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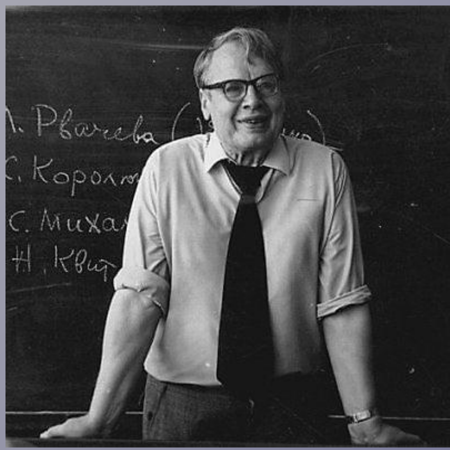

RELIABILITY:
THEORY&APPLICATIONS

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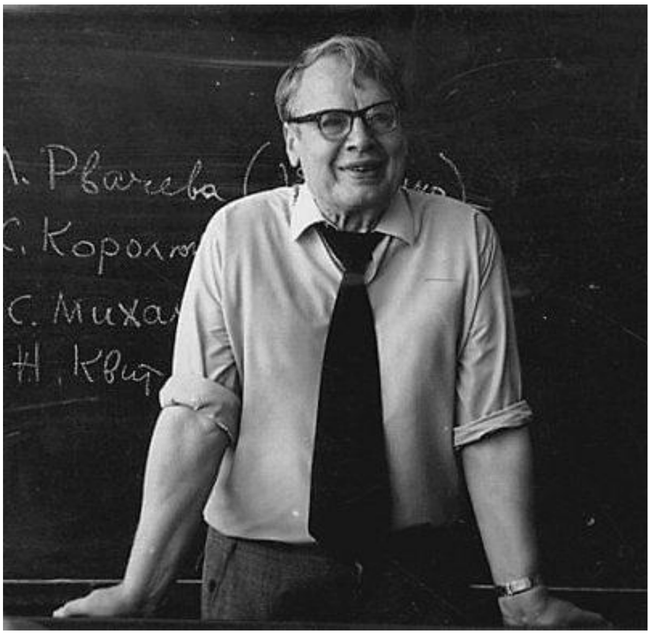

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THEORY & APPLICATIONS

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Dear Colleagues!

The past year and the beginning of the new year brought a number of sad events, which we would like to inform you about with great regret and pain. Bright scientists in the field of reliability theory, bright representatives of the best schools of the Soviet Union and Russia have passed away. They were wonderful, responsive people, with whom we and many of you were connected not only by professional relations, but also by sincere friendship. The more severe these losses are for our entire community - the Gnedenko Forum. Largely thanks to them, our Forum has gained fame and popularity in the world. All of them were active participants and members of the Editorial Board of our journal. The bright memory of these people will always remain in our hearts. We encourage you to honor their memory by publishing memoirs about them and articles using the ideas of these scientists in the upcoming issues of the journal "Reliability: Theory and Applications". We express our sincere condolences to family, relatives, friends and colleagues, to all who knew these outstanding persons.

Vladimir Korolyuk

Academician, Academy of Sciences of the Ukrainian Soviet Socialist Republic (from 1976, corresponding member from 1967). Basic research trends are calculus of probabilities and mathematical statistics, numerical mathematics and computer programming. He developed the method of time-series deflation of deficiencies relying on boundary layer effect, which appears at the change from integral, integrodifferential and finite-difference equations with small parameter to parabolic or elliptical differential equations. He suggested and developed a new approach to studying functional of Markovian and semi-Markovian processes based on inverse of linear operators perturbed on spectrum. Editor-in-Chief of the journal Probability Theory and Mathematical Statistics, Academician of the National Academy of Sciences of Ukraine Professor Volodymyr Korolyuk passed away 4th April 2020. The outstanding mathematician of worldwide recognition, Volodymyr Korolyuk was one of the founders of the Kyiv school of probability theory, mathematical statistics and cybernetics. The scientific research of Volodymyr Semenovych Korolyuk covered various areas of probability, mathematical statistics, theory of stochastic processes and cybernetics. V.S.Korolyuk was one of the first scientists in Ukraine, who assessed the theoretical and practical importance of semi-Markov processes and attracted the attention of his students for their research and application. The results of these studies launched a new direction – the theory of asymptotic phase merging and averaging of random processes. V.S.Korolyuk also launched another new direction – asymptotic analysis of random evolutions.

The mathematical heritage of V.S.Korolyuk includes 22 monographs and about 20 textbooks, most of which are reprinted in foreign languages; more than 280 scientific research papers. Under his supervision, 43 students defended their PhD dissertations, 14 – Doctoral dissertations. V.S.Korolyuk combined fruitful scientific work, teaching and scientific-organizational activities. During a long period of time he lectured the theory of programming, probability theory and mathematical statistics at the Taras Shevchenko National University of Kyiv at Faculty of

Mechanics and Mathematics. Within a group of famous experts, he was awarded the USSR State Prize (1978) for the creation of "Encyclopedia of Cybernetics". In addition, V.S.Korolyuk was awarded the Glushkov Prize (1988) and Bogolyubov Prize (1995). In 1998 he was awarded the honorary title "Honored Figure of Science and Technics of Ukraine." In 2002 V.S.Korolyuk was awarded the Prize of the National Academy of Sciences of Ukraine and Medal by the name of M.V. Ostorgradsky, and in 2003 – the State Prize of Ukraine in Science and Technology.

Volodymyr Korolyuk shared his inspiration by mathematical science and his passion to solving mathematical problems in probability and statistics with colleagues and numerous students and followers, his scientific influence has widely spread also to many specialists in applied areas who use probability and statistics methods for their research. Volodymyr Korolyuk will be always remembered as talented scientists and pedagogue, his ideas and scientific heritage has had a great impact on contemporary mathematical science, his disciples and followers will continue and develop the researches started by him.

Eliahu Gertsbakh

Doctor of Sci., Professor Emeritus

He was born in Riga (Latvia) in 1934. He wrote: "June 27, 1941, the sixth day of the war, looked like an ordinary day to me. .. My father was in the office as usual, and my mother was doing something around the house... Nothing boded ill... Around noon, the phone rang – my father from the newsroom: Troops are retreating along Brivibas. I'll be home soon. We have to leave, and we don't have much time. Of Brivibas street - the main street of Riga." The way from Riga was like a horror movie. It concludes: "So passed the day of June 27, 41. The worst day of my life. On the following day, June 28, the leading detachments of German troops were already in the suburbs of Riga. My father's brother, uncle Semyon and his family, lived in Riga. ... but on that day, uncle Semyon's wife was ill, and they did not go anywhere. They died in the Riga ghetto, uncle Semyon, his wife, and my cousins, Ruthie and Neri."

"We returned to Riga on January 3, 1945. I graduated from school in 1950 and studied mechanical engineering at the Latvian state University in 1950-1955. In 1955-1961, he worked at the WEF (state electrotechnical plant) and simultaneously studied mathematics at the same University. 1961-1964 I was a doctoral student of Chaim Borisovich Kordonsky).

During this period, Ilya Gersbach successfully worked in the field of mathematical reliability theory. An article on the choice of the optimal mode of maintenance of the groove of an identical element in automatic installations appeared in the prestigious magazine "automation and remote control", 1964, no. 3. best-Selling models of failures of I. B. Hertbach And H. B. Kordonsky, Moscow, radio, in Russian, appears in 1966. He is defending his dissertations. One of the opponents of the thesis was B. V. Gnedenko.

He wrote in 01.2017: "working at the Central research Institute of civil aviation in 1965-72 was probably the most interesting period of my professional life. It is sad that Haim Borisovich, Misha Maxim and Valery Venevtsev, the real authors and creators of the Aeroflot schedule, are no longer with us." "The computing center at the Institute of civil aviation was probably founded in 1964. He had a group that started working on computer applications at Aeroflot. The work of this group did not produce visible results, as we found out later, and was centered around hopeless attempts to apply integer programming to plotting. The administration of the Institute decided to organize a new Department in the Computer center under the leadership of Chaim Borisovich. I started working there in September 1965. we appointed a small group of people, which included Vald Lini, Valery Venevtsev, Misha Maxim and I, and we were given the task of creating the Central schedule of Aeroflot using computers."

In 1974, he was repatriated to Israel.

Ilya B. Gertsbakh received his M.Sc. in Mechanical Engineering (1955) and Mathematics

(1961) from the Latvian State University (Riga) and his Ph.D. degree in Applied Probability and Statistics from the Latvian Academy of Sciences (1964). He is Professor in the Department of Mathematics and Computer Science at Ben Gurion University of the Negev in Beersheva, Israel, where he has taught since 1975. He has published about 60 papers and three books.

Mikhail Nikulin

Ph. Doctor, Professor

UFR "Sciences and Modelling"; University Victor Segalen Bordeaux 2; 146 Rue de Leo Saignat, 33076, Bordeaux, France. The Laboratory of Statistical Methods of the Steklov Mathematical Institute at St. Petersburg, Russia. Mikhail Nikulin earned his Ph.D in the Theory of Probability and Mathematical Statistics (1973) from The Steklov Mathematical Institute in Moscow. Member of the St. Petersburg's Mathematical Society, the elected member of the International Statistics Institute, a board member of the book series "Statistics for Industry and technology", Birkhauser. From the memoirs of B. Yu. Lemeshko (Novosibirsk state technical University): In 1973, Mikhail Nikulin published 2 articles in the journal "probability Theory and its application" (1. Nikulin M. S. Chi-square Criterion for continuous distributions with shift and scale parameters // Probability theory and its application. 1973. T. XVIII. No. 3. – P. 583-591 and 2. Nikulin, M. S. On the criteria for Chi-square test for continuous distributions // Probability theory and its application. 1973. Vol. XVIII. no. 3. - P. 675-676), devoted to the Chi-square type criterion, which provided for the use of maximum likelihood estimates for the original ungrouped sample. It was shown that the asymptotic distribution of the statistics of the proposed criterion is a Chi-square distribution with the number of degrees of freedom k-1, where k is the number of intervals used. Even earlier, he reported on these results at the international conference on probability theory and mathematical statistics, held from 25 to 30 June 1973 in Vilnius. The work [1] was published in English the following year (Nikulin M. S. Chi-Square Test for Continuous Distributions with Shift and Scale Parameters // Theory Probab. Appl., 18(3), 559–568. DOI: 10.1137/1118069). In 1974 an article by Rao K. C. & Robson D. S (Rao K. C., Robson D. S. a chi-square statistical for goodness-of-fit tests within the exponential family // Communications in Statistics, 3:12, 1139-1153. DOI:10.1080/03610927408827216). Since then, international publications have referred to this criterion as the RAO-Robson criterion, and only a quarter of a century later I saw a reference to it as the RAO-Robson-Nikulin criterion.

While conducting numerical studies of the properties of hypothesis testing criteria, we naturally did not ignore the criteria proposed by Mikhail Nikulin. On the one hand, everything confirmed the good properties of the criteria, on the other – the lack of publications mentioning the use of criteria in applications was surprising. Later, it was realized that applications usually use the simplest methods that require minimal training or rely on existing software.

My acquaintance with Mikhail Nikulin took place much later. He wrote to me sometime in late 2006. A correspondence ensued. Mikhail's interest in our team was largely due to our focus on using computer technologies in research, which allows us not only to study the properties of methods, but also to apply them to solving problems in applications. In 2007, at his invitation, I visited him in Bordeaux. Mikhail's main interests at this time lay in the field of mathematical methods used in accelerated tests in problems of survival and reliability. Under his influence, we also started working in this direction. In 2008, the second International Conference on Accelerated Life Testing in Reliability and Quality Control, organized by Nikulin in Bordeaux, was attended by 3 people, and the next one in 2010, held in Clermont-Ferrand, was attended by 5 people. in 2009, Mikhail helped expand the number of international participants and our participation in the "VI International Conference Mathematical Methods in Reliability" in Moscow and in the "6th St. Petersburg Workshop on Simulation" in St. Petersburg.

From 2010 to 2013, with the participation of Mikhail Nikulin, we carried out 2 projects aimed at developing methods of statistical analysis and simulation in the study of reliability, quality control and survival. Since that time, the International Workshop "Applied Methods of Statistical Analysis" has been counting down, and Mikhail Nikulin has made a great contribution to its organization. The seminar is held regularly with a period of 2 years. In 2019, the next 5th seminar was held in Novosibirsk (http://www.amsa.conf.nstu.ru/amsa2019/).

Mikhail Nikulin was a person who was very interested in making sure that statisticians were engaged in tasks that were useful for applications, and that they were engaged in topical issues. There was such a Patriotic streak in it: the desire for Russian specialists in the field of probability theory and mathematical statistics to deal with key issues that can provide real benefits for specific areas of human activity. In his memoirs, he treated his scientific supervisor L. N. Bolshev with great warmth, especially noting how sensitive He was to questions of application. Despite all the assertiveness in defending the most promising research areas, M. S. Nikulin was understanding of the opinion of colleagues and was able to assess the importance of results in various sections of mathematical statistics. Well-known colleagues who have often contacted him, also note his kindness, and I would also add selflessness and readiness to always help.

Igor Kovalenko

Doctor of Sci., Professor

Graduated from T. Shevchenko Kyiv State University (1957). Diploma in Mathematics. Postgraduate studies at the Institute of Mathematics, Ukrainian Academy of Sciences (1957-1960) under supervision of B.V. Gnedenko, with consults of A.N. Kolmogorov. The Ukrainian and world science is mourning the loss of a brilliant scientist Professor Igor Mykolayovych Kovalenko, who died on October 19, 2019, after a difficult fight with a heart disease. Prof. Igor Kovalenko was a prominent Ukrainian mathematician in the field of probability theory and its practical applications, a disciple and associate of Boris Gnedenko and Vladimir Korolyuk. He became famous worldwide for his book "Introduction to Queuing Theory", written together with Gnedenko. He has grounded a scientific school in the theory of reliability, queueing theory and cryptography, well known in Ukraine and all over the world.

Igor Kovalenko was born on March 16, 1935 in Kyiv, Ukraine. After graduating from the Faculty of Mechanics and Mathematics of Kyiv Taras Shevchenko University, he worked at the Computing Centre of the Academy of Sciences of Ukraine. From 1962 till 1971, Kovalenko worked in Moscow, where he headed a laboratory at Moscow Institute of Electronic Engineering, and together with other Gnedenko's disciples, was the head of the seminar on queueing theory at Moscow State University. Many leading scientists of the former Soviet Union and foreign countries attended this seminar. Based on the model of piecewise linear Markov processes developed by him, Kovalenko built a mathematical model of a complex defence system reliability and developed numerical algorithms for its implementation grounded on the method of a small parameter.

In 1964, Igor Kovalenko became a Doctor of technical sciences. He formulated the principle of monotonous failures, which, while maintaining high accuracy, significantly simplified the calculations of system reliability. In 1970, Kovalenko was awarded the degree of Doctor of Physics and Mathematics for another thesis on the probabilistic theory of systems of random Boolean equations. Being a doctor twice is a very rare practice in the scientific world.

After returning to Kiev in 1971, Prof. Kovalenko founded and headed the Department of Mathematical Methods of the Theory of Complex Systems Reliability at V.M. Glushkov Institute of Cybernetics. Two areas of research formed the mainstream of investigations: approximate combined analytical and statistical methods of reliability analysis, and theoretical and applied cryptography, systems and methods for data protection. Under his guideline, it was developed the first national standard in the field of cryptographic information security in Ukraine.

Prof. Kovalenko is the author of 25 monographs and more than 200 articles. He was elected as the Academician of the National Academy of Sciences of Ukraine in 1978 (Corresponding Member since 1972). He was an extremely hard-working, honest and sincere person, a competent manager and, thanks to his human qualities, professional experience and knowledge, highly respected among his colleagues. Prof. Igor Kovalenko left many disciples, among them there are many professors and associate professors. All of them preserve in their memory the unforgettable days of joining the science and independent creativity under the guidance of a Great Scientist and Teacher, hours of direct communication with a person of great erudition and high culture.

Dear Teacher

Agasi Melikov
•
Corresponding Member of the National Academy of Sciences of
Azerbaijan, Doctor of Technical Sciences, Professor

I was lucky enough to be a student of Vladimir Semyonovich Korolyuk. I joined him in graduate school in the early 80-ies of the last century from a branch Institute with a ready-made task dedicated to the application of Queuing theory methods in oil products supply systems. At that time, I knew Vladimir Semyonovich in absentia as a world-class specialist in the field of probability theory and mathematical statistics. Therefore, I was afraid that it is unlikely that he will take the lead on this task of an applied nature.

However, he willingly agreed to the guidance and gave me very valuable advice during the work on the dissertation, carefully read the articles and read them. I later learned that Vladimir Semyonovich was one of the first programmers in the former Soviet Union and even was one of the authors of the first book on programming in Russian (co-authors were B. V. Gnedenko and E. L. Yushchenko). For this reason, he was very well versed in applied problems.

Forever A Disciple Of Vladimir Semyonovich Korolyuk

At that time, together with his probabilistic students, he intensively developed the basics of the theory of phase aggregation of States of stochastic systems and taught special courses at the Kiev State University. In the special course, he read very recent results in this area and even used his own manuscripts, which he noted in a common notebook of 96 pages. Me, too, sometimes took with him to these classes (he was a great driver, and he then had a smart car for those times GAZ 24-10). At the very beginning, 7 students attended special courses, then they became 5, then 3, and finally there were only 2 students (apparently the tasks were difficult for students to master). Then

Vladimir Semyonovich said literally the following: "By your behavior, you are discrediting the world-famous Ukrainian probability school. In 35 years of working at the University, I have never taken an exam, but I will take it from you" (I really don't know if he really took the exam or not). With this phrase, he once again (I have noticed this in private conversations before) emphasized his high assessment of the representatives of this school.

Along with theoretical problems, he highly valued the work of an applied nature and often in conversations agitated me to practically use the methods of the theory of phase aggregation of States of stochastic systems in the problems of calculation and optimization of Queuing systems. Fortunately, I and my colleague (also a student of Vladimir Semyonovich - Professor Leonid Ponomarenko) managed to realize this dream of Korolyuk. When we presented the first results in this direction, he was very happy, read them carefully, rechecked the results and submitted these works to the authoritative journal Cybernetics of the national Academy of Sciences of Ukraine (it should be noted that after us, other scientists from different countries have successfully applied this fruitful idea of phase aggregation of States of stochastic systems).

While working on his PhD and doctoral theses, Vladimir Semyonovich always read and discussed with me the results of the main works and contributed to their publication in reputable journals (fortunately, he was a member of the editorial Board of many reputable journals and in this regard, I was very lucky). Naturally, I always wanted to have a joint article with him. However, his high scientific rating and level scared me, and I could not summon the courage to ask him to be my co-author. Only a few years ago I gathered all my strength and asked him to be my co-author in articles in which he had a high share of participation. He agreed, and it was hard to describe my joy. As a result, today I have three articles co-authored with Korolyuk and I consider them my most valuable works.

He was an excellent scholar and an excellent conversationalist. After each personal meeting with him, I felt elated and happy. I wish I could see my great Teacher again. His scientific works will be a beacon for many years to come for applied scientists who use and will continue to use the methods of probability theory and mathematical statistics that he proposed.

Computer System: A Reliable Machine

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•
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Abstract

Relaibility is an essential characteristic of computer machine. Computer system have reliability that why it is used in many fields. Artificial Intelligent and Machine Learning is wide branch in which number of applications has been developed and used in real world scenario. This paper highlights the computer system reliability in various fields where different applications have employed with better accuracy and reliable results. This paper also impact of computer system in branches that is computer science, physics, chemistry, and engineering etc.

Keywords: Reliability, Information technology, artificial intelligent, machine learning, e-commerce, Internet.

I. Introduction

Information is the base of communication between two or more devices, persons and nodes. Information is made by more than one raw data which made specific meaning. By this information a person can understand and make decision in a particular domain.

Information technology (IT) generally built by two words that are information and technology. Technology refers to any device which has the capability of storing, transferring, and retrieving the information such as computer. Such type of machine is cable to handle the huge data and information. This information may be used for making decision in any organization, institutions or any company. Jim Domsic of Michigan was used information technology first time in November 1981 [1]. IT is world level acceptable tool which have many features such as reliability, high productivity at low cost and minimize time and this characteristics impact on productivity of any firms [2]. Due to this, there is no needed wait for transferring information from one place to another and also store all transaction permanently. In the modern era, IT has been used in about all organizations, institutions, colleges, firms, etc. which shows the importance of it. Government body also employed IT for spreading project information among the citizens in the country [3]. Reliable information gets by all citizens in fast speed and at low cost dint of IT.

II. Applications of IT

Computer based technology has been very popular in present time. Computer system is using almost every domain due to reliability, accuracy in result as well as time minimizing in the processing. Here we discuss some key domain of IT in which number of application has been developed and employed in real world for solving simple as well as complex task.

Online shopping, e-marking and e-commerce: Today's market has changed from past designed due to

e-marketing. Online shopping is doing at brought level by customers and also pay through online payment system. This ratio is growing day by day because Internet is available everywhere. Online shopping and e-marketing is able with Internet only.
E-marketing gives many benefits such as maintain stock position, Sales records, bill generation, worker's salary, as well as purchase records etc. This is very beneficial for small and medium business. All activities are done through compute with Internet. Such system provides online facilities with protection and security also. Some key features of e-marking are as below [4].

- Globalize area of e-market.
- Product differentiations at high level.
- Customers mobility at higher level.
- Lesser time in processing.

Customer satisfaction is primary goal of any business and reliability of IT shows importance towards increasable online shopping. This is due to interactional and distributive fairness and easiness of procedure of it [5]. Customer relationship management (CRM) is developed with the help of IT which focuses on customer relationship development [6]. CRM generally beneficial to company in term of better customer services, discovery new customers, etc.

Electronic commerce (e-commerce means buying and selling through Internet) is possible only through Internet which was started in 1969 (ARPANET). The successful internet connection throughout world is given the facility of e-commerce in which navigation tool (like Internet explorer), search engine (like Google, Yahoo) and internet services are available for providing support e-commerce (like e-mail, internet relay chat, etc.) [6].

Banking sector: Banking sector is provide the services online like payment through online (Debit, smart, credit, paytm, etc.).Generally Automated teller machines (ATM) and national electronic funds transfer (NEFT) are employed for transferring funds from one bank to another. Such type of facility gives benefits to customers to reduction in cost, increase efficiency, quality services with reliable transaction etc [7]. Apart from this facility offline facility also provide to customers which benefited to the non-literate persons. All the records and transactions are maintained systematically and stored permanently for future uses.

Railway sector: Railway sector is employed IT in which number of computers use for maintaining each and every train schedule, ticket booking and cancel etc. Computers and internet is key facilities in railway organization by which maintain whole system effectively. It is fact that computer is a reliable system which provide the better facility to railway organization. Internet of things (IOT) technology gives benefits of thorough perception, deep intelligent application which is essential support provide to the department [8].

Education: Education is a part of each and every person by which the person learn the knowledge and skills. It involves in training, teaching, discussion and research. IT is employed about all institution of the education and play vital role for teaching and learning as well as research in many disciples. As per available way of teaching and learning, education can be provided in two ways that is offline and online. Impact of online education has increased in learning process in these days but students are not taken into account it as professional model for learning the education [9]. Another part of education is distance learning in which IT shows the effectiveness and importance in it [10].

IT tools for developing curriculum are employed like as electronic performance support, repositories and knowledge management system [11]. In the present time, different kind of ICT tools are employed in education like as computer and laptop, projectors, digital camera, printer, tablets, flipped classrooms etc.

Government of India has also run the SWAYAM portal specially for providing the better education to the students (class 9 to post graduate student) which in 4 quadrants i.e. video lecture, reading materials, self-assessment tests, and online discussion [12]. NPTEL online courses also available on their portal (23 disciplines) in which is no any fee required for joining, no age limit, and learn any time for certification courses [13].

Transportation: Traffic control system should be smooth and efficient regulate traffic flow where no occur no congestion or reduced the congestion problem. Vehicle identification through number plate of the vehicle is already running in developed in different computer programming languages. As per native language of local area, owner of the vehicle designs number plates using (Hindi and English) different languages is also identified through automatic number plate recognition [12]. Such type of applications are developed through template matching (TM) [14], support vector machine (SVM) [15], and neural network methodologies etc.

IV. Discussion

IT have broad spread area in which different kind of applications have been developed and already used in whole world. As mentioned earlier section, different applications have discussed for better understand the reliability system that is computer. There is no doubt about computer system which has different features like speed processing, high storing capacity, high reliable and accurate result, versatility, diligence. Due to these feature computer is employed almost all areas in real world for solving problem as well as betterment of results.

V. Conclusions

IT has great innovation of the science which is impact almost all fields such education, transportation, railways, banking sectors, and electronic commerce etc. This paper present a brief reviews of such applications in which computer system is employed for different purposes and shows the accurate and reliable results. IT tools is used in other fields also for various purposes. This system has capability to work continuously very fast with same speed and accuracy and also capable for performing various kind works through a single system.

References

[1] Philee, H. (2010). Information Technology and Internal Controls. *Available at SSRN 1338814*.

[2] Oliveira, T., & Martins, M. F. (2011). Literature review of information technology adoption models at firm level. *Electronic Journal of Information Systems Evaluation*, *14*(1), 110.

[3] Kaushik, P. D., & Singh, N. (2004). Information technology and broad-based development: preliminary lessons from North India. *World Development*, *32*(4), 591-607.

[4] Molenaar, C. (2013). *E-Marketing: Applications of information technology and the internet within marketing*. Routledge.

[5] Chiu, C. M., Lin, H. Y., Sun, S. Y., & Hsu, M. H. (2009). Understanding customers' loyalty intentions towards online shopping: an integration of technology acceptance model and fairness theory. *Behaviour & Information Technology*, *28*(4), 347-360.

[6] Chen, I. J., & Popovich, K. (2003). Understanding customer relationship management (CRM). *Business process management journal*.

[7] Kamel, S. (2005). The use of information technology to transform the banking sector in developing nations.

[8] Liu, X., Wang, X. F., & Feng, X. (2013). Study and Application of the IOT Technology in Railway. In *Advanced Materials Research* (Vol. 694, pp. 3329-3332). Trans Tech Publications Ltd.

[9] Raju, V. (2016). Role of information technology in education sector: Scrutinizing its merits

and developments. *Matter: International Journal of Science and Technology*, *2*(1).

[10] Kumar, P. (2016). Advantages of Information and Communication Technologies in Distance Education Sector. *INTERNATIONAL JOURNAL OF ADVANCE RESEARCH AND INNOVATIVE IDEAS IN EDUCATION*, *2*(1), 1-6.

[11] McKenney, S., Nieveen, N., & Strijker, A. (2008). Information technology tools for curriculum development. In *International handbook of information technology in primary and secondary education* (pp. 195-210). Springer, Boston, MA.

[12] https://swayam.gov.in/about

[13] https://nptel.ac.in/

[14] Vaishnav, A., & Mandot, M. (2018, August). An Integrated Automatic Number Plate Recognition for Recognizing Multi Language Fonts. In *2018 7th International Conference on Reliability, Infocom Technologies and Optimization (Trends and Future Directions) (ICRITO)* (pp. 551-556). IEEE.

[15] Vaishnav, A., Ahuja, B. L. & Mandot, M., (2019). Feasibility of Efficient Number Plate Recognition using Morphological Dilation and Support Vector Machine. *International Journal of Recent Technology and Engineering*, 8(3), 345-35.

Mathematical Models of Systems with Several Lifts and Various Control Rules

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Abstract

In the present paper, several mathematical models of lifts' systems with various control rules are constructed and simulated data are used for estimating the various parameters of the systems. For systems with a rare input flow of customers, the graphs, describing the positions of the lifts, at the preceding customer's arrival instant, are given. These graphs help analytically find, the various characteristics of the lifts' systems (a customer's average waiting time, a customer's total time, energy expenses and others). There are introduced the various control rules by the lift systems, which are investigated by simulation. Using Wolfram Mathematica, the authors have prepared several programs for simulating and estimating the various operational parameters of the lifts' systems with different control rules. Using these programs, the numerical estimates of the various parameters the considered lifts' systems, have been found. These data can be used for defining the dependence of optimal lift roominess on an intensity of input customers' flow and finding the optimal number of the lifts, during planning and construction of the buildings and skyscrapers.

Keywords: simulation, rare flow, lifts systems, customer waiting, service, total time, roominess.

1. Introduction

The development of the modern cities led to an appearance of the huge metropolises like New York, Moscow, Shanghai, Istanbul and others, with a lot of skyscrapers. It is difficult to imagine today the world metropolises and the modern cities without skyscrapers, where many lifts' systems with various control rules, are used. In the process of designing a Skyscraper, one of the important problems is to find the optimal parameters of the lift systems, e.g. to find the optimal conditions (roominess, capacity, size) of a lift cabin and reduce the customer's average waiting, the service time and to save energy expenses. The effective approach for such type of investigation is a construction of mathematical models, describing lift systems. Although there are a lot of similarities between the transportation and traffic problems with lift systems, nonetheless, it is necessary to construct new mathematical models and develop the effective approaches for investigation these systems, taking into consideration their specifics, because they have different and complicated structures. The new approaches and methods can allow estimate the main operation parameters (customers' waiting and service time, energy expenses and others) and making necessary recommendations for constructors and engineers. There are a lot of publications in this field e.g. [1-5], but complicated lift systems with various control rules, are not yet investigated widely. The mathematical models, describing a behavior of lift systems, can be applied for other systems with moving servers, for instance, Shuttle and Communication Systems,

Traffic and others. In [6] the mathematical models of the systems with one lift, different control rules were introduced. One of the effective methods is a simulation by using programs, described in Wolfram Mathematica. Then, simulated statistical data can be used to find estimates of different parameters for such lift systems.
Using the given statistical data, the estimates can demonstrate, for instance, the dependence of the lifts' roominess on an intensity of the input flow. More complicated problems appear in the investigation of the systems with several lifts. The processes, describing the behavior of the lift systems have stochastic and complicated structures. It leads to constructing and investigating the new stochastic models, approaches and programs for their simulation.

In this paper, the authors consider various lift systems with different parameters and different control rules. This paper can be regarded as a continuation of the investigations presented in [6] and hence, we will follow the notations introduced in that paper.

In contrast to [6], in this paper there are considered various lift systems with several lifts, together with the average waiting and service time and other characteristics. An important parameter such as energy expenses can also be investigated. For such issues, the analytical approaches are faced with some problems. Hence, for complicated systems, the methods of collecting the simulation data can also be used. This gives the effective results and allows draw useful conclusions for practice. Some approaches are suggested in [7-9]. The behavior of lift systems can also be described by mathematical models of moving particles. Some models are suggested in [10-12] and these methods can be applied for the lift systems.

2. Various control policies for the lift systems

As it was mentioned above, the construction of skyscrapers in the modern cities requires the creation of different control rules in lifts' systems, which reduce the customer's waiting time and energy expenses. There already exist such control policies, called Odd-Even (some lifts serve customers at the odd floors and other lifts at the even floors), or some lifts serve customers, at the floors 1, 2, …, N others at 1, N+1, N+2, …, 2N. To save time, we need new more effective control policies for the lift systems, which can minimize the expenses for their construction and optimize some parameters (waiting and service time, minimization of energy resources, increase the life-time of the lifts, working without repairs and others). There exist many control policies in the world, for instance, the FIFO service (first come, first outcome), LIFO (last come, first outcome) and others. If the lift comes to a customer at the first floor, who first called it, then this lift can serve this customer plus only other customers going to upper floors than this customer (e.g. so as in Hilton hotel in Baku). An interesting unofficial control policy was created in the seventy years of the XX-th century, by the students in the dormitory of the Lomonosov Moscow State University. There are 18 floors in the student dormitory and two lifts' halls with four lifts in each.

The first lift hall operates from the 1st to 12th, 14th, 16th and 18th floors. For the lifts work more rapidly, it was skipped the odd numbered floors, after 12th. There is also a second lift hall for serving the 1st-10th floors. If in the first hall, a lift came to the first floor and the first student yelled the word "HIGHER", then, the lift would be filled by students who are going up only to the higher floors (16th and 18th) and the next lift will be filled by students who are going to the 12th, 14th, 16th and upper. If the first call had been "LOWER", then the lift would have operated between the lower floors (12th, 14th and afterward, to the other upper floors). The students called it a Higher-Lower system. In [8, 9], a comparative analysis of simple different control policies has been done.

The comparison of the control system "Higher-Lower" with "Even-Odd" showed an advantage of "Higher-Lower" control policy (the customer's average waiting time before service is shorter). For each control policy, it is possible to construct mathematical model, which show a preference of these systems.

3. The mathematical models of the lift systems

Below, if unless otherwise agreed, it will be considered the stationary regimes for the lift systems operating. Following [6], we will construct the mathematical models of the lift systems and remind the following notations:

$L_kF_nC_{xx}$ – is the systems with k lifts, n floors and control policy xx;
i - is an ordered in time identifying number of a customer during one day simulation;
f_a (i) - is the floor of appearance of the i-th customer;
f_d (i) - is the floor of destination of the i-th customer.
It is necessary to note that for different i the f_a (i) and f_d (i) can take the same value.
t_a (i) - is the instant of appearance of the i-th customer;
t_b (i) - is the instant of the beginning service of the i-th customer;
t_e(i) - is the instant of end service of the i-th customer;
$t_{c(j)}$ - is the instant when lift on j-th cycle is returning to the 1-st floor;
r - roominess, restriction on maximum possible number of customers who can be in the lift cabin;
h_f - time necessary for lift to move up or down between two neighbor floors;
h_d – time spending for opening and closing the floor's door;

Usually, in practice, h_d=2h_f. If we consider stationary input flow, then, the following parameters are used:

l_{f1f2} –intensity of customers' flow, who appear at the f_1-th floor and want to go to f_2-th
$l_1 = \sum_{k=2}^{n} l_{1k}$ - intensity of customers flow, who appeared at the first floor and are going to upper floors;
$l_2= \sum_{k=2}^{n} l_{k1}$ –intensity of customers' flow, appeared at upper {2, 3, …, n_f} floors, who want to go down to the first floor.

We would like to remind that in [6], it was also introduced the following specific characteristics for the lift systems:
CWT(f_a, f_d) – Customer average Waiting Time (f_a, f_d) – is defined as mean waiting time for customers going from the f_a floor to the f_d floor. It is measured as the time interval {t_a(i), t_b(i)) from the instant of the customers' appearance at the f_a floor, until the instant when the customer comes into the lift cabin, going (in the direction of the f_d floor) to the desired f_d floor;
CWT(f_a) – Customer average Waiting Time (f_a) is defined as the mean time for customers going from f_a floor, to any other floor (upper or lower floors). It is measured from the instant of the customers' appearance at the f_a floor, until the instant when customer comes into the lift cabin, going to desired destination floor;
CST(f_a, f_d) – Customer average Service Time (f_a, f_d), is the mean time defined for the customers coming into the cabin on the f_a floor and going to the f_d floor. It is measured from the instant when a customer comes into the cabin of the ordered lift, moving from the f_a floor (in the direction of the f_d floor), until the moment when the customer leaves the lift at the f_d floor.

Generally, if any characteristic has index i, then, it is a random variable, for instance CST (f_a(i), f_d(i)) is a random variable.

CST (f_a) – Customer average Service Time, which is going up or down, from f_a floor to the desired floor;
CST(S) – Customer average Service Time in the system S, i.e. mean time from the instant when a customer gets the lift, until the instant when the customer gets off the lift;
CTT(S) – Customer average Total Time in the system S, which is measured as a mean time from the instant when a customer arrives into the system until the customer, gets off the lift (arrival to ordered floor).

For instance, CTT($L_kF_nC_{xx}$) is a customer average total time, for a system in a building with k lifts, n floors and control policy xx.

