RT&A, No 1 (56)
Volume 15, March 2020

Transient Solution of a Heterogeneous Queuing System with Balking and Catastrophes

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Abstract

In this paper we consider a Markovian queuing system with heterogeneous servers, balking and catastrophes. The time-dependent behavior of the system is analyzed by using generating function technique. The expressions for mean and variance of the system are obtained in transient state. At last, some special cases of the model are derived and discussed.

Keywords: Transient solution, Catastrophes, Generating function, Heterogeneous servers, Balking

I. Introduction

Queuing models are playing an important role in modeling the queuing situations in computer-communication networks, hospitals, supply chain management, and in production processes. Customers’ impatience is one of the most important aspects in modeling of queuing systems. Queuing systems with customers’ impatience are comparatively less profitable than the ones without impatience. In real life many queuing situations arise in which there may be a tendency for customers to be discouraged by a long queue. As a result, the customers either decide not to join the queue (i.e. balk) or depart after joining the queue without getting service due to impatience (i.e. renege). The study of customers’ impatience in queuing theory is started in the early 1950’s. Haight (1957), Ancker and Gafarian ((1963a), (1963b)) are the pioneer researchers in the area of queueing with customers’ impatience. Barrer (1957) analyzes an M/M/c queue with customers’ impatience of constant duration. El-Paoumy and Nabwey (2011) study a Poisson queue with balking function, reneging and two heterogeneous servers. Kumar and Sharma (2012a) study a single server Markovian queuing system with balking and retention of reneging customers. They obtain the steady-state probabilities of the model. Kumar and Sharma (2012b) obtain the stationary system size probabilities of a finite capacity Markovian multi-server queuing system with balking and retention of reneging customers. Kumar and Sharma (2018) analyze the transient state probabilities of a multi-server queuing system with balking and retention of reneging customers.

The queuing systems with heterogeneous servers are more applicable as compare to their homogeneous counterparts, because in real-life situations the servers work at different rates. Morse (1958) introduces the concept of heterogeneous service. The heterogeneous service mechanisms are scheduling methods that allow customers to receive different quality of service. Most of the operations in manufacturing systems have heterogeneous service mechanism. That is why, the queuing systems with heterogeneous servers have gained significant attention in the literature. Saaty (1961) further discusses Morse's problem and derives the steady-state probabilities and the mean number in the system. Sharma and Dass (1989) analyze the initial busy period of multichannel Markovian queueing system and obtain the expression of its density function in closed form. Dharmaraja (2000) obtains the transient solution of a two-processor heterogeneous system with Poisson arrival of jobs having exponentially distributed processing times. Kumar and Sharma (2019) obtain the transient solution of a two-heterogeneous servers Markovian queuing model with retention of reneging customers.

Queuing models with catastrophes have been used in modeling a variety of real life systems, such as computer-communication networks under virus attack, manufacturing systems with sudden disasters, and call centers with sudden power breakdowns and corruption of hard disk of computer systems. Recently, due attention has been paid to the study of queuing systems with catastrophes. The occurrence of catastrophes leads to the annihilation of all the customers in the queuing system and momentarily inactivates the service facility until a new arrival occurs. In order to study the impact of noise bursts and virus on queues in computer networks Chao (1995) develops the queuing network model with catastrophes. He obtains the product-form solution of a queuing network model with catastrophes. Kumar and Arivudainambi (2000) incorporate the effect of catastrophes in a single server Markovian queuing system. They derive its transient solution using generating function technique explicitly. Di Crescenzo et al. (2003) discuss the application of M/M/1 queuing model with catastrophes in the phenomenon of muscle contraction. Jain and Kumar (2007) derive the transient solution of a queuing system with correlated arrivals, variable service capacity and catastrophes. Sudesh (2010) studies a single server queuing system with catastrophes and customers' impatience. He derives the transient solution of the model explicitly using generating function technique. Sudesh et al. (2016) derive the transient solution of a two-heterogeneous servers queuing system with catastrophes, server repair and customers' impatience. Jain and Kanethia (2006) study a single server queuing model with change in environment and catastrophes. They obtain both the transient and the steady-state solutions to the model. Tarabia (2011) performs the transient and steady-state analysis of a single server Markovian queuing system with balking, catastrophes, server failures and repairs. Yechiali (2007) studies single and multiple-server queuing models with catastrophes and impatient customers. Ammar (2014) derives the transient solution of a two-processor heterogeneous system with catastrophes, server failures and repairs. Dharmaraja and Kumar (2015) obtain the transient solution of a queuing model with multiple heterogeneous servers in presence of catastrophes. Kumar et al. (2001) obtain the transient solution of an M/M/2 heterogeneous servers queuing system in presence of catastrophes.

