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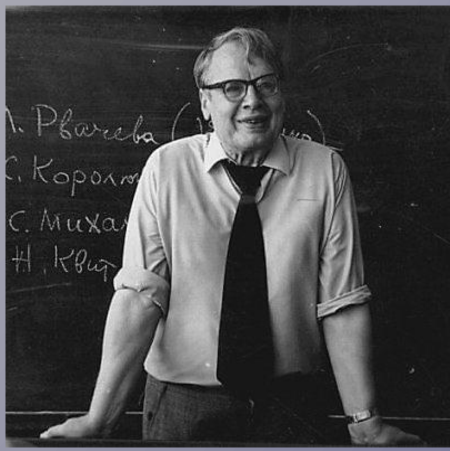

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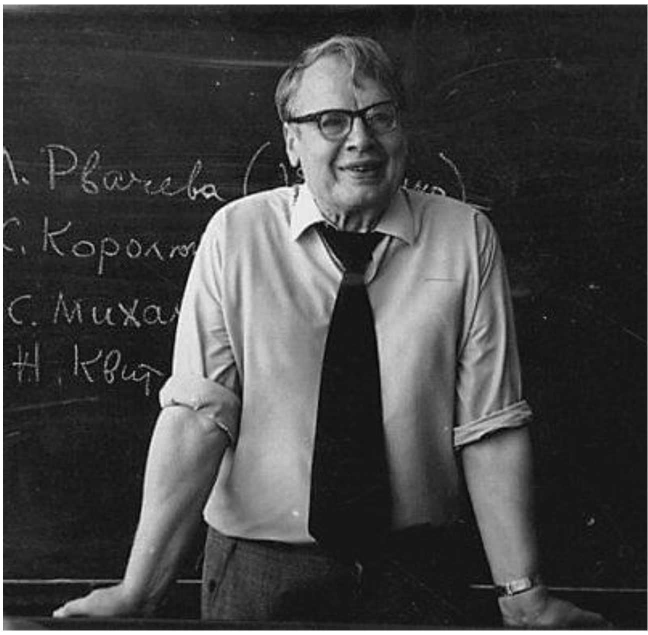

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Table of Contents

In this paper, we discuss a network model to study the queueing characeristics of nodes in a multi-hop wireless network, under the standard binary exponential back-off (BEB) contention resolution scheme. Based on the steady state distribution of the system size at a node, which was appeared in Sweta & Deepak [2], we compute the joint distribution of system size at all nodes in a multi-hop network, governed by some specific queue disciplines. Getting information on the joint queuing size distribution in the network will enable us to control the traffic (and hence congestion) in the whole network. In order to illustrate our theoretical results, a particular multi-hop network model is considered and analysed numerically.

We consider a single server queuing model with correlated reneging, balking and feedback. The time-dependent behavior of the model is studied using Runge-Kutta method. Some measures of the performance like expected system size and expected waiting time are computed.

ABOUT NEW BOOKS

The Handbook of Research on Industrial Advancement in Scientific Knowledge addresses the intersection of technology and science where engineering considerations, mathematical approaches, and management tools provide a better understanding and awareness of Industry 4.0, while also taking into account the impact on current society. This publication identifies methodologies and applications related to decision-making, risk and uncertainty, and design and development not only on scientific and industrial topics but also on social and ethical matters. It is designed for engineers, entrepreneurs, academicians, researchers, managers, and students.

This introductory book equips the reader to apply the core concepts and methods of network reliability analysis to real-life problems. It explains the modeling and critical analysis of systems and probabilistic networks, and requires only a minimal background in probability theory and computer programming. Based on the lecture notes of eight courses taught by the authors, the book is also self-contained, with no theory needed beyond the lectures. The primary focus is on essential "modus operandi," which are illustrated in numerous examples and presented separately from the more difficult theoretical material.

Reliability evaluation of radial distribution system – A case study

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Abstract

Reliability evaluation of a system or component or element is very important in order to predict its availability and other relevant indices. Reliability is the parameter which tells about the availability of the system under proper working conditions for a given period of time. In this paper reliability evaluation of an electrical power distribution system is done and different parameter are evaluated. The electrical power distribution system taken for study is radial distribution system in nature.

Keywords: Reliability, Availability, Radial distribution system, Electrical power system.

I. Introduction

Reliability evaluation of a system or component or element is very important in order to predict its availability and other relevant indices. Reliability is the parameter which tells about the availability of the system under proper working conditions for a given period of time. A Markov cut-set composite approach to the reliability evaluation of transmission and distribution systems involving dependent failures was proposed by Singh et al. [1]. The reliability indices have been determined at any point of composite system by conditional probability approach by Billinton et al. [2]. Wojczynski et al. [3] discussed distribution system simulation studies which investigate the effect of interruption duration distributions and cost curve shapes on interruption cost estimates. New indices to reflect the integration of probabilistic models and fuzzy concepts was proposed by Verma et al. [4]. Zheng et al. [5] developed a model for a single unit and derived expression for availability of a component accounting tolerable repair time. Distributions of reliability indices resulting from two sampling techniques are presented and analyzed along with those from MCS by Jirutitijaroen and Singh [6]. Dzobe et al. [7] investigated the use of probability distribution function in reliability worth analysis of electric power system. Bae and Kim [8] presented an analytical technique to evaluate the reliability of customers in a microgrid including distribution generations. Reliability network equivalent approach to distribution system reliability assessment is proposed by Billinton and Wang [9].

Evaluation of Reliability indices accounting omission of random repair time for distribution systems using Monte Carlo simulation [10]. Determination of Optimum period between Inspections for Distribution system based on Availability Accounting Uncertainties in

Inspection Time and Repair Time, Tiwary et al. [11]. Jirutitijaroen et al. [12] developed a comparison of simulation methods for power system reliability indexes and their distribution. Determination of reliability indices for distribution system using a state transition sampling technique accounting random down time omission Tiwary et al. [13]. Tiwary et al. [14] proposed a methodology based on inspection repair based availability optimization of distribution systems using Teaching Learning based Optimization. Bootstrapping based technique for evaluating reliability indices of RBTS distribution system neglecting random down time was evaluated [15]. Volkanavski et al. [16] proposed application of fault tree analysis for assessment of the power system reliability. Li et al. [17] studies the impact of covered overhead conductors on distribution reliability and safety. Reliability enhancement of distribution system using Teaching Learning based optimization considering customer and energy based indices was obtained in Tiwary et al. [18]. Self-Adaptive Multi-Population Jaya Algorithm based Reactive Power Reserve Optimization Considering Voltage Stability Margin Constraints was obtained in Tiwary et al. [19]. A smooth bootstrapping based technique for evaluating distribution system reliability indices neglecting random interruption duration is developed [20]. The impact of covered overhead conductors on distribution reliability and safety is discussed [21]. Sarantakos et al. [22] introduced a method to include component condition and substation reliability into distribution system reconfiguration. Battu et al. [23] discussed a method for reliability compliant distribution system planning using Monte Carlo simulation.

II. Reliability evaluation of series system and its implementation

Physically a system configuration will be a series reliability network if system fails even if a single component fails or system survives if all the components are working successfully.

If one assumes time independent reliability r1,r2…rn, then reliability of series system is given as

$$R_s = \prod_{i=1}^{n} r_i$$

Fig. 1, [20] consists of 7 distributor segments and 7 load points from LP-2 to LP-8. For each and every load point series path is considered from source to that load point.

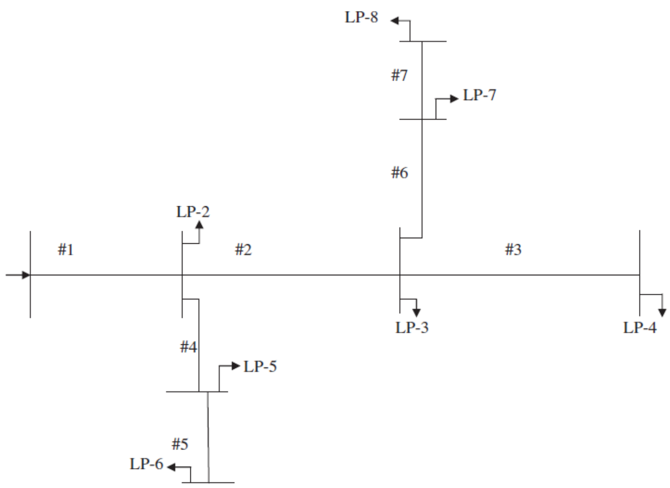

Figure 1: Eightnode distribution system

III. Results and Discussion

Table 1 shows the initial data for the radial distribution system. There are seven distribution section and the initial data for the reliability are 0.4, 0.3, 0.5, 0.7, 0.2, 0.6, 0.8 respectively. Table 2 provides the evaluated reliability for each of the load points separately. For LP-2 to LP-8 evaluated reliability value is 0.4, 0.12, 0.06, 0.28, 0.056, 0.072, 0.0576 respectively. Fig. 2 provides the magnitude of evaluated reliability at different load points.

Table 1: Initial data for the radial distribution system.

Distribution section	1	2	3	4	5	6	7
Reliability value	0.4	0.3	0.5	0.7	0.2	0.6	0.8

Table 2: Evaluated reliability for each of the load points.

Load Point	2	3	4	5	6	7	8
Evaluated Reliability	0.4	0.12	0.06	0.28	0.056	0.072	0.0576

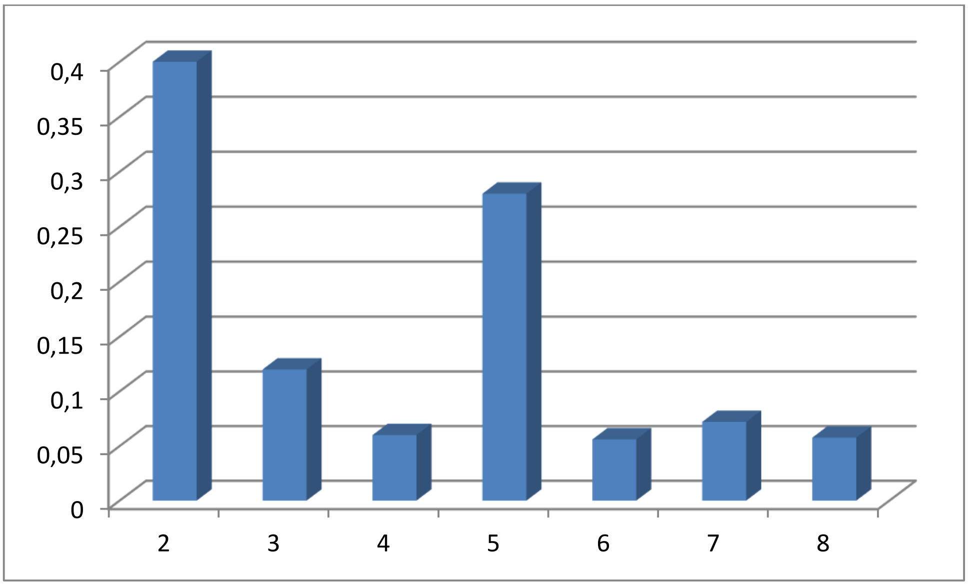

Figure 2: Magnitude of Reliability at different load points.

IV. Conclusion

Reliability evaluation of a system or component or element is very important in order to predict its availability and other relevant indices. In this paper reliability evaluation of an electrical power distribution system is evaluated. The electrical power distribution system taken for study is radial distribution system in nature. The parameter obtained is shown in Table-2 for the different Load points LP-2 to LP-8 respectively.

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A Queue Network M/M/1/∞ Model

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Abstract

In this paper we model an open queueing network of cardiac treatment section in medical sector. Assume arrival of patients follows Poisson and service times at stations have exponential distribution. The performance measures of the system are evaluated. The steady state characteristics of the network are obtained and each station solved independently by using M/M/1/∞ model while blocking and non-blocking exists. Blocking occurred when at least one service center has limited queueing space or capacity before it. An illustrative example is given.

Keywords: Queueing networks, Blocking, Steady state characteristics.

I. Introduction

Collection of interactive queueing systems is known as network of queues. Queueing networks mainly classified as open queueing networks, closed queueing networks and mixed queueing networks. Open queueing networks described as customers can arrive from outside the system at any node and depart from the system from any node. At least one service center has limited waiting space or capacity, which are classified in to restricted queueing networks. Blocking may arise in a network of queues where some or all queues have finite buffer capacity [2]. Since there is restriction in waiting space between the stations, there may occur blocks.

Many relevant studies on open restricted queue systems are done by Hunt [3], Takahashi et al.[8], Perros and Atlok [6], Koizumi et al.[5], Sreekala and Manoharan [7] and Arum Helmi Manggala Putri et al [1]. Hunt [3] used a sequential series model to obtain solution for a two station series queue with limited waiting space between stations. An approximate analysis for open queueing networks with blocking done by Takahashi et al.[8] and Perros and Atlok [6]. Koizumi et al [5] analysed blocking in open restricted queueing system by decomposition method. Recently analysis of restricted queueing networks-a blocking approach with special reference to health care system studied by Sreekala and Manoharan [7].

In this paper first we study an open queueing network of Cardiac section with infinite capacity in each station. Steady state equations and performance parameters are obtained. A brief description of the model is done in section II. Diagrammatic representation and congestion types are given in Section III. Derivation of steady state equations in without blocking and analysis of each station using decomposition approach in with blocking are given in section III. Numerical analysis is section IV. Conclusions are given in last section.

II. Model description

We can consider Out Patient (OP) section of Cardiac treatment in government medical college, as an example of open queueing network. There are five stations are defined in this queueing network. In first node S_1gives token for every customer arriving to the hospital, customers arrive according to homogeneous Poisson process. Second station stands for pressure checking which follows M/M/1/∞/FCFS schedule. Doctors are available in the third and fourth node and these nodes considered as a single node. S_3 and S_4 also follow M/M/1/K/FCFS. Fifth node is for treatment. After the diagnosis, some patients in third and fourth node leave from the system with probability α_3 and $\alpha_4$and remaining patients admit for treatment with probability 1-α_3 and 1-α_4.There are some situations where usual admission procedures cannot follow. Example: - accident cases or heart attack.

Methodological Framework

Consider an open queueing network of OP section of Cardiac treatment in medical college with five single servers. LetS_i ($i = 1,2, \dots 5$) denote stations. Arrival pattern of customers to the system according to homogeneous Poisson process with rate λ. Service times are exponentially distributed with rate μ_i ($i = 1,2, \dots 5$). Queue discipline is FCFS basis. Waiting space S_5 and between stations one and two are of infinite capacity and other stations are finite. Therefore blocking happens only between $S_3 \rightarrow S_5$ and$S_4 \rightarrow S_5$. In this paper, model the flows $S_2 \rightarrow S_3$, $S_2 \rightarrow S_4$, $S_3 \rightarrow S_5$ *and* $S_4 \rightarrow S_5$. Arrival to each node is according to Poisson process. Diagrammatic representation of the model is given in figure 1.

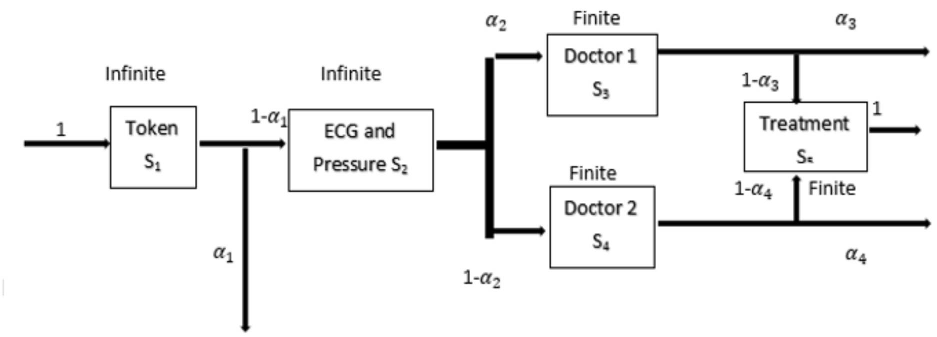

Figure 1. OP section of Cardiac treatment as queue network with blocking

Since waiting space between stations three, four and five are of finite capacity there may arise blocking between $S_3 \rightarrow S_5$ *and* $S_4 \rightarrow S_5$. We model the flows $S_3 \rightarrow S_4$and $S_4 \rightarrow S_5$. Arrival to each node is according to Poisson process. The types of congestion listed in table 1.

Table.1.Congestion types

Flow	Cause of congestion	Facing station	Congestion type
$S_1 \rightarrow S_2$	Not applicable	Not applicable	No congestion
$S_2 \rightarrow S_3$	S_3 is full	S_2	Classic Congestion
$S_2 \rightarrow S_4$	S_4 is full	S_2	Classic Congestion
$S_3 \rightarrow S_5$	S_5 is full	S_3	Blocking
$S_4 \rightarrow S_5$	S_5 is full	S_4	Blocking

III. Steady –State Analysis

In this section we first assume that every station has infinite waiting space and analyze stations without blocking. The steady state analysis of some related models can be seen in Gross and Harris [3] and Bose [2].

Steady State Analysis without blocking

The routing probability matrix generally defined as

$$P = \begin{bmatrix} r_{00} & r_{01} & r_{02} & r_{03} & r_{04} & r_{05} & r_{06} \\ r_{10} & r_{11} & r_{12} & r_{13} & r_{14} & r_{15} & r_{16} \\ r_{20} & r_{21} & r_{22} & r_{23} & r_{24} & r_{25} & r_{26} \\ r_{30} & r_{31} & r_{32} & r_{33} & r_{34} & r_{35} & r_{36} \\ r_{40} & r_{41} & r_{42} & r_{43} & r_{44} & r_{45} & r_{46} \\ r_{50} & r_{51} & r_{52} & r_{53} & r_{54} & r_{55} & r_{56} \\ r_{60} & r_{61} & r_{62} & r_{63} & r_{64} & r_{65} & r_{66} \end{bmatrix}$$

where r_{ij} is the routing probability from station i to station j (i,j= 1,2,..,6). The routing probability matrix of our model based on figure 1 is

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ \alpha_1 & 0 & 1-\alpha_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_2 & 1-\alpha_2 & 0 \\ \alpha_3 & 0 & 0 & 0 & 1 & 1-\alpha_3 \\ \alpha_4 & 0 & 0 & 0 & 0 & 1-\alpha_4 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We can find $\lambda_i : i = 1,2, \dots 5$ (Total arrival rates) by solving the traffic equations:

$$\lambda_1 = \lambda$$
$$\lambda_2 = (1-\alpha_1)\lambda$$
$$\lambda_3 = \alpha_2(1-\alpha_1)\lambda$$
$$\lambda_4 = (1-\alpha_2)(1-\alpha_1)\lambda$$
$$\lambda_5 = (1-\alpha_2)(1-\alpha_1)\lambda[(1-\alpha_3)+(1-\alpha_4)].$$

We assume there is an infinite buffer between stations. So we can solve each station independently applying M/ M/ 1/∞ queueing model.