Finally, in the loading regime, we will be interested in customers' average total time, who are going from the first floor to the upper floors. In the unloading regime, it will be the customer's average total time, who are coming from the upper floors to the first floor. This time includes possible stops on intermediate preceded floors, ordered by other customers, being in the same cabin.
LRC – Average Lift Return Cycle time, i.e. the mean time interval between two neighbor comings of the lift, to the first floor.
In this paper, in distinction of [6] we also introduce the new parameters for the lifts systems, which describe the lift energy expenses and the single race time:
LEE_j (S) – Average value of the j-th Lift Energy Expenses in the system S, measured in Kw (kilowatt);

Note that Energy Expenses in Kw depend not only on the volume cabin weight, but also on its speed, acceleration and deceleration. Empirically, electric Energy Expenses can be shown each day, on the electric counter of each lift.
SRT(t) – Average Single Rate Time, i.e. mean time when the lift is moving without customers, during time t;
SEE(S) – average value of System Energy Expenses, i.e. mean value of energy expenses of all the lifts in the system (S)
$SEE(S)=LEE_1(S)+LEE_2(S)+\ldots+LEE_n(S)$;
k_d – coefficient, defining the energy expenses of the lift, during a unit time for opening and closing the doors;
k_f – coefficient, defining the energy expenses of the lift during a unit time necessary for covering the distance between two neighbor floors.

Remind the lift system giving service for 1, 2,…, n_f floors with k lifts and control policy xx, will be denoted as $L_kF_nC_{xx}$.

Control policy xx=IL means a system without control, i.e. all the lifts are operating independently to each other. Sometimes, such a system is denoted as $L_kF_nC_{IL}$ (Independent Lifts).
For an IL control system, if at the preceding instant of a new customer's arrival, several lifts are free (empty), then, all of them will go to this customer's call. Such systems are often used in the buildings with two lifts.
$L_kF_nC_{DL}$ is the system with k Dependent Lifts, n floors (for a customer's call, the Draws the nearest Lift is going).
$L_2F_nC_{n1,n2}$ is the system with 2 lifts, n floors and after completing the customer's service, one lift must go to the n_1 floor if there is no lift, otherwise, it should go to the n_2 floor.
Below, we consider systems with a rare input flow.

4. Unloading regime in the systems with rare flows of customers. Definition

We will say that a flow of customers is rare, if at the instant of a customers' arrival, all the lifts are available. Such situations are observing:
a) in the office buildings, between 10.00-12.00 (when offices have very few visitors);
b) in the buildings, between 10.00-12.00 and 15.00-17.00 (when old people with infant children can go for a walk);
c) in the shopping malls, if, at the instant of customers' arrival into the system,
all lifts are occupied, then, the customers use escalators.

Remark 4.1 According to the above-mentioned paragraph c), such flows or point processes can be called as the flows or point processes, transforming themselves into the point processes with rare intensity (transformed flows or point processes).

We consider IL policy ($L_2F_nC_{IL}$), which means that if both lifts are free (empty), then at the next customers' call, both lifts (perhaps from different floors) are going simultaneously. Such situations can be observed in buildings where each lift has an individual call button and when those buttons are pushed simultaneously.

Then, for that system, at the preceding of a customer's instant arrival, one lift occupies the first floor, the other k-th(k=2,3,..,n) floor,(see, Fig.4.1).Below, in all Figures, the blue color means lift is free and the red color means that lift is occupied.

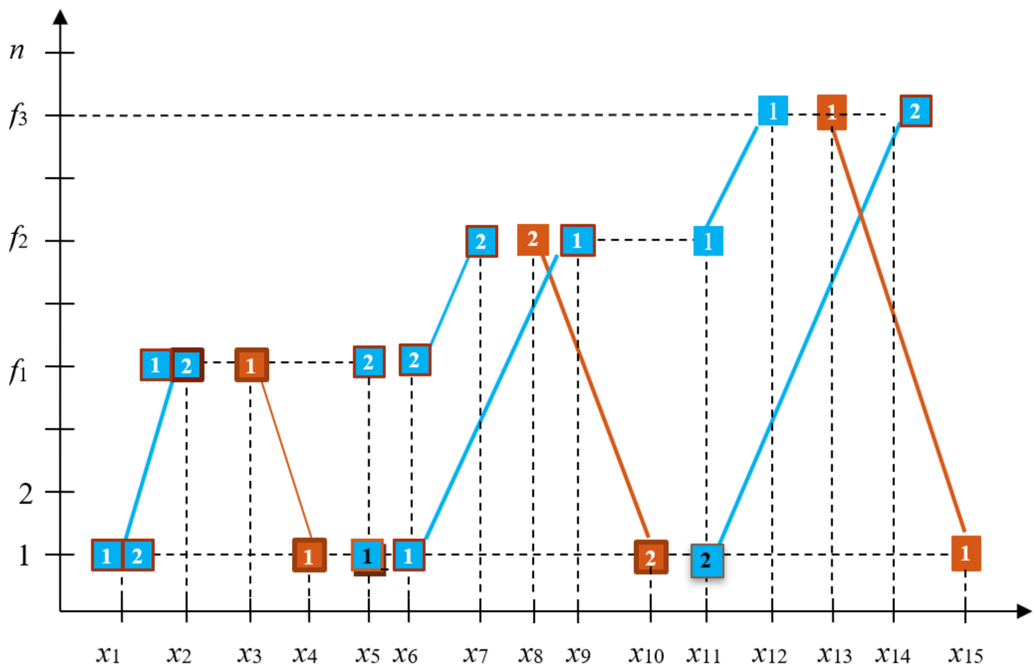

Fig.4.1

$x_1=t_a(1)$, $x_2=x_1+(f_1-1)h_f$, $x_3=x_2+h_d=t_b(1)$, $x_4=x_3+(f_1-1)h_f$, $x_5=x_4+h_d=t_e(1)$, $x_6=t_a(2)$, $x_7=x_6+(f_2-f_1)h_f$, $x_8=x_7+h_d=t_b(2)$, $x_9=x_7+(f_2-1)h_f$, $x_{10}=x_8+(f_2-1)h_f$, $x_{11}=x_{10}+h_f=t_e(2)$, $x_{12}=x_{11}+(f_2-f_1)h_f$, $x_{13}=x_{12}+h_d=t_b(3)$; $x_{14}=x_{11}+(f_3-1)h_d$, $x_{15}=x_{13}+(f_3-1)h_d$.

Thus, we get: $CWT(L_2F_nC_{IL})=h_f(n-1)/2+h_d$, $CST(L_2F_nC_{IL}) = h_f(n-1)/2+h_d$ and $CTT(L_2F_nC_{IL}) = h_f(n-1)+2h_d$; $LEE(L_2F_nC_{IL})=k_f(n-1)h_f+2k_d h_d$.

For SRT (average energy spent for one call) during the one-unit time we have $SRT=k_f h_f [(n-1)/2+(n-1)/4]+mk_d h_d = k_f h_f [3(n-1)/4]+m k_d h_d$. For Poisson input flow during the time t, an average number of arrived customers into the system, equals ◎t. Hence, $SRT(t)=$◎$(k_f h_f 3(n-1)/4+mk_d h_d)t$. As ◎T is an average number of arrivals during the time T, then ◎$Tk_f h_f (n-1)$ is an average energy which lift spends for serving the customers (motion of lift), during time T. As at each arrival instant, an average number of customers equal m, then ◎Tm is average number of customers arrived during the time T into the system. For each customer's arrival, the lift spends the time h_d for opening and closing the door. If we assume that each customer spends time h_c coming in and getting off a lift, then, the mh_c is the time, which was spent for the m customers (coming in and getting off). Hence, a customer average energy spent for opening and closing the door, for customers coming in the lift and getting off equal ◎◎T $k_d h_d$+ 2◎Tm h_c=◎◎T($k_d h_d$+m h_c). Thus, we have $LEE(L_1F_nC_{IL})$=◎$Tk_f h_f (n-1)$+2◎T($k_d h_d+mh_c$). Below, for simplicity, we assume $h_c=0$, which means that during the time h_d all customers who want to come in and get off a lift, can do it.

4.1. Unloading regime for system $L_2F_nC_{DL}$

We would like to remind that DL means that for a new customers' arrival, only the draws nearest lift which will operate. Then, at the preceding of the moment of the customer 's arrival, both lifts occupy the first floor, which means that in fact, only one lift is operating (see, Fig.4.2).

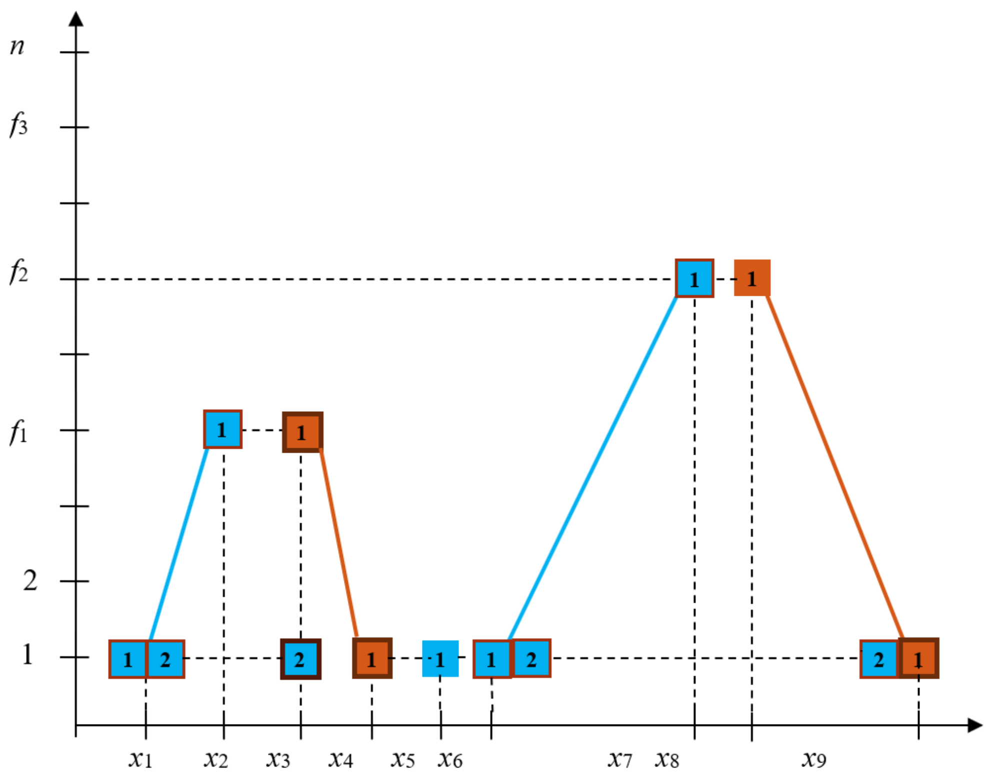

Fig.4.2

$x_1=t_a(1)$, $x_2=x_1+(f_1-1)h_f$, $x_3=x_2+h_d=t_b(1)$, $x_4=x_3+(f_1-1)h_f$, $x_5=x_4+h_d= t_e(1)$, $x_6=t_a(2)$, $x_7= x_6+(f_2-f_1)h_f$, $x_8=x_7+h_d=t_b(2)$, $x_9=x_7+(f_2-1)h_f$.
In fact, in an unloading regime, the system $L_2F_nC_{DL}$ is operating like a system with one lift, i.e.

$L_1F_nC_{IL}$. In Fig.4.2, the lifts' positions at different instants, for the system $L_2F_nC_{DL}$, are shown. Hence, we can calculate
$CTT(L_2F_nC_{DL})=(n-1)h_1/2+h_2$ and $SRT(L_2F_nC_{DL})=(n-1)h_1/2h_1+h_2$

It is easy to calculate all the characteristics and we leave it to the reader. Hence, for this case of a rare input flow (unloading regime), the system $L_2F_nC_{IL}$ is preferable to the system $L_2F_nC_{IL}$ (CTT is less).

5. Loading regime for system $L_2F_nC_{IL}$.

We assume $\lambda_{12}=\lambda_{13}=\ldots=\lambda_{1n}$ and $\lambda_1=\Sigma_{k=2}^{n}\lambda_1$ represent the intensity of the transformed flow of customers (transformed point processes) and $\lambda_{21}=\lambda_{31}=\ldots=\lambda_{n1}=0$. As it was mentioned above, such a situation can be observed in the residence building, between 10.00-12.00 hours. It is clear, at the preceding of a customer's arrival instants, one lift occupies the first flow, another j-th (j=2, 3, …).

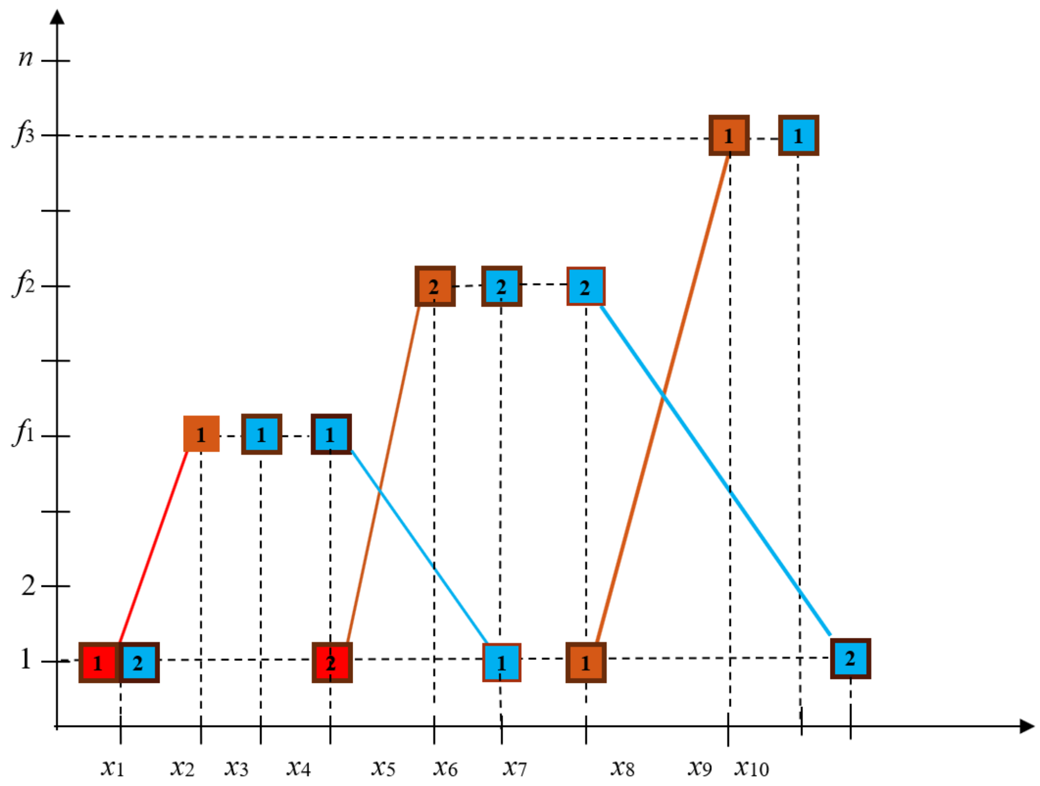

Fig.5.1

$x_1=t_a(1)=t_b(1)$, $x_2=x_1+(f_1-1)h_f$, $x_3=x_2+h_d=t_e(1)$, $x_4= t_a(2)= t_b(2)$,
$x_5= x_4+(f_2-1)h_f$, $x_6=x_5+h_d= t_e(2)$, $x_7=t_a(3)=t_b(3)$, $x_8= x_7+(f_3-1)h_f$,
$x_9=x_8+h_d=t_e(3)$, $x_{10}=x_7+(f_2-1)h_f$.

5.1 Loading regimes for system $L_2F_nC_{DL}$

At a customer's call, only one lift goes, i.e. the nearest lift. Then, starting from the third customer, at the preceding of a customer's arrival instant, based on the same probability, one lift occupies f_1-th (f_1 = 2,3,…n;) floor, another f_2 -th (f_2=2,3,..n;) floor (see Fig. 5.2).

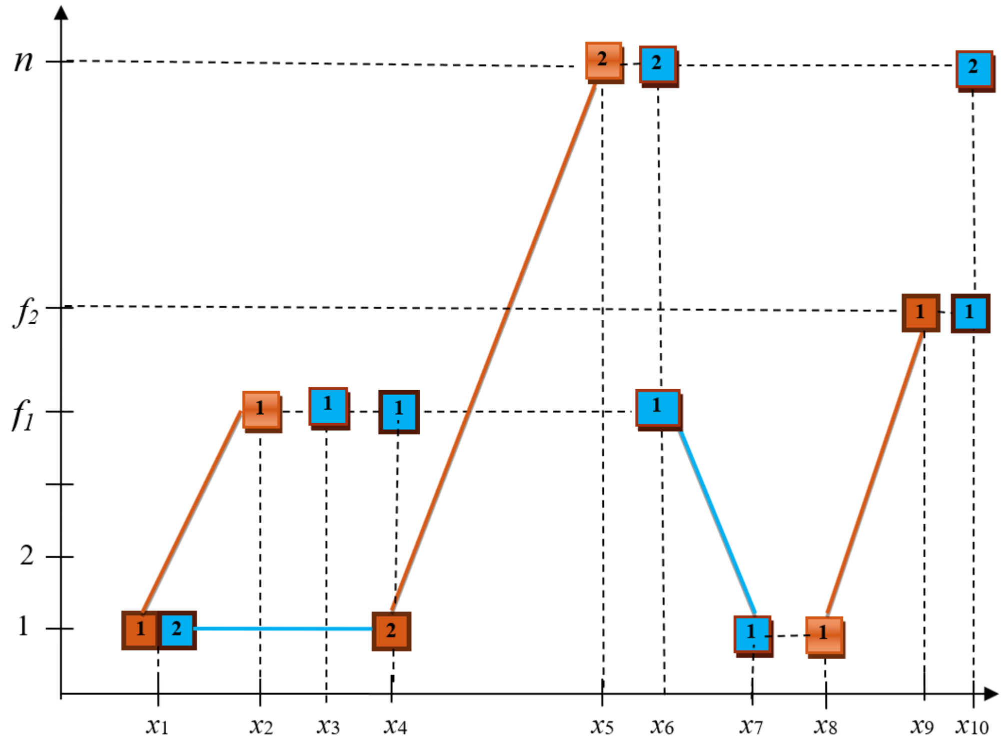

Fig.5.2

$x_1=t_a(1)=t_b(1), x_2=x_1+(f_1-1)h_f, x_3=x_2+h_d=t_e(1), x_4=t_a(2)=t_b(2), x_5=x_4+(f_2-1)h_f$, $x_6=x_5+h_d=t_e(2)$, $x_7=t_a(3)$, $x_8=x_7+(f_3-1)h_f$, $x_9=x_8+h_d=t_b(3)$, $x_{10}=x_9+(f_1-1)h_f$.

Remark 5.1. In the loading regime the behavior of the system $L_2F_n\,C_{DL}$ coincides with the behavior of the system $L_1F_nC_{IL}$, because as soon as one lift reaches the last (n-th) floor, afterward in fact, only one lift (which occupies the first floor) is operating, because it will always be a nearest lift for the arrived customer.

6. Calculating various characteristics of the lifts' system $L_2F_nC_{IL}$ (two lifts, n floors, independent lifts). Loading regime

As we consider $L_2F_nC_{IL}$ system, then, at customers' call, both lifts go independently from each other. Taking into consideration that in the loading regime, at the preceding customer's arrival instant, one lift occupies the first floor, another one j-th (j=2, 3,..., n) floor (see Fig. 5.2). Then the calculations yield to:

$CTT(L_2F_nC_{IL})=(n(n-1)(2n-1)/8+(n-1)^2(n-2)/2)h_f/(n-1)^2+h_d=(3n/4-1/8-1/8(n-1))h_f+h_d$;

$SRT(L_2F_nC_{IL})=(n(n-1)(2n-1)/24+(n-1)(n-2)(2n-3)/6)h_f/(n-1)^2+h_d=$
$= [5n/12-19/24+5/24(n-1)]h_f+h_d$.

Denote

$a=1/8-1/8\,(n-1) = (1/8)\,(1-1/(n-1))=(1/8)[(n-2)/(n-1)]<0.125$;

$b=-19/24+5/24(n-1) =(1/24)[5/(n-1)-19]/<0.8$;

As CTT and SRT are linear functions of n, with coefficients 3n/4 and 5n/12 and /a/<0.125 and /b/<0.8, then, for n>10 we can neglect a and b and hence, put:

$CTT(L_2F_nC_{IL})=(3n/4)h_f+h_d$; $SRT(L_2F_nC_{IL})=(5n/12)h_f+h_d$. (6.1)

If we put $h_f=1$, $h_d=0$, then $CTT(L_2F_nC_{IL}) = 3n/4$, $SRT(L_2F_nC_{IL}) = 5n/12$.

6.1 System $L_2F_nC_{IL}$ (two lifts, n floors, independent lifts). Mixed regime

$\lambda_{12} = \lambda_{13} =\ldots= \lambda_{1n}$, $\lambda_1 =\Sigma_{k=2}^{n}\lambda_{1k}$, $\lambda_{21} = \lambda_{31} =\ldots= \lambda_{n1}$, $\lambda_2= \Sigma_{k=2}^{n}\lambda_{k1}$ are the intensity of the transformed flows of customers. In this case, at the preceding customer's arrival instant, one lift occupies the first floor and another f-th floor (f=2, 3,…,n). The probability to have a customer at the first floor is $\lambda_1/(\lambda_1 + \lambda_2)$ and at the other (upper) floors, it equals $\lambda_2/(\lambda_1 + \lambda_2)$. Thus, for a Customer Total Time and Single Race Time after the routine calculations we have

$$CTT(L_2F_nC_{IL})=(\lambda_1/((\lambda_1+\lambda_2))((n-1)/2)+\lambda_1/(\lambda_1+\lambda_2))(1/(n-1)^2)((n(n-1)(2n-1)/8)+$$
$$+(n-1)^2(n-2)/2)h_f+h_d$$
$$SRT(L_2F_nC_{IL})=(\lambda_1/ (\lambda_1+ \lambda_2))(1/(n-1)^2)+ \lambda_1/ (\lambda_1+ \lambda_2)((n(n-1)(2n-1)/48)+$$
$$+(n-1)(n-2)(2n-3)/12)h_f+h_d.$$

If n>10 then we have the following formulas

$$CTT(L_2F_nC_{IL}) = ((n/4)(2\lambda_1+3 \lambda_2) /(\lambda_1+ \lambda_1)) h_f+h_d;$$
$$SRT(L_2F_nC_{IL}) = ((5n/24) / \lambda_2 /(\lambda_1+ \lambda_2)) h_f+h_d. \quad (6.2)$$

Corollary 6.1. If $\lambda_1 = \lambda_2$, then, it follows from formula (6.2):

$$CTT(L_2F_nC_{IL})=(5n/8)nh_f+h_d;\ SRT(L_2F_nC_{IL})=(5n/48)h_f+h_d \quad (6.3)$$

Introduce the control policy, which means that at the end of a customer's service instant, the lift should go to the f1-th floor (if there is no lift), otherwise, the lift must go to the f2-th floor. Our aim is to find f1 and f2, which minimizes the value of CTT(L2FnCxx). Then, at the preceding of a customer's arrival instant, one lift occupies f1-th floor, another f2-th floor. Similarly, to (6.3), we have:

$$CTT(L_2F_nC_{f1f2})=(\lambda_1/(\lambda_1+\lambda_2))(f_1-1+(n-1)/2)+(\lambda_2/(\lambda_1+\lambda_2))((f_1-1)^2/(n-1))+f_1+$$
$$+(f_2-f_1)/2(n-1)((f_2-f_1)/2+f_1-1)+((f_2-f_1)(f_2-1)/2(n-1)+(n-f_2+f_1-1))h_f+h_d;$$
$$SRT(L_2\ F_n\ C_{f1f2})=(\lambda_1/(\lambda_1+\lambda_2))(f_1-1)h_1+(\lambda_2/(\lambda_1+\lambda_2))(f_1-1)^2/2n+((f_2-f_1)^2 4(n-1)+$$
$$+(n - f_2)/2(n-1))h_f+h_d. \quad (6.4)$$

If h_f=1 and h_d=0, then for n>10 we have

$$f_1=\max[1,(n/4)(1-3\lambda_1/\lambda_2)];\ f_2= (f_1+2n)/3 \quad (6.5)$$

Using (6.4) and (6.5) we have

$$CTT(L_2F_nC_{f1f2})=((\lambda_1/(\lambda_1+\lambda_2))(f_1-1+(n-1)/2)+ +(\lambda_2/(\lambda_1+\lambda_2))(3(f_1-1)^2+(n-f_1)(f_1+2n-4))3(n-1)$$
$$SRT((L_2F_nC_{f1f2})=(\lambda_1/(\lambda_1+\lambda_2))(f_1-1)+(\lambda_2/(\lambda_1+\lambda_2))(3(f_1-1)^2+(n-f_1)^2)/6(n-1) \quad (6.6)$$

Remark 6.1. For a rare flow, if $\lambda_1>0$, $\lambda_2=0$, then, it follows from (6.6), that f1=1. In this case, in fact, only one lift operates, because in the end of the service, it comes to the first flow and there are no customers in another floor. Therefore, it does not matter the location of the second lift.

Such systems can be used in the residence buildings, where there are two lifts with different capacities. The lift with a larger capacity is used for the delivery of the furniture and other big goods. For n>10, it follows from (6.6)

$$CTT(L_2FnC_{1,1}) = (n-1)/2,\ SRT(L_2FnC_{1,1}) = 0.$$

Remark 6.2. For a rare flow, if ◎◎◎= 0, ◎◎◎> 0, then, for n>10, it follows from (6.5) and (6.6)
$f_1=n/4$, $f_2=3n/4$,
$CTT(L_2F_nC_{f1,f2})=5n/8$, $SRT(L_2F_nC_{f1,f2}) = n/8$ (6.7)
i.e. at the preceding customer's arrival moment, one lift should occupy (n/4) –th floor, another (3n/4)–th floor.

The comparison (6.5) and (6.6) shows that introduced control leads to an advantage in a customer's average service time 16% and regarding the single race time 10%.

Remark 6.3. If ◎◎◎◎◎ then for n>10 it follows from (6.6) and (6.7)

$f_1=1, f_2=2n/3, CTT(L_2F_nC_{f1,f2})=7n/12, SRT(L_2F_nC_{f1,f2})=(n/12)$, (6.8)

i.e. at the preceding customer's arrival moment, one lift must occupy the first floor and another, 2n/3 floor (see, Fig.6.2). We assume that 2n/3 is an integer number

$x_1=t_a(1)=t_b(1), x_2=x_1+(f_1-1)h_f, x_3=x_2+h_d=t_e(1), x_4=t_a(2)=t_b(2), x_5=x_4+(f_2-1)h_f, x_6=x_5+ +h_d=t_e(2), x_7=t_a(3), x_8=x_7+(f_3-1)h_f, x_9=x_8+h_d=t_b(3), x_{10}=x_9+(f_1-1)h_f, x_{11}=x_{10}+h_d= t_e(3)$.

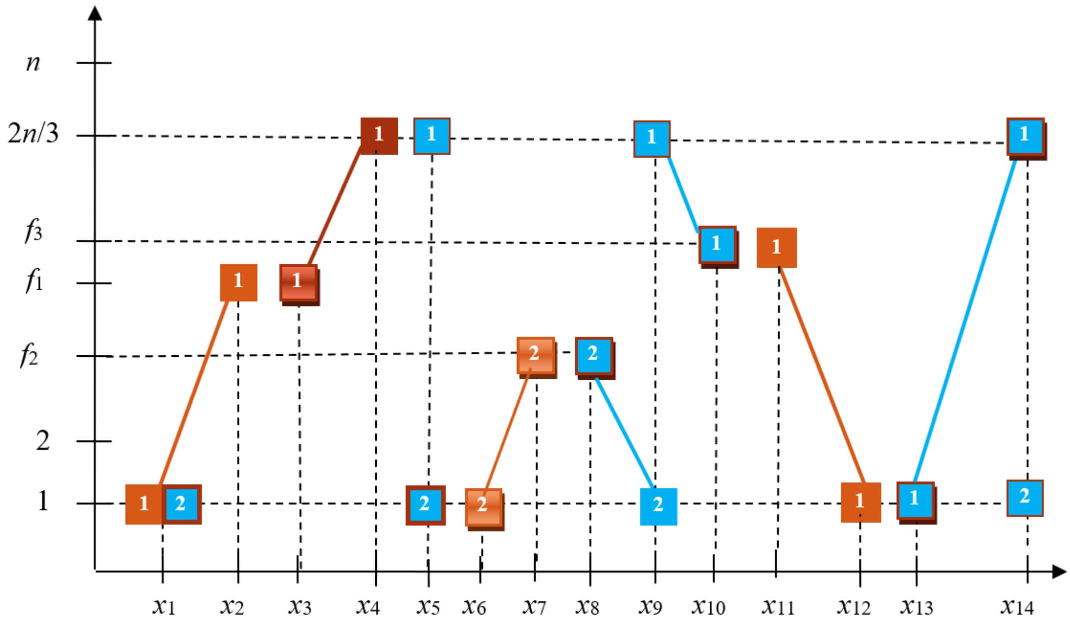

Fig.6.2

$x_1=t_a(1)=t_b(1)$, $x_2=x_1+(f_1-1)h_f$, $x_3=x_2+h_d=t_e(1)$, $x_4=x_3+(2n/3-f_1)h_d$,
$x_5 =t_a(2)$, $x_6= x_5+h_d$, $x_7=x_6+(f_2 -1)h_f$, $x_8=x_7+h_d$, $x_9= x_8+(f_2-1)h_f$,
$x_{10}= x_9+(2n/3-f_3)h_f$, $x_{11}= x_{10}+h_d$, $x_{12}= x_{11}+(f_3-1)h_f$, $x_{13}= x_{12}+h_d$, $x_{14}= x_{13}+(2n/3-1)h_f$.

In Fig.6.2, there are shown the lifts' positions at different instants, for the system $L_2F_nC_{1,2n/3}$. At the preceding of a customer's arrival instant, one lift occupies the first floor, another floor 2n/3 (see, Fig.6.2; (x_5, x_9, x_{14})). The comparison (6.8) with (6.9) shows that the control gives an advantage of the customer's average service time of 4% and regarding the single race time, of 2%.

Consider the system $L_lF_nC_{f1,f2,..,fl}$, where l – is a number of lifts in the system. Denote f_j (j=1,2,..,l) the floor, which must be occupied by j-th lift in an optimal regime.

We will show that for the system $L_lF_nC_{f1,f2,..,fl}$ in the mixed regime $f_1,f_2,\dots,f_l$, the following recurrent formulas are applied:

$$2\lambda_1(n-1)+\lambda_1(3f_1-f_2-2)=0$$
$$f_{i-1}-2f_j+f_{j+1}=0,\ j=2,3,\dots,l-1 \qquad (6.9)$$
$$2n-3f_l+f_{l-1}=0$$

Below, in Fig.6.3, there are shown the lifts' positions at the different instants, for the system $L_k F_nC_{f1,f2,\dots,fl}$. We remind that t_i is a customer's arrival instant into the system. If we put l=3 then f_1, f_2, f_3 satisfies the equation (6.9). According to an optimal control at each preceding customer's arrival instant, one lift must occupy f_1-th floor, another f_2-th floor and the third lift - f_3-th floor. Suppose that n>10. Then,

$$f_1=\max[1,\ n\lambda_2+(\lambda_2-\lambda_1(\nu-1)(2\lambda-1)/2l\lambda_2]=\max\{1,\ n/2l(1-(2l-1)\lambda_1/\lambda_2)\},$$
$$f_j=f_1+2(j-1)(n-f_1)/(2l-1)=f_1[1+2(j-1)/(2l-1)]+[2(j-1)n/(2l-1)]=$$
$$=f_1[2(l-j)+1]/(2l-1)+2n(j-1)/(2l-1), \qquad (6.10)$$

where (6.10) is the unique solution of (6.9).

Let's put l=3, $\lambda_1=\lambda_2$, then, for n>10, we have f1=1, f2=2n/5, f3=4n/5, (see Fig. 6.3)

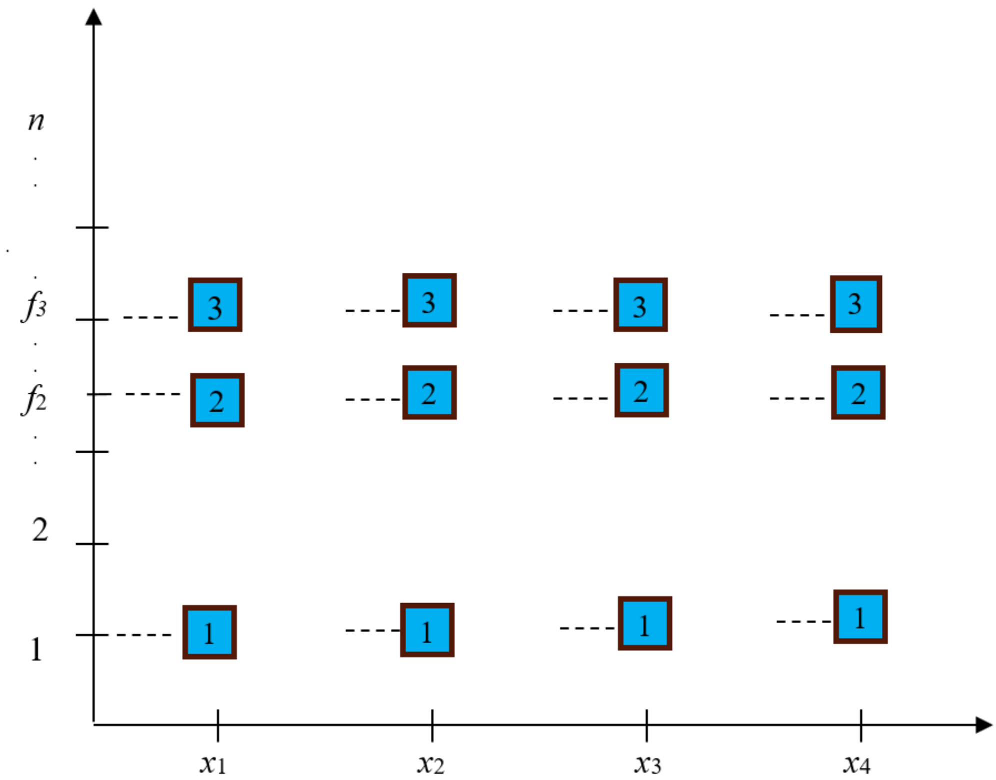

Fig. 6.3

In the Fig. 6.3, there are shown the lifts' positions at the preceding customer's arrival instant, for the system $L_3F_nC_{f1,f2,f3}$.

Remark 6.4. For the system $L_3F_nC_{f1,f2,f3}$ starting from some moment at the preceding of a customer arrival instant lifts have the position shown in the Fig.6.3. Customers arriving to the floors 1,2,…f_1/2 will be served by the first lift, customers at the f_1/2, f_1/2+1,… f_1, f_1+1,…,f_2-f_1 by second lift and so on. After finishing service i-th lift must come to the f_i floor.

7. Simulation of the systems with several lifts in loading and unloading regimes

In [6], based on Wolfram Mathematica, several programs had been developed for the simulation of the input data and imitation systems with one lift. Here we consider two lifts systems, using related developed programs. The numerical comparison between such parameters as the waiting lift times CWT and the total time CTT show the advantage of the two lifts systems. The programs for the simulation systems with two lifts had been developed and they will be applied below. Systems with more than two lifts have more complicated structures and a special approach will be realized later.

Let us consider an office building where offices are placed on floors from 2 up to n, e.g. n=10. On each of the even floor 2, 4, 6, 8, 10 there are 20 working places for customers. On each odd floor 3, 5, 7, 9, there are 30 working places for customers. Altogether there are n_c=220 working places for customers in the building. Every morning, during a time interval, e.g. half hour (t_m=1800 Sec), all the customers should be on the floors which have their offices and should start to work. In large towns, it is difficult to reach a certain building at scheduled time. Therefore, we use approximate the time when the customers reach their offices at independent times, uniformly distributed over the time interval t_m=1800 Sec.

