and-death process and discuss stability conditions. They also obtained optimization of retrial rate. Wang (2011) studied balking with delayed repairs. They investigated equilibrium threshold balking strategies for fully and partially observable single-server queues with server breakdowns and delayed repairs. Kumar (2018) studied transient and steady-state behavior of two-heterogeneous servers' queuing systems with balking retention of reneging customers. They obtained time-dependent and steady state solution of the model.

On contrary to balking Jain et. al., (2014) stated that when it comes to businesses like healthcare, restaurant, investment, service station etc a large customer base becomes an attracting factor for newly arriving customer i.e. a customer is more willing to join a firm that already has a large customer base. This behavior of customers is termed as *Reverse Balking*. Kumar (2015a, 2015b) studied queuing systems with reverse balking, reverse reneging and feedback. Further Som et. al. (2016) studied a heterogeneous queuing system with reverse balking reverse reneging. The notion of reverse balking is studied further by Kumar (2017) and Som (2018a, 2018b). A limited number of publications appeared on reverse balking as it is an evolving concept.

Reverse balking results in increasing queue length and longer waiting times. A customer waiting in queue to get served may get impatience after certain period of time and decides to abandon the queue without completion of service. This behavior of customers is termed as *reneging* Anker & Gafarian (1963a, 1963b). Reneging has gained popularity due to its practical viability. Researchers studied applications of reneging in detail. Rao (1971) studies reneging and balking in M/G/1 system. He investigated the busy period using supplementary variable technique and transforms. Abou-El-Ata et. al. (1992) studied a truncated general queue with reneging and general balk function. They derived steady-state solution of the model. Wang et. al. (2002) performed cost analysis of finite M/M/R queuing system with balking reneging and server breakdowns. They developed a cost-model of the system under study as well. Singh et. al. (2016) studied single-server finite queuing system with varying speed of server in random environment. Bakuli et. al., (2017) investigated $M/M^{(a,b)}/1$ queuing model with impatient customers. They derived solution of the model and found measures of performance. Further Kumar et. al. (2017) studied transient analysis of a multi-server queuing model with discouraged arrivals and impatient customers. Reneging has gained wide popularity due to its practically viable implication.

As reneging causes loss of customers hence it leaves a negative impact on goodwill and revenue of the firm. Kumar et. al. (2012) introduced the idea of *retention of impatient customers* in queuing literature. They mentioned that if a retention strategy is employed in form of offers and discount; a reneging customer may be retained with some probability. Kumar et. al. (2013, 2014) further performed economic analysis of M/M/c/N queue with retention of impatient customers. They obtained steady-state solution of the model and obtained various measures of effectiveness. They also optimize a queuing system with reneging and retention of impatient customers. Since then a lot of paper appeared on retention of reneged customers such as Som et. al. (2017, 2018c) discussed a various queuing system with encouraged arrivals and retention of impatient customers.

Further a serviced customer may depart from the system dissatisfied. These customers may rejoin the system for completion of incomplete or dissatisfied service. These customers are termed as feedback customers in queuing literature. Takas (1963) introduced the feedback mechanism in queues. He used instant Bernoulli's feedback in a M/G/1 queue. Nakamura (1971) studied a delayed feedback system using Bernoulli's decision process.

Further DAvignon et. al. (1976) studied state dependent M/G/1 feedback under the assumption of general state. The obtained the stationary queue length along with busy period and queue length. They applied single-server feedback queue with respect to computer time sharing system. Santhakumaran et. al. (2000) studied a single-server queue with impatient and feedback customers. They studied stationary process of the arrival distribution. Choudhary et. al. (2005) have discusses an M/G/1 queue with two phases of heterogeneous service. Further Som (2018a, 2018b) has studied a feedback queue with various queuing systems. This is also evident that servers vary in their capacity of service and provide service at heterogeneous rate.

Though the queuing models with reverse balking, reneging, retention and feedback are developed and studied but none of these models studies a facility undergoing reverse balking, feedback, and retention of impatient customers with heterogeneous service all together. Practically, all of these contemporary phenomenons occur simultaneously. Therefore it is worthy to study and measure such a system. Hence in this paper we study a feedback queuing system with reverse balking, reneging, retention and heterogeneous service. The necessary measures of performance are obtained in steady-state. The model is tested with arbitrary values. Later the cost model is developed and economic analysis of the model is performed.

2. THE MODEL

The model proposed in the paper can be presented through following state diagram;