Average queue length and Average queue delay

Average queue length of station i is obtained from the formula [3].

$$L_i^q = \frac{\rho_i^2}{1-\rho_i}, \quad (1)$$

where $\rho_i = \lambda\_i/\mu\_i$ <1 (i=1,2,…,4) and $\rho_i = \frac{\lambda_i}{\mu_i}$>1 (i=5), in the case of non-steady state, analytical model cannot be applicable to the queueing system.

By using Little's formula[3] we can obtain the average steady state waiting time,

$$W_i^q = \frac{\rho_i^2}{\lambda_i(1-\rho_i)}, i = 1,2, \dots 5. \quad (2)$$

Steady state analysis with blocking

Blocking exists between stations when some stations are finite and congestion at any particular station could potentially affect congestion levels at all upstream stations. In order to find interactions between stations, modified Jacksons approach can be used with the help of effective service time by Takahashi et al.[9]. Here we assume effective service times follow exponential distribution. The mean effective service time at station i is denoted by $\frac{1}{\tilde{\mu}_i}$. Effective waiting time is defined as the convex combination of waiting times.

$$\frac{1}{\tilde{\mu}_i} = r_{i0}\left(\frac{1}{\mu_i}\right) + \sum_j r_{ij}\left(\frac{1}{\mu_i} + W_j\right),$$

where r_{i0} is the routing probability of patients leaving from state i without facing any wait,r_{ij} is the routing probability from station i to j. In our model stations S_3 *and*, S_4 face blocking. The effective service time corresponding to S_3 *and*, S_4 are

$$\frac{1}{\tilde{\mu}_3} = r_{30}\left(\frac{1}{\mu_3}\right) + r_{35}\left(\frac{1}{\mu_3} + W_5^q\right), \quad (3)$$

$$\frac{1}{\tilde{\mu}_4} = r_{40}\left(\frac{1}{\mu_4}\right) + r_{45}\left(\frac{1}{\mu_4} + W_5^q\right) \quad (4)$$

Using equations (1) and (2), we obtain steady state queue lengths and waiting times in terms of effective service times.

Analysis of Stations

By using single node decomposition approximation by Takahashi et al. [9] the steady state of every station can solve independently from last station to first station. The steady state of each finite station (M/M/1/∞ *queue*) is analyzed using this approximation.

Analysis of station six (S_5)

In our model, station five represents who needs recheck and treatment in the clinic. The downstream node S_3 and S_4 are finite. S_5 face blocking if S_3 is full. Corresponding to S_5 the queue length and queue delay obtained by solving (1) and (2) in terms of effective service times (3). The queue length corresponding to S_5 is

$$L_{35}^q = L_5^q(\lambda_{35}/\lambda_5)$$
$$= L_5^q(\mathrm{r}_{35}\,\lambda_3/\lambda_5)$$

whereL_{35}^q is thequeue length of blocked persons at S_3 waiting to enter S_5. S_5 face blocking if S_4 is full. Corresponding to $S_5$the queue length and queue delay obtained by solving (1) and (2) in terms of effective service times (4). The queue length corresponding to S_5 is

$$L_{45}^q = L_5^q(\lambda_{45}/\lambda_5)$$
$$= L_5^q(\mathrm{r}_{45}\,\lambda_4/\lambda_5)$$

where L_{45}^q is the queue length of blocked persons at S_4 waiting to enter S_5. This method is applicable only when traffic intensity less than one.

Analysis of station four (S_4)

Station four represents patients who entered for doctors checking. The downstream node S_2 is also infinite. The queue length corresponding to S_5 is

$$L_{24}^q = L_4^q$$

where L_{24}^q is the queue length of blocked persons at S_2 waiting to enter S_4.

Analysis of station three (S_3)

Station three represents patients who entered for doctors checking. The downstream node S_2 is also infinite. The queue length corresponding to S_3 is

$$L_{23}^q = L_3^q$$

where L_{23}^q is thequeue length of blocked persons at S_2 waiting to enter S_3.

Analysis of station two (S_2)

Station two represents patients who entered for pressure checking. The downstream node S_1 is also infinite. The queue length corresponding to $S_2$is

$$L_{23}^q = L_3^q.$$

IV. Numerical Analysis

Data taken from Cardiac Section of a Medical College. To find out the Performance Parameters of with blocking and without blocking, the statistical analysis is conducted. $\lambda = 8.48$, α_1=0.85, $\alpha_2 = 0.45, \alpha_3 = 0.98, \alpha_4 = 0.97, \frac{1}{\mu_1}$=0.29, $\frac{1}{\mu_2} = 0.91$, $\frac{1}{\mu_3} = 0.83$, $\frac{1}{\mu_4} = 0.75$. Performance parameters are computed and given in table 2.

Table 2. Performance Parameters

Station	Performance Parameters	With Blocking	Without Blocking
S_2	L_2^q	1.21	1.21
	W_2^q	2.08	2.08
S_3	L_3^q	0.54	0.54
	W_3^q	0.91	0.91
S_4	L_4^q	0.66	0.66
	W_4^q	0.75	0.75

The congestion rate and waiting time of S_2 is high compared to other two stations. Blocking exists in station 5 but the traffic intensity of station 5 is greater than one. Steady state doesn't exist for this station. Normal methods cannot applicable for analyzing non-steady state queue system. Bounding of capacity of queue, Monte Carlo simulation and increase servers to the system are the possible methods for solving non-steady state queues.

V. Conclusion

We studied the Cardiac section of a Medical College as an open restricted queueing network. Blocking exists due to 5^{th} node. Steady state equations are obtained without blocking and with blocking cases assuming traffic intensity less than one. Node to node decomposition method is used to get performance measures. But the traffic intensity of 5^{th} node greater than one. We cannot analyze through steady state equations. To find performance measures we can use any of these methods-Bounding of capacity of queue, Monte Carlo simulation and increase servers to the system.

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Parametric Estimation on Constant Stress Partially Accelerated Life Tests for the Exponentiated Exponential Distribution using Multiple Censoring

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Abstract

If the items have high reliability then to check the lifetime of items under normal use condition takes more time and cost in comparison with the accelerated condition. The items put higher stress than the usual level of stresses to generate early failures in a short period to reduce the costs involved in the testing of items without any change in the quality. This study is based on constant stress partially accelerated life tests for Exponentiated Exponential distribution using multiple censoring schemes. The maximum likelihood estimates and asymptotic variance and covariance matrix are obtained. The confidence intervals for parameters are also constructed. At last, a simulation technique is used to check the performance of the estimators.

Keywords: Constant stress partially accelerated life tests, Exponentiated Exponential distribution, Multiple censoring, Fisher Information Matrix, a Simulation study.

I. Introduction

In the present market situation, the manufacturing designs are bettering day by day because there is a big change in technology. If an item has high reliability than it is too much tough to obtain information about the lifetime of items or products under normal usage condition at the time of testing. In this type of situation, the accelerated life test (ALT) is the best choice to get information on the life of the items or products. ALT is used to get information on items life or products life in a short period with a shortage of cost by testing them at accelerated conditions after this testing them on normal use conditions to induce early failures. These conditions are referred to as stresses. The stresses may be in the form of temperature, voltage, force, etc.

Normally, three types of stresses are applied in accelerated life testing, such as constant stress, step-stress, and progressive stress. Here we are focusing only on constant stress. In constant stress accelerated life test, the products or items are operated at fixed levels of stress throughout the testing. From ALT, two types of data are obtained, such as complete and censored data. In the complete data, the lifetime of each unit is known, but the lifetime of each unit is unknown in censored data. A mathematical model which is related to the lifetime of an item or product and stress is either known or can be assumed in ALT. There are many situations in which these relationships are unknown, and we can not conclude these relationships. This means that data can not be extrapolated to use conditions which are obtained from ALT. Such situations where the test items are run at both normal and higher than normal stress conditions, the partially accelerated life test (PALT) is used. In PALT, two main methods are used by reliability practitioners such as constant stress partially accelerated life test (CSPALT) and step-stress partially accelerated life test (SSPALT). The products or items are tested either usual or higher than usual condition until the test is ended in CSPALT.

In many situations, the lifetime experiment could out of control due to many reasons like components of a system may break accidentally. In type-I censoring (time censoring) scheme, the test is terminated after a fixed amount of time, and in type-II censoring (item censoring), the test is terminated after a fix proportion of items. As we know that the removal of items or components from the test during testing is possible in the progressive type censoring scheme, while type-I and type-II censoring schemes don't allow the removal of items or components from a test during testing. In this type of situation, the multiple censoring schemes are the best choice for an engineer or reliability practitioner because multiple censoring schemes allows the removal of items from the test during the testing at any situation or any time. We define multiple censoring schemes as when the testing of items or components fails because of more than one reason, then multiple censoring occurs. Tobias and Trindada [1] observed that the type-I and type-II censoring schemes are a special case of multiple censoring schemes.

There is much literature available on PALT with constant stress with many types of censoring schemes. *Abd El-Raheem et al. [2] presented a study on constant stress accelerated life test with the use of geometric process when the lifetime of test units follows Extention of Exponential distribution under the type-II progressive censoring scheme. Kamal et al. [3] presented a study on designing of partially accelerated life test when the lifetime of items follows Inverted Weibull distribution with constant stress under the type-I censoring scheme. Abdullah M. [4] dealt with parameters estimation when the lifetime of units follows Generalized Half Logistic distribution for progressive type-II censored data. Zhang and Fang [5] dealt with an estimation of acceleration factor when the lifetime of units follows Exponential distribution under CSPALT based on type-I censored data. A new approach of constructing the exact lower and upper confidence limits is proposed by them for the acceleration factor. Sadia and Islam [6] presented a study on CSPALT plans when the lifetime of units follows Rayleigh distribution based on type-II censored data. Tahani and Areej [7] dealt with an inference on CSPALT under progressive type-II censored data based on a mixture of Pareto distribution. Mohamed et al. [8] presented a study on CSPALT using progressive type-II censored data when the lifetime of items follows Modified Weibull distribution. They discussed two bootstrap confidence intervals, which are called bootstrap-t and bootstrap-p. Xiaolin and Yimin [9] presented a study on CSPALT using the masked series system when the lifetime of components follows Complementary Exponential distribution based on progressive type-II censoring. Ismail [10] presented a study on CSPALT for Weibull distribution based on hybrid censoring scheme. He makes a statistical inference by using two methods; maximum likelihood and percentile bootstrap method. Nassar and Elharoun [11] dealt with an inference on CSPALT for Exponentiated Weibull distribution in the case of multiple censored data. Amal et al. [12] presented a study on CSPALT for inverted Weibull distribution in the case of multiple censoring scheme. Cheng and Weng [13] estimated parameters under multiple censoring scheme when the lifetime of items follows Burr XII distribution.*

The paper organized as follows. The model description and test procedure are given in section II. The basic assumptions for CSPALT are also given in section II. The point Estimation is given in section III. In this section, the likelihood function of the model under multiple censoring schemes is observed, and the Fisher Information matrix is also investigated in this section. In section IV, the confidence intervals are developed. The simulation study is given in section V. Finally, the conclusions are made in section VI.

II. Model Description and Test Procedure

I. Exponentiated Exponential Model

The Exponentiated Exponential distribution is commonly known as the Generalized Exponential distribution. This distribution is a particular member of Exponentiated Weibull distribution under two parameters form [14]. It is quite effective in analyzing several lifetime data, mainly in place of Gamma and Weibull Distribution in two parameters case. The above three distributions coincide with Exponential distribution in one parameter form if the value of the shape parameter becomes one. The Exponentiated Exponential plays an important role in reliability analysis because of its simplicity. If the lifetime of the item follows the Exponentiated Exponential distribution, then the test procedure for CSPALT under multiple censoring schemes is as follows.

The probability density function (pdf) of Exponentiated Exponential distribution is given as

$$f_1(t_i) = \alpha\lambda e^{-\lambda t_i}(1-e^{-\lambda t_i})^{\alpha-1} \qquad t_i, \alpha, \lambda > 0 \; ; \; i = 1,2,\dots,n_1 \tag{1}$$

Where, α and λ are shape, scale parameters respectively. The ith observed lifetime of the test under normal condition item is denoted by t_i.

The cumulative density function (cdf) of Exponentiated Exponential distribution is given as

$$F_1(t_i) = (1-e^{-\lambda t_i})^{\alpha} \tag{2}$$

The reliability function of Exponentiated Exponential distribution is given as

$$R_1(t_i) = 1-(1-e^{-\lambda t_i})^{\alpha}$$

The hazard function of Exponentiated Exponential distribution is given as

$$H_1(t_i) = \frac{\alpha\lambda e^{-\lambda t_i}(1-e^{-\lambda t_i})^{\alpha-1}}{1-(1-e^{-\lambda t_i})^{\alpha}}$$

Under the accelerated condition, the probability density function (pdf) of a lifetime $X = \beta^{-1}T$ is given as

$$f_2(x_j) = \alpha\beta\lambda e^{-\lambda\beta x_{j_i}}(1-e^{-\lambda t\beta x_j})^{\alpha-1} \qquad t_i, \alpha, \lambda > 0, \; \beta > 1; \; j = 1,2,\dots,n_2 \tag{3}$$

Under the accelerated condition, the cumulative density function (cdf) of a lifetime $X = \beta^{-1}T$ is given as

$$F_2(x_j) = (1-e^{-\lambda t\beta x_j})^{\alpha} \tag{4}$$

The reliability function of a lifetime, $X = \beta^{-1}T$ under accelerated condition, is given as

$$R_2(x_j) = 1-(1-e^{-\lambda\beta x_j})^{\alpha}$$

The reliability function of a lifetime, $X = \beta^{-1}T$ under accelerated condition, is given as

$$H_2(x_j) = \frac{\alpha\beta\lambda e^{-\lambda\beta x_{j_i}}(1-e^{-\lambda t\beta x_j})^{\alpha-1}}{1-(1-e^{-\lambda\beta x_j})^{\alpha}}$$

Where x_j is jth observed lifetime under the case of the accelerated condition.

II. Assumptions

The basic assumptions for CSPALT are given as

- The lifetimes of items T_i $i = 1,2,..., n_1$ are independent and identically distributed random variable with probability density function given in equation (1), which is allocated to normal condition.
- The lifetimes of items X_j $j = 1,2,..., n_2$ are also independent and identically distributed random variable with probability density function given in equation (3), which is allocated to accelerated condition.
- T_i and X_i are mutually independent also.
- n_1 and n_2 are the total numbers of items at normal and accelerated condition, respectively.

III. Parameter Estimation

I. Point Estimates

In this section, we use the maximum likelihood (ML) technique for estimating parameters. ML technique is the most important technique for fitting the statistical model; it has many interesting properties like asymptotic unbiased, asymptotic efficiency and asymptotic normality, etc.

$t_{(1)} < t_{(2)} < ...t_{(n)}$ are supposed observed values of the total lifetime T at the normal condition and $t_{(1)} < t_{(2)} < ...t_{(n)}$ are the supposed observed values of the lifetime X at the accelerated condition.

Then the likelihood of Exponentiated Exponential distribution under multiple censored data is given as

$$L(t_i, \alpha, \lambda, \beta) = \prod_{i=1}^{n} \left[f_1(t_i) \right]^{\delta_i,1,f} \left[1 - F_i(t_i) \right]^{\delta_i,1,c} \times \left[f_2(x_i) \right]^{\delta_i,2,f} \left[1 - F_2(x_i) \right]^{\delta_i,2,c} \tag{5}$$

$$L(t_i, \alpha, \lambda, \beta) = \prod_{i=1}^{n} \left[\alpha\lambda e^{-\lambda t_i} (1 - e^{-\lambda t_i})^{\alpha-1} \right]^{\delta_i,1,f} \left[1 - (1 - e^{-\lambda t_i})^{\alpha} \right]^{\delta_i,1.c} \left[\alpha\beta\lambda e^{-\lambda\beta x_{j_i}} (1 - e^{-\lambda t\beta x_j})^{\alpha-1} \right]^{\delta_i,2,f} \left[1 - (1 - e^{-\lambda t\beta x_j})^{\alpha} \right]^{\delta_i,2.c}$$

$\delta_{i,1,f}$, $\delta_{i,1,c}$, $\delta_{i,2,f}$, $\delta_{i,2,c}$ are indicator functions. The values of indicator functions are given as

$$\delta_{i,1,f}, \delta_{i,2,f} = \begin{cases} 1 & the item\ failed at\ stress condition \\ 0 & otherwise \end{cases}$$

$$\delta_{i,1,c}, \delta_{i,2,c} = \begin{cases} 1 & the item censored at\ normal condition \\ 0 & otherwise \end{cases}$$

$$\sum_{i=1}^{n} \delta_{i,1,f} = n_{1f} = Number of\ failed items at\ nomal condition$$

$$\sum_{i=1}^{n} \delta_{i,2,f} = n_{2f} = Number of\ failed items at\ accelerated\ condition$$

$$\sum_{i=1}^{n}\delta_{i,1,c} = n_{1c} = Number of\ censored item at\ normal condition$$

$$\sum_{i=1}^{n}\delta_{i,2,c} = n_{2c} = Number of\ censored item at\ accelerated\ condition$$

$$n_f = n_{1f} + n_{2f}$$

The log-likelihood function is simply the natural logarithm of the likelihood function and given as

$$\ln L = \sum_{i=1}^{n}\delta_{i,1,f}\left[\ln\alpha + \ln\lambda - \lambda t_i + (\alpha-1)\ln(1-e^{-\lambda t_i})\right] + \sum_{i=1}^{n}\delta_{i,1,c}\ln\left[1-(1-e^{-\lambda t_i})^{\alpha}\right]$$
$$+\sum_{i=1}^{n}\delta_{i,2,f}\left[\ln\alpha + \ln\beta + \ln\lambda - \lambda\beta x_i + (\alpha-1)\ln(1-e^{-\lambda\beta x_i})\right] + \sum_{i=1}^{n}\delta_{i,2,c}\ln\left[1-(1-e^{-\lambda x_i})^{\alpha}\right] \tag{6}$$

Where $L(t_i,\alpha,\lambda,\beta) = \ln L$

The MLEs of α, λ and β are obtained by differentiating log-likelihood function concerning α, λ and β respectively and equating to zero. Then the equations are given as

$$\frac{\partial \ln L}{\partial\alpha} = \left[\frac{n_{1f}}{\alpha} + \sum_{i=1}^{n}\delta_{i,1,f}\ln(1-e^{-\lambda t_i})\right] - \sum_{i=1}^{n}\delta_{i,1,c}\frac{(1-e^{-\lambda t_i})^{\alpha}\ln(1-e^{-\lambda t_i})}{\left[1-(1-e^{-\lambda t_i})^{\alpha}\right]}$$
$$+\left[\frac{n_{2f}}{\alpha} + \sum_{i=1}^{n}\delta_{i,2,f}\ln(1-e^{-\lambda\beta x_i})\right] - \sum_{i=1}^{n}\delta_{i,2,c}\left[\frac{(1-e^{-\lambda\beta x_i})\ln(1-e^{-\lambda\beta x_i})}{\left[1-(1-e^{-\lambda\beta x_i})^{\alpha}\right]}\right] \tag{7}$$

$$\frac{\partial \ln L}{\partial\lambda} = \left[\frac{n_{1f}}{\lambda} - \sum_{i=1}^{n}\delta_{i,1,f}t_i + (\alpha-1)\sum_{i=1}^{n}\delta_{i,1,f}\frac{e^{-\lambda t_i}t_i}{1-e^{-\lambda t_i}}\right] + \sum_{i=1}^{n}\delta_{i,1,c}\frac{\alpha(1-e^{-\lambda t_i})^{\alpha}e^{-\lambda t_i}t_i}{\left[1-(1-e^{-\lambda t_i})^{\alpha}\right]}$$
$$+\left[\frac{n_{2f}}{\lambda} - \beta\sum_{i=1}^{n}\delta_{i,2,f}x_i + (\alpha-1)\sum_{i=1}^{n}\delta_{i,2,f}\frac{e^{-\lambda\beta x_i}\beta x_i}{1-e^{-\lambda\beta x_i}}\right] - \sum_{i=1}^{n}\delta_{i,2,c}\left[\frac{\alpha(1-e^{-\lambda\beta x_i})e^{-\lambda\beta x_i}\beta x_i}{\left[1-(1-e^{-\lambda\beta x_i})^{\alpha}\right]}\right] \tag{8}$$

$$\frac{\partial \ln L}{\partial\beta} = \left[\frac{n_{1f}}{\beta} - \lambda\sum_{i=1}^{n}\delta_{i,2,f}x_i + (\alpha-1)\sum_{i=1}^{n}\delta_{i,2,f}\frac{e^{-\lambda\beta x_i}\lambda x_i}{1-e^{-\lambda\beta x_i}}\right] - \sum_{i=1}^{n}\delta_{i,2,c}\left[\frac{\alpha(1-e^{-\lambda\beta x_i})^{\alpha-1}e^{-\lambda\beta x_i}\lambda x_i}{\left[1-(1-e^{-\lambda\beta x_i})^{\alpha}\right]}\right] \tag{9}$$

There is no closed solution of these nonlinear equations. So we use the Newton-Raphson technique for solving these equations.

