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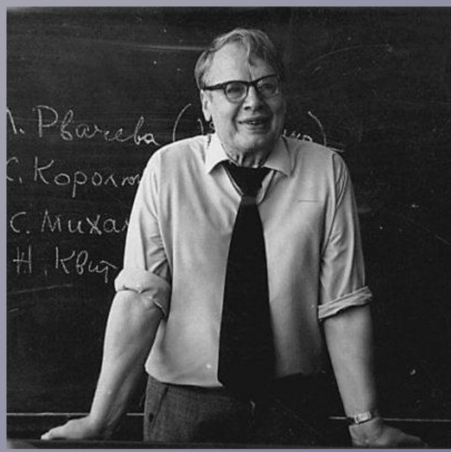

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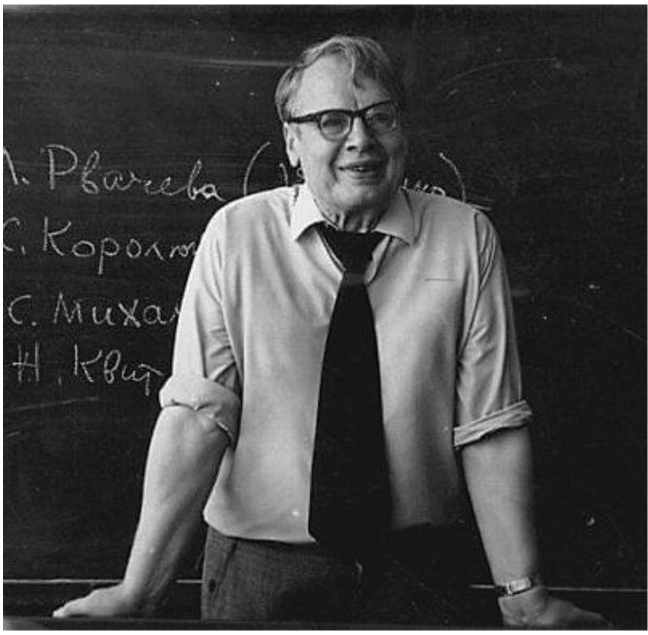

RELIABILITY:

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Table of Contents

Reliability Assessment of the Digital Relay Protection System

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Abstract

The quantitative assessment attempt of reliability indicators for the specific digital structure of the relay protection system by analogy with an assessment of similar digital systems in other industries is given in this work. The reliability models of system components are provided. The calculation sequence is shown. Calculation results give an optimistic evaluation of such protection creation and indicate the influence of the number of autonomous protection blocks reserved by the central protection and recovery time on the system availability.

Keywords: relay protection, digital system, reliability, availability.

1. Accepted Abbreviations

The list of the accepted abbreviations in article is included below,

IED Intellectual Electronic Device executes control functions and protection of substation equipment according to the specified algorithms.
SW Switch works with an Ethernet network, creating transmission channels of digital information.
MU Merging Unit accepts input analog signals from CT/PT, creates digital synchronized time sampling of the measured values and transfers them to numerous IEDs on a substation local network.
PT Potential Transformer measures analog voltage values in substation buses.
CT Current Transformer measures analog current values in substation branches.
PB Process Bus provides information exchange between connected IEDs.
BC Breaker Controller controls the power circuit-breaker.
PS Power Supply provides electronics with the electric power.
CB Circuit Breaker is intended for power network switching.

2. Introduction

One of the most important characteristics of the relay protection is its reliability. Many researches have been done in this area. However, at the present a complete digital relay protection is developed and implemented, which is completely different from the traditional protection. Nevertheless, requirements for reliability remain the same.

Mainly, typical complete digital protection system integrates the merging units, timing sources, digital protective relays and communication devices. Both relay devices and signal outputs of measuring transformers are digital in such system. These digital signals are transmitted by the digital relay via the process bus that integrates interaction of digital blocks. Complete digital protection has more components, than a traditional one that should have a certain influence on its reliability.

Example of the similar protection is the protection system for a distribution network of 110/35/10 kV with digital converting of power system [1]. In this work, the structure and functioning of protection system on substation are considered in detail. Feature of structure offered by the authors is redundancy of the autonomous digital protection for a substation segment (the transformer, bus system section) by the centralized digital protection and control device. Thus, an important function of the substituting protection reservation for joining that should increase protection reliability in general. Other feature is redundancy failure circuit current measurement by its value determination on a segment under the first Kirchhoff's law.

It is necessary to estimate reliability of such a protection structure. As its hardware basis is made by the electronic digital elements, unlike traditional protection with estimates of unnecessary, false operations and failure in operation here it is possible to estimate the protection system availability to operation, as well as at similar electronic digital systems in other industries.

3. Protection Models of Functioning Reliability

Let's consider the reliability indicators on the example of the structural diagram for the protection and control module of bus section 35 kV (fig. 1, a) and of the transformer section (fig. 1, b). As it was noted above, autonomous protection (IEDA) failure has two consequences for the centralized protection (IEDC): 1) results of failure protection measurements can be used; 2) results of failure protection measurements cannot be used. In the second case, the first Kirchhoff's law determines the current of the protected element. Reliability of such definition is connected with all *m*

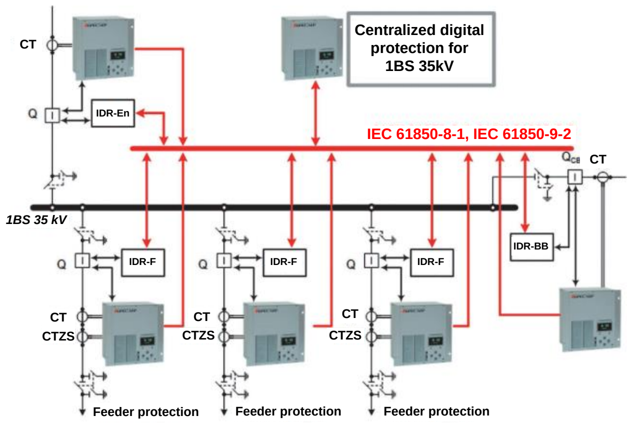

Fig. 1a. Block diagram of the protection module and control section for the bus 35 kV. IDR-interposing digital relay, F-feeder, En-entrance, BB-bus-tie breaker, CTZS-zero phase sequence current transformer.

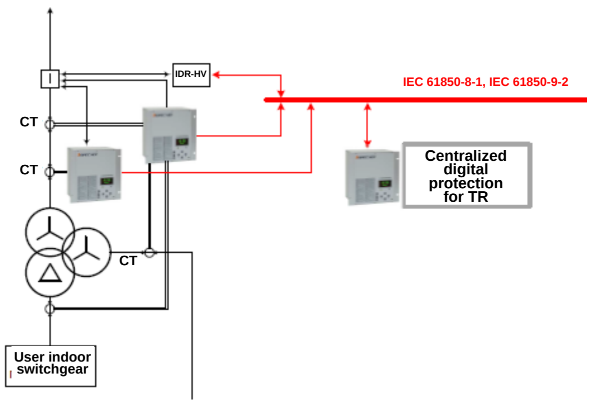

Fig. 1b. Block diagram of the protection and control module for the transformer section. IDR-interposing digital relay, HV-high voltage,

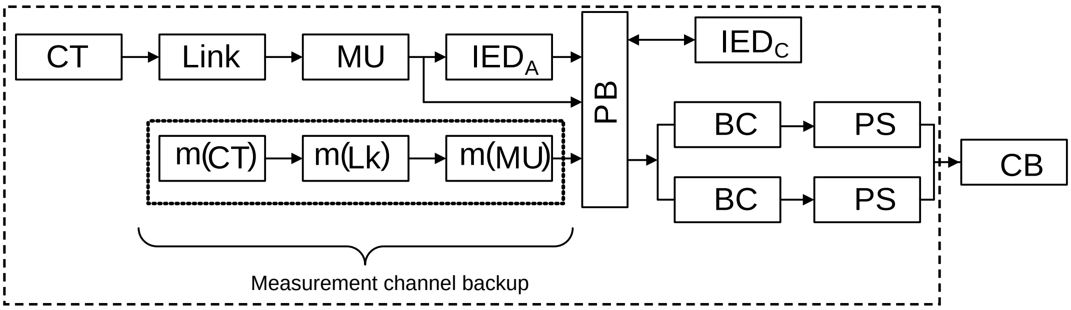

Fig. 2. Reliability block diagram of protection.

intact measuring channels. The general diagram of component communications for separate IEDA with the centralized IEDC, and measuring and executive channels is given in fig. 2 in terms of reliability. The breaker controller (BC) is entered behind the process bus (PB) and in the same place the power supply (PS) block as it is included consistently with all scheme is entered in terms of reliability.

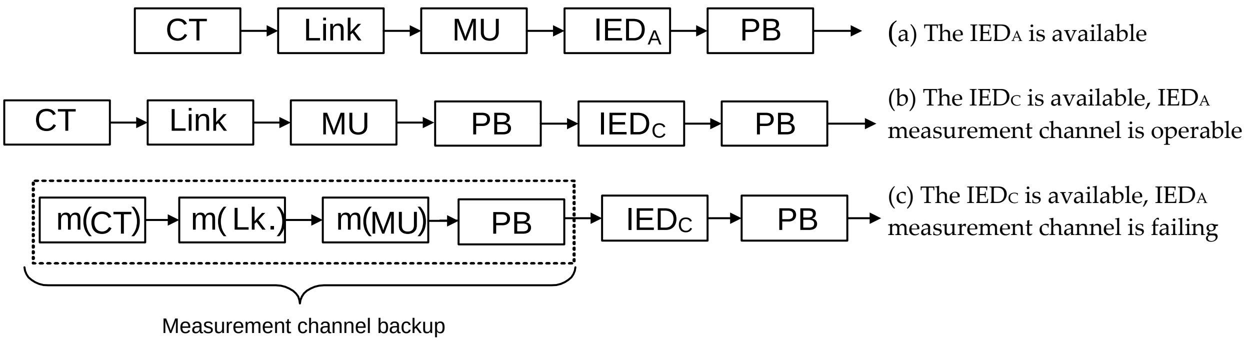

Fig. 3. Reliability block diagram for various component states.

Let's select structures of communications between components when protection functions in different situations. At operable autonomous protection, the model of its reliability is given in fig. 3a. It consists from series-connected components of the autonomous protection, and PB switch λ-s integrate with its λ-s on number of connections. Since equivalent failure rate is equal to the sum of element failure rate in series connection

$$\lambda_e = \sum_{i=1}^{n} \lambda_i, \tag{1}$$

and equivalent renewal rate is equal to mean value of separate indicators [2]

$$\mu_e = \lambda_e / \sum_{i=1}^{n} \frac{\lambda_i}{\mu_i}, \tag{2}$$

where n is component quantity in model chains, protection reliability indicators are defined for this case (fig. 3a) as:

model (a) $\lambda_{mdl\,A} = \lambda_{CT} + \lambda_{Lk} + \lambda_{MU} + \lambda_{IED} + \lambda_{PB}$,

$$\mu_{mdl\,A} = \lambda_{mdl\,A} / (\lambda_{CT}/\mu_{CT} + \lambda_{Lk}/\mu_{Lk} + \lambda_{MU}/\mu_{MU} + \lambda_{IED}/\mu_{IED} + \lambda_{PB}/\mu_{PB}). \tag{3}$$

At IEDA failure, but at its operational measurement channel (a case 1) the IEDC work model is reflected in fig. 3b. Its reliability indicators is

model (b) $\lambda_{mdl\,B} = \lambda_{CT} + \lambda_{Lk} + \lambda_{MU} + \lambda_{PB} + \lambda_{IED} + \lambda_{PB}$,

$$\mu_{mdl\,B} = \lambda_{mdl\,B} / \left(\frac{\lambda_{CT}}{\mu_{CT}} + \frac{\lambda_{Lk}}{\mu_{Lk}} + \frac{\lambda_{MU}}{\mu_{MU}} + \frac{\lambda_{PB}}{\mu_{PB}} + \frac{\lambda_{IED}}{\mu_{IED}} + \frac{\lambda_{PB}}{\mu_{PB}} \right). \tag{4}$$

In the second case, when the y measurement channel of the autonomous protection is fault, the IEDC work model corresponds fig. 3c and its reliability indicators is

model (c) $\lambda_{mdl\,C} = (\lambda_{CT} + \lambda_{Lk} + \lambda_{MU} + \lambda_{PB}) \cdot (m-1) + \lambda_{IED} + \lambda_{PB}$,

$$\mu_{mdl\,B} = \lambda_{mdl\,C} / \left[\left(\frac{\lambda_{CT}}{\mu_{CT}} + \frac{\lambda_{Lk}}{\mu_{Lk}} + \frac{\lambda_{MU}}{\mu_{MU}} + \frac{\lambda_{PB}}{\mu_{PB}} \right) \cdot (m-1) + \frac{\lambda_{IED}}{\mu_{IED}} + \frac{\lambda_{PB}}{\mu_{PB}} \right], \tag{5}$$

where m is a block number of the substation autonomous protection for section. m = 5 for bus section (fig. 1a) and m = 2 for transformer section (fig. 1b).

Let's consider a mutual work of **a** and **c** models for an availability quotient determining according to fig. 2 as the worst in reliability sense option.

The model of breaker circuit and PS block represents a redundant serial circuit, and its reliability indicators is

$$\lambda_{br.circ.} = \lambda_{BC} + \lambda_{PS}, \;\; \mu_{br.circ.} = \lambda_{br.circ.} / (\lambda_{BC}/\mu_{BC} + \lambda_{PS}/\mu_{PS}). \tag{6}$$

Then we define the Markov equations for reliability models of a protection complex **a** and **c** in state space **S1-S4** (fig. 4) [3]. Here possible statuses are defined by digit at **S**, i.e. 4 states are possible. **Up** at a letter points to up state of the corresponding model, and **Down** - disabled.

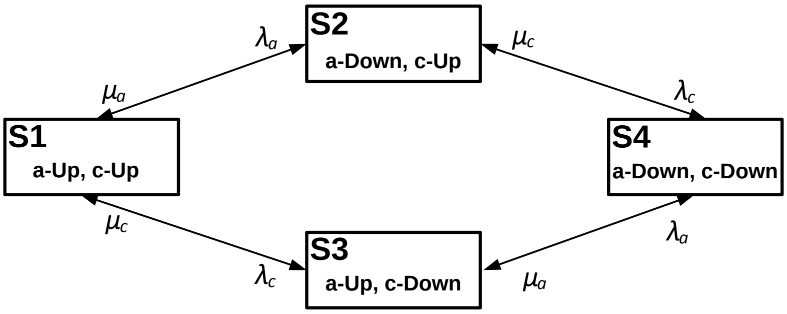

Fig. 4. State space diagram of models a and c

The corresponding state transition array given in (7).

$$\boldsymbol{T} = \begin{bmatrix} 1-(\lambda_a+\lambda_c) & \lambda_a & \lambda_c & 0 \\ \mu_a & 1-(\mu_a+\lambda_c) & 0 & \lambda_c \\ \mu_c & 0 & 1-(\mu_c+\lambda_a) & \lambda_a \\ 0 & \mu_c & \mu_a & 1-(\mu_a+\mu_c) \end{bmatrix}. \tag{7}$$

From Markov's principle that probabilities of boundary statuses do not change in further process of transition, i.e. $\boldsymbol{TP} = \boldsymbol{P}$, where P_i is probability of i-th state, $\boldsymbol{T}$ is state transition array, the equation (7) is written as

$$\begin{bmatrix} -(\lambda_a+\lambda_c) & \mu_a & \mu_c & 0 \\ \lambda_a & -(\mu_a+\lambda_c) & 0 & \mu_c \\ \lambda_c & 0 & -(\mu_c+\lambda_a) & \mu_a \\ 0 & \lambda_c & \lambda_a & -(\mu_a+\mu_c) \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{8}$$

We replace the first equation on $\sum_{i=1}^{4} P_i = 1$, i.e. the sum of all states is equal to 1, and Markov's equation system takes the form (9) where the penultimate column reflects probabilities of the corresponding states,

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ \lambda_a & -(\mu_a+\lambda_c) & 0 & \mu_c \\ \lambda_c & 0 & -(\mu_c+\lambda_a) & \mu_a \\ 0 & \lambda_c & \lambda_a & -(\mu_a+\mu_c) \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{9}$$

hence the probabilities corresponding to the states are equal

$$P_1 = \frac{\mu_a \mu_c}{(\mu_a+\lambda_a)(\mu_c+\lambda_c)}, \tag{10}$$

$$P_2 = \frac{\lambda_a \mu_c}{(\mu_a+\lambda_a)(\mu_c+\lambda_c)}, \tag{11}$$

$$P_3 = \frac{\mu_a \lambda_c}{(\mu_a+\lambda_a)(\mu_c+\lambda_c)}, \tag{12}$$

$$P_4 = \frac{\lambda_a \lambda_c}{(\mu_a+\lambda_a)(\mu_b+c)}. \tag{13}$$

For the model of breaker controller and PS block, Markov's equation system is constructed similarly.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ \lambda_{BC} & -(\mu_{BC}+\lambda_{PS}) & 0 & \mu_{PS} \\ \lambda_{PS} & 0 & -(\mu_{PS}+\lambda_{BC}) & \mu_{BC} \\ 0 & \lambda_{PS} & \lambda_{BC} & -(\mu_{BC}+\mu_{PS}) \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{14}$$

whence

$$P_1 = \frac{\mu_{BC} \mu_{PS}}{(\mu_{BC}+\lambda_{BC})(\mu_{PS}+\lambda_{PS})}, \tag{15}$$

$$P_2 = \frac{\lambda_{BC} \mu_{PS}}{(\mu_{BC}+\lambda_{BC})(\mu_{PS}+\lambda_{PS})}, \tag{16}$$

$$P_3 = \frac{\mu_{BC} \lambda_{PS}}{(\mu_{BC}+\lambda_{BC})(\mu_{PS}+\lambda_{PS})}, \tag{17}$$

$$P_4 = \frac{\lambda_{BC}\lambda_{PS}}{(\mu_{BC}+\lambda_{BC})(\mu_{PS}+\lambda_{PS})}\,. \tag{18}$$

Here, as well as in the previous case, the probability P_i is probability for the corresponding i-th state. The probability of the fourth state ($P_4^{(1)}$ in the first (13) and $P_4^{(2)}$ in the second (18) case) is necessary for determination of protection set availability at failure of all model components. Then protection set availability from the measuring transformers to trip signal output is defined as

$$A_{set} = \left(1 - P_4^{(1)}\right)\left(1 - P_4^{(2)}\right). \tag{19}$$

Since the centralized protection reserves m sets, the availability quotient A_{Mdl} of the entire protection module is equal

$$A_{Mdl} = (A_{set})^m\,. \tag{20}$$

4. Calculation of the Hardware Digital Protection Availability

The failure rate values of the components are taken as the average from several sources, since the statistics of the actual digital protection is still insufficient to determine such values, and they are taken according to the statistics of electronic equipment involved in industrial control processes [4–9]. The reliability indicators of individual model components on this basis are summarized in Table 1, where in the last column are these average values. For the process bus, λ is given in parentheses with regard to the switches.

Table 1

Component failure rates

Component	λ_i, year^{-1}	λ_i, year^{-1}	λ_i, year^{-1}	λ_i, year^{-1}	λ_i, year^{-1}	λ_i, year^{-1}	λ_e, year^{-1}
IED	0.00833	0.00100	0.00966	0.00667	0.00150	0.00330	0.005077
Software	0.00444						0.00444
Networks	0.00333	0.00300					0.003165
PT, CT	0.00200						0.002
Opt. PT, CT	0.00333			0.003			0.003165
Wire	0.00020					0.01000	0.0051
BC		0.01000	0.00333	0.00667	0.00077	0.02280	0.008714
PB		0.01000		0.01000			0.01(0.07)
PS			0.00912		0.03924		0.02418
Opt. fiber			0.00333			0.01000	0.006665
SW		0.01000	0.00869	0.02000		0.01000	0.01217
Server			0.06993				0.06993
Splitter			0.00947				0.00947
CB	0.01000			0.01000			0.01
MU	0.00200	0.01000			0.02545	0.00330	0.010188
Source	[4]	[5]	[6]	[7]	[8]	[9]	

The equivalent failure rates of the models (a and c) are summarized in Table 2, with μ_{2h} = 8760/2 = 4380 year^{-1}; μ_{48h} = 8760/48 = 182.5 year^{-1}. The failure rate of the breaker controller λ_{BC} = 0.008714 year^{-1}.

Regarding the intensity of restoration (repair or replacement of the failed component), in the known literature one of the following approaches is used, or the repair time is taken at 2 hours [11], or the replacement time is 48 hours [4-9]. In the latter case implies delivery if necessary to replace the failed module. One from cases, when all values of the components λ_i are taken equal to 0.0701 year^{-1}, is based on the time between failures of 125 thousand hours [11]. It can be seen from the first half of the Table 2, the transition from traditional devices and measurement circuits to optoelectronic slightly increases the failure rate. It is clear that an increase in the failure rates of the components to 0.0701 year^{-1} increases the equivalent intensities of the models by 3-4 times.

Table 3 shows the failure probability of the protection set $P_4^{(1)}$, the probability of failure of the switch controller set $P_4^{(2)}$ and the protection module availability A_{Mdl} for the 35 kV busbar section and for the transformer.

Table 2

Model reliability indicators

Model	$\lambda_{35кВ}$, year^{-1}	$\lambda_{тр.}$, year^{-1}	Model	$\lambda_{35кВ}$, year^{-1}	$\lambda_{тр.}$, year^{-1}
According to table 1, measurements by traditional transformers, wire communication			According to table 1, measurements by optotransformers, optical fiber communication		
a	0.092364	0.092364	**a**	0.095094	0.095094
c	0.511514	0.0249651	**c**	0.525164	0.255112
According to [11], measurements by traditional transformers, wire communication			According to [11], measurements by optotransformers, optical fiber communication		
a	0,3504	0,3504	**a**	0,3504	0,3504
c	1.5416	0.7007	**c**	1.5416	0.7007

From Table 3, it can be seen that, in general, the approach proposed in [1] ensures sufficient reliability of the digital protection operation.

Table 3

Module availability in various conditions

Measurements by traditional transformers, wire communication								
Para-metr	According to table 1				According to [11]			
	Bus 35 kV		Transformer		Bus 35 kV		Transformer	
	with μ^{-1}= 2 h	with μ^{-1}= 48 h	with μ^{-1}= 2 h	with μ^{-1}= 48 h	with μ^{-1}= 2 h	with μ^{-1}= 48 h	with μ^{-1}= 2 h	with μ^{-1}= 48 h
$P_4^{(1)}$	2.46237E-09	1.41384E-06	1.20187E-09	6.91033E-07	2.81449E-08	1.60518E-05	1.27951E-08	7.32948E-06
$P_4^{(2)}$	1.09830E-11	6.32514E-09	1.09830E-11	6.32514E-09	2.56138E-10	1.47427E-07	2.56138E-10	1.47427E-07
A_{Mdl}	0.9(7)876	0.9(5)2899	0.9(8)757	0.9(5)860	0.9(6)858	0.9(4)190	0.9(7)739	0.9(4)850
Measurements by optotransformers, optical fiber communication								
Para-metr	According to table 1				According to [11]			
	Bus 35 kV		Transformer		Bus 35 kV		Transformer	
	with μ^{-1}= 2 h	with μ^{-1}= 48 h	with μ^{-1}= 2 h	with μ^{-1}= 48 h	with μ^{-1}= 2 h	with μ^{-1}= 48 h	with μ^{-1}= 2 h	with μ^{-1}= 48 h
$P_4^{(1)}$	2.60279E-09	1.49434E-06	1.26445E-09	7.26985E-07	2.81449E-08	1.60518E-05	1.27951E-08	7.32948E-06
$P_4^{(2)}$	1.09830E-11	6.32514E-09	1.09830E-11	6.32514E-09	2.56138E-10	1.47427E-07	2.56138E-10	1.47427E-07
A_{Mdl}	0.9(7)869	0.9(5)249	0.9(8)745	0.9(5)853	0.9(6)858	0.9(4)190	0.9(7)739	0.9(4)850

A record of the form 0.9 (8) 437 indicates that after zero there are 8 nines followed by other digits, it's 437 in the example, i.e. the record would look like 0.99999999437 with a wide table column.

A certain decrease in the availability of its work causes an increase in the number of redundant devices (6 protection devices per 35 kV bus section) and recovery time (2 hours or 48 hours). However, even in the worst conditions, protection availability with redundancy is within acceptable limits. With the known repair rates and the famous formula, $1 - A_{Mdl} = \lambda/(\lambda + \mu)$ the failure rate of the protection module λ_M is defined as

$$\lambda_M = \frac{\mu_{rr}(1-A_{Mdl})}{A_{Mdl}}, \tag{21}$$

where μ_{rr} is specified repair rate. Then in the worst case, with λ_i = 0.0701 and μ_{48h} = 182.5 λ_M = 0.014784 for 35 kV buses and λ_M = 0.0027375 for a transformer. Here the number of autonomous protection blocks of a substation segment m is reflected more clearly.

5. Conclusions

In modern electric power industry, digital protection systems are widely used. One of important indicators is the reliability of its functioning. Its reliability assessment, in contrast to traditional relay protection, is performed by analogy with digital devices in other industries. In the given work, the reliability indicators of the original backup structure for a specific digital protection system were assessed, at that its hardware was assessed without taking into account the software reliability. The software reliability, unlike the technical part, does not wear out over time, but only improves. The study takes into account traditional measuring transformers with the transmission of information in analog form by wire and optoelectronic measuring transformers with information converting in digital form via optical fiber to the relay hall. The assessment did not include the circuit breaker reliability, as external to the protection component. Reliability is also not taken into account associated with communication traffic. Reliability indicators of individual protection components are mainly taken from similar electronic digital devices with built-in diagnostics, which is used in other industries, as there are not enough statistics on digital protection components.

Calculations of the availability for the considered protection system show that the proposed scheme with the stipulated conditions provides an acceptable level of the availability for its operation. It should be noted that the availability to some extent depends on the number of reserved sets m by the central protection and the recovery time t_{rr}. The measuring circuit redundancy has a small effect, worsening this indicator with an increasing m. It can be assumed that the accuracy of the current measurement, determined by the errors of all replacement sets, more affects on this parameter. The transition to fiber optic technology does not have any noticeable effect in terms of reliability. In general, the calculations show values of the availability for protection complex in the worst case at four nines after the point, which meets the requirements for relay protection.

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Method of Conversion of Double Fed Machine Into Synchronous Operation Mode and its Simulation

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Abstract

Double fed induction machines, made on the base of wound rotor machines, thanks to the rapid progress in the converter equipment (due to widespread use of fully controlled thyristors and power transistors) nowadays are widely used as generators (wind power and small hydropower) as well as the motor-where relatively small speed adjustment range (30-40%) is required, by restrictions of the frequency inverter on the installed capacity. There are cases when the technology of their application as a generator and motor mode imposes their long-term operation in sub-synchronous rotational speed, i.e, without speed control. In this case, it is proposed to use only the rectifier side of the frequency inverter feeding the rotor winding of a double fed induction machines, switch into a synchronous mode of operation. This will greatly increase the delivery of reactive power into the grid and use the generator more efficiently. Presented a developed mathematical model of double fed induction machines, which allows to study of all operation modes of double fed induction machines in single set-up–by immediate designation (sub- and super-synchronous speed control); in synchronous generator mode with a significant reactive power output into the grid, as well as in squirrel cage induction generator mode.

Keywords: double fed machine, synchronous operation mode, simulation, method of conversion.

Introduction

In recent years the double fed asynchronous machines (DFAM) are widely used as the generators of wind power plants [1, 2, 3]. They are also recommended for using as the generators in hydraulic units of small hydroelectric power plants (HPP) [4,5].

Range of their application as the motors is also extensive: they are in demand where the rotational frequency of drive mechanism needs to be controlled in relatively small ranges (30-40% of the rated one) under the limited power of frequency converter, supplying the rotor's winding of DFAM.

However, the cases often occur depending on the requirements of either electric power generation technology with DFAM operating in generator mode or the technology of drive mechanism operating with DFAM as a motor, when within a long time it is not required to control the rotational frequency of either turbine (driving motor) or operating mechanism. For example, when the hydraulic units of small HPSs are equipped with propeller turbines, it needs to control the rotational frequency of generator with essential increase (or decrease) of the water flow.

Exactly with this change of water flow (discharge) the control of rotational frequency of the hydraulic unit's shaft proportionally with the change of this water discharge by force of AMDP allows raising the efficiency of hydraulic turbine, and in addition to that the output power of hydraulic unit supplying the electric power network. When water discharge is constant (and this period may go on for a long time), it is reasonable the rotational frequency to remain constant.

With frequency converter in rotor's circuit of AMDP it can be implemented by the following ways: either to remove the frequency converter from the operating mode, and then to short-circuit the rotor windings of DFAM, thereby this machine will be converted into squirrel-cage asynchronous generator, or to leave the frequency converter in operation with adjusting it so, that the rotational frequency of generator will be nearly equal to synchronous one. In the first case it is clear that the power factor of generator will be the low one, i.e. generator will consume the significant reactive power from the network. In the second case, the power factor will be in the limits of $\cos\varphi \approx 1$, (i.e. generator doesn't consume, but also doesn't output the reactive power)..

1. Statement of a problem

To increase the output of reactive power into network a rotor winding of DFAM is offered to connect to a power source of direct current, i.e. to convert DFAM into operating mode of synchronous machine. This will allow providing with reactive power the load center of power system, to which the DFAM is connected, in addition the machine can be loaded up to full rated power.

The electrical schematic diagram of conversing of DFAM with frequency converter in rotor circuit into synchronous operating mode can be presented in the view, shown in Fig.1:

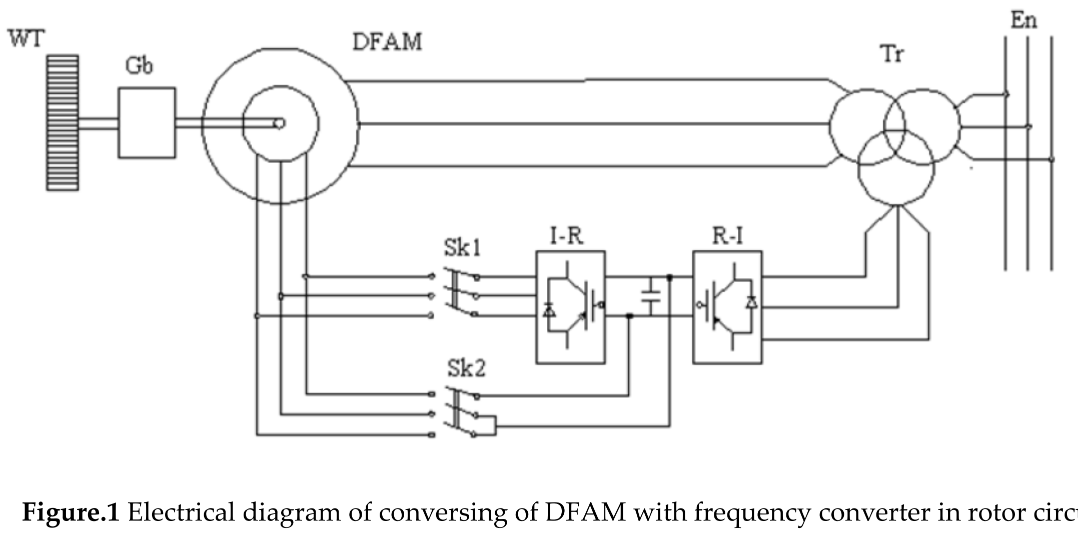

Figure.1 Electrical diagram of conversing of DFAM with frequency converter in rotor circuit into synchronous operating mode

Here WT–is a driving turbine (e.g. water one), it is aggregated by force of the gearbox Gb with the shaft of generator, carried out on the basis of double fed machine DFAM, Tr–a three-winding transformer, supplying the stator and rotor windings of DFAM, En–electric power network (system), I-R and R-I–inverter–rectifier, carried out on the basis of fully controlled IGBT–transistors, or GTO–thyristors, Sk1, Sk2–switching keys (switches).

The circuit diagram of connection of rotor windings of asynchronous machine with phase-wound rotor, shown in Fig.1, is known from [6], the originality of this diagram lies in the fact that, the rotor windings are suppled in a synchronous mode from a link of direct current of frequency converter, assigned for controlling of rotational frequency of aggregate in operating mode controlled from the rotor side of DFAM.