We also suppose that for any customer who had come at the 1st floor lift hall, at uniformly distributed time ta, in the time interval tm, and his destination floor f_d was randomly selected, without the repetition of the list of all the working places in this building, We created the program simulated for each i-th customer at his initial four-plot history {i, 1, f_d, t_a}. The i-th customer waits for the lift, during a time t_b, when the customer can come into the lift cabin. The lift is going up and reaches the destination floor f_d, at time t_e. Then, the customer goes out from the cabin. His initial history extended to {i, 1, f_d, t_a, t_b, t_e}. We have also created a program which produces these extensions of histories for all the customers. The efficiency of a lift's system can be evaluated by estimating CWT and CTT, by using the obtained extended histories. In the following table, these parameters were estimated using the data of the loading work by 1- lift system with roominess r=10.

Floor	2	3	4	5	6	7	8	9	10
CWT	54.458	63.217	40.180	56.152	56.936	65.222	66.083	47.991	55.474
CTT	64.458	79.217	62.680	82.819	90.436	102.389	108.334	95.325	109.974

Table 7.1. Estimates of mean values CWT-waiting cabin time and CTT total time destination all floors by 1-lift system with r=10 in loading process are given below.

This table shows the weak dependency parameters CWT and CTT from roominess. Therefore, we can accept r=10 as an appropriate value of the cabin's roominess. But in the case of unloading the parameters CWT and CTT, it will be too large, if r=10. This problem with 1-Lift system can be lost by increasing the cabin roominess, say to take r=16. But the most natural way to solve this problem will be the use of a 2-Lift system with r=10, for both lifts, in loading and unloading cases. Consider the system with one lift in an unloading regime with a line decreasing probability density in a time interval with a length t_m=1800 Sec. The line probability density is a hypotenuse in triangle with two orthogonal sides with lengths t_m and 1/ (2 t_m). Then, $S[x] = 1-2\,x/t_m + (x/t_mT)^2$ is a probability in the randomly taking initial history $t_a > x$. The graphic of S[x] is shown below:

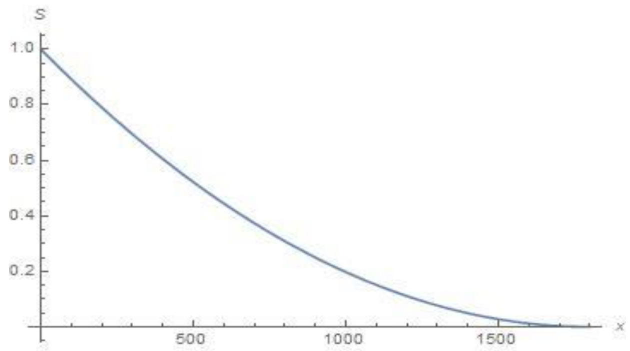

Fig.7.1. Probability function S[x]

Let the lift goes from f_1 floor down and there are some customers at the lower floors f_2 ($f_2 < f_1$), who will go down to the 1-st floor. Then the lift must stop at the f_2- floor, only if, due to the lift's roominess, there is a free space in the cabin. Otherwise, the lift passes f_2 without stopping and the customers at the f_2-floor must wait for another appearance of the lift. We are interested in the problem of finding the dependence of CWT and CTT on lift's roominess. Our programs regarding the simulation of such systems are based on a conception of the lift's cycles. The lift's cycle is defined as the time interval between two sequential instants when the lift arrives to the first floor and all the customers (if there are) leave the lift cabin. At the end of the lift's cycle, all characteristics of the system can be calculated by using of the extended six-plot histories {i, f_a, f_d, t_a, t_s, t_e}. Note that in this paper only one-day data are used, but our programs can work with many independent days' data. Then, the accuracy of the table's numbers will be exact. Below, Table 7.2, demonstrates the dependency of CWT and CTT on the value of the lift's roominess.

R	8	10	12	14	16	18	20	22
CWT	382.3	236.0	146.0	90.1	64.0	51.4	43.8	43.0
CTT	400.7	255.7	167.6	112.7	88.6	77.9	70.5	69.8

Table 7.2. The estimated mean values of CWT and CTT, over all floors in an unloading process, are given using the customers' one-day data, for one lift with different values of roominess.

From Table 7.2 we can see that by increasing the lift's roominess we obtain a reduction of CWT and CTT. Here we do not consider the problem of the accuracy s numbers in the Tables. However, note that the large values of roominess imply undesirable big sizes of the lift's cabin. Here we face the necessity to find a solution to the problem. We consider that instead of one lift with a large roominess it is worth trying to use two lifts with a smaller roominess. We suppose that these two lifts are independent. We consider systems with 2 lifts which are serving different floors. Let the 1st lift stop at odd floors 3, 5, 7, 9 and the 2nd lift stop on even floors 2, 4, 6, 8, 10. Suppose that both lifts have the same roominess r=10. For each lift, we can use our programs with initial simulation and extended histories in loading and unloading processes. Using these two lifts' similar parameters, e.g. h_f=2.5; h_d = 5; n_c = 220; t_m=1800 Sec.

Floor	3	5	7	9
CWT	38.339	33.811	33.986	37.515
CTT	43.339	47.478	56.486	68.349

Table 7.3. Customers' estimates of mean values CWT - waiting times and CTT - total serves customers' times during unloading for Lift-1 used on odd floors 3, 5, 7, 9.

Floor	2	4	6	8	10
CWT	44.425	28.631	41.062	31.321	29.299
CTT	46.925	39.381	59.812	56.821	63.049

Table 7.4. Customers' estimates of mean values CWT - waiting times and CTT - total serves customers' times during unloading for Lift-2 used on even floors 2, 4, 6, 8, 10.

In both Table 7.3 and Table 7.4, CWT and CTT take fewer values than in the system with one lift and r=10.Thus, we have found the solution to the problem. We use only one lift with a large roominess or instead of one lift with a large roominess, we use two or more lifts with less roominess. The simulation of the systems with 2 lifts shows that the cycle time is less and moreover, both CWT and CTT take smaller values than in the systems with one lift with the same roominess.

8. Conclusions

In the paper, for various lifts' systems, the universal mathematical models have been constructed and the main characteristics (CWT – a customer's waiting time before service, CTT – a customer's total time, roominess of the lift's cabin and others), are introduced. It is also introduced the new type of customer flows, called transforming flows (transforming point processes). Such processes can be transformed from any point process into processes with a rare event. For such flows of customers, by analytical approaches, the formulas for calculating the main characteristics of efficiency lift systems (CWT, CTT and others) are suggested. The various control rules, reducing the different characteristics of the lifts' systems are introduced. The several graphs, reflecting the behaviour of the different lift systems, are shown. The introduced characteristics allow find the optimal number of lifts in buildings and skyscrapers. The paper offers an important solution for investigating the lifts' systems, which are natural in modern metropolises. Simulations works of lifts' models is the base for such investigations which adequately describe the lifts' systems behaviour.

Collecting and analysing the data of the simulations allow obtain the suitable recommendations for planning the future lifts' systems. The programs for the simulation of the lifts' systems have been prepared. Numerical examples show how the optimal roominess of a lift's cabin can be ensured.

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References

[1] Haight F. Mathematical Theories of Traffic Flow.Academic Press,1966, 242p.

[2] Drew D. Traffic Flow: Theory and Control. NY, McGrow Hill, NY, 1968, 424p.

[3] Kerner B.S. Introduction to Modern Traffic Flow Theory and Control. Berlin: Springer, 2009.

[4] Gazis D. C. Traffic Theory. Berlin: Springer, 2002.

[5] Gases, Aday L. Mathematical models of traffic flows. Proceedings Of The Institute Of Radio Engineering. and Elektr., 1968, no. 4, pp. 94-109. (in Russian).

[6] BelyaevYu.K., HajiyevA.H. Mathematical models of the lift systems and their simulation. Proceed. XIII ICMSEM, Canada. Springer, Ser. Electrical Engineering, 2019. p.507-519.

[7] Gong W. and Yu Z. J. Mathematical Modeling and Simulation Analysis of Lift for Car Hydraulic System. Applied Mechanics and Materials. pp. 447-450, 2014.

[8] Belyaev Yu. K., Gadzhiev A. G., Grom Yak, Y. I., Dugina T. N. A comparative analysis of elementary systems of vertical transportation. Izv. USSR ACADEMY OF SCIENCES. Technical Cybernetics, no. 3, 1977, p. 97-103. (in Russian). (English transl.: Soviet Journal Computer and Systems Sciences).

[9] Belyaev Yu. K., Gadzhiev A. G., Dugina T. N. modeling of two Elevator systems with different control rules. State Fund of algorithms and programs. No. П002152 dated 17.9.1976. (in Russian). (Soviet State Fund of Algorithms and Programs).

[10] Newell G.F. Control of Pairing of Vehicles on a Public Transportation Route, Two Vehicles, One Control Point. // Transp. Sci., V 8, 1974, № 4, p. 248-264.

[11] Hajiyev A.H., Mammadov T.Sh. Mathematical models of moving particles and their application for queues. Theory of Probab. its Applic. 2012, v.56, №4, p.579-589.

[12] Ignall E., Kolesar P., Optimal dispatching of an infinite capacity shuttle: control at a single terminal. Oper. Res. 22, 1974, p. 1008-1025.

[13] Hajiyev A.H., Abdullayeva N.A, Ahmadazada F.A. Queues with moving servers. Simple vertical transportations. Lecture Notes in Electrical Engineering, Springer, 2012, v.2, p.p. 773-785.

Reliability Test Plan for the Marshall Olkin Length Biased Lomax Distribution

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Abstract

In this article, a generalization of the length biased Lomax distribution called the Marshall Olkin length biased Lomax distribution is introduced. A few statistical and reliability properties of the new distribution is discussed. The parameters of the introduced model are estimated by the method of maximum likelihood estimation. The suitability of the model is verified empirically by applying a real life data set. Also, establish a reliability test plan for acceptance or rejection of a lot of products submitted for inspection with lifetimes directed by this distribution.

Keywords: Marshall Olkin family, Length biased Lomax, Maximum Likelihood Estimates, Reliability Test Plan, Operating Characteristic Function.

I. Introduction

The applications in different areas such as lifetime analysis, financial modelling, business, economics, insurance, clinical trials shows that the data sets arising from various areas may require more flexible distributions. So there is an increasing trend in developing generalized families of distributions by adding one or more additional parameters to the baseline distributions.

Marshall and Olkin (1997) introduced a method of adding a new shape parameter into a family of distributions. Considering a baseline distribution with cumulative density function, G(x), of a random variable X, then the cumulative density function of the Marshall and Olkin - G (MO-G) family of distributions is

$$F(x) = \frac{G(x)}{1-(1-\gamma)(1-G(x))} \tag{1}$$

The corresponding probability density function of (1) is given by,

$$f(x) = \frac{\gamma g(x)}{[1-(1-\gamma)(1-G(x))]^2} \tag{2}$$

where $\gamma > 0$ is a shape parameter. Clearly, for $\gamma = 1$, we obtain the baseline distribution, i.e., F(x) = G(x). Many authors have proposed various distributions belonging to the Marshall Olkin family of distributions such as Marshall Olkin Weibull by Ghitany et al. (2005), Marshall Olkin Pareto by Alice et al. (2003), Marshall Olkin semi-Weibull by Alice et al. (2005), Marshall Olkin Lomax by Ghitany et al. (2007), Marshall Olkin semi-Burr and Marshall Olkin Burr by Jayakumar et al. (2008) and Marshall Olkin q-Weibull by Jose et al. (2010), Marshall Olkin Frechet distribution by Krishna et al. (2013), Marshall Olkin gamma by Ristic et al. (2007), Marshall Olkin Lomax by Ghitany et al. (2007), Marshall Olkin linear failure-rate by Ghitany et al. (2007), Marshall Olkin log-logistic by Gui (2013), and Marshall Olkin exponential Weibull by Pogny et al. (2015).

The Lomax distribution has so many applications in life testing, reliability modeling and is also used to fit business failure data. This distribution was proposed by Lomax (1954). Using generalized probability weighted moment, Abdullah and Abdullah (2010) estimated the parameters of Lomax distribution.

A continuous random variable X is said to have a Lomax distribution with parameters θ and β if its probability density function (pdf) is,

$$f(x\mid\theta,\beta)=\frac{\theta\left(\frac{\beta+x}{\beta}\right)^{-\theta}}{\beta+x}. \tag{3}$$

for $0<x<\infty, \theta>0$ and $\beta>0.$

In the rest of the paper, we define the new model and the statistical properties of the proposed distribution are discussed in Section 2. The failure rate function is given in section 3. Estimation of parameters by using maximum likelihood estimation method is presented in Section 4. An application of proposed model on real life data set are provided in Section 5. In Section 6, we develop reliability test plans to decide whether to accept or reject a submitted lot of products whose lifetime is assumed to be a Marshall Olkin length biased lomax distribution. Section 7 gives the description of the tables and applications. In Section 8, concluding remarks are presented.

II Marshall Olkin Length Biased Lomax distribution

The concept of length-biased distributions originated from the study of weighted distributions. The weighted or moment distributions arise in the context of unequal probability sampling; have a great importance in reliability, biomedicine, and ecology, among others. The weighted distributions are first considered by Fisher (1934) and they have been found applications in several areas such as ecology, fisheries etc. In this regard, Gupta and Akman(1995) considered certain applications of the weighted distributions in biomedical.

To introduce the concept of a weighted distribution[Patil (2002)], suppose that X is a random variable(r.v.) with its natural probability density function (pdf) f(x|c), where the natural parameter $\theta\in\Theta$ (Θ is the parameter space). A weighted distribution with kernel f(x|θ) and weight function w(x, β) is defined as g(x|θ, β) = w(x, β)f(x|θ)/C(θ, β).

where, w(x, β) > 0 and C(θ, β) = E_f [w(X, β)]. When X is a non-negative random variable and w(x, β) = x, the resultant weighted distribution is known as length-biased distribution

$$f_{X_w}(x)=\frac{w(x)f_X(x)}{E[w(x)]} \tag{4}$$

Assuming that E[w(x)] $<\infty$.

The length biased distribution is obtained by taking weight as w(x) = x which can be denoted by a random variable X_L which has pdf expressed as

$$f_{X_L}(x) = \frac{x f_X(x)}{\mu} \tag{5}$$

Where μ = E(x) $< \infty$.

Afaq Ahmad et al. (2016) proposed the length biased weighted lomax distribution (here we will call it Length Biased Lomax (LBL) distribution) through assigning weight to the lomax distribution by following the idea of Fisher (1934). Afaq Ahmad et al. (2016) proved that the LBL is more flexible than the lomax distribution.

The Probability Density Function (PDF) and Cumulative Distribution Function (CDF) of LBL distribution are;

$$g(x) = \frac{(\theta-1)\theta x\left(\frac{x}{\beta}+1\right)^{-\theta-1}}{\beta^2} \tag{6}$$

for $x > 0, \theta, \beta > 0$

and

$$G(x) = 1-\beta^{\theta-1}(\beta+x)^{-\theta}(\beta+\theta x) \tag{7}$$

In this article, we proposed a new extension of the LBL distribution called the Marshall Olkin Length Biased Lomax (MOLBL) distribution. The new model is obtained using the Marshall Olkin family of distributions. Some of its statistical properties are constructed with the hope that it will bring wider applications in reliability and other areas of research. The new distribution can be defined by inserting (7) in (1). Then, the CDF of the MOLBL model, say F(x) = $F(x;\alpha,\beta,\gamma)$ is;

$$F(x) = \frac{1-\beta^{\theta-1}(\beta+\mathrm{x}\theta)(\beta+x)^{-\theta}}{1-(1-\gamma)\left(\beta^{\theta-1}(\beta+\mathrm{x}\theta)(\beta+x)^{-\theta}\right)} \tag{8}$$

where $\theta > 0$ and $\gamma > 0$ are shape parameters and $\beta >$ 0 is the scale parameter.

By putting Eqs. (6) and (7) in Eq.(2) , we obtain the PDF of the MOLBL distribution

$$f(x) = \frac{\gamma(\theta-1)\theta x\left(\frac{x}{\beta}+1\right)^{-\theta-1}}{\beta^2\left(1-(1-\gamma)\beta^{\theta-1}(\beta+x)^{-\theta}(\beta+\theta x)\right)^2} \tag{9}$$

For γ = 1, the MOLBL reduces to the LBL distribution. The Survival Function (SF) of X is;

$$S(x) = \frac{\gamma\beta^{\theta}(\beta+\mathrm{x}\theta)}{\beta(\beta+x)^{\theta}+(\gamma-1)\beta^{\theta}(\beta+\mathrm{x}\theta)}$$

Figure 1 shows some possible shapes of the PDF of the MOLBL distribution for selected values of parameters.

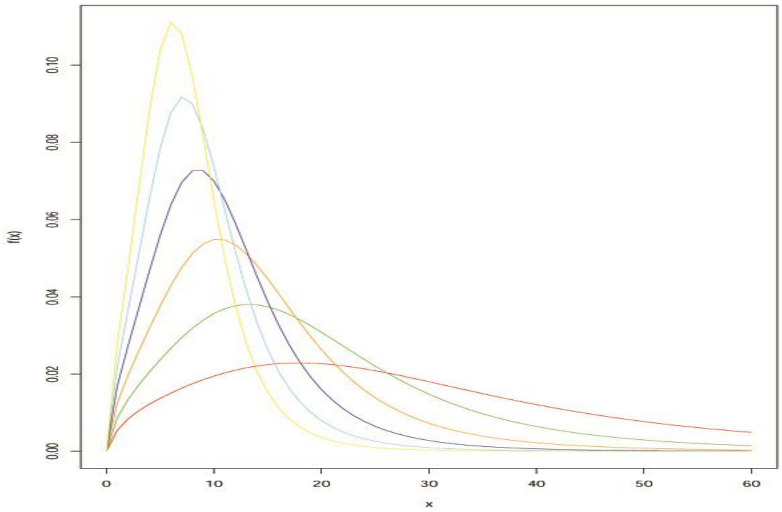

Figure 1: Graph for MOLBL density function at different parameter values. (θ, β, γ) = red (4, 10,15), Green (5,10, 15), Orange (6, 10 , 15), Blue (7, 10, 15), Skyblue(8,10,15), Yellow(9,10,15).

III. Failure Rate Function

The failure rate or hazard rate function of the MOLBL distribution is given by,

$$h(x) = \frac{\gamma(\theta-1)\theta x\left(\frac{x}{\beta}+1\right)^{-\theta-1}}{\beta^2\left(1-(1-\gamma)\beta^{\theta-1}(\beta+x)^{-\theta}(\beta+\theta x)\right)^2\left(1-\frac{1-\beta^{\theta-1}(\beta+x\theta)(\beta+x)^{-\theta}}{1-(1-\gamma)\beta^{\theta-1}(\beta+x\theta)(\beta+x)^{-\theta}}\right)}$$

where $\theta > 0$, $\gamma > 0$ and $\beta > 0$.

Figs. 2 and 3 shows some possible shapes of the hazard rate function of the MOLBL distribution for selected values of parameters.

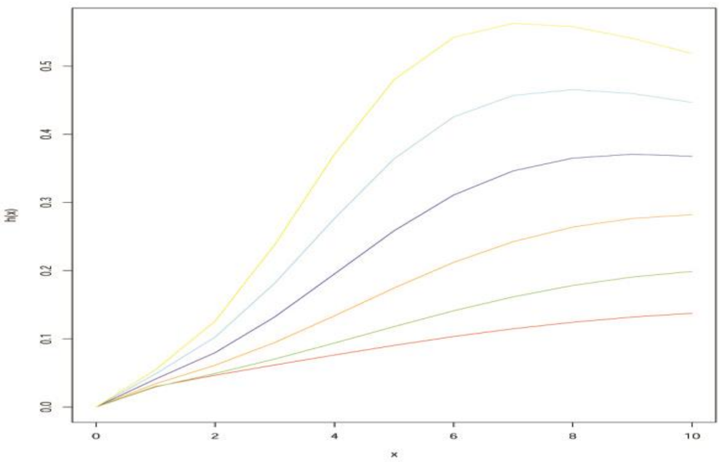

Figure 2 : Graph for hazard rate function at different parameter values. (θ, β, γ) = red (5, 6,10), Green (6,6, 15), Orange (7, 6 , 18), Blue (8, 6, 20), Skyblue(9,6,22), Yellow(10,6,25).

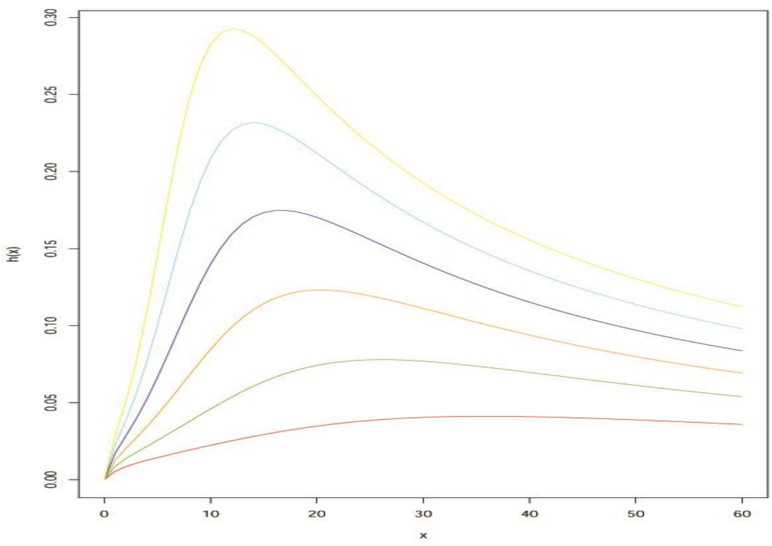

Figure 3 **: Graph for hazard rate function at different parameter values.** , (θ, β, γ) = red (4, 10,15), Green (5,10, 15), Orange (6, 10 , 15), Blue (7, 10, 15), Skyblue(8,10,15), Yellow(9,10,15).

IV Maximum Likelihood Estimation

In this section, the unknown parameters of the MOLBL distribution are estimated by using the maximum likelihood method. Let $x_1; x_2; \ldots x_n$ be a random sample of size n from the MOLBL distribution. The likelihood function for the MOLBL distribution is;

$$L = \prod_{i=1}^{n} \frac{\gamma(\theta-1)\theta x_i \left(\frac{x_i}{\beta}+1\right)^{-\theta-1}}{\beta^2\left(1-(1-\gamma)\beta^{\theta-1}(\beta+x_i)^{-\theta}(\beta+\theta x_i)\right)^2}$$

The log-likelihood function reduces to

$$LogL = -2\sum_{i=1}^{n}\log\left(1-(1-\gamma)\beta^{\theta-1}(\beta+x_i)^{-\theta}(\beta+\theta x_i)\right)-(\theta+1)\sum_{i=1}^{n}\log\left(\frac{x_i}{\beta}+1\right)$$
$$+\sum_{i=1}^{n}\log(x_i)+n\log(\gamma)+n\log(\theta-1)+n\log(\theta)-2n\log(\beta).$$

Now, differentiate the above equation with respect to parameters,

$$\frac{\partial \log L}{\partial\theta} = -\sum_{i=1}^{n}\log\left(\frac{x_i}{\beta}+1\right)+\frac{n}{\theta-1}+\frac{n}{\theta}$$
$$-2\sum_{i=1}^{n}\frac{-(1-\gamma)\beta^{\theta-1}x_i(\beta+x_i)^{-\theta}-(1-\gamma)\beta^{\theta-1}\log(\beta)(\beta+x_i)^{-\theta}(\beta+\theta x_i)+(1-\gamma)\beta^{\theta-1}(\beta+x_i)^{-\theta}(\beta+\theta x_i)\log(\beta+x_i)}{1-(1-\gamma)\beta^{\theta-1}(\beta+x_i)^{-\theta}(\beta+\theta x_i)}$$

$$\frac{\partial \mathrm{logL}}{\partial \beta} = -(\theta+1)\sum_{i=1}^{n} - \frac{x_i}{\beta^2\left(\frac{x_i}{\beta}+1\right)} - \frac{2n}{\beta} - 2\sum_{i=1}^{n} \frac{\begin{array}{l}(1-\gamma)\theta\beta^{\theta-1}(\beta+\theta x_i)(\beta+x_i)^{-\theta-1} - \\ (1-\gamma)\beta^{\theta-1}(\beta+x_i)^{-\theta} - \\ (1-\gamma)(\theta-1)\beta^{\theta-2}(\beta+\theta x_i)(\beta+x_i)^{-\theta}\end{array}}{1-(1-\gamma)\beta^{\theta-1}(\beta+x_i)^{-\theta}(\beta+\theta x_i)}$$

And

$$\frac{\partial \mathrm{logL}}{\partial \gamma} = \frac{n}{\gamma} - 2\sum_{i=1}^{n} \frac{\beta^{\theta-1}(\beta+x_i)^{-\theta}(\beta+\theta x_i)}{1-(1-\gamma)\beta^{\theta-1}(\beta+x_i)^{-\theta}(\beta+\theta x_i)}$$

The maximum likelihood estimator of the parameters can be obtained by solving the above equations. This solution can also be obtained by using R software with *nlm* or *maxLik* package.

V Data Analysis

In this section, we provide an application to real data to illustrate the importance of the MOLBL distribution. The MLEs of the model parameters are computed and goodness-of-fit statistics for this model are compared with the Length biased Lomax model introduced by Afaq Ahmad et al. (2016). For this, we consider the data set from Lee and Wang (2003). This data set consists of 128 observations on remission times (in months) of a random sample of bladder cancer patients.

Data set: 0.08, 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 0.26, 0.31, 0.73, 0.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 11.98, 4.51, 2.07, 0.22, 13.8, 25.74, 0.50,2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 19.13, 6.54, 3.36, 0.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 1.76, 8.53, 6.93, 0.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 3.25, 12.03, 8.65, 0.39, 10.34, 14.83, 34.26, 0.90,2.69, 4.18, 5.34, 7.59, 10.66, 4.50, 20.28, 12.63, 0.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 6.25, 2.02, 22.69, 0.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 8.37, 3.36, 5.49, 0.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 12.02, 6.76, 0.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.1, 1.46, 4.40, 5.85, 2.02, 12.07.

In order to compare the two models, we consider the criteria like AIC (Akaike information criterion) and BIC (Bayesian information criterion). The better distribution corresponds to lesser AIC, and BIC values.

AIC = 2k-2logL and BIC = klogn-2logL

where k is the number of parameters in the statistical model, n is the sample size and -2logL is the maximized value of the log-likelihood function under the considered model.

We have fitted Length biased Lomax and Marshal Olkin Length biased lomax models to this data. These two distributions are fitted to the subject data using maximum likelihood estimation. The MLEs of the parameters with standard errors(SE) and the corresponding loglikelihood values, AIC, BIC are displayed in Table 1.

Table 1: MLEs and some goodness of fit measures for Comparison.

Distribution	Parameter	Estimates	SE	-2logL	AIC	BIC
LBL	$\hat{\alpha}$	3.1142	0.5056	814.94	818.95	824.65
	$\hat{\beta}$	5.2372	1.6951			
MOLBL	$\hat{\alpha}$	2.6495	0.2230	808.02	814.01	822.57
	$\hat{\beta}$	0.6115	0.5824			
	$\hat{\gamma}$	15.2889	17.6980			

From Table 1, it has been observed that the Marshal Olkin Length biased Lomax distribution has the lesser AIC and BIC values as compared to Length biased Lomax Distribution. Hence we can conclude that the Marshal Olkin Length biased Lomax distribution leads to a better fit than the Length biased Lomax distribution.

VI Reliability Test Plan

The acceptance sampling plan is affected with the inspection of a sample of products taken from a lot and the decision whether to accept or to reject the lot based on the quality of the product in statistical quality control . Here we discuss the reliability test plan for accepting or not to accepting a lot when the lifetime of the product follows the Marshal olkin length biased Lomax distribution. In a life testing experiment, the process is to restrict the test by a predetermined time t and notice the number of failures. If the number of failures at the end of time t does not exceed a given number c (acceptance number), then we accept the lot with a given probability of at least p*. But if the number of failures exceeds c before time t, then the test is terminated, and the lot is rejected. For such an event like a truncated life test and the associated decision rule, we are interested in finding the smallest sample size to make at a decision. Assume that the lifetime of a product follows the Marshal olkin length biased Lomax distribution with cumulative distribution function (cdf)

$$F(t) = \frac{1-\beta^{\theta-1}(\beta+t\theta)(\beta+t)^{-\theta}}{1-(1-\gamma)\left(\beta^{\theta-1}(\beta+t\theta)(\beta+t)^{-\theta}\right)}; \quad 0<t<\infty \tag{10}$$

where $\theta>0$ and $\gamma>0$ are shape parameters and $\beta>0$ is the scale parameter. If θ and γ are known, then the average lifetime depends only on β. Let β_0 be the required minimum average lifetime. Then

$$F(t,\theta,\gamma,\beta) \le F(t,\theta,\gamma,\beta_0) \Leftrightarrow \beta \ge \beta_0.$$

A sampling plan is specified by the following quantities:

- the number of units (n) on test
- the acceptance number (c)
- the maximum test duration (t)
- the minimum average lifetime (β_0).

The consumers risk, which is the probability of accepting a bad lot should not exceed the value 1-

p*,where p* is a lower bound for the probability that a lot of true value β below β_0 is rejected by the sampling plan. For fixed p* the sampling plan is characterized by (n, c, $\frac{t}{\beta_0}$). By considering the large lots, the binomial distribution may be used to determine the acceptance probability. The problem is to determine the smallest positive integer n for given values of c and $\frac{t}{\beta_0}$ such that

$$L(p_0) = \sum_{i=0}^{c} \binom{n}{i} p_0^i (1-p_0)^{n-i} \leq 1-p^* \tag{11}$$

where p_0 = $F(t,\theta,\gamma,\beta_0)$ given by (10) indicates the failure probability before time t which depends only on the ratio $\frac{t}{\beta_0}$. The function L(p) is known as the operating characteristic(OC) function of the sampling plan, i.e. the probability of acceptance of the lot as a function of the probability of failure $p(\beta)$ = $F(t;\theta,\gamma,\beta)$. The average lifetime of the product is increasing with β, and therefore the failure probability p(β) decreases, implying that the OC function is increasing in β. The minimum values of n satisfying eq (11) are obtained for θ =3 , γ = 2, p* = 0.75,0.90,0.95,0.99 and $\frac{t}{\beta_0}$ = 0.75, 1, 1.25, 1.75, 2, 3 and 3.5. The results are displayed in Table 2. If p_0 = $F(t,\theta,\gamma,\beta_0)$ is very small and n is large, the binomial probability may be approximated by the Poisson probability with parameter $\lambda = np_0$, so eq (11) becomes

$$L_1(p_0) = \sum_{i=0}^{c} \frac{\lambda^i}{i!} e^{-\lambda} \leq 1-p^* \tag{12}$$

The minimum values of n satisfying eq (12) are obtained for the same combination of values of θ,γ and $\frac{t}{\beta_0}$ for various values of p* and is presented in Table 3. The operating characteristic function of the sampling plan $(n,c,\frac{t}{\beta_0})$ gives the probability of accepting the lot and is given by

$$L(p) = \sum_{i=0}^{c} \binom{n}{i} p_0^i (1-p_0)^{n-i}$$

where p = F(t, β) is considered as a function of β. The values of n and c are obtained by means of operating characteristics function for given value of p* and $\frac{t}{\beta_0}$ are displayed in Table 4 by considering the fact that p = F($\frac{t}{\beta_0}$ / $\frac{\beta}{\beta_0}$)

Table 2: Minimum sample size for the specified ratio t/β_0, confidence level p^*, acceptance number c, θ =3 and γ = 2 using binomial approximation.

p^*	c	t/β_0						
		0.75	1	1.25	1.75	2	3	3.5
.75	0	5	4	3	2	2	2	1
	1	11	8	6	5	4	3	3
	2	15	11	9	7	6	5	4
	3	20	15	12	9	8	6	6
	4	25	19	15	11	10	8	8
	5	30	23	18	13	12	9	9
	6	35	27	21	15	14	11	10
	7	40	29	24	17	16	13	11
	8	45	31	27	19	18	14	13
	9	50	35	28	21	20	15	14
	10	52	38	30	23	21	16	15
.90	0	9	6	5	3	3	2	2
	1	15	11	8	6	6	4	4
	2	20	15	12	9	8	6	6
	3	28	19	16	12	10	8	7
	4	33	22	20	15	12	10	8
	5	38	26	22	17	14	11	10
	6	43	30	24	18	16	12	11
	7	48	34	26	20	18	14	13
	8	53	37	29	22	20	16	15
	9	58	41	32	24	22	18	16
	10	61	44	35	26	24	20	17
.95	0	11	8	6	4	4	3	3
	1	18	14	10	7	6	5	4
	2	24	17	14	10	9	6	6
	3	32	21	18	13	11	8	8
	4	35	25	22	15	13	10	9
	5	41	29	26	17	15	12	11
	6	45	33	29	19	17	14	13
	7	51	37	31	22	19	15	14
	8	56	41	32	24	21	16	15
	9	60	45	35	26	23	18	16
	10	66	48	38	28	25	19	18
.99	0	18	12	9	6	6	4	4
	1	27	19	14	10	9	6	6
	2	31	22	17	12	11	8	7
	3	44	27	21	16	13	10	10
	4	49	31	25	18	16	12	12
	5	50	35	29	20	19	14	13
	6	55	40	32	23	21	15	14
	7	61	44	35	26	24	17	15
	8	67	49	38	28	25	19	17
	9	74	52	41	30	27	21	18
	10	77	55	44	32	29	22	20

Table 3: Minimum sample size for the specified ratio t/β_0, confidence level p^*, acceptance number c, θ =3 and γ = 2 using poisson approximation.

p^*	c	t/β_0						
		0.75	1	1.25	1.75	2	3	3.5
.75	0	6	5	4	3	3	2	2
	1	11	9	7	6	5	4	4
	2	17	13	10	8	7	6	6
	3	22	17	13	10	9	8	7
	4	26	21	17	12	11	9	9
	5	31	25	20	14	13	11	11
	6	35	29	22	16	15	13	13
	7	40	32	24	19	17	15	14
	8	45	35	27	21	19	16	15
	9	49	37	30	23	21	17	16
	10	54	40	32	25	23	18	17
.90	0	10	7	6	5	4	4	3
	1	16	13	10	8	7	6	6
	2	22	20	14	10	10	8	7
	3	28	23	18	13	12	10	9
	4	33	25	20	15	14	11	11
	5	39	28	23	18	16	13	13
	6	44	32	26	20	18	15	15
	7	49	36	29	22	21	17	16
	8	55	39	32	25	24	19	17
	9	59	43	36	27	26	21	19
	10	63	47	38	29	27	22	20
.95	0	13	9	8	6	6	5	4
	1	20	15	12	9	9	7	7
	2	26	20	16	12	11	9	10
	3	32	24	20	15	15	11	11
	4	39	28	23	18	17	13	12
	5	43	32	27	20	18	15	14
	6	49	36	30	23	21	17	16
	7	54	40	32	25	24	19	18
	8	59	44	36	27	26	20	19
	9	65	48	40	30	28	22	21
	10	70	52	42	32	29	24	22
.99	0	19	15	12	9	8	7	6
	1	28	20	17	13	12	10	9
	2	36	26	21	16	16	13	12
	3	42	32	25	19	18	15	14
	4	49	37	29	22	20	17	15
	5	54	41	33	26	23	18	17
	6	60	44	36	28	25	20	19
	7	66	49	39	30	29	23	21
	8	72	53	43	33	31	26	23
	9	77	58	47	36	34	27	25
	10	83	61	49	38	35	28	26

Table 4: Values of the operating characteristic function of the sampling plan $(n, c, \frac{t}{\beta_0})$

				$\frac{\beta}{\beta_0}$						
p^*	n	c	$\frac{t}{\beta_0}$	2	4	6	8	10	12	14
.75	15	2	0.75	0.8142	0.9862	0.9979	0.9995	0.9998	0.9999	0.9999
	11	2	1	0.7820	0.9804	0.9967	0.9992	0.9997	0.9999	0.9999
	9	2	1.25	0.7445	0.9730	0.9951	0.9987	0.9995	0.9998	0.9999
	7	2	1.75	0.6687	0.9541	0.9905	0.9973	0.9990	0.9996	0.9998
	6	2	2	0.6803	0.9535	0.9899	0.997	0.9989	0.9995	0.9997
	5	2	3	0.5387	0.9016	0.9739	0.9913	0.9965	0.9984	0.9992
	4	2	3.5	0.6287	0.9233	0.9793	0.9929	0.9971	0.9987	0.9993
.90	20	2	0.75	0.6743	0.9695	0.9952	0.9988	0.9996	0.9998	0.9999
	15	2	1	0.6088	0.9542	0.9918	0.9979	0.9993	0.9997	0.9998
	12	2	1.25	0.5668	0.9405	0.9884	0.9969	0.9989	0.9995	0.9998
	9	2	1.75	0.4887	0.9089	0.9794	0.9939	0.9978	0.9991	0.9995
	8	2	2	0.4682	0.8965	0.9753	0.9925	0.9972	0.9988	0.9994
	6	2	3	0.3833	0.8383	0.9535	0.9839	0.9935	0.9970	0.9985
	6	2	3.5	0.2757	0.7619	0.9225	0.9711	0.9878	0.9942	0.9970
.95	24	2	0.75	0.5612	0.9512	0.9918	0.998	0.9993	0.9997	0.9998
	17	2	1	0.5247	0.9369	0.9883	0.9969	0.9990	0.9996	0.9998
	14	2	1.25	0.4570	0.9122	0.9820	0.9950	0.9983	0.9993	0.9997
	10	2	1.75	0.4088	0.8815	0.9722	0.9916	0.9969	0.9987	0.9994
	9	2	2	0.3771	0.8614	0.9653	0.9892	0.9959	0.9982	0.9991
	6	2	3	0.3833	0.8383	0.9535	0.9839	0.9935	0.9970	0.9985
	6	2	3.5	0.2757	0.76199	0.9225	0.9711	0.9878	0.9942	0.9970
.99	31	2	0.75	0.3854	0.909	0.9834	0.9958	0.9986	0.9994	0.9997
	22	2	1	0.3432	0.883	0.9762	0.9936	0.9978	0.9991	0.9996
	17	2	1.25	0.3183	0.8615	0.9693	0.9913	0.9969	0.9987	0.9994
	12	2	1.75	0.2765	0.8194	0.9540	0.9857	0.9947	0.9977	0.9989
	11	2	2	0.2341	0.7820	0.9401	0.9804	0.9925	0.9967	0.9984
	8	2	3	0.1734	0.6909	0.8965	0.9614	0.9838	0.9925	0.9962
	7	2	3.5	0.1692	0.6687	0.8823	0.9541	0.9801	0.9905	0.9951

VII Illustration of Table and application of sampling plan

Assume that the life time distribution is Marshall-Olkin length biased lomax distribution with θ =3 and γ = 2 . Suppose that the experimenter is interested in constituting that the correct unknown average lifetime is at least 1000 hours. Suppose that it is desired to stop the the experiment at t=1000 hours. So if consumers risk is is set to be $1-p^* = .25$ then from Table 2 sampling plan is $(n=11, c=2, t/\beta_0 = 1)$. ie; if during 1000 hours,not more than 2 failures out of 11 are observed then the experimenter can say with confidence limit 0.75 that the average life is at least 1000 hours. If we use Poisson approximation to binomial the corresponding value is n=13.