II. Fisher Information Matrix

The Fisher Information matrix is the composition of negative second partial derivatives of log-likelihood function and can be expressed as

$$I = \begin{bmatrix} -\dfrac{\partial^2 \ln L}{\partial\alpha^2} & -\dfrac{\partial^2 \ln L}{\partial\alpha\partial\lambda} & -\dfrac{\partial^2 \ln L}{\partial\alpha\partial\beta} \\ -\dfrac{\partial^2 \ln L}{\partial\lambda\partial\alpha} & -\dfrac{\partial^2 \ln L}{\partial\lambda^2} & -\dfrac{\partial^2 \ln L}{\partial\lambda\partial\beta} \\ -\dfrac{\partial^2 \ln L}{\partial\beta\partial\alpha} & -\dfrac{\partial^2 \ln L}{\partial\beta\partial\lambda} & -\dfrac{\partial^2 \ln L}{\partial\beta^2} \end{bmatrix} \tag{10}$$

The elements of Fisher-Information matrix is given as

$$\frac{\partial^2 \ln L}{\partial \alpha^2} = \sum_{i=1}^{n} \delta_{i,1,f}\left[-\frac{1}{\alpha^2}\right] - \sum_{i=1}^{n} \delta_{i,1,c} \frac{(1-e^{-\lambda t_i})^{\alpha}\ln(1-e^{-\lambda t_i})}{\left\{1-(1-e^{-\lambda t_i})^{\alpha}\right\}}\left[\ln(1-e^{-\lambda t_i}) + \frac{(1-e^{-\lambda t_i})^{\alpha}\ln(1-e^{-\lambda t_i})}{\left\{1-(1-e^{-\lambda t_i})^{\alpha}\right\}}\right]$$

$$+\sum_{i=1}^{n} \delta_{i,2,f}\left[-\frac{1}{\alpha^2}\right] - \sum_{i=1}^{n} \delta_{i,2,c} \frac{\ln(1-e^{-\lambda\beta x_i})}{\left\{1-(1-e^{-\lambda\beta x_i})\right\}}\left[\ln(1-e^{-\lambda\beta x_i}) + \frac{(1-e^{-\lambda\beta x_i})\ln(1-e^{-\lambda\beta x_i})}{\left[1-(1-e^{-\lambda\beta x_i})^{\alpha}\right]}\right]$$

$$\frac{\partial^2 \ln L}{\partial \alpha \partial \lambda} = \sum_{i=1}^{n} \delta_{i,1,f}\left[-\frac{e^{-\lambda t_i} t_i}{1-e^{-\lambda t_i}}\right]$$

$$+\sum_{i=1}^{n} \delta_{i,1,c}\left[\frac{\left\{1-(1-e^{-\lambda t_i})^{\alpha}\right\}\left\{(1-e^{-\lambda t_i})^{\alpha} e^{-\lambda t_i} t_i + (1-e^{-\lambda t_i})^{\alpha-1}\ln(1-e^{-\lambda t_i}) e^{-\lambda t_i} t_i\right\} + \left\{t_i (1-e^{-\lambda t_i})^{2\alpha-1}\ln(1-e^{-\lambda t_i}) e^{-\lambda t_i}\right\}}{\left\{1-(1-e^{-\lambda t_i})^{\alpha}\right\}^2}\right]$$

$$-\sum_{i=1}^{n} \delta_{i,2,f}\left[\frac{e^{-\lambda\beta x_i}}{(1-e^{-\lambda\beta x_i})}\right] + \sum_{i=1}^{n} \delta_{i,2,c}\left[\frac{\left\{1-(1-e^{-\lambda\beta x_i})^{\alpha}\right\}\left\{(1-e^{-\lambda\beta x_i})^{\alpha-1} e^{-\lambda\beta x_i} + (1-e^{-\lambda\beta x_i})^{\alpha-1}\ln(1-e^{-\lambda\beta x_i}) e^{-\lambda\beta x_i}\beta_i\right\}}{\left\{1-(1-e^{-\lambda\beta x_i})^{\alpha}\right\}^2}\right.$$

$$\left. + \frac{\left\{\beta x_i (1-e^{-\lambda\beta x_i})^{2\alpha-1}\ln(1-e^{-\lambda\beta x_i}) e^{-\lambda\beta x_i}\right\}}{\left\{1-(1-e^{-\lambda\beta x_i})^{\alpha}\right\}^2}\right]$$

$$\frac{\partial^2 \ln L}{\partial \alpha \partial \beta} = \sum_{i=1}^{n} \delta_{i,2,f}\left[\frac{e^{-\lambda\beta x_i}\lambda x_i}{(1-e^{-\lambda\beta x_i})}\right] - \sum_{i=1}^{n} \delta_{i,2,c}\left[\frac{\left\{1-(1-e^{-\lambda\beta x_i})^{\alpha}\right\}\left\{1-(1-e^{-\lambda\beta x_i})^{\alpha-1} e^{-\lambda\beta x_i} - (1-e^{-\lambda\beta x_i})^{\alpha-1}\lambda x_i\right\}}{\left\{1-(1-e^{-\lambda\beta x_i})^{\alpha}\right\}^2}\right.$$

$$\left. + \frac{\left\{(1-e^{-\lambda\beta x_i})^{2\alpha-1}\ln(1-e^{-\lambda\beta x_i})\lambda x_i e^{-\lambda\beta x_i}\right\}}{\left\{1-(1-e^{-\lambda\beta x_i})^{\alpha}\right\}^2}\right]$$

$$\frac{\partial^2 \ln L}{\partial \lambda \partial \alpha} = \sum_{i=1}^{n} \delta_{i,1,f}\left[-\frac{e^{-\lambda t_i} t_i}{1-e^{-\lambda t_i}}\right] + \sum_{i=1}^{n} \delta_{i,1,c}\left[\frac{t_i e^{-\lambda t_i}(1-e^{-\lambda t_i})^{\alpha-1}}{\left\{1-(1-e^{-\lambda t_i})^{\alpha}\right\}}\left\{\alpha^{-1} + \ln(1-e^{-\lambda t_i}) + \frac{(1-e^{-\lambda t_i})^{\alpha}\ln(1-e^{-\lambda t_i})}{\left\{1-(1-e^{-\lambda t_i})^{\alpha}\right\}}\right\}\right]$$

$$+\sum_{i=1}^{n} \delta_{i,2,f}\left[\frac{e^{-\lambda\beta x_i}\beta x_i}{(1-e^{-\lambda\beta x_i})}\right] + \sum_{i=1}^{n} \delta_{i,2,c}\left[\frac{\beta x_i e^{-\lambda\beta x_i}(1-e^{-\lambda\beta x_i})}{1-(1-e^{-\lambda\beta x_i})^{\alpha}}\left\{\alpha^{-1} + \ln(1-e^{-\lambda t\beta x_i}) + \frac{(1-e^{-\lambda\beta x_i})^{\alpha}\ln(1-e^{-\lambda t\beta x_i})}{\left\{1-(1-e^{-\lambda\beta x_i})^{\alpha}\right\}}\right\}\right]$$

$$\frac{\partial^2 \ln L}{\partial \lambda^2} = -\sum_{i=1}^{n} \delta_{i,1,f}\left[\frac{1}{\lambda^2} + (\alpha-1)\frac{e^{-\lambda t_i} t_i}{1-e^{-\lambda t_i}}\left\{\frac{t_i + e^{-\lambda t_i}}{1-e^{-\lambda t_i}}\right\}\right]$$

$$+\sum_{i=1}^{n} \delta_{i,1,c}\alpha t_i\left[\frac{(1-e^{-\lambda t_i})e^{-\lambda t_i}}{1-(1-e^{-\lambda t_i})^{\alpha}}\left\{\frac{(\alpha-1)(1-e^{-\lambda t_i})^{\alpha-2} t_i e^{-\lambda t_i}}{(1-e^{-\lambda t_i})^{\alpha-1}} - t_i + \frac{\alpha e^{-\lambda t_i}(1-e^{-\lambda t_i})}{1-(1-e^{-\lambda t_i})^{\alpha}}\right\}\right]$$

$$-\sum_{i=1}^{n} \delta_{i,2,f}\left[\frac{1}{\lambda^2} + (\alpha-1)\frac{e^{-\lambda\beta x_i} t_i}{1-e^{-\lambda\beta x_i}}\left\{\frac{\beta x_i + e^{-\lambda\beta x_i}}{1-e^{-\lambda\beta x_i}}\right\}\right]$$

$$+\sum_{i=1}^{n} \delta_{i,2,c}\alpha\beta x_i\left[\frac{(1-e^{-\lambda\beta x_i})e^{-\lambda\beta x_i}}{1-(1-e^{-\lambda\beta x_i})^{\alpha}}\left\{\frac{(\alpha-1)(1-e^{-\lambda\beta x_i})^{\alpha-2} - \lambda\beta x_i e^{-\lambda\beta x_i}}{(1-e^{-\lambda\beta x_i})^{\alpha-1}} - \lambda\beta x_i + \frac{\alpha e^{-\lambda\beta x_i}(1-e^{-\lambda\beta x_i})}{1-(1-e^{-\lambda\beta x_i})^{\alpha}}\right\}\right]$$

$$\frac{\partial^2 \ln L}{\partial \lambda \partial \beta} = \sum_{i=1}^{n} \delta_{i,2,f}\left[-x_i + (\alpha-1)\frac{e^{-\lambda\beta x_i}\beta x_i}{1-e^{-\lambda\beta x_i}}\left\{-\lambda x_i + \frac{1}{\beta} - \frac{\lambda x_i e^{-\lambda\beta x_i}}{1-e^{-\lambda\beta x_i}}\right\}\right]$$

$$+\sum_{i=1}^{n} \delta_{i,2,c}\frac{\alpha(1-e^{-\lambda\beta x_i})e^{-\lambda\beta x_i}\beta x_i}{1-(1-e^{-\lambda\beta x_i})^{\alpha}}\left[\frac{e^{-\lambda\beta x_i}}{(1-e^{-\lambda\beta x_i})} - \lambda x_i + \frac{1}{\beta} + \frac{\alpha(1-e^{-\lambda\beta x_i})^{\alpha-1}e^{-\lambda\beta x_i}\lambda x_i}{1-(1-e^{-\lambda\beta x_i})^{\alpha}}\right]$$

$$\frac{\partial^2 \ln L}{\partial\beta\partial\alpha} = \sum_{i=1}^{n}\delta_{i,2,f}\left[\frac{e^{-\lambda\beta x_i}\lambda x_i}{1-e^{-\lambda\beta x_i}}\right]$$
$$+\sum_{i=1}^{n}\delta_{i,2,c}e^{-\lambda\beta x_i}\lambda x_i\left[\left\{\frac{\alpha(1-e^{-\lambda\beta x_i})^{\alpha-1}}{1-(1-e^{-\lambda\beta x_i})}\right\}\left\{\frac{1}{\alpha}+\ln(1-e^{-\lambda\beta x_i})+\frac{(1-e^{-\lambda\beta x_i})^{\alpha}\ln(1-e^{-\lambda\beta x_i})}{1-(1-e^{-\lambda\beta x_i})^{\alpha}}\right\}\right]$$

$$\frac{\partial^2 \ln L}{\partial\beta\partial\lambda} = \sum_{i=1}^{n}\delta_{i,2,f}\left[-x_i+(\alpha-1)x_i\frac{\lambda x_i}{1-e^{-\lambda\beta x_i}}\left\{\frac{1}{\lambda}-\beta x_i-\frac{e^{-\lambda\beta x_i}\beta x_i}{1-e^{-\lambda\beta x_i}}\right\}\right]$$
$$+\sum_{i=1}^{n}\delta_{i,2,c}\alpha x_i\left[\left\{\frac{\lambda(1-e^{-\lambda\beta x_i})^{\alpha-1}e^{-\lambda\beta x_i}}{1-(1-e^{-\lambda\beta x_i})}\right\}\left\{\frac{1}{\lambda}-\beta x_i+\frac{(\alpha-1)(1-e^{-\lambda\beta x_i})^{\alpha-2}e^{-\lambda\beta x_i}\beta x_i}{(1-e^{-\lambda\beta x_i})^{\alpha-1}}+\frac{\alpha(1-e^{-\lambda\beta x_i})^{\alpha-1}e^{-\lambda\beta x_i}\beta x_i}{1-(1-e^{-\lambda\beta x_i})^{\alpha-1}}\right\}\right]$$

$$\frac{\partial^2 \ln L}{\partial\beta^2} = \sum_{i=1}^{n}\delta_{i,2,f}\left[-\frac{1}{\beta^2}+(\alpha-1)\lambda x_i\frac{e^{-\lambda\beta x_i}}{1-e^{-\lambda\beta x_i}}\left\{-\lambda x_i-\frac{e^{-\lambda\beta x_i}\lambda x_i}{1-e^{-\lambda\beta x_i}}\right\}\right]$$
$$+\sum_{i=1}^{n}\delta_{i,2,c}\alpha\lambda x_i\left[\left\{\frac{(1-e^{-\lambda\beta x_i})e^{-\lambda\beta x_i}}{1-(1-e^{-\lambda\beta x_i})^{\alpha-1}}\right\}\left\{\frac{\lambda x_i e^{-\lambda\beta x_i}}{(1-e^{-\lambda\beta x_i})}-\lambda x_i+\frac{\alpha(1-e^{-\lambda\beta x_i})^{\alpha-1}e^{-\lambda\beta x_i}\lambda x_i}{(1-e^{-\lambda\beta x_i})^{\alpha}}\right\}\right]$$

The asymptotic variance-covariance is simply obtained by taking the inverse of the Fisher Information matrix. The asymptotic variance-covariance is given as

$$\Sigma = I^{-1} = \begin{bmatrix} -\frac{\partial^2 \ln L}{\partial\alpha^2} & -\frac{\partial^2 \ln L}{\partial\alpha\partial\lambda} & -\frac{\partial^2 \ln L}{\partial\alpha\partial\beta} \\ -\frac{\partial^2 \ln L}{\partial\lambda\partial\alpha} & -\frac{\partial^2 \ln L}{\partial\lambda^2} & -\frac{\partial^2 \ln L}{\partial\lambda\partial\beta} \\ -\frac{\partial^2 \ln L}{\partial\beta\partial\alpha} & -\frac{\partial^2 \ln L}{\partial\beta\partial\lambda\partial} & -\frac{\partial^2 \ln L}{\partial\beta^2} \end{bmatrix}^{-1} = \begin{bmatrix} AVar(\hat{\alpha}) & ACov(\hat{\alpha}\hat{\lambda}) & ACov(\hat{\alpha}\hat{\beta}) \\ ACov(\hat{\lambda}\hat{\alpha}) & AVar(\hat{\lambda}) & ACov(\hat{\lambda}\hat{\beta}) \\ ACov(\hat{\beta}\hat{\alpha}) & ACov(\hat{\beta}\hat{\lambda}) & AVar(\hat{\beta}) \end{bmatrix} \quad (11)$$

Where, $AVar$ and $ACov$ stand for asymptotic variance, asymptotic covariance respectively.

IV. Interval Estimates

A confidence interval for parameters is a type of interval estimate, computed from the statistics of the observed data, that consist of the accurate value of an unknown population parameter. In other words, a confidence interval is simply the probability. So, a confidence interval means the probability that the value of a parameter will fall between the lower and upper bound of a probability distribution. Mostly, 90%, 95%, and 99% confidence levels are used.

The two-sided confidence limits can be constructed as

$$p\left[-z \le \frac{\hat{\varphi}-\varphi}{\sigma(\hat{\varphi})} \le z\right] = 1-\kappa \quad (13)$$

This construction of two-sided confidence limits is for the maximum likelihood estimate $\hat{\varphi}$ of a population parameter $\psi = (\alpha, \lambda, \beta)$. In the above equation (13), z stands for $100(1-\kappa/2)$ the standard normal percentile and κ stands for the significance level. So, for a population parameter

φ, an appropriate confidence limits can be obtained, such that

$$p[\hat{\varphi} - z\sigma(\hat{\varphi}) \leq \varphi \leq \hat{\varphi} + z\sigma(\hat{\varphi})] = 1 - \kappa$$

Where, lower confidence limit $L_\varphi = \hat{\varphi} - z\sigma(\hat{\varphi})$ and upper confidence limit $U_\varphi = \hat{\varphi} + z\sigma(\hat{\varphi})$

V. Simulation Study

In this section, we perform a simulation study to check the performance of the estimators having Exponentiated Exponential distribution using multiple censored data. This simulation study is done Monte Carlo Simulation technique by using R-Software. The means square error and bias are estimated to check the performance of estimators. The following steps are made for this simulation study.