2. Mathematical model for study

Let's demonstrate the performance of above proposal on previously developed by us the mathematical model of DFAM (system is supplemented with the expressions for active and reactive powers of stator and rotor) [1]. Equations of DFAM, frequency controlled from the rotor side, are presented in the view [in relative units]:

$$\left.\begin{aligned}
&p\Psi_{ds} = -U_s \cdot sin\theta + \psi_{qs}(1-s) - r_s \cdot i_{ds}\\
&p\Psi_{qs} = U_s \cdot cos\theta - \psi_{ds}(1-s) - r_s \cdot i_{qs}\\
&p\psi_{dr} = -k_{ur} \cdot sin(k_{fr} \cdot \tau) - r_r \cdot i_{dr}\\
&p\psi_{qr} = \pm k_{ur} \cdot cos(k_{fr} \cdot \tau) - r_r \cdot i_{qr}\\
&ps = \frac{1}{T_j} m \frac{1}{T_j}{}_{em_{WT}}\\
&p\theta = s\\
&m_{em} = \psi_{ds} \cdot i_{qs} - \psi_{qs} \cdot i_{ds}\\
&p_s = U_{ds} \cdot i_{ds} + U_{qs} \cdot i_{qs}\\
&q_s = U_{qs} \cdot i_{ds} - U_{ds} \cdot i_{qs}\\
&p_r = U_{dr} \cdot i_{dr} + U_{qr} \cdot i_{qr}\\
&q_r = U_{qr} \cdot i_{dr} - U_{dr} \cdot i_{qr}\\
&i_{ds} = k_s \cdot \psi_{ds} - k_m \cdot \psi_{dr}\\
&i_{qs} = k_s \cdot \psi_{qs} - k_m \cdot \psi_{qr}\\
&i_{dr} = k_r \cdot \psi_{dr} - k_m \cdot \psi_{ds}\\
&i_{qr} = k_r \cdot \psi_{qr} - k_m \cdot \psi_{qs}\\
&p_{tot} = p_s + p_r\\
&q_{tot} = q_s + q_r
\end{aligned}\right\} \quad (1)$$

In the system of equations (1) the following designations: Ψ_{ds}, Ψ_{qs}, Ψ_{dr}, Ψ_{qr} – are accordingly the flux linkages of stator and rotor circuits on direct and quadrature axes; i_{ds}, i_{qs}, i_{dr}, i_{qr} – currents of stator and rotor windings on *d* and *q* axes; U_s – amplitude of voltage applied to the stator winding of machine; k_{ur}, k_{fr} – amplitude and frequency of controlled by frequency converter voltage, supplied to the rotor windings of machine; *s* – slip of the machine equal to s=1– ω_r; ω_r – angular frequency of revolution; θ – angle between the axis of rotor and synchronously rotating axis (with speed $\omega_S = 1$); m_{WT}, m_{em} – driving torque of the driven motor (e.g. water turbine) and electromagnetic torque of DFAM; $U_{ds}= -U_s \cdot \sin(\theta)$ and $U_{qs}=U_s \cdot \cos(\theta)$ – components of stator voltage on *d* and *q* axes; $U_{dr}= -k_{ur} \cdot \sin(k_{fr} \cdot \tau)$, $U_{qr}=k_{ur} \cdot \cos(k_{fr} \cdot \tau)$ – components of rotor voltage on *d* and *q* axes; p_s, p_r – values of active powers of stator and rotor circuits; q_s, q_r – values of reactive powers of stator and rotor; p_{tot}, q_{tot} – total active and reactive powers of DFAM; T_j – inertia constant of the rotating parts of driving motor and DFAM; $\tau=314 \cdot t$ – synchronous time [in rad.].

Furthermore, the factors k_s, k_r and k_m are determined from the following correlations:

$$k_s = \frac{x_r}{x_r \cdot x_s - x_m^2};\ k_r = \frac{x_s}{x_r \cdot x_s - x_m^2};\ k_m = \frac{x_m}{x_r \cdot x_s - x_m^2} \quad (2)$$

DFAM parameters: r_s, r_r – resistances of stator and rotor windings; x_s, x_r – full inductive reactances of stator and rotor circuits; x_m – mutual induction reactance between stator and rotor circuits (they are the analogous of corresponding inductivities).

It should be noted that the system of equations (1) is written in *d, q,* axes, rotating with rotor speed ω_r of the machine. Exactly this circumstance allows realizing in one structure of mathematical model the operating modes of all conversions of double fed machine into squirrel-cage asynchronous machine, synchronous machine with the implementation of excitation system on one of axes (d axis).

The calculations have been performed for DFAM with the following parameters: P_n=110 kW; M_n=727.25 Nm; cosφ=0.9; η=0.95; U_{base}=311 V; I_{base}=285 A; Z_{base}=1.09 ohm; J=0.86 kgm^2. The winding data (in relative units) r_s=0.01; r_r=0.03; x_s=4.878; x_r=4.9; x_m=4.8; $x_{\sigma s}$=0.078; $x_{\sigma r}$=0.1 (resistances and reactances of leakage of stator and rotor windings).

Algorithm of solution (Mathcad program) with the numerical data is presented below:

$$\boldsymbol{D(\tau,Y)} = \begin{bmatrix} -1\cdot sin(Y_6)+Y_2-Y_5\cdot Y_2-0.01\cdot(5.69\cdot Y_1-5.56\cdot Y_3) \\ 1\cdot cos(Y_6)-Y_1+Y_5\cdot Y_1-0.01\cdot(5.69\cdot Y_2-5.56\cdot Y_4) \\ -k_{ur}\cdot sin(k_{fr}\cdot\tau)-0.03\cdot(5.66\cdot Y_3-5.56\cdot Y_1) \\ \pm k_{ur}\cdot cos(k_{fr}\cdot\tau)-0.03\cdot(5.66\cdot Y_4-5.56\cdot Y_2) \\ 0.005\cdot m[Y_1\cdot(5.69\cdot Y_2-5.56\cdot Y_4)-Y_2\cdot(5.69\cdot Y_1-5.56\cdot Y_3)]_{WT} \\ Y_5 \end{bmatrix} \tag{3}$$

where: $Y_1=\Psi_{ds}$; $Y_2=\Psi_{qs}$; $Y_3=\Psi_{dr}$; $Y_4=\Psi_{qr}$; $Y_5=s$; $Y_6=\theta$. The initial values of all variables Y_0=0, besides Y_{05}=1 (slip s_0=1).

3. Study of Double Fed Induction Machine in Synchronous Operation Mode

As it was mentioned above, in a long-time operating mode two options are possible in a range near synchronous rotational frequency. In the first option the frequency converter can be removed from the operation (Fig.1) and rotor windings short-circuited, thus DFAM converts into squirrel-cage asynchronous generator. In this case, equations (3) and (4) of the system (1) will appear as:

$$\left.\begin{aligned} p\psi_{dr} &= -r_r\cdot i_{dr} \\ p\psi_{qr} &= -r_r\cdot i_{qr} \end{aligned}\right\} \tag{4}$$

Since U_{dr} and U_{qr}, k_{ur} and k_{fr} are equal to zero, the equations for p_r and q_r will also disappear from the system (1) (i.e. equations 9 and 10).

In the second option, which is the most reasonable one, DFAM could be converted into the synchronous generator, thereto in the system of equations (1) the same equations (3) and (4) should be written as:

$$\left.\begin{aligned} p\psi_{dr} &= U_{df}-r_{df}i_{dr} \\ p\psi_{qr} &= -r_{qr}\cdot i_{qr} \end{aligned}\right\} \tag{5}$$

That is, the system (1) as a whole is transformed into the Park-Gorev equations with the implementation of excitation U_{df} on direct axis d of the machine .

According to Fig.1, a constant voltage is supplied to start U_{df} of rotor winding of phase A, and to joined together the starts of phases B and C, and the ends of these windings are joined together (zero point), i.e. phases B and C are connected in parallel to each other and serially with the phase A. When aligning the direct axis d with axis of winding of A phase, it can be considered with a certain error, that windings' axes of B and C phases are on the quadrature axis q.

In consequence of such connection the resistances r_{df} = r_{dr} of the expression (4) should be increased by 1.5 times, i.e. r_{rf} =r_{dr}=1.5·r_r, and the resistance r_{qr} will be equal to r_{rq}=2·r_r. With a fractional error it can be considered that the leakage reactances of rotor circuits $x_{\sigma dr}$=1.5·$x_{\sigma r}$ change in the same ratio; $x_{\sigma qr}$=2·$x_{\sigma r}$. This naturally will entail the changes of rotor circuits' impedances x_{dr} and x_{qr}, and values k_s, k_r and k_m in the system of equations (1).

With taking into account the parameters of machine and circuit diagrams of rotor winding according to Fig.1 the connection of currents with flux linkages in this mode will appear in the following digital form:

$$\left.\begin{aligned} i_{ds} &= 4.5\cdot\psi_{ds}-4.36\cdot\psi_{dr} \\ i_{qs} &= 3.7\cdot\psi_{qs}-3.55\cdot\psi_{qr} \\ i_{dr} &= 4.43\cdot\psi_{dr}-4.36\cdot\psi_{ds} \\ i_{qr} &= 3.61\cdot\psi_{qr}-3.55\cdot\psi_{qs} \end{aligned}\right\} \tag{6}$$

Let's demonstrate the above calculations on the same structure of mathematical model of DFAM.

There are presented in the Fig.2 (*a, b, c, d, e, f, g, h*) accordingly the electromagnetic torque of the machine m_{em}, its rotational frequency ω_r, active and reactive powers of generator p_s and q_s and stator's currents i_{ds} and i_{qs} and rotor's ones i_{dr} and i_{qr}.

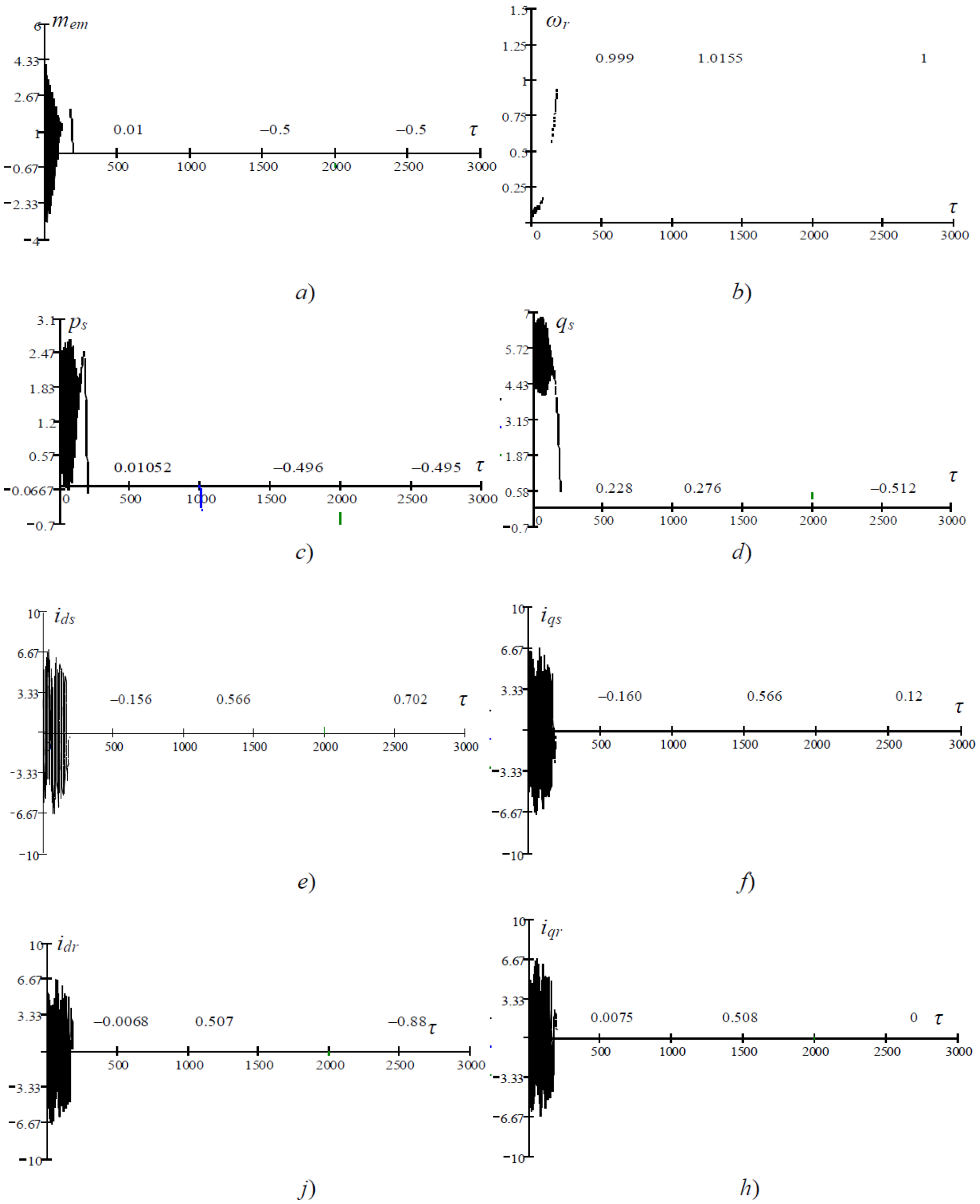

Figure.2 The fluktogrammas of change of double fed asynchronous machine's operating conditions when operating in synchronous mode.

Startup is carried out with taking into account the friction torque equal to m_{WT}=0.01 (i.e. practically without load) in the time period of from τ=0 to τ=1000 radian. Wherein the rotation frequency ω_r=0.999. From τ=1000 rad. to τ=2000 rad. the machine operates in asynchronous generator mode, with short-circuited rotor's windings when the driving torque of driven motor (turbine) is equal to m_{WT}= –0.5 (minus sign indicates generator mode). In this mode the electromagnetic torque m_{em}= –0.5 (Fig.2, a), rotational frequency ω_r=1.0155 (Fig.2, *b*), (ω_r>1which indicates the machine operation in generator mode). Active and reactive powers p_s and q_s equal to p_{tot} and q_{tot} in this mode reach the values p_s= –0.496 and q_s=0.276, and the reactive power is positive, i.e. generator consumes reactive power from the network (Fig.2, *c* and *d*). Stator currents i_{ds} and i_{qs} and rotor ones i_{dr} and i_{qr} in this mode are variables, the amplitude of stator currents does not exceed the values i_{ds}=i_{qs}=0.566, and the rotor ones i_{dr}=i_{qr}=0.508 (Fig.2, *e*, *f* and *i*, *j*).

On the fluktogrammas of the same figure in the time range of from τ =2000 rad. to τ =3000 rad. conversion into synchronous operating mode of the machine is carried out, i.e. equations (2) and (3) of the system (1) are formed according to the ratios (4) and (5). In this range the drive torque remains the same, i.e. m_{WT}= –0.5, according to it m_{em} = –0.5, rotational frequency is strictly equal to ω_r=1, which indicates the synchronous mode. For this machine the constant value of excitation voltage in the equations (4) is chosen to be equal to U_{df} = –0.04. In this process the active power value is equal to p_s= –0.495 (Fig.2, *c*) and reactive one is q_s= –0.512 (Fig.2, *d*). That is, the machine operates in synchronous mode with output to the network both an active and the reactive powers, and the value of reactive power is a little bit more than active one. Power factor has capacitive character and reaches the value of cosφ_{st}≈0.7.

In synchronous mode the stator and rotor currents (Fig.2, *e*, *f*) does not exceed the permissible limits. Excitation current i_{dr}=i_{df} sets at a level of i_{dr}=i_{df}= –0.889 (Fig.2, *g*), and current i_{dr} in this mode is naturally equal to i_{qr}=0 (Fig.2, *h*). It must be noted that operating mode of DFIM in near synchronous mode, values of the control parameters will be k_{ur}=k_{fr}=0.01 when m_{WT}=–0.5.

The electromagnetic torque sets at the value of m_{em}= –0.5, the rotational frequency is equal to ω_r=1.01, the active and reactive powers are accordingly equal to p_{tot}= –0.49, q_{tot} ≈ –0.03. Thus, DFIM operates in a design mode, and output of reactive power is almost equal to zero, i.e. generator operates with power factor equal to cosφ ≈1.

Summarizing the above-stated, the following algorithm of DFAM operation can be recommended under the long-term operating conditions (month, season) in a range of near synchronous speed for the average values of driving torque of driven motor: when a considerable reactive power output to network is required, AMDP should be transfered into operating mode of a purely synchronous generator with excitation from controlled rectifier (Fig.1), in this process the inverter (I-R) is removed from the circuit; when DFAM operating on partially compensated with reactive power the electric power network it is necessary to leave the circuit of frequency converter unchanged; to secure the near synchronous rotational frequency by the values of the control parameters (k_{ur}=k_{fr}), in this process the reactive power (cosφ≈1) isn't output to the network and isn't consumed from the network; and finally with significant compensation of reactive power to the electric network it is necessary to remove completely the frequency converter from the operating mode and to convert DFAM into squirrel-cage asynchronous generator mode, in this case the reactive power will be consumed from the network.

Conclusions

1. The presented notation of the equations of controlled double fed asynchronous machine allows relatively easy studying of the mathematical model of DFAM in one structure, conversion of the machine into the modes of synchronous generator and squirrel-cage asynchronous generator.
2. When DFAM operating on the uncompensated electric power networks and under the appropriate processing conditions it is advisable to convert its operation into a synchronous mode, connecting and powering the rotor windings according to the diagram on Fig.1 from the rectifier

part of frequency converter. This allows significant increasing of output of reactive power by generator into the network, while the value of the leading cosφ constitutes cosφ ≈0.7.

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Maintenance of Reliability of Methodical Support of the Management of Objects EPS

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Abstract

One of the basic problems of development of intellectual control systems of maintenance service and repair of the equipment and devices of electro power systems is increase of reliability of methodical recommendations. The risk of the erroneous decision exists, first of all, because of presence among statistical data of operation of gross blunders, abnormal values. If to that still to add difference not casual samples statistical data of operation from theoretical representative samples random variables from a general data set, to consider multivariate character of statistical data of operation and absence of methods of the analysis small samples multivariate data, difficulty of the decision of this problem becomes obvious. The method which on the basis of fiducially the approach and theories of check of statistical hypotheses is capable to reveal abnormal realizations is developed. And application the express train-methods of calculation of critical fiducially values an interval for the chosen significance value, allows to solve this problem without special tables and the COMPUTER.

Keywords: Reliability, faultlessness, statistical data, methods, fiducially approach, abnormal realizations, the importance.

I. Introduction

Statement of the problem

Development of computer technologies, transition from information to intellectual systems, an objective quantitative estimation of individual reliability, profitability and safety of objects of electro power systems are indissolubly connected with the requirement of a safety and a faultlessness of initial data [1].

The problem of safety of data as a whole can be solved by reservation of objects of a database with the closed access [2]. Presence of abnormal data, i.e. infringement of faultlessness, caused mainly human factor and is concrete mistakes of operators. Process of recognition of abnormal data is many-sided and, first of all, depends on type of the equipment and devices that is why demands development of specialized methods of the statistical analysis. The general, thus, multidimensionality and small number of statistical data of operation is. In clause the analysis of the traditional approach is resulted, inadmissibility of neglect marked by conditions of application of existing criteria, the new method offered. The method based on fiducially approach, the theory of check of statistical hypotheses, express train-methods of calculation of critical values of analyzed parameters.

II. Methods of recognition of abnormal data in small samples.

In [3,4,5] as a result of the analysis for number of realizations of sample n≤10 are allocated as the most effective three methods: N.V.Smirnov, Shovenes and Dickson. We shall take advantage of this recommendation.

First of all, we shall note, that these three methods assume all, that considered small samples concern to representative samples of set of normally distributed random variables. It is known, that characteristic for EPS small samples of statistical data about reliability are received from set of multivariate data, by classification of this set on some versions of an attribute, for example, on a class of a pressure. And, as it is often done, we "shall close eyes" to a degree of their conformity to the normal law. And three criteria are chosen because the last century B.V.Gnedenko recommended to apply at check of hypotheses not less than two criteria [6].

It is established [7], that each of criteria usually reflects the importance of concrete statistical property of sample. We shall result a known example: samples can differ casually on size of average arithmetic value of realizations and essentially to differ on disorder and on the contrary.

We shall consider sample of monthly average values of the charge of the electric power (W) for own needs (o.n.) boiler installations of power units 300 MWt on gas black oil fuel, in %, with number of realizations n=5. It {30,4; 2,38; 2,14; 2,44; 2,48}. We shall estimate with a significance value α =0,1 presence of abnormal realizations. For this purpose:

- range realizations W_{on} in ascending order. Sample gets a kind {2,14; 2,32; 2,44; 2,48; 3,04};
- define average arithmetic value of realizations $M^*(W_{on})$=2,5%
- define average quadratic value of realizations of sample $\sigma^*(W_{on})$=0,318%

N.V Smirnov's criterion provides calculation:

- the greatest deviation $\Delta^* W_{on}$=0,54%
- Statistics $\rho^*(W_{on}) = \Delta^*(W_{on}) \big/ \sigma^*(W_{on}) = 1,7$ (in relative units)
- definition of critical value $\varrho_{0,1}$=1,79

As $\varrho^*(W_{on})<\varrho_{0,1}$, with a significance value even α=0,1, the assumption of presence of abnormal realizations is rejected

Criterion Shovenes also assumes calculation of statistics $\varrho^*(W_{on})$. Further:

- on tabulated values of integrated function of standard normal distribution value of function is defined $F[\varrho^*(W_{on})]$=0.955
- Shovenes statistics is calculated $Sh^*(W_{on})=2\cdot n\{1- F[\varrho^*(W_{on})]\}$=0.45
- critical value of statistics Sh_c is defined. For α=0,1 and n=5 value Sh_c=0.40

Since $Sh^*(W_{on})>Sh_c$, that is accepted the assumption of presence of abnormal realization $W_{on,5}$=3,04%

The criterion of Dickson provides:

- calculation of statistics $r(W_{on})=\max\{r_1(W_{on}); r_2(W_{on})\}$

where:
$$r_1(W_{on}) = \frac{W_{on,2}-W_{on,1}}{W_{on,5}-W_{on,1}} = \frac{2,32-2,14}{3,04-2,14} = 0,2$$
$$r_2(W_{on}) = \frac{W_{on,5}-W_{on,4}}{W_{on,5}-W_{on,1}} = \frac{3,04-2,48}{3,04-2,14} = 0,62$$

- definition of critical value r_k. For α=0,1 and n=5 value r_c=0.40
- comparison $r(W_{on})$ and r_c. As $r(W_{on})$ is more r_c, that is accepted the assumption of presence of abnormal supervision $W_{on,5}$.

Thus, from three criteria two confirm presence of abnormal supervision.

III. The recommended method.

The recommended method based on criterion of recognition of the importance of distinction of parameters of reliability calculated on set of multivariate data and not casual sample of this set [5]. For example, there is some data set about reliability of switches 110-500 kV. After it possible to estimate average duration idle time in emergency repair (as average temperature on hospital). We shall assume, that us switches 110 kV interest. In the first this sample not random, and in the second, on former, includes multivariate data (as average temperature of patients in surgical branch). These comments in brackets we, first of all, wish to pay attention to that fact, that both to a data set, and to sample it is impossible to apply methods of recognition of the mistakes, stipulated for one-dimensional data representative samples.

The enlarged block diagram of algorithm of the control of a faultlessness of sample is present on fig.1.

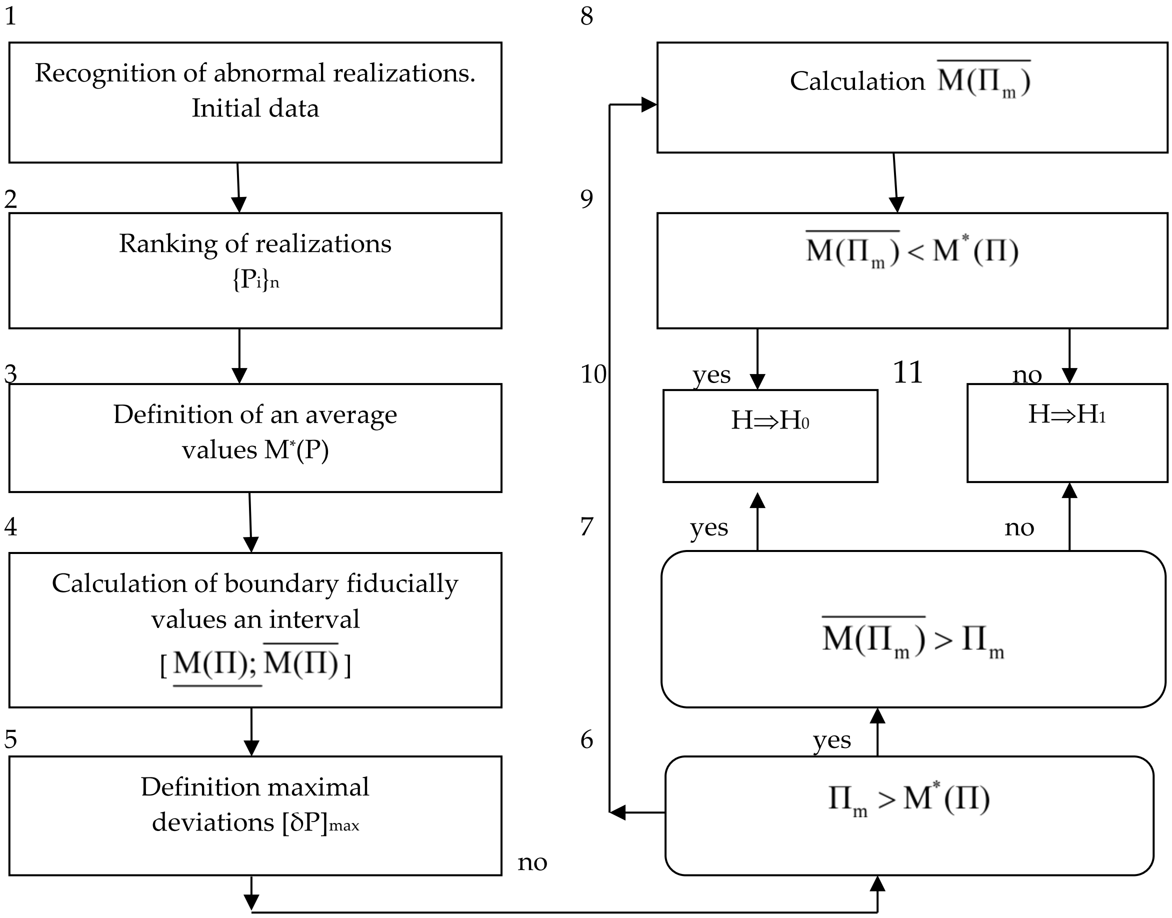


Fig.1. The integrated block diagram of algorithm of recognition of abnormal realizations

The algorithm (sequence of calculations) provides:

The block 1. As initial data sample of random variables (parameters) in volume n serves. Such sizes can be time of a finding in emergency repair, monthly average relative value of the charge of the electric power for own needs of boiler installation of the power unit, size of the fifth harmonic on trunks of substation, etc. It is necessary to evaluate the accuracy of these data;

The block 2. First of all, it is necessary to arrange these data in ascending order (to range);
The block 3. Average arithmetic value of realizations is calculated. We shall designate it as M*(P);
The block 4. Boundary values of fiducially an interval are defined $[\underline{M(\Pi)}; \overline{M(\Pi)}]$.They can be calculated a method of imitating modelling on the COMPUTER for any significance value [7]. But, is much easier for of some the parameters calculated as an average arithmetic random variables, or the probability of occurrence of event, or relative duration of a condition, boundary values fiducially an interval to define the express train-method under the formula approximating interrelation of critical values, a significance value and number of realizations [8]. For example, for M*(P) the top boundary value fiducially interval is defined under the formula $\overline{M(P)} = M^*(P)(1 + A/\sqrt{n})$, and the bottom boundary value - under the formula

$\underline{M(P)} = M^*(P)(1 - A/\sqrt{n})$. If n=5, that for α=0,1; 0,05 and 0,01 relative deviation $A/\sqrt{n}$ be accordingly equal 42,4%; 50,4 and 63,5%;

The block 5. The maximal relative deviation 1-th and n-th realizations of random variables of sample from average value under the formula pays off: $[\delta P]_{max}=\max\{\delta(P_1);\delta(P_2)\}$, where $\delta P_1=|[P_1-M^*(P)]|$; $\delta P_n=|[P_n-M^*(P)]|$;

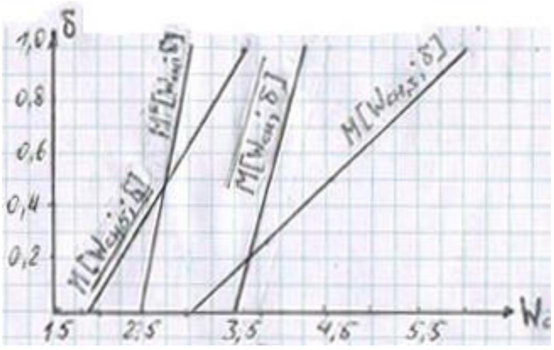
The block 6. The parity of realization P_m and M*(P) is checked Here P_m realization on which the size $[\delta P]_{max}$ defined. If $P_m>M^*(P)$ in the block 7 check of an accessory of realization P_m to set of possible values {M(P)} is spent, casually differing M*(P) and being an interval $[M^*(P)\,;\overline{M(P)}]$. If P_m belongs to this set, our assumption (H) about it abnormal (H_1) is erroneous and $H \Rightarrow H_0$ (*see the block 10*). Otherwise - P_m it is possible to consider abnormal, and assumption H_0 - erroneous (*see the block 11*) with risk of the erroneous decision α. If $P_m < M^*(P)$, management is transferred the block 8 where the size $\overline{M(P_m)}$pays off;

The block 9. Here the accessory of $\underline{M^*(P)}$ to set of possible realizations $\{M(P)_m\}$, casually differing from P_m and being in an interval $[P_m; \overline{M(P_m)}]$ is checked. Therefore, if $M^*(P)$ does not enter into this interval $H \Rightarrow H_1$. Otherwise, $H \Rightarrow H_0$, i.e. it is possible to consider the assumption of abnormal character P_m erroneous. In conformity with the block diagram of algorithm, the criterion of check of assumption $H \Rightarrow H_1$ looks like:

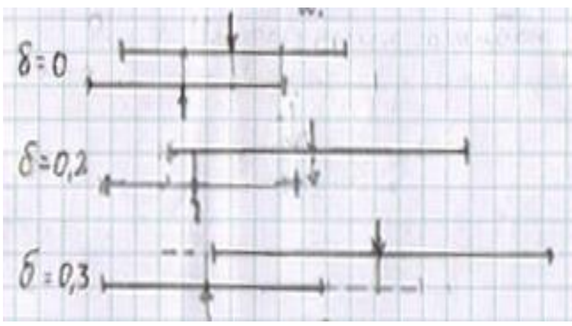
$$\left.\begin{array}{l} \text{if } P_m>M^*(P), \text{ and } \overline{M(P)} < P_m, \text{ then } H \Rightarrow H_1 \\ \quad\quad \text{and if } P_m<M^*(P) \text{ и } \overline{M(P_m)} < M^*(P), \text{ then } H \Rightarrow H_1 \\ \text{otherwise } H \Rightarrow H_0 \end{array}\right\} \quad (1)$$

But, naturally, the expert has a question: why conditions of the control differ at $P_m>M^*(P)$ and $П_m<M^*(P)$? The answer to this question is given in the graphic form on fig. 2 and 3. As an example is used the same sample of realizations of the charge of the electric power for own needs of boiler installation W_{in} in volume n=5 a kind {2,14; 2,38; 2,44; 2,48; 3,04}.

On fig. 2 the illustration of the decision of a question on a faultlessness of value W_{on} of the power unit with W_{on}=3,04% and with artificial increase W_{on} at value δ, which changes in an interval [0,1] is resulted.

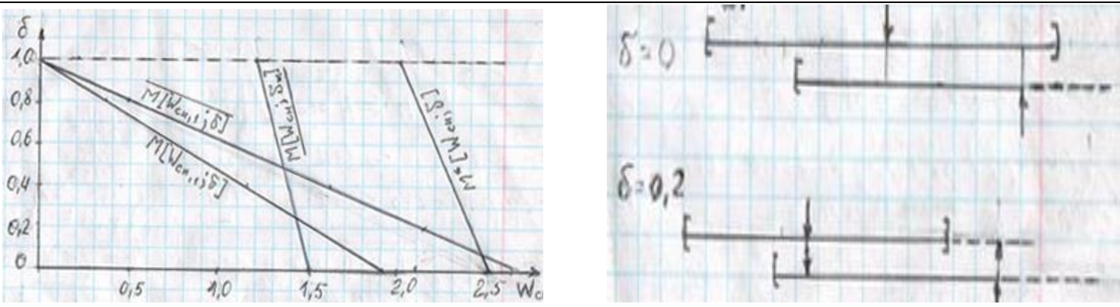


a)



b)

Fig. 2. An illustration of parities fiducially intervals at $M^*(W_{on},\delta)<W_{on,5}(\delta)$.
a) - dependence of boundary values fiducially intervals from δ;
b) - parities of boundary values fiducially intervals



a) b)

Fig. 3. An illustration of parities fiducially intervals at $M^*(W_{on};\delta)>W_{on,1}(\delta)$;
a) - dependence of boundary values fiducially intervals from δ;
b) - parities of boundary values fiducially intervals

First of all, these figures evidently testify what even at α=0,1 W_{on}=3,04% does not concern to gross blunders. Erroneous decisions at use of criteria Shovenes and Dickson are natural, since we "have closed eyes" and have broken conditions of their application. Therefore important not only to apply in calculations not less than two criteria but also in any a measure not break a condition of application of the chosen criteria.