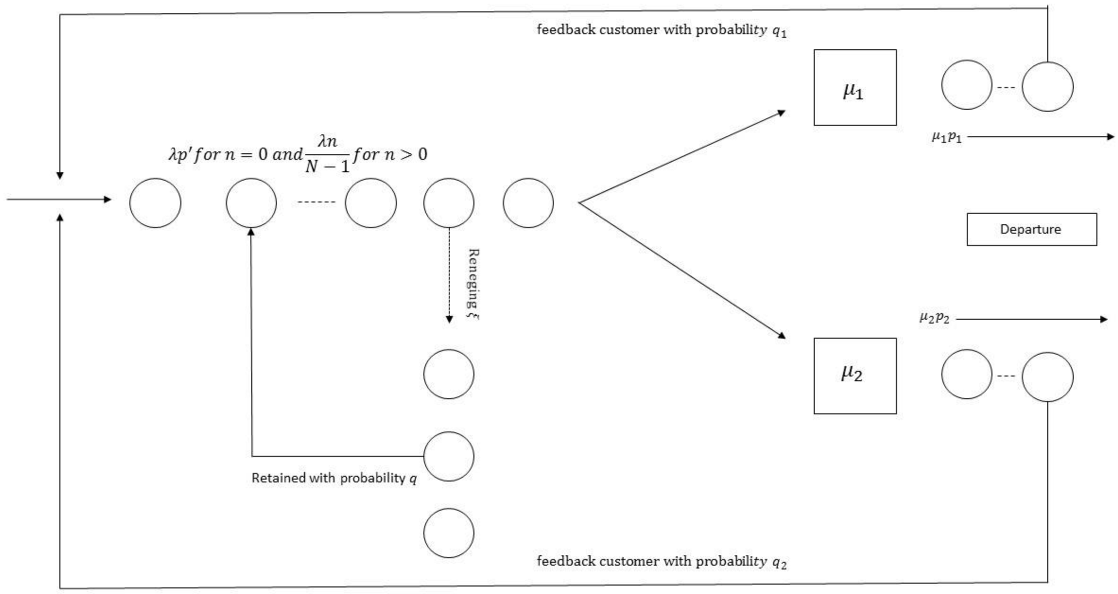

Figure -1

Consider the arrivals occur one by one in accordance with Poisson process. Inter-arrival times are exponentially distributed with parameter $1/\lambda$. Customers are serviced through two servers with heterogeneous service times distributed exponentially with parameters μ_1 and μ_2. An arriving customer joins the system in front of an empty server i.e. with probability π_1 in front of server 1 and with π_2(=1- π_1) in front of server two. Capacity of system is finite say, N. When system is empty, an arriving customers reverse balks with probability q' and joins the system with probability $q' = (1-p')$. When number of customers in the system are ≥0, an arriving customer reverse balks with probability $\left(1-\frac{n}{N-1}\right)$and does not reverse balk with probability $\left(\frac{n}{N-1}\right)$. A customer waiting for service in queue may get impatient after time T and decides to abandon the queue with an exponentially distributed parameter ξ. Arrivals are served in order of their arrival i.e. the queue discipline is first come first serve. A reneging customer may be retained with probability $q = (1-p)$. A serviced customer may not get satisfied with the service of first server μ_1 and rejoin the system as a feedback customer with probability $q_1 = (1-p_1)$. While a serviced customer may not be satisfied with the service of second server $\mu_2$and rejoin to the system as a feedback customer with probability $q_2 = (1-p_2)$.

2.1 BALANCE EQUATIONS AND STEADY STATE SOLUTION

Let $P_n(t)$ = probability of n customers in the system at time t. $P_{ij}(t)$= probability that there are i customers in front of first server one and j customers in front of second server at time t. In steady-state as $t \to \infty$, $P_n(t) = P_n, P_{ij}(t) = P_{ij}$ and $P_n'(t) = P_{ij}'(t) = 0$. The system of steady-state equations governing the model is given by;

$$\lambda p' P_{00} = \mu_1 p_1 P_{10} + \mu_2 p_2 P_{01}; n = 0 \ (1)$$
$$\mu_2 p_2 P_{11} = \left(\frac{\lambda}{N-1} + \mu_1 p_1\right) P_{10} - \lambda \pi_1 p' P_{00}; n = 1(2)$$
$$\mu_1 p_1 P_{11} = \left(\frac{\lambda}{N-1} + \mu_2 p_2\right) P_{01} - \lambda \pi_2 p' P_{00}; n = 1(3)$$
$$(\mu_1 p_1 + \mu_2 p_2 + \xi p) P_3 = \left(\frac{2\lambda}{N-1} + \mu_1 p_1 + \mu_2 p_2\right) P_2 - \frac{\lambda}{N-1} P_1; n = 2 \ (4)$$
$$\{\mu_1 p_1 + \mu_2 p_2 + n\, \xi p\} P_{n+1} = \left\{\frac{n\lambda}{N-1} + \mu_1 p_1 + \mu_2 p_2 + (n-2)\xi p\right\} P_n - \frac{\lambda(n-1)}{N-1} P_{n-1}; n \leq N-1 \ (5)$$
$$\{\mu_1 p_1 + \mu_2 p_2 + (N-2)\xi p\} P_N = \lambda P_{N-1}; n = N \ (6)$$

Steady-state solution

On solving (1) – (6), we get

$$P_{10} = \left\{\frac{\lambda + (\mu_1 p_1 + \mu_2 p_2)\pi_1 (N-1)}{2\lambda + (\mu_1 p_1 + \mu_2 p_2)(N-1)}\right\} \left(\frac{\lambda}{\mu_1 p_1}\right) p' P_{00} \ (7)$$
$$P_{01} = \left\{\frac{\lambda + (\mu_1 p_1 + \mu_2 p_2)\pi_2 (N-1)}{2\lambda + (\mu_1 p_1 + \mu_2 p_2)(N-1)}\right\} \left(\frac{\lambda}{\mu_2 p_2}\right) p' P_{00} \ (8)$$

Adding (7) and (8)

$$P_1 = \left\{\frac{\lambda + (\pi_1 \mu_2 p_2 + \pi_2 \mu_1 p_1)(N-1)}{2\lambda + (\mu_1 p_1 + \mu_2 p_2)(N-1)}\right\} \frac{\lambda(\mu_1 p_1 + \mu_2 p_2)}{\mu_1 p_1 u_2 p_2} p' P_{00} \ (9)$$

Adding equation (2) and (3) and using (4)

$$P_2 = \frac{1}{N-1} \left\{\frac{\lambda + (\pi_1 \mu_2 p_2 + \pi_2 \mu_1 p_1)(N-1)}{2\lambda + (\mu_1 p_1 + \mu_2 p_2)(N-1)}\right\} \frac{\lambda}{\mu_1 p_1} \frac{\lambda}{\mu_2 p_2} p' P_{00} \ (10)$$

Using equation (5)

$$P_n = \frac{(n-1)!}{(N-1)^{n-1}}\left\{\frac{\lambda+(\pi_1\mu_2 p_2+\pi_2\mu_1 p_1)(N-1)}{2\lambda+(\mu_1 p_1+\mu_2 p_2)(N-1)}\right\}\frac{\lambda}{\mu_1 p_1}\frac{\lambda}{\mu_2 p_2}\prod_{k=3}^{n}\frac{\lambda}{\mu_1 p_1+\mu_2 p_2+(k-2)\xi p}p' P_{00} \quad (11)$$

Using (6) and (11)

$$P_N = \frac{(N-2)!}{(N-1)^{N-2}}\left\{\frac{\lambda+(\pi_1\mu_2 p_2+\pi_2\mu_1 p_1)(N-1)}{2\lambda+(\mu_1 p_1+\mu_2 p_2)(N-1)}\right\}\frac{\lambda}{\mu_1 p_1}\frac{\lambda}{\mu_2 p_2}\prod_{k=3}^{N}\frac{\lambda}{\mu_1 p_1+\mu_2 p_2+(k-2)\xi p}p' P_{00} \quad (12)$$