For the sampling plan $(n = 11, c = 2, t/\beta_0 = 1)$ with the consumer risk 0.25 under the Marshall-olkin length biased lomax distribution the operating characteristic values from Table 4 are,

$\frac{\beta}{\beta_0}$	2	4	6	8	10	12	14
L(p)	0.7820	0.9804	0.9967	0.9992	0.9997	0.9999	0.9999

This shows that when $\frac{\beta}{\beta_0} = 4$ producers risk is 0.02 and when $\frac{\beta}{\beta_0} = 12$ it is negligible.

Application: Consider the following ordered failure times of the product, gathered from a software development project. The data set regarding software reliability was presented by Wood (1996), analyzed via the acceptance sampling viewpoint by Rosaiah and Kantam(2005), Rao, Ghitany and Kantam (2008,2009), Lio, Tsai and Wu (2010) ,Rao and Kantam (2010) and Jose et al. (2018). This data consists of a sample of 18 observations.

Data Set : 519, 968, 1430, 1893, 2490, 3058, 3625, 4422, 5218, 5823, 6539, 7083, 7487,7846,8205,8564,8923,9282.

Let the specified average lifetime be 1000 hours, and let the testing time be 750 hours. This leads to the ratio t/β_0 = 0.75 with corresponding n = 18 and c = 1 for p* = 0.95. Therefore, the sampling plan for the above sample data is (n = 18, c = 1, t/β_0 = 0.75). We accept the lot only if the number of failures not after 750 hours is less than or equal to one. However, the confidence level is assured by the sampling plan only if the given lifetimes follow the MOLBL distribution.

In the above sample, there is only one failure at 519 hours before termination t = 750 hours. Hence, we accept the product. Here, we notice that termination time t is smaller when using this sampling plan compaired with the sampling plan suggested by Krishna et al. (2013) and Rosaiah et al.(2005). So, when we consider this present sampling plan, we can reduce the cost and the experimental time.

VIII Conclusion

In this paper, We consider the Marshal Olkin length biased Lomax distribution and discuss some properties. We fit the distribution to a real-life data set. Also, a reliability test plan is developed when the lifetimes of the items follow the Marshal Olkin length biased Lomax distribution. The results are illustrated by a numerical example. Table 2 provides the minimum sample size for the specified ratio t/β_0 , confidence level p*, acceptance number c for θ=3 and γ = 2 using binomial approximation. Table 3 provides the minimum sample size using Poisson approximation. Table 4 provides values of the operating characteristic function of the sampling plan (n, c, t/β_0) for given confidence level p* with θ =3 , γ = 2 and acceptance number c = 2. The new test plans deliver positive results in comparison with the existing works and can be activated for making optimal decisions.

References

1. Abdullah, M.A. ; Abdullah, H.A., (2010).Estimation of Lomax parameters based on Generalized Probability Weighted Moment. *JKAU*, Sci (22), 171-184.
2. Afaq Ahmad; S.P Ahmad; A. Ahmed, (2016). Length-Biased Weighted Lomax Distribution: Statistical Properties and Application. *Pak.j.stat.oper.res.*, Vol.XII No.2 245-255.
3. E. Krishna; K. K. Jose ; M. M. Ristic, (2013). Applications of Marshall Olkin Frachet distribution. *Communications in Statistics. Simulation and Computation,* 42, 76-89.
4. E. Krishna; K. K. Jose; T. Alice ; M. M. Ristic, (2013). Marshall Olkin Frachet distribution. *Communications in Statistics. Theory and Methods,* 42 , 4091-4107.
5. Fisher, R.A , (1934). The effects of methods of ascertainment upon the estimation of frequencies. *Ann. Eugenics*, 6, 13-25.
6. G. S. Rao ; R. R. L. Kantam, (2010). Acceptance sampling plans from truncated life tests based on the log-logistic distribution for percentiles. *Econ. Qual. Control,* 25 , 153-167.
7. G. S. Rao; M. E. Ghitany; R. R. L. Kantam, (2008). Acceptance sampling plans for Marshall Olkin extended Lomax distribution. *Int. J. Appl. Math.*, 21 , 315-325.
8. G. S. Rao; M. E. Ghitany ; R. R. L. Kantam,(2009). Marshall Olkin extended Lomax distribution: An economic reliability test plan. *Int. J. Appl. Math.*, 22 , 139-148.
9. G. S. Rao ; M. E. Ghitany ;R. R. L. Kantam, (2009). Reliability test plans for Marshall Olkin extended exponential distribution. *Appl. Math. Sci.*, 3 , 2745-2755.
10. Ghitany, M.E.; AL-Hussaini; E.K., AL-Jarallah, R.A., (2005). Marshall-Olkin extended Weibull distribution. And its application to censored data. *J. Appl. Statist.*, 32, 1025-1034.
11. Ghitany, M.E. ; Kotz, S., (2007). Reliability properties of extended linear failure-rate distributions. *Prob. Eng. Inf. Sci*., 21, 441-450.
12. Gui, W, (2013). Marshall-Olkin extended log- logistic distribution and its application in mimification processes. *Appl. Math. Sci*., 7, 3396-3947.
13. Gupta, R.C. ; Akman, H.O. (1995). On the reliability studies of a weighted inverse Gaussian model. *Journal of Statistical Planning and Inference*, 48, 69-83.
14. Jose, K.K., Lishamol Tomy and Sophia P. Thomas.(2018). On a Generalization of the Weibull Distribution and Its Application in Quality Control. *Stochastics and Quality Control*, **33(2)**, 113-124.
15. K. Jayakumar; M. Thomas, (2008). On a generalization of Marshall Olkin scheme and its application to Burr type XII distribution. *Statistical Papers* ,49 , 421-439.
16. K. K. Jose; S. R. Naik ; M. M. Ristic, (2010). Marshall Olkin q-Weibull distribution and maximin processes. *Statistical Papers* ,51 , 837-851.
17. K. Rosaiah ; R. R. L. Kantam, (2005). Acceptance sampling based on the inverse Rayleigh distribution. *Econ. Qual. Control* ,20 , 277-286.
18. Lee, E.T. and Wang, J.W., Statistical Methods for Survival Data Analysis, 3 edn, John Willey, New York , 2003.
19. Lomax, H.S, (1954). Business Failures, Another example of the analysis of Failure data. *JASA*, (49), 847-852.
20. M. E. Ghitany; F. A. Al-Awadhi ; L. A. Alkhalfan, (2007). Marshall Olkin extended Lomax distribution and its applications to censored data.*Communications in Statistics. Theory and Methods* , 36 , 1855-1856.
21. Marshall, A.W.; Olkin, I, (1997).A new method for adding a parameter to a family of distributions with application to the exponential and Weibull families. *Biometrika*, 84 (3), 641-652.
22. Pogny, T.K.; Saboor, A; Provost, S., (2015). The Marshall-Olkin exponential Weibull distribution. *Hacettepe J. Math. Stat*., 44, 1579-1594.
23. Ristic, M.M. ; Jose, K.K.; Ancy, J, A(2007). Marshall-Olkin gamma distribution and minification process. *STARS: Stress Anxiety Res. Soc.* , 11, 107-117 .

24. T. Alice ; K. K. Jose, (2003). Marshall Olkin Pareto process. *Far East Journal of Theoretical Statistics* , 9 , 117-132.
25. T. Alice ; K. K. Jose, (2005). Marshall Olkin semi-Weibull minification processes. *In: Recent Advances in Statistical Theory and Applications*, Vol. I, STC Publications, 6-17.
26. Wood, (1996). Predicting software reliability. *IEEE Trans. Softw. Eng.* , 22 , 69-77.
27. Y. L. Lio; T. R. Tsai ; S. J. Wu, (2010). Acceptance sampling plans from truncated life tests based on the Birnbaum Saunders distribution for percentiles. *Comm. Statist. Simulation Comput*., 39, 119-136

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Operative Benchmarking Same Type Technical Objects

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Abstract

Objects, with the same identical varieties of attributes belong to the same-type. One of the main tasks of benchmarking is objective comparison and ranging of work efficiency. In modern representation, work efficiency is the integrated property consisting of profitability, reliability and safety. In general, the work efficiency is characterized by tens technical and economic indicators. For new objects reliability and safety of work guaranteed at relevant rules and guidelines. And the work efficiency is characterized by one of economic parameters. For example, on thermal power stations this is the specific consumption of equivalent fuel. If the of service life of the main equipment exceeds of normative value, the guaranteed term of compliance of the reliability and safety with the requirements is completed. Comparison and ranging of the same technical objects in these conditions without change of methodology leads to increase in risk of erroneous decisions and occurrence of system failures with inadmissible consequences. Decrease in risk of the erroneous decision reached by transition to calculation of an integrated parameter of work efficiency. At the same time, it turns out to be necessary to overcome the numerous difficulties caused by distinction of dimensions and scale of individual parameters, presence of interrelation and differences in the direction of change, multidimensionality and small number of realizations, bulkiness and laboriousness. An indispensable condition is the development of the automated systems of information and methodical support of the personnel and the Management of objects.

Keywords. Benchmarking, same type of technical object, work efficiency, integral parameter, profitability, reliability, safety.

I. Introduction

The opportunity of objective ranging of the same technical objects for their work efficiency defines one of the most important problems of the organization of operation, maintenance service and repair [1]. For example, the ranking of power units of thermal power stations (TPS) allows you to reduce the risk of the erroneous decisions when distributing the load, planning of diagnostics of a technical condition, conducting routine and major overhauls and requires the ability to compare objects, i.e. conduct their benchmarking [2].

Today, as many years ago, comparison of the same type objects spent mainly on one of technical and economic indicators (TEI). For example, on TPS - this is a specific consumption of equivalent fuel. But, the specific consumption of equivalent fuel partially characterizes only one of components of work efficiency - economic efficiency. Accepted, that two others components - reliability of work and safety of service guaranteed when the relevant Rules and guidelines are execute. And this guarantee operates, as always, only during normative service life. The consequences from "neglect" to reliability and safety, when exceeding the service life of objects of normative value, shown in system failures, accompanied by the death and injury the personnel, infringement of ecology, greater material inputs [3].

There are two ways of maintenance of work efficiency of technical objects, which service life exceeds calculated are possible. The first and simplest - to replace object on modern and effective. But input of modern objects of the big capacity demands scale concentration of financial and other resources, otherwise it is beyond the strength even to many economically developed countries. The second way is modernization. Though modernization demands in (3-4) times of less cost, than replacement, the absolute value of these costs is still extremely great.

It is supposed to "legitimize" the third method, which provides for partial modernization of objects (replacement of auxiliary equipment and devices) and rapid assessment not only profitability of object, but also its reliability and safety. It also requires the comprehension that process of ageing (technical and moral) cannot stopped. The most important task in this is development of computer technology for operational (average monthly) estimation of integrated parameters of work efficiency and methodical support of a Management by way of increase of work efficiency of object [4].

Multidimensional character of the numerous indicators characterizing work efficiency of technical objects, determines multidimensionality of difficulties of objective comparison and ranging of work efficiency of objects (difficulty in benchmarking a technical object). Some of these difficulties overcome, the some people are overcome, and up to the some people still «do not reached the hands». The method of calculation integral indicator of operative work efficiency among of overcome difficulties. Recommendations by calculation of an integral indicator is based on summation TEI and calculation of their average arithmetic value [5, 6] in view of distinction of their units of measure, scale, the importance, directivity and other factors.

At the same time, it known, that as TEI can significantly differ from each other, their arithmetic mean value not always objectively reflects character of distribution composed [7]. Consequently, the risk of the erroneous decision when comparing and ranking such integral indicators of operative work efficiency is unacceptable. In these conditions, recommended to use not the arithmetic mean, but the geometric mean of the normalized TEI values. This recommendation has found the greatest reflection in a method of curves of Harrington's desirability [8]. The method has found wide application in psychology, ecologies, economy, and medicine and in other fields. The estimation of character of a divergence of integrated indicators concerns to number of difficulties (calculated as average geometrical mean) at classification of statistical data on versions of attributes.

II. Estimation integrated indicators of operative work efficiency.

Suppose, that in a considered interval of time tj (possible intervals: shift, day, month, etc.) operative work efficiency *n* the same type objects is characterized *m* TEI. According, on these data the integrated indicators [$Ip(t_j)$] operative work efficiency as an mean geometrical [$G^*(t_j)$] $m \cdot n$ relative deviations TEI from normative (factory) value $\delta P_{i,k}(t_j)$ can be calculated by the formula

$$Ip^*(t_j) = G^*(t_j) = \sqrt[m]{\prod_{i=1}^{n}\prod_{k=1}^{m}[1-\delta P_{i,k}(t_j)]} \quad (1)$$

According, on the same data as a result of their classification on *n* to objects, we calculate the integrated indicators of an operative work efficiency of everyone i-th object $[Ip^*_{o,i}]$ and the integrated indicators characterizing the operative importance of everyone TEI of objects $Ip_{n,m}(t_j)$ by the formula:

$$Ip^*_{o,i}(t_j) = G^*_{o,i}(t_j) = \sqrt[m]{\prod_{k=1}^{m}[1-\delta P_{i,k}(t_j)]} \quad (2)$$

$$Ip^*_{п,к}(t_j) = G^*_{п,к}(t_j) = \sqrt[n]{\prod_{i=1}^{n}[1-\delta P_{i,k}(t_j)]} \quad (3)$$

The relative deviations of the TEI are calculated by the formulas:

$$\left.\begin{aligned} \delta P_{i,k}(t_j) &= \frac{\overline{P_k^f} - P_{i,k}}{\overline{P_k^f} - \underline{P_k^f}} \quad \text{at } A=0 \\ \delta P_{i,k}(t_j) &= \frac{P_{i,k} - \underline{P_k^f}}{\overline{P_k^f} - \underline{P_k^f}} \quad \text{at } A=1 \end{aligned}\right\} \tag{4}$$

where: $\overline{P_k^f}$ and $\underline{P_k^f}$ - accordingly the upper and lower boundary values of the fiducial (f) interval of possible values of realizations of the k-th TEI ($P_{i,k}$); A - the indicator of the direction of change of work efficiency.

It is necessary to note, that factors $[1-\delta P_{i,k}(t_j)]$ with i=1,n and k=1,m characterize value of a residual resource. The values $G^*(t_j)$ and $G^*_{o,i}(t_j)$ - are the geometric mean of the residual resource of the average and concrete object. $G^*_{P,k}(t_j)$ - the importance concrete TEI. They have a visual physical sense.

Since the realizations TEI in an interval t_j are a result of influence of some attributes and their varieties (serial numbers of objects and them TEI, loading, a season, service life, etc. attributes), they considered as random variables. Consequently, casual character of change accompanies also to integrated parameters $Ip^*(t_j)$, $Ip^*_{o,i}(t_j)$, $Ip^*_{P,k}(t_j)$. In these conditions observable between objects of a divergence, also as well as the divergence in estimations of importance TEI may be insignificant and classification by a given type of sign is inexpedient.

III. Results of calculating of boundary values the fiducial interval of integral indicators

The method and algorithm for calculation the fiducial distributions of the indicators, characterizing the dispersion *r* of random variables with uniform distribution in an interval [0, 1] given in [9]. The method and algorithm of calculation the fiducial distribution of the geometric mean of these random variables $G^*_r(\xi)$ is practically similar, c that difference, that the parameter $G^*_r(\xi)$ is calculated under by the formula $G^*_r(\xi) = \sqrt[r]{\prod_{i=1}^{r} \xi_i}$. The method is reduced to multiple modelling of possible realizations $G^*_r(\xi)$, ranging of these realizations in ascending order and estimations of probability of occurrence of everyone *(i)* realizations under by the formula $F^*_i[G^*_r(\xi)] = i/N$, where N - number of modeled realizations $G^*_r(\xi)$.

The analysis of results of modelling allows to conclude:

3.1. As one would expect, with increase *r* the spread of realizations $G^*_r(\xi)$ decreases, in other words, the sizes of the fiducial interval decrease. The following relation pays attention: if $r_1 < r_2$, then $\underline{G^f_{r1}(\xi_{\alpha 2})} < \underline{G^f_{r2}(\xi_{\alpha 2})}$, and $\overline{G^f_{r1}(\xi_{\alpha 2})} > \overline{G^f_{r2}(\xi_{\alpha 2})}$, where $\alpha_1+\alpha_2=\alpha$; α - a significance level of the fiducial interval; $\overline{G^f}$ and $\underline{G^f}$ - respectively, the upper and lower boundary values of the fiducial interval (f). Let us remind that *fiducial intervals* are intervals of change of concrete possible realizations of multivariate random variables, which law of distribution not known. And *confidential intervals* are intervals of theoretical random variables, which law of distribution is known;

3.2. Distributions $F^*[G_r^*(\xi)]$ are asymmetric, which confirmed by the histograms of the distributions. Thus, asymmetry of the geometric mean value exceeds asymmetry of the arithmetic mean value;

3.3. Laws of change of boundary values of the fiducial interval $[\underline{G_r^*(\xi)};\overline{G_r^*(\xi)}]_\alpha$ with the significance level α are "nonlinear";

3.4. Considering, that tabulated values r far not always correspond to actual number of versions of attributes, the curves of changes in the boundary values of the fiducial interval of realizations - $G_r^*(\xi)$ are approximated as a function of the number of realizations t;

Conclusion

1. The estimation of an integral indicator of an operative work efficiency of technical objects, of course, is the significant result, allowing comparing and ranging objects. Thus casual character of these indicators and an opportunity of their casual distinction not considered;
2. Carried out researches allowed us to obtain receive statistical functions of distribution of average geometrical mean of random variables. These distributions called fiducial;
3. On these functions of distribution boundary values of the fiducial interval with a given significance level established;
4. Approximation of laws of change of critical values of an integral indicator in function of number of realizations allowed implement "express method" of calculation for any number of realizations.

References

[1] Voropay N.I., Kovalev G.F. et al. The concept of ensuring reliability in the electric power industry - M.: LLC ID "Energy", 2013, 304 p.
[2] Fontsova L.V. Benchmarking as the tool of an estimation of a system effectiveness of the internal control. Ed. Prospect 2016, 135 p.
[3] Dyakov A.F., Isamuhammedov Y.S., Molodyuk V.V. Problems and ways to improve the reliability of the UES of Russia. // Methodical questions of research of reliability of large energy systems: Issue 64. Reliability of energy systems: achievements, problems, prospects // Res. ed. N.I. Voropai - ISEM SB RAS, 2014, p. 8-16.
[4] Farhadzadeh E.M., Muradaliyev A.Z., Rafieva T.K., Rustamova A.A. Maintenance of reliability of methodical support of a management of objects EPS. M:, Electricity, No.2, 2020, p. 4-9
[5] Kiryanov B.F., Kiryanov D.V. Theory of building integral indicators of the quality of systems based on linear mathematical models. M: Modern high technology. No. 4, 2008, p. 73-74
[6] Vasilieva L.V. Analysis of methodological approaches to the construction of integrated economic indicators. - M, "Economic research and development", 2017, No. 12. Access: http://edrj.ru/article/18-12-17/
[7] Falin G.N., Falin A.L. Inequality for the middle. - M., Publishing House "First of September", 2006, No. 10, p. 25-36
[8] Zaznobina N.I. Assessment of the environmental situation in a large industrial center using the generalized Harrington desirability function. // Bulletin of the University of Nizhny Novgorod, 2007, No. 2, pp. 115-118
[9] Farhadzade E.M., Muradaliev A.Z., Farzaliev Yu.Z. Sampling distribution of a continuous random variable. Kiev, Electronic Modeling No. 6, 2015, p.69-82

The Poisson-Shukla Distribution and its Applications

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Abstract

In this paper, a study on Poisson-Shukla distribution (PSD), a Poisson mixture of Shukla distribution introduced by Shukla and Shanker (2019), has been carried out. The expression for r *th factorial moment about origin has been derived. The expressions for its mean and variance have been given. Maximum likelihood estimation for estimating the parameters have been discussed. The goodness of fit of the proposed distribution has been explained with two count datasets and its fit was found quite satisfactory over Poisson distribution (PD), Poisson-Lindley distribution (PLD), Poisson-weighted Lindley distribution (P-WLD), Generalized Poisson Lindley distribution (GPLD) and Negative Binomial distribution (NBD).*

Keywords: Shukla distribution, Moments based measures, Estimation of parameter, Goodness of fit

1. Introduction

The probability density function (pdf) and the cumulative distribution function (cdf) of Shukla distribution proposed by Shukla and Shanker (2019) are given by

$$f\left(x;\theta,\alpha\right)=\frac{\theta^{\alpha+1}}{\theta^{\alpha+1}+\Gamma\left(\alpha+1\right)}\left(\theta+x^{\alpha}\right)e^{-\theta x};x>0,\theta>0,\alpha\geq 0 \qquad (1.1)$$

$$F\left(x;\theta,\alpha\right)=1-\frac{\theta^{\alpha}\left(\theta+x^{\alpha}\right)e^{-\theta x}+\alpha\,\Gamma\left(\alpha,\theta x\right)}{\theta^{\alpha+1}+\Gamma\left(\alpha+1\right)};x>0,\theta>0,\alpha\geq 0 \qquad (1.2)$$

This distribution, being a convex combination of exponential (θ) distribution and gamma (α,θ) distribution, has been proposed for modeling lifetime data form engineering and biomedical science. The statistical properties of Shukla distribution including its shapes, moments, skewness, kurtosis, hazard rate function, mean residual life function, stochastic ordering, mean deviations along with estimation of parameters using Maximum likelihood estimation are given in Shukla and Shanker (2019). The main motivation behind this new discrete distribution arises from the fact that it has been observed by Shukla and Shanker (2019) that Shukla distribution provides a better fit for modeling lifetime data than exponential distribution, Lindley distribution of Lindley (1958), gamma distribution, generalized Lindley distribution of Zakerzadeh and Dolati (2009) and weighted Lindley distribution given by Ghitanay et al (2011), and it is expected that the Poisson mixture Shukla distribution would provide a better fit for modeling count data than the Poisson mixture of these distributions namely Poisson-Lindley distribution (PLD) of Sankaran (1970), Negative binomial distribution (NBD) of Greenwood and Yule (1920), generalized Poisson-Lindley distribution (GPLD) of Mahmoudi and Zakerzadeh (2010) and Poisson-Weighted Lindley distribution (P-WLD) introduced by El-Monsef and Sohsah (2014) and studied by Shanker and Shukla (2019). The probability mass function of PLD, NBD, GPLD and P-WLD are presented in the following table

Table 1: The pmf of Poisson Lindley distribution (PLD), Negative binomial distribution (NBD) , Generalized Poisson Lindley distribution (GPLD) and Poisson weighted Lindley distribution (PLWD)

Distributions	Probability mass function (pmf)	Introducer (Year)
PLD	$P_1(x;\theta)=\dfrac{\theta^2(x+\theta+2)}{(\theta+1)^{x+3}}\quad ;x=0,1,2,...,\theta>0$	Sankaran (1970)
NBD	$P(X=x)=(x+1)\left(\dfrac{\theta}{\theta+1}\right)^p\left(\dfrac{1}{\theta+1}\right)^x;x=0,1,2,...$	Greenwood and Yule (1920)
GPLD	$P(x;\theta,\alpha)=\dfrac{\theta^3\left\{\alpha x^2+(\theta+3\alpha+1)x+(\theta^2+3\theta+2\alpha+2)\right\}}{(\theta^2+\theta+2\alpha)(\theta+1)^{x+3}}$	Mahmoudi, and Zakerzadeh (2010)
P-WLD	$P(x;\theta,\alpha)=\dfrac{\Gamma(x+\alpha)}{\Gamma(x+1)\Gamma(\alpha)}\dfrac{\theta^{\alpha+1}}{(\theta+\alpha)}\dfrac{x+\theta+\alpha+1}{(\theta+1)^{x+\alpha+1}};\theta>0,\alpha>0$	El-Monsef and Sohsah (2014)

In the present paper Poisson-Shukla distribution (PSD) by compounding Poisson distribution with Shukla distribution has been proposed. The paper is divided into six sections. Second section deals with derivation of the pmf of PSD and shapes of PSD for varying values of parameters. Moments and moments based measures of the proposed distribution have been discussed in section 3. Sections four and five deal the discussion of the estimation of parameters using maximum likelihood estimation and the goodness of fit of the proposed distribution, respectively. Finally, conclusions of the paper have been given in section six.

2. Poisson-Shukla Distribution

Assuming the parameter λ of the Poisson distribution following Shukla distribution (1.1), the Poisson mixture of Shukla distribution can be obtained as

$$P_1(x,\theta,\alpha)=\int_0^\infty\frac{e^{-\lambda}\lambda^x}{x!}\frac{\theta^{\alpha+1}}{\theta^{\alpha+1}+\Gamma(\alpha+1)}\left(\theta+\lambda^\alpha\right)e^{-\theta\lambda}d\lambda \tag{2.1}$$

$$=\frac{\theta^{\alpha+1}}{\left\{\theta^{\alpha+1}+\Gamma(\alpha+1)\right\}\Gamma(x+1)}\int_0^\infty e^{-(\theta+1)\lambda}\left(\theta\lambda^x+\lambda^{x+\alpha}\right)d\lambda$$

$$=\left(\frac{\theta}{\theta+1}\right)^{\alpha+1}\left(\frac{1}{\theta+1}\right)^x\frac{1}{\Gamma(x+1)}\frac{\theta(\theta+1)^\alpha\Gamma(x+1)+\Gamma(x+\alpha+1)}{\theta^{\alpha+1}+\Gamma(\alpha+1)}\;;x>0,(\theta,\alpha,\beta)>0 \tag{2.2}$$

We name this distribution "Poisson-Shukla distribution (PSD)". The behavior of the pmf of PSD for different values of parameter is shown in the figure 1

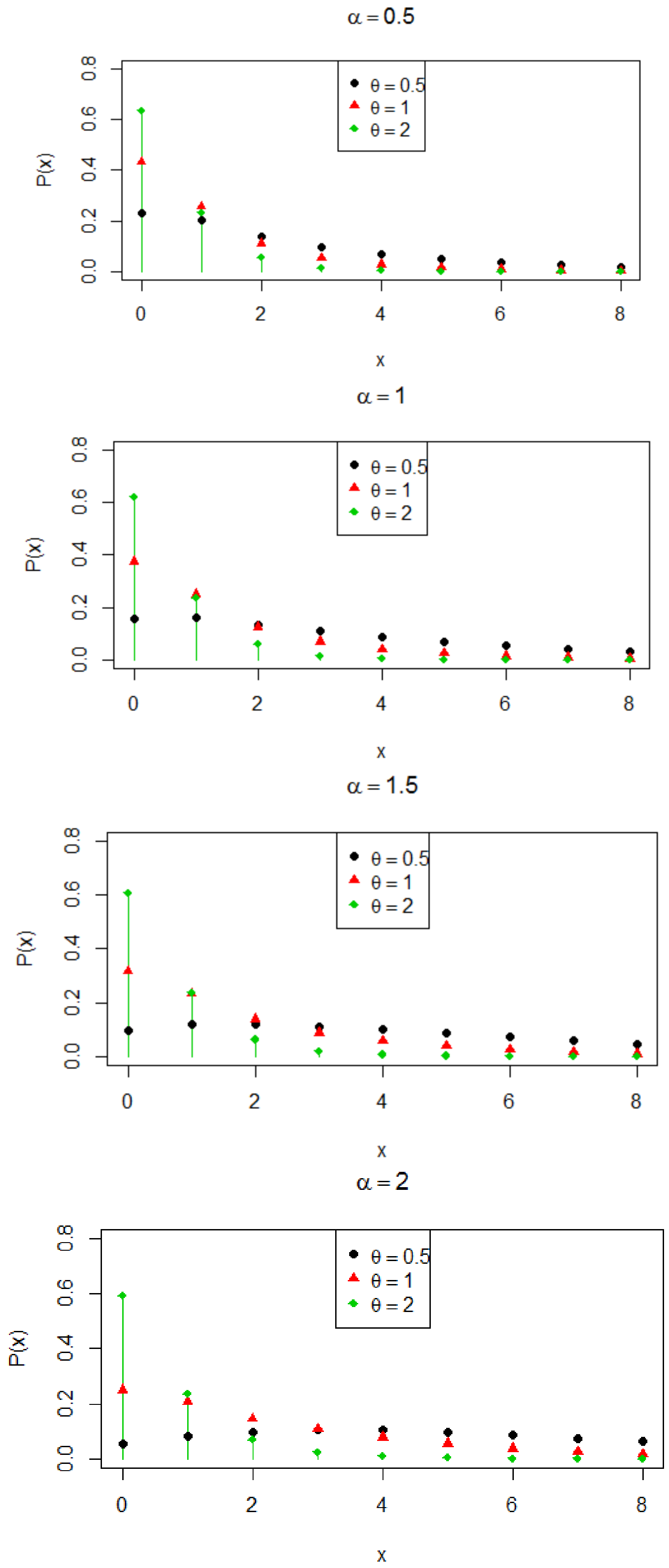

Fig.1: Behavior of the pmf of PSD for different values of the parameters θ and α

3. Moments

The r[th] factorial moment about origin of PSD (2.2) can be obtained as

$$\mu_{(r)}' = E\left[E\left(X^{(r)} \mid \lambda\right)\right], \qquad \text{where} \quad X^{(r)} = X\left(X-1\right)\left(X-2\right)...\left(X-r+1\right)$$

Using (2.1), the r th factorial moment about origin of PSD (2.2) can be obtained as

$$\mu_{(r)}' = E\left[E\left(X^{(r)} \mid \lambda\right)\right] = \frac{\theta^{\alpha+1}}{\theta^{\alpha+1}+\Gamma(\alpha+1)}\int_0^\infty \left[\sum_{x=0}^\infty x^{(r)} \frac{e^{-\lambda}\lambda^x}{x!}\right]\left(\theta+\lambda^\alpha\right)e^{-\theta\lambda}d\lambda$$

$$= \frac{\theta^{\alpha+1}}{\theta^{\alpha+1}+\Gamma(\alpha+1)}\int_0^\infty \lambda^r \left[\sum_{x=r}^\infty \frac{e^{-\lambda}\lambda^{x-r}}{(x-r)!}\right]\left(\theta+\lambda^\alpha\right)e^{-\theta\lambda}d\lambda$$

Taking $x+r$ in place of x within the bracket, we get

$$\mu_{(r)}' = \frac{\theta^{\alpha+1}}{\theta^{\alpha+1}+\Gamma(\alpha+1)}\int_0^\infty \lambda^r \left[\sum_{x=0}^\infty \frac{e^{-\lambda}\lambda^{x}}{x!}\right]\left(\theta+\lambda^\alpha\right)e^{-\theta\lambda}d\lambda$$

The expression within the bracket is clearly unity and hence we have

$$\mu_{(r)}' = \frac{\theta^{\alpha+1}}{\theta^{\alpha+1}+\Gamma(\alpha+1)}\int_0^\infty \lambda^r \left(\theta+\lambda^\alpha\right)e^{-\theta\lambda}d\lambda$$

Using gamma integral and a little algebraic simplification, we get finally, a general expression for the r th factorial moment about origin of PSD (2.2) as

$$\mu_{(r)}' = \frac{\theta^{\alpha+1}\Gamma\left(r+1\right)+\Gamma\left(\alpha+r+1\right)}{\theta^r\left\{\theta^{\alpha+1}+\Gamma\left(\alpha+1\right)\right\}}\ ; r=1,2,3,.... \tag{3.1}$$

The first four factorial moments about origin can thus be obtained as

$$\mu_{(1)}' = \frac{\theta^{\alpha+1}+\Gamma\left(\alpha+2\right)}{\theta\left\{\theta^{\alpha+1}+\Gamma\left(\alpha+1\right)\right\}}$$

$$\mu_{(2)}' = \frac{2\theta^{\alpha+1}+\Gamma\left(\alpha+3\right)}{\theta^2\left\{\theta^{\alpha+1}+\Gamma\left(\alpha+1\right)\right\}}$$

$$\mu_{(3)}' = \frac{6\theta^{\alpha+1}+\Gamma\left(\alpha+4\right)}{\theta^3\left\{\theta^{\alpha+1}+\Gamma\left(\alpha+1\right)\right\}}$$

$$\mu_{(4)}' = \frac{24\theta^{\alpha+1}+\Gamma\left(\alpha+5\right)}{\theta^4\left\{\theta^{\alpha+1}+\Gamma\left(\alpha+1\right)\right\}}.$$

Using the relationship between moments about origin and factorial moments about origin, the first two moments about origin of PSD can be expressed as

$$\mu_1' = \mu_{(1)}' = \frac{\theta^{\alpha+1}+\Gamma\left(\alpha+2\right)}{\theta\left\{\theta^{\alpha+1}+\Gamma\left(\alpha+1\right)\right\}}$$

$$\mu_2' = \mu_{(2)}' + \mu_{(1)}' = \frac{\theta^{\alpha+2}+2\theta^{\alpha+1}+\theta\Gamma\left(\alpha+2\right)+\Gamma\left(\alpha+3\right)}{\theta^2\left\{\theta^{\alpha+1}+\Gamma\left(\alpha+1\right)\right\}}$$

Similarly, third and fourth moments about origin can be obtained. The variance of PSD can thus be obtained as

$$\mu_2 = \mu_2' - \left(\mu_1\right)^2 = \left(\frac{\theta^{\alpha+2}+2\theta^{\alpha+1}+\theta\Gamma\left(\alpha+2\right)+\Gamma\left(\alpha+3\right)}{\theta^2\left\{\theta^{\alpha+1}+\Gamma\left(\alpha+1\right)\right\}}\right) - \left(\frac{\theta^{\alpha+1}+\Gamma\left(\alpha+2\right)}{\theta\left\{\theta^{\alpha+1}+\Gamma\left(\alpha+1\right)\right\}}\right).$$

The expressions for third and fourth moments about the mean are lengthy and hence are not being given. However, if they are required, can be obtained easily.