In fig. 2 and 3 average value of realizations for set δ was calculated under the formula

$$M^*[W_{on},\delta] = M^*(W_{on})[1+\frac{\delta}{n}) \tag{2}$$

at δ=0, $M^*(W_{on})$=2,5%

As the greatest deviation W_{on} with i=1,5 from $M^*(W_{on})$ took place for W_{on}=3,04%, i.e. $M^*(W_{on})<W_{on,5}$ (as it was required to establish), dependence of this size from δ analogy to $M^*(W_{on};)$ it is linear and pays off under the formula:

$$W_{on,5}(\delta)= W_{on,5}(1+\delta) \tag{3}$$

The top boundary values fiducially interval for $M^*(W_{on})$ at δ=0 and α=0,1 equaled

$$\overline{M^*(W_{on})} = (1+\delta_c)\cdot M^*(W_{on}).$$

And at δ>0 - under the formula:

$$\overline{M^*(W_{on},\delta)_c} = (1+\delta_c)\cdot M^*(W_{on},\delta) \tag{4}$$

The bottom boundary value фидуциального an interval for $M^*(W_{on};\delta)=W_{on,5}(\delta)$ it is calculated under the formula:

$$\underline{M^*(W_{on,5};\delta)} = (1-\delta_c)\cdot M^*(W_{on,5},\delta) \tag{5}$$

The analysis of a parity fiducially intervals of possible realizations of $M^*(W_{on};)$ and $M^*(W_{on,5};\delta)$ shows, that at performance of conditions $W_{on,5}>M^*(W_{on})$ (see the block 6) and $\overline{M^*(W_{on},\delta)} > W_{on,5}(\delta)$ (see the block 7) is necessarily carried out also a condition $\underline{M^*(W_{on,5};\delta)} < M^*(W_{on},\delta)$, but not on the contrary. It is distinctly presented on fig. 2b. Here top fiducially the interval is an interval $\left\{\underline{M[W_{on,5};\delta]};\overline{M[W_{on,5};\delta]}\right\}$, and bottom is an interval $\left\{\underline{M[W_{on};\delta]};\overline{M[W_{on};\delta]}\right\}$. For δ=0,25 condition $\overline{M(W_{on},\delta)} > W_{on,5}(\delta)$ it is not carried out, and the condition $\underline{M^*(W_{on,5};\delta)} < M^*(W_{on},\delta)$ is carried out.

Therefore, to check this condition there is no necessity. We shall pass now to a case, when $P_m<M^*(P)$. Calculations we shall lead for sample {3,04; 2,42; 1,90; 2,64; 2,47} Here too the $M^*(W_{on})$=2,5, and abnormal is supposed size $W_{on,1}$=1,9%. Formulas for calculation of boundary values fiducially interval in functions from it will a little transform and looks like:

$$M^*[W_{on,\delta}]= M^*(W_{on})\cdot(1 - \delta/n)$$

$$W_{on,1}(\delta)= W_{on,1}\cdot(1 - \delta)$$

$$\underline{M[W_{on},\delta]} = M^*[W_{on},\delta]\cdot(1-\delta_c)$$

$$\overline{M[W_{on,1};\delta]} = M^*[W_{on,1};\delta]\cdot(1+\delta_c)$$

According to data fig. 3a and 3b, if a condition of $M^*[W_{on};\delta]$ it is less, than the top boundary value fiducially interval $\overline{M[W_{on,1};\delta]}$ that corresponds to assumption H⇒H_0 conformity to this assumption it is observed and for a parity $\underline{M^*[W_{on};\delta]}$ and $W_{on,1}(\delta)$. Therefore, to check it there is no necessity. On fig. 3b δ=0 both conditions are satisfied, and already δ=0,2 condition of $M^*[W_{on};\delta]<\overline{M[W_{on,1};\delta]}$ it is not carried out, and condition $W_{on}(\delta)>\underline{M^*[W_{on};\delta]}$ is carried out, that confirms sufficiency of the control of the first condition.

The algorithm of transition to correct sample is simple:

- if presence of abnormal supervision is established, it is replaced in sample by an average arithmetic estimation of M*(P);
- considering probability of presence more than one abnormal realization, in sample control check (n-1) realizations spent also. Calculation comes to the end at performance of a condition of criterion 1, at which H⇒H_0

The automated system so simplifies the decision of a question on presence of abnormal supervision in small sample of multivariate data that allows to hope for development of one more step of a problem of a faultlessness of methodical recommendations.

Conclusion

The urgency of a problem of maintenance of a faultlessness of a database of intellectual systems and their methodical recommendations in due course increases:
1. The quality monitoring of presence in small samples of the multivariate given abnormal realizations is developed. The method based on fiducially approach and the theory of check of statistical hypotheses;
2. Basis of a method there is a recommended criterion of check of uniformity of sample;
3. Application the express train-methods of calculation of critical values fiducially interval allows to translate the decision of a problem on absence of abnormal supervision group of problems successfully solved on calculators.

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Study of Stochastic Model of a Two Unit System with Inspection and Replacement Under Multi Failure

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Abstract

The present paper studies a two non-identical units system model arranged in parallel with inspection and preparation time for replacement under multi failures. Initially, first unit (A) is in operative mode and other unit (B) is kept as warm standby. The first unit is subjected to two types of failures, i.e. minor failure and major failure. On failure of the first unit, it will be sent for inspection to check the type of failure i.e. whether minor or major failure. If some minor failure is found, it will be repaired and on major failure, the unit will be replaced by the new unit. However, the system will take some preparation time for replacement. Further, the standby unit may also fail during the standby mode. There is a single repairman which is always available with the system. Different measures of reliability have been obtained to study the effectiveness of the system such as transition probabilities, mean time to system failure, availability, busy period of repairman and net profit incurred and various system parameters are analysed graphically.

Keywords: inspection, preparation time, replacement, minor failure, major failure.

1. Introduction

Reliability is considered as important characteristic for the system design and plays a vital role in the planning of system expansion, operation and maintenance. Quality of supply can be improved by reliability. To obtain useful results from system reliability assessments, reasonable values of component reliability parameters need to be used. However, the required accuracy of the reliability depends on the system design, its performance and the failure phenomenon of the system components. However, the components failure rates may vary with component, time and the environmental conditions. Therefore, it is sometimes not accurate to assign identical failure rate to all components of a particular type. Each component is treated as an individual with a unique failure rate. Many authors had worked in the reliability modelling field with different failure rates and disciplines. Rander, Kumar and Tuteja [8] have discussed a two unit cold standby system with major and minor failure and preparation time in case of major failure. El-Damcese and Temraz[7] carried out the analysis for a parallel repairable system with different failure modes and Chander, Chand and Singh[2] has studies stochastic analysis of an operating system with two types of inspection subject to degradation. Further Bhatti, Chitkara and Bhardwaj [1] studied the profit analysis of two unit cold standby system with two types of failure under inspection policy and discrete distribution and Dhankhar and Malik[5] analyse the cost-benefit analysis of a system reliability models with server failure during inspection and repair while Chib, Joorel and Sharma [3,4] have worked on MTSF and profit analysis of a two unit warm standby system with inspection

and they also worked on the analysis of a two non-identical unit cold standby system with partial and total failure and priority and El-Damcese and Sharma [6] investigated reliability and availability analysis of a repairable system with two type of failure.
The present paper studies a two non-identical units system model arranged in parallel with inspection and preparation time for replacement under multi failures. Initially, first unit (A) is in operative mode and other unit (B) is kept as warm standby. The first unit is subjected to two types of failures, i.e. minor failure and major failure. On failure of the first unit, it will be sent for inspection to check the type of failure i.e. whether minor or major failure. If some minor failure is found, it will be repaired and on major failure, the unit will be replaced by the new unit. However, the system will take some preparation time for replacement. Further, the standby unit may also fail during the standby mode. There is a single repairman which is always available with the system. Different measures of reliability have been obtained to study the effectiveness of the system such as transition probabilities, mean time to system failure, availability, busy period of repairman and net profit incurred and various system parameters are analysed graphically.

2. Assumptions

1. All the times associated with different events are random variables and independent.
2. Failure time distribution of both the units is exponential but with different parameters.
3. Inspection time distribution is also exponential.
4. Repair time distribution of both the units is taken as general but with different cdfs and replacement time distribution of first unit is also general.
5. On failure of both the units, the system will break down.
6. Switch over time is negligible.

3. Notations

α	inspection rate for unit A
β	failure rate of unit B
α_1	rate of minor / major failure in unit A with probability p and q
α_2	rate of completion of preparation for replacement
$h_1(t)/H_1(t)$	p.d.f and c.d.f of repair time of unit A
$h_2(t)/H_2(t)$	p.d.f and c.d.f of replacement time of unit A
$g(t)/G(t)$	p.d.f and c.d.f of repair time of unit B
$\pi_I(\cdot)$	c.d.f of time to system failure when $S_i \epsilon E$.
$A_i(t)$	Pr [starting from $S_i \epsilon E$, the system is up at time t].
$B_i(t)$	Pr [Repairman is busy at time $t \setminus E_0 = S_i \epsilon E$].
$V_i(t)$	Expected number of visits by repairman in (0,t].
μ_i	Mean sojourn time in state $S_i \epsilon E$.

Following Symbols are used to study the proposed model:

A_o/B_o	unit A/B is operative
A_r/B_r	unit A/B is under repair
A_I	unit A is under inspection and
A_{PR}/A_R	unit A is under replacement or preparation for replacement
B_{ws}	unit B is in warm standby mode
B_{wr}	unit B is waiting for repair

Different possible states of the system are described and shown by Fig.: 1

Up states

$S_0: (A_0, B_{ws})$ $S_2: (A_r, B_o)$ $S_4: (A_R, B_o)$

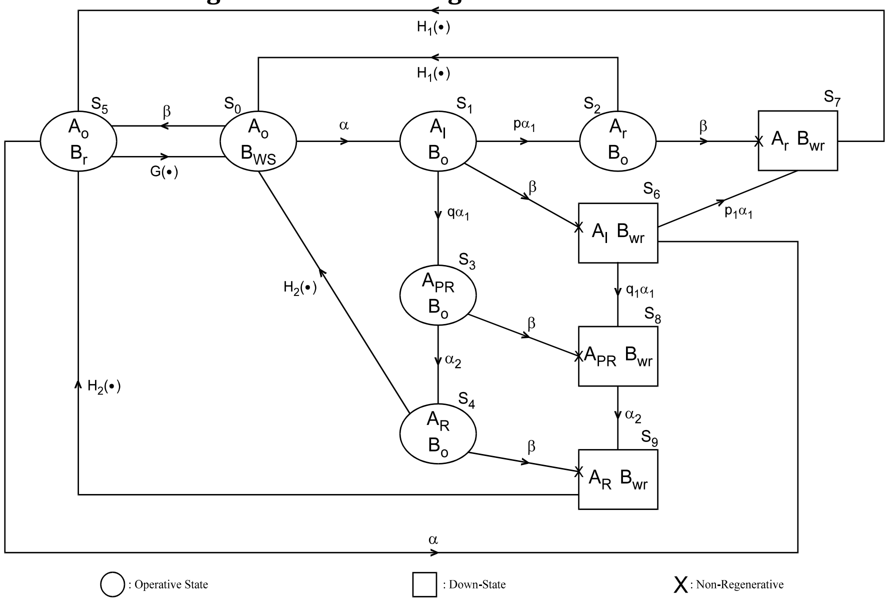
$S_1: (A_I, B_o)$ $S_3: (A_{PR}, B_o)$ $S_5: (A_o, B_r)$

Down states

$S_6 = (A_I, B_{wr})$ $S_8: (A_{PR}, B_{wr})$

$S_7: (A_r, B_{wr})$ $S_9: (A_R, B_{wr})$

Fig.: 1 Transition Diagram



4. Transition Probabilities and Mean Sojourn Time

If T_1, T_2, T_3 … denote the epochs at which the system enter any state and X_n denotes the state visited at point T_{n+}, i.e. just after the transition at T_n then the transient and steady state transition probabilities are defined as $Q_{ij}(t) = P[X_{n+1} = j, T_{n+1} - T_n \leq t | X_n = i]$ and $p_{ij} = \lim_{t\to\infty} Q_{ij}(t)$ respectively. The following steady state transition probabilities of the system are obtained:

$p_{01} = \frac{\alpha}{\alpha+\beta}$ $p_{17}^{6} = \frac{\beta p_1}{\alpha_1+\beta}$ $p_{34} = \frac{\alpha_2}{\alpha_2+\beta}$ $p_{45}^{9} = 1 - h_2^*(\beta)$

$p_{05} = \frac{\beta}{\alpha+\beta}$ $p_{18}^{6} = \frac{\beta q_1}{\alpha_1+\beta}$ $p_{38} = \frac{\beta}{\alpha_2+\beta}$ $p_{50} = g^*(\alpha)$

$p_{12} = \frac{p\alpha_1}{\alpha_1+\beta}$ $p_{20} = h_1^*(\beta)$ $p_{39}^{8} = \frac{\beta}{\alpha_2+\beta}$ $p_{56} = 1 - g^*(\alpha)$

$p_{13} = \frac{q\alpha_1}{\alpha_1+\beta}$ $p_{27} = 1 - h_1^*(\beta)$ $p_{40} = h_2^*(\beta)$ $p_{67} = p_1$

$p_{16} = \frac{\beta}{\alpha_1+\beta}$ $p_{25}^{7} = 1 - h_1^*(\beta)$ $p_{49} = 1 - h_2^*(\beta)$ $p_{68} = q_1$

$$p_{75} = p_{89} = p_{95} = 1 \tag{1}$$

It may be noted that $\sum_j p_{ij} = 1, \quad for\ all\ possible\ values\ of\ i$ (2)

Further, if T_i denotes the sojourn time in state S_i then mean sojourn time is defined as the time of stay in state S_i before transiting to any other state and is denoted by μ_i. Following are the expressions for mean sojourn time:

$\mu_0 = \frac{1}{\alpha+\beta}$ $\mu_3 = \frac{1}{\alpha_2+\beta}$ $\mu_6 = \frac{1}{\alpha_1}$ $\mu_9 = \int_0^\infty \bar{H}_2(u)\, du$

$\mu_1 = \frac{1}{\alpha_1+\beta}$ $\mu_4 = \frac{1}{\beta}[1 - h_2^*(\beta)]$ $\mu_7 = \int_0^\infty \bar{H}_1(u)\, du$

$$\mu_2 = \frac{1}{\beta}[1 - h_1^*(\beta)] \qquad \mu_5 = \frac{1}{\alpha}[1 - g^*(\alpha)] \qquad \mu_8 = \frac{1}{\alpha_2} \tag{3}$$

5. Mean Time To System Failure

The distribution function of time to system failure is obtained by considering the failed states as absorbing state and the time taken by the system to reach in the failed state for the first time is known as time to system failure and is denoted by T_i and $\pi_i(t)$ denotes its expected value which is known as the mean time to system failure.
The following result for mean time to system failure is obtained by using Laplace transformation.

$$\text{MTSF}=\frac{\alpha[(\alpha_2+\beta)\beta+p\alpha_1\beta(\theta+\alpha)\{1-h_1^*(\beta)\}(\alpha_2+\beta)+q\alpha_1\{\beta+\alpha_2\{1-h_1^*(\beta)\}\}(\theta+\alpha)]+\beta(\alpha_2+\beta)\{1-g^*(\beta)\}(\alpha_1+\beta)}{(\alpha_1+\beta)(\alpha_2+\beta)(\beta+\alpha)-\alpha\{p\alpha_1h_1^*(\alpha)(\alpha_2+\beta)+q\alpha_1\alpha_2h_2^*(\beta)\}-\beta(\alpha_1+\beta)g^*(\alpha)(\alpha_2+\beta)} \tag{4}$$

6. Availability Analysis

$A_i\,(t)$ is defined as the probability that a system will be in operational service during a scheduled operating period i.e., probability that the system is up at epoch 't' given that initially it starts from state S_i without transiting to any non-regenerative state. By using simple probabilistic concepts, recurrence relations among $A_i\,(t)'s$ are obtained and on solving those equations by using the Laplace- transformation following results hold:

$$A_0^*(s)=\frac{N_2(s)}{D_2(s)}$$

where,

$$N_2(s)=[M_0^*+q_{01}^*(M_1^*+q_{12}^*M_2^*+q_{13}^*M_3^*+q_{13}^*q_{34}^*M_4^*)](1-q_{56}^*q_{67}^*q_{75}-q_{56}^*q_{68}^*q_{89}^*q_{95}^*)+ \\ M_5^*[q_{05}^*+q_{01}^*(q_{12}^*q_{25}^{(7)*}+q_{13}^*q_{34}^*q_{45}^{(9)*}+q_{13}^*q_{39}^{(8)*}q_{95}^*+q_{18}^{(6)*}q_{89}^*q_{95}^*+q_{17}^{(6)*}q_{75}^* \tag{5}$$

$$D_2(s)=(1-q_{56}^*q_{67}^*q_{75}-q_{56}^*q_{68}^*q_{89}^*q_{95}^*)[1-q_{01}^*(q_{12}^*q_{20}^*+q_{13}^*q_{34}^*q_{40}^*)]-q_{50}^*[q_{05}^*+q_{01}^* \\ (q_{12}^*q_{25}^{*(7)}+q_{13}^*q_{39}^{*(8)}q_{95}^*+q_{18}^{*(6)}q_{89}^*q_{95}^*+q_{17}^{*(6)}q_{75}^*)] \tag{6}$$

Steady state availability of the system starting from state s_0 is obtained as follows:

$$A_0=\lim_{t\to\infty}A_0\,(\infty)=\lim_{s\to 0}sA_0^*\,(s)=\frac{N_2(0)}{D_2'(0)} \tag{7}$$

$$A_0=\frac{\begin{array}{c}[(\alpha_1+\beta)(\alpha_2+\beta)\beta+\alpha(\alpha_1+\beta)(\alpha_2+\beta)\beta+p\alpha_1(\alpha_1+\beta)\alpha(\alpha_2+\beta)\{1-h_2^*(\beta)\}+q\alpha_1\beta(\alpha+\beta)+q\alpha_1\alpha_2(\alpha+\beta)\{1-h_2^*(\beta)\} \\ g^*(\alpha)\alpha+\{1-g^*(\alpha)\}\beta[(\alpha_1+\beta)(\alpha_2+\beta)+\alpha\{p\alpha_1\beta(\alpha_2+\beta)\{1-h_1^*(\beta)\}(\alpha+\beta)+q\alpha_1\alpha_2\beta\{1-h_2^*(\beta)\}(\alpha+\beta)+q\beta\alpha_1(\alpha+\beta) \\ +(\alpha+\beta)(\alpha_2+\beta)\beta\}]\theta_2\,\alpha_2\end{array}}{\begin{array}{c}g^*(\alpha)[(\alpha_1+\beta)(\alpha_2+\beta)\beta\alpha\alpha_2\theta_2+\alpha(\alpha_2+\beta)\alpha\beta\alpha_2\theta_2+p\alpha_1(\alpha_2+\beta)\alpha\{1-h^*(\beta)\}\alpha_2\theta_2\alpha+q\alpha\alpha_1\theta_2\alpha\beta+q\alpha\alpha_1\theta_2\alpha_2^2\theta_2 \\ \alpha\{1-h_1^*(\beta)\}]+[\{1-g^*(\alpha)\}\alpha+\beta][(\alpha+\beta)\beta(\alpha_1+\beta)(\alpha_2+\beta)\alpha-\alpha\{p(\alpha_2+\beta)h_1^*(\alpha)\,\alpha_1+q\beta\alpha_1\alpha_2h_2^*(\alpha)\{1-g^*(\alpha)\}]\theta_2\alpha_2 \\ +\int\bar{H}_1(u)[\alpha p_1\{1-g^*(\alpha)\}(\alpha_1+\beta)(\alpha_2+\beta)(\alpha+\beta)-\alpha\{p\alpha_1h_2^*(\beta)(\alpha_2+\beta)+q\alpha_1\alpha_2\beta h_2^*(\beta)\}+\alpha g^*(\alpha)\beta p_1\alpha_1\alpha(\alpha_2+\beta) \\ +\alpha q_1\{1-g^*(\alpha)\}[(\alpha_1+\beta)(\alpha_2+\beta)(\alpha+\beta)-p\alpha\alpha_1g^*(\alpha)(\alpha_2+\beta)+q\beta\alpha_1\alpha_2\{1-h_2^*(\beta)\}+\alpha^2q_1\beta(\alpha_2+\beta)]+\alpha^2\beta\int\bar{H}_1(u) \\ [q_1(\alpha_1+\beta)\{1-g_2^*(\alpha)\}(\alpha_2+\beta)(\alpha+\beta)-\alpha\{p\alpha_1(\alpha_2+\beta)g_2^*(\alpha)h_2^*(\beta)+q\alpha_1\alpha_2\beta\}+\alpha\{1-g^*(\alpha)\}\{q\beta\alpha_1+\beta p_1(\alpha_2+\beta)\}]\end{array}} \tag{8}$$

7. Busy Period Analysis

$B_i\,(t)$ is the probability that the repairman is busy due to repair of the unit at an instant 't' given that system entered state S_i at t=0. Now we will determine these probabilities. To illustrate the calculations we consider $B_0(t)$ and similar arguments may be employed for other probabilities.

$$B_0^*(s)=\frac{N_3(s)}{D_2(s)}$$

$$N_3(s)=q_{01}^*\big(M_1^*+q_{12}^*M_2^*+q_{13}^*q_{34}^*M_4^*+q_{13}^8q_{39}^{(8)*}M_9^*+q_{16}^{(8)*}M_8^*+q_{16}^{(8)*}q_{89}^*M_9^*\big)(1-q_{56}^*q_{67}^* \\ q_{75}^*-q_{56}^*q_{68}^*q_{89}^*q_{95}^*)+(M_5^*+q_{56}^*M_6^*+q_{56}^*q_{68}^*M_8^*+q_{56}^*q_{68}^*q_{89}^*M_9^*)[q_{01}^*(q_{12}^*q_{25}^{*(7)}+ \\ q_{13}^*q_{34}^*q_{45}^{*(9)}+q_{13}^*q_{39}^{*(8)}q_{95}^*+q_{18}^{*(6)}q_{89}^*q_{95}^*+q_{17}^{*(6)}q_{75}^*)+q_{05}^*] \tag{9}$$

$$D_2(s)=(1-q_{56}^*q_{67}^*q_{75}^*-q_{56}^*q_{68}^*q_{89}^*q_{95}^*)[1-q_{01}^*(q_{12}^*q_{20}^*+q_{13}^*q_{34}^*q_{40}^*)]-q_{50}^*[q_{05}^*+q_{01}^* \\ (q_{12}^*q_{25}^{*(7)}+q_{13}^*q_{39}^{*(8)}q_{95}^*+q_{18}^{*(6)}q_{89}^*q_{95}^*+q_{17}^{*(6)}q_{75}^*)] \tag{10}$$

Therefore, busy period analysis for repairman is given by:

$$\mathrm{B}_0(0) = \frac{N_3(0)}{D_2'(0)} \tag{11}$$

$$\mathrm{B}_0 = \frac{\begin{array}{c}\alpha[\{(\alpha_1+\beta)\beta(\alpha_2+\beta)\alpha_2+(\alpha+\beta)(\alpha_2+\beta)p\alpha_2\alpha_1\{1-h_1^*(\beta)\}+(\alpha+\beta)\{1-h_2^*(\beta)\}q\alpha_1\alpha_2^2\}+\alpha_1\beta\alpha_2(\alpha+\beta)\int\bar{H}_2(u)q+q_1\beta \\ (\alpha+\beta)(\alpha_2+\beta)+q_1\beta(\alpha_1+\beta)(\alpha_2+\beta)\int\bar{H}_2(u)(\alpha+\beta)]g^*(\alpha)\alpha+[\alpha[\{\alpha_1\alpha_2\theta_2+\alpha_2\alpha\theta_2+\alpha\alpha_1q_1\theta_2+\alpha q_1\}+(\alpha_1+\beta) \\ (\alpha_2+\beta)(\alpha+\beta)][\alpha\{p\alpha_1\beta\theta_2(\alpha_2+\beta)\{1-h_1^*(\beta)\}+q\beta\alpha_1\alpha_2\alpha_2\{1-h_2^*(\beta)\}+q\beta\alpha_1+q_1\beta(\alpha_2+\beta)\alpha_2+p_1\beta(\alpha_2+\beta)\alpha_2] \\ +(\alpha_1+\beta)\beta(\alpha_2+\beta)]\end{array}}{\begin{array}{c}g^*(\alpha)[(\alpha_1+\beta)(\alpha_2+\beta)\beta\alpha\alpha_2\theta_2+\alpha(\alpha_2+\beta)\alpha\beta\alpha_2\theta_2+p\alpha_1(\alpha_2+\beta)\alpha\{1-h^*(\beta)\}\alpha_2\theta_2\alpha+q\alpha\alpha_1\theta_2\alpha\beta+q\alpha\alpha_1\theta_2\alpha_2^2\theta_2 \\ \alpha\{1-h_1^*(\beta)\}]+[\{1-g^*(\alpha)\}\alpha+\beta][(\alpha+\beta)\beta(\alpha_1+\beta)(\alpha_2+\beta)\alpha-\alpha\{p(\alpha_2+\beta)h_1^*(\alpha)\ \alpha_1+q\beta\alpha_1\alpha_2h_2^*(\alpha)\{1-g^*(\alpha)\}]\theta_2\alpha_2 \\ +\int\bar{H}_1(u)[\alpha p_1\{1-g^*(\alpha)\}(\alpha_1+\beta)(\alpha_2+\beta)(\alpha+\beta)-\alpha\{p\alpha_1h_2^*(\beta)(\alpha_2+\beta)+q\alpha_1\alpha_2\beta h_2^*(\beta)\}+\alpha g^*(\alpha)\beta p_1\alpha_1\alpha(\alpha_2+\beta) \\ +\alpha q_1\{1-g^*(\alpha)\}[(\alpha_1+\beta)(\alpha_2+\beta)(\alpha+\beta)-p\alpha\alpha_1g^*(\alpha)(\alpha_2+\beta)+q\beta\alpha_1\alpha_2\{1-h_2^*(\beta)\}+\alpha^2q_1\beta(\alpha_2+\beta)]+\alpha^2\beta\int\bar{H}_1(u) \\ [q_1(\alpha_1+\beta)\{1-g_2^*(\alpha)\}(\alpha_2+\beta)(\alpha+\beta)-\alpha\{p\alpha_1(\alpha_2+\beta)g_2^*(\alpha)h_2^*(\beta)+q\alpha_1\alpha_2\beta\}+\alpha\{1-g^*(\alpha)\}\{q\beta\alpha_1+\beta p_1(\alpha_2+\beta)\}]\end{array}} \tag{12}$$

8. Expected Number of Visits by Repairman

$V_i(t)$ is the expected number of visits by the repairman to the system to repair the failed unit, when the system initially starts from regenerative state S_i. By probabilistic reasoning the recurrence relations for $V_i(t)$ are obtained and solving those relations by using Laplace transformation, we have,

$$V_0^*(s) = \frac{N_4(s)}{D_2(s)}$$

$$\mathrm{N}_4(\mathrm{s}) = \left[(q_{01}^* + q_{05}^*) + q_{01}^*\left(q_{13}^*q_{34}^* + q_{18}^{*(6)}q_{89}^*\right)\right][1 - q_{56}^*q_{67}^*q_{75}^* - q_{56}^*q_{68}^*q_{89}^*q_{95}^*] + q_{56}^* \\ q_{67}^*q_{75}^*[q_{05}^* + q_{01}^*\left(q_{12}^*q_{25}^{*(7)} + q_{13}^*q_{34}^*q_{45}^{*(9)} + q_{13}^*q_{39}^{*(8)}q_{95}^* + q_{16}^{*(8)}q_{89}^*q_{95}^* + q_{17}^{*(6)}q_{75}^*\right)] \tag{13}$$

$$D_2(s) = (1 - q_{56}^*q_{67}^*q_{75}^* - q_{56}^*q_{68}^*q_{89}^*q_{95}^*)[1 - q_{01}^*(q_{12}^*q_{20}^* + q_{13}^*q_{34}^*q_{40}^*)] - q_{50}^*[q_{05}^* + q_{01}^* \\ (q_{12}^*q_{25}^{*(7)} + q_{13}^*q_{39}^{*(8)}q_{95}^* + q_{18}^{*(6)}q_{89}^*q_{95}^* + q_{17}^{*(6)}q_{75}^*)] \tag{14}$$

In steady state, the number of times the repairman visits the system is given by:

$$V_0^* = \lim_{S\to 0} sV_0(s) = \frac{N_4(0)}{D_2'(0)} \tag{15}$$

$$V_0 = \frac{\begin{array}{c}[(\alpha_1+\beta)(\alpha+\beta)(\alpha_2+\beta)+\alpha\alpha_1\alpha_2q+q_1\beta(\alpha+\beta)(\alpha_2+\beta)]+q_1\{1-g^*(\alpha)\}[\beta(\alpha_1+\beta)(\alpha_2+\beta)+\alpha\{(\alpha_1+\beta)(\theta_1+\beta)-p\alpha_1 \\ g_2^*(\alpha)\}-q\alpha_1\alpha_2h_2^*(\beta)]g^*(\alpha)\end{array}}{\begin{array}{c}g^*(\alpha)[(\alpha_1+\beta)(\alpha_2+\beta)\beta\alpha\alpha_2\theta_2+\alpha(\alpha_2+\beta)\alpha\beta\alpha_2\theta_2+p\alpha_1(\alpha_2+\beta)\alpha\{1-h^*(\beta)\}\alpha_2\theta_2\alpha+q\alpha\alpha_1\theta_2\alpha\beta+q\alpha\alpha_1\theta_2\alpha_2^2\theta_2 \\ \alpha\{1-h_1^*(\beta)\}]+[\{1-g^*(\alpha)\}\alpha+\beta][(\alpha+\beta)\beta(\alpha_1+\beta)(\alpha_2+\beta)\alpha-\alpha\{p(\alpha_2+\beta)h_1^*(\alpha)\ \alpha_1+q\beta\alpha_1\alpha_2h_2^*(\alpha)\{1-g^*(\alpha)\}]\theta_2\alpha_2 \\ +\int\bar{H}_1(u)[\alpha p_1\{1-g^*(\alpha)\}(\alpha_1+\beta)(\alpha_2+\beta)(\alpha+\beta)-\alpha\{p\alpha_1h_2^*(\beta)(\alpha_2+\beta)+q\alpha_1\alpha_2\beta h_2^*(\beta)\}+\alpha g^*(\alpha)\beta p_1\alpha_1\alpha(\alpha_2+\beta) \\ +\alpha q_1\{1-g^*(\alpha)\}[(\alpha_1+\beta)(\alpha_2+\beta)(\alpha+\beta)-p\alpha\alpha_1g^*(\alpha)(\alpha_2+\beta)+q\beta\alpha_1\alpha_2\{1-h_2^*(\beta)\}+\alpha^2q_1\beta(\alpha_2+\beta)]+\alpha^2\beta\int\bar{H}_1(u) \\ [q_1(\alpha_1+\beta)\{1-g_2^*(\alpha)\}(\alpha_2+\beta)(\alpha+\beta)-\alpha\{p\alpha_1(\alpha_2+\beta)g_2^*(\alpha)h_2^*(\beta)+q\alpha_1\alpha_2\beta\}+\alpha\{1-g^*(\alpha)\}\{q\beta\alpha_1+\beta p_1(\alpha_2+\beta)\}]\end{array}} \tag{16}$$

9. Profit Analysis

The profit in steady state generated by proposed model may be obtained as follows:
The expected profits incurred in (0,t] = expected total revenue in (0,t] – expected total repair in (0,t] –expected cost of visit by repairman in (0,t]
Therefore, profit analysis of the system can be written as:

$$P_1 = K_0A_0 - K_1B_0 - K_2V_0$$

where,
$\mathrm{K}_0 =$ revenue per unit up time of the system,
$\mathrm{K}_1 =$ Cost per unit time for which the repair is busy
$\mathrm{K}_2 =$ Cost per unit visits by the repairman
The expressions for $\mathrm{A}_0, \mathrm{B}_0$ and V_0 are given by equations (8), (12)and (16) respectively.