Using condition of normality $\sum_{n=0}^{N} P_n = 1$ we get,

$$P_0 = \left[\left\{\frac{\lambda+(\pi_1\mu_2 p_2+\pi_2\mu_1 p_1)(N-1)}{2\lambda+(\mu_1 p_1+\mu_2 p_2)(N-1)}\right\}\frac{\lambda(\mu_1 p_1+\mu_2 p_2)}{\mu_1 p_1 u_2 p_2}p' + \frac{1}{N-1}\left\{\frac{\lambda+(\pi_1\mu_2 p_2+\pi_2\mu_1 p_1)(N-1)}{2\lambda+(\mu_1 p_1+\mu_2 p_2)(N-1)}\right\}\frac{\lambda}{\mu_1 p_1}\frac{\lambda}{\mu_2 p_2}p' \right.$$
$$+\sum_{n=3}^{N-1}\frac{(n-1)!}{(N-1)^{n-1}}\left\{\frac{\lambda+(\pi_1\mu_2 p_2+\pi_2\mu_1 p_1)(N-1)}{2\lambda+(\mu_1 p_1+\mu_2 p_2)(N-1)}\right\}\frac{\lambda}{\mu_1 p_1}\frac{\lambda}{\mu_2 p_2}\prod_{k=3}^{n}\frac{\lambda}{\mu_1 p_1+\mu_2 p_2+(k-2)\xi p}p'$$
$$\left.+\frac{(N-2)!}{(N-1)^{N-2}}\left\{\frac{\lambda+(\pi_1\mu_2 p_2+\pi_2\mu_1 p_1)(N-1)}{2\lambda+(\mu_1 p_1+\mu_2 p_2)(N-1)}\right\}\frac{\lambda}{\mu_1 p_1}\frac{\lambda}{\mu_2 p_2}\prod_{k=3}^{N}\frac{\lambda}{\mu_1 p_1+\mu_2 p_2+(k-2)\xi p}p'\right]^{-1}$$

3 MEASURES OF PERFORMANCE

In this section necessary measures of performance are derived. Apart from these other measures of performance such as average waiting time in queue, average queue length, and average waiting time in the system can be drawn by using classical queuing theory relations.

3.1 EXPECTED SYSTEM SIZE

$$L_s = \sum_{n=1}^{N} nP_n$$

$$L_s = \sum_{n=1}^{N} nP_n = P_1 + 2P_2 + \sum_{n=3}^{N-1} nP_n + NP_N$$

$$L_s = \left\{\frac{\lambda+(\pi_1\mu_2 p_2+\pi_2\mu_1 p_1)(N-1)}{2\lambda+(\mu_1 p_1+\mu_2 p_2)(N-1)}\right\}\frac{\lambda(\mu_1 p_1+\mu_2 p_2)}{\mu_1 p_1 u_2 p_2}p' P_{00}$$
$$+2\frac{1}{N-1}\left\{\frac{\lambda+(\pi_1\mu_2 p_2+\pi_2\mu_1 p_1)(N-1)}{2\lambda+(\mu_1 p_1+\mu_2 p_2)(N-1)}\right\}\frac{\lambda}{\mu_1 p_1}\frac{\lambda}{\mu_2 p_2}p' P_{00}$$
$$+\sum_{n=3}^{N-1} n\frac{(n-1)!}{(N-1)^{n-1}}\left\{\frac{\lambda+(\pi_1\mu_2 p_2+\pi_2\mu_1 p_1)(N-1)}{2\lambda+(\mu_1 p_1+\mu_2 p_2)(N-1)}\right\}\frac{\lambda}{\mu_1 p_1}\frac{\lambda}{\mu_2 p_2}\prod_{k=3}^{n}\frac{\lambda}{\mu_1 p_1+\mu_2 p_2+(k-2)\xi p}p' P_{00}$$
$$+N\frac{(N-2)!}{(N-1)^{N-2}}\left\{\frac{\lambda+(\pi_1\mu_2 p_2+\pi_2\mu_1 p_1)(N-1)}{2\lambda+(\mu_1 p_1+\mu_2 p_2)(N-1)}\right\}\frac{\lambda}{\mu_1 p_1}\frac{\lambda}{\mu_2 p_2}\prod_{k=3}^{N}\frac{\lambda}{\mu_1 p_1+\mu_2 p_2+(k-2)\xi p}p' P_{00}$$

3.2 AVERAGE RATE OF REVERSE BALKING

$$R_b' = q'\lambda P_0 + \sum_{n=1}^{N-1}\left(1-\frac{n}{N-1}\right)\lambda P_n$$

$$R_b' = q'\lambda P_0 + \frac{N-2}{N-1}\lambda P_1 + \frac{N-3}{N-1}\lambda P_2 + \sum_{n=3}^{N-1}\left(1-\frac{n}{N-1}\right)\lambda P_n$$

$$R_b' = q'\lambda P_0 + \left[\frac{N-2}{N-1}\lambda\left\{\frac{\lambda+(\pi_1\mu_2 p_2+\pi_2\mu_1 p_1)(N-1)}{2\lambda+(\mu_1 p_1+\mu_2 p_2)(N-1)}\right\}\frac{\lambda(\mu_1 p_1+\mu_2 p_2)}{\mu_1 p_1 u_2 p_2}p'\right.$$

$$+\frac{N-3}{N-1}\lambda\frac{1}{N-1}\left\{\frac{\lambda+(\pi_1\mu_2 p_2+\pi_2\mu_1 p_1)(N-1)}{2\lambda+(\mu_1 p_1+\mu_2 p_2)(N-1)}\right\}\frac{\lambda}{\mu_1 p_1}\frac{\lambda}{\mu_2 p_2}p'$$

$$\left.+\left(1-\frac{n}{N-1}\right)\sum_{n=3}^{N-1}\frac{(n-1)!}{(N-1)^{n-1}}\left\{\frac{\lambda+(\pi_1\mu_2 p_2+\pi_2\mu_1 p_1)(N-1)}{2\lambda+(\mu_1 p_1+\mu_2 p_2)(N-1)}\right\}\frac{\lambda}{\mu_1 p_1}\frac{\lambda}{\mu_2 p_2}\prod_{k=3}^{n}\frac{\lambda}{\mu_1 p_1+\mu_2 p_2+(k-2)\xi p}p'\right]$$

3.3 AVERAGE RAE OF RENEGING

$$R_r = \sum_{n=1}^{N}(n-2)\xi p P_n$$

$$R_r = \sum_{n=3}^{N-1}(n-2)\xi p P_n + (N-2)\xi p P_N$$

$$R_r = \left[(n-2)\xi p\sum_{n=3}^{N-1}\frac{(n-1)!}{(N-1)^{n-1}}\left\{\frac{\lambda+(\pi_1\mu_2 p_2+\pi_2\mu_1 p_1)(N-1)}{2\lambda+(\mu_1 p_1+\mu_2 p_2)(N-1)}\right\}\frac{\lambda}{\mu_1 p_1}\frac{\lambda}{\mu_2 p_2}\prod_{k=3}^{n}\frac{\lambda}{\mu_1 p_1+\mu_2 p_2+(k-2)\xi p}p'\right.$$

$$\left.+(N-2)\xi p\frac{(N-2)!}{(N-1)^{N-2}}\left\{\frac{\lambda+(\pi_1\mu_2 p_2+\pi_2\mu_1 p_1)(N-1)}{2\lambda+(\mu_1 p_1+\mu_2 p_2)(N-1)}\right\}\frac{\lambda}{\mu_1 p_1}\frac{\lambda}{\mu_2 p_2}\prod_{k=3}^{N}\frac{\lambda}{\mu_1 p_1+\mu_2 p_2+(k-2)\xi p}p'\right]P_0$$