Yaseen and Tarabia (2017) analyze the transient and steady-state behavior of Markovian queuing system with balking and reneging subject to catastrophes and server failures. Suranga Sampath and Liu (2018) study an M/M/1 queuing system with reneging, catastrophes, server failures and repairs. They obtained the transient as well as steady-state solution of the model. The applicability of our queuing model can be seen in hospital emergency departments and computer communication system.
The remainder of the paper is structured as follows. In section 2, the queuing model is described. A mathematical model is formulated in section 3. In section 4 transient solution of the model is

studied. The time-dependent mean and variance of the model are obtained in section 5. Section 6 deals with the special cases of the model. Finally, the paper is concluded in section 7.

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II. Queuing Model Description

In this section, we describe the queueing model. The model is based on following assumptions:

1. In accordance with a Poisson process, the arrivals occur one by one with intensity λ.
2. The system has multi-servers (say, c) having distinct service rates and the service times at each server are exponential distributed. This means that the customers are always served by the fastest servers. That is, when such a server becomes available, a customer may switch to a fastest server.
3. On arrival customer either decides to enter the queue with probability p or balk with probability $1-p$.
4. Apart from this, the catastrophes may also occur at the service facility as a Poisson process with rate ψ, when the system is not empty. At the moment when catastrophe occurs at the system, all the customers are destroyed, all the servers get inactivated momentarily and after the catastrophe, the servers become ready for service immediately.
5. The queue discipline is FCFS and the capacity of the system is infinite.
6. Initial condition: $P_0(0)=1$.

III. Mathematical Formulation of the Model

Define, $\mathrm{P_n(t) = P\{X(t) = n\}}$, $n = 0, 1, \ldots$. The queuing model under investigation is governed by the following differential-difference equations:

$$\frac{dP_0(t)}{dt} = -(\lambda+\psi)P_0(t) + \mu_1 P_1(t) + \psi \tag{1}$$

$$\frac{dP_n(t)}{dt} = -(\lambda+\psi+\textstyle\sum_{i=1}^{n}\mu_i)P_n(t) + \sum_{i=1}^{n+1}\mu_i\, P_{n+1}(t) + \lambda P_{n-1}(t), \quad 1 \le n < c \tag{2}$$

$$\frac{dP_c(t)}{dt} = -(\lambda p+\psi+\textstyle\sum_{i=1}^{c}\mu_i)P_c(t) + \sum_{i=1}^{c}\mu_i\, P_{c+1}(t) + \lambda P_{c-1}(t),\ n = c \tag{3}$$

$$\frac{dP_n(t)}{dt} = -(\lambda p+\psi+\textstyle\sum_{i=1}^{c}\mu_i)P_n(t) + \sum_{i=1}^{c}\mu_i\, P_{n+1}(t) + \lambda p P_{n-1}(t),\ n > c \tag{4}$$

IV. Transient solution of the model

Theorem 1. The transient state probabilities of a Markovian queuing system with multi heterogeneous servers, balking and catastrophes which is governed by the differential-difference equations (1) − (4) are given by:

$$P_k(t) = b_{k,0}(t) + \psi\int_0^t b_{k,0}(u)du + \gamma\int_0^t b_{k,c-1}(u)P_c(t-u)du, \quad k = 0, 1, \ldots, c-1$$

$$P_c(t) = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{(-1)^m}{\gamma}\left(\frac{\alpha}{2\lambda p}\right)^{n+1}(n+1)\binom{n}{m}\Big[\int_0^t A(t-u)\int_0^u B^{C(m)}(u-v)\exp\{-(\lambda p+\gamma+\psi)v\}\frac{I_{n+1}(\alpha v)}{v}dudv + \psi\int_0^t H(t-u)\int_0^u B^{C(m)}(u-v)\exp\{-(\lambda p+\gamma+\psi)v\}\frac{I_{n+1}(\alpha v)}{v}dudv\Big]$$

and, for n = 1, 2, ...