4. Maximum Likelihood Estimate (mle)

Let $(x_1, x_2, ..., x_n)$ be a random sample of size n from the PSD (2.2) and let f_x be the observed frequency in the sample corresponding to $X = x\ (x = 1, 2, 3, ..., k)$ such that $\sum_{x=1}^{k} f_x = n$, where k is the largest observed value having non-zero frequency. The likelihood function L of the PSD (2.2) is given by

$$L = \left(\frac{\theta}{\theta+1}\right)^{n(\alpha+1)} \left(\frac{1}{\theta^{\alpha+1}+\Gamma(\alpha+1)}\right)^{n} \frac{1}{(\theta+1)^{\sum_{x=1}^{k} x f_x}} \prod_{x=1}^{k} \left[\frac{\theta(\theta+1)^{\alpha}\Gamma(x+1)+\Gamma(x+\alpha+1)}{\Gamma(x+1)}\right]^{f_x}.$$

The log- likelihood function is thus obtained as

$$\log L = n(\alpha+1)\log\left(\frac{\theta}{\theta+1}\right) - n\log\left[\theta^{\alpha+1}+\Gamma(\alpha+1)\right] - n\bar{x}\log(\theta+1)$$
$$+\sum_{x=1}^{k} f_x\left[\log\left\{\theta(\theta+1)^{\alpha}\Gamma(x+1)+\Gamma(x+\alpha+1)-\log\Gamma(x+1)\right\}\right],$$

where $\bar{x}$ is the sample mean.

The maximum likelihood estimates (MLE's $(\hat{\theta}, \hat{\alpha})$ of the parameter (θ, α) of PSD (2.2) are the solutions of the following log-likelihood equations

$$\frac{\partial \log L}{\partial \theta} = \frac{n(\alpha+1)}{\theta(\theta+1)} - \frac{n(\alpha+1)\theta^{\alpha}}{\theta^{\alpha+1}+\Gamma(\alpha+1)} - \frac{n\bar{x}}{(\theta+1)} + \sum_{x=1}^{k} \frac{\left[(\theta\alpha+1)(\theta+1)^{\alpha}\Gamma(x+1)\right] f_x}{\theta(\theta+1)^{\alpha}\Gamma(x+1)+\Gamma(x+\alpha+1)} = 0$$

$$\frac{\partial \log L}{\partial \alpha} = n\log\left(\frac{\theta}{\theta+1}\right) - \frac{n\theta^{\alpha+1}\log\theta+\psi(\alpha+1)}{\theta^{\alpha+1}+\Gamma(\alpha+1)} + \sum_{x=1}^{k} \frac{\left[\theta(\theta+1)^{\alpha}\log(\theta+1)\Gamma(x+1)\right] f_x}{\left[\theta(\theta+1)^{\alpha}\Gamma(x+1)+\Gamma(x+\alpha+1)\right]} = 0,$$

where $\psi(\alpha) = \frac{d}{d\alpha}\ln\Gamma(\alpha)$ is the digamma function. These two log likelihood equations do not seem to be solved directly because they do not have closed forms. Therefore, to find the maximum likelihood estimates of parameters an iterative method such as Fisher Scoring method, Bisection method, Regula Falsi method or Newton-Raphson method can be used. In this paper Newton-Raphson method has been used using R-software.

5. Goodness of fit

The PSD has been fitted to a number of datasets to test its goodness of fit along with PD, PLD, NBD, GPLD and P-WLD., because PLD, NBD, GPLD and P-WLD are always over-dispersed and hence their comparison regarding goodness of fit is justifiable. The maximum likelihood estimate has been used to fit PD, PLD, NBD, GPLD, P-WLD and PSD for two examples of observed count datasets. The first dataset in table 2 is due to the data regarding the number of European red mites on apple leaves, available in Bliss (1953), the second dataset in table 3 is the Mammalian Cytogenetic dosimetry Lesions in Rabbit Lymphoblast induced by streponigrin (NSC-45383), available in Catcheside et al (1946).

Table 2: Observed and Expected number of European red mites on Apple leaves, available in Catcheside et al (1946)

Number of European red mites per leaf	Observed frequency	Expected frequency					
		PD	PLD	NBD	GPLD	P-WLD	PSD
0	70	47.6	67.2	69.5	69.8	69.8	71.3
1	38	54.6	38.9	37.6	36.7	36.8	35.0
2	17	31.3	21.2	20.1	20.1	20.1	19.5
3	10	11.9	11.1	10.7	10.9	10.9	11.1
4	9	3.4	5.7	5.7	5.8	5.8	6.2
5	3	0.8	2.8	3.0	3.1	3.0	3.4
6	2	0.2	1.4	1.6	1.6	1.6	1.7
7	1	0.1	0.9	0.9	0.8	0.8	0.8
8	0	0.1	0.8	0.9	1.2	1.2	1.0
Total	150	150.0	150.0	150.0	150.0	150.0	150.0
ML estimate		$\hat{\theta}=1.14666$	$\hat{\theta}=1.26010$	$\hat{\alpha}=1.02459$ $\hat{p}=0.52811$	$\hat{\theta}=1.09620$ $\hat{\alpha}=0.78005$	$\hat{\theta}=1.09141$ $\hat{\alpha}=0.82194$	$\hat{\theta}=1.8444$ $\hat{\alpha}=3.1231$
Standard $\hat{\theta}$ Errors $\hat{\alpha}$		0.08743	0.11390	0.42097 0.40136	0.25400 0.31550	0.26231 0.25230	0.6409 2.0978
χ^2		26.50	2.49	2.91	2.43	2.41	2.09
d.f		2	4	3	3	3	3
p-value		0.0000	0.5595	0.4057	0.4880	0.4917	0.5539
$-2\log L$		485.61	445.02	469.68	444.62	425.35	444.09
AIC		487.61	447.02	447.02	448.62	429.35	448.09

Table 3: Mammalian Cytogenetic dosimetry Lesions in Rabbit Lymphoblast induced by streponigrin (NSC-45383), Exposure- 90 $\mu g \mid kg$

Class/Exposure ($\mu g \mid kg$)	Observed frequency	Expected frequency					
		PD	PLD	NBD	GPLD	P-WLD	PSD
0	155	127.8	158.3	155.1	155.3	155.9	158.9
1	83	109.0	77.2	80.6	80.1	80.0	76.8
2	33	46.5	35.9	36.7	36.9	36.7	35.5
3	14	13.2	16.1	15.9	16.0	15.9	16.0
4	11	2.8	7.1	6.7	6.7	6.7	7.1
5	3	0.5	3.1	2.8	2.8	2.7	3.2
6	1	0.2	2.3	2.2	2.2	2.1	2.5
Total	300	300.0	300.0	300.0	300.0	300.0	300.0
ML estimate		$\hat{\theta}=0.85333$	$\hat{\theta}=1.61761$	$\hat{\theta}=1.56009$ $\hat{\alpha}=1.33128$	$\hat{\theta}=1.80860$ $\hat{\alpha}=1.18743$	$\hat{\theta}=1.82011$ $\hat{\alpha}=1.16320$	$\hat{\theta}=1.4384$ $\hat{\alpha}=0.7011$
SE($\hat{\theta}$) SE ($\hat{\alpha}$)		0.05333	0.11327	0.41479 0.33752	0.40045 0.37007	0.41992 0.32483	0.3398 1.1181
χ^2		24.969	1.51	1.60	1.69	1.78	1.40
d.f		2	3	2	2	2	2
p-value		0.0000	0.6799	0.4488	0.42955	0.4106	0.4965
$-2\log L$		800.92	766.10	765.86	765.79	834.51	766.32
AIC		802.92	768.10	769.86	769.79	838.51	770.32

The fitted plots of the distributions for dataset in tables 2 and 3 have been presented in figure 3 and 4 respectively.

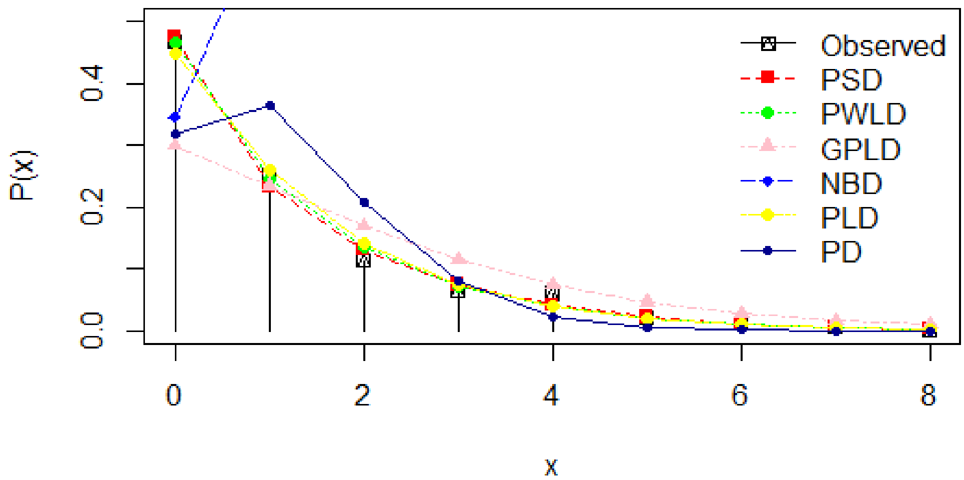

Fig.3: Fitted plot of distributions for datasets in table 2

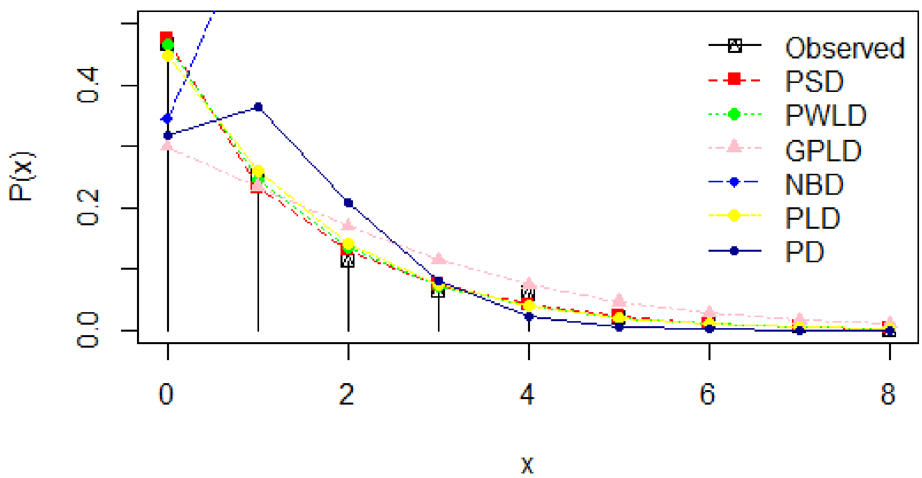

Fig. 4: Fitted plot of distributions for datasets in table 3

It is clear from the goodness of fit of the considered distribution in tables 3 and 4 and the fitted plots of the distributions that PSD is gives much closer fit.

6. Concluding Remarks

In this paper Poisson-Shukla distribution (PSD) has been proposed. The expression for the r th factorial moment about origin has been derived and hence its mean and variance have been obtained. The maximum likelihood estimation for estimating its parameter has been discussed. The goodness of fit of PSD over PD, PLD, NBD, GPLD and P-WLD have been discussed with two examples of observed real datasets. It is observed that PSD gives much better fit than PD, PLD, NBD, GPLD and P-WLD on all the datasets and hence it can be considered an important discrete distribution for modeling count data over these distributions.

References

1. Bliss, C.I. (1953): Fitting negative binomial distribution to biological data, *Biometrics*, 9, 177 – 200.
2. Catcheside, D.G., Lea, D.E. & Thoday, J.M. (1946). The production of chromosome structural changes in Tradescantia microspores in relation to dosage, intensity and temperature, *Journal of Genetics*, 47, 137-149.
3. El-Monsef, M.M.E. and Sohsah, N.M. (2014): Poisson-Weighted Lindley Distribution, *Jokull Journal*, 64(5), 192 – 202
4. Ghitany ,M.E., Alqallaf ,F., Al-Mutairi, D.K., Husain, H.A. (2011): A two-parameter weighted Lindley distribution and its applications to survival data, *Mathematics and Computers in simulation*, 81, 1190-1201.
5. Greenwood, M. and G.U., Yule, (1920) "An inquiry into the nature of frequency distributions representative of multiple happenings with particular reference to the multiple attacks of disease or of repeated accidents", *Journal of the Royal Statistical Society*, 83(2), 115 – 121
6. Lindley, D.V. (1958): Fiducial distributions and Bayes theorem, *Journal of the Royal Statistical Society*, 20 (1), 102- 107.
7. Mahmoudi, E. and Zakerzadeh, H. (2010): Generalized Poisson-Lindley distribution, *Communications in Statistics-Theory & Methods*, 39, 1785 – 1798.
8. Sankaran, M. (1970): The discrete Poisson-Lindley distribution, *Biometrics*, 26, 145- 149.
9. Shanker, R. and Shukla, K. K. (2019): On Poisson weighted Lindley distribution and its Applications *Journal of Scientific Research,* 11(1), 1-13.
10. Shukla, K.K. and Shanker, R (2019): Shukla distribution and its Application*, R.T.&A,*14(3), 46-55
11. Zakerzadeh, H. and Dolati, A. (2009): Generalized Lindley distribution, *Journal of Mathematical extension*, 3 (2), 13 – 25.

Critical Infrastructure: the probability and duration of national and regional power outages

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Abstract

Power generation and distribution systems are part of a nation's critical infrastructure. Power losses or outages are random with a learning trend of declining size with increasing experience or risk exposure, with the largest outages being rare events of low probability. Data have been collected for power losses and outage duration affecting critical infrastructure for a wide range of events in Belgium, Canada, Eire, France, Sweden, New Zealand and USA. A new correlation has been obtained for the probability of large regional power losses for outage scales up to nearly 50,000 MW(e) for events without additional infrastructure damage that have been generally fully restored in less than 24 hours. For more severe events, including damage due to natural hazards (floods, fire, ice storms, hurricanes etc.), the observed variation in the duration of the outage up to more than 500 hours depends on the degree of difficulty. The irreparable fraction data range (the "tail" of the distribution) indicates that the chance of remaining unrestored is small but finite, even after several hundred hours. Therefore, explicit expressions have been given and validated for both the probability and duration for the full range from "normal" large power loss out to extended outages for rare and more "severe" events with greater access and repair difficulty.

Keywords: power, outages, critical infrastructure, severe events, prediction

I. Introduction

Power generation and distribution systems are part of a nation's critical infrastructure. Everyone everywhere on losing their electricity supply want to now how long the power will be "out". The chance of a large blackout, power loss or supply interruption is key to planning adequate supply margins, undertaking emergency response and protecting other critical infrastructure [1]. We are interested in the prediction of the probability of a large rare outage or power loss event and its duration. There is a gap in the knowledge between overall reliability studies of electric grid reliability and concerns and the impact of rare and record events and natural disasters (e.g. recent Hurricanes Katrina, Sandy, Harvey and Florence in USA) whose extensive flooding and damage caused multiple power outages and delayed restoration. It is such major disasters that are of concern for infrastructure fragility, and we need to estimate their probability or chance of occurrence and the timescales for restoration [2].

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II. Method

I. Power loss and restoration data

The national power loss data is derived from Kearsley [3] for large blackouts in France, Sweden and Belgium, being for a range of 28,000, 11,400 and 2,400 MW(e) initial losses, respectively.

For the entire USA for the period 1984-2000, the IRGC report [4] gives a plot of the exceedence probability of an outage, P, versus the size, Q, in MW(e). Basically, a sample of power loss or outages were observed and counted; and similar plots by sub-region have also been presented and fitted using empirical binomial, Weibull and lognormal distributions [5]. These distributions are of course heavily weighted by the many "normal" or everyday outages, not rare catastrophic events. Since we wish to predict the low probability "tail" of the distribution such standard statistical methods are not applicable, as clearly evident in their Figure S-28. In addition, Murphy et al [5] also looked to see if outages were linked, and concluded: "…that the largest correlated failure instances were caused by extreme weather". This observation is precisely what we should expect given the large geographic scale and impact of natural hazards (storms, hurricanes, floods, ice storms and wildfires) and the consequent universal power restoration characteristics [2]. Large NH events do not respect or recognize human-drawn boundaries or arbitrary grid distribution regions, and cause event-related damage and destruction over wide swathes.

The original IRGC data were from the NERC database, and were shown as a graph with dots and lines on a log-log plot; but because of the unavailability of the data[1], we were forced to hand transcribe using enlarged images. The error so incurred is a maximum of about 5% in probability for exceeding a given power loss or outage, P_i, which is sufficient accuracy for the present purposes of rare event prediction (see below). For the observed sample total outages each has a probability, P_i, we define the likely mean or probable average outage as,

$$\bar{Q} = \overline{\sum_{\iota} P_{\iota}\, Q_{\iota}}$$

The IRGC [4] data then have an average expected outage of $\bar{Q} = 95\ MW(e)$.

Our earlier work examined the duration of very large outages or power losses at the national level [6]. The extensive power restoration data included many severe events e.g. storms, ice storms, fires, hurricanes, cyclones and floods, causing outages lasting from 24 to 800 hours over a wide range of urban, regional and international scales [2].

In all cases, the affected power companies, emergency management organizations and government agencies deployed vast numbers (sometimes many thousands) of staff, repair crews, equipment and procedures to address power recovery, evacuate people and repair damage. Essentially restoration only can and does proceed "as fast as humanly possible", limited by damage, access and social disruption issues caused by flooding, storms, fires, wind, ice and snow, and as stated by DHS [7] "the restoration of the grid is generally the same across all hazards".

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[1] We have asked a lead author for access to the original data files and numbers forming the basis for the plot. Surprisingly for such important records, the actual NERC data for the USA are proprietary (privately owned) so the line drawings are apparently all that are publically or openly accessible.

II Theory

The first key assumption is that the power outages are indeed random, whatever the cause. Secondly, there should exist a learning trend declining with increasing experience or risk exposure. Thirdly, as shown by the data, we should also expect the largest outages to be rare events of lower probability. The learning hypothesis theory, where the rate of decrease of the failure rate, λ, at any risk exposure or experience, ε, is proportional to the rate, has a failure rate given by [8]:

$$\lambda(\varepsilon) = \lambda_m + (\lambda_0 - \lambda_m)e^{-k\varepsilon} \quad (1)$$

Here, λ_0 and λ_m are the initial and smallest attainable rates, respectively, and k is the proportionality constant. Hence, as usual, the probability is

$$P_i(loss) = 1 - e^{-\int \lambda(\varepsilon)d\varepsilon} = 1 - e^{-\left(\frac{(\lambda-\lambda_m)}{k}\right)} \quad (2)$$

Therefore, we take the failure rate, λ, as equivalent to the power loss or outage rate. The measure of the relative risk exposure or experience measure is self-evidently actually directly proportional to the power outage magnitude, ε=f(Q), which we can scale relative to the average outage magnitude, $\bar{Q}$. We may assume the outages are always essentially completely restored, so we may take, $\lambda_m \ll \lambda \approx \lambda_0\, e^{-k\varepsilon}$.

As a first approximation, we adopt the following simplest values consistent with the physical situation. The risk exposure depends on the relative outage magnitude, $\varepsilon \equiv Q/\bar{Q}$, with $\lambda_0 = 1/\varepsilon$, implying all outage events are independent. Therefore, the probability of any power loss or outage, becomes simply the intriguing double exponential,

$$P_i(loss) = 1 - e^{-\frac{\bar{Q}}{kQ}\left\{1 - e^{-\frac{kQ}{\bar{Q}}}\right\}} \quad (3)$$

Obvious limits are:

(a) small outage or loss $kQ/\bar{Q} \to 0,\ P_i(loss) = 1$;

(b) infinitely large loss $kQ/\bar{Q} \to \infty,\ P_i(loss) = 0$;

(c) average loss, assuming that k=1,

$$Q/\bar{Q} \to 1,\ P_i(loss) = 1 - e^{-\{1 - e^{-1}\}} = 0.74$$

III. Outage duration

The probability of any outage of any size lasting duration, D, is then simply given by multiplying the probability of loss by the probability of non-restoration, P(NR).

The probability of any individual outage being restored is actually random, and being observed as outcomes follows the well-known and established laws of statistical physics as described in [8] and [9]. Therefore, the data for electric power non-restoration probability, P(NR), for all outage events are all well correlated by simple exponential functions, dependent on and grouped by the degree of difficulty as characterized by the extent of infrastructure damage, social disruption and concomitant access issues [2]. A typical generalized best fit was derived from a wide range of severe events , including hurricanes, ice storms, floods, and wildfires [2] , with h measured in hours after the peak outage,

$$P(NR) = P_m + P_0^* e^{-\beta h} \approx 0.007 + e^{-0.014h} \tag{4}$$

Therefore, in general ,

$$P(D>h) = P_i(loss)\ x\ P(NR) \tag{5}$$

Substituting Equations 3) and (4) into (5),

$$P(D > h)) = (1 - e^{-\frac{\bar{Q}}{kQ}\left\{1 - e^{-\frac{kQ}{\bar{Q}}}\right\}})\left(P_m + P_0^* e^{-\beta h}\right) \tag{6}$$

To a good approximation, since $P_0^* \approx 1 \gg P_m$,

$$P(D > h)) \approx (1 - e^{-\frac{\bar{Q}}{kQ}\left\{1 - e^{-\frac{kQ}{\bar{Q}}}\right\}})\ e^{-\beta h} \tag{7}$$

The limits are:

(a) small outage or loss ; $P(D > h)) \approx e^{-\beta h}$
(b) infinitely large loss ; $P(D > h)) \approx 0$
(c) average loss with k=1 ; $P(D > h)) \approx 0.74\ e^{-\beta h}$

IV General equation for rare events

The more general form of this new EVD Equation (3) is, for any variable, x, where the over bar is the relevant or selected average value:

$$P_i(x) = 1 - e^{-\frac{\bar{x}}{kx}\left\{1 - e^{-\frac{x}{k\bar{x}}}\right\}} \tag{8}$$

There are just two “adjustable” parameters, the average ,$\bar{x}$, and the learning constant, k, where both have physical significance. This equation can be compared to typical arbitrary three-parameter Generalized Extreme Value Distributions (GEVD) quoted elsewhere [10] of the general form:

$$P_i(x) = 1 - e^{-1+\xi\left(x - \frac{\psi}{\beta}\right)^{-1/\xi}} \tag{9}$$

For the conventionally-named distributions:

- Gumbell Type 1 ξ=0
- Frechet Type 2 ξ>0
- Weibull Type 3 ξ<0

Ockham’s Razor suggests using the simplest. The reader is of course free to adopt whatever best suits the purpose and represents appropriately the physics and logic of the situation.

IV Results

I. National outage data 1984-2000 compared to theory

The demonstration of using this simple theory, Equation (3), as a basis for correlation and comparison to data is shown in Figure 1, obtained by simply adjusting the single parameter, k=2. The theoretically-based probability then has an uncertainty of order ±20% compared to the transcribed data, sufficient for present estimating purposes where the predictive larger losses of Q_M > 40,000 MW(e) have a probability of approximately 0.003 or less. The probability, of having an average system outage, P_i(loss) = 0.74 in this case is ~97MW(e), compared to the P_i (loss) =0.86 observed. This result is sufficiently encouraging to examine comparisons with other loss data as follows.

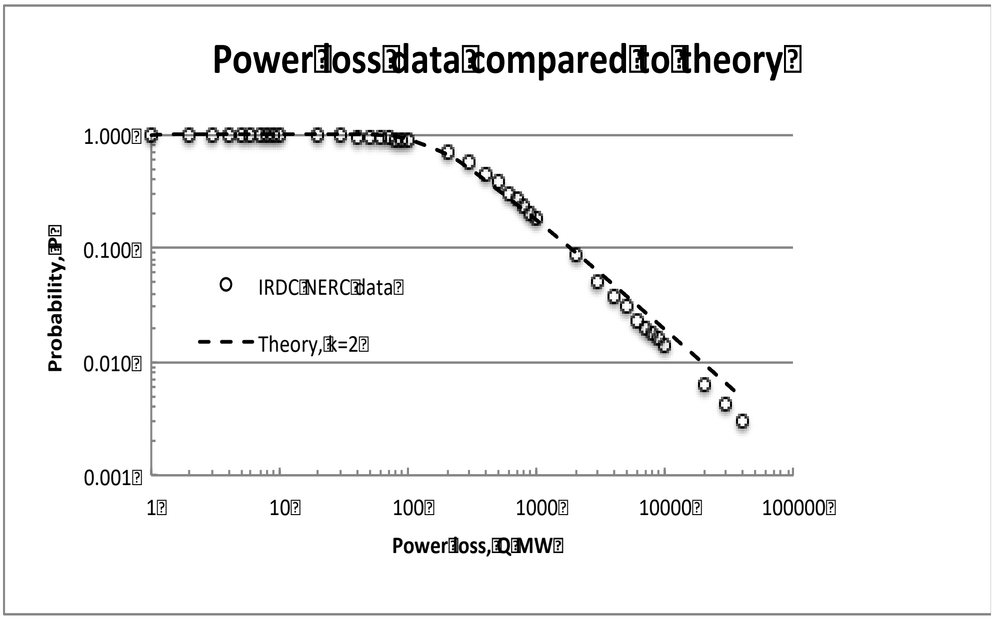

Figure 1 Initial test o USA power loss probabaility and theory

II. Regional power loss data compared to theory

A recent paper has data plots in 2016 for all eight NERC regions [5]. The individual probabilities are naturally one order lower for the largest recorded regional power losses, suggesting that the average outage, $\bar{Q}$ and the best-fit k-value change significantly.

Table 1 Regional largest power losses (data from EIA and Murphy et al [5])

Region	**Capacity, MW(e) (EIA)**	**Largest loss, MW(e) (NERC)**	**Fractional loss**
FRCC	47700	5000	0.1
MRO	33000	7000	0.21
NPCC	57700	15000	0.26
RFC	NA	49000	
SERC	129000	40000	0.31
SPP	51800	9000	0.17
TRE	71100	11000	0.15
WECC	139400	14000	0.1

These large region–to-region variations in losses are evident in the Table 1, where the maximum outages that have been experienced in 2016 are compared to the corresponding regional capacities reported by the EIA for 2016 [5; and www.eia.gov/electricity/data/eia411/]. It can be seen that the fraction of power lost in a region can range from 10% up to 30% of the total capacity, with the magnitudes differ by nearly a factor of ten (an order of magnitude).

Previous experience with learning theory suggests that it should be possible to compare the regional data in a non-dimensional manner, normalizing to the maximum outage sizes, Q/Q_m. Once again hand transcribing from the plots in [5], a comparison calculation is shown in Figure 2 for just two randomly chosen NERC regions, using average losses, $\bar{Q}$, of 5500 MW(e) and 1900 MW(e) for RFC and NPCC, respectively. In both cases, k=0.001, this low value implying no discernable learning from prior outages, or in this case no evidence of fundamental differences between the two distribution systems.

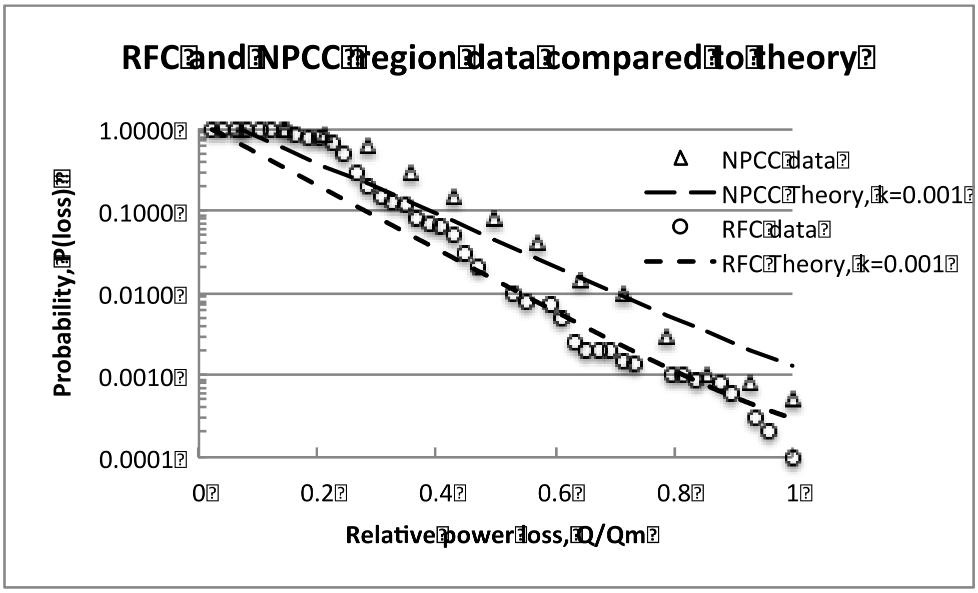

Figure 2 Regional power loss data compared to theory

Furthermore, given this new theory, we can now predict the probability of a total (100%) regional blackout, being "a catastrophic power outage of a magnitude beyond modern experience" [1]. As an example, for the NPPC case, this probability is P (total loss 57700 MW (e)) = 0.00015, and represent a pure *quantitative* prediction of an unimaginable and not previously experienced outage.

V Predicting the duration of the outage

I National Data

We can define the probability of non-restoration, *P(NR)*, at any time as the ratio of the power loss remaining, *Q(h)*, to the initial power loss, Q_0 . When there is no additional disruption, access difficulty, or grid damage the restoration is rapid..
Every national restoration data in [3] follows an exponential learning curve, with its own e-folding rate of between 0.3-0.8 per hour, and coefficients of determination all of, R^2 =0.9. As an average estimate, in Figure 3 the best fit to the overall pooled data for four events in three countries , with a coefficient of determination, R^2 = 0.69, is

$$P(NR) = e^{-0.43h} \quad (10)$$

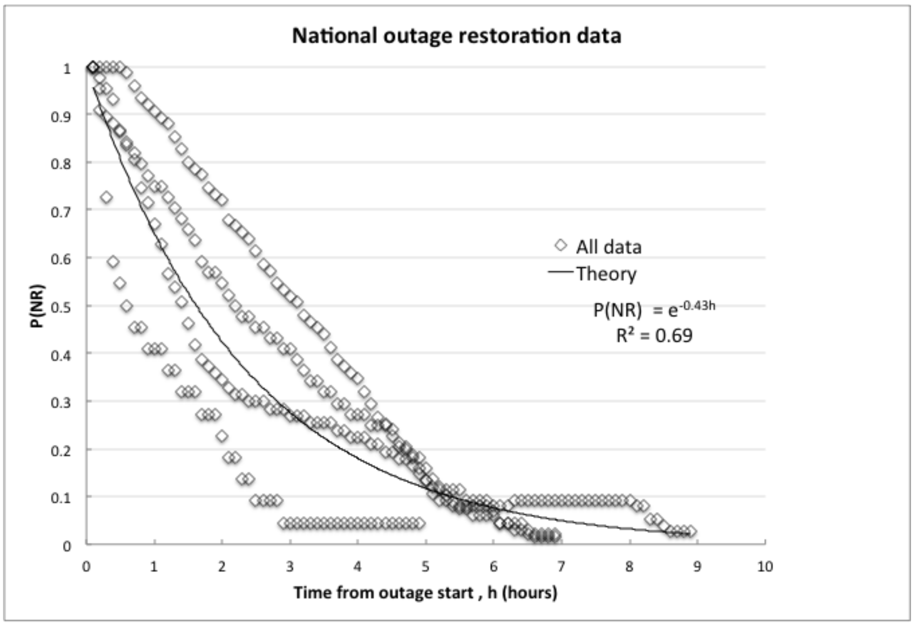

Figure 3 National duration data compared to present theory

It can be seen that even for these massive blackouts restoration was accomplished in less than 10 hours, despite the factor of ten differences in MW(e) size or scale of the initial outage. The causes were generally overall transmission and distribution failures or overloads, cascading through the system but with no additional physical damage due flooding, fires or hurricanes etc.

To compare the different power loss and outage number data sets, clearly, on average the number of outages at any time, $n(h)$, is proportional to the size or scale of the overall power loss at that time, so $n(h) \propto Q$. The probability of power system non-recovery is, $P(NR) = n(h)/N_0$, the ratio of the outages remaining, $n(h)$, to the total (initial or maximum) number,N_0, being the complement of the usual reliability, $R(t) = 1-P(NR)$.

II Regional data and severe events

As opposed to traditional plots of the numbers of outages versus time for different events (see e.g. [11]), the present formulation normalizes all the events, and demonstrates it is not the number of outages that affects characteristic recovery timescales. The data clearly show groupings between "normal" and "extreme" events restoration, with the "normal" group being faster; and events with more extreme damage and/or access difficulty clearly have much slower restoration and longer durations, by at least a factor of ten to twenty.

This key issue of the extent of damage is reflected in and by the characteristic or e-folding "degree of difficulty" parameter, β per hour; and the minimum achievable or even not restorable by, P_m. For system design and recovery planning purposes from the actual data we define the loss event categories as (see Figures 3 and 4):

- Type 0: Ordinary, 0.8>β>0.3, due to an effectively instantaneous outage with essentially no additional damage, which we classify as outage restorations that are relatively rapid, taking less than a day with simple equipment replacement, breaker resetting, line/grid repairs, and/or reconnection.

- Type 1: Normal baseline, β ~ 0.2, when outage numbers quickly peak due to finite but relatively limited additional infrastructure damage. Repairs are still fairly straightforward and all outages are restored over timescales of 20 to about 200 hours.
- Type 2: Delayed, β ~ 0.1–0.02, progressively reaching peak outages in 20 plus hours, as extensive but repairable damage causes lingering repair timescales of 200–300 hours before almost all outages are restored.
- Type 3: Extended, β ~ 0.01, with perhaps 50 or more hours before outage numbers peak due to continued damage and significant loss of critical infrastructure. Restoration repair timescales last for 300–500 hours or more with residual outages lasting even longer.
- Type 4: Extraordinary, β ~ 0.001 or less, for a cataclysmic event with the electric distribution system being essentially completely destroyed and not immediately repairable (e.g. Haiti, Costa Rica, and NAIC "catastrophic outages").

The data for Superstorm Sandy are shown (open circles) purely as an example, because it represents a "long term outage" as specifically defined by FEMA [1, p32]. The exponential form and trends do not change with overall duration.