3.4 AVERAGE RATE OF RETENTION

$$R_r = \sum_{n=1}^{N}(n-2)\xi q P_n$$

$$R_r = \sum_{n=3}^{N-1}(n-2)\xi p P_n + (N-2)\xi p P_N$$

$$R_R = \left[(n-2)\xi q\sum_{n=3}^{N-1}\frac{(n-1)!}{(N-1)^{n-1}}\left\{\frac{\lambda+(\pi_1\mu_2 p_2+\pi_2\mu_1 p_1)(N-1)}{2\lambda+(\mu_1 p_1+\mu_2 p_2)(N-1)}\right\}\frac{\lambda}{\mu_1 p_1}\frac{\lambda}{\mu_2 p_2}\prod_{k=3}^{n}\frac{\lambda}{\mu_1 p_1+\mu_2 p_2+(k-2)\xi p}p'\right.$$

$$\left.+(N-2)\xi q\frac{(N-2)!}{(N-1)^{N-2}}\left\{\frac{\lambda+(\pi_1\mu_2 p_2+\pi_2\mu_1 p_1)(N-1)}{2\lambda+(\mu_1 p_1+\mu_2 p_2)(N-1)}\right\}\frac{\lambda}{\mu_1 p_1}\frac{\lambda}{\mu_2 p_2}\prod_{k=3}^{N}\frac{\lambda}{\mu_1 p_1+\mu_2 p_2+(k-2)\xi p}p'\right]P_0$$

4 IMPACT ANALYSIS AND NUMERICAL ILLUSTRATION

In this section we, measure the impact of feedback, impatience and retention on various measures of performance by varying one parameter at a time. Figure -1 and 2 studies the impact of reneging on the system. Figure -3 studies the impact of retention, while figure -4 studies the impact of feedback on relevant measures of performance.

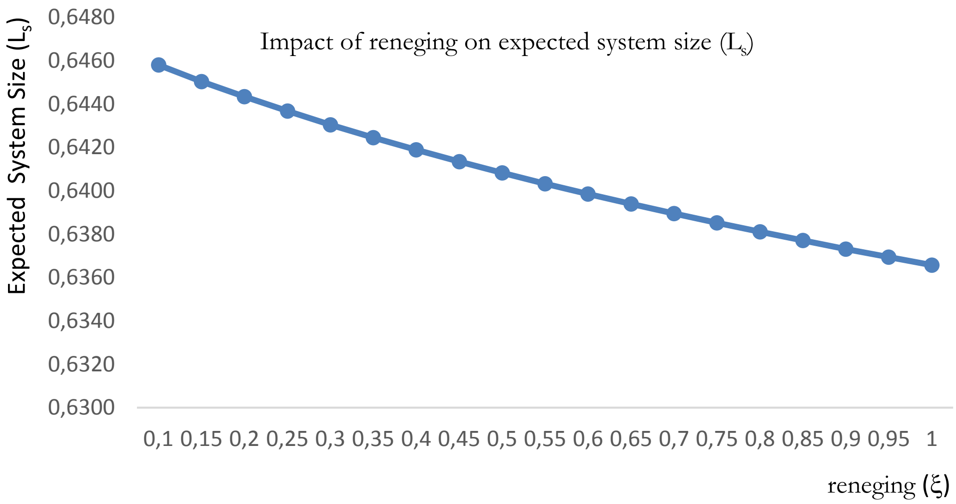

Figure -1 ($\lambda = 4, \mu_1 = 2, \mu_2 = 3, q' = 0.6, q_1 = 0.2, q_2 = 0.2, q = 0.8, N = 15$)

It can be observed from figure-1 that increasing rate of reneging leaves a negative impact on expected system size. The expected system size gradually reduces as, increasing rate of reneging leads to more and more customers leaving the system without completion of service. Reducing system size is not good for any system.

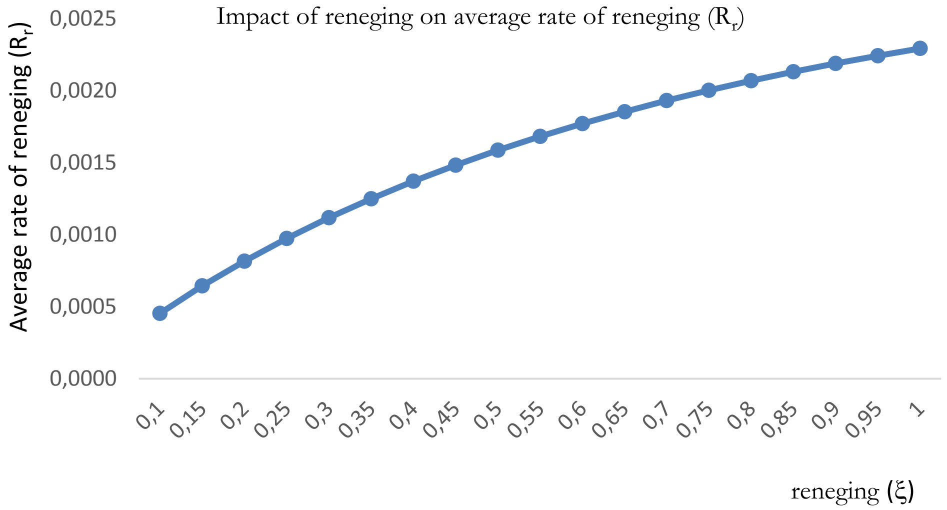

Figure -2 ($\lambda = 4, \mu_1 = 2, \mu_2 = 3, q' = 0.6, q_1 = 0.2, q_2 = 0.2, q = 0.8, N = 15$)

From figure-2 it is clear that with increase in rate of reneging average rate of reneging (R_r)

increases.