$$P_{n+c}(t) = n\beta^n\int_0^t \exp\{-(\lambda p+\psi+\gamma)(t-u)\}\frac{I_n(\alpha(t-u))}{(t-u)}P_c(u)du$$

where $H(t) = \int_0^t A(u)du$ and $B^{C(m)(t)}$ is m – fold convolution of B(t) with itself with $B^{C(0)} = \delta(t)$, the Dirac - delta function.

Proof. Define the pgf P(z, t) for the transient state probabilities $P_n(t)$ by

$$P(z,t) = q_c(t) + \sum_{n=1}^{\infty}P_{n+c}(t)z^{n+1}; \qquad P(z,0) = 1 \tag{5}$$

with

$$\sum_{n=0}^{c}P_n(t) = q_c(t) \tag{6}$$

Adding the equations (1) - (3), we get

$$\frac{d}{dt}\big(q_c(t)\big) = -\lambda p P_c(t) + \sum_{i=1}^{c}\mu_i\, P_{c+1}(t) - \psi q_c(t) + \psi \tag{7}$$

On multiplying equation (4) by z^n and summing, we get

$$\frac{d}{dt}\left[\sum_{n=1}^{\infty} P_{n+c}(t)z^{n+1}\right] = \left[(\lambda p + \psi + \textstyle\sum_{i=1}^{c}\mu_i) + \left(\lambda pz + \frac{\sum_{i=1}^{c}\mu_i}{z}\right)\right]\sum_{n=1}^{\infty} P_{n+c}(t)z^n + \lambda pzP_c(t) - \sum_{i=1}^{c}\mu_i P_{c+1}(t) \quad (8)$$

By adding (7) and (8) the following differential equation is obtained:

$$\frac{\partial P(z,t)}{\partial t} = \left[\left(\lambda pz + \frac{\sum_{i=1}^{c}\mu_i}{z}\right) - (\lambda p + \psi + \textstyle\sum_{i=1}^{c}\mu_i)\right]P(z,t) - \left[\left(\lambda pz + \frac{\sum_{i=1}^{c}\mu_i}{z}\right) - (\lambda p + \textstyle\sum_{i=1}^{c}\mu_i)\right]q_c(t) + \lambda p(z-1)P_c(t) + \psi \quad (9)$$

On solving (9), we get

$$P(z,t) = \exp\left\{\left[\left(\lambda pz + \frac{\sum_{i=1}^{c}\mu_i}{z}\right) - (\lambda p + \psi + \textstyle\sum_{i=1}^{c}\mu_i)\right]t\right\} + \int_0^t\left[\lambda p(z-1)P_c(u) - \left(\left(\lambda pz + \frac{\sum_{i=1}^{c}\mu_i}{z}\right) - (\lambda p + \textstyle\sum_{i=1}^{c}\mu_i)\right)q_c(u)\right] \times \exp\left\{\left[\left(\lambda pz + \frac{\sum_{i=1}^{c}\mu_i}{z}\right) - (\lambda p + \psi + \textstyle\sum_{i=1}^{c}\mu_i)\right](t-u)\right\}du + \psi\int_0^t \exp\left\{\left[\left(\lambda pz + \frac{\sum_{i=1}^{c}\mu_i}{z}\right) - (\lambda p + \psi + \textstyle\sum_{i=1}^{c}\mu_i)\right](t-u)\right\}du \quad (10)$$

If $\gamma = \sum_{i=1}^{c}\mu_i$, $\alpha = 2\sqrt{\lambda p\gamma}$ and $\beta = \sqrt{\frac{\lambda p}{\gamma}}$, then using the modified Bessel function of first kind $I_n(.)$ and the Bessel function properties, we get

$$\exp\left\{\left(\lambda pz + \frac{\gamma}{z}\right)t\right\} = \sum_{n=-\infty}^{\infty}(\beta z)^n I_n(\alpha t) \quad (11)$$