- First, we divide the total sample n into two parts, n_1 and n_2. $n_1 = n\pi$ and $n_2 = n(1-\pi)$
- Generate $t_{1,1} < t_{2,2} < \ldots < t_{n_1,1}$ and $t_{2,1} < t_{2,2} < \ldots < t_{n_2,2}$ random samples of size n_1 and n_2 in normal and stress condition respectively from Exponentiated Exponential distribution.
- We generate 1000 random of size 50, 100, 150 and 200 and choose the values of the parameters as Case (I) $(\alpha = 0.6, \lambda = 0.6, \beta = 1.6)$, Case (II) $(\alpha = 0.6, \lambda = 0.6, \beta = 1.8)$ Case (III) $(\alpha = 0.4, \lambda = 0.8, \beta = 1.6)$, Case (IV) $(\alpha = 0.4, \lambda = 0.8, \beta = 1.8)$
- The acceleration factor and the distribution parameters are obtained for each sample and each set of parameters. The asymptotic variance and covariance matrix are also obtained for each set of parameters.
- Finally, for confidence levels $\gamma = 95\%, 99\%$ of acceleration factor, the two sides confidence limits and two parameters are constructed with the use of equation (13) for parameters α, λ and β.

Table 1: The values of Bias and MSE under the different size of samples for multiple censored data

n	Parameters	Case I $(\alpha = 0.6, \lambda = 0.6, \beta = 1.6)$			Case II $(\alpha = 0.6, \lambda = 0.6, \beta = 1.8)$		
		Estimates	Bias	MSE	Estimates	Bias	MSE
50	α	0.638	0.321	0.082	0.712	0.302	0.098
	λ	0.812	0.083	0.023	0.912	0.098	0.036
	β	1.321	0.068	0.235	1.543	0.076	0.243
100	α	0.616	0.310	0.092	0.743	0.298	0.094
	λ	0.823	0.078	0.019	0.843	0.088	0.034
	β	1.313	0.576	0.206	1.654	0.0702	0.224
150	α	0.602	0.297	0.076	0.765	0.287	0.087
	λ	0.801	0.065	0.014	0.921	0.784	0.045
	β	1.304	0.521	0.184	1.432	0.687	0.286
200	α	0.602	0.288	0.071	0.700	0.301	0.900
	λ	0.792	0.075	0.011	0.933	0.654	0.028
	β	1.297	0.543	0.098	1.876	0.765	0.198

Table 2: The values of Bias and MSE under the different size of samples for multiple censored data

n	Parameters	Case III $(\alpha=0.4, \lambda=0.8, \beta=1.6)$			Case IV $(\alpha=0.4, \lambda=0.8, \beta=1.8)$		
		Estimates	Bias	MSE	Estimates	Bias	MSE
50	α	0.543	0.289	0.064	0.612	0.598	0.078
	λ	0.865	0.265	0.054	0.923	0.336	0.067
	β	1.323	0.086	0.342	1.257	0.089	0.476
100	α	0.564	0.265	0.608	0.645	0.566	0.065
	λ	0.843	0.200	0.046	0.946	0.289	0.065
	β	1.456	0.076	0.298	1.345	0.081	0.398
150	α	0.486	0.286	0.586	0.596	0.500	0.054
	λ	0.802	0.202	0.065	0.897	0.288	0.058
	β	1.487	0.065	0.299	1.446	0.076	0.411
200	α	0.598	0.254	0.566	0.665	0.456	0.066
	λ	0.843	0.198	0.0421	0.886	0.328	0.048
	β	1.543	0.076	0.256	1.225	0.067	0.356

Table 3: Asymptotic Variance and Covariance Matrix of Estimators for Different Size of Samples under Multiple Censored Data

n	Parameters	Case I $(\alpha=0.6, \lambda=0.6, \beta=1.6)$			Case II $(\alpha=0.6, \lambda=0.6, \beta=1.8)$		
		α	λ	β	α	λ	β
50	α	0.00632	0.00226	0.00456	0.00776	0.00211	0.00509
	λ	0.00321	-0.00784	0.00387	0.00224	0.00449	0.00277
	β	0.00437	0.00298	0.04541	0.00443	0.00109	0.00118
100	α	0.00576	0.00276	0.00432	0.00654	0.00210	0.00498
	λ	0.00227	-0.00876	0.00267	0.00221	0.00265	0.00176
	β	0.00338	0.00234	0.00453	0.00343	0.00025	0.00101
150	α	0.00465	0.00199	0.00365	0.00554	0.00189	0.00334
	λ	0.00176	-0.00998	0.00223	0.00176	0.00228	0.00116
	β	0.00225	0.00178	0.00116	0.00225	-0.00987	-0.00554
200	α	0.00356	0.00113	0.00294	0.00445	0.00156	0.00223
	λ	0.00114	-0.00887	0.00132	0.00114	0.00189	0.00115
	β	0.00115	0.00117	0.00101	0.00112	-0.00998	-0.00776

Table 4: Asymptotic Variance and Covariance Matrix of Estimators for Different Size of Samples under Multiple Censored Data

n	Parameters	Case III $(\alpha=0.4,\lambda=0.8,\beta=1.6)$			Case IV $(\alpha=0.4,\lambda=0.8,\beta=1.8)$		
		α	λ	β	α	λ	β
50	α	0.00332	0.00098	0.00543	0.00376	0.00076	0.00432
	λ	0.00254	0.00221	0.00065	0.00577	0.00981	0.00087
	β	0.00443	0.00545	0.00334	-0.00654	0.00443	-0.00043
100	α	0.00224	0.00065	0.00332	0.00331	0.00054	0.00224
	λ	0.00223	0.00188	0.00045	0.00443	0.00076	0.00066
	β	0.00376	0.00332	0.00224	-0.00765	0.00411	-0.00066
150	α	0.00202	0.00043	0.00223	0.00269	0.00044	0.00187
	λ	0.00123	0.00117	0.00032	0.00332	0.00387	0.00054
	β	0.00321	0.00212	-0.00987	-0.00799	0.00332	-0.00098
200	α	0.00187	0.00011	0.00165	0.00211	0.00012	0.00112
	λ	0.00115	0.00076	0.00011	0.00287	0.00225	0.00043
	β	0.00234	0.00133	-0.00999	-0.00998	0.00225	-0.00076

Table 5: At Confidence Level $\kappa = 95\%, 99\%$, the Confidence Bounds of Estimates at Different Size of Samples

n	Parameters	Case I $(\alpha=0.4,\lambda=0.8,\beta=1.6)$				σ	Case I $(\alpha=0.4,\lambda=0.8,\beta=1.8)$				σ
		Confidence Interval $z=1.96$		Confidence Interval $z=2.58$			Confidence Interval $z=1.96$		Confidence Interval $z=2.58$		
		Lower Bound	Upper Bound	Lower Bound	Upper Bound		Lower Bound	Upper Bound	Lower Bound	Upper Bound	
50	α	0.57	0.73	0.53	0.89	0.08	0.51	0.78	0.61	0.93	0.07
	λ	0.68	0.89	0.57	0.76	0.04	0.55	0.86	0.67	0.83	0.10
	β	0.88	1.32	0.66	0.91	0.38	0.79	1.89	0.87	1.90	0.32
100	α	0.59	0.67	0.55	0.84	0.09	0.57	0.84	0.73	0.99	0.09
	λ	0.61	0.75	0.66	0.80	0.06	0.61	0.82	0.62	0.79	0.06
	β	0.77	1.34	0.73	0.88	0.43	0.98	1.56	0.97	2.11	0.35
150	α	0.64	0.71	0.67	0.81	0.06	0.44	0.60	0.65	0.76	0.05
	λ	0.64	0.76	0.58	0.72	0.09	0.87	0.93	0.56	0.69	0.08
	β	0.79	1.22	0.69	0.81	0.48	0.78	1.23	0.67	1.36	0.42
200	α	0.59	0.67	0.71	0.79	0.08	0.56	0.65	0.54	0.67	0.06
	λ	0.55	0.63	0.69	0.82	0.03	0.74	0.82	0.65	0.76	0.09
	β	0.88	1.01	0.61	0.73	0.35	0.77	1.02	0.68	1.11	0.36

Table 6: At Confidence Level $\kappa = 95\%, 99\%$, the Confidence Bounds of Estimates at Different Size of Samples

n	Parameters	Case I ($\alpha=0.4, \lambda=0.8, \beta=1.6$) Confidence Interval $z=1.96$ Lower Bound	Case I $z=1.96$ Upper Bound	Case I Confidence Interval $z=2.58$ Lower Bound	Case I $z=2.58$ Upper Bound	σ	Case I V ($\alpha=0.4, \lambda=0.8, \beta=1.8$) Confidence Interval $z=1.96$ Lower Bound	Case I V $z=1.96$ Upper Bound	Case I V Confidence Interval $z=2.58$ Lower Bound	Case I V $z=2.58$ Upper Bound	σ
50	α	0.61	0.78	0.56	0.94	0.04	0.57	0.78	0.58	0.84	0.09
	λ	0.65	0.86	0.61	0.79	0.06	0.55	0.78	0.69	0.95	0.12
	β	0.78	1.22	0.61	0.85	0.32	0.72	1.86	0.75	1.60	0.39
100	α	0.55	0.69	0.61	0.79	0.08	0.68	0.94	0.64	0.79	0.08
	λ	0.56	0.71	0.77	0.89	0.07	0.68	0.87	0.62	0.79	0.10
	β	0.66	1.23	0.63	0.78	0.37	0.78	1.36	0.97	2.11	0.42
150	α	0.61	0.72	0.63	0.71	0.05	0.55	0.68	0.69	0.81	0.03
	λ	0.54	0.65	0.61	0.69	0.08	0.78	0.85	0.56	0.69	0.13
	β	0.68	1.10	0.79	0.85	0.42	0.67	1.11	0.67	1.36	0.47
200	α	0.55	0.62	0.65	0.72	0.02	0.64	0.71	0.65	0.69	0.07
	λ	0.59	0.66	0.65	0.76	0.05	0.64	0.69	0.68	0.79	0.02
	β	0.72	0.81	0.69	0.76	0.37	0.86	1.02	0.79	1.19	0.39

VI. Conclusions

This paper presented an inference on constant stress partially accelerated life tests for the Exponentiated Exponential distribution using multiple censoring schemes. The following observations are made based on the simulation study. The observations are

- In the table (1) and (2), the MSE and bias of estimators are obtained in four cases, and we can observe that the sample size increases the values of bias and MSEs decreases. The maximum likelihood estimates have good statistical properties for all sets of parameters because this set has the smallest biases for all sample sizes.
- In the table (3) and (4), the asymptotic variance and covariance matrix are obtained, and we can observe that the asymptotic variance-covariance of estimators decreases as sample size increases for the all sets of parameters.
- In the table (5) and (6), the confidence limits of the intervals for the parameters and the acceleration factor at 95% and 99% are obtained. The standard deviation (σ) of estimators is also obtained. We can observe that the width of the interval decreases as sample size increases for all sets of parameters.

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A Review on Reusability of Component Based Software Development

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Abstract

People across software community have marked the true endeavor to go round development of software through component based software development (CBSD) practices. Reusability of software component has a very positive impact on development time, cost, reliability and marketability of the software. In this paper we are discussing about bottlenecks of software component reusability and trying to provide some useful guidelines to improve the reusability for the component based software developer which can further improve the productivity, reduce the development cost. This paper also discusses the steps for effectively storing the software component in component repository that is helping the component user in finding the most eligible component for reuse which will progress the component based software development.

Keywords: CBSD, Component Repository System (CRS), Component Classification (CC), COTS.

I Introduction

Reusability of software Component is considered as best answer to multiple problems faced by software developer community. It is highly accepted and appreciated for faster development of large and multifaceted system, reducing development expenditure and improving overall quality of software. Many software developing organizations that are functioning in this domain have already started developing their own component library and claimed considerable benefits from it. They are trying to develop such software component which is reusable in multiple implementations without any change or only with minor changes. Such components are thoroughly documented and certified in order to make it portable across different hardware and operating system. Other software developing organizations have supported their reuse policy with Commercial off-The Shelf (COTS). Customer for such software component has no access to source code in most of the cases therefore they are only looking for certified software component which is reliable and trustworthy. As the size of component library is increasing due to their increasing demand, component selection process is another big challenge for component reusability. Therefore many software organizations are spending a large amount of time in development of appropriate component repository and efficient retrieval of reusable component. Since the selection of the suitable components has a foremost impact on the software project and finally the resulting products. Researchers in this domain have suggested classifying the component repository into three different categories based on availability of source code for individual component to improve the reusability. This will also make the retrieval and selection of software component easier and faster. A global survey conducted recently by the research community has observed that almost two third of the software developing organizations are practicing component based approach in course of software development.

Component integrator lacks in trust for a software component developed by a third party vendor during deployment of Component as per defined component architecture. This is again a huge challenge for component reuse. The reason behind mistrust and lack of confidence among component integrator is that majority of the component is available without source code and without proper documentation. A component developed in a particular programming language is available for reuse in different operating environment with different prospective of component user. This characteristic of the component is again causing a big hurdle for component reuse. A component of large size is again not very friendly to component user due to its complexity and interoperability.

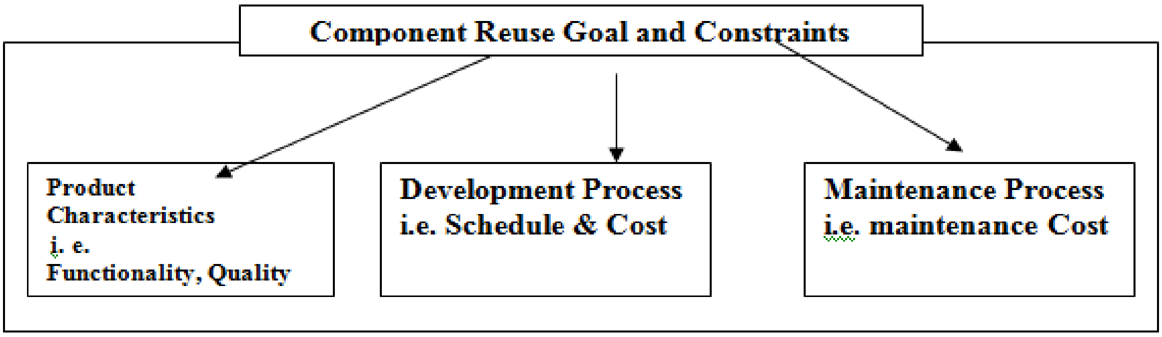

Figure: 1 (Reuse Goal and Constraints)

II Identifying reusable software Component

Concept of software reuse is continuously improving due to rapid demand of new software in less time and less cost. In general Software Company is continuously facing both technical and economical challenges due to ever changing business requirement. Effective software reuse can always help in improving business profitability and reducing the development time of the software. Component based software development is solely depending on reuse of preexisting software component. The purpose of CBSD is to built once and reuses any number of times with no modifications or minor modifications. Therefore reuse policy for Component Based software development can be generally categorized into:

Reuse of preexisting software component without any change.

Reuse of preexisting software component with change.

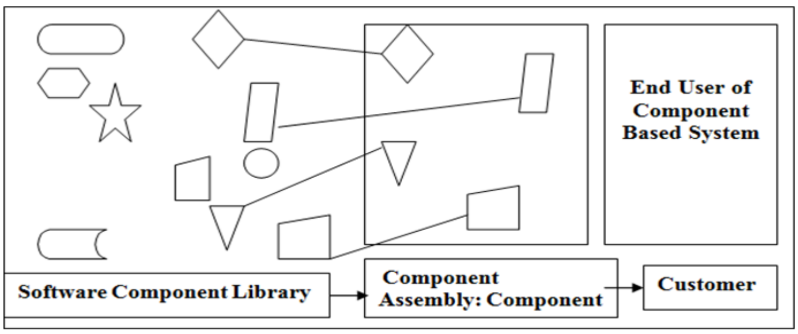

Figure 2: Component assembly process used by component integrator

As we already discussed about unavailability of source code for most of the software component due to its black box nature. Therefore its implementation is entirely hidden from component user. It has to be used as plug and play devices. At the time of reuse we cannot make any change into it. They are directly used into the application and integrated via component interface.

On the other hand if the software component considered for reuse needs some minor modifications due to little discrepancy between component functional and non functional characteristics and characteristics of component based software which is presently going to develop then we prefer to have software component which are developed in-house. In that case source code of the component is available for required modifications. Even after change the modified components need to be thoroughly tested before plugging it into new system.

III Guidelines for improving reusability

Reuse goals of software component and its process of integration for development of component based software is mentioned in Figure 1 & 2.Component reusability has very high impact on overall quality of the component based software. On the basis of review of various aspects of component reuse we are suggesting following guidelines to the different stakeholders of software component which can improve component reusability:

Table 1: Findings and Observations for improving component reusability

S. No	Guidelines
1	Language used for development of new component must be platform independent.
2	Code structure used for development of component must be of moderate size and easy to test.
3	During component development it must be thoroughly documented for easy to use and future implementation.
4	Certification of each component must be considered on a serious note by component developer
5	Design specification of component must be optimized for next implementation.
6	Development of new component must be based on its portability across different hardware and operating system .
7	Applying suitable reuse metrics for better understanding of reusable components
8	Standardization of component retrieval process

IV Repository Development of software component

Most of the component users are facing plenty of challenges in effectively retrieving the reusable components to develop component based software as per specific user requirements. Large number of components in the component repository makes the component retrieval process more and more challenging and it takes lots of time and consumes more resources. Proper classification, cataloging and certification of the component need to be performed for every component before placing it into component repository. It supports the component user not only in faster retrieval of component but also in finding most suitable component in application development. Component repository is most valuable asset in the process of developing component based software. Repository for the software component must be upgraded on regular basis. Budgeting is another important aspect which must be communicated to the component used time to time. A systematic approach for developing component repository is mentioned in figure 3. Software component information is stored into the repository using tools for Component Repository System (CRS) as mentioned in the figure. Searching of component is based on using suitable keyword. This method of searching helps in faster retrieval of component without much knowledge about it.