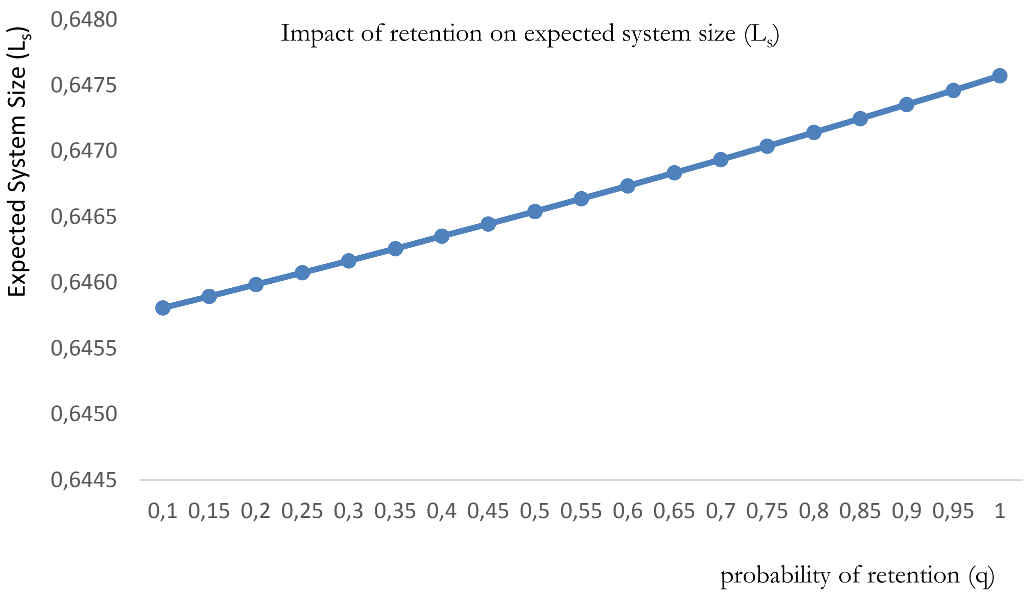

Figure -3 ($\lambda = 4, \mu_1 = 2, \mu_2 = 3, q' = 0.6, q_1 = 0.2, q_2 = 0.2, \xi = 0.1, N = 15$)

From figure -3, it is clear that with increasing rate of retention more and more customers get retained and system size increases gradually. Increasing system size is good for health of any organization as they can earn larger revenue.

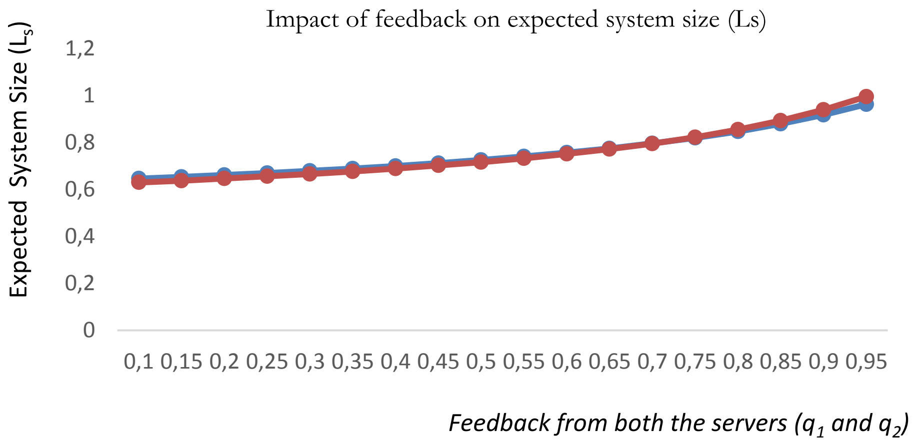

Figure -4 ($\lambda = 4, \mu_1 = 2, \mu_2 = 3, q' = 0.6, \xi = 0.1, q_1 = 0.2\ (varying\ q_2), q_2 = 0.2\ (varying\ q_1), q = 0.8, N = 15$)

From figure -4 it can be observed that more and more customers retiring in to the system. This results in increasing system size.

It can be observed here that both retention of reneging customers and feedback of customers result in increasing system size. Increasing in system size due to retention is good because a customer is retained which otherwise was lost, on other hand increasing system size due to feedback indicates dissatisfaction in service.

Now we will present numerical illustration of the model. Let us consider facility in which arrivals occur in accordance to Poisson process with an average rate of arrival 5 customers per unit time, there are two servers providing service in accordance to exponential distribution with average service rate of 2 and 3 units per unit time. Reneging times are exponentially distributed with a rate of 0.1 per unit time. Firms employ different strategies to retain reneging customers and a reneging customer may be retained with a 60% chance. While due to unsatisfactory service 20% customers rejoin the system from each servers per unit time. Initially, an arriving customer shows least interest in the facility due to n =0 and it may not join (reverse balk) the system with a probability of 0.8. An arriving customer may join on server one with probability 0.4 and server two with a probability 0.6.

The cost of service is Rs 4 per server per customer, holding cost id Rs 2, reverse balking cost is Rs 7, cost of retaining a reneging customer is Rs 2, and cost of a reneging customer is Rs 3 while feedback cost of a customer is Rs 2. The facility earns a revenue of Rs 50 on each customer on an average.

Calculate;

(i) Probability of zero customers in the system
(ii) Expected System Size
(iii) Expected waiting time in the system
(iv) Average rate of reverse balking
(v) Average rate of reneging
(vi) Average rate of retention
(vii) Total Expected Cost
(viii) Total Expected Revenue
(ix) Total Expected Profit

Solution

Measure of Performance	**Numerical Output**
Probability of zero customers in the system (P_0)	0.65885
Expected System Size (L_s)	0.533313
Expected waiting time in the system (W_s)	0.0067
Average rate of reverse balking (R_b')	1.140397
Average rate of reneging (R_r)	0.0002
Average rate of retention (R_R)	0.00013
Total Expected Cost (TEC)	Rs 71.93
Total Expected Revenue (TER)	Rs 128.59
Total Expected Profit (TEP)	Rs 56.67

In next session we develop cost model for the system discusses above and perform cost-profit analysis.

5 COST-PROFIT ANALYSIS

In this section economic analysis of the model is presented. The cost-model is developed with the functions of total expected cost, total expected revenue and total expected profit. The hypothetical values are taken to test the model. Here, TEC = Total Expected Cost, TER = Total Expected Revenue , TEP = Total Expected Profit C_s = Cost of service per unit, C_h = Holding cost per unit , C_f = Feedback cost per unit , C_r = Reneging cost per unit, C_R = Retention cost per unit

The functions of total expected cost, revenue and profit are described as under;

Total expected cost of the model is given by;

$$TEC = C_s(\mu_1 + \mu_2) + C_h L_s + C_b R_b' + C_r R_r + C_R R_R + C_f(\mu_1 p_1 + \mu_2 p_2)$$

Total expected revenue if given by;

$$TER = R \times \mu \times (1 - P_0)$$

Total expected profit is given by;

$$TEP = TER - TEC$$

Following tables present sensitivity analysis of the model with respect to arbitrary inputs of variables.