10. Particular cases

As we have assumed that the repair time and replacement time distribution is general, so firstly we

convert it into exponential with parameters θ, θ_1 and θ_2. We have assumed
$G(t) = \theta e^{-\theta t}, H_1(t) = \theta_1 e^{-\theta_1 t}, H_2(t) = \theta_2 e^{-\theta_2 t}$
So under these assumptions the expressions for different transitions with their mean sojourn time, MTSF, availability and profit function are obtained as:

Transition probabilities

$p_{20} = \frac{\theta_1}{\theta_1+\beta}$ $\quad p_{27} = \frac{\beta}{\theta_1+\beta}$ $\quad p_{25}^{7} = \frac{\beta}{\theta_1+\beta}$ $\quad p_{40} = \frac{\theta_2}{\theta_2+\beta}$ $\quad p_{49} = \frac{\beta}{\theta_2+\beta}$

$p_{45}^{9} = \frac{\beta}{\theta_2+\beta}$ $\quad p_{50} = \frac{\theta}{\theta+\alpha}$ $\quad p_{56} = \frac{\alpha}{\theta+\alpha}$ $\quad p_{95} = 1$

Mean Sojourn Time

$\mu_2 = \frac{1}{\theta_1+\beta}$ $\quad \mu_4 = \frac{1}{\theta_2+\beta}$ $\quad \mu_5 = \frac{1}{\theta+\alpha}$ $\quad \mu_7 = \frac{1}{\theta_1}$ $\quad \mu_9 = \frac{1}{\theta_2}$

Mean time to system failure

$$\text{MTSF}=\frac{\alpha[(\theta_1+\beta)(\alpha_2+\beta)(\theta_2+\beta)+p\alpha_1\beta(\theta+\alpha)(\alpha_2+\beta)(\theta_2+\beta)+q\alpha_1\{(\theta_2+\beta)+\alpha_2\beta\}(\theta+\alpha)(\theta_2+\beta)]+\alpha\beta(\theta_1+\beta)(\alpha_2+\beta)(\alpha_1+\beta)(\theta_2+\beta)}{(\alpha_1+\beta)(\theta_1+\beta)(\alpha_2+\beta)(\theta_2+\beta)(\theta+\alpha)(\beta+\alpha)-\alpha\{p\alpha_1\theta_1(\alpha_2+\beta)(\theta_2+\beta)+q\alpha_1\alpha_2\theta_2(\theta_1+\beta)(\theta+\alpha)\}-\beta\theta(\alpha_1+\beta)(\theta_1+\beta)(\alpha_2+\beta)(\theta_2+\beta)}$$

Availability

$$A_0 = \frac{[(\alpha_1+\beta)(\alpha_2+\beta)(\theta_1+\beta)(\theta_2+\beta)+\alpha(\alpha_1+\beta)(\theta_1+\beta)(\theta_2+\beta)(\alpha_2+\beta)\beta+p\alpha_1(\alpha_1+\beta)\alpha(\theta_2+\beta)(\alpha+\beta)+q\alpha_1(\alpha+\beta)(\theta_1+\beta)(\theta_2+\beta)\beta+q\alpha_1\alpha_2(\alpha+\beta)\alpha]\theta+[(\alpha_1+\beta)(\alpha_2+\beta)(\theta_1+\beta)(\theta_2+\beta)+\alpha\{p\alpha_1\beta(\alpha_2+\beta)(\theta_2+\beta)(\alpha+\beta)+q\alpha_1\alpha_2\beta(\alpha+\beta)(\theta_1+\beta)+q\beta\alpha_1(\alpha+\beta)(\theta_1+\beta)(\theta_2+\beta)+(\alpha+\beta)(\theta_1+\beta)(\theta_2+\beta)(\alpha_2+\beta)\beta\}]}{\alpha_2\theta\theta_2[(\alpha_1+\beta)(\theta_1+\beta)(\theta_2+\beta)(\alpha_2+\beta)+\alpha(\alpha_2+\beta)(\theta_1+\beta)(\theta_2+\beta)\theta_2\alpha+p\alpha_1(\alpha_2+\beta)(\theta_2+\beta)+(\theta_2+\beta)q(\theta_1+\beta)\alpha\alpha_2\theta_2\alpha+(\theta_1+\beta)q\alpha\alpha_1\theta_2\alpha_2^2\theta_2\alpha]+[2\alpha+\beta][(\alpha+\beta)\beta(\alpha_1+\beta)(\alpha_2+\beta)(\theta_1+\beta)(\theta_2+\beta)-\alpha\{p(\theta_2+\beta)(\alpha_2+\beta)\alpha_1+q\beta\alpha_1\alpha_2(\theta_1+\beta))]\theta_2\alpha_2+[\alpha p_1(\alpha_1+\beta)(\theta_1+\beta)(\theta_2+\beta)(\alpha_2+\beta)(\alpha+\beta)-\alpha\{p\alpha_1(\alpha_2+\beta)\theta_1(\theta_2+\beta)+(\theta_2+\beta)q\alpha_1\alpha_2\beta\}+\alpha\beta p_1\alpha_1\alpha\theta(\alpha_2+\beta)(\theta_1+\beta)(\theta_2+\beta)+\alpha q_1[(\alpha_1+\beta)(\alpha_2+\beta)(\alpha+\beta)(\theta_1+\beta)(\theta_2+\beta)(\theta+\alpha)-p\alpha\alpha_1\theta_1(\alpha_2+\beta)(\theta_2+\beta)+q\beta\alpha_1\alpha_2(\theta_1+\beta)+\alpha^2q_1\beta(\alpha_2+\beta)(\theta_1+\beta)(\theta_2+\beta)]+\alpha^2\beta[q_1(\alpha_1+\beta)(\alpha_2+\beta)(\theta_1+\beta)(\theta_2+\beta)(\alpha+\beta)-\alpha\{p\alpha_1\theta_1(\alpha_2+\beta)+q\alpha_1\alpha_2\beta(\theta_1+\beta)\}+\alpha\{q\beta\alpha_1+\beta p_1(\alpha_2+\beta)\}]}$$

Busy period

$$B_0 = \frac{\alpha[(\alpha_1+\beta)\beta(\alpha_2+\beta)(\theta_1+\beta)(\theta_2+\beta)\alpha_2+(\alpha+\beta)(\theta_2+\beta)(\alpha_2+\beta)p\alpha_2\alpha_1+(\theta_1+\beta)(\alpha+\beta)q\alpha_1\alpha_2^2+\alpha_1(\theta_1+\beta)(\theta_2+\beta)\beta\alpha_2(\alpha+\beta)q+q_1\beta(\alpha+\beta)(\alpha_2+\beta)(\theta_2+\beta)+q_1\beta(\alpha_1+\beta)(\theta_1+\beta)(\theta_2+\beta)(\alpha_2+\beta)(\alpha+\beta)]\theta+[\{\alpha_1\alpha_2\theta_2+\alpha_2\alpha\theta_2+\alpha\alpha_1q_1\theta_2+\alpha q_1\}(\theta_1+\beta)(\theta_2+\beta)(\alpha_1+\beta)(\alpha_2+\beta)(\alpha+\beta)][\alpha\alpha_2\{p\alpha_1\beta\theta_2(\alpha_2+\beta)(\theta_2+\beta)+q\beta\alpha_1\alpha_2\alpha_2(\theta_1+\beta)+q\beta\alpha_1(\theta_1+\beta)(\theta_2+\beta)+q_1\beta(\alpha_2+\beta)(\theta_1+\beta)(\theta_2+\beta)\alpha_2+p_1\beta(\theta_1+\beta)(\theta_2+\beta)(\alpha_2+\beta)\alpha_2\theta_2]}{\alpha_2\theta\theta_2[(\alpha_1+\beta)(\theta_1+\beta)(\theta_2+\beta)(\alpha_2+\beta)+\alpha(\alpha_2+\beta)(\theta_1+\beta)(\theta_2+\beta)\theta_2\alpha+p\alpha_1(\alpha_2+\beta)(\theta_2+\beta)+(\theta_2+\beta)q(\theta_1+\beta)\alpha\alpha_2\theta_2\alpha+(\theta_1+\beta)q\alpha\alpha_1\theta_2\alpha_2^2\theta_2\alpha]+[2\alpha+\beta][(\alpha+\beta)\beta(\alpha_1+\beta)(\alpha_2+\beta)(\theta_1+\beta)(\theta_2+\beta)-\alpha\{p(\theta_2+\beta)(\alpha_2+\beta)\alpha_1+q\beta\alpha_1\alpha_2(\theta_1+\beta))]\theta_2\alpha_2+[\alpha p_1(\alpha_1+\beta)(\theta_1+\beta)(\theta_2+\beta)(\alpha_2+\beta)(\alpha+\beta)-\alpha\{p\alpha_1(\alpha_2+\beta)\theta_1(\theta_2+\beta)+(\theta_2+\beta)q\alpha_1\alpha_2\beta\}+\alpha\beta p_1\alpha_1\alpha\theta(\alpha_2+\beta)(\theta_1+\beta)(\theta_2+\beta)+\alpha q_1[(\alpha_1+\beta)(\alpha_2+\beta)(\alpha+\beta)(\theta_1+\beta)(\theta_2+\beta)(\theta+\alpha)-p\alpha\alpha_1\theta_1(\alpha_2+\beta)(\theta_2+\beta)+q\beta\alpha_1\alpha_2(\theta_1+\beta)+\alpha^2q_1\beta(\alpha_2+\beta)(\theta_1+\beta)(\theta_2+\beta)]+\alpha^2\beta[q_1(\alpha_1+\beta)(\alpha_2+\beta)(\theta_1+\beta)(\theta_2+\beta)(\alpha+\beta)-\alpha\{p\alpha_1\theta_1(\alpha_2+\beta)+q\alpha_1\alpha_2\beta(\theta_1+\beta)\}+\alpha\{q\beta\alpha_1+\beta p_1(\alpha_2+\beta)\}]}$$

Expected number of repairs

$$V_0 = \frac{\theta[(\alpha_1+\beta)(\alpha+\beta)(\alpha_2+\beta)+\alpha\alpha_1\alpha_2q+q_1\beta(\alpha+\beta)(\alpha_2+\beta)](\theta_1+\beta)(\theta_2+\beta)+q_1\alpha[\beta(\alpha_1+\beta)(\theta_1+\beta)(\theta_2+\beta)(\alpha_2+\beta)+\alpha\{(\alpha_1+\beta)(\theta_1+\beta)-p\alpha_1\theta_1\}-q\alpha_1\alpha_2\theta_2]}{\alpha_2\theta\theta_2[(\alpha_1+\beta)(\theta_1+\beta)(\theta_2+\beta)(\alpha_2+\beta)+\alpha(\alpha_2+\beta)(\theta_1+\beta)(\theta_2+\beta)\theta_2\alpha+p\alpha_1(\alpha_2+\beta)(\theta_2+\beta)+(\theta_2+\beta)q(\theta_1+\beta)\alpha\alpha_2\theta_2\alpha+(\theta_1+\beta)q\alpha\alpha_1\theta_2\alpha_2^2\theta_2\alpha]+[2\alpha+\beta][(\alpha+\beta)\beta(\alpha_1+\beta)(\alpha_2+\beta)(\theta_1+\beta)(\theta_2+\beta)-\alpha\{p(\theta_2+\beta)(\alpha_2+\beta)\alpha_1+q\beta\alpha_1\alpha_2(\theta_1+\beta))]\theta_2\alpha_2+[\alpha p_1(\alpha_1+\beta)(\theta_1+\beta)(\theta_2+\beta)(\alpha_2+\beta)(\alpha+\beta)-\alpha\{p\alpha_1(\alpha_2+\beta)\theta_1(\theta_2+\beta)+(\theta_2+\beta)q\alpha_1\alpha_2\beta\}+\alpha\beta p_1\alpha_1\alpha\theta(\alpha_2+\beta)(\theta_1+\beta)(\theta_2+\beta)+\alpha q_1[(\alpha_1+\beta)(\alpha_2+\beta)(\alpha+\beta)(\theta_1+\beta)(\theta_2+\beta)(\theta+\alpha)-p\alpha\alpha_1\theta_1(\alpha_2+\beta)(\theta_2+\beta)+q\beta\alpha_1\alpha_2(\theta_1+\beta)+\alpha^2q_1\beta(\alpha_2+\beta)(\theta_1+\beta)(\theta_2+\beta)]+\alpha^2\beta[q_1(\alpha_1+\beta)(\alpha_2+\beta)(\theta_1+\beta)(\theta_2+\beta)(\alpha+\beta)-\alpha\{p\alpha_1\theta_1(\alpha_2+\beta)+q\alpha_1\alpha_2\beta(\theta_1+\beta)\}+\alpha\{q\beta\alpha_1+\beta p_1(\alpha_2+\beta)\}]}$$

11. Graphical Study of the System Model

In order to have a graphical analysis of the above discussed model, we graphed these characteristics i.e., MTSF, availability and profit function. Firstly we have obtained the values of MTSF, availability and profit function with respect to failure and repair rates using C++ language and then we have plotted those values using STATISTICA. Firstly graphs are plotted for MTSF, Availability and Profit with respect to failure rate α_1 for different values of repair rate θ_1, θ_2 & θ keeping all other parameters constant as $\alpha = 0.5$, $\alpha_2 = 0.25, \beta = 0.35$, $k_0 = 1000, k_1 = 300, k_2 = 200, p = 0.5, q = 0.5, p_1 = 0.5, q_1 = 0.5$

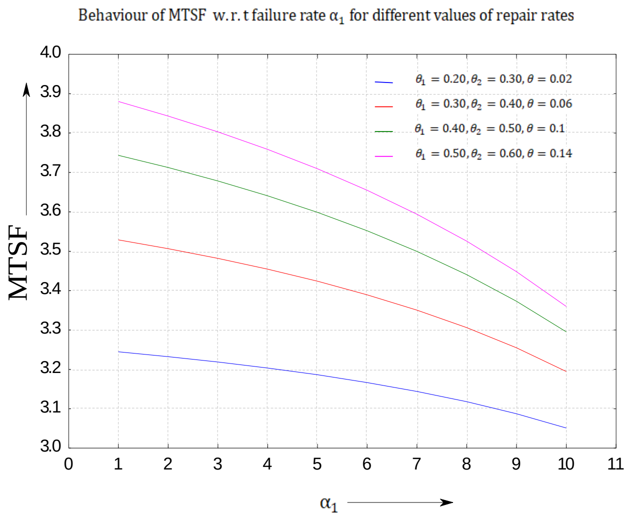


Fig.: 2

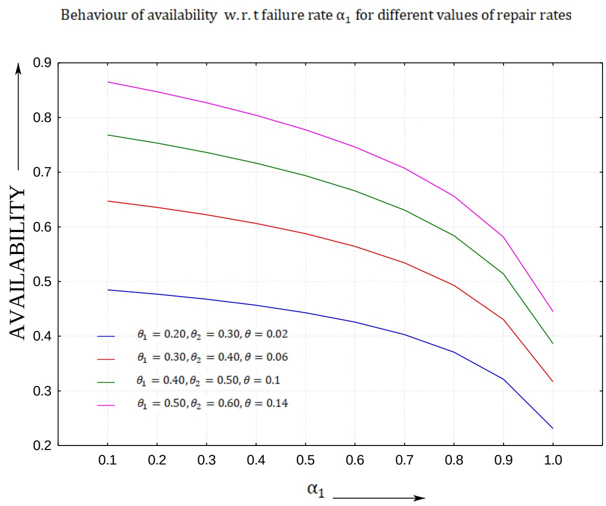


Fig.: 3

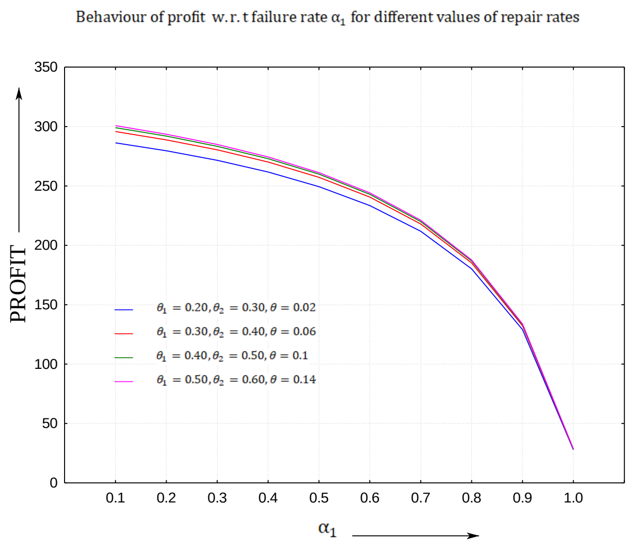


Fig.: 4

From Fig 2, 3 & 4, we have observed that MTSF, availability and profit function respectively, decreases with the increase in the failure rate of the system and these characteristics shows an increase, as we increase the repair rate of the system. Therefore, we can conclude here that the expected lifetime of the system can be increased by providing the proper repair facility to the system, as regular repair of the units improves the reliability and effectiveness of the system.

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Reliability Modelling and Assessment of Multi Standby Hybrid System

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Abstract

Systems connected to an external supporting device for their operations viewed as hybrid systems have been manufactured to meet the demand of industries, economic growth and populace in general. Companies and organizations heavily rely on these systems to conduct their business. The paper deals with the reliability and availability characteristics of four different systems requiring external supporting device for their operation. The system consists of main unit connected to the cold standby supporting devices. The failure and time of both main unit and supporting device are assumed to be exponentially distributed. Markov models are developed and differential difference equations are derived to obtain explicit expressions for the steady-state availability and mean time to failure and perform analytical and numerical comparisons. Comparisons show that system with five cold standby supporting devices is the most reliable system.

Keywords: Availability, mean time to failure, supporting device, single unit

Mathematics Subject Classification (2000): 90B25

1. Introduction

High system reliability and availability play a vital role towards industrial growth as the profit is directly dependent on production volume which depends upon system performance. Thus the reliability and availability of a system may be enhanced by proper design, optimization at the design stage and by maintaining the same during its service life. Because of their prevalence in power plants, manufacturing systems, and industrial systems, many researchers have studied reliability and availability problem of different systems. Hajeeh (2012) deals with availability of a system with different repair options. Hu et al. (2012) presents availability analysis and design optimisation for a repairable series-parallel system with failure dependencies. Jain and Rani (2013) studied the availability analysis for repairable system with warm standby, switching failure and reboot delay. Wang et al. (2012) performed comparative analysis of availability between two systems with warm standby units and different imperfect coverage. Wang and Chen (2009) performed comparative analysis of availability between three systems with general repair times, reboot delay and switching failures.

In real-life situations we often encounter cases where the systems that cannot work without the help of external supporting devices connect to such systems. These external supporting devices are systems themselves that are bound to fail. Such systems are found in power plants, manufacturing systems, and industrial systems. Large volumes of literature exist on the issue relating to prediction of various systems performance connected to an external supporting device for their operations. Yusuf (2014) performed comparative analysis of profit between three dissimilar repairable redundant systems using supporting external device for operation. Yusuf et al (2016) performed reliability computation of a linear consecutive 2-out-of-3 system in the presence of supporting device. Yusuf (2016) presents reliability evaluation of a parallel system with a supporting device and two types of preventive maintenance. The problem considered in this paper is different from the work of discussed authors above. In this paper, a single unit system connected to cold standby external supporting device is considered. The objectives of this paper are: to derive the explicit expressions for the availability and mean time to failure, to determine the optimal system. The organization of the paper is as follows. Section 2 contains a description of the system under study. Section 3 presents formulations of the models. The results of our analytical comparison between the systems are presented in section 4. Numerical examples are presented in section 5. Finally, we make some concluding remarks in Section 6.

2. Description and States of the System

In this paper, a single unit system connected to an external cold standby supporting devices is considered. It is assumed that the system most work with one supporting device. System I has main unit with five cold standby supporting devices, system II has four cold standby supporting devices, system III has three cold standby supporting devices, system IV has two cold standby supporting devices. It is also assumed that the switching from standby to operation is perfect. Both the unit and supporting devices are assumed to be repairable. Each of the primary supporting devices fails independently of the state of the other and has an exponential failure distribution with parameter λ_1. Whenever a primary supporting device fails, it is immediately sent to repair with parameter μ_1 and a standby supporting device is switch to operation. System failure occurs when the unit has failed with parameter λ_0 and it is sent for repair with parameter with parameter μ_0 or the failure of all supporting device.

Table 1: Transition rate table

	S_0	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}
S_0	0	λ_1	0	0	0	λ_0	0	0	0	0	0
S_1	μ_1	0	λ_1	0	0	0	0	λ_0	0	0	0
S_2	0	μ_1	0	λ_1	0	0	0	0	λ_0	0	0
S_3	0	0	μ_1	0	λ_1	0	0	0	0	λ_0	0
S_4	0	0	0	μ_1	0	0	λ_1	0	0	0	λ_0
S_5	μ_0	0	0	0	0	0	0	0	0	0	0
S_6	0	0	0	0	μ_1	0	0	0	0	0	0
S_7	0	μ_0	0	0	0	0	0	0	0	0	0
S_8	0	0	μ_0	0	0	0	0	0	0	0	0
S_9	0	0	0	μ_0	0	0	0	0	0	0	0
S_{10}	0	0	0	0	0	0	0	0	0	0	0

3. Formulation of the Models

In order to analyse the system availability of system I, we define $P_i(t)$ to be the probability that the system at $t \geq 0$ is in state $S_i, i = 0,1,2,3,...,10$. Also let $P(t)$ be the row vector of these probabilities at time t. The initial condition for this problem is:

$$P(0) = [p_0(0), p_1(0), p_2(0), \dots, p_{10}(0)] = [1,0,0,0,0,0,0,0,0,0,0]$$

Following Trivedi (2007), Wang et al. (2000), and Wang et al. (2006), we obtain the following differential equations from Figure 1:

$$\begin{aligned}
&p_0' = (\lambda_0 + \lambda_1)p_0(t) + \mu_1 p_1(t) + \mu_0 p_5(t) \\
&p_1'(t) = -(\lambda_0 + \lambda_1 + \mu_1)p_1(t) + \lambda_1 p_0(t) + \mu_1 p_2(t) + \mu_0 p_7(t) \\
&p_2'(t) = -(\lambda_0 + \lambda_1 + \mu_1)p_2(t) + \lambda_1 p_1(t) + \mu_1 p_3(t) + \mu_0 p_8(t) \\
&p_3'(t) = -(\lambda_0 + \lambda_1 + \mu_1)p_3(t) + \lambda_1 p_2(t) + \mu_1 p_4(t) + \mu_0 p_9(t) \\
&p_4'(t) = -(\lambda_0 + \lambda_1 + \mu_1)p_4(t) + \lambda_1 p_3(t) + \mu_1 p_6(t) + \mu_0 p_{10}(t) \\
&p_5'(t) = -\mu_0 p_1(t) + \lambda_0 p_0(t) \\
&p_6'(t) = -\mu_1 p_6(t) + \lambda_1 p_4(t) \\
&p_7'(t) = -\mu_0 p_7(t) + \lambda_0 p_1(t) \\
&p_8'(t) = -\mu_0 p_8(t) + \lambda_0 p_2(t) \\
&p_9'(t) = -\mu_0 p_9(t) + \lambda_0 p_3(t) \\
&p_{10}'(t) = -\mu_0 p_{10}(t) + \lambda_0 p_4(t)
\end{aligned} \tag{1}$$

This can be written in the matrix form as

$$\boldsymbol{P}' = \boldsymbol{QP}, \tag{2}$$

where

$$Q = \begin{pmatrix}
-(\lambda_0+\lambda_1) & \mu_1 & 0 & 0 & 0 & \mu_0 & 0 & 0 & 0 & 0 & 0 \\
\lambda_1 & -(\lambda_0+\lambda_1+\mu_1) & \mu_1 & 0 & 0 & 0 & 0 & \mu_0 & 0 & 0 & 0 \\
0 & \lambda_1 & -(\lambda_0+\lambda_1+\mu_1) & \mu_1 & 0 & 0 & 0 & 0 & \mu_0 & 0 & 0 \\
0 & 0 & \lambda_1 & -(\lambda_0+\lambda_1+\mu_1) & \mu_1 & 0 & 0 & 0 & 0 & \mu_0 & 0 \\
0 & 0 & 0 & \lambda_1 & -(\lambda_0+\lambda_1+\mu_1) & 0 & \mu_1 & 0 & 0 & 0 & \mu_0 \\
\lambda_0 & 0 & 0 & 0 & 0 & -\mu_0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \lambda_1 & 0 & -\mu_1 & 0 & 0 & 0 & 0 \\
0 & \lambda_0 & 0 & 0 & 0 & 0 & 0 & -\mu_0 & 0 & 0 & 0 \\
0 & 0 & \lambda_0 & 0 & 0 & 0 & 0 & 0 & -\mu_0 & 0 & 0 \\
0 & 0 & 0 & \lambda_0 & 0 & 0 & 0 & 0 & 0 & -\mu_0 & 0 \\
0 & 0 & 0 & 0 & \lambda_0 & 0 & 0 & 0 & 0 & 0 & -\mu_0
\end{pmatrix}$$

Let T denote the time-to-failure of the system. We use the following procedure to develop the steady-state availability. The steady-state availability (the proportion of time the system is in a functioning condition or equivalently, the sum of the probabilities of operational states) is given by

$$A_{T1}(\infty) = p_0(\infty) + p_1(\infty) + p_2(\infty) + p_3(\infty) + p_4(\infty) = \frac{\mu_0\mu_1^5 + \mu_0\mu_1^4\lambda_1 + \mu_0\mu_1^3\lambda_1^2 + \mu_0\mu_1^2\lambda_1^3 + \mu_0\mu_1\lambda_1^4}{\mu_1\lambda_0\lambda_1^4 + \mu_1^2\lambda_0\lambda_1^3 + \mu_1^3\lambda_0\lambda_1^2 + \mu_1^4\lambda_0\lambda_1 + \mu_1^5\lambda_0 + \mu_0\lambda_1^5 + \mu_0\mu_1\lambda_1^4 + \mu_0\mu_1^2\lambda_1^3 + \mu_0\mu_1^3\lambda_1^2 + \mu_0\mu_1^4\lambda_1 + \mu_0\mu_1^5} \tag{3}$$

To develop the explicit expression for mean time to failure, we use the concept of Trivedi (2002), Wang and Kuo (2000) and Wang et al. (2006) as follows:

The procedures require deleting rows and columns of absorbing states of matrix Q and take the transpose to produce a new matrix, say M. The expected time to reach an absorbing state is obtained from

$$E\left[T_{P(0)\to P(absorbing)}\right] = P(0)(-M^{-1})(1,1,1,1,1)^t \tag{4}$$

where the initial conditions are given by $P(0) = [p_0(0), p_1(0), p_2(0), p_3(0), p_4(0)] = [1,0,0,0,0]$ and

$$M = \begin{pmatrix} -(\lambda_0+\lambda_1) & \lambda_1 & 0 & 0 & 0 \\ \mu_1 & -(\lambda_0+\lambda_1+\mu_1) & \lambda_1 & 0 & 0 \\ 0 & \mu_1 & -(\lambda_0+\lambda_1+\mu_1) & \lambda_1 & 0 \\ 0 & 0 & \mu_1 & -(\lambda_0+\lambda_1+\mu_1) & \lambda_1 \\ 0 & 0 & 0 & \mu_1 & -(\lambda_0+\lambda_1+\mu_1) \end{pmatrix}$$

The explicit expression for is given by $MTTF_1$

$$MTTF_1 = \frac{N_1}{H_1} \tag{5}$$

where

$$\begin{aligned} N_1 = {} & (\mu_1^4 + 4\lambda_0\mu_1^3 + \lambda_1\mu_1^3 + \lambda_1^2\mu_1^2 + 6\lambda_0\lambda_1\mu_1^2 + 6\lambda_0^2\mu_1^2 + 6\lambda_0\lambda_1^2\mu_1 + \lambda_1^3\mu_1 + 4\lambda_0^3\mu_1 + 9\lambda_0^2\lambda_1\mu_1 + \lambda_0^4 \\ & + 4\lambda_0\lambda_1^3 + \lambda_1^4 + 6\lambda_0^2\lambda_1^2) + \\ & \lambda_1(\lambda_0^3 + 3\lambda_0^2\lambda_1 + 3\lambda_0^2\mu_1 + 3\lambda_0\lambda_1^2 + 4\lambda_0\lambda_1\mu_1 + 3\lambda_0\mu_1^2 + \lambda_1^3 + \lambda_1^2\mu_1 + \lambda_1\mu_1^2 + \mu_1^3) \\ & + \lambda_1^2(\lambda_0^2 + 2\lambda_0\lambda_1 + 2\lambda_0\mu_1 + \lambda_1^2 + \lambda_1\mu_1 + \mu_1^2) \\ & + \lambda_1^3(\lambda_0 + \lambda_1 + \mu_1) \end{aligned}$$

$$\begin{aligned} H_1 = {} & (\lambda_0\mu_1^4 + 2\lambda_0\lambda_1\mu_1^3 + 4\lambda_0^2\mu_1^3 + 6\lambda_0^3\mu_1^2 + 3\lambda_0\lambda_1^2\mu_1^2 + 9\lambda_0^2\lambda_1\mu_1^2 + 4\lambda_0\lambda_1^3\mu_1 + 12\lambda_0^3\lambda_1\mu_1 + 12\lambda_0^2\lambda_1^2\mu_1 \\ & + 4\lambda_0^4\mu_1 + \\ & 10\lambda_0^3\lambda_1^2 + \lambda_0^5 + \lambda_1^5 + 5\lambda_0^4\lambda_1 + 10\lambda_0^2\lambda_1^3 + 5\lambda_0\lambda_1^4) \end{aligned}$$

Special Cases:

Case I: Availability and mean time to failure of system requiring four cold standbys supporting devices

$$A_{T2}(\infty) = \frac{\mu_0\mu_1^4+\mu_0\mu_1^3\lambda_1+\mu_0\mu_1^2\lambda_1^2+\mu_0\mu_1\lambda_1^3}{\mu_1\lambda_0\lambda_1^3+\mu_1^2\lambda_0\lambda_1^2+\mu_1^3\lambda_0\lambda_1+\mu_1^4\lambda_0+\mu_0\lambda_1^4+\mu_0\mu_1\lambda_1^3+\mu_0\mu_1^2\lambda_1^2+\mu_0\mu_1^3\lambda_1+\mu_0\mu_1^4} \tag{6}$$

$$MTTF_2 = \frac{N_2}{H_2} \tag{7}$$

where

$$\begin{aligned} N_2 = {} & (\lambda_0^3 + 3\lambda_0^2\lambda_1 + 3\lambda_1^2\mu_1 + 3\lambda_0\lambda_1^2 + 4\lambda_0\lambda_1\mu_1 + 3\lambda_0\mu_1^2 + \lambda_1^3 + \lambda_1^2\mu_1 + \lambda_1\mu_1^2 + \mu_1^3) \\ & + \lambda_1(\lambda_0^2 + 2\lambda_0\lambda_1 + 2\lambda_0\mu_1 + \lambda_1^2 + +\lambda_1\mu_1 + \mu_1^2) + \lambda_1^2(\lambda_0 + \lambda_1 + \mu_1) + \lambda_1^3 \end{aligned}$$

$$H_2 = \lambda_0^4 + 4\lambda_0^3\lambda_1 + 3\lambda_0^3\mu_1 + 6\lambda_0^2\lambda_1^2 + 6\lambda_0^2\lambda_1\mu_1 + 3\lambda_0^2\mu_1^2 + 4\lambda_0\lambda_1^3 + 3\lambda_0\lambda_1^2\mu_1 + 2\lambda_0\lambda_1\mu_1^2 + \lambda_0\mu_1^2 + \lambda_1^4$$

Case II: Availability and mean time to failure of system requiring three cold standbys supporting devices

$$A_{T3}(\infty) = \frac{\mu_0\mu_1^3+\mu_0\mu_1^2\lambda_1+\mu_0\mu_1\lambda_1^2}{\mu_1\lambda_0\lambda_1^2+\mu_1^2\lambda_0\lambda_1+\mu_1^3\lambda_0+\mu_0\lambda_1^3+\mu_0\mu_1\lambda_1^2+\mu_0\mu_1^2\lambda_1+\mu_0\mu_1^3} \tag{8}$$

$$MTTF_3 = \frac{\lambda_0^2+2\lambda_0\lambda_1+2\lambda_0\mu_1+2\lambda_1^2+\lambda_1\mu_1+\mu_1^2+\lambda_1(\lambda_0+\lambda_1+\mu_1)}{\lambda_0^3+3\lambda_0^2\lambda_1+2\lambda_0^2\mu_1+3\lambda_0\lambda_1^2+2\lambda_0\lambda_1\mu_1+\lambda_0\mu_1^2+\lambda_1^3} \tag{9}$$

Case III: Availability and mean time to failure of system requiring two cold standbys supporting devices

$$A_{T4}(\infty) = \frac{\mu_0\mu_1^2+\mu_0\mu_1\lambda_1}{\mu_1\lambda_0\lambda_1+\mu_1^2\lambda_0+\mu_0\lambda_1^2+\mu_0\mu_1\lambda_1+\mu_0\mu_1^2} \tag{10}$$

$$MTTF_4 = \frac{2\lambda_1+\lambda_0+\mu_1}{\lambda_0^2+2\lambda_0\lambda_1+\lambda_0\mu_1+\lambda_1^2} \tag{11}$$