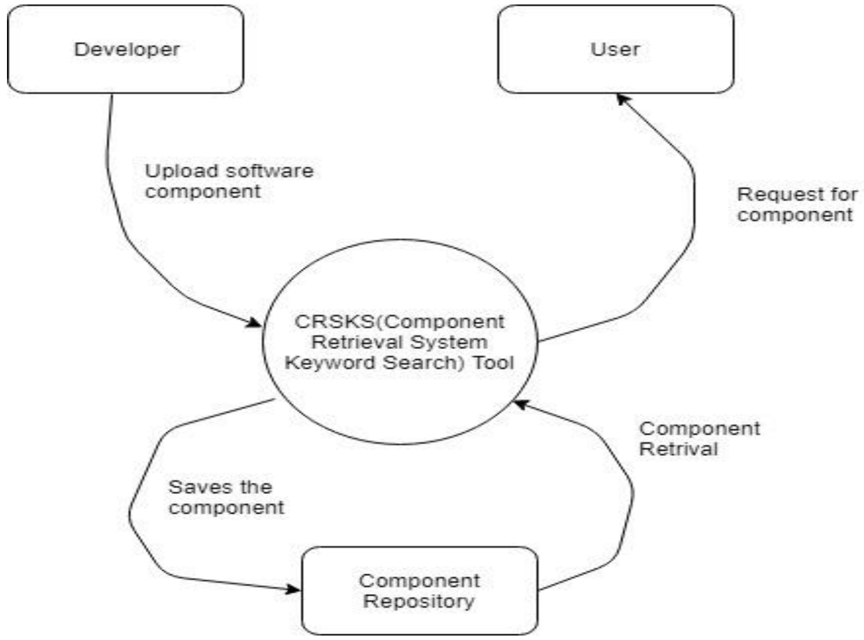

Figure 3: Repository Development for Software Component

V Conclusion and Future Work

In this paper we have discussed about various reusability constraints of component based software development and further made effort to provide some useful tips about improving component reuse to the different stakeholders of component-based software. This paper also gives an insight view of development of component repository which is helping the component user in effective and faster retrieval of software component without much knowledge about it. Retrieval of component is based on keyword search technique which can be further improved based on some advance search algorithm. This algorithm may help the component user in improving component reusability and faster retrieval of components.

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An Approximation to Joint System Size Distribution at Nodes in Some Multi-hop Wireless Networks

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Abstract

In this paper, we discuss a network model to study the queueing characeristics of nodes in a multi-hop wireless network, under the standard binary exponential back-off (BEB) contention resolution scheme. Based on the steady state distribution of the system size at a node, which was appeared in Sweta & Deepak [2], we compute the joint distribution of system size at all nodes in a multi-hop network, governed by some specific queue disciplines. Getting information on the joint queuing size distribution in the network will enable us to control the traffic (and hence congestion) in the whole network. In order to illustrate our theoretical results, a particular multi-hop network model is considered and analysed numerically.

Keywords: queuing networking model , multi-hop wireless network, joint system size distribution

I Introduction

Sharing data files among nodes in a network without a port, and hence by using less expensive infrastructure is one of the major attractions for switching over to wireless network from wired one. In ad hoc wireless networks, each node acts not only as a host but also as a relay of packets for forwarding packets to another nodes which are not in the direct transmission range of source nodes.

In order to illustrate the dynamics and behaviour of nodes in a wireless network, we consider a simple network, having 4 nodes with gateway GW, as shown in figure 1. Among the 4 nodes, assume that the nodes 1 and 3 are source nodes. That is, external arrivals can be generated only at these nodes. The entire route of the packets generated at each of the source nodes is also shown in figure 1.

A circle centred at a node defines the transmission range of that node. All nodes that are lying inside the transmission range of a paricular node are called the one hop neighbours of that node. All other nodes that are lying inside the circles centred at all one-hop neighbours of a node are called its two-hop neighbours. In figure 1, node 1 has only one one-hop neighbour which is node 2 but it has two two-hop neighbours namely, node 3 and 4. Node 2 has 3 one-hop neighbors 1,3 and 4, but it has no two-hop neighbour. Similarly for node 3, one-hop neighbour is node 2 and two hop neighbours are 1 and 4. If a transmission is being taken place between two nodes, all one-hop neighbours of those two nodes will sense the channel as busy, but all two-hop nodes, being not in the transmission range of source node, will not be able to sense the channel and hence there can be a chance of collision due to the possibile simultaneous transmission by these nodes.

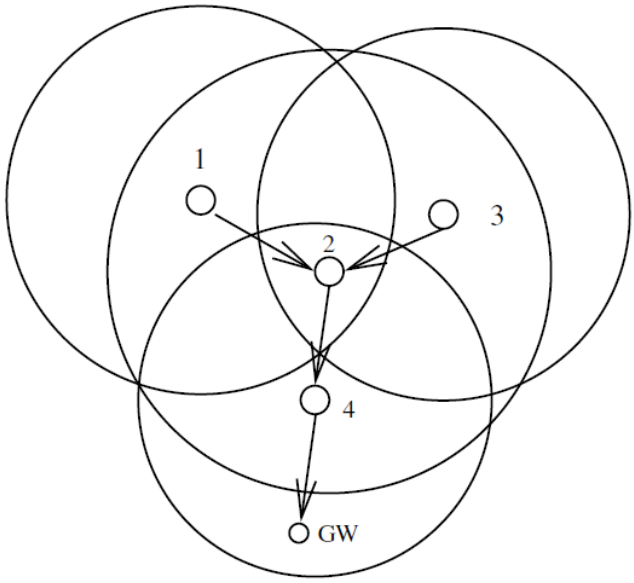

Figure 1: A general network

Many protocols have been proposed to reduce the chances of collission resulted from simultaneous transmissions by several nodes in multi-hop routing networks. Among them IEEE 802.11 [5] has been accepted as international standard, where the fundamental mechanism to access the medium is the distributed co-ordination function (DCF). According to the DCF basic access mechanism, a node with a packet for transmission monitors the channel activity and if the channel is found idle for a predetermined period called DIFS (distributed inter frame space), it transmits the packet. If the channel is found busy, the node undergoes a random back-off period- a random number of time slots- and initializes a back-off counter. At each instant at which the channel is monitored, the back-off counter is decremented if the channel is found idle for a period longer than DIFS, else it is frozen. The node, of which back-off counter expires first, begins transmission and all of its neighbouring nodes freeze their counters. Once the current transmission gets completed, back-off processes of all neighbours of transmitting node resume as explained above.

In order to minimize the possibility of collisions due to multiple simultaneous transmissions, DCF employs several contention resolution schemes namely, binary exponential back-off (BEB) rule, LIMD (linear increase multiple decrease) rule and so on. In our work (here and in our earlier problem), BEB rule is used as it is the most standard one. The rule is explained briefly as given below: If a packet is ready for transmission from a node, contention window size is chosen as W and a random value from 0, 1,2, $W-1$ is uniformly selected as its back-off counter. If the packet does not get transmittied successfully, that is, it meets with a collision in that attempt, the contention window size will be doubled so that it is set as $W_1 = 2W$. A value for back-off counter is selected uniformly from 0, 1,2, $W_1 - 1$. If it further meets with a collision on its next attempt, the contention window size will be doubled again and this will continue up to a maximum of m collisions. After m unsuccessful attempts, if it again meets with a collision, the contention window size will be fixed as $W_m = 2^m W$. If an attempt results in successful transmission, the contention window size for that node will be reset as W. Hence

$$\begin{aligned} CW_{min} &= W, \\ CW_{max} &= 2^m W. \end{aligned}$$

As an attempt to learn some major characteristics of waiting packets at an arbitrary node in a wireless network, Sweta & Deepak [2] proposed a model and analysed it by matrix theoretic approach to get some important statistical characteristics such as probability distributions of system size, waiting time of packets, number of collisions experienced by a packet at a single node, and their moments in a rigorous manner. However, Sweta & Deepak [2] couldn't take up the problem of computing the joint distribution of system size at all nodes in the entire network due to a large dimensional state space. Here, we use the theoretical approach developed by Kelly [3] to address this for a network, governed by some specific queue disciplines. A summary of the assumptions and results that appeared in Sweta & Deepak [2], and relevant to the present problem too, is given in the next section.

II Some of our earlier results

The major assumptions in Sweta & Deepak [2] were:

(i) Packets are generated at a node according to a Poisson rule of rate λ, and join a waiting line till they are being considered for transmission.

(ii) At an epoch at which a packet is considered for transmission, the back-off period for the node commences if it senses the channel as idle, and if so the node selects a back-off counter uniformly from 0, 1, 2, $W-1$. If the packet has already experienced j collisions, then the back off counter will be from 0, 1, 2, $W_j - 1$. Also, time spent on counters are assumed to be independent and identically distributed exponential variates having mean $1/\mu$.

(iii) If the channel is found busy after completion of a back off counter time, the back off timer will be frozen and will commence again only after the channel is sensed as idle. The channel idle periods and busy periods are taken as independent Phase type (PH) variates with representations (α_1, T_1) and (α_2, T_2) of order n_1 and n_2 respectively. For details on PH variates, see Neuts [4].

(iv) When the back-off counter at a particular back-off stage becomes zero, the node starts transmission. Packet transmission times are assumed to be independent and identical exponential variates having mean $1/\gamma$.

(v) A transmission results in collision with probability p and is successful with probability $1-p$.

The underlying Markov process in connection with the dynamics of a specific node could be seen as a Quasi Birth-Death (QBD) process and hence its steady state analysis could be carried out by the matrix analytical approach (See Neuts[4]).

Among major results in Sweta & Deepak [2], the one which is relevant to the present model is given below:

3.1 Distribution of the time between the instant at which a packet is considered for transmission and the instant at which it is successfully transmitted

If U represents the duration of time from the epoch at which a packet is chosen for transmission till the epoch at which it is successfully transmitted, we proved that U is a continuous phase type variate having representation (β, S). Here,

$$\beta = \begin{bmatrix} \frac{1}{W} & \frac{\alpha_1}{W} & \cdots & \frac{\alpha_1}{W} & 0 & 0 & \cdot & 0 \end{bmatrix}$$

and

$$S = \begin{bmatrix} D_0 & B_1 & 0 & \cdot & 0 & 0 & 0 & \cdot & 0 \\ 0 & D_1 & B_2 & \cdot & 0 & 0 & 0 & \cdot & 0 \\ 0 & 0 & D_2 & \cdot & 0 & 0 & 0 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & D_m + B_m & 0 & 0 & \cdot & 0 \\ F_0 & 0 & 0 & \cdot & 0 & G_0 & 0 & \cdot & 0 \\ 0 & F_1 & 0 & \cdot & 0 & 0 & G_1 & \cdot & 0 \\ \cdot & \cdot & F_2 & \cdot & 0 & 0 & 0 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & F_m & 0 & 0 & \cdot & G_m \end{bmatrix}$$

where

$$B_i = \begin{bmatrix} p\gamma/W_i & p\gamma\alpha_1/W_i & p\gamma\alpha_1/W_i & \cdots & p\gamma\alpha_1/W_i \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

for $i = 1,2, \dots m$ and

$$D_i = \begin{bmatrix} -\gamma & 0 & 0 & \cdots & 0 \\ \mu e & T_1 - \mu I & 0 & \cdots & 0 \\ 0 & \mu I & T_1 - \mu I & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & T_1 - \mu I \end{bmatrix}$$

for $i = 0,1,2, \dots m$.
Also,

$$F_i = \begin{bmatrix} 0 & T_2^0\alpha_1 & 0 & \cdots & 0 \\ 0 & 0 & T_2^0\alpha_1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & T_2^0\alpha_1 \end{bmatrix}$$

and

$$G_i = \begin{bmatrix} T_2 & 0 & 0 & \cdots & 0 \\ 0 & T_2 & 0 & \cdots & 0 \\ 0 & 0 & T_2 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & T_2 \end{bmatrix}$$

for $i = 0,1, \dots, m$.

Hence the density of U is

$$f(u) = \beta e^{Su}(-S)e, \qquad 0 < u < \infty \tag{1}$$

and

$$E[U] = \beta(-S)^{-1}e. \tag{2}$$

Note that in the above e represents a column vector, having all entries 1, of appropriate dimension.

III Joint system size distribution

Network queues corrrespond to systems which consist of many queues with different types of customers moving from one queue to another in their routes. The route of a customer through the queues of the system may be fixed or random. Several researchers produced equalibrium system size distribution in product form for such networks based on the assumption that amounts of service required by a customer at successive queues along its route are independent and exponentially distributed. This assumption forced the said authors to demand that knowledge of the past route of a customer in a queue is of no use in predicting its future route. However, Kelly[3] conjectured that if the queues of the network were of a certain form, then even with the assumptions that the amount of service required by a customer at a queue in its route was almost arbitrarily distributed and depend on its route and the amount of service required by it at other queues along its route, the equalibrium system size distribution could be found in an analytical form. Later Barbour [1] proved this conjecture.

Kelly [3] dealt with an open system and used a customer's type to determine not only its route through the system but also the distribution of the amount of service it requires at each queue along that route. The following are the main assumptions made by Kelly [3] and Barbour [1].

- Queueing network consists of J nodes.
- Customers of type i (i=1, 2, .. , I) enter the system in a Poisson stream at rate $\nu(i)$ and pass through the sequence of queues $r(i,1), r(i,2), -., r(i,S(i))$ before leaving the system ,where S(i) denotes the number of stages a customer of type i visits along its route.

- A type i customer at its stage s (when $r(i,s) = j$) needs a random amount of service Q_{is}.
- Total service effort offered by a single server when there are n_j customers in queue j is $\phi_j(n_j)$.
- A customer in mth position of jth queue will be given a proportion $\gamma_j(m,n_j)$ of this effort, where $1 \le m \le n_j$.
- When a customer arrives at queue j, it moves into position m $(1 \le m \le n_j + 1)$ with probability $\gamma_j(m, n_j + 1)$.

Then Kelly [3] conjuctured and Barbour [1] later proved that $n(t) \equiv \{n_1(t), n_2(t), \ldots., n_J(t)\}$ has a limitting distribution $P(n)$ such that

$$P(n) \propto \prod_{j=1}^{J} \frac{a_j^{n_j}}{\prod_{m=1}^{n_j} \phi_j(m)}, \tag{3}$$

where

$$a_j = \sum_{n=1}^{I} \nu(i) \sum_{s=1}^{S(i)} I_{[r(i,s)=j]} EQ_{is}, \tag{4}$$

provided

$$M = \sum_n P(n) < \infty.$$

Note that the usage of the same fuction γ in the last two assumptions listed above is very essential, without which the existence of the equalibrium distribution of the joint system size given by eqns (3) and (4) will not be valid for network models bearing non-exponential service time distributional assumptions. For a detailed discussion on this, refer Kelly [3] and Barbour [1].

Now we use eqns (3) and (4) to determine the joint distribution of the number of packets waiting at nodes in some special type of wireless networks. Let us consider a network with nodes having identical features like the same number of one-hop and two-hop neighbours. Because of this, we can assume that the distribution of the amount of time the channel is sensed as busy by each of the nodes are identically distributed. In a similar manner, channel idle times sensed by all nodes can also be assumed to be disributed identically. Hence, the distribution of the time from the instant at which a packet is ready to the instant at which it is successfully transmitted from each node are also identically distributed. Its density and mean are defined by eqns (1) and (2) respectively. Hence EQ_{is} corresponding to our model can assumed to be the same for all i and s, and is given by

$$E[Q_{is}] = \beta(-S)^{-1}e. \tag{5}$$

Now consider the routing probability matrix as

$$R = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix}.$$

As we assumed earlier, type of a customer will be decided by the route along which it may traverse. Hence, we can have a maximum of $I = n!$ types of customers. Suppose that the total external packet generation to the system obey a Poisson rule of parameter λ and a q_i proportion of these is of type i for $i = 1,2,\dots,I$ so that $\sum_{i=1}^{I} q_i = 1$. As we assumed, average time that any type of customer takes at any node in its route is $E[Q_{is}]$, and is given by eqn (5).

As per the the two important assumptions made by Kelly [3] and Barbour [1], which are listed as the last two assumptions, given above in this section, we should also use the same fuction γ in our model due to the non-exponential variate Q_{is}. Hence, we assume two cases here namely,

case 1

Selection of packets for transmission at nodes is done by LCFS and the new packet always joins at the end of the queue.

Then we have

$$\gamma_j(m, n_j) \;= 1 \quad if m = n_j$$
$$= 0 \quad if m \neq n_j.$$

case 2

Selection of packets for transmission is done uniformly from the waiting line and also the customer joins a position randomly (as per uniform law) upon its arrival at a node along its route. In this case, we have

$$\gamma_j(m, n_j) \;= \frac{1}{n_j} \quad \text{for} m = 1,2,3,\dots n_j.$$

In both cases, we have

$$a_j = \sum_{i=1}^{I} q_i \lambda \sum_{s=1}^{S(i)-1} p_{r(i,s),r(i,s+1)=j} \beta(-S)^{-1}e. \tag{6}$$

Hypothetically, since we have only one server at each node, $\phi_j(m) = 1$ for $m = 1,\cdots,n_j$ and $j = 1,\cdots,J$.

Therefore,

$$P(n) \equiv \prod_{j=1}^{n} \frac{a_j^{n_j}}{\prod_{m=1}^{n_j} \phi_j(m)}$$

$$\equiv \prod_{j=1}^{n} (\lambda\beta(-S)^{-1}e \sum_{i=1}^{I} q_i \sum_{s=1}^{S(i)-1} p_{r(i,s),r(i,s+1)=j})^{n_j}. \tag{7}$$

In the above, $p_{r(i,s),r(i,s+1)=j}$ represents the routing probability of a packet of type i, which is currently at the s th stage of its route, moving to node j at the next stage.

IV Numerical illustration

In order to illustrate the theoretical results established in the previous section numerically, we consider a network model with nodes having equal number of one-hop and two-hop neighbours, as shown in figure 2.

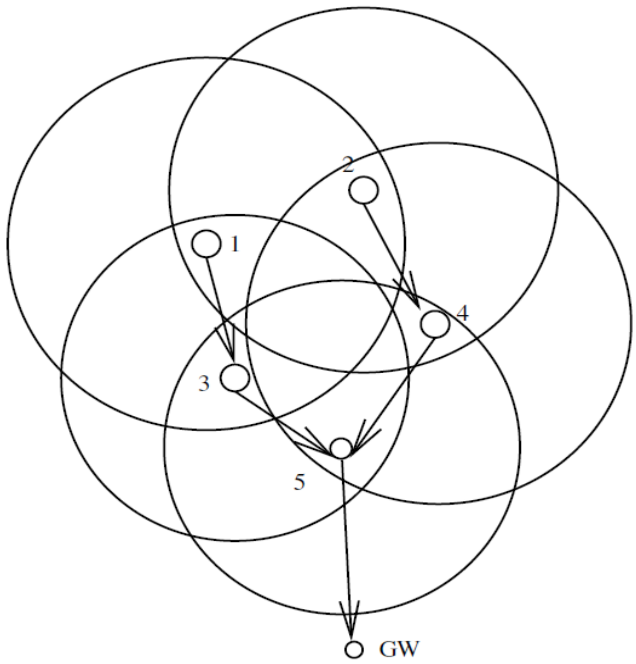

Figure 2: A particular network

Here node 1 and 2 are assumed as source nodes and GW is the gateway. The matrices R, F and N exhibit the details of routing of packets, one-hop, and two-hop neighbours of each node respectively.