Table -1
(System performance with change in rate of reneging ξ)
$\lambda = 5, \mu_1 = 2, \mu_2 = 3, q' = 0.6, q_1 = 0.2, q_2 = 0.2, q = 0.8, N = 15$
$C_s = 4, C_h = 2, C_R = 2, C_r = 3, C_b = 7, C_f = 4, R = 50$

Rate of Reneging (ξ)	Expected System Size (L_s)	Average Rate of Reneging (R_r)	Total Expected Cost (TEC)	Total Expected Revenue (TER)	Total Expected Profit (TEP)
0.1	0.548	0.0003	71.928	128.564	56.637
0.2	0.547	0.0006	71.927	128.512	56.585
0.3	0.545	0.0008	71.926	128.465	56.538
0.4	0.544	0.0010	71.926	128.421	56.496
0.5	0.544	0.0012	71.925	128.380	56.456
0.6	0.543	0.0014	71.924	128.342	56.418
0.7	0.542	0.0015	71.923	128.305	56.382
0.8	0.541	0.0017	71.922	128.270	56.348
0.9	0.541	0.0018	71.922	128.237	56.315
1.0	0.540	0.0019	71.921	128.205	56.284

Table -1, shows that reneging leaves a negative impact on the system, as more and more customers leave the system without completion of service. Expected system size with TER, TEC and TEP reduces. And average rate of reneging increases gradually.

Table -2

(System performance with change in probability of reverse balking when n=0)
$\lambda = 5, \mu_1 = 2, \mu_2 = 3, \xi = 0.2, q_1 = 0.2, q_2 = 0.2, q = 0.8, N = 15$
$C_s = 4, C_h = 2, C_R = 2, C_r = 3, C_b = 7, C_f = 4, R = 50$

Probability of Reverse Balking at n=0 (q')	Expected System Size (L_s)	Average Rate of Reneging ($R_{b'}$)	Total Expected Cost (TEC)	Total Expected Revenue (TER)	Total Expected Profit (TEP)
0.1	0.7489	3.401	69.31	176.04	106.73
0.2	0.7222	3.458	69.66	169.76	100.10
0.3	0.6905	3.526	70.07	162.32	92.25
0.4	0.6524	3.607	70.56	153.35	82.79
0.5	0.6056	3.707	71.17	142.35	71.18
0.6	0.5467	3.833	71.93	128.51	56.58
0.7	0.4705	3.996	72.91	110.60	37.68
0.8	0.3679	4.215	74.24	86.48	12.24
0.9	0.2224	4.525	76.12	52.29	-23.84
1.0	0.0000	5.000	79.00	0.00	-79.00

Table -2 shows that, increasing probability of reverse balking when system is empty leaves a bad effect on revenue. We can see that system goes under loss when probability of reverse balking raises a certain limit. The system size obviously reduces to zero as no customer joins the system.

Table -3
(System performance with change in probability of reverse balking when n=0)
$\lambda = 5, \mu_1 = 2, \mu_2 = 3, \xi = 0.2, q' = 0.6, q_1 = 0.2, q_2 = 0.2, N = 15,$
$C_s = 4, C_h = 2, C_R = 2, C_r = 3, C_b = 7, C_f = 4, R = 50$

Probability of Retention (q)	Expected System Size (L_s)	Average Rate of Retention (R_R)	Total Expected Cost (TEC)	Total Expected Revenue (TER)	Total Expected Profit (TEP)
0.1	0.5440	0.0001	71.92	128.40	56.48
0.2	0.5445	0.0003	71.92	128.42	56.50
0.3	0.5449	0.0004	71.92	128.44	56.52
0.4	0.5455	0.0006	71.92	128.46	56.54
0.5	0.5461	0.0007	71.93	128.49	56.56
0.6	0.5467	0.0009	71.93	128.51	56.58
0.7	0.5474	0.0011	71.93	128.54	56.61
0.8	0.5483	0.0014	71.93	128.56	56.64
0.9	0.5494	0.0017	71.93	128.59	56.66
1.0	0.5507	0.0021	71.93	128.63	56.69

Table -3, discusses the effect of retention on system. As retention pulls the customers back to system the system size gets higher and higher and firms make more profit and revenue.

Table -4
(System performance with change in probability of feedback from first server)
$\lambda = 5, \mu_1 = 2, \mu_2 = 3, \xi = 0.2, q_1 = 0.2, q = 0.8, q' = 0.6, N = 15$
$C_s = 4, C_h = 2, C_R = 2, C_r = 3, C_b = 7, C_f = 4, R = 50$

Probability of feedback on server one (q_1)	Expected System Size (L_s)	Total Expected Cost (TEC)
0.1	0.5310	70.93
0.2	0.5483	71.92
0.3	0.5691	72.96
0.4	0.5948	74.06
0.5	0.6274	75.23
0.6	0.6704	76.50
0.7	0.7299	77.94
0.8	0.8167	79.64
0.9	0.9487	70.93

Feedback is a negative process. Increasing feedback depicts poor quality of service table-4 shows increasing probability of feedback from server 1 and hence the system size increases. The facility is crowded with people on which either no or very less revenue is earned. We can observe the rising cost with increase in probability of feedback. Figure-1, shows change in total expected cost with respect to increasing probability of feedback at second server.

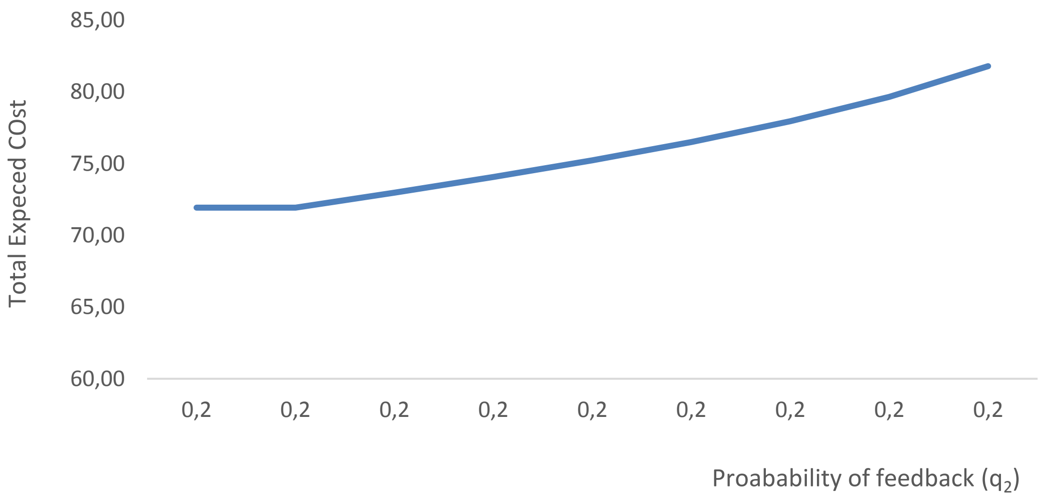

Figure -1
Total Expected Cost w.r.t probability of feedback on server 2 (q_2)
$\lambda = 5, \mu_1 = 2, \mu_2 = 3, \xi = 0.2, q_2 = 0.2, q = 0.8, q' = 0.6, N = 15$
$C_s = 4, C_h = 2, C_R = 2, C_r = 3, C_b = 7, C_f = 4, R = 50$

8 CONCLUSION AND FUTURE WORK

In this paper a feedback queuing model with heterogeneous service, reverse balking, and retention of impatient customers is formulated. The model is solved in steady-state. Necessary measures of performance, numerical illustration and cost-profit analysis of the model is performed. The model is useful for firms that are going through mentioned contemporary challenges. The model can be used for designing effective administrative strategies. The future scope of the work is to test the model in real time environment. The optimization of the model with respect to various parameters can also be obtained thereafter.