4. Comparison between the systems

MAPLE software package was used to program the analytical comparison in this study. The results are presented below.

$$A_{T1}(\infty) - A_{T2}(\infty) = \frac{\mu_0^2\mu_1^5\lambda_1^4}{D_1D_2} \tag{12}$$

$$A_{T2}(\infty) - A_{T3}(\infty) = \frac{\mu_0^2\mu_1^4\lambda_1^3}{D_2D_3} \tag{13}$$

$$A_{T3}(\infty) - A_{T4}(\infty) = \frac{\mu_0^2\mu_1^3\lambda_1^2}{D_3D_4} \tag{14}$$

$$MTTF_1 - MTTF_2 = \frac{\lambda_1^4(\lambda_0^4 + 4\lambda_0^3\lambda_1 + 4\lambda_0^3\mu_1 + 6\lambda_0^2\lambda_1^2 + 9\lambda_0^2\lambda_1\mu_1 + 6\lambda_0^2\mu_1^2 + 4\lambda_0\lambda_1^3 + 6\lambda_0\lambda_1^2\mu_1 + 6\lambda_0\lambda_1\mu_1^2 + 4\lambda_0\mu_1^3 + \lambda_1^4 + \lambda_1^3\mu_1 + \lambda_1^2\mu_1^2 + \lambda_1\mu_1^3 + \mu_1^4)}{H_1H_2} \tag{15}$$

$$MTTF_2 - MTTF_3 = \frac{\left(\lambda_1^3\left(\lambda_0^3+3\lambda_0^2\lambda_1+3\lambda_0^2\mu_1+3\lambda_0\lambda_1^2+4\lambda_0\lambda_1\mu_1+3\lambda_0\mu_1^2+\lambda_1^3+\lambda_1^2\mu_1+\lambda_1\mu_1^2+\mu_1^3\right)\right)}{H_2\left(\lambda_0^3+3\lambda_0^2\lambda_1+2\lambda_0^2\mu_1+3\lambda_0\lambda_1^2+2\lambda_0\lambda_1\mu_1+\lambda_0\mu_1^2+\lambda_1^3\right)} \quad (16)$$

$$MTTF_3 - MTTF_4$$
$$= \frac{\lambda_1^2(\lambda_0^2 + 3\lambda_0^2\lambda_1 + 3\lambda_0^2\mu_1 + 3\lambda_0\lambda_1^2 + 4\lambda_0\lambda_1\mu_1 + 3\lambda_0\mu_1^2 + \lambda_1^3 + \lambda_1^2\mu_1 + \lambda_1\mu_1^2 + \mu_1^3)}{(\lambda_0^3 + 3\lambda_0^2\lambda_1 + 2\lambda_0^2\mu_1 + 3\lambda_0\lambda_1^2 + 2\lambda_0\lambda_1\mu_1 + \lambda_0\mu_1^2 + \lambda_1^3)(\lambda_0^2 + 2\lambda_0\lambda_1 + \lambda_0\mu_1 + \lambda_1^2)} \quad (17)$$

where

$$D_4 = \mu_1\lambda_0\lambda_1 + \mu_1^2\lambda_0 + \mu_0\lambda_1^2 + \mu_0\mu_1\lambda_1 + \mu_0\mu_1^2$$
$$D_3 = \mu_1\lambda_0\lambda_1^2 + \mu_1^2\lambda_0\lambda_1 + \mu_1^3\lambda_0 + \mu_0\lambda_1^3 + \mu_0\mu_1\lambda_1^2 + \mu_0\mu_1^2\lambda_1 + \mu_0\mu_1^3$$
$$D_2 = \mu_1\lambda_0\lambda_1^3 + \mu_1^2\lambda_0\lambda_1^2 + \mu_1^3\lambda_0\lambda_1 + \mu_1^4\lambda_0 + \mu_0\lambda_1^4 + \mu_0\mu_1\lambda_1^3 + \mu_0\mu_1^2\lambda_1^2 + \mu_0\mu_1^3\lambda_1 + \mu_0\mu_1^4$$
$$D_1 = \mu_1\lambda_0\lambda_1^4 + \mu_1^2\lambda_0\lambda_1^3 + \mu_1^3\lambda_0\lambda_1^2 + \mu_1^4\lambda_0\lambda_1 + \mu_1^5\lambda_0 + \mu_0\lambda_1^5 + \mu_0\mu_1\lambda_1^4 + \mu_0\mu_1^2\lambda_1^3 + \mu_0\mu_1^3\lambda_1^2 + \mu_0\mu_1^4\lambda_1 + \mu_0\mu_1^5$$

From (12) to (17)

$$A_{T1}(\infty) > A_{T2}(\infty) > A_{T3}(\infty) > A_{T4}(\infty)$$
$$MTTF_1(\infty) > MTTF_2(\infty) > MTTF_3(\infty) > MTTF_4(\infty)$$
$$\forall \lambda_0, \lambda_1, \mu_0, \mu_1 > 0$$

5. Numerical example

Numerical examples are presented to demonstrate the impact of repair and failure rates on steady-state availability and mean time to failure of the system based on given values of the parameters. MATLAB software package was used to program the numerical comparison in this study. The results are presented below. For the purpose of numerical example, the following sets of parameter values are used: $\lambda_1 = 0.3$, $\lambda_0 = 0.2$, $\mu_1 = 0.6$, $\mu_1 = 0.6$ for Figures 2 and 3 and $\lambda_1 = 0.4$, $\lambda_0 = 0.1$, $\mu_1 = 0.05$ for Figures 4 and 5.

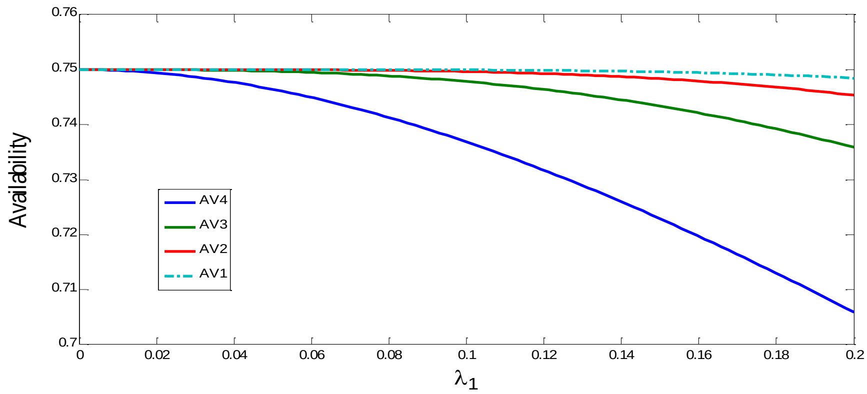

Figure 1: Availability against supporting device failure rate λ_1

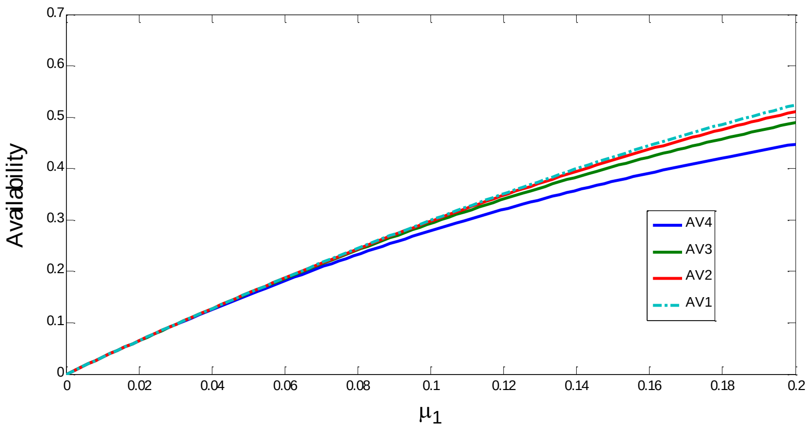

Figure 2: Availability against supporting device repair rate μ_1

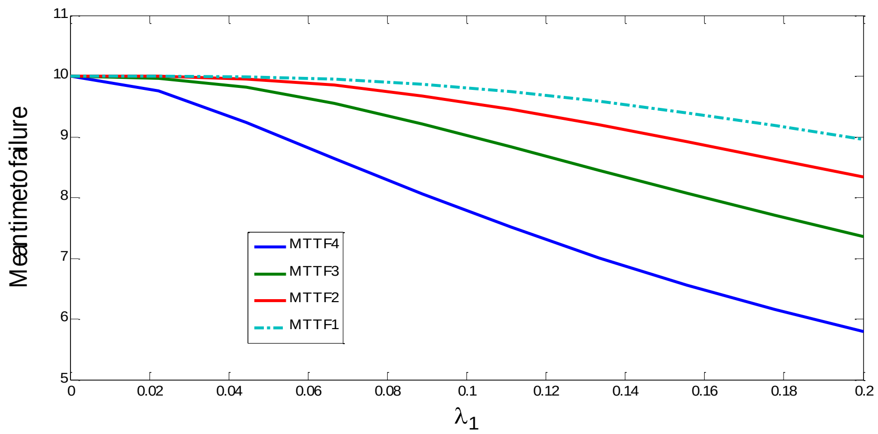

Figure 3: mean time to failure against supporting device failure rate λ_1

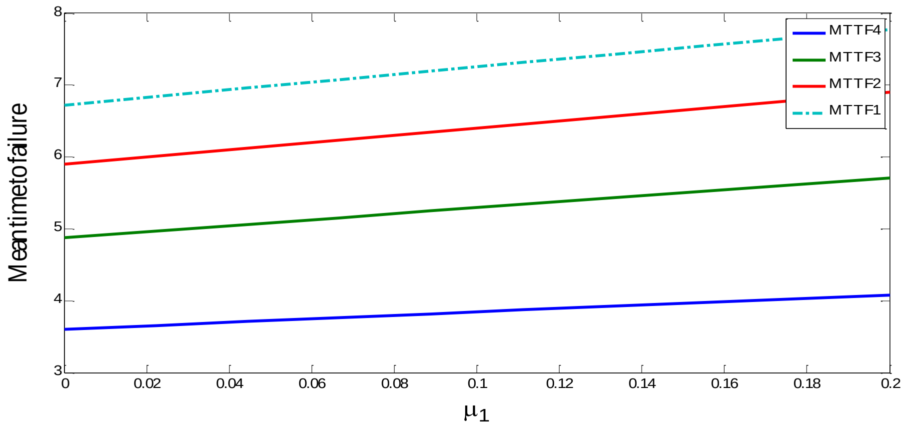

Figure 4: mean time to failure against supporting device repair rate μ_1

Figures 1 and 3 show the results availability and mean time to failure for the four systems against the failure rate λ_1. It is clear from the figures that system I (system with five standby supporting device) has higher availability and mean time to failure as compared to the other three systems. Similar observation is also depicted in Figures 2 and 4 with respect to repair rate μ_1. It is evident from these figures that system I (system with five standby supporting device) has higher availability and mean time to failure as compared to the other three systems. These tend to suggest that system I is better than the other systems.

Conclusion

This paper studied a single system connected to two types of supporting device type I and II for its operation. Explicit expression for the steady-state availability was derived. Comparative analysis was performed analytically along with numerically example in this study. It is enough to mention first that the optimal system is system with five cold standbys supporting devices.

Thus,

$$A_{T1}(\infty) > A_{T2}(\infty) > A_{T3}(\infty) > A_{T4}(\infty)$$
$$MTTF_1(\infty) > MTTF_2(\infty) > MTTF_3(\infty) > MTTF_4(\infty)$$

On the basis of the analytical and numerical results obtained f, it is suggested that the system reliability can be improved significantly by:

(i) Adding more cold standby units.
(ii) Increasing the repair rate.
(iii) Reducing the failure rate of the system by hot or cold duplication method.
(iv) Exchange the system when old with new one before failure.

The system can further be developed into system with more standbys in solving reliability and availability problems.

The present study is important to system designers, engineers, maintenance managers and plant management for proper maintenance analysis, decision and safety of the system as a whole. The study will also assist engineers, decision makers and plant management to avoid an incorrect reliability assessment and consequent erroneous decision-making, which may lead to unnecessary expenditures, incorrect maintenance scheduling and reduction of safety standards.

Conflict of Interests
The authors declare that there is no conflict of interests.

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Shukla Distribution And Its Application

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Abstract

In this paper a two-parameter lifetime distribution named, 'Shukla distribution' which includes several one parameter lifetime distributions including exponential, Shanker, Ishita, Pranav ,Rani and Ram Awadh as particular cases, has been proposed and investigated. Its moments have been obtained. The hazard rate function, mean residual life function and stochastic ordering of the distribution have been discussed. Maximum likelihood estimation has been explained for estimating the parameters of the distribution. Applications of the distribution have been explained through real life time data and its fit has been found satisfactory over well-known one parameter and two-parameter lifetime distributions.

Keywords: Lifetime distributions, Moments, Hazard rate function, Mean residual life function, Maximum likelihood estimation, Goodness of fit.

1. Introduction

In the new era of the world, it is important to study through the model for systematic approach and statistical approach. In this case approach of distribution theory is crucial to develop statistical model for knowing the occurrence of some event and their interest for some populations of individuals in almost every field of knowledge. The statistical modeling and their studies along with lifetime data has been drawn interest to researchers in engineering, biomedical science, insurance, finance, amongst others. Applications of lifetime distributions range from investigations into the endurance of manufactured items in engineering to research involving human diseases in biomedical sciences.

In the recent past years, a number of one parameter and two-parameter lifetime distributions for modeling lifetime data have been proposed by different statisticians. As we know that classical one parameter exponential distribution including other popular distribution such as Lindley, Akash, Shanker, Ishita, Pranav, Rani, Ram Awadh distributions are proposed and applied on life time data from various field. The two-parameter lifetime distributions popular in statistics are gamma, Weibull, Power Lindley, Quasi Lindley and Exponentiated exponential.

The probability density function (pdf) along with introducer (year) of exponential, Lindley, Akash, Shanker, Pranav and Ram Awadh distributions are presented in table 1.

Table 1: The pdf of exponential, Lindley, Shanker, Pranav, Rani and Ram Awadh distributions

Distributions	pdf	Introducer (Year)
Exponential	$f(x;\theta)=\theta e^{-\theta x}$	
Lindley	$f(x)=\frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}$	Lindley (1958)
Shanker	$f(x)=\frac{\theta^2}{\theta^2+1}(\theta+x)e^{-\theta x}$	Shanker (2015 a)
Akash	$f(x;\theta)=\frac{\theta^3}{\theta^2+2}(1+x^2)e^{-\theta x}$	Shanker (2015 b)
Ishita	$f(x;\theta)=\frac{\theta^3}{\theta^3+2}(\theta+x^2)e^{-\theta x}$	Shanker and Shukla (2017)
Pranav	$f(x;\theta)=\frac{\theta^4}{\theta^4+6}(\theta+x^3)e^{-\theta x}$	Shukla (2018)
Rani	$f(x;\theta)=\frac{\theta^5}{\theta^5+24}(\theta+x^4)e^{-\theta x}$	Shanker (2017)
Ram Awadh	$f(x;\theta)=\frac{\theta^6}{\theta^6+120}(\theta+x^5)e^{-\theta x}$	Shukla(2018)

Ghitany *et al* (2008) have discussed various statistical properties, estimation of parameter and application of Lindley distribution to model waiting time data in a bank and showed that Lindley distribution is a suitable model over exponential distribution. Shanker *et al* (2015) have detailed comparative and critical study on applications of exponential and Lindley distributions for modeling real lifetime datasets from biomedical science and engineering and showed that in majority of datasets exponential distribution shows satisfactory fit over Lindley distribution.

Recently, Shanker and Shukla (2019) proposed a two-parameter lifetime distribution named Rama-Kamlesh distribution (RKD) defined by its pdf and survival function as

$$f(x;\theta,\alpha)=\frac{\theta^{\alpha+1}}{\theta^{\alpha}+\Gamma(\alpha+1)}(1+x^{\alpha})e^{-\theta x};x>0,\theta>0,\alpha\geq 0 \tag{1.1}$$

$$S(x;\theta,\alpha)=\frac{\theta^{\alpha}(1+x^{\alpha})e^{-\theta x}+\alpha\Gamma(\alpha,\theta x)}{\theta^{\alpha}+\Gamma(\alpha+1)};x>0,\theta>0,\alpha\geq 0, \tag{1.2}$$

where $\Gamma(\alpha,\theta x)$ is the lower incomplete gamma function defined as

$$\Gamma(\alpha,z)=\int_0^z e^{-t}t^{\alpha-1}\,dt \tag{1.3}$$

It has been mentioned by Shanker and Shukla (2019) that RKD includes several one parameter lifetime distributions. Various interesting properties, estimation of parameters and application of the distribution have been given in Shanker and Shukla (2019).

The main aim of the present paper is to introduce two-parameter lifetime distribution named Shukla distribution (SD)' which includes many one parameter distributions including exponential distribution as particular case. Several other one parameter lifetime distributions can also be generated from SD. Its moments about origin and the variance have been obtained. The hazard rate function and stochastic ordering have been discussed. Maximum likelihood estimation has been discussed for estimating the parameters of the distribution. Applications of the distribution have been discussed with real lifetime dataset and the goodness of fit of the distribution has been compared with well known one parameter and two-parameter lifetime distributions.

2. Shukla Distribution

The pdf of Shukla distribution (SD) having parameters θ and α can be defined as

$$f(x;\theta,\alpha)=\frac{\theta^{\alpha+1}}{\theta^{\alpha+1}+\Gamma(\alpha+1)}(\theta+x^{\alpha})e^{-\theta x};x>0,\theta>0,\alpha\geq 0 \tag{2.1}$$

It can be easily verified that exponential, Shanker, Ishita, Pranav, Rani and Ram Awadh distributions are particular cases of SD for $\alpha=0,\alpha=1,\alpha=2,\alpha=3,$ $\alpha=4$ and $\alpha=5$ respectively. The pdf (2.1) can be shown as a convex combination of exponential (θ) and gamma (α,θ) distributions. We have

$$f(x;\theta,\alpha)=pg_1(x;\theta)+(1-p)g_2(x;\alpha,\theta),$$

where

$$p=\frac{\theta^{\alpha+1}}{\theta^{\alpha+1}+\Gamma(\alpha+1)},\ g_1(x;\theta)=\theta e^{-\theta x},\ g_2(x;\alpha,\theta)=\frac{\theta^{\alpha+1}}{\Gamma(\alpha+1)}e^{-\theta x}x^{\alpha+1-1}.$$

$$S(x;\theta,\alpha)=P(X>x)=\int_x^{\infty}f(t;\theta,\alpha)dt=\frac{\theta^{\alpha+1}}{\theta^{\alpha+1}+\Gamma(\alpha+1)}\int_x^{\infty}(\theta+t^{\alpha})\,e^{-\theta t}dt$$

$$=\frac{\theta^{\alpha+1}}{\theta^{\alpha+1}+\Gamma(\alpha+1)}\left[\theta\int_x^{\infty}e^{-\theta t}dt+\int_x^{\infty}e^{-\theta t}\,t^{\alpha}dt\right]$$

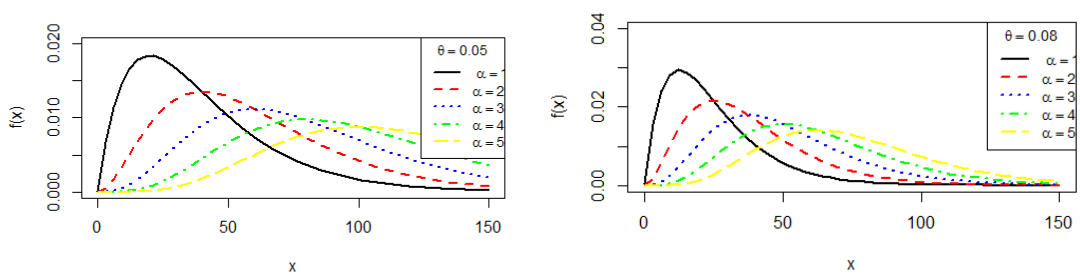
$$=\frac{\theta^{\alpha+1}}{\theta^{\alpha+1}+\Gamma(\alpha+1)}\left[\frac{e^{-\theta x}}{1}+\frac{e^{-\theta x}(\theta x)^{\alpha}+\alpha\Gamma(\alpha,\theta x)}{\theta^{\alpha+1}}\right]$$

$$=\frac{\theta^{\alpha}(\theta+x^{\alpha})e^{-\theta x}+\alpha\Gamma(\alpha,\theta x)}{\theta^{\alpha+1}+\Gamma(\alpha+1)},$$

Thus the corresponding cdf of SD can be obtained as

$$F(x;\theta,\alpha)=1-S(x;\theta,\alpha)=1-\frac{\theta^{\alpha}(\theta+x^{\alpha})e^{-\theta x}+\alpha\Gamma(\alpha,\theta x)}{\theta^{\alpha+1}+\Gamma(\alpha+1)};x>0,\theta>0,\alpha\geq 0 \tag{2.2}$$

Behaviors of pdf and survival function of SD for varying values of parameters θ and α have been shown in figures 1 and 2, respectively.



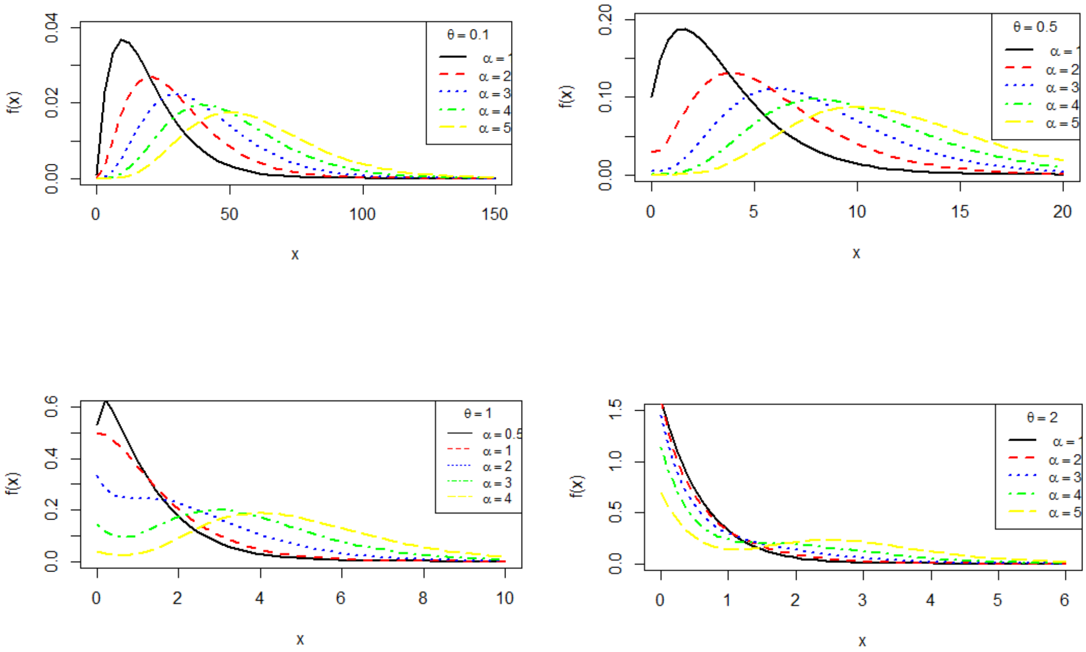


Fig.1: Behavior of the pdf of SD for varying values of parameters θ and α

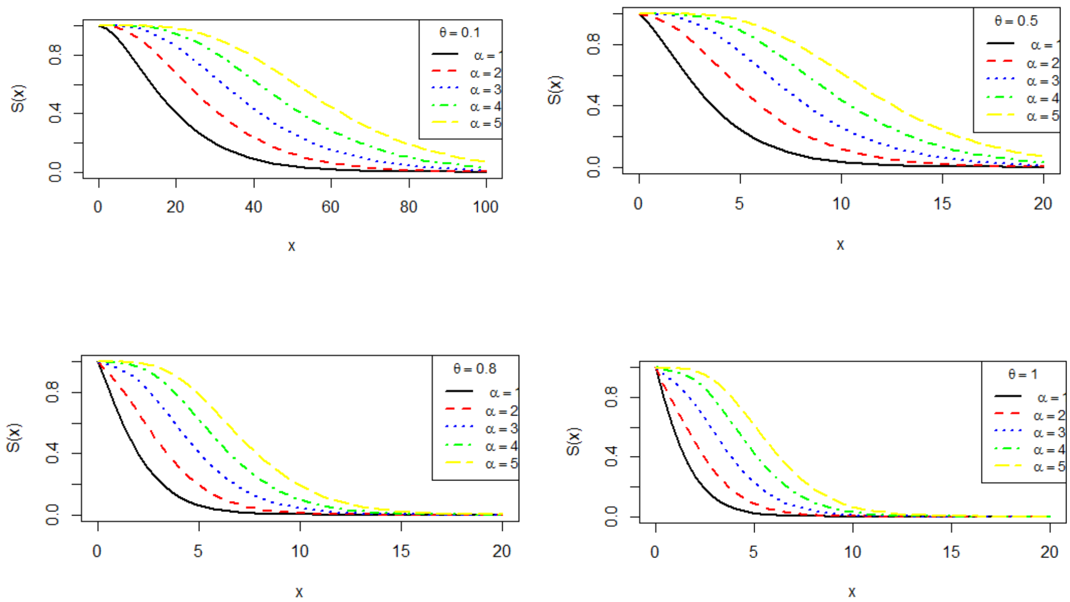


Fig.2: Behavior of the S(x) of SD for varying values of parameters θ and α

3. Moments

The r th moment about origin, μ_r' of Shukla distribution(SD) can be obtained as

$$\mu_r' = \frac{\theta^{\alpha+1}}{\theta^{\alpha+1}+\Gamma(\alpha+1)}\int_0^{\infty} x^r\,(\theta+x^{\alpha})e^{-\theta x}dx$$

$$= \frac{\theta^{\alpha+1}\Gamma(r+1)+\Gamma(\alpha+r+1)}{\theta^r\{\theta^{\alpha+1}+\Gamma(\alpha+1)\}};r=1,2,3,\ldots$$

Thus the first four moments about origin of SD are obtained as

$$\mu_1' = \frac{\theta^{\alpha+1}+\Gamma(\alpha+2)}{\theta\{\theta^{\alpha+1}+\Gamma(\alpha+1)\}}$$

$$\mu_2' = \frac{2\theta^{\alpha+1}+\Gamma(\alpha+3)}{\theta^2\{\theta^{\alpha+1}+\Gamma(\alpha+1)\}}$$

$$\mu_3' = \frac{6\theta^{\alpha+1}+\Gamma(\alpha+4)}{\theta^3\{\theta^{\alpha+1}+\Gamma(\alpha+1)\}}$$

$$\mu_3' = \frac{24\theta^{\alpha+1}+\Gamma(\alpha+5)}{\theta^4\{\theta^{\alpha+1}+\Gamma(\alpha+1)\}}.$$

The variance of SD can be obtained as

$$\mu_2 = \mu_2' - (\mu_1)^2 = \frac{\{2\theta^{\alpha+1}+\Gamma(\alpha+3)\}\{\theta^{\alpha+1}+\Gamma(\alpha+1)\}-\{\theta^{\alpha+1}+\Gamma(\alpha+2)\}^2}{\theta^2\{\theta^{\alpha+1}+\Gamma(\alpha+1)\}^2}$$

Taking $r=1,2,3$ and 4, the first four moments about origin, μ_r' of SD can be obtained. It should be noted that the r th moment about origin, μ_r' of exponential, Shanker, Ishita, Pranav, Rani and Ram Awadh distribution can be obtained from the μ_r' of SD by taking $\alpha=0,1,2,3,4,$ and 5.

4. Hazard Rate Function and Mean Residual Life Function

For a continuous random variable X having pdf $f(x)$ and cdf $F(x)$, the hazard rate function (also known as the failure rate function) , $h(x)$, is defined as

$$h(x) = \lim_{\Delta x\to 0}\frac{P(X<x+\Delta x|X>x)}{\Delta x} = \frac{f(x)}{1-F(x)}.$$

Thus, hazard rate function, $h(x)$ of Shukla distribution can be expressed as $h(x) = h(x;\theta,\alpha) =$

$$\frac{f(x;\theta,\alpha)}{1-F(x;\theta,\alpha)} = \frac{\theta^{\alpha+1}(\theta+x^{\alpha})e^{-\theta x}}{\theta^{\alpha}(\theta+x^{\alpha})e^{-\theta x}+\alpha\Gamma(\alpha,\theta x)};x>0,\theta>0,\alpha\geq 0$$

The mean residual life function, $m(x)$ of Shukla distribution can be obtained as

$$m(x;\theta,\alpha) = \frac{1}{S(x;\theta,\alpha)}\int_x^{\infty} tf(t;\theta,\alpha)\,dt - x$$

$$= \frac{\theta^{\alpha+1}+\Gamma(\alpha+1)}{\theta^{\alpha}(\theta+x^{\alpha})e^{-\theta x}+\alpha\Gamma(\alpha,\theta x)}\int_x^{\infty} t\frac{\theta^{\alpha+1}}{\theta^{\alpha+1}+\Gamma(\alpha+1)}(\theta+t^{\alpha})e^{-\theta t}\,dt - x$$

$$= \frac{\theta^{\alpha+1}}{\theta^{\alpha}(\theta+x^{\alpha})e^{-\theta x}+\alpha\Gamma(\alpha,\theta x)}\left[\theta\int_x^{\infty} e^{-\theta t}tdt + \int_x^{\infty} e^{-\theta t}t^{\alpha+1}\,dt\right] - x$$

$$= \frac{\theta^{\alpha+1}}{\theta^{\alpha}(\theta+x^{\alpha})e^{-\theta x}+\alpha\Gamma(\alpha,\theta x)}\left[\frac{e^{-\theta x}(\theta x+1)}{\theta}+\frac{e^{-\theta x}(\theta x)^{\alpha}(\theta x+\alpha+1)+\alpha(\alpha+1)\Gamma(\alpha,\theta x)}{\theta^{\alpha+2}}\right]-x$$

$$=\frac{e^{-\theta x}\{\theta^{\alpha+1}+(\alpha+1)(\theta x)^{\alpha}\}+\alpha(\alpha+1-\theta x)\Gamma(\alpha,\theta x)}{\theta\{\theta^{\alpha}(\theta+x^{\alpha})e^{-\theta x}+\alpha\Gamma(\alpha,\theta x)\}}.$$

Note that $h(0)=\frac{\theta^{\alpha+2}}{\theta^{\alpha+1}+\Gamma(\alpha+1)}=f(0)$ and $m(0)=\frac{\theta^{\alpha+1}+\Gamma(\alpha+2)}{\theta\{\theta^{\alpha+1}+\Gamma(\alpha+1)\}}=\mu_1{}'$. The behaviors of $h(x)$ and $m(x)$ of SD for varying values of parameters θ and α have been shown in figures 3 and 4 respectively.