Using (11) in (10), we get

$$P(z,t) = \exp\{-(\lambda p + \psi + \gamma)t\}\sum_{n=-\infty}^{\infty}(\beta z)^n I_n(\alpha t) + \lambda p\int_0^t P_c(u)\exp\{-(\lambda p + \psi + \gamma)(t-u)\}\sum_{n=-\infty}^{\infty}(\beta z)^n\left[\beta^{-1}I_{n-1}(\alpha(t-u)) - I_n(\alpha(t-u))\right]du + \int_0^t q_c(u)\exp\{-(\lambda p + \psi + \gamma)(t-u)\}\sum_{n=-\infty}^{\infty}(\beta z)^n\left[-\lambda p\beta^{-1}I_{n-1}(\alpha(t-u)) + (\lambda p + \gamma)I_n(\alpha(t-u)) - \beta\gamma I_{n+1}(\alpha(t-u))\right]du + \psi\int_0^t \exp\{-(\lambda p + \psi + \gamma)(t-u)\}\sum_{n=-\infty}^{\infty}(\beta z)^n I_n(\alpha(t-u))du \quad (12)$$

Now, comparing the coefficients of z^n on either side of (12), we obtain for $n = 1, 2, \ldots$

$$P_{n+c}(t) = \exp\{-(\lambda p + \psi + \gamma)t\}(\beta)^n I_n(\alpha t) + \lambda p\int_0^t \exp\{-(\lambda p + \psi + \gamma)(t-u)\}\left[I_{n-1}(\alpha(t-u))\beta^{n-1} - I_n(\alpha(t-u))\beta^n\right]P_c(u)du - \int_0^t \exp\{-(\lambda p + \psi + \gamma)(t-u)\}q_c(u)\left[\lambda pI_{n-1}(\alpha(t-u))\beta^{n-1} - (\lambda p + \gamma)I_n(\alpha(t-u))\beta^n + \gamma I_{n+1}(\alpha(t-u))\beta^{n+1}\right]du + \psi\int_0^t \exp\{-(\lambda p + \psi + \gamma)(t-u)\}\beta^n I_n(\alpha(t-u))du \quad (13)$$

Comparing the terms free of z on either side of equation (12), that is, for $n = 0$, we get

$$q_c(t) = \exp\{-(\lambda p + \psi + \gamma)tI_0(\alpha t)\} + \lambda p\int_0^t \exp\{-(\lambda p + \psi + \gamma)(t-u)\}\left[I_1(\alpha(t-u))\beta^{-1} - I_0(\alpha(t-u))\right]P_c(u)du - \int_0^t \exp\{-(\lambda p + \psi + \gamma)(t-u)\}q_c(u)\left[\alpha I_1(\alpha(t-u)) - (\lambda p + \gamma)I_0(\alpha(t-u))\right]du + \psi\int_0^t \exp\{-(\lambda p + \psi + \gamma)(t-u)\}I_0(\alpha(t-u))du \quad (14)$$

After simplifying (13), we obtain

$$\int_0^t \exp\{-(\lambda p + \psi + \gamma)(t-u)\}q_c(u)\left[\lambda pI_{n+1}(\alpha(t-u))\beta^{n-1} - (\lambda p + \gamma)I_n(\alpha(t-u))\beta^n + \gamma I_{n-1}(\alpha(t-u))\beta^{n+1}\right]du$$
$$= \exp\{-(\lambda p + \psi + \gamma)t\}(\beta)^n I_n(\alpha t) + \lambda p\int_0^t \exp\{-(\lambda p + \psi + \gamma)(t-u)\}\left[I_{n+1}(\alpha(t-u))\beta^{n-1} - I_n(\alpha(t-u))\beta^n\right]P_c(u)du + \psi\int_0^t \exp\{-(\lambda p + \psi + \gamma)(t-u)\}\beta^n I_n(\alpha(t-u))du \quad (15)$$

Substituting (15) in (13), we get

$$P_{n+c}(t) = n\beta^n\int_0^t \exp\{-(\lambda p + \psi + \gamma)(t-u)\}\frac{I_n(\alpha(t-u))}{(t-u)}P_c(u)du, \quad n = 1, 2, \ldots \quad (16)$$