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Transient Analysis of a Single-Server Queuing System with Correlated Inputs and Reneging

Rakesh Kumar, Bhavneet Singh Soodan

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School of Mathematics
Shri Mata Vaishno Devi University, Katra-182320, J & K (India)
rakesh.kumar@smvdu.ac.in and bhavneet5678@gmail.com

Abstact

In this paper, we study a continuous time single-server queuing system, wherein the arrivals at two consecutive transition marks are correlated. The service times and the reneging times are exponential distributed. The time-dependent behavior of the model is studied using Runge-Kutta method.

Keywords: Correlated input, Exponential, Queuing model, Reneging, Transient analysis

1 Introduction

Queuing modelling has been playing a very vital role since its inception. It has a great role in modelling and designing communication systems. A lot of work has been done in queuing theory with reference to its applications in inventory management, manufacturing, supply chain management, population studies, genetic studies and in transportation management. Mohan and Murari [9] obtained the transient solution for a correlated queuing system with variable capacity. Murari [10] studied the steady-state behavior of single server queuing system in which both the arrivals and phase-type service were correlated. Andrade Parra [2] studied the correlated nature of cell traffic in broadband communications. Kamoun and Ali [7] considered a two-node tandem network with correlated arrivals and discussed its application in ATM networks. Takine, Suda and Hasegawa [11]studied the ATM switching nodes with the correlated cell arrivals. They also proved that the cell loss and output process characteristics are affected by correlation and variation of cell arrivals. Drezner [4] obatined the performance measures of for $M^c/G/1$ queuing system with dependent arrivals. Jain and Kumar [5] considered the correlated queuing problem with variable capacity and catastrophes and obtained the transient solution by probability generating technique. Jain and Kumar [6] incorporated the concept of restoration in a queuing system with correlated arrivals, variable capacity and catastrophes. Kumar [8] studied the correlated queuing system with catastrophe, restoration and customer impatience. Banerjee [3] studied a workload dependent service queuing system with Markovian Arrival Process. Vishnevskii and Dudin [12] did the review of the queuing systems with correlated inputs with their applications to modeling telecommunication networks.

In this paper, we obtain transient solution of a single-server queuing system with correlated inputs and reneging where the service times are exponentially distributed. Rest of the paper is as follows: In section 2, we described the queuing model. In section 3, the differential-difference equation of the model is presented. Section 4 deals with transient analysis of the model. In section 5 we concluded our paper.

2 Queuing Model Description

The queuing model considered is based on the following assumptions: The customers arrive at a service facility and form a queue. The arrivals can occur only at the transition marks $t_0, t_1, t_2, \ldots$ where $\theta_r = t_r - t_{r-1}, r = 1,2,3\ldots$, are negative exponentially distributed random variables with parameter λ. The arrivals of customers at the two consecutive transition marks t_{r-1} and $t_r, r = 1,2,3\ldots$, are governed by the following transition probability matrix:

$$\begin{array}{cc} & t_r \\ & \begin{array}{cc} 0 & 1 \end{array} \\ t_{r-1} \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{bmatrix} \end{array}$$

where $p_{00} + p_{01} = 1$ and $p_{10} + p_{11} = 1$, where 0 refers to no arrival and 1 refers to the occurrence of arrival. Hence, the arrivals are correlated The system has finite capacity, say N. There is a single server and the customers are served one by one on FCFS basis. The service time distribution is negative exponential with parameter μ. Every customer that enters the system will wait for a certain period of time after which he becomes impatient and leaves the queue. This behaviour of a customer is known as reneging. The reneging times of the customers are assumed to be distributed according to negative exponential distribution with parameter ξ.

3 Mathematical Model

Defining the following probabilities

$Q_{0,0}(t)$ = Probability that at time t the queue length is empty, the server is idle and no arrival has occurred at the previous transition mark.

$Q_{0,1}(t)$ = Probability that at time t the queue length is empty, the server is idle and an arrival has occurred at the previous transition mark.

$P_{0,0}(t)$ = Probability that at time t the queue length is empty, the server is not idle and no arrival has occurred at the previous transition mark.

$P_{0,1}(t)$ = Probability that at time t the queue length is empty, the server is not idle and an arrival has occurred at the previous transition mark.

$P_{n,0}(t)$ = Probability that at time t the queue length is equal to n $(1 \leq n < N)$, the server is not idle and no arrival has occurred at the previous transition mark.

$P_{n,1}(t)$ = Probability that at time t the queue length is equal to n $(1 \leq n < N)$, the server is not idle and an arrival has occurred at the previous transition mark.

$P_{N,0}(t)$ = Probability that at time t the queue length is equal to N, the server is not idle and no arrival has occurred at the previous transition mark.

$P_{N,1}(t)$ = Probability that at time t the queue length is equal to N, the server is not idle and an arrival has occurred at the previous transition mark.

The differential-difference equations of the model are:

$$\frac{d}{dt}Q_{0,0}(t) = -\lambda Q_{0,0}(t) + \mu P_{0,0}(t) + \lambda[p_{00}Q_{0,0} + p_{10}Q_{0,1}]$$
$$\frac{d}{dt}Q_{0,1}(t) = -\lambda Q_{0,1}(t) + \mu P_{0,1}(t)$$
$$\frac{d}{dt}P_{0,0}(t) = -(\lambda+\mu)P_{0,0}(t) + (\mu+\xi)P_{1,0}(t) + \lambda[p_{00}P_{0,0} + p_{10}P_{0,1}]$$
$$\frac{d}{dt}P_{0,1}(t) = -(\lambda+\mu)P_{0,1}(t) + (\mu+\xi)P_{1,1}(t) + \lambda[p_{01}Q_{0,0} + p_{11}Q_{0,1}]$$
$$\frac{d}{dt}P_{n,0}(t) = -(\lambda+\mu+n\xi)P_{n,0}(t) + [\mu+(n+1)\xi]P_{n+1,0}(t) + \lambda[p_{00}P_{n,0}(t) + p_{10}P_{n,1}(t)]$$
$$\frac{d}{dt}P_{n,1}(t) = -(\lambda+\mu+n\xi)P_{n,1}(t) + [\mu+(n+1)\xi]P_{n+1,1}(t) + \lambda[p_{01}P_{n-1,0}(t) + p_{11}P_{n-1,1}(t)]$$
$$\frac{d}{dt}P_{N,0}(t) = -(\mu+N\xi)P_{N,0}(t) + \lambda[p_{00}P_{N,0}(t) + p_{10}P_{N,1}(t)]$$
$$\frac{d}{dt}P_{N,1}(t) = -(\mu+N\xi)P_{N,1}(t) + \lambda[p_{01}P_{N-1,0}(t) + p_{11}P_{N-1,1}(t)]$$