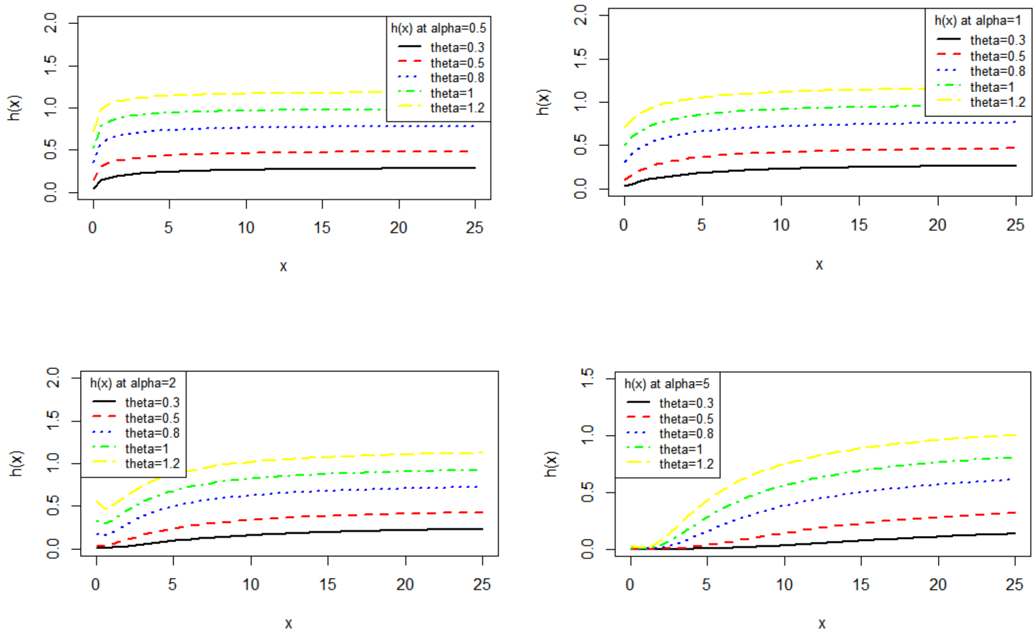


Fig.3: Behavior of $h(x)$ of SD for varying values of parameters θ and α

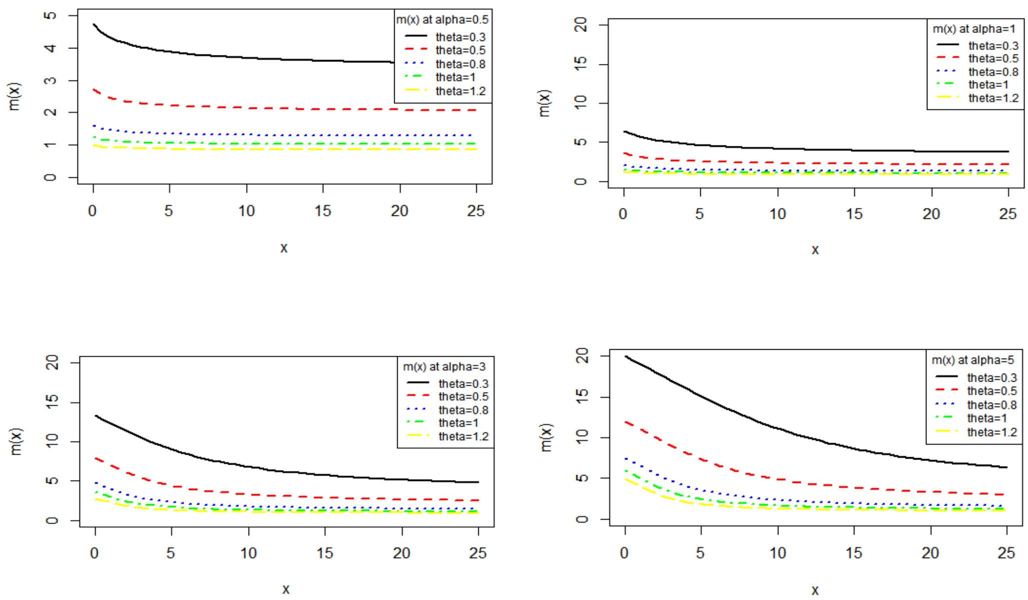


Fig. 4: Behavior of $m(x)$ of SD for varying values of parameters θ and α

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5. Stochastic Ordering

Stochastic ordering of positive continuous random variables is an important tool for judging their comparative behavior. A random variable X is said to be smaller than a random variable Y in the

(i) stochastic order $(X \leq_{st} Y)$if $F_X(x) \geq F_Y(x)$for all x

(ii) hazard rate order $(X \leq_{hr} Y)$if $h_X(x) \geq h_Y(x)$ for all x

(iii) mean residual life order $(X \leq_{mrl} Y)$if $m_X(x) \leq m_Y(x)$for all x

(iv) likelihood ratio order $(X \leq_{lr} Y)$if $\frac{f_X(x)}{f_Y(x)}$ decreases in x.

The following results due to Shaked and Shanthikumar (1994) are well known for establishing stochastic ordering of distributions

$$\begin{array}{l} X \leq_{lr} Y \Rightarrow X \leq_{hr} Y \Rightarrow X \leq_{mrl} Y \\ \Downarrow \\ X \leq_{st} Y \end{array}$$

KRD is ordered with respect to the strongest 'likelihood ratio' ordering as shown in the following theorem:

Theorem: Let X ~ RKD(θ_1, α_1) and Y Y~ RKD$(\theta_2, \alpha_2)(\theta_2, \alpha_2)$. If $\alpha_1 \leq \alpha_2$ and $\theta_1 > \theta_2$, then $X \leq_{lr} Y$and hence$X \leq_{hr} Y$, $X \leq_{mrl} Y$and$X \leq_{st} Y$.

Proof: We have

$$\frac{f_X(x;\theta_1,\alpha_1)}{f_Y(x;\theta_2,\alpha_2)} = \frac{\theta_1^{\alpha_1+1}\left(\theta_2^{\alpha_2+1}+\Gamma(\alpha_2+1)\right)}{\theta_2^{\alpha_2+1}\left(\theta_1^{\alpha_1+1}+\Gamma(\alpha_1+1)\right)}\left(\frac{\theta_1+x^{\alpha_1}}{\theta_2+x^{\alpha_2}}\right)e^{-(\theta_1-\theta_2)x};\ x > 0$$

Now

$$\ln\frac{f_X(x;\theta_1,\alpha_1)}{f_Y(x;\theta_2,\alpha_2)} = \ln\left[\frac{\theta_1^{\alpha_1+1}\left(\theta_2^{\alpha_2+1}+\Gamma(\alpha_2+1)\right)}{\theta_2^{\alpha_2+1}\left(\theta_1^{\alpha_1+1}+\Gamma(\alpha_1+1)\right)}\right] + \ln\left(\frac{\theta_1+x^{\alpha_1}}{\theta_2+x^{\alpha_2}}\right) - (\theta_1-\theta_2)x$$

.

This gives $$\frac{d}{dx}\ln\frac{f_X(x;\theta_1,\alpha_1)}{f_Y(x;\theta_2,\alpha_2)} = \frac{\alpha_1\theta_2 x^{\alpha_1-1} - \theta_1\alpha_2 x^{\alpha_2-1} + (\alpha_1-\alpha_2)x^{\alpha_1+\alpha_2-1}}{(1+x^{\alpha_1})(1+x^{\alpha_2})} - (\theta_1-\theta_2)$$

Thus, for$\alpha_1 \leq \alpha_2$ and $\theta_1 > \theta_2$,$\frac{d}{dx}\ln\frac{f_X(x;\theta_1,\alpha_1)}{f_Y(x;\theta_2,\alpha_2)} < 0$. This means that $X \leq_{lr} Y$and hence$X \leq_{hr} Y$, $X \leq_{mrl} Y$and$X \leq_{st} Y$. This shows flexibility of SD over one parameter exponential, Shanker, Ishita, Pranav, Rani and Ram Awadh distributions.

6. Maximum Likelihood Estimation

Let $(x_1, x_2, x_3, \ldots, x_n)$ be a random sample from SD (2.1). The likelihood function, L of (2.1) can be expressed as

$$L = \left(\frac{\theta^{\alpha+1}}{\theta^{\alpha+1}+\Gamma(\alpha+1)}\right)^n \prod_{i=1}^{n}\left(\theta + {x_i}^{\alpha}\right) e^{-n\theta\bar{x}}$$

The natural log likelihood function is thus obtained as

$$\ln L = n\ln\left(\frac{\theta^{\alpha+1}}{\theta^{\alpha+1}+\Gamma(\alpha+1)}\right) + \sum_{i=1}^{n}\ln\left(\theta + {x_i}^{\alpha}\right) - n\theta\bar{x}$$

$$= n\left[(\alpha+1)\ln\theta - \ln\left(\theta^{\alpha+1}+\Gamma(\alpha+1)\right)\right] + \sum_{i=1}^{n}\ln\left(\theta + {x_i}^{\alpha}\right) - n\theta\bar{x}.$$

The maximum likelihood estimates (MLEs) $\left(\hat{\theta}, \hat{\alpha}\right)$ of parameters $\left(\theta, \alpha\right)$ of SD are the solution of the following nonlinear log likelihood equations

$$\frac{\partial \ln L}{\partial\theta} = \frac{n(\alpha+1)}{\theta} - \frac{n(\alpha+1)\theta^{\alpha}}{\theta^{\alpha+1}+\Gamma(\alpha+1)} + \sum_{i=1}^{n}\frac{1}{\theta + {x_i}^{\alpha}} - n\bar{x} = 0$$

$$\frac{\partial \ln L}{\partial \alpha} = n \ln \theta - \frac{n[\theta^{\alpha+1} \ln \theta + \psi(\alpha+1)]}{\theta^{\alpha+1} + \Gamma(\alpha+1)} + \sum_{i=1}^{n} \frac{x_i{}^{\alpha} \ln(x_i)}{\theta + x_i{}^{\alpha}} = 0$$

where $\overline{x}$ is the sample mean and $\psi(\alpha+1) = \frac{d}{d\alpha} \ln \Gamma(\alpha+1)$ is the digamma function. These two natural log likelihood equations do not seem to be solved directly, because they cannot be expressed in closed forms. The (MLE's) $(\hat{\theta}, \hat{\alpha})$ of (θ, α) can be computed directly by solving the natural log likelihood equation using Newton-Raphson iteration available in R-software till sufficiently close values of $\hat{\theta}$ and $\hat{\alpha}$ are obtained.

7. Data Analysis

The applications of SD have been discussed with the following dataset relating to engineering from Fuller et al (1994).This data set is the strength data of glass of the aircraft window reported by Fuller *et al* (1994):

18.83	20.8	21.657	23.03	23.23	24.05	24.321	25.5	25.52	25.8	26.69	26.77
26.78	27.05	27.67	29.9	31.11	33.2	33.73	33.76	33.89	34.76	35.75	35.91
36.98	37.08	37.09	39.58	44.045	45.29	45.381					

For the above dataset, SD has been fitted along with two- parameter distributions including Power Lindley distribution (PLD) proposed by Ghitany et al (2013), Weibull distribution suggested by Weibull distribution(1951), gamma distribution, Quasi Lindley distribution introduced by Shanker and Mishra (2013) and generalized exponential distribution proposed by Gupta and Kundu (1999), RKD and one parameter lifetime distributions including exponential, Lindley, Shanker, Akash, Ishita, Pranav, Rani and Ram Awadh . The ML estimates, value of $-2\log L$, Akaike Information criteria (AIC), K-S statistics and p-value of the fitted distributions are presented in tables 2 and 3. The AIC and K-S Statistics are computed using the following formulae: $AIC = -2 \ln L + 2k$ and K-S $= \underset{x}{\text{Sup}}|F_n(x) - F_0(x)|$, where k = the number of parameters, n = the sample size , $F_n(x)$is the empirical (sample) cumulative distribution function, and $F_0(x)$ is the theoretical cumulative distribution function. The best distribution is the distribution corresponding to lower values of $-2\log L$, AIC, and K-S statistics and higher p-value

Table 2: MLE's, Standard Errors, - 2ln L, AIC, K-S and p-values of the fitted distributions for dataset 1

Distributions	ML Estimates	$-2\log L$	AIC	BIC	K-S	p-value
SD	$\hat{\theta} = 0.6144$	**208.23**	**212.23**	**216.05**	**0.134**	**0.580**
	$\hat{\alpha} = 17.9299$					
PLD	$\hat{\theta} = 0.00243$	220.14	224.14	226.13	0.198	0.152
	$\hat{\alpha} = 1.9439$					
RKD	$\hat{\theta} = 0.61361$	208.23	212.23	216.05	0.134	0.580
	$\hat{\alpha} = 17.9060$					
Gamma	$\hat{\theta} = 0.61482$	208.22	212.22	216.05	0.134	0.578
	$\hat{\alpha} = 18.9433$					
Weibull	$\hat{\theta} = 0.00203$	241.61	245.61	247.61	0.353	0.000
	$\hat{\alpha} = 1.80566$					

QLD	$\hat{\theta}=0.03416$	274.45	278.45	281.32	0.458	0.000
	$\hat{\alpha}=18.9393$					
GED	$\hat{\theta}=0.16531$	208.27	212.27	215.13	0.135	0.581
	$\hat{\alpha}=92.0017$					
Exponential	$\hat{\theta}=0.0325$	274.53	276.53	277.96	0.459	0.000
Lindley	$\hat{\theta}=0.0629$	253.99	255.99	257.42	0.333	0.000
Akash	$\hat{\theta}=0.0970$	240.68	242.68	244.11	0.296	0.006
Shanker	$\hat{\theta}=0.06471$	252.35	254.35	255.78	0.357	0.000
Ishita	$\hat{\theta}=0.09732$	240.48	242.48	243.48	0.297	0.006
Pranav	$\hat{\theta}=0.1298$	232.77	234.77	235.77	0.253	0.030
Rani	$\hat{\theta}=0.1623$	277.25	229.25	230.24	0.220	0.080
Ram Awadh	$\hat{\theta}=0.19471$	223.07	225.07	226.07	0.197	0.155

It is obvious from the goodness of fit given in tables 2 that SD competes well with considered one-parameter and two-parameter lifetime distributions. Therefore, SD can be considered an important two-parameter lifetime distribution as.

8. Conclusions

In this paper a two-parameter lifetime distribution named, 'Shukla distribution (SD)' which includes one parameter lifetime distributions including exponential, Shanker, Ishita, Pranav, Rani and Ram Awadh as particular cases, has been proposed and studied. Its moments have been obtained. The hazard rate function, mean residual life function and stochastic ordering have been discussed. The estimation of its parameters using maximum likelihood estimation has been discussed. Goodness of fit has been presented with a real lifetime dataset and fit found quite satisfactory over all well- known considered lifetime distributions.

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Fuzzy Reliability of a System by Converting Trapezoidal Intervalued Fuzzy Number to Pentagonal Triangular Intervalued Fuzzy Number

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Abstract

In classical set theory there exist only two possibility of any element belonging to the set yes or no, that is its probability of belonging to the set either 0 or 1, but this theory is fail to predictable in many system where the possibility of an element belonging to set is not exact, that is there exist some vagueness about the element affecting the system. Therefore L. A. Zadeh gives a new theory of fuzzyness, where the belongingness of an element can except 0 or 1 and take any value between [0, 1]. This new approach give us much benefit to modelling the real situation and find the reliability of any system. This theory also useful to find the most critical event in any fault tree model. Fuzzy theory are applicable in many areas industrial, technical, engineering, medical etc.

Keywords: Healthcare system, Fault tree, Pentagonal-triangular intervalued fuzzy numbers, α - cut, signed distance, COG.

I. Introduction

In this article we consider a new intervalued pentagonal-triangular fuzzy number with the help of converting intervalued trapezoidal fuzzy numbers and find out reliability of mixed system. According to WHO (World Health Organization) 'Healthcare system goals are good health for citizens responsiveness to the expectation of the population and fair means of funding operation. Other dimension for the evaluation of health system include quality, efficiency, acceptability and equity. Butnariu developed a neuron model with the help of fuzzy analysis, Acoustico-vestibulary nerve as a fuzzy automation describe with this help. Similarly Rocha has been developed nervous system using fuzzy logic. The most extensive application of fuzzy theory in the area of medical diagnosis, in diagnosis process we mapped symptoms with diseases, the relation between symptom and disease are imprecise due to various stage of disease We know that in healthcare system there are many uncertainty, To determine reliability of whole system we use COG and signed distance method as defuzzification. Here we use interval valued fuzzy numbers which belongs to (λ,ρ), $0<\lambda<\rho$. This study used level (λ,ρ) interval valued fuzzy numbers to determine fuzzy reliability of mixed healthcare system. Fault tree analysis (FTA) have been applied for patient safety risk modelling in healthcare [1-2],[7],[10],[12]. Fault tree analysis also extensively used as a powerful technique in health related risk analysis from both qualitative and quantitative perspectives [2],[6].[12]. Hyman and Johnson [9] present a FTA of the patient harm-related clinical alarms failures. Park and Lee [2] constructed a FTA of hand washing process to investigate the causes for faults in hygiene management, the possibility of failure of the top event is calculated from the possibilities of failure of its components according to the extension principle [3],[6].

II. Fuzzy Sets

A fuzzy set is defined by a membership function from the universal set to the interval [0,1], as given below;

$$\mu_A(x): X \to [0,1] \tag{1}$$

, here $\mu_A(x)$ gives the degree of belongingness of X in the set A. A fuzzy set A can be expressed as follows:

$$\tilde{A} = \{(x, \mu_A(x)) : x \in X\} \tag{2}$$

III. Level (λ, ρ)Inter-Valued trapezoidal Fuzzy Numbers

The i-v fuzzy set $\tilde{A}$ indicates that, when the membership grade of x belongs to the interval $\lfloor \mu_{\tilde{A}^L}(x), \mu_{\tilde{A}^U}(x) \rfloor$ the largest grade is $\mu_{\tilde{A}^U}(x)$ and the smallest grade is $\mu_{\tilde{A}^L}(x)$

$$\mu_{\tilde{A}^L}(x) = \begin{cases} \frac{\lambda(x-b)}{c-b} b \le x \le c \\ \lambda c \le x \le d \\ \frac{\lambda(e-x)}{e-d} d \le x \le e \\ 0 otherwise \end{cases} \tag{3}$$

Therefore, $\tilde{A}^L = (b, c, d, e : \lambda) b < c < d < e$

$$\mu_{\tilde{A}^U}(x) = \begin{cases} \frac{\rho(x-a)}{c-a} a \le x \le c \\ \rho c \le x \le d \\ \frac{\rho(f-x)}{(f-d)} d \le x \le f \\ 0 otherwise \end{cases} \tag{4}$$

Therefore $\tilde{A}^U = (a, c, d, f : \rho), a < c < d < f$, Consider the case in which $0 < \lambda \le \rho \le 1$ and $a < b < c < d < e < f$.

From (3) and (4) we obtain $\tilde{A} = [\tilde{A}^L, \tilde{A}^U] [b, c, d, e; \lambda), (a, c, d, f; \rho)][b, c, d, e; \lambda), (a, c, d, f; \rho)]$, Which is called the level$(\lambda, \rho) i -$ v trapezoidal fuzzy number. The intervalued trapezoidal fuzzy numbers shown in fig1.

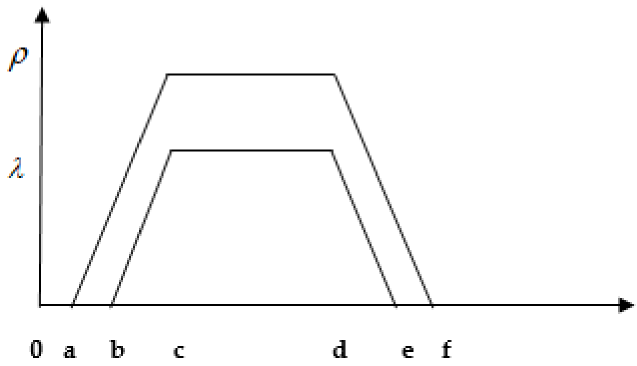

Fig1. Intervalued trapezoidal fuzzy numbers

IV. Level (λ, ρ)Inter-Valued Pentagonal-Triangular Fuzzy Numbers

The i-v pentagonal-triangular fuzzy numbers indicates that, when the membership grade of x belongs to the interval $[\mu_{\tilde{A}^L}(x), \mu_{\tilde{A}^U}(x)]$the largest grade is $\mu_{\tilde{A}^U}(x)$ and the smallest grade is $\mu_{\tilde{A}^L}(x)$ is given by following equations.

$$\mu_{\tilde{A}^L}(x) = \begin{cases} \frac{\lambda(x-a)}{c-a} a \leq x \leq c \\ \frac{\lambda(e-x)}{e-c} c \leq x \leq e \\ 0 otherwise \end{cases} \tag{5}$$

Therefore, $\tilde{A}^L = (a, c, e{:}\, \lambda) a < c < e$

$$\mu_{\tilde{A}^U}(x) = \begin{cases} \frac{\lambda(x-a)}{b-a} a \leq x \leq b \\ \lambda + \frac{x-b}{c-b}(\rho - \lambda) b \leq x \leq c \\ \lambda + \frac{x-d}{c-d}(\rho - \lambda) c \leq x \leq d \\ \frac{\lambda(e-x)}{(e-d)} d \leq x \leq e \\ 0, otherwise \end{cases} \tag{6}$$

Therefore $\tilde{A}^U = (a, b, c, d, e,{:}\, \rho), a < b < c < d < e,$ .Consider the case in which $0 < \lambda \leq \rho \leq 1$ and $a < b < c < d < e.$ from (5) and (6) we obtain $\tilde{A} = [\tilde{A}^L, \tilde{A}^U] = [(a, c, e; \lambda), (a, b, c, d, e; \rho)]$, Which is called the level (λ, ρ) $i-$v pentagonal-triangular intervalued fuzzy numbers.

The intervalued pentagonal-triangular fuzzy numbers is shown in fig2. $A_l^U(\alpha)$ indicate left upper α - cut $A_l^L(\alpha)$ for left lower α - cut , $A_r^L(\alpha)$ for right lower α - cut and $A_r^U(\alpha)$ indicate right upper α - cut.

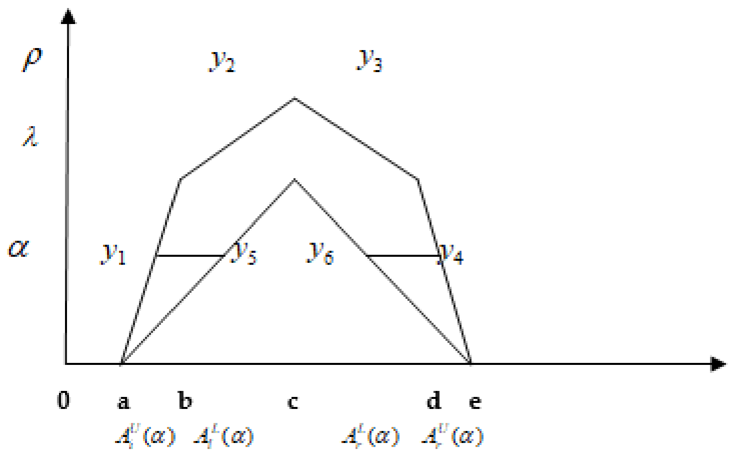

Fig2. Intervalued pentagonal-triangular fuzzy numbers

Corresponding to each curve, the x coordinate corresponding to α - cut and y coordinate given by

$$
\begin{aligned}
x_1 &= a + \tfrac{\alpha}{\lambda}(b-a) & y_1 &= \lambda\left(\tfrac{x-a}{b-a}\right)\\
x_2 &= b + \tfrac{\alpha-\lambda}{\rho-\lambda}(c-b) & y_2 &= \left\{\lambda + \tfrac{x-b}{c-b}(\rho-\lambda)\right\}\\
x_3 &= d + \tfrac{\alpha-\lambda}{\rho-\lambda}(c-d) & y_3 &= \left\{\lambda + \tfrac{x-d}{c-d}(\rho-\lambda)\right\}\\
x_4 &= e + \tfrac{\alpha}{\lambda}(d-e) & y_4 &= \lambda\left(\tfrac{x-e}{d-e}\right)\\
x_5 &= a + \tfrac{\alpha}{\lambda}(c-a) & y_5 &= \lambda\left(\tfrac{x-a}{c-a}\right)\\
x_6 &= e + \tfrac{\alpha}{\lambda}(c-e) & y_6 &= \lambda\left(\tfrac{x-e}{c-e}\right)
\end{aligned}
\qquad (7)
$$

V. α - cut and Signed Distance of Pentagonal Triangular Intervalued Fuzzy Numbers [11]:

if $0 \le \alpha$ $0 \le \alpha < \lambda$ then α cut of $\widetilde{A}$ is $A(\alpha) = \{x/\mu_{\tilde{A}^U}(x) \ge \alpha\} - \{x/\mu_{\tilde{A}^L}(x) \ge \alpha\} = [A_l^U(\alpha), A_l^L(\alpha)] \cup [A_r^L(\alpha), A_r^U(\alpha)]$; Otherwise, for $\lambda \le \alpha \le \rho$, the α cut of $\widetilde{A}$ is $[A_l^U(\alpha), A_r^U(\alpha)]$

$d^*(a,0) = A_l^U(\alpha), d^*(A_l^L(\alpha),0) = A_l^L(\alpha), d^*(A_r^L(\alpha),0) = A_r^L(\alpha), d^*(A_r^U(\alpha),0) = A_r^U(\alpha)$.

Therefore the signed distance [11] of the interval $\left[A_l^U(\alpha), A_l^L(\alpha)\right]$ from 0 can be defined as follows:

$$d^*([A_l^U(\alpha), A_l^L(\alpha)],0) = \tfrac{1}{2}\left(d^*(A_l^U(\alpha),0) + d^*(A_l^L(\alpha),0)\right) = \tfrac{1}{2}(A_l^U(\alpha), A_l^L(\alpha)) = \tfrac{1}{2}\left[a + (b-a)\tfrac{\alpha}{\lambda} + a + (c-a)\tfrac{\alpha}{\rho}\right] = \tfrac{1}{2}\left[2a + \tfrac{\alpha}{\lambda}(b+c-2a)\right] \qquad (8)$$

Similarly $d^*([A_r^L(\alpha), A_r^U(\alpha),0]) = \tfrac{1}{2}\left[e + \tfrac{\alpha}{\lambda}(d-e) + e + \tfrac{\alpha}{\lambda}(c-e)\right] = \tfrac{1}{2}\left[2e + \tfrac{\alpha}{\lambda}(d+c--2e)\right]$

(9)

When $[A_l^U(\alpha), A_l^L(\alpha)] \cap [A_r^L(\alpha), A_r^U(\alpha)] = \Phi$, the signed distance of $[A_l^U(\alpha), A_l^L(\alpha)] \cup [A_r^L(\alpha), A_r^U(\alpha)]$ from 0 can be defined as $d^*([A_l^U(\alpha), A_l^L(\alpha)] \cup [A_r^L(\alpha), A_r^U(\alpha)], 0) = \tfrac{1}{2}[d^*([A_l^U(\alpha), A_l^L(\alpha)], 0) + d^*([A_r^L(\alpha), A_r^U(\alpha)], 0)]$ $= \tfrac{1}{4}\left[2(a+e) + \tfrac{\alpha}{\lambda}(d + 2c - 2a + b - 2e)\right]$

(10)

For $\lambda \le \alpha \le \rho$, then the signed distance from $\widetilde{A}$ to 0 is

$d([A_l^U(\alpha), A_r^U(\alpha): \alpha], 0) = \tfrac{1}{2}\left[(b+d) + \tfrac{\alpha-\lambda}{\rho-\lambda}(2c-b-d)\right]$

(11)

V(I). Defintion Let $\tilde{A} = [(a,b,c,d,e;\rho),(a,c,e;\lambda)] \in F_{IV}(\lambda,\rho), \tilde{0} \in F_p$, the signed distance of $\tilde{A}$ from $\tilde{O}$ is defined as follows:

For $0 \le \lambda < \rho \le 1$

$$d(\tilde{A},\tilde{0}) = \frac{1}{\lambda}\int_0^{\lambda}\frac{1}{4}\left[2(a+e)+\frac{\alpha}{\lambda}(d+2c-2a+b-2e)\right]d\alpha + \frac{1}{\rho-\lambda}\int_{\lambda}^{\rho}\frac{1}{2}\left[(b+d)+\frac{\alpha-\lambda}{\rho-\lambda}(2c-b-d)\right]d\alpha$$

$$= \frac{1}{4}\left[2a+2e+(d+2c+b-2a-2e)\frac{\lambda}{2}\right]+\frac{1}{2}\left[(b+d)+\frac{1}{2}(2c-b-d)(\rho-\lambda)\right]$$

$$= \frac{1}{4}\left[2(a+e+b+d)+(3b+3d-2a-2e)\lambda+2(c-b-d)\rho\right] \tag{12}$$

Now we set $\frac{1}{2}d(\tilde{A},\tilde{0})$ as the defuzzified value of fuzzy numbers

Now using definition we obtained the following estimate of the reliability of system is

$$= \frac{1}{8}[2(a+e+b+d)+(3b+3d-2a-2e)\lambda+2(c-b-d)\rho] \tag{13}$$

V(II). COG method: COG method is one of the most applicable method to defuzzified the fuzzy numbers and is given by

$$x^* = \frac{\int x\mu_{\tilde{A}}(x)}{\int \mu_{\tilde{A}}(x)}dx$$

$$x^{*U} = \left\{\frac{\int_e^b x.\lambda\left(\frac{x-a}{b-a}\right)dx+\int_b^c x\left\{\lambda+\frac{x-b}{c-b}(\rho-\lambda)\right\}dx+\int_c^d x\left\{\lambda+\frac{x-d}{c-d}(\rho-\lambda)\right\}dx+\int_d^e x.\lambda\left(\frac{x-e}{d-e}\right)dx}{\int_e^b .\lambda\left(\frac{x-a}{b-a}\right)dx+\int_b^c \left\{\lambda+\frac{x-b}{c-b}(\rho-\lambda)\right\}dx+\int_c^d \left\{\lambda+\frac{x-d}{c-d}(\rho-\lambda)\right\}dx+\int_d^e .\lambda\left(\frac{x-e}{d-e}\right)dx}\right\}$$

$$x^{*L} = \left\{\frac{\int_a^c x.\lambda\left(\frac{x-a}{c-a}\right)dxdx+\int_c^e x.\lambda\left(\frac{x-e}{c-e}\right)dx}{\int_a^c .\lambda\left(\frac{x-a}{c-a}\right)dxdx+\int_c^e .\lambda\left(\frac{x-e}{c-e}\right)dx}\right\}$$

$$x^{*U} = \frac{\frac{1}{6}[\lambda(3c^2+e^2+bc+ed-a^2-3d^2-ab-cd)+\rho(d^2+cd-b^2-bc)]}{\frac{1}{2}[\lambda(c+e-d-a)+\rho(d-b)]} \tag{14}$$

$$x^{*L} = \frac{\frac{1}{6}\lambda(e^2-a^2+ec-ac)}{\frac{1}{2}\lambda(e-a)} \tag{15}$$

, simplify this we obtain

$$x^{*L} = \frac{1}{3}(a+c+e) \tag{16}$$

Then mean of both defuzzified value is the estimate failure probability and is given by

$$x^* = \frac{1}{2}\left(x^{*U}+x^{*L}\right) \tag{17}$$

$$\begin{pmatrix} 0.001551120, 0.00205181, 0.0030079695, 0.003964129, 0.004705971 & : 1 \\ 0.001551120, 0.0030079695, 0.004705971 & : 0.8 \end{pmatrix}$$

TABLE 1: Fuzzy operation of two intervalued pantagonal-triangular fuzzy numbers

OPERATION	**PENTAGONAL-TRIANGULAR FUZZY INTERVALUED NUMBERS**
MULTIPPLICATION	$\begin{pmatrix} \boldsymbol{a_1, b_1, c_1, d_1, e_1: \rho} \\ \boldsymbol{a_1,, c_1, e_1: \lambda} \end{pmatrix} \times \begin{pmatrix} \boldsymbol{a_2, b_2, c_2, d_2, e_2: \rho} \\ \boldsymbol{a_2, c_2, e_2: \lambda} \end{pmatrix} = \begin{pmatrix} \boldsymbol{a_1.a_2, b_1b_2, c_1c_2, d_1d_2, e_1e_2: \rho} \\ \boldsymbol{a_1.a_2, c_1c_2, e_1e_2: \lambda} \end{pmatrix}$
COMPLEMENT	$\boldsymbol{1} - \begin{pmatrix} \boldsymbol{a, b, c, d, e: \rho} \\ \boldsymbol{a, c, e: \lambda} \end{pmatrix} = \begin{pmatrix} \boldsymbol{1-e, 1-d, 1-c, 1-b, 1-a: \rho} \\ \boldsymbol{1-e, 1-c, 1-a: \lambda} \end{pmatrix}$

Definition 1 Let $\tilde{A} = (a_1, b_1, c_1, d_1, e_1 : \rho),(a_1, c_1, e_1 : \lambda)$ and $\tilde{B} = (a_2, b_2, c_2, d_2, e_2 : \rho),(a_2, c_2, e_2 : \lambda)$ be two i-v pentagonal-tringular fuzzy numbers then the failure possibility $F(\tilde{A} \cup \tilde{B})$ for $\tilde{A} > 0$ and $\tilde{B} > 0$ can be defined using OR operator [8] as

$$F(\tilde{A} \cup B) = 1\Theta_{T_w} \left[(1\Theta_{T_w} F(\tilde{A})) \otimes_{T_w} (1\Theta_{T_w} F(\tilde{B}))\right]$$

Definition 2 Let $\tilde{A} = (a_1, b_1, c_1, d_1, e_1 : \rho),(a_1, c_1, e_1 : \lambda)$ and $\tilde{B} = (a_2, b_2, c_2, d_2, e_2 : \rho),(a_2, c_2, e_2 : \lambda)$ be two i-v pentagonal-tringular fuzzy numbers then the failure possibility $F(\tilde{A} \cap \tilde{B})$ for $\tilde{A} > 0$ and $\tilde{B} > 0$ can be defined using AND operator [8] as

$$F(\tilde{A} \cap B) = F(\tilde{A}) \otimes_{T_w} F(\tilde{B})$$

VI. Example

FTA of a medication pump failing to deliver medication [4]

The FTA of a medication pump failing to deliver medication to a patient is shown in Fig.3[4]. This fault tree has four combination of failures i.e. medication not delivered to patient, immediately below the top event is an OR gate meaning that any individual item below the gate is sufficient by itself to cause the next higher level failure state. For example, pump failure, clamp not removed from tube, pump not activated, and tubing kinked by patient movement are each independently work. In this example, the pump and the alarm work together. Pump failure event occurs due to two events (the pump stops and the alarm does not alert to the practitioner regarding the pump stopping) connected by an AND gate. The pump stops due to either an electrical power failure, a pump motor failure, or tubing occlusion. In this fault tree, we have considered three human errors plus one patient factor. Marx and slonim[1] considered the values of failure probabilities of all the basic events as 0.001(column 3 of table 4) However, this could not be possible for real system, and so we have considered these values as different pentagonal triangular intervalued fuzzy numbers as given in table 4(column 4).