On solving (1) and (2), we obtain the remaining probabilities $P_n(t), n = 0, 1, 2, \ldots, c$. Equations (1) and (2) can be written in matrix form as:

$$\frac{d\boldsymbol{P}(t)}{dt} = A\boldsymbol{P}(\boldsymbol{t}) + \gamma P_c(t)\boldsymbol{e_1} + \psi\boldsymbol{e_2} \quad (17)$$

where the matrix $A = (a_{i,j})_{c\times c}$ is given as:

$$A = \begin{bmatrix} -(\lambda+\psi) & \mu_1 & \cdot\ \cdot\ \cdot & 0 \\ \lambda & -(\lambda+\psi+\mu_1) & \cdot\ \cdot\ \cdot & 0 \\ \cdot & \cdot & \cdot\ \cdot\ \cdot & \cdot \\ \cdot & \cdot & \cdot\ \cdot\ \cdot & \cdot \\ \cdot & \cdot & \cdot\ \cdot\ \cdot & \sum_{i=1}^{c-1}\mu_i \\ 0 & 0 & \cdot\ \cdot\ \cdot & -(\lambda+\psi+\sum_{i=1}^{c-1}\mu_i) \end{bmatrix}$$

$\boldsymbol{P(t)} = (P_0(t)\ P_1(t) \dots P_{c-1}(t))^T$, $\boldsymbol{e_1} = (0\ 0\ \dots 1)^T$ and $\boldsymbol{e_2} = (1\ 0\ \dots 0)^T$ are vectors of order c. Let the Laplace Transform (LT) of $\mathbf{P(t)}$ is $\boldsymbol{P^*(s)} = (P_0{}^*(s)\ P_1{}^*(s) \dots P_{c-1}{}^*(s))^T$. Taking the Laplace Transform of equation (17) we get

$$\boldsymbol{P^*(s)} = (sI - A)^{-1}\left\{\gamma P_c^*(s)\boldsymbol{e_1} + \boldsymbol{P(0)} + \frac{\psi}{s}\boldsymbol{e_2}\right\} \tag{18}$$

with $\mathbf{P(0)} = (1\ 0\ \dots 0)^{\mathrm{T}}$. If $e = (1\ 1\ \dots 1)^{\mathrm{T}}{}_{c\times 1}$, then

$$e^T\boldsymbol{P^*(s)} + P_c^*(s) = q_c^*(s) \tag{19}$$

Define

$$f(s) = [(s+\lambda p+\gamma+\psi) - \sqrt{(s+\lambda p+\gamma+\psi)^2 - \alpha^2}]$$

Taking LT of (14), we obtain

$$s(s+\psi)q_c^*(s) = (s+\psi) + sP_c^*(s)\frac{1}{2}[f(s) - \alpha\beta] \tag{20}$$

Using (20) in (19), we get

$$P_c^*(s) = \left(\frac{s+\psi}{s}\right)\frac{1 - se^T(sI-A)^{-1}\left(\boldsymbol{P(0)} + \frac{\psi}{s}\boldsymbol{e_2}\right)}{\left\{(s+\lambda p+\psi) - \frac{1}{2}\{f(s)\} + (s+\psi)\gamma e^T(sI-A)^{-1}\boldsymbol{e_1}\right\}} \tag{21}$$

Let us assume that $(sI - A)^{-1} = \left(b_{ij}^*(s)\right)_{c\times c}$

We observe that $(sI - A)^{-1}$ is almost lower triangular. Following Raju and Bhat (1982), we obtain, $i = 0,1,\dots,c-1$

$$b_{ij}^*(s) = \begin{cases} \frac{1}{\sum_{k=1}^{j+1}\mu_k}\frac{\mu_{c,j+1}(s)\mu_{i,0}(s) - \mu_{i,j+1}(s)\mu_{c,0}(s)}{\mu_{c,0}(s)}, & j = 0,1,\dots,c-2 \\ \frac{\mu_{i,0}(s)}{\mu_{c,0}(s)}, & j = c-1 \end{cases} \tag{22}$$