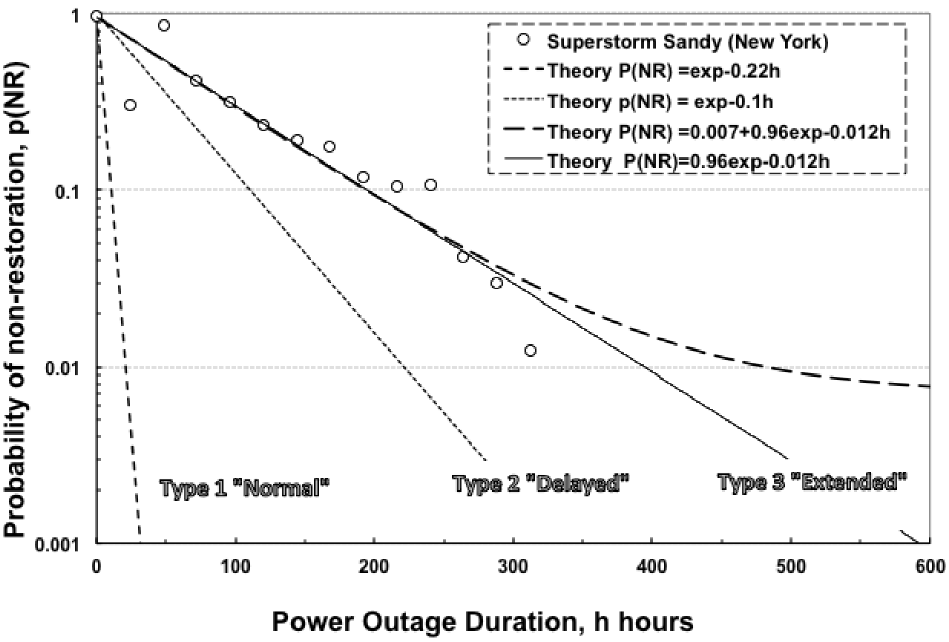

Figure 4 Simplified categories of outage restoration difficulty and timescales.

These categories allow for more refined emergency response and communication, and more realistic restoration planning. This observed variation in the degree of difficulty (0.01 <β <0.2) implies an average repair rate spread of 20 simply due to the damage extent. The irreparable fraction data range (the "tail" of the distribution) indicates that the chance of remaining unrestored is small but finite, say $0.003<P_m<0.01$,even after several hundred hours. As an example, for every million outages at first, despite achieving over 99% restoration after 600 hours several thousand could still be left without power.

III Cyber attacks

The US DHS [11,12] makes the not unreasonable assumption that the restoration curve for power outages or "virtual" damage due to cyber attacks is similar to that for known severe events, like hurricanes and ice storms. By this analogy, cyber attacks causing power outages are postulated to simply increase the restoration timescales and numbers, which we would interpret as reflecting an increased "degree of difficulty" with β reducing further. The publically available data [13] shows a cyber attack caused power outages by disconnecting networks and operator control before being restored after "several hours" .We would now classify this event as a Type 1 "normal" outage, with a P(NR) range of "cyber degree of difficulty" $0.1<\beta<0.22$, because there was no concomitant or additional access, physical damage, or societal disruption affecting recovery of the power system infrastructure and associated computing/communication networks.

Ockham's Razor suggests using the simplest. The reader is of course free to adopt whatever best suits the purpose and represents appropriately the physics and logic of the situation.

VI Conclusions

Power generation and distribution systems are part of a nation's critical infrastructure. Power losses or outages are random with a learning trend of declining size with increasing experience or risk exposure, with the largest outages being rare events of low probability. Data have been collected for power losses and outage duration affecting critical infrastructure for a wide range of events in Belgium, Canada, Eire, France, Sweden, New Zealand and USA.

Using simple theory, a new correlation has been obtained for the probability of large regional power losses for outage scales up to nearly 50,000 MW(e) for events without additional infrastructure damage that have been generally fully restored in less than 24 hours.

For more severe events , including damage due to natural hazards (floods, fire, ice storms, hurricanes etc.), the observed variation in the duration of the outage up to more than 500 hours depends on the degree of difficulty. The irreparable fraction data range (the "tail" of the distribution) indicates that the chance of remaining unrestored is small but finite, even after several hundred hours.

Therefore, explicit expressions have been given and validated for both the probability and duration for the full range from "normal" large power loss and to extended outages in rare and more "severe" events with greater access and repair difficulty. These expressions enable prediction and planning for large-scale unprecedented outages of interest for emergency planning and national response actions.

References

[1] NIAC , Surviving a catastrophic power outage, President's National Infrastructure Advisory Council, Washington, DC, 2018 (accessed at www.dhs.gov/national-infrastructure-advisory-council
[2] Duffey, R.B. (2019) Power restoration prediction following extreme events and disasters, *Int J Disaster Risk Science*, 10(1) 134-148, Springer, DOI: 10.1007/s13753-018-0189-2
[3] Kearsley, R. (1987) Restoration in Sweden and experience gained from the blackout of 1983, *IEEE Trans. Power Syst*., vol. 2, no. 2, pp. 422–428, May .
[4] IRGC, Managing and Reducing Social Vulnerabilities From Coupled Critical Infrastructures, White paper #3, Geneva, Switzerland, Figure 5 p28. ,2006
[5] Murphy, S, J Apt, J Moura and F Sowell (2017) Resource adequacy risks to the bulk power system of North America, *Applied Energy*, (212)1360-1376 DOI: 10.1016/j.apenergy.2017.12.097
[6] Duffey, R.B. and T Ha (2013) The probability and timing of power system restoration, *IEEE*

Trans Power Systems, (28)3-9, Paper # TPWRS-00826-2010, DOI 10.1109/TPWRS.2012.2203832

[8] Duffey, R.B. and J.W. Saull, Managing Risk, J.Wiley and Sons, ISBN 978 0 470 69976 8, 2008

[9] Greiner,W., L.Neise and H. Stocker, Thermodynamics and Statistical Mechanics, Springer , NY, ISBN 0 387 94299 8 ,1995

[10] Espinoza, S., M Panteli, P Mancarella, and H Rudnick (2016) Multi-phase assessment and adaptation of power systems resilience to natural hazards, *Electric Power Systems Research,* (136)352-361, DOI: 10.1016/j.epsr.2016.03.019

[11] DHS, 2018, U.S. Department of Homeland Security (DHS), Strengthening the cyber security of Federal networks and critical infrastructure, Section 2(e): Assessment of Electricity Disruption Incident Response Capabilities. August 9. (accessed at www.dhs.gov/sites/default/files/publications/EO13800-electricity-subsector-report.pdf.)

[12] DHS, 2017, Power Outage Incident Annex (POIA) to the Response and Recovery Federal Interagency Operational Plans, Managing the Cascading Impacts from a Long-Term Power Outage, US Department of Homeland Security, Final, June 2017 (accessed at www.fema.gov/media-library-data/POIA_Final_7 2017v2.508.pdf.)

[13] Lee, R M, M J Assante and T Conway, 2016, Analysis of the Cyber Attack on the Ukrainian Power Grid: Defense Use Case, E-ISAC Report, Electricity Information Sharing and Analysis Center, Industrial Control Systems, Washington, DC, (accessed at ics.sans.org/media/E-ISAC_SANS_Ukraine_DUC_5.pdf)

McCMOS Based Low Power and High Speed 32 x 32 Bit Nikhilum Multiplier

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Abstract

Leakage power is the significant component of total power dissipation in nano-scale devices. Leakage power is inversely proportional of channel length. So as the device dimensions are reduced, to incorporate more no of devices, for more function to be performed, subthreshold leakage current & hence leakage power will increases. To reduce the leakage power, channel length has to be increased. So we use non-minimum length transistors to reduce the leakage current thus power. This technique is called Multiple Channel CMOS (McCMOS). Vedic mathematics is the branch of mathematics which depends upon 16 sutra and 13 up-sutra, given between 1911 and 1918 by Sri Bharati Krisna Tirthaji (1884-1960). In this paper we had used McCMOS technique as well as ancient Vedic technique Nikhilum Sutra to reduce the power dissipation and increase the speed of the multiplier. The designed 32 x 32 bit multiplier dissipates a power of 0.556 mW and a propagation delay of 27.82 nsec. 60510 transistors were used in this design. These results are improvements over power dissipations and delays reported in literature for Vedic and Booth Multiplier.

Keywords: McCMOS, Vedic Mathematics, CMOS, Leakage current, Urdhva Tiryakbhyam, Nikhilum Sutra, Leakage Power consumption.

1. Introduction:

In order to increase the functionality of any IC, more no of devices are embedded on the same chip. For this individual transistor has to be occupied less area. Hence their dimension has to be decreased. But at same time, Leakage current and leakage power will increases, as these are inversely proportional to the channel length. So there is a compromise. If we want to increase the functionality of the chip we need to embed more no of devices. To embed more no of devices in the same area of chip, we need to scale down the dimension of individual devices. But for nano scale devices, second order effects like DIBL are more pronounced. Hence subthreshold current and power will be more in these devices. To reduce the power we have to increase the channel length. So we use NMOS as increased channel length to control the subthreshold current. Pmos are made with smaller channel length devices. In Vedic mathematics, Nikhilum Sutra, is the method to produce the calculation in less time. Let us discuss these two methods in details.

1.1 McCMOS Technique

We designed a McCMOS inverter and other basic building block. In this all NMOS are made with a channel length of 29 nm and all PMOS are made with 16.5 nm. These data are taken by repeatedly choosing different length and measuring power delay product. The channel length of the PMOSs are taken 0.5 nm higher than the 16nm technology to incorporate the lateral diffusion. The effect of channel length on threshold voltage (and leakage) is very well documented demonstrating that V_{TH} decreases rapidly as effective channel length (L_{EFF}) is reduced [10].

First we optimize the basic gates using McCMOS technique. Fig 1 shows the design of inverter using CMOS, whereas fig 2 shows the inverter circuit using McCMOS technique. Table 1 shows power delay product difference in CMOS inverter and McCMOS inverter. Table 2 shows how we arrive to the specific length & width of the transistor used in McCMOS technique.

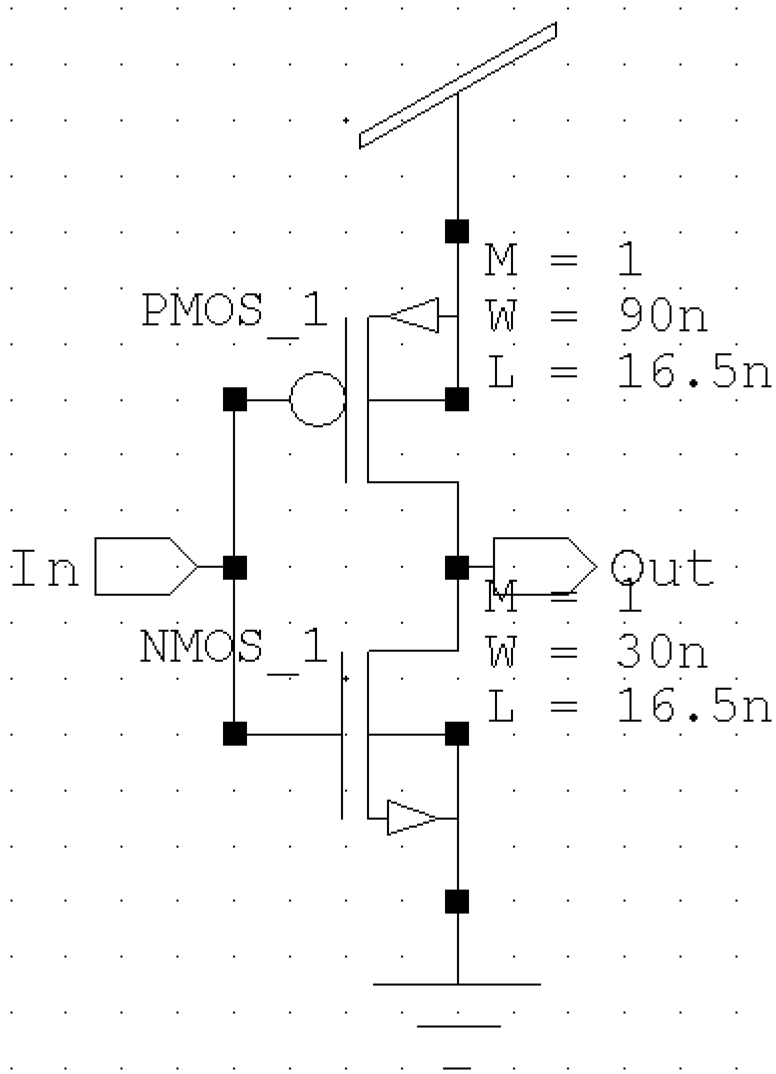

Fig 1: CMOS inverter circuit using 16nm technology

From table 1.1 & 1.2 it is clear that McCMOS inverter design with NMOS W=29n L=29n PMOS W=99n L=16.5n is best as far as PDP is concern.

Table 1.1: Comparison of power delay product of CMOS & McCMOS inverter

Technique	**W & L**	**Power (nW)**	**Delay (nsec)**	**PDP (nW-nsec)**
CMOS	**NMOS** W=30n L=16.5n **PMOS** W=90n L=16.5n	45.4	0.294	13.3
McCMOS	**NMOS** W=29n L=29n **PMOS** W=99n L=16.5n	42.4	0.286	12.1

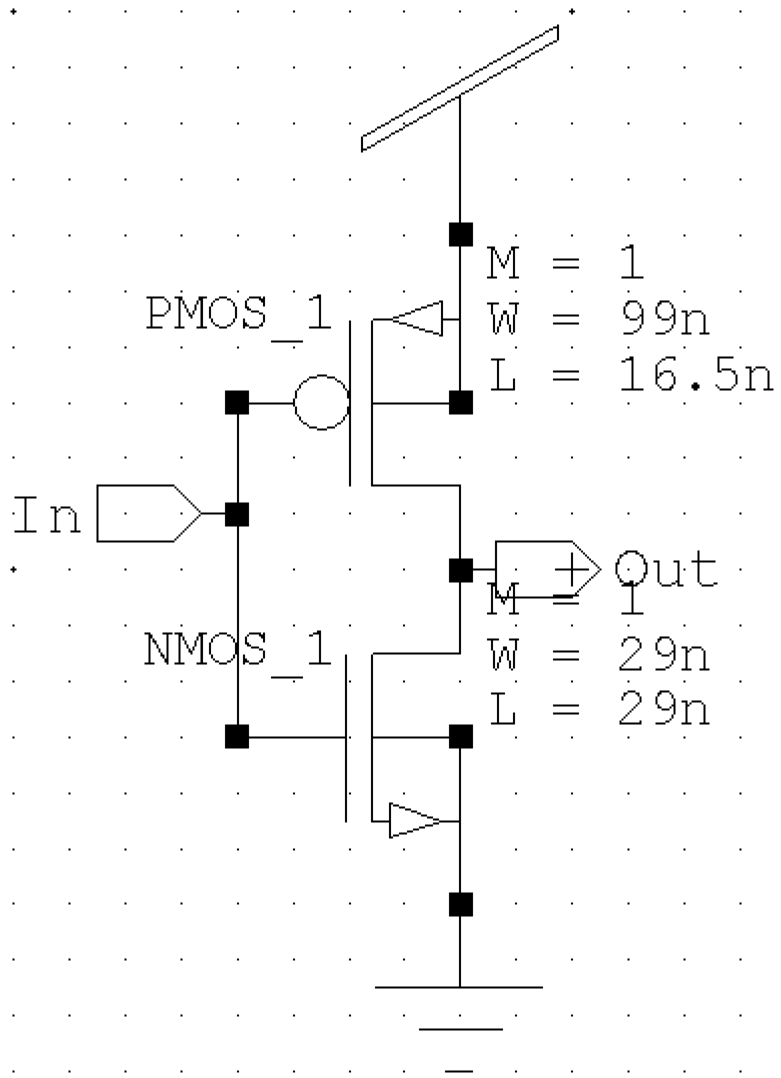

Fig 2: McCMOS inverter circuit using 16nm technology

Table 1.2: Comparison of PDP in McCMOS inverter using different lengths & Widths

W & L	Power (nWatt)	Delay (n sec)	PDP (nW-nsec)
NMOS W=25n L=25n PMOS W=25n L=16.5n	39.05	0.4760	18.5878
NMOS W=25n L=25n PMOS W=35n L=16.5n	40.26	0.4060	16.3456
NMOS W=25n L=25n PMOS W=50n L=16.5n	41.10	0.3510	14.4261
NMOS W=25n L=25n PMOS W=80n L=16.5n	42.23	0.3040	12.8379
NMOS W=33n L=33n PMOS W=80n L=16.5n	42.27	0.3040	12.8501
NMOS W=20n L=20n PMOS W=80n L=16.5n	43.23	0.3030	13.0987
NMOS W=40n L=40n PMOS W=80n L=16.5n	42.74	0.3050	13.0357
NMOS W=22n L=22n PMOS W=80n L=16.5n	42.91	0.3030	13.0017
NMOS W=28n L=28n PMOS W=80n L=16.5n	41.92	0.3040	12.7437
NMOS W=30n L=30n PMOS W=80n L=16.5n	41.94	0.3040	12.7498
NMOS W=29n L=29n PMOS W=80n L=16.5n	41.90	0.3040	12.7376
NMOS W=27n L=27n PMOS W=80n L=16.5n	41.98	0.3040	12.7619
NMOS W=28n L=28n PMOS W=99n L=16.5n	42.38	0.2860	12.1207
NMOS W=29n L=29n PMOS W=99n L=16.5n	42.36	0.2860	12.1150

Fig 3 & 4 shows the circuit of CMOS and McCMOS based nand circuit.

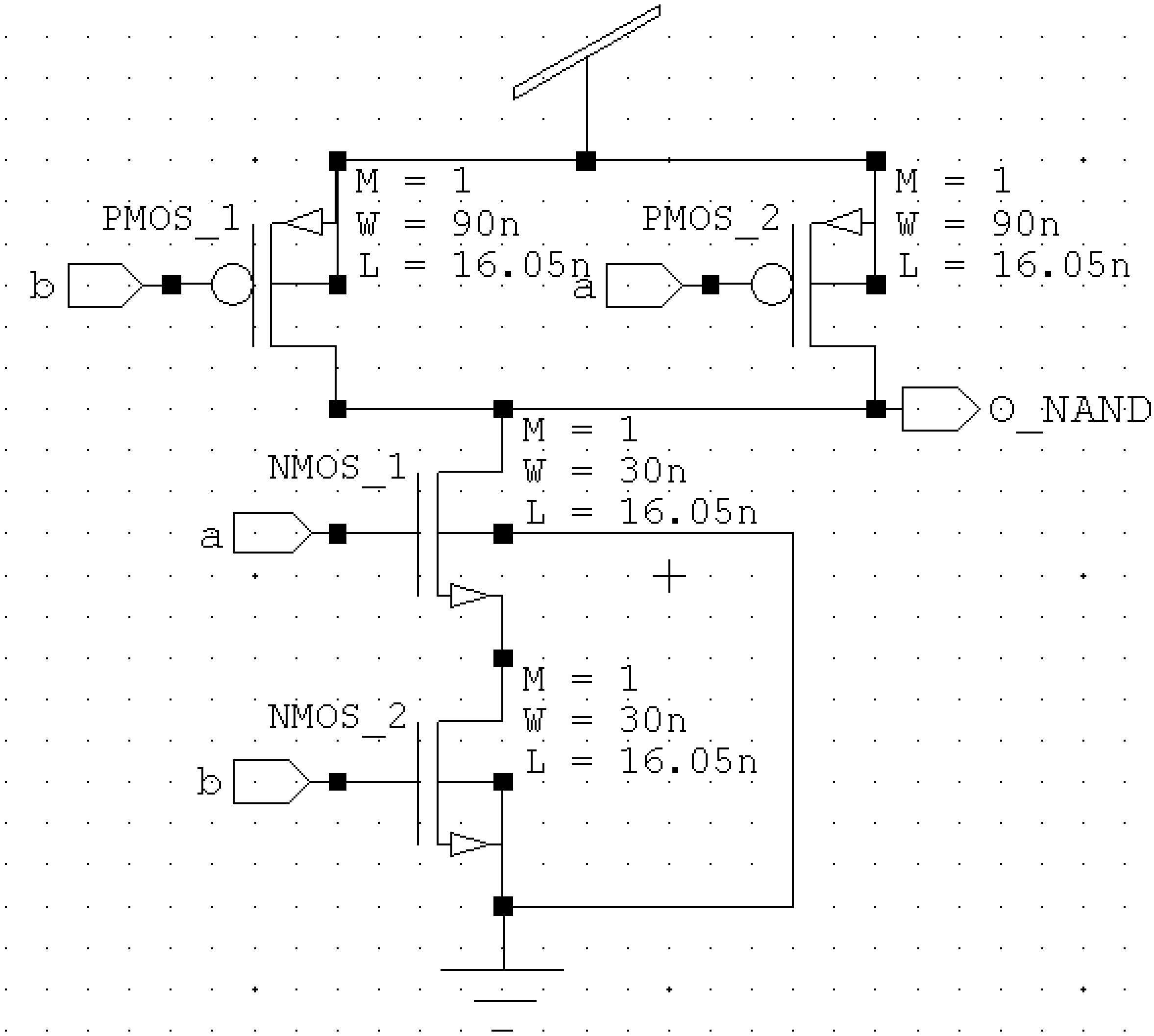

Fig 3: CMOS 2 input NAND circuit using 16nm technology

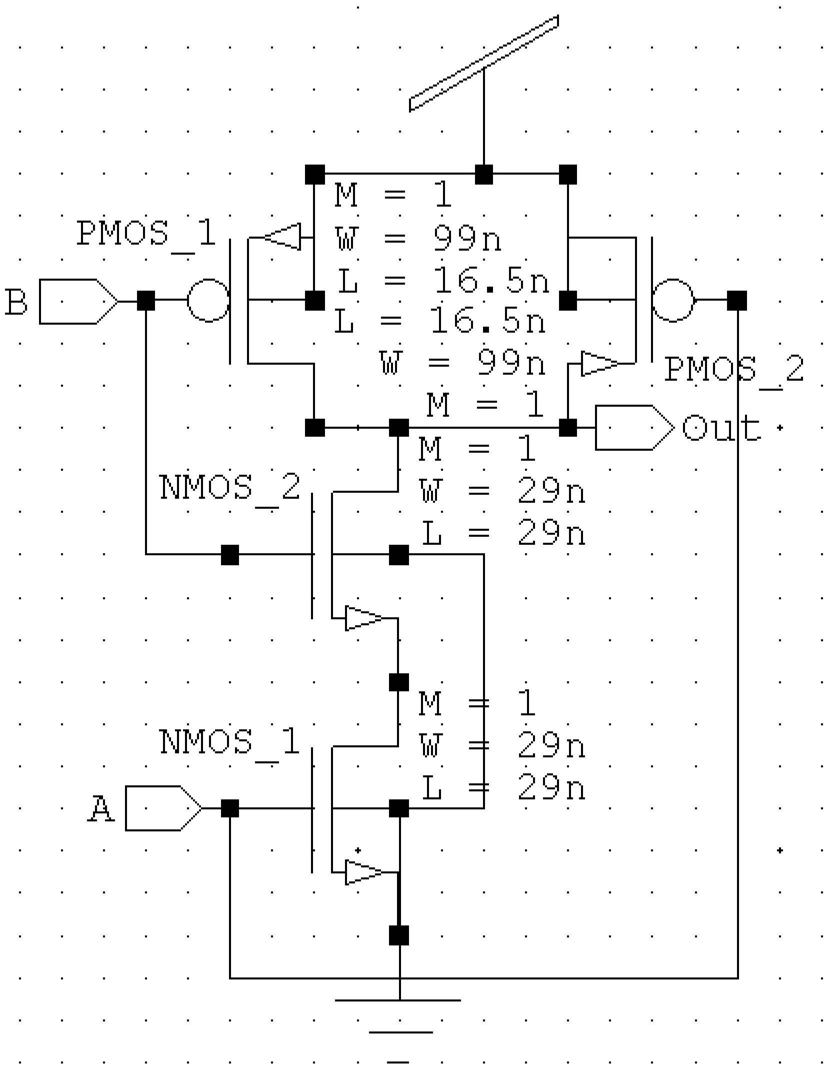

Fig 4: McCMOS 2 input NAND circuit using 16nm technology

From the table 1.3 it is clear that in McCMOS based nand circuit power & delay are much improved in comparison to its counterpart of CMOS nand circuit.
Similarly other basic component are simulated using McCMOS and CMOS and it has been verified that McCMOS based circuit has less power as well as less delay.

Table 1.3: Comparison of power & delay in CMOS & McCMOS based nand circuit.

Technique	Length & Width	Power (nWatt)	Delay (n Sec)	PDP (atta Ws)
CMOS	NMOS W=30n L=16.05n PMOS W=90n L=16.05n	92.30	0.40	37.20
McCMOS	NMOS W=29n L=29n PMOS W=99n L=16.5n	64.08	0.34	21.53

2. Vedic Mathematics

The ancient system of Vedic Mathematics was rediscovered from the Indian Sanskrit texts known as the Vedas, between 1911 and 1918 by Sri Bharati Krisna Tirthaji (1884-1960) from the Atharva Vedas. According to his research all of mathematics is based on sixteen Sutras, or word-formulas [11]. The description of urthatryakbhyam sutra and nighilum suta is given in [

DESIGN OF THE 32x32 BIT UT MULTIPLIER

In the design for 32x32 bit UT multiplier using McCMOS technique, we designed fundamental blocks like

- 2x2 bit UT multiplier using McCMOS technique (fig. 5 shows schematic design & fig. 6 shows waveform)
- 4x4 bit UT multiplier using McCMOS technique (fig. 7 shows schematic design)
- 8x8 bit UT multiplier using McCMOS technique (fig. 8 shows schematic design)
- 16x16 bit UT multiplier using McCMOS technique (fig.9 shows schematic design)
- 32x32 bit UT multiplier using McCMOS technique (fig. 10 shows schematic design)

The comparison of delay, power, no of transistor used in different designed multipliers is illustrated in table 2.1.

Table 2.1 Comparison of Designed UT Multipliers

Multiplier	Delay	Power	Power-delay product	No. of transistor used
2x2 CMOS	30.12 nsec	0.7114 µW	21.0746 nsec- µW	60
4x4 McCMOS	29.68 nsec	4.97 µW	21.42737 nsec- µW	618
4x4 CMOS	30.14 nsec	4.96 µW	147.5096 nsec- µW	618
8x8 McCMOS	29.67 nsec	26.31 µW	149.4944 nsec- µW	3222
8x8 CMOS	30.14 nsec	26.92 µW	780.6177 nsec- µW	3222
16x16 McCMOS	29.68 nsec	0.108 mW	811.3688 nsec- mW	14382
16x16 CMOS	30.14 nsec	0.130 mW	3.20544 nsec- mW	14382
32x32 McCMOS	29.68 nsec	0.564 mW	3.9182 nsec- mW	60510
32x32 CMOS	30.12 nsec	0.575 mW	16.73952 nsec- mW	60510

The comparison of delay, power, no of transistor used in reported work and proposed multiplier is illustrated in table 2.2.

Table 2.2 Comparison of UT Multiplier with Reported Work

UT Multiplier	Delay	Power	Power-delay product	No. of transistor used
Paper [2]	59 nsec	500 mW	16615.2 nsec- mW	23600
Paper [3]	15 ns	277 mW	1179.38 nsec- mW	27704
Paper [6] with CSA	96.5 ns	22.1 mW	29.5 nsec- mW	-
Paper [6] with CLA	54.1 ns	21.8 mW	2132.65 nsec- mW	-
Paper [7]	29.67 nsec.	0.56 mW	17.319 nsec- mW	60510
Proposed 32x32 CMOS	30.12 nsec	0.575 mW	16.73952 nsec- mW	60510
Proposed 32x32 McCMOS	29.68 nsec	0.564 mW	3.9182 nsec- mW	60510

Our proposed 32x32 bit multiplier using McCMOS and UT Sutra of Vedic mathematics is much far better as compared to reported work in terms of Power-Delay Product.