Table 4. Failure probability in pentagonal triangular intervalued fuzzy numbers

Basic event	Failure possibility	Crisp value	TPFNs representation
A	$\tilde{q}_A$	0.001	$\begin{pmatrix} 0.0006, 0.0008, 0.001, 0.0012, 0.0015 : 1 \\ 0.0006, 0.001, 0.0015,: 0.8 \end{pmatrix}$
B	$\tilde{q}_B$	0.001	$\begin{pmatrix} 0.0006, 0.0008, 0.001, 0.0012, 0.0015 : 1 \\ 0.0006, 0.001, 0.0015,: 0.8 \end{pmatrix}$
C	$\tilde{q}_C$	0.001	$\begin{pmatrix} 0.00055, 0.0007, 0.001, 0.0013, 0.0014 : 1 \\ 0.00055, 0.001, 0.0014 : 0.8 \end{pmatrix}$
D	$\tilde{q}_D$	0.001	$\begin{pmatrix} 0.0006, 0.0007, 0.00095, 0.0012, 0.00145 : 1 \\ 0.0006, 0.00095, 0.00145 : 0.8 \end{pmatrix}$
E	$\tilde{q}_E$	0.001	$\begin{pmatrix} 0.0005, 0.0007, 0.001, 0.0013, 0.0016 : 1 \\ 0.0005, 0.001, 0.0016 : 0.8 \end{pmatrix}$
F	$\tilde{q}_F$	0.001	$\begin{pmatrix} 0.0005, 0.0007, 0.001, 0.0013, 0.0016 : 1 \\ 0.0005, 0.001, 0.0016 : 0.8 \end{pmatrix}$
G	$\tilde{q}_G$	0.001	$\begin{pmatrix} 0.00055, 0.00065, 0.000975\ 0.0013, 0.0015 : 1 \\ 0.00055, 0.000975, 0.0015 : 0.8 \end{pmatrix}$
H	$\tilde{q}_H$	0.001	$\begin{pmatrix} 0.0005, 0.0007, 0.001, 0.0013, 0.0016 : 1 \\ 0.0005, 0.001, 0.0016 : 0.8 \end{pmatrix}$

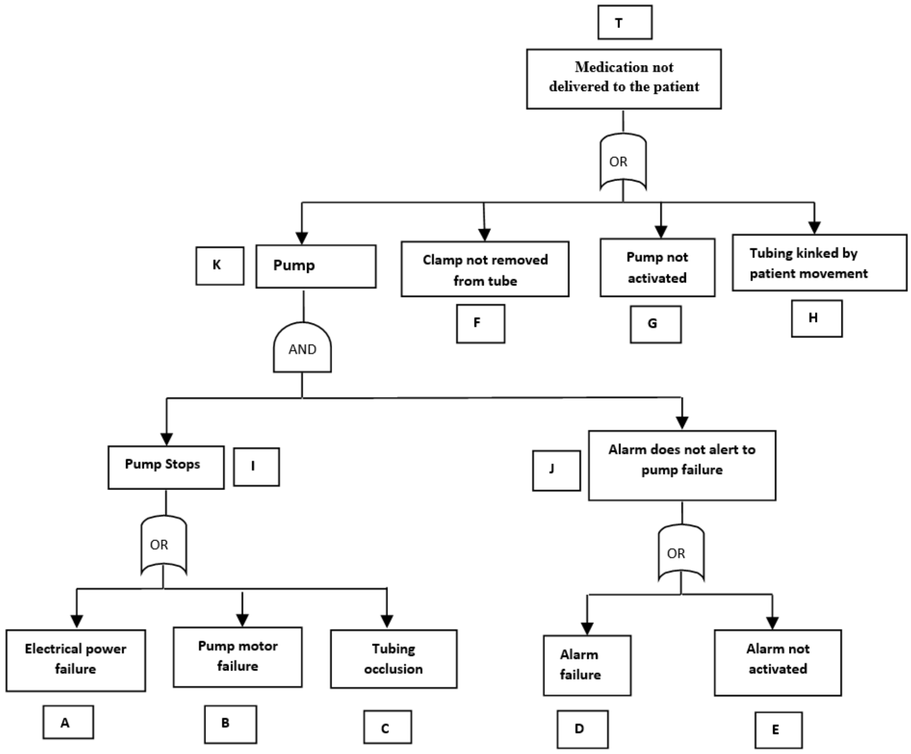

Fig.3. A medication pump fault tree with human error factor failing to deliver medication [1]

Mathematical expression of event is given by

$$\begin{aligned} T &= K \cup F \cup G \cup H \\ &= (I \cap J) \cup F \cup G \cup H \\ &= ((A \cup B \cup C) \cap (D \cup E)) \cup F \cup G \cup H \end{aligned} \tag{14}$$

And mathematical formula of this expression is given as :

$$\begin{aligned} q_{T_1} &= 1-[(1-q_K)\times(1-q_F)\times(1-q_G)\times(1-q_H)] \\ &= 1-[(1-q_I\times q_J)\times(1-q_F)\times(1-q_G)\times(1-q_H)] \\ &= 1-[(1-(1-(1-q_A)\times(1-q_B)\times(1-q_C)) \\ &\qquad \times(1-(1-q_D)\times(1-q_E)))\times(1-q_F)\times(1-q_G)\times(1-q_H)] \end{aligned} \tag{15}$$

VII. Result

By the fuzzy operation with the help of table 1 and table 2 we have the failure probability of top event is

$$\begin{pmatrix} 0.001551120, 0.00205181, 0.0030079695, 0.003964129, 0.004705971 : 1 \\ 0.001551120, 0.0030079695, 0.004705971 \qquad : 0.8 \end{pmatrix}$$

$$\begin{pmatrix} 0.9955294029,\ 0.996035871,\ 0.9969920305,\ 0.99794819,\ 0.99844888;1 \\ 0.9955294029,\quad 0.9969920305,\quad 0.99844888;0.8 \end{pmatrix}$$

VII(I). Conclusion1: Defuzzification by signed distance method, we obtain failure probability of top event from equation 13 is

$x^* = 0.0028696286$ and reliability of top event is 0.97130371

And by COG method we obtain failure probability of top event from equation (15), (16),and (17) is

$x^{*L} = 0.0016765622$

$x^{*U} = 0.0022540335$

Now $x^* = \frac{1}{2}(0.0016765622 + 0.0022540335)$ (18)

VII(II). Conclusion2.: Therefore by COG method the failure probability of top event $x^* = 0.0039305957$ and reliability of top event is 0.996069404

VII(III). Difference Error: the difference in both method is about 0.1060967 % which imply that the COG method and Signed distance method are give similar result.

The fuzzy failure probability and fuzzy reliability in pentagonal-triangular intervalued fuzzy numbers are in fig4 and fig5 respectively.

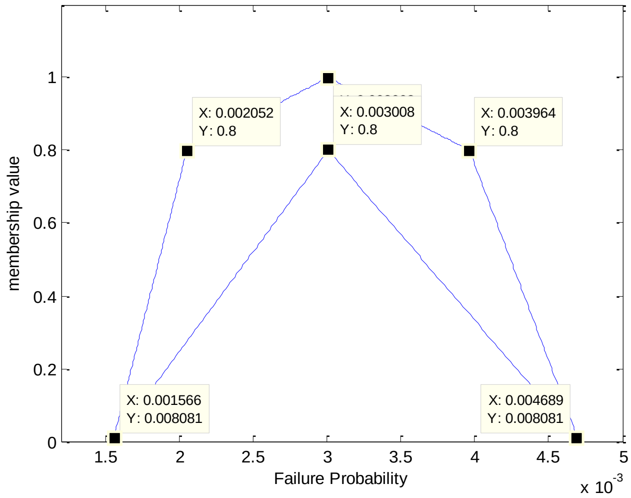

Fig 4.Fuzzy pentagonal-Triangular failure probability

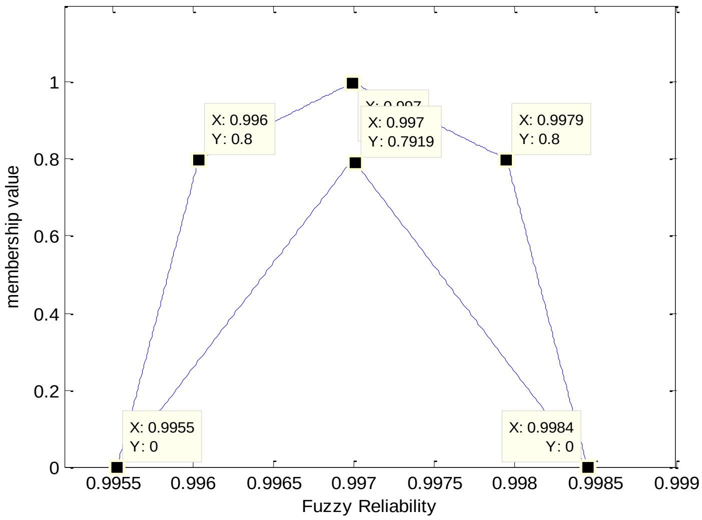

Fig 5.Fuzzy pentagonal-Triangular reliability probability

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Calculating the Variance of the Linear Regression Coefficient

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Abstract

In this paper, we choose such a particular formulation of the problem of calculating linear regression coefficient, when the moments of observation form an arithmetic progression. It is proved that the variance of the trend estimation in this case decreases proportionally to the third degree of the length of the series of observations. If the estimation of a linear trend is based on several independent samples, the integral estimation of the trend is constructed and its variance is determined by special optimization procedure. This procedure is based on simple geometric consideration.

Keywords: linear regression coefficient, variance calculating, independent samples of observations.

1 Introduction

The problem of studying the variance of the linear trend estimation and its dependence on the length of the time series on which this estimate is based is of both theoretical and practical interest. This problem is closely related to the problem of small samples in mathematical statistics. In reliability theory, this problem arises when using linear regression analysis to predict the safety margin of a technical system (see, for example, [1], [2]). This task can be extended to the case when there are several time series, in particular for small-scale production. In this paper we choose such a particular formulation of this problem when the moments of observation form an arithmetic progression. It is proved that the variance of the trend estimation in this case decreases proportionally to the third degree of the length of the series of observations. This makes it possible to use short series of observations to estimate the linear trend. If the estimation of a linear trend is based on several independent samples, the integral estimation of the trend is constructed and its variance is determined.

2 The variance of the estimate of linear trend for a single series of observations

Consider the following linear regression model $x(t) = y(t) + \varepsilon(t),\ y(t) = at + b$. Assume that at times $t_1, \dots, t_n,\ 0 \le t_1 < t_2 < \dots < t_N$, measured values are $y(t_1), \dots, y(t_N)$ with random errors $\varepsilon_1, \dots, \varepsilon_N$. The random variables $\varepsilon_1, \dots, \varepsilon_N$ are assumed to be independent, equally distributed with zero mean and variance σ^2.

To solve this problem, replace the variable $\tilde{t} = t - T_N,\ T_N = \frac{\sum_{k=1}^N t_k}{N}$, and define a linear function

$$\tilde{y}(t) = y(t + T_n) = at + b + aT_N = at + \tilde{b}, \quad \sum_{k=1}^{N} \tilde{t}_k = 0, \quad \tilde{b} = b + aT_N.$$

To do this, we compute $\tilde{t}_k,\ k = 1, \dots, N$, and construct the least squares [3], [4] estimates of the coefficients $a,\ \tilde{b}$ of the linear regression function $\tilde{y}(t) = at + \tilde{b}$ from observations

$$x_1 = \tilde{y}(\tilde{t}_1) + \varepsilon_1, \dots, x_N = \tilde{y}(\tilde{t}_N) + \varepsilon_N.$$

The solution to this problem is a random vector consisting of estimates

$$\hat{a}_N = \frac{\sum_{k=1}^{N} x_k \tilde{t}_k}{\sum_{k=1}^{N} \tilde{t}_k^2},\ \hat{b}_N = \frac{\sum_{k=1}^{N} x_k}{N}$$

of coefficients $a,\ \tilde{b}$ of linear function $\tilde{y}(t)$. The components of this vector have the following averages, variances, and covariance coefficient:

$$M\hat{a}_N = a,\ M\hat{b}_N = \tilde{b},\ D\hat{a}_N = \frac{\sigma^2}{\sum_{k=1}^{N} \tilde{t}_k^2},\ D\hat{b}_N = \frac{\sigma^2}{N},\ cov(\hat{a}_N, \hat{b}_N) = 0. \quad (1)$$

Of greatest interest to us is the denominator $S(N) = \sum_{K=1}^{N} \tilde{t}_k^2$ in Formula (1) in the definition of the variance $D\hat{a}_N$. To simplify the calculations, assume that $\tilde{t}_{k+1} - \tilde{t}_k = 1, \dots N - 1$. By induction at $n = 1,2, \dots$, it is easy to obtain equalities

$$S(2n + 1) = \frac{2n^3}{3} + n^2 + \frac{n}{3},\ S(2n) = \frac{2n^3}{3} - \frac{n}{6}. \quad (2)$$

Indeed, the definition implies that $S(2n + 1) = 2R(n),\ R(n) = \sum_{k=1}^{n} k^2$. Looking for $R(n)$ in the form $R(n) = a_0 + a_1 n + a_2 n^2 + a_3 n^3$. Then we have the equality $R(n + 1) = R_n + (n + 1)^2$ and so obtain the relation

$$a_0 + a_1 n + a_2 n^2 + a_3 n^3 + (n + 1)^2 = a_0 + a_1(n + 1) + a_2(n + 1)^2 + a_3(n + 1)^3.$$

Removing the parentheses and leading like that, we get the following equalities:

$$a_0 + 1 = a_0 + a_1 + a_2 + a_3,\ a_1 + 2 = a_1 + 2a_2 + 3a_3,\ a_2 + 1 = a_2 + 3a_3.$$

The solution of this system of linear algebraic equations is $a_1 = \frac{1}{6},\ a_2 = \frac{1}{2},\ a_3 = \frac{1}{3}$. Since $R(1) = 1$, then $a_0 = 0$. The first equality in Formula (2) is proved.

Now calculate $S(2n) = \frac{2Q(n)}{2^2}$, $Q(n) = 1^2 + 3^2 + \dots + (2(n - 1) + 1)^2$. Looking for $Q(n)$ in the form of $Q(n) = b_0 + b_1 n + b_2 n^2 + b_3 n^3$. Then the equalities $Q(n + 1) = Q(n) + (2n + 1)^2$ are true and the following equalities are fulfilled

$$b_0 + b_1 n + b_2 n^2 + b_3 n^3 + (2n + 1)^2 = b_0 + b_1(n + 1) + b_2(n + 1)^2 + b_3(n + 1)^3.$$

Removing the parentheses and leading like that, we get the following equality:

$$b_0 + 1 = b_0 + b_1 + b_2 + b_3,\ b_1 + 4 = b_1 + 2b_2 + 3b_3,\ b_2 + 4 = b_2 + 3b_3.$$

The solution to this system of linear algebraic equations is $b_1 = -\frac{1}{3},\ b_2 = 0,\ b_3 = \frac{4}{3}$. Because $Q(1) = 1$, then $b_0 = 0$ and means

$$b_0 = 0,\ b_1 = -\frac{1}{3},\ b_2 = 0,\ b_3 = \frac{4}{3}.$$

The second equality in the formula (2) is also proved.

Formula (2) leads to asymptotic relations:

$$S(2n) \sim S(2n+1) \sim \frac{2n^3}{3},\ n \to \infty. \tag{3}$$

However, in the applied plan the values $S(2n)$, $S(2n+1)$ are of great interest for small values n.

We now give the results of numerical calculations in the following tables.

N	2	3	4	5	6	7	8	9	10	11	12	13
$S(N)$	0.5	2	5	10	17.5	28	42	60	82.5	110	143	182

N	14	15	16	17	18	19	20	21	22	23
$S(N)$	227.5	280	340	408	484.5	570	665	770	885.5	1012

Tables of $S(N)$

4 Estimation of linear regression coefficient on several independent samples

Assume that there are m independent samples $i = 1, \dots, m$. The sample i has volume N_i. The estimate $\hat{a}^{(i)}$ of linear regression coefficient a satisfies the equalities

$$\hat{a}^{(i)} = a,\ D\hat{a}^{(i)} = d_i = \frac{\sigma^2}{S(N_i)},\ i = 1, \dots, m. \tag{4}$$

We will look for an estimate of $\hat{a}$ of the linear regression coefficient a in the form

$$\hat{a} = \sum_{i=1}^{m} c_i \hat{a}^{(i)},\ \sum_{i=1}^{m} c_i = 1,\ D\hat{a} = \sum_{i=1}^{m} c_i^2 d_i. \tag{5}$$

Choice of coefficients c_i, $i = 1, \dots, M$, are produced from the minimum condition

$$\min\left(\sum_{i=1}^{m} c_i^2 d_i:\ \sum_{i=1}^{m} c_i = 1\right). \tag{6}$$

Make the change of variables $f_i = c_i\sqrt{d_i}$, $i = 1, \dots, m$ and we can rewrite the optimization problem (6) in the form

$$\min\left(\sum_{i=1}^{m} f_i^2 = F:\ \sum_{i=1}^{m} \frac{f_i}{\sqrt{d_i}} = 1\right). \tag{7}$$

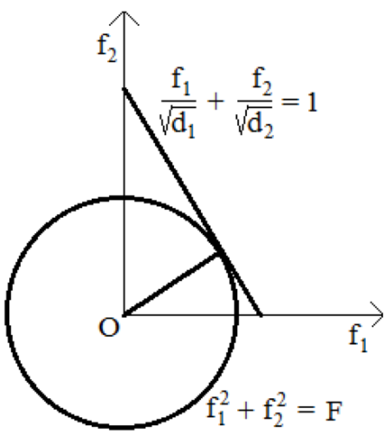

Fig 1. Geometric interpretation of the optimization problem.

From simple geometric considerations we obtain the following solution to the optimization problem (7): for $i = 1, \dots, m$

$$f_i = d_i^{-1/2}\left(\sum_{k=1}^{m} d_k^{-1}\right)^{-1},\ c_i = d_i^{-1}\left(\sum_{k=1}^{m} d_k^{-1}\right)^{-1},\ \sum_{i=1}^{m} c_i^2 d_i = \left(\sum_{k=1}^{m} d_k^{-1}\right)^{-1}. \quad (8)$$

Thus, from Formulas (5), (8) we finally obtain:

$$\hat{a} = \sum_{i=1}^{m} d_i^{-1}\left(\sum_{k=1}^{m} d_k^{-1}\right)^{-1}\hat{a}^{(i)} = \frac{\sum_{i=1}^{m} S(N_i)\hat{a}^{(i)}}{\sum_{k=1}^{m} S(N_k)},\ D\hat{a} = \left(\sum_{k=1}^{m} d_k^{-1}\right)^{-1} = \frac{\sigma^2}{\sum_{k=1}^{m} S(N_k)}. \quad (9)$$

4 Conclusion

The results show that the coefficient of linear regression has a variance significantly lower than the variance of the free term. This makes it possible to raise the question of the evaluation of this coefficient separately from the evaluation of the free member. The resulting estimate can be used to predict relatively short series of observations.

The results of the estimation of the linear regression coefficient for several independent series of observations suggest that it is possible to estimate the linear trend coefficient fairly economically and accurately for a small group of series of observations. This result is obtained by simple geometric considerations that are based on the basic properties of the variance of a random variable.

5 Acknowledgements

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Refining Stochastic Models of Critical Infrastructures by Observation

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Abstract

The simulation of cascading effects in networks of critical infrastructures (CIs) can be approached in various ways, all of which at some point call for the specification of (numeric) model parameters. Taking stochastic models as one popular class of methods, finding proper settings for the values that determine the stochastic models can be a challenge. In this work, we describe a method of graphical specification of a probability value on a qualitative scale, and how to convert and use the obtained value as a prior for Bayesian statistics. The connection is made to the point of having the initial value specified only as an "initial guess", which can be refined using Bayesian statistics. Eventually, under consistency conditions depending on the application, this amounts to an online learning approach that takes the parameter to convergence towards their true values, based on the user's subjective initial guess, but never challenging a person to give a reliable number for a probabilistic parameter.

Keywords: simulation, cascading effect, risk management, stochastic model, security

I. Introduction

Among the biggest challenges in stochastic models is probability. Scientists often provide people with sophisticated model having beautiful theoretical properties, but left with the highly nontrivial challenge of finding proper values for a set of parameters, many of which are probabilities. What if the person simply does not have these values or cannot reliably estimate them? This work proposes to avoid the issue of pulling numbers "out of the air", by instead resorting to purely graphical method and machine learning to poll or estimate probabilities.

Probabilistic models have the appeal of being often easy to define and plausible to use, yet the intuitiveness of the model specification turns into a difficulty when creating a model instance in many cases. Suppose that the model includes some probability parameter p that "simply" quantifies the likelihood of some event to occur; for example, the impact of an incident on related parts in a system (e.g., a dependent infrastructure). Likewise, we may use a parameter (probability) p to describe the likelihood of a threat along risk analysis, or call p the likelihood for human error to bring the human element into a model. How do we set such values in practice? It is tempting to use them in a model because they are easy to argue and statistics enjoys a solid mathematical fundament, but the practitioner facing the challenge of assigning some reasonable value to the variable p may find this to be an almost impossible task to accomplish reliably. In many cases, the setting of such parameters thus resorts to choices on qualitative scales, say, defining the probability just to be "low" or "high", with the meaning of these values remaining vague or defined by representative standard values specified elsewhere. In many practical cases when people try to *apply* or *use* (not define or invent) stochastic models, the choice of probability parameters is a matter of asking experts for numbers that they simply do not have. This can practically limit the applicability of such models despite any theoretical beauty.

Statistics has lots to offer to people seeking to estimate parameters of stochastic models, since the whole theory of point- and interval estimation is dedicated to the problem of finding values or ranges of values for unknown quantities. Common to most of these techniques is their use of empirical data to compute the estimators. In risk management, and particularly in the context of critical infrastructures (CIs), the situation is just not satisfying the assumptions: data is scarce, and we cannot expect having hundreds of data samples from past incidents in a critical infrastructure (simply because the CI would not have survived the necessary lot of incidents to gather enough data for a statistically reliable estimation).

Instead, we need to come up with a reasonable initial guess for the probabilistic parameters and look for a way to refine that value upon continuous experience. Bayesian estimation thus appears as a reasonable way to go, and this work describes a very straightforward and easy to implement version of such a Bayesian estimation approach, where we explicitly exploit the absence of much prior knowledge as an advantage. Indeed, if there is not too much robust prior knowledge about how a probability parameter should be set numerically, this also means that any choice is as good as the other. While it would not make sense to step forward by just picking parameter values at random, the Bayesian method is much more elegant in letting us choose a prior *distribution* to our own convenience, and – realistically reflecting the uncertainty of the person instantiating the model – leaving the parameter p actually unspecified in the beginning. The actual value for p is then obtained from the prior distribution in first place, and iteratively refined by bringing in experience about the model performance to continuously refine it towards an accurate setting for the real model.

To the end of using that method for model parameterization, we thus have to devise (i) a method to pick a reasonable initial guess for some (generic) probability parameter p (Section II will describe an example model for illustration), and (ii) describe a method to define that guess, which assures that we will eventually end up with the correct value for p over the long run. We dedicate Section III to this.

As a running example, we will pick a specific model to describe critical infrastructure dependencies, to study cascading effects by simulation. Our choice of the CERBERUS model [1] is arbitrary here, and can be replaced by any other stochastic model based on Markov chains, percolation theory, or others. The palette is rich, and we refer the reader to [2–11] for models to which our work may offer an aid to get a practical instance, meaning concrete numeric settings, for the involved probability parameters.

II. The CERBERUS Risk Simulation Model

Consider a network of interdependent critical infrastructures that we represent as a directed graph $G = (V, E)$ with edges $A \to B$ meaning that CI B somehow depends on CI A. For example, A could provide energy, water, food, transport, etc. for B. To all infrastructures in the (node) set V, we assign one out of k possible operational states, reflecting their degree of "health". Typically, this state ranges from "fully functional" (state 1) to "outage" (state k), with intermediate states from 2 to $k-1$ corresponding to ascending limitations in a CI's service(s). The dependency of a CI B on one or more of its providers may be of arbitrary form and dynamic. For example, a CI may have providers that it vitally depends on, or whose service can be substituted for a limited period of time (e.g., emergency power generators can cover a power outage for some time, until they run out of fuel). Other dynamics of dependency may involve the kind of service more explicitly, say, if CI B relies on online-services of A (e.g., an outsourced data center) in order to coordinate the shipping of goods from another provider C to B.

Commonly, authors distinguish the type of dependency here, dividing it into physical dependencies (e.g., supply with physical utilities), cyber-dependencies (e.g., communication and data exchange), geographic dependency (often physical proximity or reachability), and others (cf. [6,12–15]), including temporal dependencies (that are outside our scope here since we look for the setting of probability parameters).

To study cascading effects in such models, we thus need to describe what happens to an infrastructure if its providers fail. While there is lots of work on understanding dependencies (see [16] for a considerable collection of respective references), quantitative studies on how to describe the parameter value for some stochastic model are rare (not so the models themselves; see the references in the introduction). In this context, we want to highlight the work in [16], where an empirical study on how strong the impact of several critical infrastructures may be on others is provided.

The CERBERUS model uses precisely such information to describe an infrastructure model and cascading effects therein in the following way:

- The behavior of a CI B is described by a bipartite graph (see Figure 1):
    - The top layer has exactly k nodes, one for each operational state in which the CI can be
    - The bottom layer has k nodes per CI A that CI B depends on. That is, each supplier CI A is represented in the graph model as its own set of k nodes, one per operational state of CI A, and every other supplier of B having its own copy of these k input nodes.
- The bipartite inner graph is complete, meaning that there is an edge from each state node of each supplier to the overall state node of CI B. These edges are annotated by probabilities, indicating how likely it is that CI B moves into state j, if infrastructure A is in state ℓ. For each $\ell \in \{1,2,\dots,k\}$, we thus have to specify a probability $p_{\ell j} = \Pr(\text{CI B is put into state } j |\text{ CI } A \text{ is in state } \ell)$. If the change is a (deterministic) fixed consequence, we can put $p_{\ell j} := 1$ to model this.
- Since the edges connect only two nodes at a time (the model is a graph, not a hypergraph), the effects of a supplier on B are independent on what other suppliers do. Moreover, B can be put into distinct operational states upon different of its providers changing their state individually. Intuitively, this reflects the real world quite well, since a problem at provider A_1 may cause only slight stress for CI B, while another (independent) problem at provider A_2 may have a substantial impact on B's functionality. Thus, there is an aggregation function being applied on the states that probabilistically follow from the supplier states, which in the simplest case is just the maximum of all possible states that the suppliers may put B into. For example, if provider A_1's failure puts B into state "normal" (i.e., no immediate effect), but supplier A_2's outage causes severe problems in B, the overall state of B is the worst of the two, set to be "severe problems".

This kind of maximum-aggregation assumes that higher state indices correspond to more severe problems (taking the lowest state as the best). Logically, it corresponds to an OR, since B has troubles if at least one of its critical providers fails. This logic can be changed into an AND by resorting to a minimum-value aggregation, causing the state of B to remain "healthy", unless all of its providers fail. The proper choice per infrastructure is up to the application.

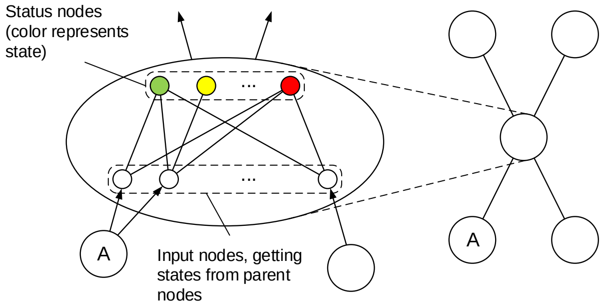


Figure 1 *CERBERUS Model (picture adapted from [1])*

The CERBERUS model includes this simplification to avoid a combinatorial explosion of parameters that would need specification otherwise. For example, the most powerful description of dependency (that includes the above OR/AND dependencies as trivial special cases) is that of a Bayesian network [17]. This approach is similar to the CERBERUS model, however, requires a worst-case exponential number of parameters specified to describe the dependency as a full-fledged conditional *distribution*. The above reduces that number to "only" polynomially many (exactly $k \cdot n$ conditional probability *values*, if k states are used and the CI depends on n other CIs). Since both, A and B have a common set of possible states, the transition regime can be described as a matrix of the general form:

		State of CI B (depending on A's state)			
		1	2	...	k
State if CI A	1	p_{11}^A	p_{12}^A	...	p_{1k}^A
	2	p_{21}^A	p_{22}^A	$\ddots$	p_{2k}^A
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	k	p_{k1}^A	p_{k2}^A	...	p_{kk}^A

The superscript A is here only a reminder that these transitions relate to infrastructure A, and more such matrices would be required to describe the dependency of B on other CIs. The specification is very much like (though not identical) to a transition matrix of a Markov chain, since in each row, there has to be one target state for CI B. Our problem in the following will thus be the specification of these (many) values, using an initial guess and online learning to refine it.

Again, we stress that the choice of this model for illustration is arbitrary, and replaceable by others. The reader feeling more familiar with Markov chains or other models is safe to think along these lines during the remainder of this work. Indeed, we will become more general than the above in considering the estimation of a whole vector of probability values, constrained to form a probability distribution (thus covering the more complex case of Bayesian network specification too).

III. Model Parameterization: Initial Guesses

In absence of empirical data, the best that we can do is resorting to domain expertise, subjective experience and empirical studies as far as they are available (e.g., [16]). However, the problem remains one where experts have to provide (qualitative or better quantitative) values that are usually hard to obtain. One possibility is getting domain experts into discussion to agree on a common assessment (e.g., using systematic methods such as Delphi and/or opinion pooling [18]), which generally means aggregating different assessments into an object (number) that we can start with – an initial guess. Lossless aggregation into a distribution is also possible and has been described for general risk management in [19]; however, this method is out of our scope here, but mentioned as another option to get a prior distribution for Bayesian updating (met later in Section IV)

I. Graphical Specification of Parameters

To avoid asking people for numbers, graphical ways of specifying probabilities and general risk parameters have been developed. One method aiming to help with the quantification of risk as the product of "likelihood" and "impact" is to let experts draw a "risk rectangle", whose horizontal length reflects the person's (subjective) assessment on a range for the unknown likelihood, and the vertical breadth acts as an interval estimate for the potential impacts; see Figure 2 for an illustration. The area of the rectangle *can*, but with care, be associated with the usual formula $likelihood \times impact = risk$, where both inputs are ranges reflecting uncertainty. Intuitively, the larger the rectangle is, the more uncertain would the specification be, stressing that even for small areas, the width and height still need consideration in their own meaning of uncertainty (a very thin rectangle has small area, yet may express large uncertainty about one of the coordinates).
As an initial guess for a parameter, such a graphical method may serve as a replacement for a number, since the actual numeric value is easy for a computer to compile from the rectangle's coordinates.

In any case, this is just a heuristic and there is no formal or scientific reason (so far) why any such graphical method should deliver more reliable results than a direct specification. It is as such a matter of usability and convenience to specify values in this way. This potential benefit becomes even more evident if we transfer the idea to the specification of a whole matrix of values, say, a transition matrix of a Markov chain. Why not think of the matrix as a rectangular grid, on which our task is to place masses, proportionally to as how likely it is that state i will take the chain into the target state j. Returning to the CERBERUS model above, we would, for each supplier CI A, have one such matrix to tell B's target state based on A's current state.