where $\mu_{i,j}(s)$ are recursively given as

$$\mu_{i,i}(s) = 1, \qquad i = 0,1,\dots,c-1$$

$$\mu_{i+1,i}(s) = \frac{s+\lambda+\psi+\sum_{k=1}^{i}\mu_k}{\sum_{k=1}^{i+1}\mu_k} \qquad i = 0,1,\dots,c-2$$

$$\mu_{i+1,i-j}(s) = \frac{\left(s+\lambda+\psi+\sum_{k=1}^{i}\mu_k\right)\mu_{i,i-j} - \lambda\mu_{i-1,i-j}}{\sum_{k=1}^{i+1}\mu_k}, \quad j \le i, \qquad i = 1,2,\dots,c-2$$

$$\mu_{c,j}(s) = \begin{cases} [s+\lambda+\psi+\sum_{k=1}^{c-1}\mu_k]\mu_{c-1,j} - \lambda\mu_{c-2,j}, & j = 0,1,\dots,c-2 \\ s+\lambda+\psi+\sum_{k=1}^{c-1}\mu_k, & j = c-1 \end{cases} \tag{23}$$

and $\mu_{i,j}(s) = 0$, for other i and j. We have suppressed the argument s to facilitate computation. The advantage in using these relations is that we do not evaluate any determinant. Using these in equation (21), we get

$$P_c^*(s) = \left(\frac{s+\psi}{s}\right)\frac{1 - (s+\psi)\sum_{i=0}^{c-1}b_{i,o}^*(s)}{\left\{(s+\lambda p+\psi) - \frac{1}{2}\{f(s)\} + (s+\psi)\gamma\sum_{j=0}^{c-1}b_{j,c-1}^*(s)\right\}} \tag{24}$$

and for $k = 0,1,\dots,c-1$ from equation (18), we get

$$P_k^*(s) = \left(1 + \frac{\psi}{s}\right)b_{k,0}^*(s) + \gamma b_{k,c-1}^*(s)P_c^*(s) \tag{25}$$

We observe that $b_{i,j}^*(s)$ are all rational algebraic functions in s. So, by partial fraction decomposition the inverse transform $b_{i,j}(t)$ of $b_{i,j}^*(s)$ can be obtained. Let $s_i, i = 0,1,\dots,c-1$, be the characteristic roots of the matrix A. Then after simplification, $P_c^*(s)$ equals to

$$\frac{\left(1+\frac{\psi}{s}\right)A^*(s)}{\frac{1}{2}[(s+\lambda p+\gamma+\psi)+\sqrt{(s+\lambda p+\gamma+\psi)^2-\alpha^2}]\left[1-\frac{2\gamma(1-B^*(s))}{[(s+\lambda p+\gamma+\psi)+\sqrt{(s+\lambda p+\gamma+\psi)^2-\alpha^2}]}\right]} \tag{26}$$

where

$$A^*(s)=\sum_{i=0}^{c-1}\frac{A_i}{s-s_i} \tag{27}$$

$$B^*(s)=\sum_{i=0}^{c-1}\frac{B_i}{s-s_i} \tag{28}$$

with constants A_i and B_i given by

$$A_i=\lim_{s\to s_i}(s-s_i)\left[1-\sum_{l=0}^{c-1}(s+\psi)b_{l,0}^*(s)\right] \tag{29}$$

$$B_i=\lim_{s\to s_i}(s-s_i)\left[\sum_{l=0}^{c-1}(s+\psi)b_{l,c-1}^*(s)\right] \tag{30}$$

Hence, (26) simplifies into

$$P_c^*(s)=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{(-1)^m}{\gamma}\left(\frac{\alpha}{2\lambda p}\right)^{n+1}(n+1)\binom{n}{m}\left(1+\frac{\psi}{s}\right)A^*(s)\left(B^*(s)\right)^m\frac{[(s+\lambda p+\gamma+\psi)+\sqrt{(s+\lambda p+\gamma+\psi)^2-\alpha^2}]^{n+1}}{(n+1)\alpha^{n+1}} \tag{31}$$