4 Transient Analysis of the Model

In this section, the transient analysis of the model is carried out. Runge -Kutta method of fourth order is used o obtain the solution. The $''ode45''$ function of MATLAB software is used to find the transient numerical results corresponding to the differential-difference equation of the model.

Here we take $N = 6, \lambda = 1.8, \mu = 2.5, \xi = 0.15, p_{00} = 0.2, p_{01} = 0.8, p_{10} = 0.3$ and $p_{11} = 0.7$. In Fig. 1, we plot the system size probabilities with time. We observed that initially $P_{0,0}$ is higher and with the passage of time it decreases becomes steady. The probabilities of the system have lower values initially but they increase gradually and after sometime these become steady.

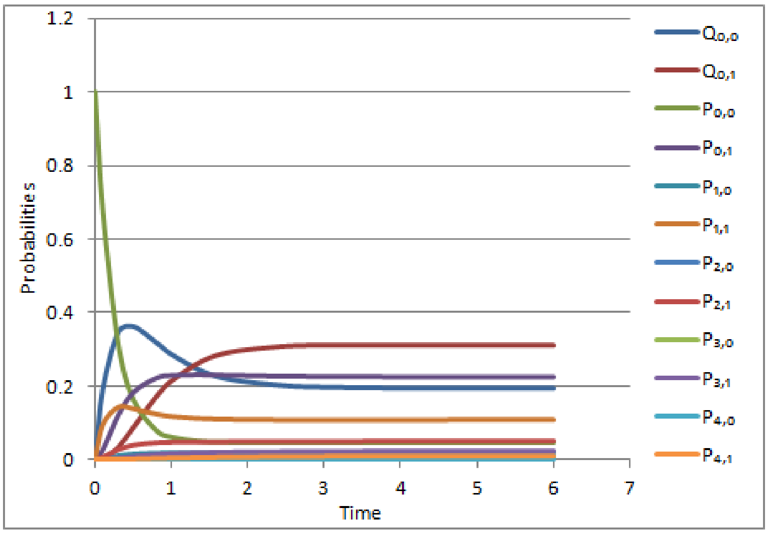

Figure 1: Time dependent behavior of probabilities.

In Fig. 2, we show a graph between expected system size and time. Further we consider two queuing models: one with correlated arrivals and reneging and the other with

Poisson arrivals and reneging. It can be seen from the graph that the expected system size is relatively lower in case of correlated queuing model than the simple model.

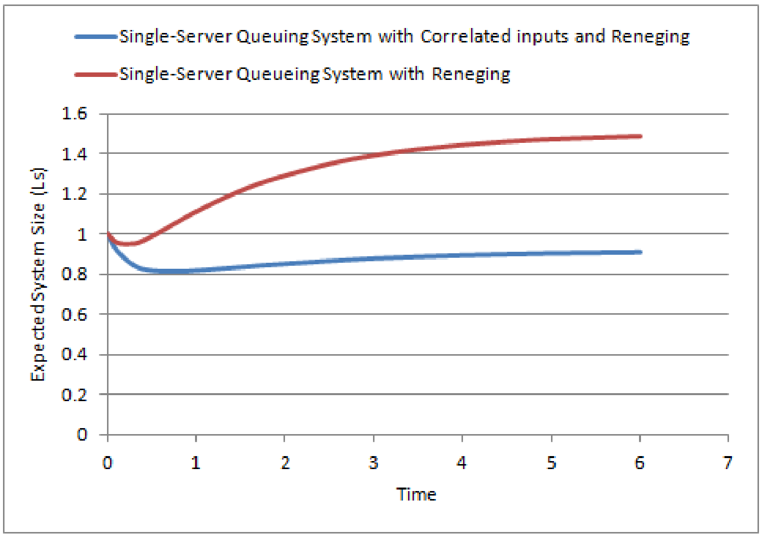

Figure 2: Expected system size vs time

In Fig. 3, the variation in expected waiting time with time is shown. We can see that the expected waiting time of customers is lower in case of correlated queuing system then the system with simple poisson arrivals. This sort of comparison indicates that the correlated input queuing system performs better than the one without correlated arrivals.

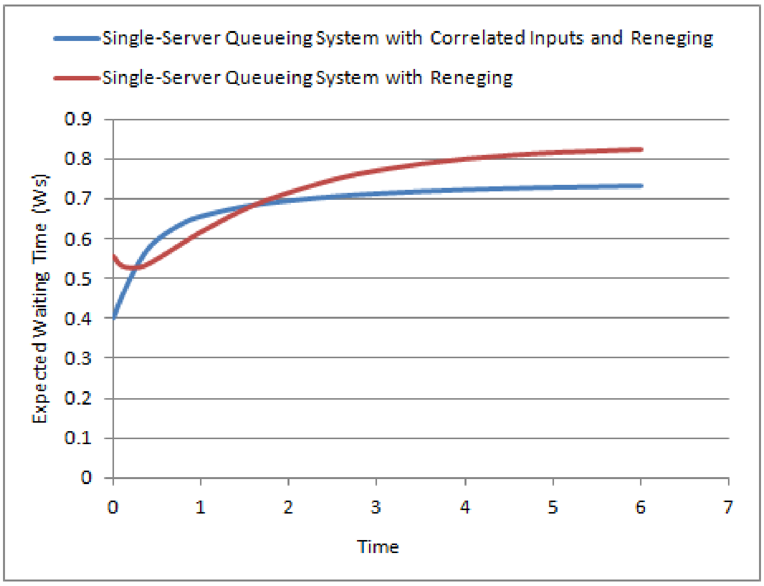

Figure 3: Expected waiting time vs time

5 Conclusion

In this paper we have performed the transient numerical analysis of a single server queuing model with correlated inputs and reneging. We have compared our model with a single server queuing model with reneging and have observed that our model performs better than the other.

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