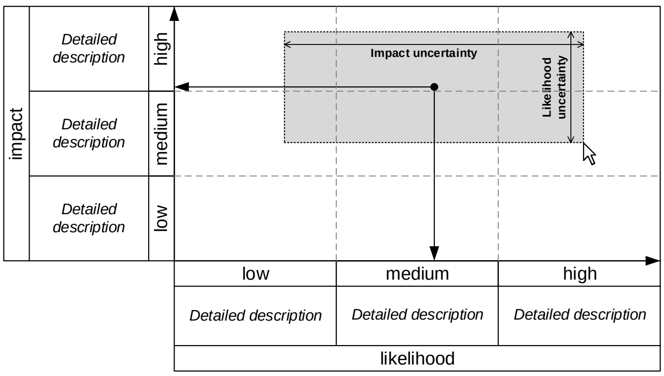
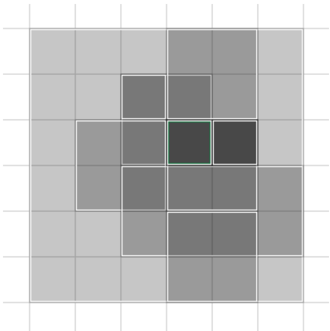


Figure 2 *Graphical Risk Specification Method (picture adapted from* **[20]***)*

The idea is a straightforward extension of the graphical specification from before: assuming that the states are ordered (in ascending or descending levels of criticality), we can go and draw a bunch of rectangles into the grid, which may even overlap, and each of which places some mass onto a cell in the grid, i.e., element in the matrix. The amount of weight being placed is then a matter of how much the rectangle overlaps the respective region. Intuitively, if we draw a rectangle over several cells (horizontally and vertically), we may express something like "any state between i_1 and i_2 may put the dependent CI B into some state between j_1 and j_2" – not becoming too specific on how likely a specific transition is, but only telling what one may think is possible. The more such possibilities are supplied, the more weight accumulates on a cell, and the more likelihood is assigned accordingly. Figure 3 displays the idea, with some example values compiled from the cumulative areas.



$$\rightarrow \begin{pmatrix} \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{4} & \frac{1}{4} & \frac{1}{8} \\ \frac{1}{11} & \frac{1}{11} & \frac{3}{11} & \frac{3}{11} & \frac{2}{11} & \frac{1}{11} \\ \frac{1}{15} & \frac{2}{15} & \frac{1}{5} & \frac{4}{15} & \frac{4}{15} & \frac{1}{15} \\ \frac{1}{14} & \frac{1}{7} & \frac{3}{14} & \frac{3}{14} & \frac{3}{14} & \frac{1}{7} \\ \frac{1}{12} & \frac{1}{12} & \frac{1}{6} & \frac{1}{4} & \frac{1}{4} & \frac{1}{6} \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{4} & \frac{1}{4} & \frac{1}{8} \end{pmatrix}$$

Figure 3 *Graphical Specification of a transition matrix*

II. Prior Distribution for Online Learning

Suppose that we have a set of probabilities $p_1, \dots, p_k$ that jointly form a distribution, i.e., satisfy $p_1 + p_2 + \cdots + p_k = 1$. For the example of the CERBERUS model, given a dependency of CI B on A, such a set would be a matrix as outlined above, or at least a single row in it.

Most likely, the initial guess is inaccurate, subjective, not well founded on empirical data or experience, or suffers from other sources of vagueness. This is most naturally so, since we cannot expect an(y) expert to have precise or objectively reliable figures for likelihoods in a quality better than to the best of her/his knowledge.

It is, however, possible to refine and "correct" these initial guesses in the long run by observing the system, tracking the real state changes, and refine our hypothesis iteratively, knowing that it will converge to the "objective" and hence correct probabilities. The mechanism is Bayesian updating of a properly chosen prior distribution, which makes the whole process even computationally efficient and trivial to implement.

Our choice is the Dirichlet distribution, having $k \geq 2$ parameters $(\alpha_1, \dots, \alpha_k)$ satisfying $\alpha_i > 0$ for all $i = 1, \dots, k$, and the probability density function

$$f_{Dirichlet}(x_1, \dots, x_k | \alpha_1, \dots, \alpha_k) = \frac{\Gamma\left(\sum_{i=1}^{k} \alpha_i\right)}{\prod_{i=1}^{k} \Gamma(\alpha_i)} \prod_{i=1}^{k} x_i^{\alpha_i - 1}.$$

The interesting point for our purpose is the fact that this distribution relates to a vector $\boldsymbol{X} = (X_1, \dots, X_k) \in (0,1)^k$ constrained by $X_1 + \cdots + X_k = 1$, so that it can be used to describe a probability distribution. That is, our sought probability vector, the distribution to be specified, is viewable as a sample of the random vector $\boldsymbol{X}$, whose distribution is Dirichlet with the density as above. Under that perspective, we can equate the desired likelihoods $p_i \coloneqq E(X_i)$ with X_i being the i-th coordinate in $\boldsymbol{X}$.

For the Dirichlet distribution, this expectation is simply

$$E(X_i) = \frac{\alpha_i}{\sum_{i=1}^{k} \alpha_i}$$

Now, suppose that we have an initial guess for the values $p_1, \dots, p_k$; then even without those normalizing to unit sum, we can plainly specify the parameters α_i as $\alpha_i \coloneqq p_i$ to start with, since the denominator in the above expression is nothing else than a normalization, so that the soinstantiated Dirichlet density, encodes our initial guess for the probability parameters by the component-wise expectations.

Remark: The case for a single parameter is treated only slightly different; noting that above, we require at least two values. If there is only a single probability parameter in question, the prior would be the Beta distribution, having the density $f_{Beta}(x|\alpha_1,\alpha_2) := f_{Dirichlet}(x, 1-x|\alpha_1,\alpha_2)$, with the expectation following the same formula as given above. The major (only) difference is that while the Dirichlet distribution describes a set of k probability values, the Beta distribution describes only a single value that is also a probability; in both cases, the last value ($x_2 = 1-x$ or $x_k = 1-x_1-x_2-\cdots-x_{k-1}$) is fixed by its predecessors (not surprisingly so, since we have the constraint of all these values to sum up to 1).

IV. Bayesian Updating

On a level of abstraction, the CERBERUS model is a set of Markov chain instances, where a state transition of a CI triggers another state transition of a dependent CI. Suppose that this switch is observable, i.e., we would note the change in reality, and can relate it to an edge in the model (see Figure 1).

Adopting a Bayesian statistics perspective, the observation is nothing else than data sampled from a distribution whose parameters we seek to estimate. More specifically, consider only the i-th row $\boldsymbol{p}_{i,\cdot}$ in a transition matrix $\boldsymbol{P}$, telling us that if the current state is i, then the next possible states $j \in \{1,2,\dots\}$ will occur with probabilities $p_{i1}, p_{i2}, \dots$. This single row is a categorical distribution, and the values in it are exactly the parameters (the distribution is, in a way, not only determined, but actually directly represented by its parameter set). Now, suppose that an observation is made, which tells that out of the current state i, our system has (physically, in reality) moved into the state j. Formally, this is $\boldsymbol{x} = (0,0,\dots,1,0,0,\dots)$, with only the j-th entry being 1, sampled from the aforementioned categorical distribution $\boldsymbol{p}_{i,\cdot}$ (which in turn is just the i-th row in the transition matrix $\boldsymbol{P}$).

More importantly, this view takes the incoming observations as samples from a 0/1-valued random variable. Such a variable is an indicator, and the expectation of an indicator variable is a probability, thus making the approach meaningful to estimate probability parameters.

Now, let us put this to practice: suppose that we observed the event of our system to have undergone a transition from state i into state j. If the Bayesian prior distribution is a Dirichlet (or Beta), with parameters $\alpha_1,\dots,\alpha_k$ (in the case of a single parameter p to be estimated, we would only have α_1 and α_2, with $p = \frac{\alpha_1}{\alpha_1+\alpha_2}$), the Bayesian update of the row $\boldsymbol{p}_{i,\cdot}$ in the transition matrix P, which is described by a prior distribution with parameter vector $(\alpha_1,\dots,\alpha_k)$, proceeds via the assignment

$$(\alpha_1,\dots,\alpha_{j-1},\alpha_j,\alpha_{j+1},\dots,\alpha_k) \leftarrow (\alpha_1,\dots,\alpha_{j-1},\alpha_j+1,\alpha_{j+1},\dots,\alpha_k),$$

i.e., only the j-th parameter gets increased by 1. What could be simpler? It essentially amounts to counting the occurrences of each transition! Even if several observations are collected in a data vector, say, $\boldsymbol{d} = (n_1, n_2, \dots, n_k)$ with n_1 observed transitions into state 1, another n_2 transitions observed into state n_2, etc., the update to $\boldsymbol{\alpha} = (\alpha_1,\dots,\alpha_k)$ would simply be $\boldsymbol{\alpha} \leftarrow \boldsymbol{\alpha} + \boldsymbol{d}$.

The current estimate $\hat{p}_j$ of the j-th (not precisely known) probability parameter p_j vector is for each $j = 1,2,\dots,k$ given as

$$\hat{p}_j = E(X_j) = \frac{\alpha_j}{\alpha_1 + \alpha_2 + \cdots + \alpha_k}$$

Now, let us suppose that we started from initial values (guesses) $\alpha_1^*, \dots, \alpha_k^*$. What would happen in the long run? If we observe the transition into the j-th state for N times out of $M \gg N$ cases and let $M \to \infty$, then the estimator $\hat{p}_j$ after a total of M updates is

$$\frac{\alpha_j^* + N}{\left(1 - \alpha_j^* + M - N\right) + \left(\alpha_j^* + N\right)};$$

this is easy to see from the fact that we increase the pseudo-count[1] α_j^* for N times, whilst increasing any of the other parameters for the remaining $M - N$ times (whose totality is collected in the term $1 - \alpha_j^* + M - N$). Overall, since the initial guess does not change, the limit is

$$\frac{\alpha_j^* + N}{(1 - \alpha_j^* + M - N) + (\alpha_j^* + N)} \to \frac{N}{M} = p_j,$$

Since this is merely the fraction of "good cases" among "all cases", i.e., by definition the sought probability. The key insight here is that this limit does not depend on the initial guess! That is, no matter if we were wrong with our initial parameter choice (and in most cases, we may have been wrong), the long-run updating will asymptotically "correct" our error automatically. Of course, the speed of convergence depends on how far off the inaccuracy of the initial guess put us away from the real value of p_j. The closer our initial guess has been, the earlier we get into a reasonable proximity of the true value p_j.

Let's also take a closer look at the case of a single parameter: if we don't have a whole Markov chain, but rather a single parameter that describes an event by a probabilistic value, there is no conceptual change to the above. The respective prior has two parameters (α_1, α_2), which we update to $(\alpha_1 + 1, \alpha_2)$ if the event has been observed, or into $(\alpha_1, \alpha_2 + 1)$ if the event did not occur; both cases assume that the parameter p in question describes the probability of the event's occurrence (otherwise, the update would be done with the roles of α_1 and α_2 being switched). We refer to [21] for a fully detailed elaboration of this prior idea, which we here generalized. The reference cited treats the topic in the different direction of using the idea for predictive analytics (see [22] for a survey).

Example for the CERBERUS Model

The application of the above scheme in the CERBERUS model is straightforward, based on what we have: suppose that a history of cascading effects was recently observed in the network of critical infrastructures, or is available from documented cases of incidents or experience. Then, we can consider each part of the chain of events described in the following form: "CI A changed its state from x to y, causing CI B to change its state from u to v". To update our model, we look into the inner model for CI B, which embodies a transition matrix $\boldsymbol{P}_A$ that tells us how likely a change into state v is for CI B, provided that CI A is in state y. Taking that row i of P_A that corresponds to state y, and associating it with its (Bayesian) Dirichlet prior $\boldsymbol{\alpha}_{i,\cdot}^A = (\alpha_1, \dots, \alpha_k)$, where k ranges over the possible states of CI B, the update is simply an addition of 1 to the j-th coordinate in the vector α, relating to the state v that CI B turned into. The Bayesian update on this set of transitions is done

[1] A pseudo-count is a fractional count value; this term is technically exact here since we may start from a fractional value α_j, but add 1 upon an observation of the respective transition. Thus, although we do count, the counter's value remains fractional at all times; hence it is called "pseudo".

by that point.

Note that here we did not make any use of the previous states x of CI A or u for CI B. This is due to the fact that the change of state for CI A would be subject to an according update of the inner model for CI A (just as described). The prior state of CI B plays indeed no role here.

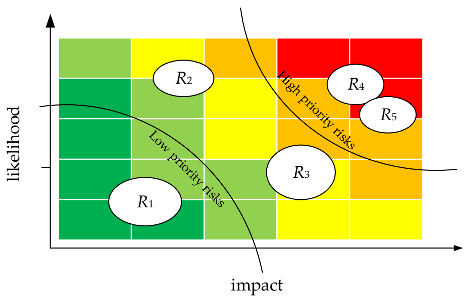
V. Avoiding Numbers – Asking for Ranks

In some cases, we may be able to completely avoid the specification of numbers, and poll people only for ranks. As with CERBERUS, risk management has such cases that we will look into now. One use case (among other possible ones) for the CERBERUS model relates to decision making towards risk mitigation. Considering a framework like ISO31000, a risk manager may roughly follow these steps:

1. Identification of context: this means a clear delineation of what assets we are concerned with, what level of protection is required, and seeks an understanding of external and internal factors with impact on the assets.
2. Risk identification: this is the identification of all threats with the potential of realizing themselves as risks to the previously defined assets.
3. Risk analysis: this is the actual challenge that we are concerned with here, being an *estimation* of impacts and likelihoods, so as to "quantify" risks by the well known rule of thumb

   $$\text{risk} = \text{impact} \times \text{likelihood}$$

   This formula has the statistical appeal of resembling an expected value, since it is easily extended into a weighted sum of impacts, each related to another threat on the list from the risk identification step. Asking for precise numbers here is the same problem as asking for a general probability parameter, and asking for a value for "impact" is even more difficult, since this can be any number (such as for financial losses), but also just an indicator (such as loss of human lives, where a quantification of "damage" would induce substantial ethical issues beforehand).
4. Risk Evaluation: with risks identified, the evaluation step asks for a ranking of those to assign priorities to risks whose mitigation is more urgent than for others. Here, we actually do not need to evaluate the risk formula from above, as all we require is a ranking of risks, based on impact and likelihood. This degree of freedom is important to stress, since we can create the familiar risk bubble charts like shown in Figure 4 without numeric information up to this point, even though the risk management process itself may be quantitative in the end. All we need is ranks, rather than precise numbers!



Impact ranking: $R_1 < R_2 < R_3 < R_4 < R_5$
Likelihood ranking: $R_1 < R_3 < R_5 < R_4 < R_2$

Figure 4 *Risk Bubble Chart with Induced Rankings (Example)*

5. Risk treatment: this is the point where we actually need to become "somewhat numeric",

since decisions about mitigation actions will in most cases depend on the expected efficacy, or equivalently said, the *return on investment* for a security control. Based on the risk after mitigation, obtained from the formula above, we can step forward by taking decisions for those actions that optimize the impacts or likelihoods (or both), so as to optimize the risk. This is what risk treatment is about, and what the parallel task of "monitoring and review" prescribes as part of ISO31000, as well as a continuous communication of all these steps to the outside (see Figure 5).

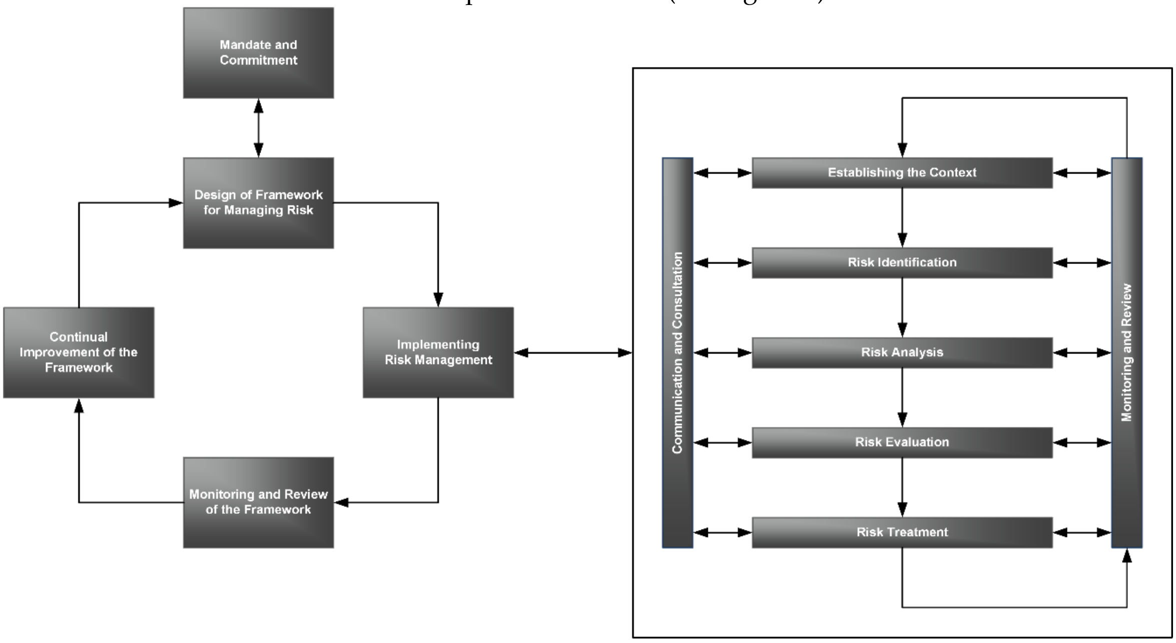


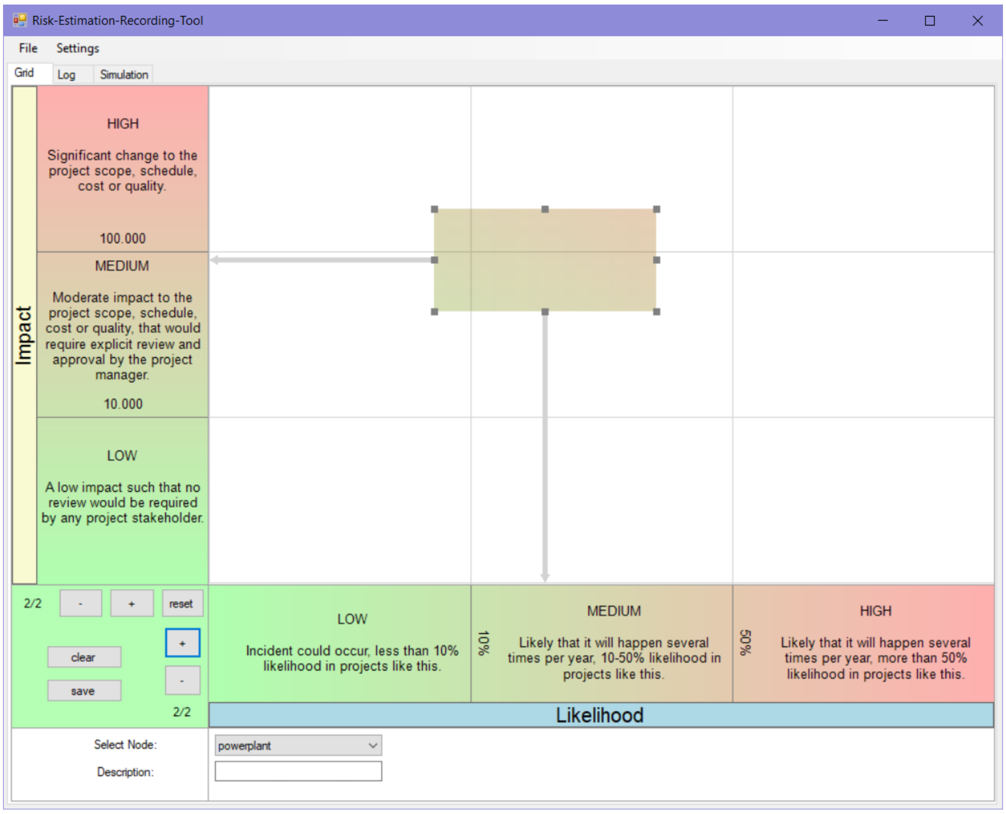
Figure 5 *ISO 31000 Risk Management Process [23]*

Following Peter Drucker's famous quote that "you can't manage what you can't measure", evaluating the risk formula eventually becomes necessary. Or doesn't it? The perhaps surprising (though well known) answer is no!

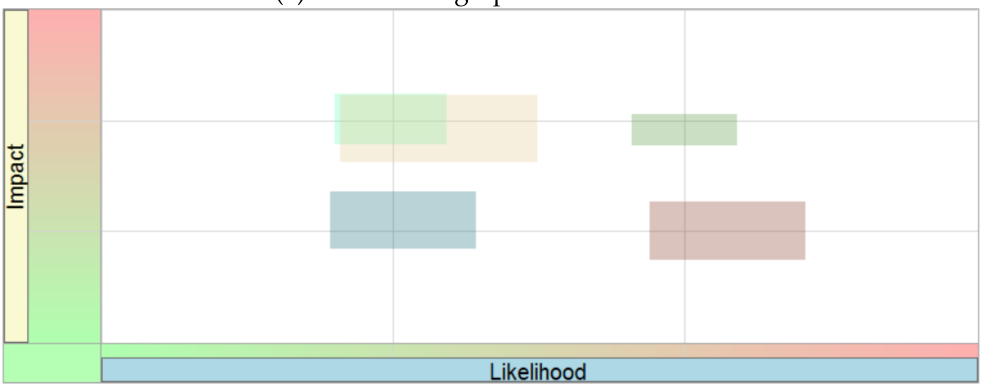
Suppose that we ask a domain expert for an assessment of several risks at the same time, specifically, allowing for doing the ranking of impacts and likelihood relative to one another (and in an order of risks that is up to the expert's own choice). Figure 6(b) shows how the results of such an assessment may look like, with four five boxes being drawn on the grid, at positions that were dependent on one another.

This representation resembles that of a usual risk matrix, only offering new and interesting possibilities: first, we can visually inspect the picture for outliers, and remove them (manually) if necessary. Second, and more importantly, we could aggregate those values into a single representative value, which in the simplest case amounts to taking an average, or in a more sophisticated form, takes the variations, i.e. uncertainties (reflected in the height and width of the boxes) into account to weigh each value inverse proportionally to the "certainty" in the final average (see [18,24] for several proposals in this direction).

Our goal, however, is not on getting numbers from the image, but rather on decision making, for which numbers are an aid, but not a necessity. Game theory offers the answer on how to make decisions based on rankings between actions (only), if we recall the very fundamental starting points laid by von Neumann and Morgenstern themselves [25] (and later extended by Debreu [26]): The important insight of these pioneers was that certain ordering relations can be expressed by real-valued functions, which we commonly call *utility functions* in game theory.



(a) Individual graphical risk assessment



(b) Assessment of several risks relative to one another

Figure 6 *Subjective relative risk ratings*

The application of game theory to matters of decision making in risk management is simple, but instructive:

- Suppose that we have a status quo in a system, and several threats in question of which one is most urgent to address. This (simple) decision problem only asks for ranks, not numbers, and a graphical specification is all we need.
- Likewise, if there is a single threat to be addressed now by several possible countermeasures, their efficacy is equally well specifiable on a ranking scale, and does not require numbers per se. If we seek for a balance between investment and efficacy, a two-dimensional ranking such as in Figure 2, Figure 4 or Figure 6 is already sufficient.
- The combination of several threats and several countermeasures to address them is the nontrivial case, where randomized decisions are often unavoidable. Let us consider the simplest example from game theory, Rock-Scissors-Paper, which despite its triviality, is

nonetheless a valid "template" for a risk management decision making process (just think of the column and row labels to be replaced with threats and countermeasures).

Payoffs: (player 1, player 2)

		Player 2		
		Rock	Scissors	paper
Player 1	Rock	(0, 0)	(1, -1)	(-1, 1)
	Scissors	(-1, 1)	(0, 0)	(1, -1)
	Paper	(1, -1)	(-1, 1)	(0, 0)

This game is straightforward to specify in the sense that we just "assign" a payoff of +1 or -1 to a player depending on whether it wins or loses. For security risk management, and generally many security models, including probabilistic ones in particular, the assignment of risks, based on impacts and likelihoods, also requires numbers, but which are much more difficult to obtain or argue. The numbers +1 and -1 in the above game matrix are just a direct specification of a utility function, and many (if not most) game theoretic models do give a direct such specification.

The axiomatic roots of game theory, however, start with a proof of existence of such functions, which merely demands a ranking of actions, and from this starting point, constructs utility values to represent this ordering. The important point is that these utility functions are constructed explicitly by the theory, so we can repeat the steps of the proof to get utility values. Irrespectively of whether we are seeking values for impacts, likelihoods or other (probabilistic) variables, all we need is a specification of values relative to each other. Formally, let us thus generically consider a space $(R, \leq)$, where R can be a set of impact, likelihood, or other values that we ought to specify. The simple ingredients enabling us to assign a value only concern the ordering on R; more generally, on its convex hull, with the ordering naturally extended; following the exposition of [27]: the ordering relation should be *total* and *transitive*. Furthermore, we require *conservation of the order under indifferent alternatives*, meaning that

$$r_1 \leq r_2 \text{ and } \alpha r_1 + (1-\alpha) r_2 \leq \alpha r_2 + (1-\alpha) r \text{ should hold for all } r \in R,$$

and *connectedness* (or closedness), meaning that for every $r_1 \leq r_2 \leq r_3$, there are two values $\alpha, \beta \in (0,1)$ such that $\alpha r_1 + (1-\alpha) r_3 \leq r_2 \leq \beta r_1 + (1-\beta) r_3$. The last assumption implies that for any two $r_1 < r_2$ that enclose some r as $r_1 \leq r \leq r_2$, there is a unique value $v \in [0,1]$ with $r \sim v \cdot r_1 + (1-v) \cdot r_2$, where the $\sim$-relation is induced by $\leq$ in the canonic way ($r \sim s \Leftrightarrow [r \leq s] \wedge [s \leq r]$). If the ordering satisfies these axioms, we can define a utility value as

(1) $U(r) = v$ if $r_2 \leq r \leq r_1$ and $r \sim v r_1 + (1-v) r_2$
(2) $U(r) = -\frac{v}{1-v}$ if $r \leq r_2$ and $r_2 \sim v r_1 + (1-v) r$
(3) $U(r) = \frac{1}{v}$ if $r_1 \leq r$ and $r_1 \sim v r + (1-v) r_2$

In particular, $U(r_1) = 1$ and $U(r_2) = 0$, and U preserves the ordering $\leq$ on R. Indeed, U is a linear function, since if $r_2 = \alpha r_1 + (1-\alpha) r_3$, we have $U(r_2) = \alpha U(r_1) + (1-\alpha) U(r_3)$. Extended to the convex hull of R, we need one more assumption to admit *linearization*: fix the range on which we need to specify our parameters as the interval $[r_1, r_2]$ and let P be a probability measure on this range with $P([r_1, r_2]) = 1$. With $\alpha(r) := \big(U(r) - U(r_1)\big) / \big(U(r_2) - U(r_1)\big)$ and $\beta = \int_{r_1}^{r_2} \alpha(r) dP(r)$, we need to assume that $P \sim \beta \delta_{r_2} + (1-\beta) \delta_{r_1}$ (where δ is the Dirac mass), i.e., if some value $r \sim \alpha(r) r_1 + \big(1 - \alpha(r)\big) r_2$, then this equivalence holds on average. This is nothing else than the assumption of "linearity" on the scale between two ratings (ranks) r_1 and r_2, and we have $\beta = \frac{E_P[U(r)] - U(r_1)}{U(r_2) - U(r_1)}$ Mapping this back to our graphical specification, all this just formalizes that the axes defining the 2D area on which the rectangles are drawn are *linearly scaled*. Under (all these) assumptions, we can use the so-constructed function U to express the ordering (ranking) for us, since for any two $P_1, P_2 \in conv(R)$ (where $conv$ is the convex hull, or equivalently, set of randomized decisions), we have $P_1 \leq P_2$ if and only if $E_{P_1}[U(r)] \leq E_{P_2}[U(r)]$. Moreover, this function U is unique up to affine transformations, i.e., any alternative valuation U' would take the form $U'(r) = a \cdot U(r) + b$ for some real values $a > 0$ and b.

How do we make use of all these (old and well known) facts for our actual challenge of specifying numbers in absence of precise knowledge about them? The idea is to use precisely the "three-case" definition of U above based on the ranking, not values, of the actions by making use of "linear interpolation" between them. More concretely, assuming that the axes are linearly scaled, we can just go ahead and take the graphical (visual) coordinates of the graphical range, mapped to the utility values U and letting us compute optimal decisions by standard methods and algorithms from game theory. The graphical specification is herein an aid to get the utility function for the decision making, and backed up by the axiomatic foundation of game theory. The crucial point, however, is that all of this, namely

1. The graphical ranking of actions, risks, etc.
2. The retrieval of (graphical) coordinates of those to play the role of the utility function
3. And the decision making as a matter of optimization (over finite sets in our example case even),

works without ever asking an expert for any number!

It is not surprising that this theoretical possibility has a number of caveats. First of all, our thoughts cannot be taken as formal argument or are mathematically rigorous here; the axiomatic approach to the existence of utilities hereby only plays the role of making our heuristic plausible, but do not lend themselves to proving any correctness. Numeric values obtained in this way do not necessarily have any particularly better semantic or accuracy than any other educated guess, but the main point of all this is to *ease guessing*, but without claiming to improve it.

Essentially, all of the axioms to define a utility function as such can be put to question, and the whole field of bounded rationality [28,29] deals with observations on human decision making to violate one or more of these assumptions (an excellent essay about this is that of Starmer [30]). Propsect theory [31] for example, accounts for phenomena of over- and underrating values near the end of the scale. For probabilities, this amounts to the effect that, subjectively, low probabilities are overrated, while large probabilities are underrated potentially. Probability weighting functions like that of Prelec [32] try to annihilate this effect. The research prototype shown in Figure 6(a) allows for a similar such correction by letting subjects individually adjust the grid towards smaller or wider ranges of the scales (visually).

A theoretical limitation concerns the use of multiple goals, as often occur in risk management applications. The existence of a utility function like the above is known under a variety of alternative conditions, often summarized as *Debreu representation theorems*. The common denominator therein is the *continuity* of the ordering, meaning that whenever a sequence $(r_n)_{n\in\mathbb{N}}$ with limit $r = \lim_{n\to\infty} r_n$ satisfies $s \leq r_n$, then the limit r should also satisfy $s \leq r$. This holds for orders on real values, and is indeed a major reason for game theory to be done mostly within $\mathbb{R}$. In higher dimensions, we can resort to weighted sums of utilities for different goals, which leads to Pareto-optimal decisions. However, if the ordering among the goals is "more explicit" in the sense of being lexicographic, then continuous utility functions no longer exist. The lexicographic order is indeed not continuous in the sense just stated. To see this, consider any sequence $a_n \to 0$, and take the limit of $(0, a_n)$ as $n \to \infty$. Then, obviously, $(a_n, 0) >_{lex} (0,1)$, but $\lim_{n\to\infty}(a_n, 0) = (0,0) \leq_{lex} (0,1)$, so the ordering is discontinuous. This leads to variations of decision theory, based on non-standard calculus and extension fields to $\mathbb{R}$, such as has been done in [19,33], in which some discontinuities naturally "disappear" by virtue of the richer algebraic structures. Those methods are applicable when multiple goals are relevant for a simultaneous optimization, or if the optimization shall be w.r.t. a lexicographic order (see Figure 4, where the lexicographic order would first consider "impact" and break ties using the "likelihood"). See [34] for an implementation of such methods in the **R** software [35].

VI. Conclusion

The ideas laid out here are applicable whenever a probabilistic parameter describes an observable event, so that data for a Bayesian update is collectible. A practical issue can indeed be the speed of convergence, since the above argument is nonetheless asymptotic, and the true value is reached only after a hypothetic infinitude of updates. Therefore, we may need to update upon every incoming ticket at the IT administration office, or as often as we can, in practice.

We also stress that the above model does not serve too well as a model of human trust: the updating is in some sense "symmetric" and "self-stabilizing", meaning that (i) the likelihood changes eventually become smaller as more updates come in (self-stabilization), and the likelihoods will update with roughly comparable magnitudes in both directions. The latter is contrary to human subjective changes to trust, since confidence in an event to occur may substantially change upon recent experience and differently in the direction towards zero or towards one. In other words, if the probabilistic parameter is interpreted as a "trust value", say, if we take it as the expectation of some event (that we rely on) to occur, then subjective trust may be lost upon a single incident, but may be regained only over a much longer period of positive experience. On the contrary, the above model would not reflect such asymmetry due to human pessimism. This leads to the advice of applying the above model only for the estimation of parameters that describe *physical* processes, and *not* subjective human factors. The latter are subject to much deeper psychological mechanisms for whose capture the above model may be overly simplistic.

If the parameter in question, however, relates to a physical event that can be observed, then the Bayesian updating as described above offers a computationally efficient and elegant way of online learning parameters in absence of reliable domain expertise to specify a (more) accurate model or prior guess.

Finally, the methods outlined here are so far conceptual and lack an empirical study on accuracy, subjective comfort felt in the specification methods as such and similar. While they are certainly viable to make a start for a Bayesian updating, open questions relate to the accuracy of any such "guess", which on the one hand determines the speed of convergence as further Bayes updates come in, and on the other hand, have a direct influence on the accuracy for decision making in risk management.

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