Taking Laplace inverse of (31), we obtain

$$P_c(t)=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{(-1)^m}{\gamma}\left(\frac{\alpha}{2\lambda p}\right)^{n+1}\binom{n}{m}\left[\int_0^t A(t-u)\int_0^u B^{C(m)}(u-v)\exp\{-(\lambda p+\gamma+\psi)v\}\frac{I_{n+1}(\alpha v))}{v}dudv+\psi\int_0^t H(t-u)\int_0^u B^{C(m)}(u-v)\exp\{-(\lambda p+\gamma+\psi)v\}\frac{I_{n+1}(\alpha v))}{v}dudv\right] \tag{32}$$

where $H(t)=\int_0^t A(u)du$ and $B^{C(m)(t)}$ is m – fold convolution of B(t) with itself with $B^{C(0)}=\delta(t)$, the Dirac - delta function. Now, the Laplace inverse of equation (25) yields,

$$P_k(t)=b_{k,0}(t)+\psi\int_0^t b_{k,0}(u)du+\gamma\int_0^t b_{k,c=1}(u)P_c(t-u)du,\ k=0,1,\dots,c-1 \tag{33}$$

where $P_c(u)$ is given in (32). Thus, the equations (16), (32) and (33) determine all the transient state probabilities. Hence, the time-dependent probabilities of the model are obtained explicitly.

V. Mean and Variance

In this section we derive the expressions for time-dependent mean and variance of the queuing system.

Mean, M(t): The mean number of customers in the system at time t is given by:

$$E[X(t)\}=M(t)=m(t)+r(t)=\sum_{n=1}^{c-1}nP_n(t)+\sum_{n=c}^{\infty}nP_n(t) \tag{34}$$

$$M(0)=m(0)+r(0)=\sum_{n=1}^{c-1}nP_n(0)+\sum_{n=c}^{\infty}nP_n(0)$$

$$M'(t)=m'(t)+r'(t)=\sum_{n=1}^{c-1}nP_n'(t)+\sum_{n=c}^{\infty}nP_n'(t)$$

Multiplying (1)-(3) by n and summing over the range of n, we get

$$M'(t)=\lambda\sum_{n=0}^{c-1}P_n(t)-\psi[m(t)+r(t)]-\lambda p\sum_{n=c}^{\infty}nP_n(t)+\lambda p\sum_{n=c+1}^{\infty}nP_{n-1}(t)+\sum_{i=1}^{n+1}\mu_i\sum_{n=1}^{c-1}nP_{n+1}(t)-\sum_{i=1}^{n}\mu_i\sum_{n=1}^{c-1}nP_n(t)-\sum_{i=1}^{c}\mu_i\sum_{n=c}^{\infty}nP_n(t)+\sum_{i=1}^{c}\mu_i\sum_{n=c}^{\infty}nP_{n+1}(t)$$

On solving above equation we get

$$M'(t)=-\psi M(t)+\sum_{n=1}^{c-1}P_n(t)(\lambda-\sum_{i=1}^{n}\mu_i)+\lambda P_0(t)+\sum_{n=c}^{\infty}P_n(t)\,(\lambda p-\sum_{i=1}^{c}\mu_i)$$

The above equation is of the form $y'+Py=Q$ whose solution is

$$E[X(t)\}=M(t)=(\lambda-\sum_{i=1}^{n}\mu_i)\sum_{n=1}^{c-1}\int_0^t P_n(u)\exp\left(-\psi(t-u)\right)du+\lambda\int_0^t P_0(u)\exp\left(-\psi(t-u)\right)du+$$

$(\lambda p - \sum_{i=1}^{c} \mu_i) \sum_{n=c}^{\infty} \int_0^t P_n(u) \exp\big(-\psi(t-u)\big)\, du$ (35)

Variance, V(t): The variance of number of customers in the system at time t is given by:

$$V(t) = K(t) - [M(t)]^2 \quad (36)$$

$K(t) = E[X^2(t)] = k(t) + l(t) = \sum_{n=1}^{c-1} n^2 P_n(t) + \sum_{n=c}^{\infty} n^2 P_n(t)$

$K'(t) = k'(t) + l'(t) = \sum_{n=1}^{c-1} n^2 P'_n(t) + \sum_{n=c}^{\infty} n^2 P'_n(t)$