$R =$

	1	2	3	4	5
1	0	0	1	0	0
2	0	0	0	1	0
3	0	0	0	0	1
4	0	0	0	0	1
5	0	0	0	0	0

$F =$

	1	2	3	4	5
1	0	1	1	0	0
2	1	0	0	1	0
3	1	0	0	0	1
4	0	1	0	0	1
5	0	0	1	1	0

$N =$

	1	2	3	4	5
1	0	0	0	1	1
2	0	0	1	0	1
3	0	1	0	1	0
4	1	0	1	0	0
5	1	1	0	0	0

As approximate Phase-type representations of the distribitions for channel busy time and idle time sensed by each node, we use the same representations that used in Sweta & Deepak [2]. These had been obtained by collecting around 300000 observations from a network which is being governed by BEB scheme under 802.11 MAC [5] specification. Activities at one of the nodes were monitored and the observations corresponding to the events like arrivals of data packets at the node and amount of time the channel was idle/busy sensed by that node were used to get an approximate phase type fit for the said variates. Also, the same observations were used for estimating the packet arrival rate. The representation thus obtained for channel busy time was

$$\alpha_1 = \begin{bmatrix} 0.7530 & 0.0766 & 0.1704 \end{bmatrix};$$

$$T_1 = \begin{bmatrix} -1.8097 & 0.2397 & 0.6790 \\ 0.1939 & -1.3306 & 0.5483 \\ 0.1847 & 0.5014 & -1.1274 \end{bmatrix},$$

and that for channel idle time was

$$\alpha_2 = \begin{bmatrix} 0.8823 & 0.0307 & 0.0870 \end{bmatrix};$$

$$T_2 = \begin{bmatrix} -7.2175 & 0.1960 & 0.5386 \\ 0.2835 & -1.5407 & 0.5142 \\ 0.2729 & 0.5297 & -1.3043 \end{bmatrix}.$$

Packet arrival rate is estimated as $\lambda = 1.0629$. Also, we have $E[Q_{is}]$=0.6034.

In the present example, there are two types of packets namely, the one that traverses the route 1 → 3 → 5 → GW and the other having the route 2 → 4 → 5 → GW. Suppose that the inflow of packets to the system obey Poisson rule of rate $\lambda = 1.0629$, of which both types claim the same proportion. That is, $q_i = \frac{1}{2}$ for $i = 1,2$. Table 1 presents a few values for the joint system size probabilities of packets at nodes in our model, under both case 1 and case 2 discussed in the previous section.

Table 1: Joint System Size Probabilities

n	P(n)	n	P(n)
(1,2,1,1,2)	0.000106	(1,1,1,3,2)	0.000034
(1,1,2,2,1)	0.000053	(3,1,1,2,1)	0.000017
(2,2,1,1,2)	0.000034	(1,1,2,2,3)	0.000022
(1,1,1,3,3)	0.000022	(3,1,1,1,2)	0.000034
(1,1,3,2,1)	0.000017	(1,1,1,1,2)	0.000332
(1,4,1,1,3)	0.000007	(2,1,1,2,2)	0.000034
(1,1,2,2,2)	0.000034	(3,1,1,1,3)	0.000022
(2,1,1,2,3)	0.000022	(1,2,2,1,2)	0.000034
(1,1,2,2,1)	0.000053	(1,1,1,3,3)	0.000022
(2,1,1,2,3)	0.000022	(1,1,1,3,1)	0.000053

For computing the joint system size probabilities, as displayed in table 1, the normalization constant is taken as the sum of the probabilities corresponding to state vectors $n = (n_1, n_2, n_3, n_4, n_5)$ for each n_i varies over 0 to 50.

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Transient Numerical Analysis of a Queueing Model with Correlated Reneging, Balking and Feedback

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Abstract

We consider a single server queuing model with correlated reneging, balking and feedback. The time-dependent behavior of the model is studied using Runge-Kutta method. Some measures of the performance like expected system size and expected waiting time are computed.

Keywords: Queuing model, Correlated reneging, Balking, Feedback, Numerical analysis

I Introduction

Queuing models with reneging have attracted the attention of many researchers for their real-life applications in industry and communication networks. The study of the reneging behavior of customers plays an important role in the design of queuing systems for various production and services systems. The pioneer work dealing with customers' impatience was initiated by Haight [8, 9], Ancker and Gafarian [1, 2], and Subba Rao [21, 22]. They have developed the basic queuing models with renging and balking. Since then, a number of researchers have worked on various queuing models with reneging and balking. Recenty, Kumar and Sharma [14] put forth a new concept of retention of reneging customers in queuing theory. They derived the steady-state solution and computed some performance measures, and also showed the effect of probability of customers' retention on expected system size. Kumar and Sharma [15] obtained the transient solution with the probability generating technique for the single server queuing model with retention of reneging customers. Kumar and Sharma [16] obtained the transient and steady-state probabilities for a two-heterogenous servers' Markovian queuing system with retention of reneging.

Mohan [17] was the first to introduce the concept of correlation in gambler's ruin problem. Conolly [4] considered a queuing system having services depending on inter-arrival times. Conolly and Hadidi [5] considered a model having arrival pattern impacting the service pattern. They examined the initial busy period, state and output processes. Murari [19] studied a queuing system with correlated arrivals and general service time distribution. Mohan and Murari [18] obtained the transient solution of a queuing model with correlated arrivals and variable service capacity. Cidon et al. [3] considered a queue in which service time is correlated to inter-arrival time. They studied this correlation in case of communication systems and showed the impact through numerical results by comparing with less reliable models. Patuwo et al. [20] worked on serial correlation in the arrivals. He studied the consequences of correlation on mean queuing performances. Kamoun [13] considered a single server queuing model with finite capacity and correlated arrival in which the packets are submitted to random interruptions. Drezner [7] performed the performance analysis of $M^c/G/1$ queues. Iravani and Luangkesorn [12] studied a model of parallel queues with correlated arrivals and bulk services. To get the performance measures they used the matrix geometric method. Hwang and Sohraby [11] considered a correlated queue of packets moving in transmission line with finite capacity. Numerical examples are illustrated to exhibit the importance of correlation on system performances. Hunter [10] studied the consequences of correlated arrivals on the steady-state queue length process for single server queuing model.

The concept of feedback in queuing theory is used to model the situations when the customers are not satisfied with their first service. A dissatisfied customer retries for service with certain probability. Takacs [24] studied a single server queuing model with feedback mechanism. Davignon and Disney [6] considered an $M/G/1$ queuing system where the served customer either joins the queue again with some probability or depart permanently. They studied the stationary queue length and departure process. Santhakumaran and Thangaraj [23] studied a single server queuing system with feedback and impatient customers.

The conventional reneging considered in the literature so far has the assumption that the reneging times happen to follow certain probability distribution and the reneging of the customers occur with certain rate. But, this assumption may not hold where the behavior of reneging customers can be bursty as this case may be possible in many practical scenarios. For example, consider a central system of an online shopping company where all the orders as well as the cancellation requests of orders are received. The arrival of orders is analogous to the arrival of customers, the dispatching of orders is analogous to the service of customers, and the orders cancelled before dispatching can be considered as reneging customers. A customer who visits a shopping site and does not find a satisfactory product may not place any order. This situation is similar to balking behavior of customers. The cancellation of orders could be abrupt or bursty at times because of the reasons like delay in delivery, some other online shopping companies start offering discounts, bad reviews about the products become viral etc. That is, if an order is cancelled at any time instant, then there is a probability that an order may or may not be cancelled at the next time instant. Similarly, if an order is not cancelled at any time instant, then there is a probability that an order may or may not be cancelled at the next time instant. This kind of reneging is referred to as correlated reneging, and is better than conventional reneging to capture the burntness. Sometimes it happens that the received product is below the expectations of the buyer and he feels unsatisfied, so he may put a request for re-order of the same product to get a new one. This situation resembles with the feedback in queuing theory and re-order of the same product can be considered as a feedback customer.

The literature survey shows that no work has appeared on correlated reneging till date. Moreover, because of the usefulness of the concept of correlated reneging as discussed in the previous paragraph we develop a single server queuing model with correlated reneging, balking, and feedback. We perform the transient numerical analysis of the queuing model. Rest of the paper is as follows: In section 2, the stochastic queuing model is described. In section 3, the mathematical model is presented. Section 4 deals with the transient analysis of the model. The sensitivity analysis of the model is presented in section 5. Finally, the paper is concluded in section 6.

II Queuing Model Description

The queueing model considered is based on the following assumptions: The customers arrive at a service facility one by one in accordance with Poisson process with parameter λ. There is a single queue and a single server. The service-times are independently, identically and exponentially distributed with parameter μ. On arrival, an incoming customer may decide not to join the queue (i.e. balk) with certain probability (say, $1-\beta$). This means that the arrival customer may join the queue with probability β. After being served, a customer either leaves the system with probability q or rejoins the queue as a feedback customer with complementary probability p=(1-q). The capacity of the system is finite (say, N). After joining the queue and waiting for sometime, a customer may get impatient and leave the queue(renege) without getting the service. The reneging of the customers can take place only at the transition marks $t_0, t_1, t_2, \ldots$ where $\theta_r = t_r - t_{r-1}, r = 1,2,3\ldots$, are random variables with $P[\theta_r \leq x] = 1 - exp(-\xi x); \xi > 0, r = 1,2,3, \ldots$ That is, the distribution of inter-transition marks is negative exponential with parameter ξ. The reneging at two consecutive transition marks is governed by the following transition probability matrix:

$$\text{from } t_{r-1} \begin{matrix} \\ 0 \\ 1 \end{matrix} \overset{\text{to } t_r}{\begin{Vmatrix} \overset{0}{p_{00}} & \overset{1}{p_{01}} \\ p_{10} & p_{11} \end{Vmatrix}} \text{ where } p_{00} + p_{01} = 1 \text{ and } p_{10} + p_{11} = 1$$

0 refers to no reneging and 1 refers to the occurrence of reneging.

Thus, the reneging at two consecutive transition marks is correlated.

III Mathematical Model

Defining the probabilities:

$Q_{0,0}(t)$ = Probability that at time t the queue is empty, the server is idle, and a customer has not reneged at the previous transition mark.

$Q_{0,1}(t)$ = Probability that at time t the queue is empty, the server is idle, and a customer has reneged at the previous transition mark.

$P_{0,0}(t)$ = Probability that at time t the queue is empty, the server is not idle, and a customer has not reneged at the previous transition mark.

$P_{0,1}(t)$ = Probability that at time t the queue is empty, the server is not idle, and a customer has reneged at the previous transition mark.

$P_{n,0}(t)$ = Probability that at time t the queue length is n , the server is not idle, and a customer has not reneged at the previous transition mark.

$P_{n,1}(t)$ = Probability that at time t the queue length is n, the server is not idle, and a customer has reneged at the previous transition mark.

$P_{N,0}(t)$ = Probability that at time t the queue length is N, the server is not idle, and a customer has not reneged at the previous transition mark.

$P_{N,1}(t)$ = Probability that at time t the queue length is N, the server is not idle, and a customer has reneged at the previous transition mark.

The differential equations of the model are:

$$\frac{d}{dt}Q_{0,0}(t) = -\lambda Q_{0,0}(t) + \mu q P_{0,0}(t) \quad (1)$$

$$\frac{d}{dt}P_{0,0}(t) = -(\lambda + \mu q)P_{0,0}(t) + \mu q P_{1,0} + \lambda Q_{0,0}(t) \quad (2)$$

$$\frac{d}{dt}P_{1,0}(t) = -(\lambda\beta + \mu q + n\xi)P_{1,0}(t) + \mu q P_{2,0}(t) + \lambda P_{0,0}(t)$$
$$+\xi[p_{00}P_{1,0}(t) + p_{10}P_{1,1}(t)] \quad (3)$$

$$\frac{d}{dt}P_{n,0}(t) = -(\lambda\beta + \mu q + n\xi)P_{n,0}(t) + \mu q P_{n+1,0}(t) + \lambda\beta P_{n-1,0}(t)$$
$$+n\xi[p_{00}P_{n,0}(t) + p_{10}P_{n,1}(t)], 1 < n < N \quad (4)$$

$$\frac{d}{dt}P_{N,0}(t) = -(\mu q + N\xi)P_{N,0}(t) + \lambda\beta P_{N-1,0}(t) + N\xi[p_{00}P_{N,0}(t)$$
$$+p_{10}P_{N,1}(t)] \quad (5)$$

$$\frac{d}{dt}Q_{0,1}(t) = -\lambda Q_{0,1}(t) + \mu q P_{0,1}(t) \quad (6)$$

$$\frac{d}{dt}P_{0,1}(t) = -(\lambda + \mu q)P_{0,1}(t) + \mu q P_{1,1} + \lambda Q_{0,1}(t) + \xi[p_{11}P_{1,1}(t)$$

$$+p_{01}P_{1,0}(t)] \quad (7)$$

$$\frac{d}{dt}P_{1,1}(t) = -(\lambda\beta + \mu q + n\xi)P_{1,1}(t) + \mu q P_{2,1}(t) + \lambda P_{0,1}(t)$$
$$+2\xi[p_{01}P_{2,0}(t) + p_{11}P_{2,1}(t)] \quad (8)$$

$$\frac{d}{dt}P_{n,1}(t) = -(\lambda\beta + \mu q + n\xi)P_{n,1}(t) + \mu q P_{n+1,1}(t) + \lambda\beta P_{n-1,1}(t)$$
$$+(n+1)\xi[p_{01}P_{n+1,0}(t) + p_{11}P_{n+1,1}(t)], 1 < n < N \quad (9)$$

$$\frac{d}{dt}P_{N,1}(t) = -(\mu q + N\xi)P_{N,1}(t) + \lambda\beta P_{N-1,1}(t) \quad (10)$$

IV Transient Analysis of the Model

In this section we perform the transient analysis of the model. We use the Runge-Kutta method of fourth order to obtain the transient solution as it is quite difficult to obtain analytical solution explicitly. The $''ode45''$ function of MATLAB software is used to compute the transient numerical results.

4.1 Performance measures

We study the following performance measures:

1. Expected system Size ($L_s(t)$):
$$L_s(t) = \sum_{n=0}^{N} (n+1)[P_{n,0}(t) + P_{n,1}(t)]$$

2. Expected waiting time in the system ($W_s(t)$):
$$W_s(t) = \frac{L_s(t)}{\mu(1 - Q_{0,0}(t) - Q_{0,1}(t))}$$

Where $L_s(t)$ is mean system size at time t.

Now, we illustrate the transient behaviour of the model with the help of a numerical example. We take $\lambda = 2.3, \mu = 2.9, \xi = 0.3, \beta = 0.8, q = 0.7, p_{00} = 0.8, p_{01} = 0.2, p_{10} = 0.7,\ p_{11} = 0.3, N = 6$. In figures 1 and 2 the system size probabilities are plotted against time. We can observe that all the probabilities increase to a certain extent and after sometime they become stationary. However, the probability $P_{0,0}(t)$ has highest value in the beginning and it decreases to a certain extent and after sometime it becomes stationary. This behaviour of $P_{0,0}(t)$ is due to initial condition, that is, $P_{0,0}(0) = 1$. In figure 3, the variation in expected system size is plotted against time. The expected system size gradually increases from the initial state and achieves a constant value after some time. In figure 4, the variation in expected system size is plotted against time. The expected system size gradually increases from the initial state and achieves a constant value after some time.