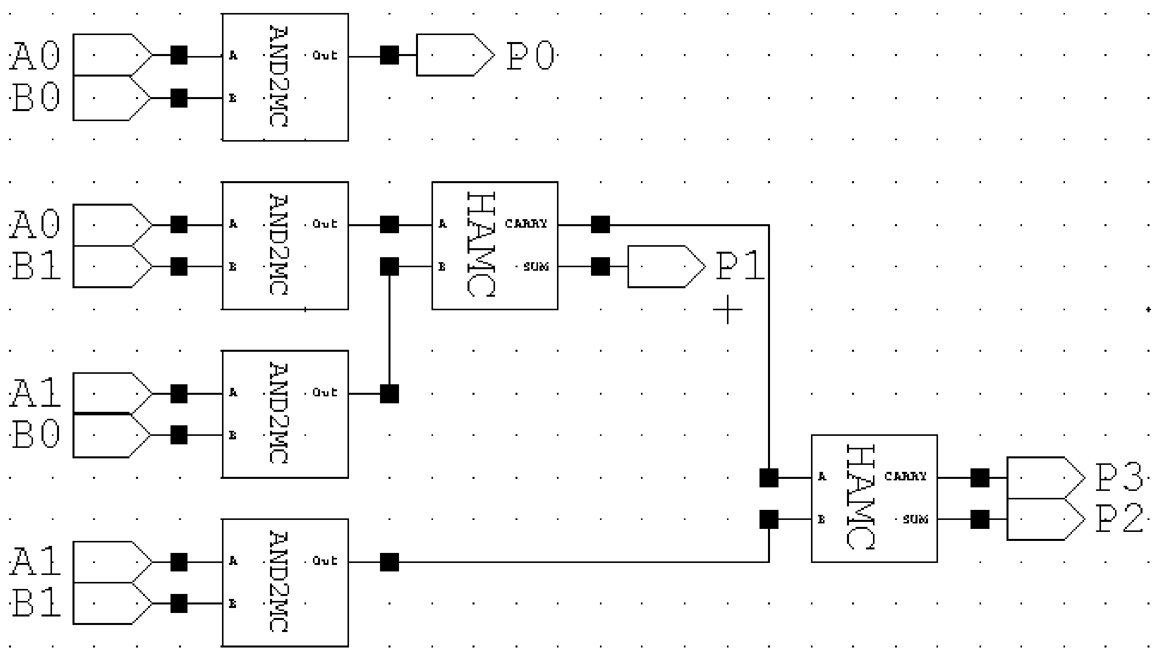

Figure5) 2 x 2 Bit Multiplier and & its symbol

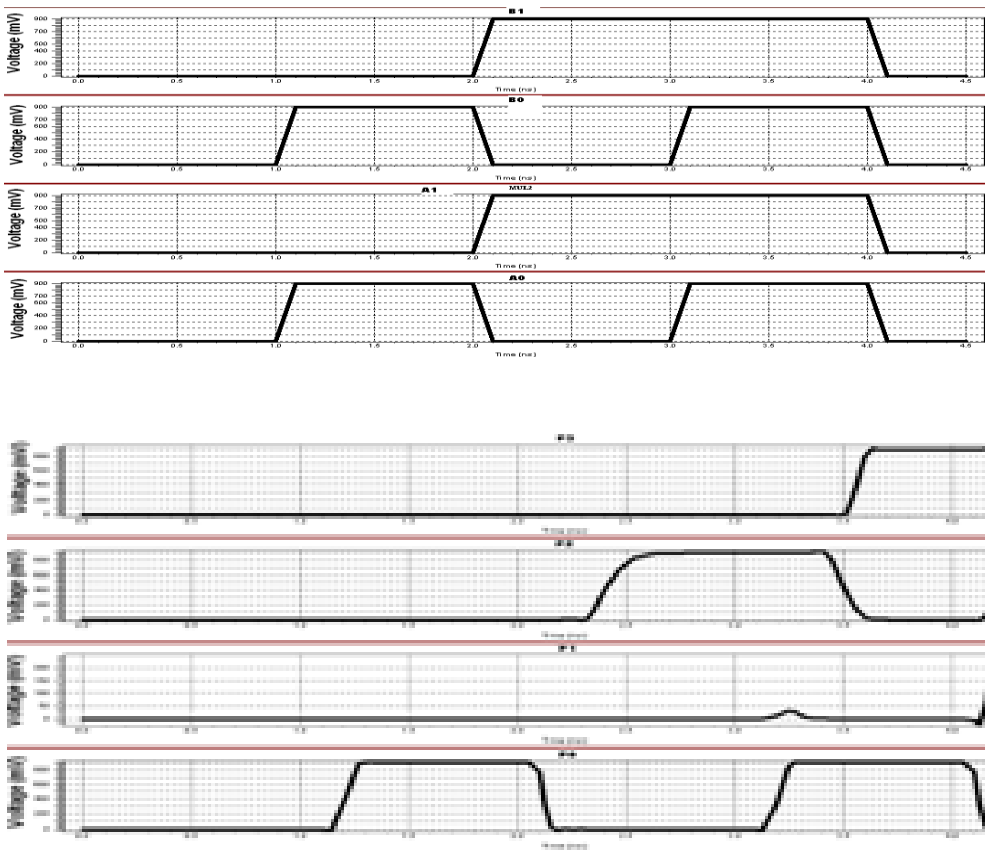

Figure 6 Input Output waveforms of 2x2 Bit multiplier

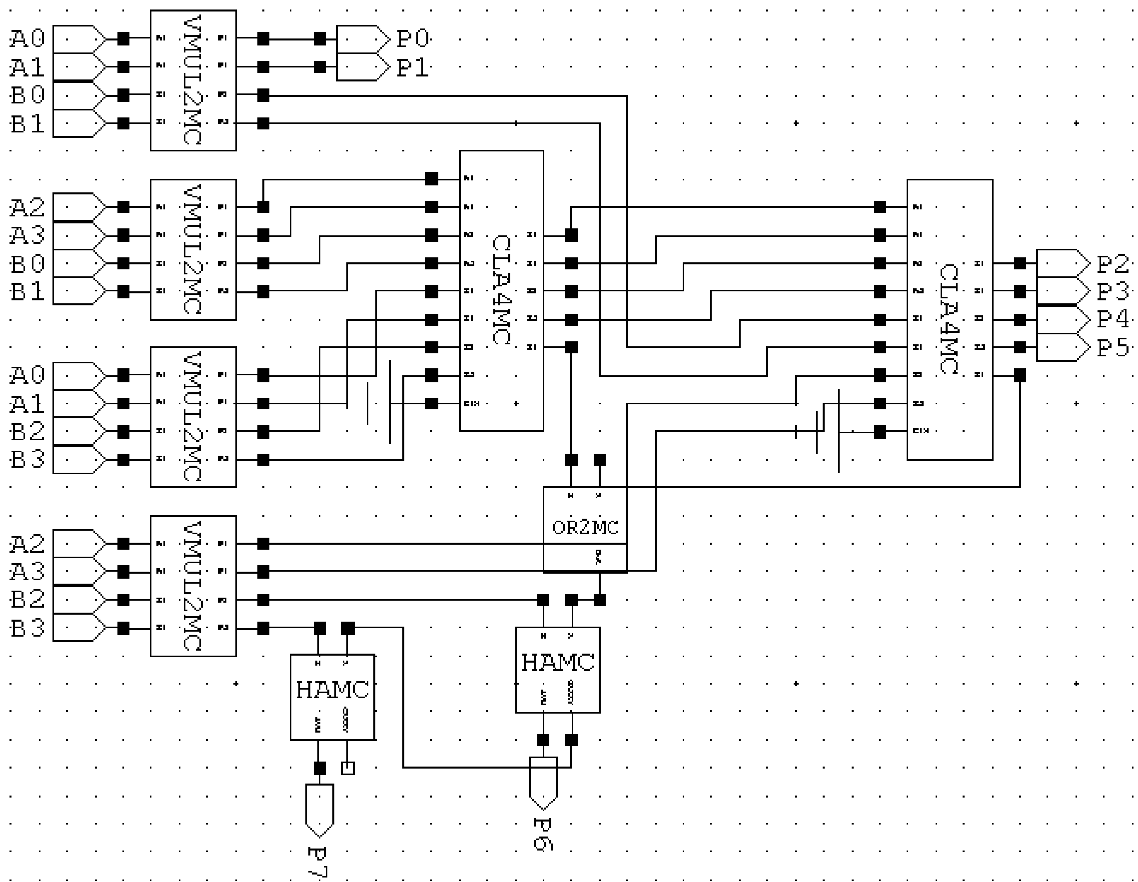

Figure7) 4 x 4 Bit Multiplier

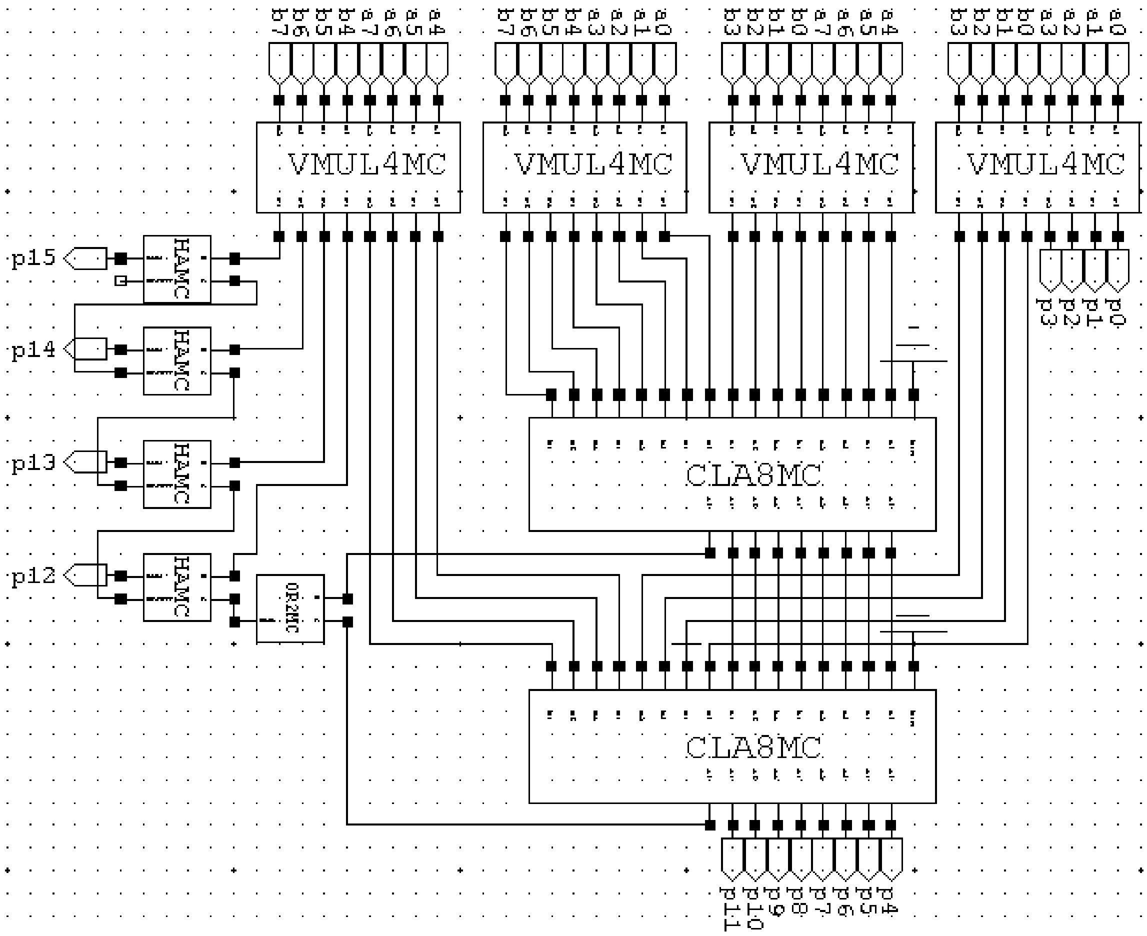

Figure8) 8 x 8 Bit Multiplier

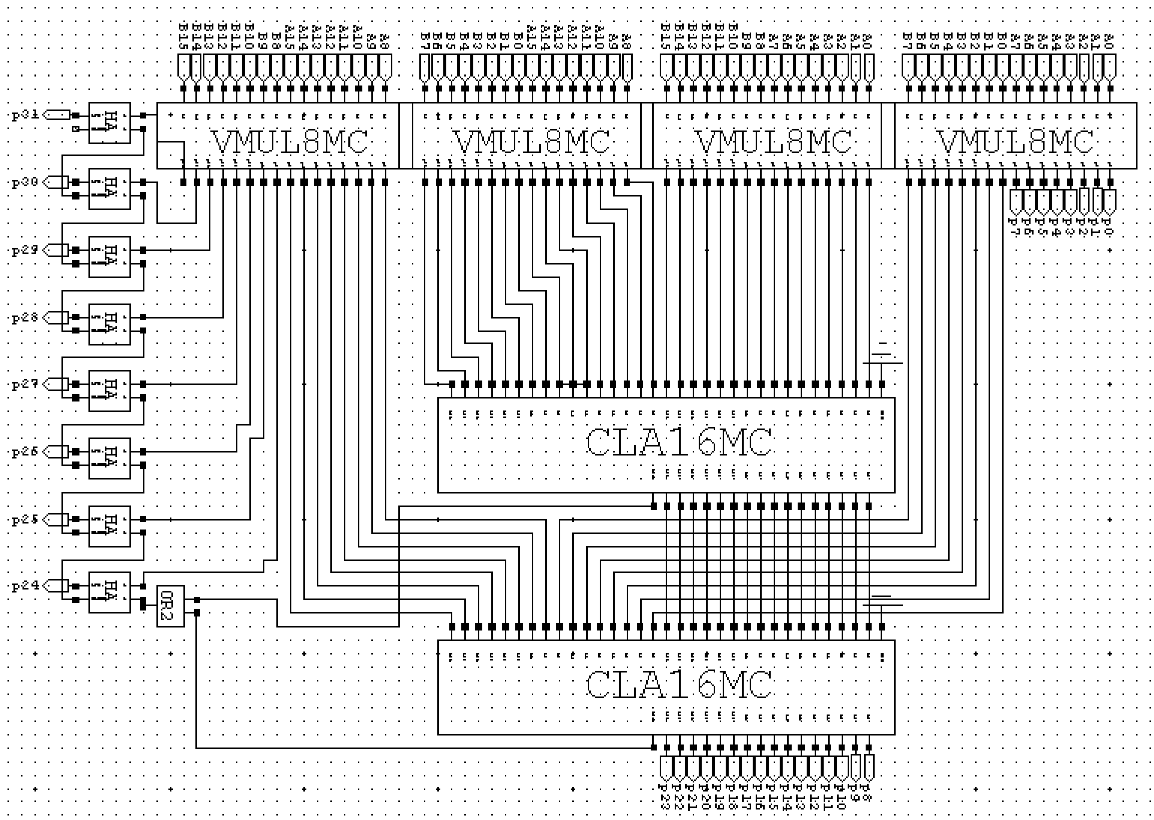

Figure9) 16 x 16 Bit Multiplier

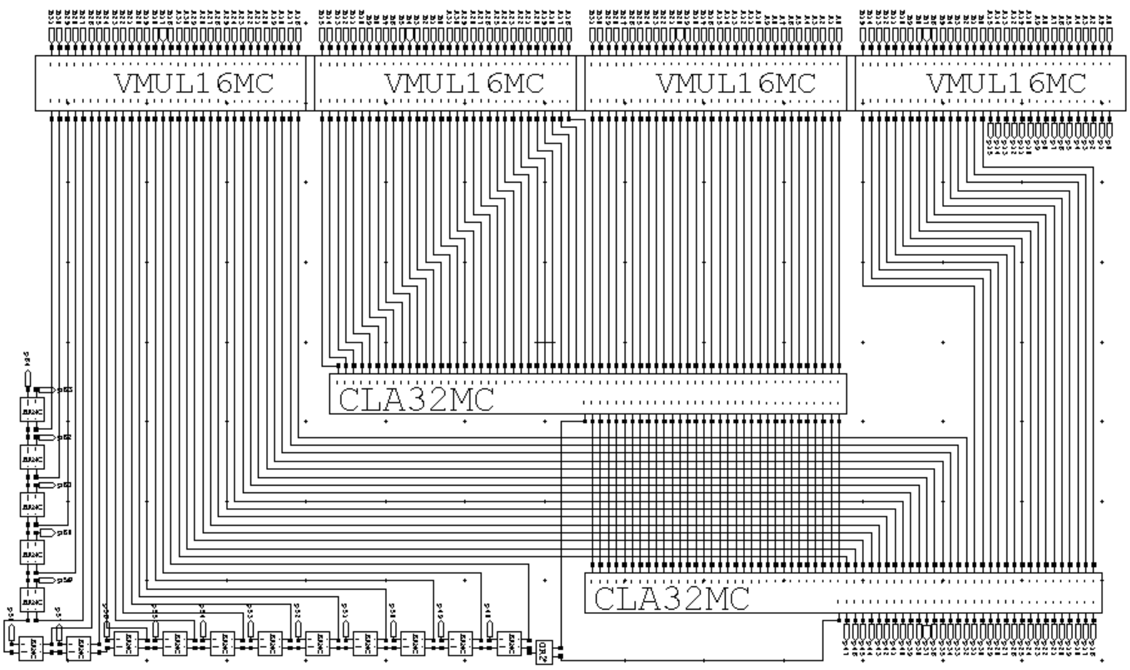

Figure10) 32 x 32 Bit Multiplier

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2. Design Of The 32x32 Bit Nikhilum Multiplier

The basic structure of the Nikhilum Multiplier is shown in fig11. In the design for 32x32 bit **Nikhilum (NM) Multiplier** *using McCMOS technique, we designed following fundamental blocks:*

- 2x2 bit NM multiplier using McCMOS technique (fig. 12 shows schematic design)
- 4x4 bit NM multiplier using McCMOS technique (fig. 13 shows schematic design)
- 8x8 bit NM multiplier using McCMOS technique (fig. 14 shows schematic design)
- 16x16 bit NM multiplier using McCMOS technique (fig.15 shows schematic design)
- 32x32 bit NM multiplier using McCMOS technique (fig. 16 shows schematic design)

STEPS INVOLVED FOR MULTIPLICATION USING NIKHILUM (NM) SUTRA

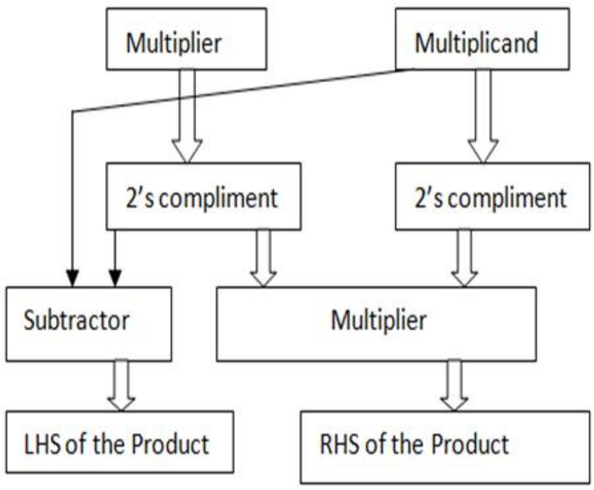

Fig 11 Block Diagram of Nikhilum Sutra

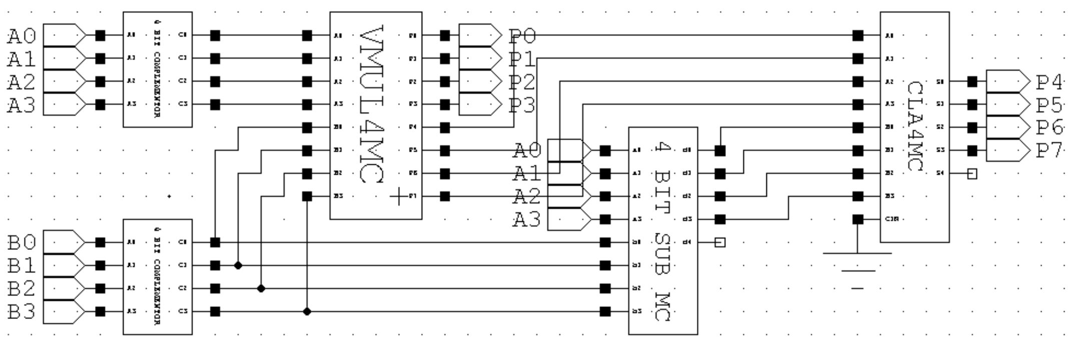

Figure12) 4 x 4 Bit Multiplier

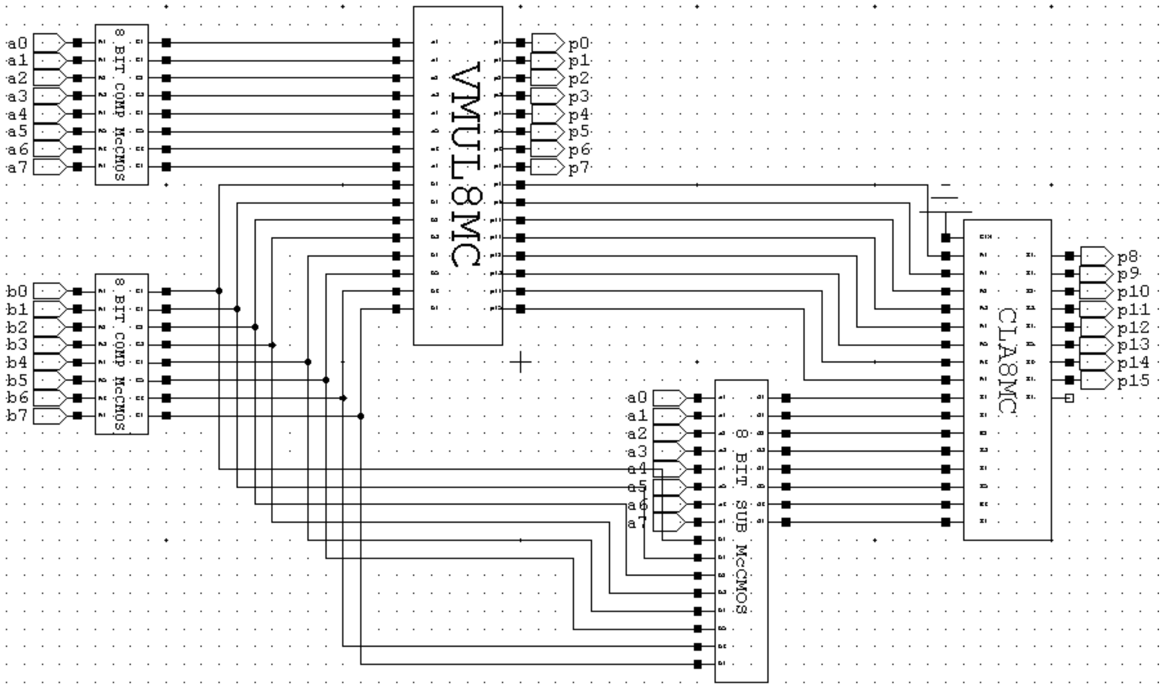

Figure13) 8 x 8 Bit Multiplier

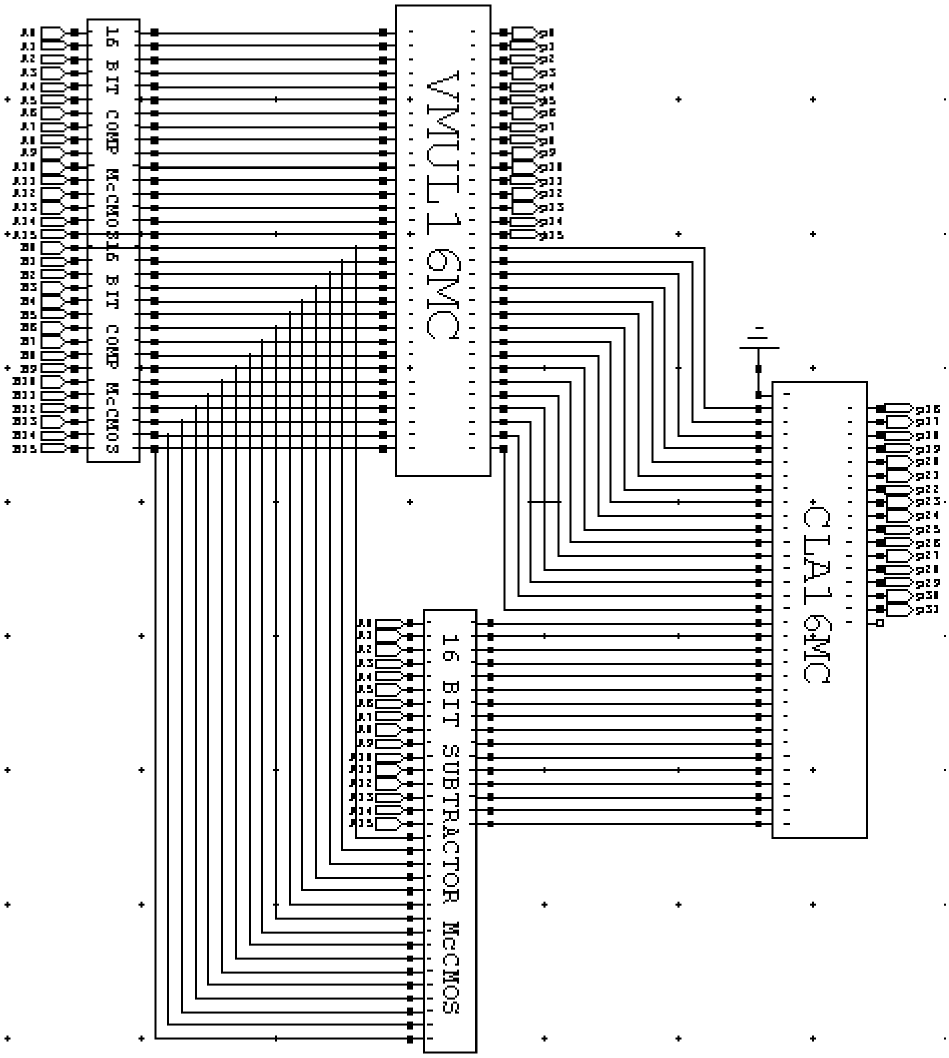

Figure14) 16 x 16 Bit Multiplier

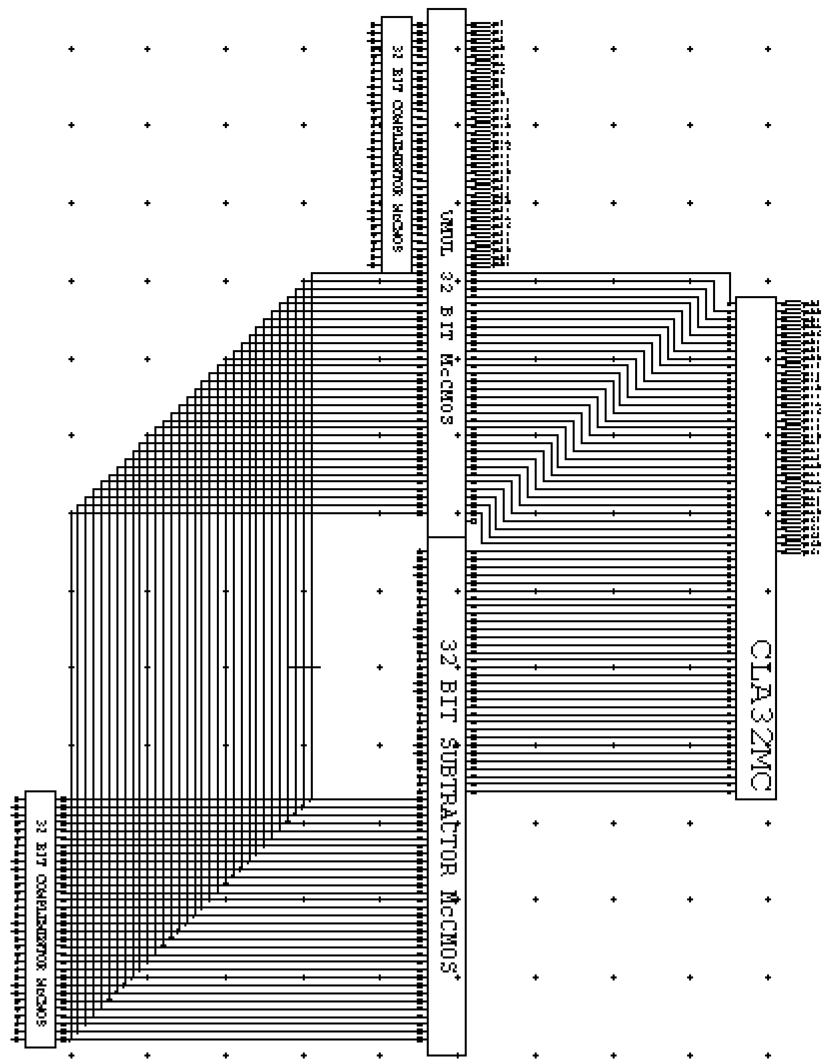

Figure 15) 32 x 32 Bit Multiplier

3. Conclusion

The proposed Vedic multiplier (discussed in section 3) is simulated using Tanner Tool v14.1. The Comparison between proposed multiplier and other reported multiplier is shown in table I.

Table 4.1: Comparison of Vedic multiplier

POWER

	UT	UTMcCMOS	NM	NMMcCMOS
4-BIT	4.96E-06	4.98E-06	8.39E-06	4.78E-06
8-BIT	2.69E-05	2.63E-05	1.94E-05	1.87E-05
16-BIT	1.30E-04	1.09E-04	1.20E-04	1.17E-04
32-BIT	5.75E-04	5.64E-04	5.60E-04	5.56E-04

DELAY

	UT	UTMcCMOS	NM	NMMcCMOS
4-BIT	3.01E-08	2.97E-08	2.90E-08	2.84E-08
8-BIT	3.01E-08	2.97E-08	2.89E-08	2.83E-08
16-BIT	3.01E-08	2.97E-08	2.88E-08	2.81E-08
32-BIT	3.01E-08	2.97E-08	2.78E-08	2.78E-08

Table 4.2: Comparison of 32x32 bit multipliers

S. No.	Parameter of comparison	Paper [2]	Paper [3]	Paper [5]	Paper [6]	Nikhilum Design
1	Delay (ns)	59	15	625.292	54.1	27.82
2	Power (mW)	500	277	29.34	21.8	0.556
3	Power-Delay Product (mW-nsec)	29500	4155	18346.067	1179.38	15.46792
4	Transistors used	23600	27704	-	-	64862

As shown in the table 4.2 our designed is much better in terms of Power dissipation. Also delay is improved. No of transistor is large but this is the sacrifice we have to pay for a less power delay product. The **99% improvement in PDP with respect to the lowest paper [6] is achieved.**

4. Acknowledgement:

We sincerely thanks to the PTM website to provide the Technology file of 16 nm.

References

[1] YOSHIHISA HARATA, et al., "A High-Speed Multiplier Using a Redundant Binary Adder Tree", IEEE JOURNAL OF SOLID-STATE CIRCUITS, VOL. SC-22,NO. 1, FEBRUARY1987
[2] SHOJI KAWAHITO,et. al., "A 32 X 32-bit Multiplier Using Multiple-Valued MOS Current-Mode Circuits", IEEE JOURNAL OF SOLID-STATE CIRCUITS, VOL. 23, NO. 1, FEBRUARY 1988
[3] MASATO NAGAMATSU, "A 15-ns 32 X 32-b CMOS Multiplier with an Improved Parallel Structure", IEEE JOURNAL OF SOLID-STATECIRCUITS, VOL. 25, NO. 2, APRIL 1990
[4] Hanho LEE, "Power Aware Scalable Pipelined Booth Multiplier" IEICE Transaction on faudamentals, Vol. E-88-A, NO. 11, Nov. 2005.
[5] Hasan Krad and Aws Yousif Al-Taie, "Performance Analysis of a 32-Bit Multiplier with a Carry-Look-Ahead Adder and a 32-bit Multiplier with a Ripple Adder using VHDL", Journal of Computer Science 4 (4): 305-308, 2008 ISSN 1549-3636© 2008 Science Publications
[6] Raminder Preet Pal Sing1, Parveen Kumar, Balwinder Singh, "Performance Analysis of 32-Bit Array Multiplier with a Carry Save Adder and with a Carry-Look-Ahead Adder", International Journal of Recent Trends in Engineering, Vol 2, No. 6, November 2009.
[7] D.Venu Gopal1* and M. Mohan Reddy, "Design and Implementation of a Fast Unsigned 32-bit Multiplier Using Verilog HDL", International Journal of Computer Sciences and Engineering,Volume-2, Issue-5 E-ISSN: 2347-2693, 2014.
[8] Sumit Vaidya and Deepak Dandekar. "delay-power performance comparison of multipliers in vlsi circuit design". International Journal of Computer Networks & Communications (IJCNC), Vol.2, No.4, July 2010

[9] Dr. K.S. Gurumurthy, M.S Prahalad "Fast and Power Efficient 16×16 Array of Array Multiplier using Vedic Multiplication",
[10] M. Ramalatha, K. Deena Dayalan, P. Dharani, S. Deborah Priya," High Speed Energy Efficient ALU Design using Vedic Multiplication Techniques ", ACTEA 2009
[11] Abhijit Asati and Chandrashekhar "A High-Speed, Hierarchical 16×16 Array of Array Multiplier Design", IMPACT 2009.
[12] Kevin Biswas, "Multiplexer Based Array Multipliers," A Ph.D. Dissertation, University of Windsor, Electrical and Computer Engineering, Apr. 2005.
[13] Himanshu Thapliyal and Hamid R. Arabnia, "A time area power efficient multiplier and square architecture based on ancient Indian Vedic mathematics,www.vedicmathsindia.org.
[14] Vishal Verma and Himanshu Thapliyal , "High Speed Efficient N X N Bit Multiplier Based On Ancient Indian Vedic Mathematics", Proceedings International Conference On VLSI, Las Vegas, June 2003.

Estimation of Stress-Strength Reliability Model Using Finite Mixture of M-Transformed Exponential Distributions

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Abstract

In this paper Stress –Strength reliability is studied where various cases have been considered for stress (Y) and strength (X) variables viz., the strength follows finite mixture of M-Transformed Exponential distributions and stress follows exponential, Lindley and M-Transformed Exponential distributions. The reliability of a system and the parameters are obtained by the method of maximum likelihood method. At the end results are illustrated with the help of numerical evaluations and real life data.

Key words: Reliability, M-Transformed Exponential Distribution, Stress, Strength.

1. Introduction

The term "stress-strength reliability" specify the quantity R=P(Y < X), where X is a random strength and Y random stress such that the system fails if the stress Y exceeds the strength Y. Church and Harris imported the term stress-strength for the first time. The stress-strength reliability and its problems for considerable distributions have been discussed by Church and Harris [1970], Woodward and Kelley [1977], Beg and Singh [1979], Awad and Gharraf [1986], Surles and Padgett [1998, 2001], Raqab and Kundu [2005], Mokhlis [2005], and Saraçoğlu et al. [2011]. Kotz et al. [2003] have presented an inspection of all approaches and results on the stress-strength reliability in the last four decades. Adil H. khan and T.R Jan [2014] have studied the stress-strength reliability for two parameter Lindley distribution, Exponential distribution and Gamma distribution. And have considered different conditions for stress and strength variables.

In the present paper, we discuss stress strength reliability and have considered that the strength variable follows finite mixture of M-Transformed Exponential distribution and stress variables follows finite mixture of exponential or Lindley or M-Transformed Exponential distributions. The structural properties and uses of M-Transformed Exponential distribution as a lifetime distribution are studied by Dinesh Kumar et al. [2017]. We debate the estimation method for finite mixture of M-Transformed Exponential distribution by the method of maximum likelihood estimation. The M-Transformed Exponential distribution with parameters α is defined by its probability density function (p.d.f.)

$$\mathrm{f}(\mathrm{x};\alpha) = \frac{2e^{-\frac{x}{\alpha}}}{\alpha\left(2-e^{-\frac{x}{\alpha}}\right)^2} \quad ; \quad \alpha, \mathrm{x} > 0 \tag{1}$$

In the present paper, we consider three cases

1) Stress follows exponential distribution and strength follows finite mixture of M-Transformed Exponential distribution.
2) Stress follows Lindley distribution and strength follows finite mixture of M-Transformed Exponential distribution.
3) Stress follows finite mixture of Lindley distributions and strength follows finite mixture of M-Transformed Exponential distributions.
4) Stress and strength both follows finite mixture of M-Transformed Exponential distribution.

2. Statistical Model

In this model we assume that random variables X (strength) and Y (stress) are independent and the values of X and Y are non-negative. The reliability of a component with strength X and stress Y imposed on it is given by

$$R = P(X > Y) = \int_{o}^{\infty} \left[\int_{0}^{x} g(y)dy \right] f(x)dx \tag{2.1}$$

where $f(x)$ and $g(y)$ are pdf of strength and stress respectively.
A finite mixture of M-Transformed Exponential distributions with v components can be expressed as

$$\mathrm{f(x)} = \mathrm{p_1 f_1(x)} + \mathrm{p_2 f_2(x)} + \cdots + \mathrm{p_v f_v(x)}\,;\, \mathrm{p_i} > 0, i = 1,2,\ldots, v \quad \sum_{\mathrm{i=1}}^{\mathrm{v}} \mathrm{p_i} = 1 \tag{2.2}$$

3. Reliability Computations

Let X be the strength of the v-components with probability density functions $\mathrm{f_i(x)}; \mathrm{i} = 1,2, \ldots, \mathrm{v}$. The pdf of X which follows finite mixture of M-Transformed Exponential distributions is

$$\mathrm{f_i(x)} = \mathrm{p_i} \frac{2e^{-\frac{x}{\alpha_i}}}{\alpha_i\left(2-e^{-\frac{x}{\alpha_i}}\right)^2}\;;\;\; \mathrm{x} > 0,\;\; \alpha_\mathrm{i} > 0, \mathrm{p_i} > 0, \mathrm{i} = 1,2,\ldots, v, \textstyle\sum_{\mathrm{i=1}}^{\mathrm{v}} \mathrm{p_i} = 1$$

Case I: The stress Y follows exponential distribution

As Y follows exponential distribution, pdf of Y is given by

$$\mathrm{g(y)} = \lambda \mathrm{e}^{-\lambda \mathrm{y}}, \lambda > 0. y > 0$$

For two components $v = 2$, and

$$\mathrm{f(x)} = \mathrm{p_1} \frac{2e^{-\frac{x}{\alpha_1}}}{\alpha_1\left(2 - e^{-\frac{x}{\alpha_1}}\right)^2} + \mathrm{p_2} \frac{2e^{-\frac{x}{\alpha_2}}}{\alpha_2\left(2 - e^{-\frac{x}{\alpha_2}}\right)^2}\;\;;\;\; \mathrm{p_1} + \mathrm{p_2} = 1, \alpha_1, \alpha_2, \mathrm{x} > 0$$

As X and Y are independent then from (2), Reliability function $\mathrm{R_2}$ is

$$\mathrm{R_2} = \int_0^{\infty}\int_0^{\mathrm{x}} \lambda \mathrm{e}^{-\lambda \mathrm{y}} \left[\mathrm{p_1} \frac{2e^{-\frac{x}{\alpha_1}}}{\alpha_1\left(2 - e^{-\frac{x}{\alpha_1}}\right)^2} + \mathrm{p_2} \frac{2e^{-\frac{x}{\alpha_2}}}{\alpha_2\left(2 - e^{-\frac{x}{\alpha_2}}\right)^2} \right] \mathrm{dxdy}$$

$$= \int_0^{\infty}\int_0^{\mathrm{x}} \lambda \mathrm{e}^{-\lambda \mathrm{y}} \left[\mathrm{p_1} \sum_{m_1=1}^{\infty} \frac{m_1}{2^{m_1}\alpha_1} e^{-\frac{m_1 x}{\alpha_1}} + \mathrm{p_2} \sum_{m_2=1}^{\infty} \frac{m_2}{2^{m_2}\alpha_2} e^{-\frac{m_2 x}{\alpha_2}} \right] \mathrm{dxdy}$$

$$\mathrm{R_2} = \int_0^{\infty} \left(1 - \mathrm{e}^{-\lambda \mathrm{x}}\right) \sum_{i=1}^{2} \sum_{m_i=1}^{\infty} \frac{\mathrm{p_i} m_i}{2^{m_i}\alpha_i} e^{-\frac{m_i x}{\alpha_i}} \mathrm{dx}$$

$$= 1 - \sum_{i=1}^{2} \sum_{m_1=1}^{\infty} \frac{p_i m_i}{2^{m_i}(\lambda\alpha_i + m_i)}\;; \qquad \sum_{\mathrm{i=1}}^{2} \mathrm{p_i} = 1 \tag{3.1}$$

For three components $v = 3$, we have

$$f(x) = p_1 \frac{2e^{-\frac{x}{\alpha_1}}}{\alpha_1\left(2-e^{-\frac{x}{\alpha_1}}\right)^2} + p_2 \frac{2e^{-\frac{x}{\alpha_2}}}{\alpha_2\left(2-e^{-\frac{x}{\alpha_2}}\right)^2} + p_3 \frac{2e^{-\frac{x}{\alpha_3}}}{\alpha_3\left(2-e^{-\frac{x}{\alpha_3}}\right)^2}$$

$$; \ p_1 + p_2 + p_3 = 1, \alpha_1, \alpha_2, \alpha_3, x > 0$$

$$R_3 = \int_0^\infty \int_0^x \lambda e^{-\lambda y}\left[p_1 \frac{2e^{-\frac{x}{\alpha_1}}}{\alpha_1\left(2-e^{-\frac{x}{\alpha_1}}\right)^2} + p_2 \frac{2e^{-\frac{x}{\alpha_2}}}{\alpha_2\left(2-e^{-\frac{x}{\alpha_2}}\right)^2} + p_3 \frac{2e^{-\frac{x}{\alpha_3}}}{\alpha_3\left(2-e^{-\frac{x}{\alpha_3}}\right)^2}\right]dxdy$$

$$R_3 = \int_0^\infty \int_0^x \lambda e^{-\lambda y}\left[p_1 \sum_{m_1=1}^{\infty} \frac{m_1}{2^{m_1}\alpha_1} e^{-\frac{m_1 x}{\alpha_1}} + p_2 \sum_{m_2=1}^{\infty} \frac{m_2}{2^{m_2}\alpha_2} e^{-\frac{m_2 x}{\alpha_2}} + p_3 \sum_{m_3=1}^{\infty} \frac{m_3}{2^{m_3}\alpha_3} e^{-\frac{m_3 x}{\alpha_3}}\right]dxdy$$

$$R_3 = \int_0^\infty \left(1 - e^{-\lambda x}\right) \sum_{i=1}^{3} \sum_{m_i=1}^{\infty} \frac{p_i m_i}{2^{m_i}\alpha_i} e^{-\frac{m_i x}{\alpha_i}} dx$$

$$R_3 = 1 - \sum_{i=1}^{3} \sum_{m_i=1}^{\infty} \frac{p_i m_i}{2^{m_i}(\lambda\alpha_i + m_i)} \quad ; \quad \sum_{i=1}^{3} p_i = 1 \tag{3.2}$$

In general for v-components, $f(x) = p_1 f_1(x) + p_2 f_2(x) + \cdots + p_v f_v(x)$; $\sum_{i=1}^{v} p_i = 1$, we hav

$$R_v = 1 - \sum_{i=1}^{v} \sum_{m_i=1}^{\infty} \frac{p_i m_i}{2^{m_i}(\lambda\alpha_i + m_i)} ; \quad \sum_{i=1}^{v} p_i = 1 \tag{3.3}$$