Multiplying (1)-(3) by n^2 and summing over the range of n, we get

$K'(t) = \lambda \sum_{n=0}^{c-1}(2n+1)P_n(t) - \psi K(t) + \lambda p[\sum_{n=c}^{\infty} P_n(t)] + 2\sum_{n=c}^{\infty} nP_n(t) + \sum_{i=1}^{c} \mu_i (c-1)^2 P_c(t) + 2\sum_{i=1}^{c} \mu_i P_{c-1}(t) - \sum_{n=1}^{c-1}\sum_{i=1}^{n} \mu_i (2n+1)P_n(t) + \sum_{i=1}^{c} \mu_i [\sum_{n=c+1}^{\infty} P_n(t) - c^2 P_c(t) + 2cP_c(t) - 2\sum_{n=c}^{\infty} nP_n(t)]$

On solving above equation we get

$K'(t) = -\psi K(t) + [\lambda p + \sum_{i=1}^{c} \mu_i] \sum_{n=c}^{\infty} P_n(t) + 2\sum_{i=1}^{c} \mu_i P_{c-1}(t) + 2[\lambda p - \sum_{i=1}^{c} \mu_i] r(t) + [\lambda + \sum_{i=1}^{n} \mu_i] \sum_{n=1}^{c-1} P_n(t) + 3\lambda P_0(t) + 2[\lambda - \sum_{i=1}^{n} \mu_i] m(t)$

The above equation is of the form $y' + \mathrm{P}y = \mathrm{Q}$ whose solution is

$K(t) = [\lambda p + \sum_{i=1}^{c} \mu_i] \sum_{n=c}^{\infty} \int_0^t P_n(u) \exp\big(-\psi(t-u)\big)\, du + 2\sum_{i=1}^{c} \mu_i \int_0^t P_{c-1}(u) \exp\big(-\psi(t-u)\big)\, du + 2[\lambda p - \sum_{i=1}^{c} \mu_i] \int_0^t (M(u) - m(u)) \exp\big(-\psi(t-u)\big)\, du + 2[\lambda - \sum_{i=1}^{n} \mu_i] \int_0^t m(u) \exp\big(-\psi(t-u)\big)\, du + [\lambda + \sum_{i=1}^{n} \mu_i] \sum_{n=1}^{c-1} \int_0^t P_n(u) \exp\big(-\psi(t-u)\big)\, du + 3\lambda \int_0^t P_0(u) \exp\big(-\psi(t-u)\big)\, du$

Therefore,

$$V(t) = [\lambda p + \sum_{i=1}^{c} \mu_i] \sum_{n=c}^{\infty} \int_0^t P_n(u) \exp\big(-\psi(t-u)\big)\, du + 2\sum_{i=1}^{c} \mu_i \int_0^t P_{c-1}(u) \exp\big(-\psi(t-u)\big)\, du + 2[\lambda p - \sum_{i=1}^{c} \mu_i] \int_0^t (M(u) - m(u)) \exp\big(-\psi(t-u)\big)\, du + 2[\lambda - \sum_{i=1}^{n} \mu_i] \int_0^t m(u) \exp\big(-\psi(t-u)\big)\, du + [\lambda + \sum_{i=1}^{n} \mu_i] \sum_{n=1}^{c-1} \int_0^t P_n(u) \exp\big(-\psi(t-u)\big)\, du + 3\lambda \int_0^t P_0(u) \exp\big(-\psi(t-u)\big)\, du - [M(t)]^2 \quad (37)$$

where $M(t)$ is given in equation (35).

VI. Special Cases

Case 1 When there is no balking (i.e. $p = 0$), then the transient state probabilities are same as that of the model studied by Dharmaraja and Kumar (2015).
Case 2 If we remove the catastrophe from the model (i.e. $\psi = 0$), then the results of our model resemble with the model studied by Kumar and Arivudainambi (2001).

V. Conclusions

In this paper the transient analysis of a Markovian queuing system with heterogeneous servers, balking and catastrophes is performed. The time-dependent mean and variance of the number of customers in the system are also obtained. Some important queuing models are derived as the special cases.

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