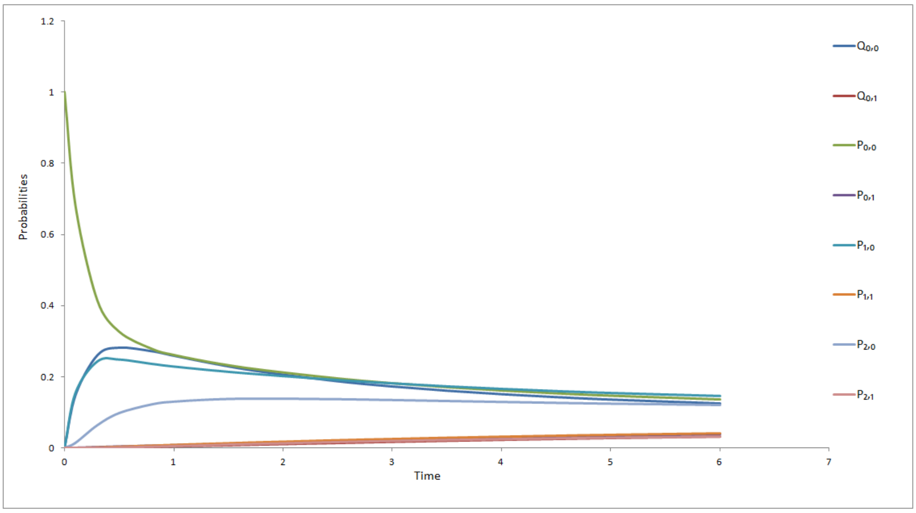

Figure 1: Probabilities vs Time

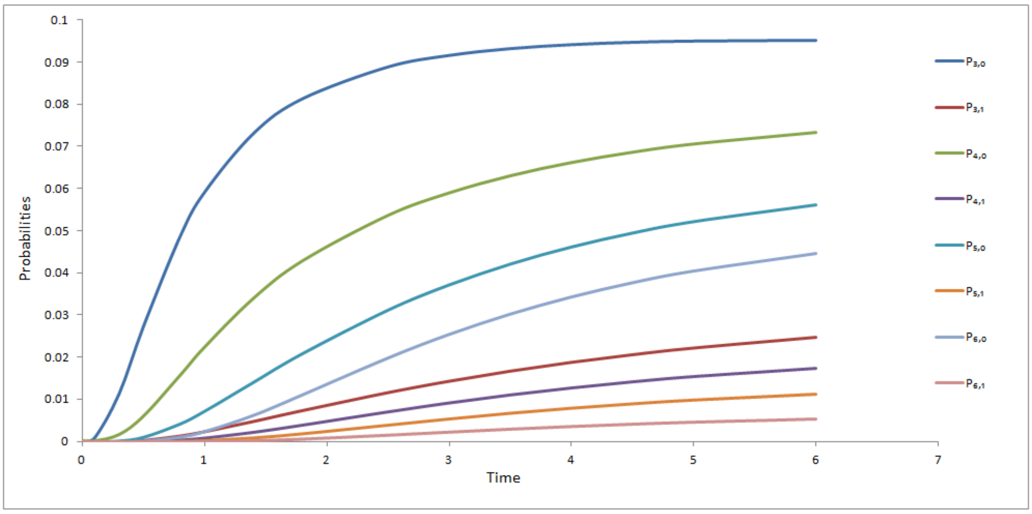

Figure 2: Probabilities vs Time

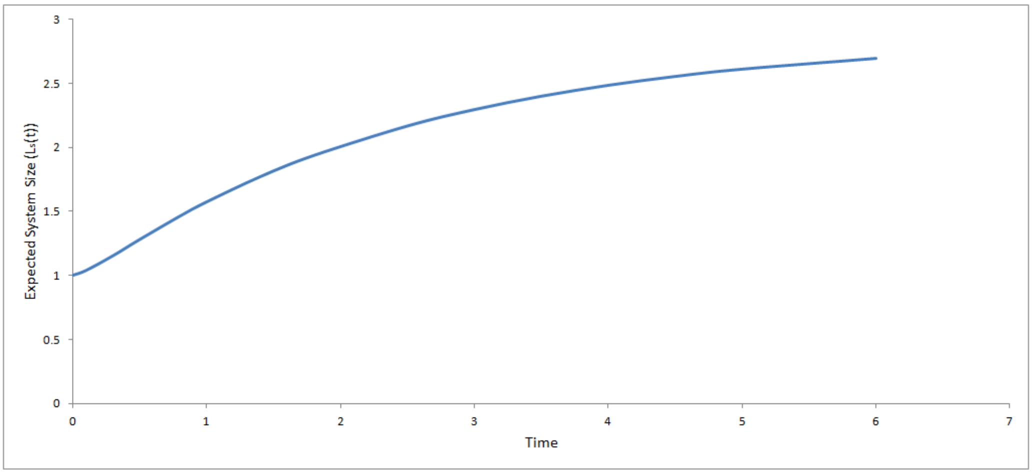

Figure 3: Expected system size vs Time

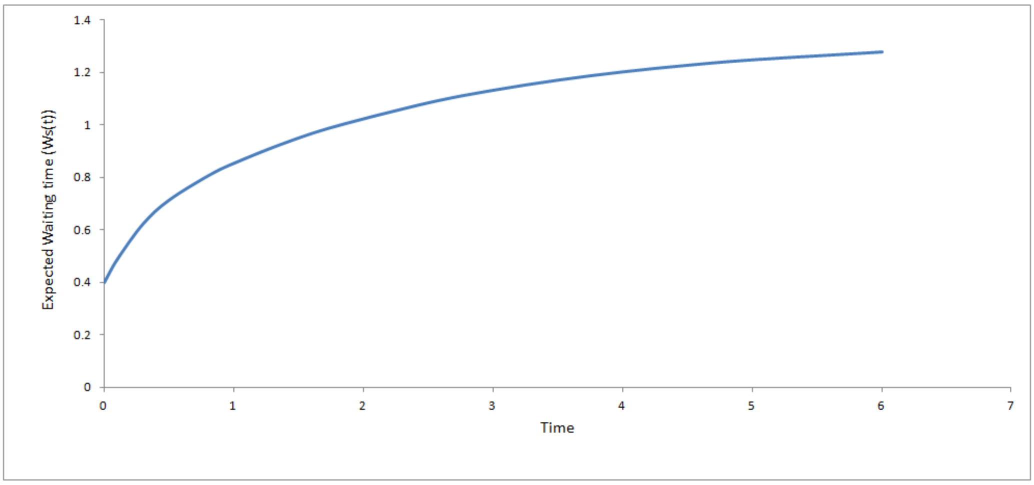

Figure 4: Expected waiting time vs Time

V Sensitivity analysis of the model

In this section, we study the variation in performance measures with respect to the change in system parameters. In table 1, the variation in expected system size and in expected waiting time with respect to mean arrival rate is presented. One can see that the performance measures decreases with the increase in mean arrival rate. The variation in performance measures with respect to mean service rate is shown in table 2. With the increase in the mean service rate the expected system size increases. Similar is the case with expected waiting time. In table 3, the variation in performance measure with respect to the probability p_{00} is studied. An increase in p_{00} leads to the increase in performance measures $L_s(t)$ and $W_s(t)$. Since $p_{01} = 1 - p_{00}$, the variation in performance measures is reverse for p_{01}. Table 4 deals with the changes in $L_s(t)$ and $W_s(t)$ with respect to change in the probability p_{10}. One can observe that the increase in p_{10} increases $L_s(t)$ and $W_s(t)$. The variations are in reverse order for probability $p_{11}(= 1 - p_{10})$. The variations in performance measures with respect to the change in feedback probability are presented in table 5. One can see that with the increase in feedback probability the measures of performance $L_s(t)$ and $W_s(t)$ show increasing trend. The increase in feedback probability means more number of feedback customers join the queue and thus increase the system size and hence the waiting time in the system also increases. The variations in performance measures with respect to the change in balking probability are presented in table 6. One can see that with the increase in balking probability the measures of performance $L_s(t)$ and $W_s(t)$ show decreasing trend. The increase in balking probability means more number of customers do not join the queue and thus decreases the system size and hence the waiting time in the system also decreases. The numerical results discussed in tables 1-6 describe the functioning of our model. font=normalsize,sf

Table 1: Variation in performance measures w.r.t. mean arrival rate
Here, $\mu = 3.3, \xi = 0.8, \beta = 0.8, q = 0.7, p_{00} = 0.8, p_{01} = 0.2, p_{10} = 0.7, p_{11} = 0.3, N = 6, t = 4$.

S. No.	Mean arrival rate (λ)	Expected system size ($L_s(t)$)	Expected waiting time ($W_s(t)$)
1	1.3	0.8674	0.5182
2	1.5	1.0563	0.5627
3	1.7	1.259	0.6102
4	1.9	1.4743	0.6607
5	2.1	1.7005	0.7139
6	2.3	1.9358	0.7694
7	2.5	2.1777	0.8268

Table 2: Variation in performance measures w.r.t. mean service rate
Here, $\lambda = 3.3, \xi = 0.8, \beta = 0.8, q = 0.7, p_{00} = 0.8, p_{01} = 0.2, p_{10} = 0.7, p_{11} = 0.3, N = 6, t = 4$.

S. No.	Mean service rate (μ)	Expected system size ($L_s(t)$)	Expected waiting time ($W_s(t)$)
1	3.1	2.3282	0.9161
2	3.3	2.1777	0.8268
3	3.5	2.0384	0.7502
4	3.7	1.9098	0.6839
5	3.9	1.7914	0.6265
6	4.1	1.6825	0.5762
7	4.3	1.5824	0.5321

Table 3: Variation in performance measures w.r.t. probability (p_{00})
Here, $\lambda = 2.5, \mu = 3.3, \xi = 0.9, \beta = 0.8, q = 0.7, p_{10} = 0.7, p_{11} = 0.3, N = 6, t = 4$.

S. No.	Probability (p_{00})	Expected system size ($L_s(t)$)	Expected waiting time ($W_s(t)$)
1	0.1	1.7244	0.6893
2	0.2	1.7549	0.6985
3	0.3	1.7917	0.7097
4	0.4	1.8367	0.7233
5	0.5	1.8925	0.7403
6	0.6	1.9633	0.7618
7	0.7	2.0551	0.7897
8	0.8	2.1777	0.8268
9	0.9	2.3465	0.8777

Table 4: Variation in performance measures w.r.t. probability (p_{10})
Here, $\lambda = 2.5, \mu = 3.3, \xi = 0.8, \beta = 0.8, q = 0.7, p_{00} = 0.8, p_{01} = 0.2, N = 6, t = 4$.

S. No.	Probability (p_{10})	Expected system size ($L_s(t)$)	Expected waiting time ($W_s(t)$)
1	0.1	1.9361	0.7541
2	0.2	1.9899	0.7705
3	0.3	2.0371	0.7848
4	0.4	2.0788	0.7973
5	0.5	2.1157	0.8083
6	0.6	2.1484	0.8181
7	0.7	2.1777	0.8268
8	0.8	2.2038	0.8346
9	0.9	2.2273	0.8416

Table 5: Variation in performance measures w.r.t. probability of feedback p
Here, $\mu = 3.3, \xi = 0.8, \beta = 0.8, q = 0.7, p_{00} = 0.8, p_{01} = 0.2, p_{10} = 0.7, p_{11} = 0.3, N = 6, t = 4$.

S. No.	Probability of feedback (p)	Expected system size ($L_s(t)$)	Expected waiting time ($W_s(t)$)
1	0.1	1.6101	0.6966
2	0.2	1.8664	0.7572
3	0.3	2.1777	0.8268
4	0.4	2.5511	0.9107
5	0.5	2.9909	1.0112
6	0.6	3.4952	1.1303
7	0.7	4.0537	1.2687
8	0.8	4.6453	1.4246
9	0.9	5.2388	1.5914

Table 6: Variation in performance measures w.r.t. probability of balking $1-\beta$
Here, $\lambda = 3.3, \xi = 0.8, \beta = 0.8, q = 0.7, p_{00} = 0.8, p_{01} = 0.2, p_{10} = 0.7, p_{11} = 0.3, N = 6, t = 4.$

S. No.	Probability of balking $(1-\beta)$	Expected system size $(L_s(t))$	Expected waiting time $(W_s(t))$
1	0.1	2.4152	0.8890
2	0.2	2.1777	0.8268
3	0.3	1.9561	0.7581
4	0.4	1.7547	0.6943
5	0.5	1.5764	0.6369
6	0.6	1.4228	0.5866
7	0.7	1.2935	0.5439
8	0.8	1.1868	0.5082
9	0.9	1.0099	0.4791

VI Conclusion

In this paper we have performed the transient numerical analysis of a single server queuing model with correlated reneging, balking and feedback. Sensitivity analysis has also been performed.

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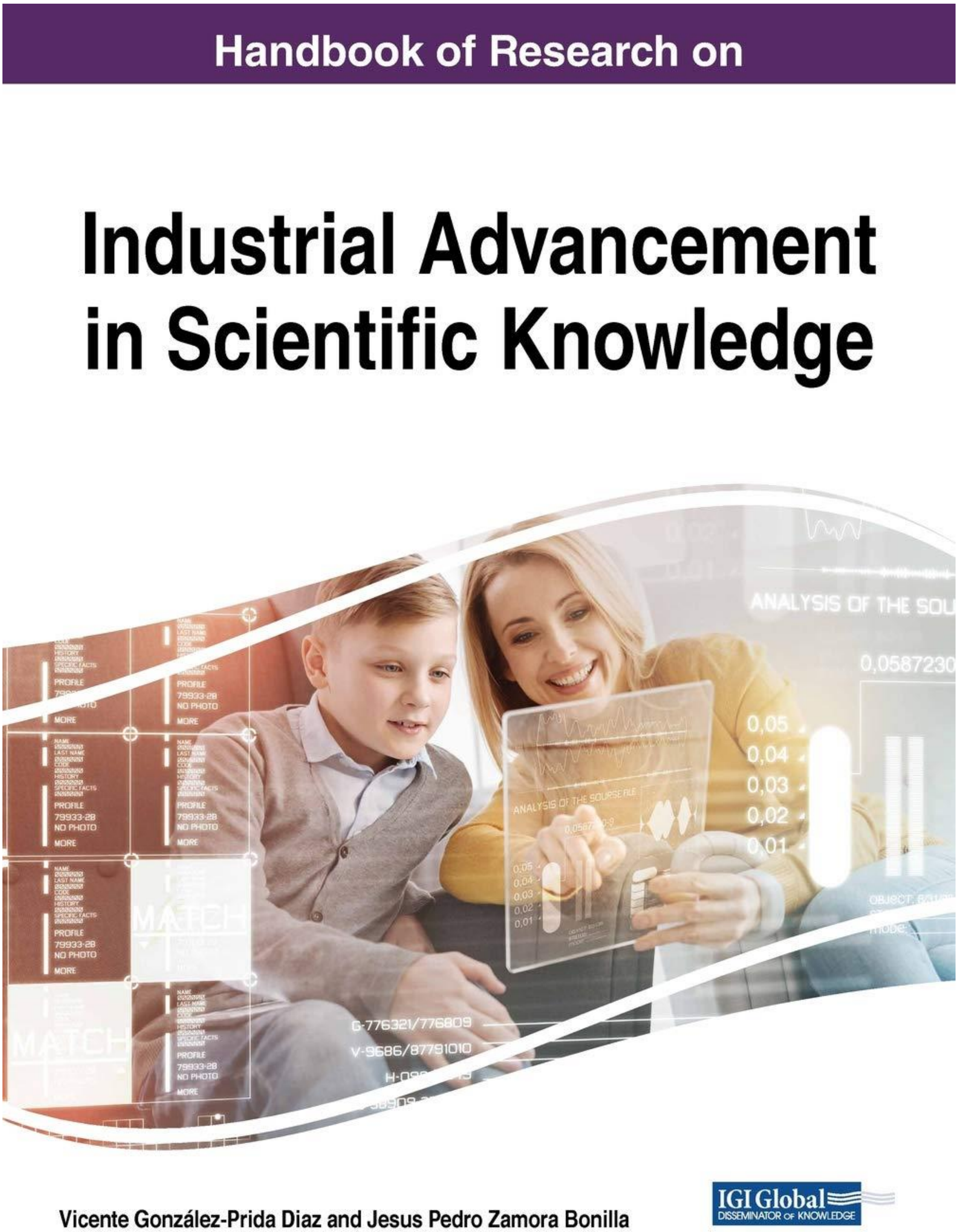
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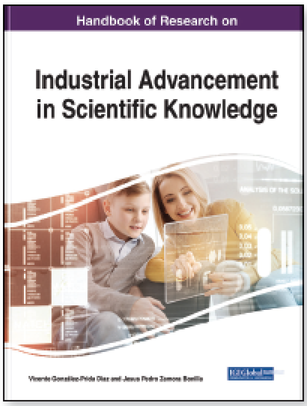

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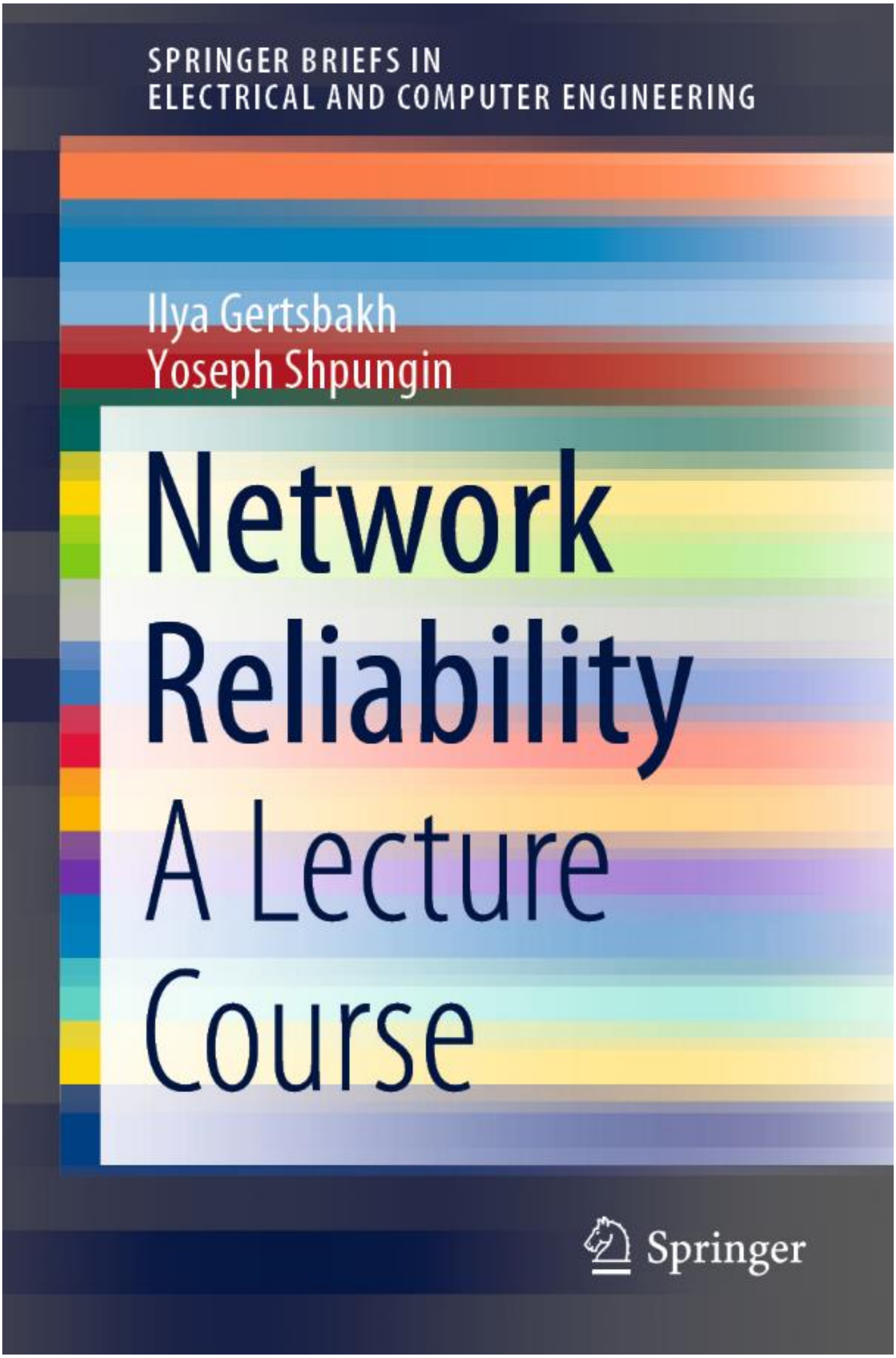
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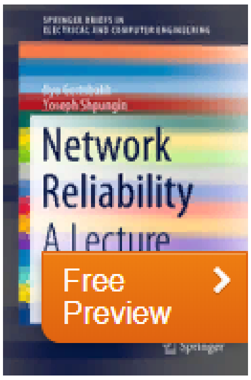

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Network Reliability

A Lecture Course

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This introductory book equips the reader to apply the core concepts and methods of network reliability analysis to real-life problems. It explains the modeling and critical analysis of systems and probabilistic networks, and requires only a minimal background in probability theory and computer programming.

Based on the lecture notes of eight courses taught by the authors, the book is also self-contained, with no theory needed beyond the lectures. The primary focus is on essential "modus operandi," which are illustrated in numerous examples and presented separately from the more difficult theoretical material.

Chapter 8

Examples of network analysis

Abstract This chapter presents three applications of the theory developed in previous chapters to practical analysis of network-type systems. The first section is a comparison of networks resilience under random attack on their nodes. It is shown that regular networks are considerably more resilient than randomly created networks having the same number of nodes and edges. The second section presents an example of predisaster reinforcement of highway/railway system based on locating and using the most influential (important) edges of the system. The third section presents reliability analysis of flow networks with randomly failing edges.

Keywords Regnet, Ternet, Prefnet · Predisaster reinforcement · Flow in random networks

8.1 Network structure and resilience against node attack

Suppose we have a network which is subject to an attack on its nodes. The attacker chooses randomly a node in the network and destroys it. The attacked node becomes isolated and all edges incident to it are erased. This situation may reflect action of secret services aimed at the discovery and destruction of a terrorist network. Our first example is exactly this case and was borrowed from the report of Valdis Krebs published on his website *www.orgnet.com.* It describes the terrorist network in USA preparing their attack on 9/11/2001. We decided to compare the resilience of the terrorist network (we call it "ternet") to networks of approximately the same size but having different structure.

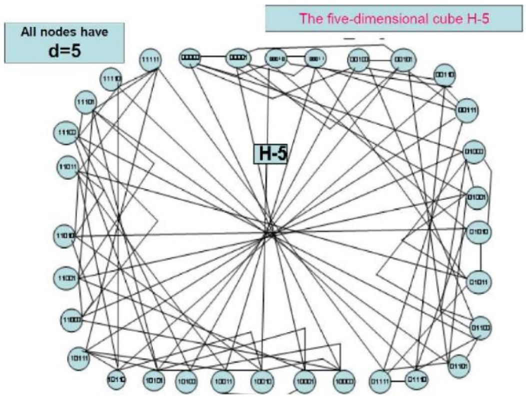

Fig. 8.1: Regnet

The ternet has been "created" in special circumstances, the nodes of it are the people, terrorists, and the edges - connections between them. Secret character of the whole situation made the connection quite random. We will compare the resilience of the ternet to the resilience of a more "organized", i.e. less randomly created network. We took two networks which we call regnet - regular network and prefnet - a network obtained by preferential assignment approach described in [1]. As a regular network we took a five dimensional cube H-5 with 32 nodes and 80 edges. Each node is incident to 5 edges and has degree 5. The nodes of regnet are 5-digit binary numbers from 00000 to 11111. If two nodes differ by only one binary digit, then they are connected by an edge. For example, nodes 00000 and 01000 are connected by an edge.

Prefnet is a network with the same number of 32 nodes and 80 edges. This network is made by having a small initial kernel of nodes and edges and adding new nodes and new edges following the principle: any new edge from a new node goes with higher probability to an existing node having more edges incident to it. This principle is called "preferential assignment" [1].

We designed our prefnet with 80 edges and 32 nodes. It has one node with 15 edges and one - with ten, and many nodes with degree 4-5. The

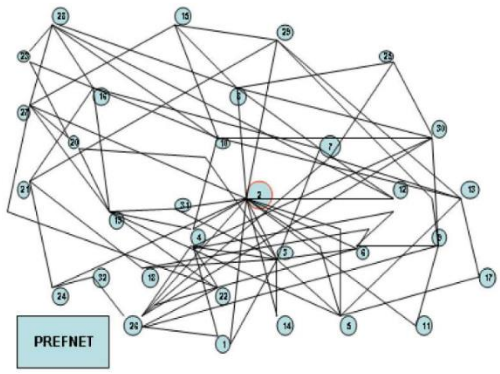

Fig. 8.2: Prefnet

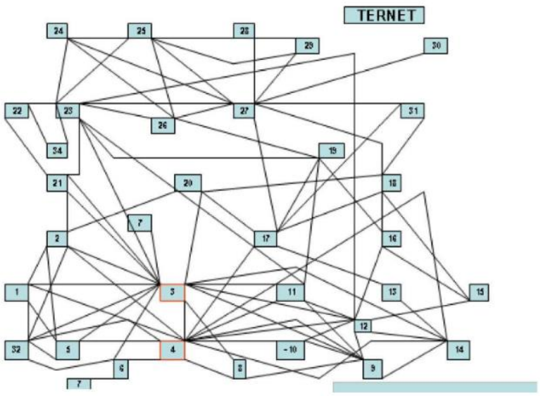

Fig. 8.3: Ternet

average node degree is 5. The "real" ternet had 34 nodes and 91 edges. It has been modified randomly to a network with 32 nodes and 80 edges. The original ternet and the modified ternet have one hub with degree 16.

Probably, it might correspond to the connections of the leader of the terrorist group (Muhammed Atta). Regnet, Prefnet and Ternet are shown on Fig. 8.1, 8.2 and 8.3 respectively.

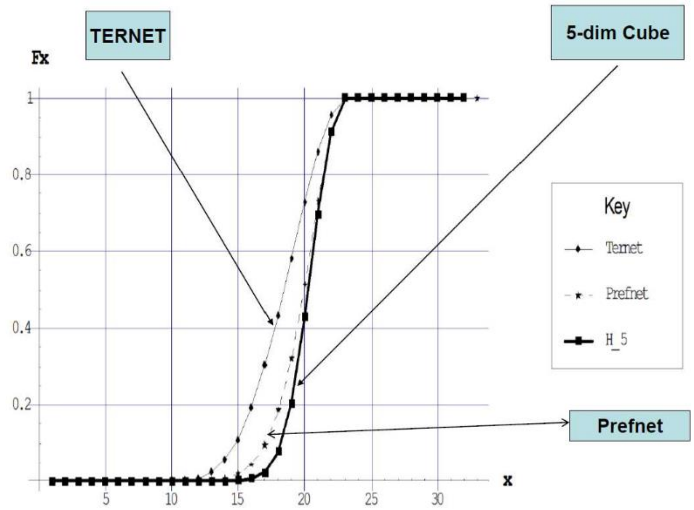

Fig. 8.4: CD-spectra of Regnet, Prefnet and Ternet.

So, we have a "most organized" Regnet, a "less organized" Prefnet and completely random "Ternet", all three having the same size and the same average degree.

Of crucial importance is the choice of the network *UP/ DOWN* criterion. We assumed that the networks become *DOWN* if their *largest connected component becomes of size* $L \leq 10$. We assume that any system designed to carry out a special mission is not more capable of doing so if it greatest **connected component** of the whole system is less than one third of the original number of nodes in the system.

The results of our analysis are presented on Fig. 8.4. It shows the CD-spectra for all three networks.

From Fig. 8.4 it becomes evident that the CD-spectrum *F(x)* of Ternet dominates two other spectra and that the CD-spectrum of Prefnet dominates the CD-spectrum of Ternet. In other words, most resilient is Regnet and less resilient is Ternet. Compare, for example, the *DOWN* probabilities after failure of 17 nodes. For Ternet it is about 0.32, for Prefnet - 0.1 and only 0.03 for Regnet.

8.2 Road/highway reinforcement

In this section we consider a transportation network reinforcement problem. By reinforcement we mean network reliability improvement achieved by replacing a certain number of its "weak" edges, by more reliable ones. If there are several ways to make the network "stronger", we will prefer the less costly way.

We show how this works taking as an example the ring way network with 15 nodes and 22 edges shown on Fig. 8.5.

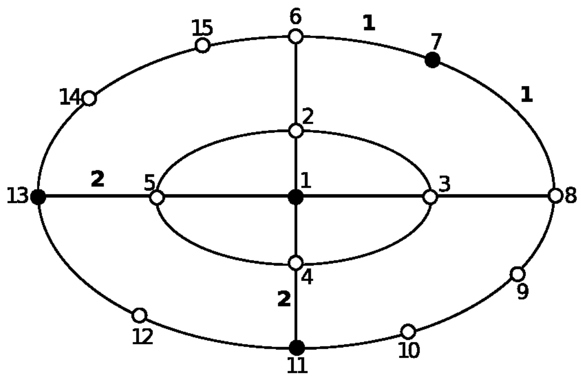

Fig. 8.5: Ring network. The network is *UP* if all four terminals (shown bold) are connected. Copy of Fig. 6.2

All edges have the same reliability $p(e) = 0.7$. Edge reinforcement means its replacement by more reliable one, say by an edge having $p = 0.9$. Suppose that edge replacement has the same cost for all edges.
The initial network reliability calculated using CMC is $R = 0.505$. Our goal is to raise this probability to a level $R^* = 0.85$, for minimal cost. The solution of our problem is equivalent to choosing the *minimal* number of

edges to reinforce.

The analysis of BIM-spectra allows to divide the edges into the following six groups. (The edges belonging to the first and the second groups are marked in Fig. 8.5, by 1 and 2, respectively.)

$$(6,7) = (7,8) >$$
$$(4,11) = (5,13) >$$
$$(11,12) = (12,13) >$$
$$(2,6) = (3,8) >$$
$$(8,9) = (9,10) = (10,11) = (13,14) = (14,15) = (6,15) >$$
$$other$$

Table 8.1: Ring BIM-spectrum. Edges unreliable. Terminals are T=(1,7,11, 13).

i	$z_{i,(1,2)}$ group 6	$z_{i,(2,6)}$ group 4	$z_{i,(4,11)}$ group 2	$z_{i,(6,7)}$ group 1	$z_{i,(8,9)}$ group 5	$z_{i,(11,12)}$ group 3
2	0	0	0	.0042	0	0
3	.0005	.0026	.0033	.0145	.0022	.0035
4	.0042	.0141	.0216	.0362	.0129	.0193
5	.0163	.0442	.0562	.0732	.0347	.0501
6	.0436	.0898	.1070	.1200	.0719	.0950
7	.0967	.1532	.1771	.1824	.1254	.1579
8	.1788	.2257	.2562	.2574	.1960	.2368
9	.2843	.3138	.3415	.3436	.2845	.3210
10	.3813	.3939	.4153	.4100	.3729	.4031
11	.4607	.4684	.4788	.4781	.4567	.4777
12	.5292	.5315	.5362	.5400	.5278	.5350

BIM-spectra of the edges, each of which represents a separate group, are shown in table 8.1. The first row is zero, since the minimum cut of the network is 2. Spectrum values after $i = 12$ are not shown. These values are almost the same, since the probability of network failure starting from step 13 is very close to 1. Choosing for replacement one by one the edges with highest BIMs, we arrive at the reliability $R = 0.851$, using the following 6 edges:

$$(6,7), (7,8), (4,11), (5,13), (11,12), (12,13).$$

Remark 8.1 Strictly speaking, after replacing the first edge (6,7), not all edges will have the same reliability, and that will affect the BIM's of other edges. But from a practical point of view, our heuristic approach justifies itself.

Table 8.2: Edges, costs and BIM's values

i	edge e	$p(e)$	cost $c(e)$	BIM_i	$\alpha = BIM_i \cdot (0.9 - p(e))$	$\alpha/c(e)$
1	(1,2)	0.6	2	0.064	0.019	0.01
2	(2,6)	0.6	2	0.167	0.050	0.025
3	(3.8)	0.6	2	0.167	0.050	0.025
4	(4,11)	0.6	2	0.240	0.072	.0.036
5	(5,13)	0.6	2	0.242	0.073	0.036
6	(6,7)	0.7	1	0.261	0.052	0.052
7	(7,8)	0.7	1	0.253	0.051	0.051
8	(8,9)	0.7	1	0.104	0.023	0.023
9	(11,12)	0.7	1	0.218	0.044	0.044
10	(12,13)	0.7	1	0.216	0.043	0.043

Let us now assume that **not all edges** are equally reliable and also the costs of edge reinforcement are not equal too. Namely, the reliability of edges located on a large ring is 0.7 and the cost of reinforcing an edge equals 1. For the remaining edges, the reliability is 0.6 and reinforcement cost equals 2. The initial network reliability calculated using CMC is $R = 0.479$. Our goal remains the same: to increase the reliability to 0.85, for minimal cost. Table 8.2 presents edge reliability, as well as the cost of replacing an edge by new one having $p = 0.9$. (The table shows only a part of all edges.)

The BIM's in this table are computed using the turnip algorithm (see Remark 7.4). Note that they can also be calculated by definition 5.1.4. Now the criterion for choosing an edge for reinforcement will be determined by the largest value of $BIM_e \cdot (0.9 - p(e))/c(e)$, which represents the increase in edge reliability per unit cost. Such an approach resembles the Knapsack problem algorithm. After calculations, we get the following sequence of 7 edges, resulting in network reliability $R = 0.846$.

$$(6,7),(7,8),(11,12),(12,13),(4,11),(5,13),(2,6)$$

The corresponding cost is equal to $C = 10$.
Remark 8.2 Recalculating the BIMs after choosing each edge, according to the values $BIM_e \cdot (0.9 - p(e))/c(e)$, we could get a solution by a slightly lower cost.

More about various strategies of roads system predisaster design the reader can find in [4, 2].

8.3 Flow in network with unreliable edges

In this section we consider flow in a network with unreliable edges. Flow networks are very important and widely used models in transportation networks and various types of supply networks, see [5, 3].

We will consider networks with independent randomly failing **directed** edges. For each edge $e = (a, b)$ directed from the node a to node b, we define the maximal flow $c(e)$ (capacity) which can be delivered from a to b along this edge. We say that the network state is UP if the maximal flow from source to sink is not less than some prescribed value Φ.
Fig. 8.6 represents very simple flow network with 4 nodes and 5 edges.

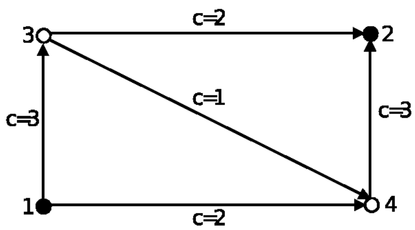

Fig. 8.6: Flow network with 4 nodes and 5 edges

It is easy to check that the maximal flow from source $s = 1$ to sink $t = 2$ equals 5. For example, it may be obtained by the following flows:

$$w(1,3) = 3, w(1,4) = 2, w(3,2) = 2, w(3,4) = 1, w(4,2) = 3.$$

Suppose that we define the UP state as the state with maximal flow no less than $\Phi = 4$. Then, if edge $(3, 2)$ is *down*, the maximal flow equals 3, and the network is $DOWN$.

Suppose now that any edge $e(x, y)$ fails with probability p and its flow capacity drops from maximal value $c(e)$ to zero. Our goal will be to estimate the probability that the maximal flow between the source node s and the sink node t will be not less than some prescribed value Φ.
Fig 8.7 presents the flow network with 35 edges and 15 nodes. 1 and 2 are the source node and the sink node, respectively.

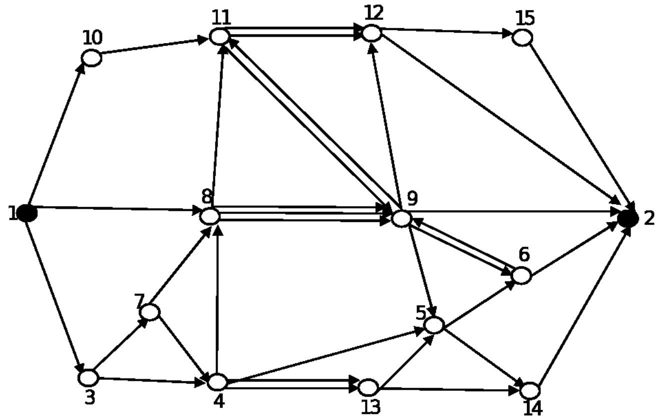

Fig. 8.7: Flow network with 15 nodes and 35 edges

Edge capacities are presented in table 8.3 below. There are several pairs of adjacent nodes connected by two or three edges. They are denoted in the table as $(i, j-1)$ or $(i, j)-2$ or $(i, j)-3$. The arrows on the edges denote the direction in which the flow is allowed to go. Network failure ($DOWN$) is defined as the flow reduction below $\Phi = 12$. The probability of edge failure is denoted by q.

We investigate this network by means of Monte Carlo based on estimating the CD-spectrum of the network (see chapter 6). Let us remind in short the estimation procedure. First, we consider the set of randomly ordered network element numbers - the permutations. Each simulated permutation is "destroyed" from left to right by erasing one element (edge) after another. After each destruction (edge elimination) the network state is checked by a special algorithm and the position of D-anchor is registered. After repeating this procedure M times we remember the numbers M_i of cases when the D-anchor was on the i-th position and compute the cumulative spectrum (CD-spectrum) $y_1, ..., y_n$ as

$$y_1 = \frac{M_1}{M}, y_2 = \frac{M_1 + M_2}{M}, ..., y_n = \frac{M_1 + M_2 + ... + M_n}{M}$$

An important fact is that there is *no need* to check the network state on each step of the destruction process. The position of the anchor can be

Table 8.3: edge capacities

$e(i,j)$	$c(i,j)$	$e(i,j)$	$c(i,j)$	$e(i,j)$	$c(i,j)$
(1,3)	8	(1,8)	9	(1,10)	8
(3,4)	6	(3,7)	6	(4,5)	6
(4,8)	6	(4,13 − 1)	5	(4,13 − 2)	2
(5,6)	6	(5,14)	5	(6,2)	6
(6,9)	3	(7,4)	5	(7,8)	4
—	—	(8,4)	4	(8,9 − 1)	5
(8,9 − 2)	4	(8,9 − 3)	3	(8,11)	4
(9,5)	6	(9,6)	4	(9,2)	5
(9,11)	4	(9,12)	5	(10,11)	5
(11,9)	4	(11,12 − 1)	4	(11,12 − 2)	2
(12,2)	6	(12,15)	5	(13,14)	5
(13,5)	5	(14,2)	5	(15,2)	5

efficiently found by applying **bisection** algorithm which works as follows. Erase the $[n/2]$ edges of the permutation. Check the network state. If it is already *DOWN*, the anchor must be in the first $[n/2]$ positions. If not, the anchor is within remaining positions. Proceed by bisecting the part of the permutation until you locate the anchor. On the average, the location of the anchor will be found by $O(\log_2 n)$ network state checks.

Finally, we note that the network state can be checked by applying the well-known Ford-Fulkerson classic algorithm(see [5, 3] for calculating the max flow in a network. If this max flow is less then our limit Φ, we say that the network is $DOWN$. The CD-spectra for our network for $\Phi = 12$ and $\Phi = 10$ are presented on Fig. 8.8.

If the maximum flow is less than 10, then it is for sure less than 12, that is, the probability of $DOWN$ for $\Phi = 12$ is higher. This explains why in Fig. 8.8 the left curve dominates the right one.

After knowing the CD-spectrum, network reliability is calculated by the well-known formula 6.2.3 based on the fundamental property 6.2.2 of the CD-spectrum.

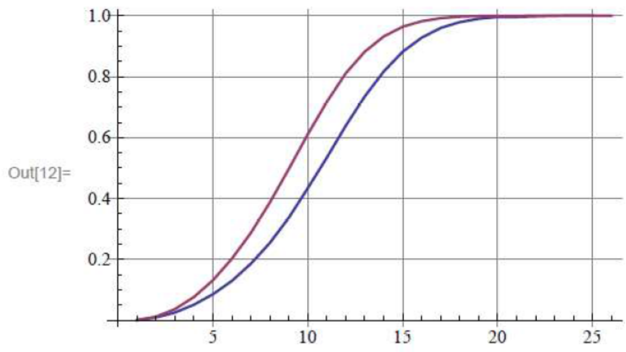

Fig. 8.8: CD-spectra for $\Phi = 12$ - left curve, and $\Phi = 10$ - right curve

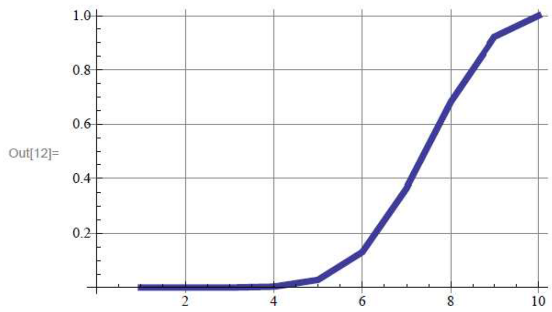

Fig. 8.9: Horizontal axis shows $x = 10p$.The vertical axis is network reliability $P(UP)$ for $\Phi = 12$

The graph on Fig. 8.9 shows how depends network reliability $R = P(UP)$ on edge *up probability p*.

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