Case II: The stress Y follows two parameter Lindley distribution

As Y follows Lindley distribution, pdf of Y is given by

$$g(y) = \frac{\theta^2}{\theta + \lambda}(1 + \lambda y)e^{-\theta y} \quad ; \quad y > 0, \theta > 0, \alpha > -\theta$$

For two components $v = 2$

$$f(x) = p_1 \frac{2e^{-\frac{x}{\alpha_1}}}{\alpha_1\left(2-e^{-\frac{x}{\alpha_1}}\right)^2} + p_2 \frac{2e^{-\frac{x}{\alpha_2}}}{\alpha_2\left(2-e^{-\frac{x}{\alpha_2}}\right)^2} \ ; \ p_1 + p_2 = 1, \alpha_1, \alpha_2, x > 0$$

As X and Y are independent then from (2), Reliability function R_2 is

$$R_2 = \int_0^\infty \int_0^x \frac{\theta^2}{\theta + \lambda}(1 + \lambda y)e^{-\theta y}\left(p_1 \frac{2e^{-\frac{x}{\alpha_1}}}{\alpha_1\left(2-e^{-\frac{x}{\alpha_1}}\right)^2} + p_2 \frac{2e^{-\frac{x}{\alpha_2}}}{\alpha_2\left(2-e^{-\frac{x}{\alpha_2}}\right)^2}\right)dxdy$$

$$R_2 = \int_0^\infty \int_0^x \frac{\theta^2}{\theta + \lambda}(1 + \lambda y)e^{-\theta y}\left(p_1 \sum_{m_1=1}^{\infty} \frac{m_1}{2^{m_1}\alpha_1} e^{-\frac{m_1 x}{\alpha_1}} + p_2 \sum_{m_2=1}^{\infty} \frac{m_2}{2^{m_2}\alpha_2} e^{-\frac{m_2 x}{\alpha_2}}\right)dxdy$$

$$R_2 = \int_0^\infty \left(1 - \frac{\theta + \lambda + \lambda\theta x}{\theta + \lambda} e^{-\theta x}\right)\left(\sum_{i=1}^{2} \sum_{m_i=1}^{\infty} p_i \frac{m_i}{2^{m_i}\alpha_i} e^{-\frac{m_i x}{\alpha_i}}\right) dx$$

$$R_2 = 1 - \sum_{i=1}^{2} \sum_{m_i=1}^{\infty} p_i \frac{m_i}{2^{m_i}\alpha_i}\left(\int_0^\infty e^{-\left(\theta + \frac{m_i}{\alpha_i}\right)x} dx + \frac{\lambda\theta}{\theta + \lambda}\int_0^\infty x e^{-\left(\theta + \frac{m_i}{\alpha_i}\right)x} dx\right)$$

$$R_2 = 1 - \sum_{i=1}^{2} \sum_{m_i=1}^{\infty} p_i \frac{m_i}{2^{m_i}}\left(\frac{1}{m_i + \theta\alpha_i} + \frac{\lambda\theta\alpha_i}{(\theta + \lambda)(m_i + \theta\alpha_i)^2}\right) \tag{3.4}$$

For three components $v = 3$, we have

$$f(x) = f(x) = p_1 \frac{2e^{-\frac{x}{\alpha_1}}}{\alpha_1\left(2-e^{-\frac{x}{\alpha_1}}\right)^2} + p_2 \frac{2e^{-\frac{x}{\alpha_2}}}{\alpha_2\left(2-e^{-\frac{x}{\alpha_2}}\right)^2} + p_3 \frac{2e^{-\frac{x}{\alpha_3}}}{\alpha_3\left(2-e^{-\frac{x}{\alpha_3}}\right)^2}$$

$$; \ p_1 + p_2 + p_3 = 1, \alpha_1, \alpha_2, \alpha_3, x > 0$$

$$R_3 = \int_0^\infty \int_0^x \frac{\theta^2}{\theta+\lambda}(1+\lambda y)e^{-\theta y}\left[p_1 \frac{2e^{-\frac{x}{\alpha_1}}}{\alpha_1\left(2-e^{-\frac{x}{\alpha_1}}\right)^2} + p_2 \frac{2e^{-\frac{x}{\alpha_2}}}{\alpha_2\left(2-e^{-\frac{x}{\alpha_2}}\right)^2} + p_3 \frac{2e^{-\frac{x}{\alpha_3}}}{\alpha_3\left(2-e^{-\frac{x}{\alpha_3}}\right)^2}\right] dxdy$$

$$R_3 = \int_0^\infty \left(1 - \frac{\theta+\lambda+\lambda\theta x}{\theta+\lambda} e^{-\theta x}\right)\left(\sum_{i=1}^{3}\sum_{m_1}^{\infty} p_i \frac{m_i}{2^i \alpha_i} e^{-\frac{m_i x}{\alpha_i}}\right) dx$$

$$R_3 = 1 - \sum_{i=1}^{3}\sum_{m_i=1}^{\infty} p_i \frac{m_i}{2^{m_i}\alpha_i}\int_0^\infty \left(e^{-\left(\frac{m_i}{\alpha_i}+\theta\right)x} + \frac{\lambda\theta}{\theta+\lambda} x e^{-\left(\frac{m_i}{\alpha_i}+\theta\right)x}\right) dx$$

$$R_3 = 1 - \sum_{i=1}^{3}\sum_{m_i=1}^{\infty} p_i \frac{m_i}{2^{m_i}}\left(\frac{1}{m_i+\theta\alpha_i} + \frac{\lambda\theta\alpha_i}{(\theta+\lambda)(m_i+\theta\alpha_i)^2}\right) \tag{3.5}$$

In general for v-components, $f(x) = p_1 f_1(x) + p_2 f_2(x) + \cdots + p_v f_v(x)$; $\sum_{i=1}^{v} p_i = 1$, we have

$$R_v = 1 - \sum_{i=1}^{v}\sum_{m_i=1}^{\infty} p_i \frac{m_i}{2^{m_i}}\left(\frac{1}{m_i+\theta\alpha_i} + \frac{\lambda\theta\alpha_i}{(\theta+\lambda)(m_i+\theta\alpha_i)^2}\right) \tag{3.6}$$

Special case

When $\lambda = 1$, two parameter Lindley distribution reduces to one parameter Lindley distribution and then the reliability function is given as

$$R_v = 1 - \sum_{i=1}^{v}\sum_{m_i=1}^{\infty} p_i \frac{m_i}{2^{m_i}}\left(\frac{1}{m_i+\theta\alpha_i} + \frac{\theta\alpha_i}{(\theta+1)(m_i+\theta\alpha_i)^2}\right) \tag{3.7}$$

Case III: The stress Y follows mixture of two parameter Lindley distributions

As Y follows mixture of M-Transformed Exponential distributions, pdf of X and Y is given by
For two components, $v = 2$

$$f(x) = p_1 \frac{2e^{-\frac{x}{\alpha_1}}}{\alpha_1\left(2-e^{-\frac{x}{\alpha_1}}\right)^2} + p_2 \frac{2e^{-\frac{x}{\alpha_2}}}{\alpha_2\left(2-e^{-\frac{x}{\alpha_2}}\right)^2} \;;\; p_1 + p_2 = 1, \alpha_1, \alpha_2, x > 0$$

$$g(y) = p_3 \frac{\theta_3^2}{\theta_3+\lambda_3}(1+\lambda_3 y)e^{-\lambda_3 y} + p_4 \frac{\theta_4^2}{\theta_4+\lambda_4}(1+\lambda_4 y)e^{-\theta_4 y} \;;\; p_3 + p_4 = 1,$$

As X and Y are independent then from (2), Reliability function R_2 is

$$R_2 = \int_0^\infty \int_0^x \left(p_1 \frac{2e^{-\frac{x}{\alpha_1}}}{\alpha_1\left(2-e^{-\frac{x}{\alpha_1}}\right)^2} + p_2 \frac{2e^{-\frac{x}{\alpha_2}}}{\alpha_2\left(2-e^{-\frac{x}{\alpha_2}}\right)^2}\right)\left(p_3 \frac{\theta_3^2}{\theta_3+\lambda_3}(1+\lambda_3 y)e^{-\lambda_3 y} + p_4 \frac{\theta_4^2}{\theta_4+\lambda_4}(1+\lambda_4 y)e^{-\theta_4 y}\right) dxdy$$

$$R_2 = \sum_{j=i+2}^{4}\sum_{i=1}^{2}\int_0^\infty\int_0^x p_i \sum_{m_i=1}^{\infty} \frac{m_i}{2^{m_i}\alpha_i} e^{-\frac{m_i}{\alpha_i}x} p_j \frac{\theta_j^2}{\theta_j+\lambda_j}(1+\lambda_j y)e^{-\lambda_j y} dxdy$$

$$R_2 = \sum_{j=i+2}^{4}\sum_{i=1}^{2}\int_0^\infty p_i p_j \sum_{m_i=1}^{\infty} \frac{m_i}{2^{m_i}\alpha_i} e^{-\frac{m_i}{\alpha_i}x}\left(1 - \frac{\theta_j+\lambda_j+\lambda_j\theta_j x}{\theta_j+\lambda_j} e^{-\theta_j x}\right) dx$$

$$R_2 = 1 - \sum_{j=i+2}^{4}\sum_{i=1}^{2}\sum_{m_i=1}^{\infty} \frac{p_i p_j m_i}{2^{m_i}\alpha_i}\int_0^\infty \left(e^{-\left(\frac{m_i}{\alpha_i}+\theta_j\right)x} + \frac{\lambda_j\theta_j}{\theta_j+\lambda_j} x e^{-\left(\frac{m_i}{\alpha_i}+\theta_j\right)x}\right) dx$$

$$R_2 = 1 - \sum_{j=i+2}^{4}\sum_{i=1}^{2}\sum_{m_i=1}^{\infty} \frac{p_i p_j m_i}{2^{m_i}}\left[\frac{1}{m_i+\theta_j\alpha_i} + \frac{\lambda_j\theta_j\alpha_i}{(\theta_j+\lambda_j)(m_i+\theta_j\alpha_i)^2}\right] \tag{3.8}$$

For three components $v = 3$, we have

$$f(x) = p_1 \frac{2e^{-\frac{x}{\alpha_1}}}{\alpha_1\left(2-e^{-\frac{x}{\alpha_1}}\right)^2} + p_2 \frac{2e^{-\frac{x}{\alpha_2}}}{\alpha_2\left(2-e^{-\frac{x}{\alpha_2}}\right)^2} + p_3 \frac{2e^{-\frac{x}{\alpha_3}}}{\alpha_3\left(2-e^{-\frac{x}{\alpha_3}}\right)^2}$$

$$;\; p_1 + p_2 + p_3 = 1, \alpha_1, \alpha_2, \alpha_3, x > 0$$

$$g(y) = p_4 \frac{\lambda_4^2}{\lambda_4 + \alpha_4}(1 + \alpha_4 y)e^{-\lambda_4 y} + p_5 \frac{\lambda_5^2}{\lambda_5 + \alpha_5}(1 + \alpha_5 y)e^{-\lambda_5 y} + p_3 \frac{\lambda_6^2}{\lambda_6 + \alpha_6}(1 + \alpha_6 y)e^{-\lambda_6 y}$$
$$;\ p_4 + p_5 + p_6 = 1, \lambda_4, \lambda_5, \lambda_6, y > 0$$

X and Y are independent then from (I), Reliability function R is

$$R_3 = \int_0^\infty \int_0^x \left(p_1 \frac{2e^{-\frac{x}{\alpha_1}}}{\alpha_1 \left(2 - e^{-\frac{x}{\alpha_1}}\right)^2} + p_2 \frac{2e^{-\frac{x}{\alpha_2}}}{\alpha_2 \left(2 - e^{-\frac{x}{\alpha_2}}\right)^2} + p_3 \frac{2e^{-\frac{x}{\alpha_3}}}{\alpha_3 \left(2 - e^{-\frac{x}{\alpha_3}}\right)^2} \right)$$
$$\times \left(\frac{p_4 \lambda_4^2}{\lambda_4 + \alpha_4}(1 + \alpha_4 y)e^{-\lambda_4 y} + \frac{p_5 \lambda_5^2}{\lambda_5 + \alpha_5}(1 + \alpha_5 y)e^{-\lambda_5 y} + \frac{p_3 \lambda_6^2}{\lambda_6 + \alpha_6}(1 + \alpha_6 y)e^{-\lambda_6 y} \right) dxdy$$

$$R_3 = \sum_{j=i+3}^{6} \sum_{i=1}^{3} \int_0^\infty \int_0^x p_i \sum_{m_i=1}^{\infty} \frac{m_i}{2^{m_i}\alpha_i} e^{-\frac{m_i}{\alpha_i}x} p_j \frac{\theta_j^2}{\theta_j + \lambda_j}(1 + \lambda_j y)e^{-\lambda_j y} dxdy$$

$$R_3 = \sum_{j=i+3}^{4} \sum_{i=1}^{2} \int_0^\infty p_i p_j \sum_{m_i=1}^{\infty} \frac{m_i}{2^{m_i}\alpha_i} e^{-\frac{m_i}{\alpha_i}x} \left(1 - \frac{\theta_j + \lambda_j + \lambda_j \theta_j x}{\theta_j + \lambda_j} e^{-\theta_j x}\right) dx$$

$$R_3 = 1 - \sum_{j=i+3}^{6} \sum_{i=1}^{3} \sum_{m_i=1}^{\infty} \frac{p_i p_j m_i}{2^{m_i}\alpha_i} \int_0^\infty \left(e^{-\left(\frac{m_i}{\alpha_i} + \theta_j\right)x} + \frac{\lambda_j \theta_j}{\theta_j + \lambda_j} x e^{-\left(\frac{m_i}{\alpha_i} + \theta_j\right)x} \right) dx$$

$$R_3 = 1 - \sum_{j=i+3}^{6} \sum_{i=1}^{3} \sum_{m_i=1}^{\infty} \frac{p_i p_j m_i}{2^{m_i}} \left[\frac{1}{m_i + \theta_j \alpha_i} + \frac{\lambda_j \theta_j \alpha_i}{(\theta_j + \lambda_j)(m_i + \theta_j \alpha_i)^2} \right] \quad (3.9)$$

In general for v-components,

$$f(x) = p_1 f_1(x) + p_2 f_2(x) + \cdots + p_v f_v(x)\ ;\ \sum_{i=1}^{v} p_i = 1$$

$$g(y) = p_{v+1} f_{v+1}(y) + p_{v+2} f_{v+2}(y) + \cdots + p_{2v} f_{2v}(y)\ ;\ \sum_{i=v+1}^{2v} p_i = 1$$

$$R_v = \int_0^\infty \int_0^x \left(p_1 f_1(x) + p_2 f_2(x) + \cdots + p_k f_k(x)\right)\left(p_{v+1} f_{v+1}(y) + p_{v+2} f_{v+2}(y) + \cdots + p_{2v} f_{2v}(y)\right) dxdy$$

$$R_v = \sum_{j=i+v}^{2v} \sum_{i=1}^{v} \int_0^\infty \int_0^x p_i \sum_{m_i=1}^{\infty} \frac{m_i}{2^{m_i}\alpha_i} e^{-\frac{m_i}{\alpha_i}x} p_j \frac{\theta_j^2}{\theta_j + \lambda_j}(1 + \lambda_j y)e^{-\lambda_j y} dxdy$$

$$R_v = \sum_{j=i+v}^{2v} \sum_{i=1}^{v} \int_0^\infty p_i p_j \sum_{m_i=1}^{\infty} \frac{m_i}{2^{m_i}\alpha_i} e^{-\frac{m_i}{\alpha_i}x} \left(1 - \frac{\theta_j + \lambda_j + \lambda_j \theta_j x}{\theta_j + \lambda_j} e^{-\theta_j x}\right) dx$$

$$R_v = 1 - \sum_{j=i+v}^{2v} \sum_{i=1}^{v} \sum_{m_i=1}^{\infty} \frac{p_i p_j m_i}{2^{m_i}\alpha_i} \int_0^\infty \left(e^{-\left(\frac{m_i}{\alpha_i} + \theta_j\right)x} + \frac{\lambda_j \theta_j}{\theta_j + \lambda_j} x e^{-\left(\frac{m_i}{\alpha_i} + \theta_j\right)x} \right) dx$$

$$R_v = 1 - \sum_{j=i+v}^{2v} \sum_{i=1}^{v} \sum_{m_i=1}^{\infty} \frac{p_i p_j m_i}{2^{m_i}} \left[\frac{1}{m_i + \theta_j \alpha_i} + \frac{\lambda_j \theta_j \alpha_i}{(\theta_j + \lambda_j)(m_i + \theta_j \alpha_i)^2} \right] \quad (3.10)$$

Special case

When $\lambda_j = 1$, two parameter Lindley distribution reduces to one parameter Lindley distribution and then R_k is the reliability function when X follows mixture of one parameter Lindley distribution and Y follows mixture of one M-Transformed Exponential distribution and is given as

$$R_v = 1 - \sum_{j=i+v}^{2v} \sum_{i=1}^{v} \sum_{m_i} \frac{p_i p_j m_i}{2^{m_i}} \left[\frac{1}{m_i + \theta_j \alpha_i} + \frac{\theta_j \alpha_i}{(\theta_j + 1)(m_i + \theta_j \alpha_i)^2} \right] \quad (3.11)$$

Case IV: The stress Y follows mixture of M-Transformed Exponential distributions

As Y follows mixture of M-Transformed Exponential distributions, pdf of X and Y is given by

For two components, $v = 2$

$$f(x) = p_1 \frac{2e^{-\frac{x}{\alpha_1}}}{\alpha_1\left(2-e^{-\frac{x}{\alpha_1}}\right)^2} + p_2 \frac{2e^{-\frac{x}{\alpha_2}}}{\alpha_2\left(2-e^{-\frac{x}{\alpha_2}}\right)^2} \ ; \ p_1 + p_2 = 1, \alpha_1, \alpha_2, x > 0$$

$$g(y) = p_3 \frac{2e^{-\frac{y}{\alpha_3}}}{\alpha_3\left(2-e^{-\frac{y}{\alpha_3}}\right)^2} + p_4 \frac{2e^{-\frac{y}{\alpha_4}}}{\alpha_4\left(2-e^{-\frac{y}{\alpha_4}}\right)^2} \ ; \ p_3 + p_4 = 1,$$

As X and Y are independent then from (2), Reliability function R_2 is

$$R_2 = \int_0^\infty \int_0^x \left(p_1 \frac{2e^{-\frac{x}{\alpha_1}}}{\alpha_1\left(2-e^{-\frac{x}{\alpha_1}}\right)^2} + p_2 \frac{2e^{-\frac{x}{\alpha_2}}}{\alpha_2\left(2-e^{-\frac{x}{\alpha_2}}\right)^2} \right)\left(p_3 \frac{2e^{-\frac{y}{\alpha_3}}}{\alpha_3\left(2-e^{-\frac{y}{\alpha_3}}\right)^2} + p_4 \frac{2e^{-\frac{y}{\alpha_4}}}{\alpha_4\left(2-e^{-\frac{y}{\alpha_4}}\right)^2} \right) dxdy$$

$$R_2 = \sum_{j=i+2}^{4} \sum_{i=1}^{2} \int_0^\infty \int_0^x p_i \sum_{m_i=1}^{\infty} \frac{m_i}{2^{m_i}\alpha_i} e^{-\frac{m_i}{\alpha_i}x} \, p_j \sum_{m_j=1}^{\infty} \frac{m_j}{2^{m_j}\alpha_j} e^{-\frac{m_j}{\alpha_j}x} \, dxdy$$

$$R_2 = \sum_{j=i+2}^{4} \sum_{i=1}^{2} \int_0^\infty p_i p_j \sum_{m_i=1}^{\infty} \frac{m_i}{2^{m_i}\alpha_i} e^{-\frac{m_i}{\alpha_i}x} \left(1 - \sum_{m_j=1}^{\infty} \frac{1}{2^{m_j}} e^{-\frac{m_j}{\alpha_j}x} \right) dx$$

$$R_2 = 1 - \sum_{j=i+2}^{4} \sum_{i=1}^{2} p_i p_j \int_0^\infty \left(\sum_{m_i=1}^{\infty} \frac{m_i}{2^{m_i}\alpha_i} e^{-\frac{m_i}{\alpha_i}x} \right)\left(\sum_{m_j=1}^{\infty} \frac{1}{2^{m_j}} e^{-\frac{m_j}{\alpha_j}x} \right) dx$$

$$R_2 = 1 - \sum_{j=i+2}^{4} \sum_{i=1}^{2} p_i p_j \left(\frac{\alpha_j}{\alpha_i + \alpha_j} \right) \tag{3.12}$$

For three components, $v = 3$

$$f(x) = p_1 \frac{2e^{-\frac{x}{\alpha_1}}}{\alpha_1\left(2-e^{-\frac{x}{\alpha_1}}\right)^2} + p_2 \frac{2e^{-\frac{x}{\alpha_2}}}{\alpha_2\left(2-e^{-\frac{x}{\alpha_2}}\right)^2} + p_3 \frac{2e^{-\frac{x}{\alpha_3}}}{\alpha_3\left(2-e^{-\frac{x}{\alpha_3}}\right)^2} \ ; \ p_1 + p_2 + p_3 = 1$$

$$g(y) = p_4 \frac{2e^{-\frac{y}{\alpha_4}}}{\alpha_4\left(2-e^{-\frac{y}{\alpha_4}}\right)^2} + p_5 \frac{2e^{-\frac{y}{\alpha_5}}}{\alpha_5\left(2-e^{-\frac{y}{\alpha_5}}\right)^2} + p_6 \frac{2e^{-\frac{x}{\alpha_6}}}{\alpha_6\left(2-e^{-\frac{x}{\alpha_6}}\right)^2} \ ; \ p_4 + p_5 + p_6 = 1,$$

As X and Y are independent then from (2), Reliability function R_2 is

$$R_3 = \int_0^\infty \int_0^x \left(p_1 \frac{2e^{-\frac{x}{\alpha_1}}}{\alpha_1\left(2-e^{-\frac{x}{\alpha_1}}\right)^2} + p_2 \frac{2e^{-\frac{x}{\alpha_2}}}{\alpha_2\left(2-e^{-\frac{x}{\alpha_2}}\right)^2} + p_3 \frac{2e^{-\frac{x}{\alpha_3}}}{\alpha_3\left(2-e^{-\frac{x}{\alpha_3}}\right)^2} \right)\left(p_4 \frac{2e^{-\frac{y}{\alpha_4}}}{\alpha_4\left(2-e^{-\frac{y}{\alpha_4}}\right)^2} + p_5 \frac{2e^{-\frac{y}{\alpha_5}}}{\alpha_5\left(2-e^{-\frac{y}{\alpha_5}}\right)^2} + p_6 \frac{2e^{-\frac{x}{\alpha_6}}}{\alpha_6\left(2-e^{-\frac{x}{\alpha_6}}\right)^2} \right) dxdy$$

$$R_3 = \sum_{j=i+3}^{6} \sum_{i=1}^{3} \int_0^\infty \int_0^x p_i \sum_{m_i=1}^{\infty} \frac{m_i}{2^{m_i}\alpha_i} e^{-\frac{m_i}{\alpha_i}x} \, p_j \sum_{m_j=1}^{\infty} \frac{m_j}{2^{m_j}\alpha_j} e^{-\frac{m_j}{\alpha_j}x} \, dxdy$$

$$R_3 = \sum_{j=i+3}^{6} \sum_{i=1}^{3} \int_0^\infty p_i p_j \sum_{m_i=1}^{\infty} \frac{m_i}{2^{m_i}\alpha_i} e^{-\frac{m_i}{\alpha_i}x} \left(1 - \sum_{m_j=1}^{\infty} \frac{1}{2^{m_j}} e^{-\frac{m_j}{\alpha_j}x} \right) dx$$

$$R_3 = 1 - \sum_{j=i+3}^{6} \sum_{i=1}^{3} p_i p_j \int_0^\infty \left(\sum_{m_i=1}^{\infty} \frac{m_i}{2^{m_i}\alpha_i} e^{-\frac{m_i}{\alpha_i}x} \right)\left(\sum_{m_j=1}^{\infty} \frac{1}{2^{m_j}} e^{-\frac{m_j}{\alpha_j}x} \right) dx$$

$$R_3 = 1 - \sum_{j=i+3}^{6} \sum_{i=1}^{3} p_i p_j \left(\frac{\alpha_j}{\alpha_i + \alpha_j} \right) \tag{3.13}$$

In general for v-components,

$$f(x) = p_1 f_1(x) + p_2 f_2(x) + \cdots + p_v f_v(x)\ ; \quad \sum_{i=1}^{v} p_i = 1$$

$$g(y) = p_{v+1} f_{v+1}(y) + p_{v+2} f_{v+2}(y) + \cdots + p_{2v} f_{2v}(y)\ ; \quad \sum_{i=v+1}^{2v} p_i = 1$$

$$R_v = \int_0^\infty \int_o^x \left(p_1 f_1(x) + p_2 f_2(x) + \cdots + p_v f_v(x)\right)\left(p_{v+1} f_{v+1}(y) + p_{v+2} f_{v+2}(y) + \cdots + p_{2v} f_{2v}(y)\right) dxdy$$

$$R_v = \sum_{j=i+v}^{2v} \sum_{i=1}^{v} \int_0^\infty \int_0^x p_i \sum_{m_i=1}^{\infty} \frac{m_i}{2^{m_i}\alpha_i} e^{-\frac{m_i}{\alpha_i}x}\, p_j \sum_{m_j=1}^{\infty} \frac{m_j}{2^{m_j}\alpha_j} e^{-\frac{m_j}{\alpha_j}x}\, dxdy$$

$$R_v = \sum_{j=i+v}^{2v} \sum_{i=1}^{v} \int_o^\infty p_i p_j \sum_{m_i=1}^{\infty} \frac{m_i}{2^{m_i}\alpha_i} e^{-\frac{m_i}{\alpha_i}x} \left(1 - \sum_{m_j=1}^{\infty} \frac{1}{2^{m_j}} e^{-\frac{m_j}{\alpha_j}x}\right) dx$$

$$R_v = 1 - \sum_{j=i+2v}^{2v} \sum_{i=1}^{v} p_i p_j \int_o^\infty \left(\sum_{m_i=1}^{\infty} \frac{m_i}{2^{m_i}\alpha_i} e^{-\frac{m_i}{\alpha_i}x}\right)\left(\sum_{m_j=1}^{\infty} \frac{1}{2^{m_j}} e^{-\frac{m_j}{\alpha_j}x}\right) dx$$

$$R_v = 1 - \sum_{j=i+v}^{2v} \sum_{i=1}^{v} p_i p_j \left(\frac{\alpha_j}{\alpha_i + \alpha_j}\right) \tag{3.14}$$

4. Maximum Likelihood Estimation

$$L(\alpha_1, \alpha_2 p_1/\hat{y}) = \prod_{j=1}^{n} \left[p_1 \frac{2e^{-\frac{x_i}{\alpha_1}}}{\alpha_1 \left(2 - e^{-\frac{x_i}{\alpha_1}}\right)^2} + p_2 \frac{2e^{-\frac{x_i}{\alpha_2}}}{\alpha_2 \left(2 - e^{-\frac{x_i}{\alpha_2}}\right)^2} \right]$$

$$L(\alpha_1, \alpha_2 p_1/\hat{y}) = \frac{2^n n!}{n_1!\, n_2!} \left(\frac{p_1}{\alpha_1}\right)^{n_1} \left(\frac{p_2}{\alpha_2}\right)^{n_2} \prod_{i=1}^{n_1} \frac{e^{-\frac{x_i}{\alpha_1}}}{\left(2 - e^{-\frac{x_i}{\alpha_1}}\right)^2} \prod_{i=1}^{n_2} \frac{e^{-\frac{x_i}{\alpha_2}}}{\left(2 - e^{-\frac{x_i}{\alpha_2}}\right)^2}$$

$$\log L(\alpha_1, \alpha_2 p_1/\hat{y}) = log\left(\frac{2^n n!}{n_1!\, n_2!}\right) + n_1 log p_1 - n_1 log \alpha_1 + n_2 \log(1 - p_1) - n_2 log \alpha_2 + \sum_{i=1}^{n_1}\left[-\frac{x_i}{\alpha_1} - 2log\left(2 - e^{-\frac{x_i}{\alpha_1}}\right)\right] + \sum_{i=1}^{n_2}\left[-\frac{x_i}{\alpha_2} - 2log\left(2 - e^{-\frac{x_i}{\alpha_2}}\right)\right]$$

Now,

$$\frac{\partial \log L}{\partial \alpha_1} = 0$$

$$-\frac{n_1}{\alpha_1} + \sum_{i=1}^{n_1}\left[\frac{x_i}{{\alpha_1}^2} + \frac{2x_i e^{-\frac{x_i}{\alpha_1}}}{{\alpha_1}^2\left(2 - e^{-\frac{x_i}{\alpha_1}}\right)}\right] = 0$$

$$\alpha_1 = \frac{1}{n_1}\sum_{i=1}^{n_1}\left[x_i + \frac{2x_i e^{-\frac{x_i}{\alpha_1}}}{\left(2 - e^{-\frac{x_i}{\alpha_1}}\right)}\right] \tag{4.1}$$

$$\frac{\partial \log L}{\partial \alpha_2} = 0$$

$$-\frac{n_2}{\alpha_2} + \sum_{i=1}^{n_2}\left[\frac{x_i}{{\alpha_2}^2} + \frac{2x_i e^{-\frac{x_i}{\alpha_2}}}{{\alpha_2}^2\left(2 - e^{-\frac{x_i}{\alpha_2}}\right)}\right] = 0$$

$$\alpha_2 = \frac{1}{n_2}\sum_{i=1}^{n_2}\left[x_i + \frac{2x_i e^{-\frac{x_i}{\alpha_2}}}{\left(2 - e^{-\frac{x_i}{\alpha_2}}\right)}\right] \tag{4.2}$$

$\frac{\partial \log L}{\partial p_1} = 0$

$$\frac{n_1}{p_1} - \frac{n_2}{1-p_1} = 0$$

$$\hat{p}_1 = \frac{n_1}{n}\ ; n_1 + n_2 + 1 \tag{4.3}$$

Generalizing the above results for k-components we get

$$\alpha_i = \frac{1}{n_i}\sum_{j=1}^{n_1}\left[x_j + \frac{2x_i e^{-\frac{x_j}{\alpha_i}}}{\left(2 - e^{-\frac{x_j}{\alpha_i}}\right)}\right] \quad and \quad \hat{p}_i = \frac{n_i}{n}\ ; \quad n = \sum_{i=1}^{v} n_i \tag{4.4}$$

It is oblivious that first equation is nonlinear in α_j and is not easily solvable, therefore for obtaining their estimates we propose to use numerical iteration method. Therefore α_j can be obtained by using the nonlinear equation

$$h(\alpha_i) = \alpha_i \tag{4.5}$$

where

$$h(\alpha_i) = \frac{1}{n_i}\sum_{j=1}^{n_1}\left[x_j + \frac{2x_i e^{-\frac{x_j}{\alpha_i}}}{\left(2 - e^{-\frac{x_j}{\alpha_i}}\right)}\right] \tag{4.6}$$

since $\hat{\alpha}_i$ is a fixed point solution of $h(\alpha_i) = \alpha_i$, therefore it can be obtained by using simple iteration procedure as $h(\alpha_{i(k)}) = \alpha_{i(k+1)}$, where $\alpha_{i(k)}$ and $\alpha_{i(k+1)}$ are k[th] and (k+1)[th] iterations of $\hat{\alpha}_i$. When the difference between $\alpha_{i(k)}$ and $\alpha_{i(k+1)}$ is very small we stop the iteration and once we obtain the estimates $\hat{\alpha}_i$ *and* $\hat{p}_i$ the M.L.E of reliabilities for different cases are given as

1. M.L.E of reliability function when stress follows exponential distribution with known parameter λ and strength follows finite mixture of M-Transformed Exponential distribution with parameter α_i is
$$\hat{R}_v = 1 - \sum_{i=1}^{v}\sum_{m_i=1}^{\infty}\frac{\hat{p}_i m_i}{2^{m_i}(\lambda\hat{\alpha}_i + m_i)}$$
2. M.L.E of reliability function when stress follows Lindley distribution with known parameters λ and θ and strength follows finite mixture of M-Transformed Exponential distributions with parameter α_i is
$$\hat{R}_v = 1 - \sum_{i=1}^{v}\sum_{m_i=1}^{\infty}\hat{p}_i\frac{m_i}{2^{m_i}}\left(\frac{1}{m_i + \theta\hat{\alpha}_i} + \frac{\lambda\theta\hat{\alpha}_i}{(\theta+\lambda)(m_i+\theta\hat{\alpha}_i)^2}\right)$$
3. M.L.E of reliability function when stress follows finite mixture of Lindley distributions with known parameters λ_j *and* θ_j and strength follows finite mixture of M-Transformed Exponential distributions with parameter α_i is
$$\hat{R}_v = 1 - \sum_{j=i+v}^{2v}\sum_{i=1}^{v}\sum_{m_i=1}^{\infty}\frac{\hat{p}_i\hat{p}_j m_i}{2^{m_i}}\left[\frac{1}{m_i + \theta_j\hat{\alpha}_i} + \frac{\lambda_j\theta_j\hat{\alpha}_i}{(\theta_j+\lambda_j)(m_i+\theta_j\hat{\alpha}_i)^2}\right]$$
4. M.L.E of reliability function when stress and strength both follows finite mixture of M-Transformed Exponential distributions with parameters α_i and α_j is
$$\hat{R}_v = 1 - \sum_{j=i+v}^{2v}\sum_{i=1}^{v}\hat{p}_i\hat{p}_j\left(\frac{\hat{\alpha}_j}{\hat{\alpha}_i + \hat{\alpha}_j}\right)$$

5. Numerical Evaluation

In this section, we have evaluated reliability of a system with two components for specific values of a parameters involved in the reliability expressions in section 3 using graphical approach. Here we will discuss how reliability of two component system behaves when stress, strength or probability parameters are changed.

Case I: Stress follows exponential distribution and strength follows finite mixture of M-Transformed Exponential distribution

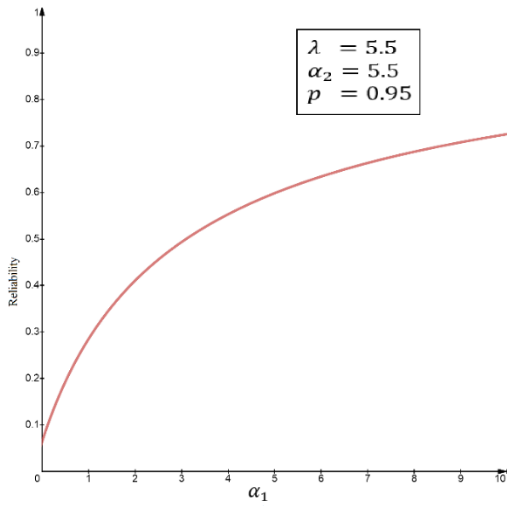

Figure 1

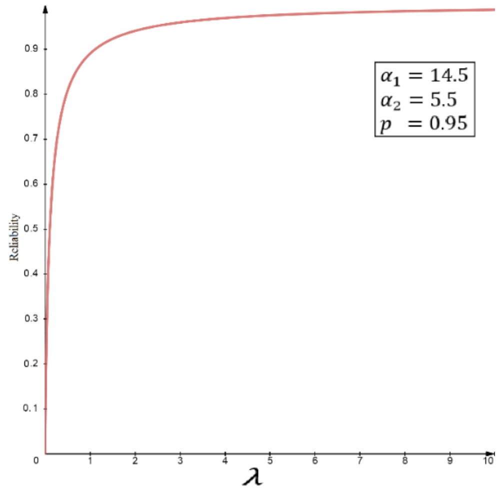

Figure 2

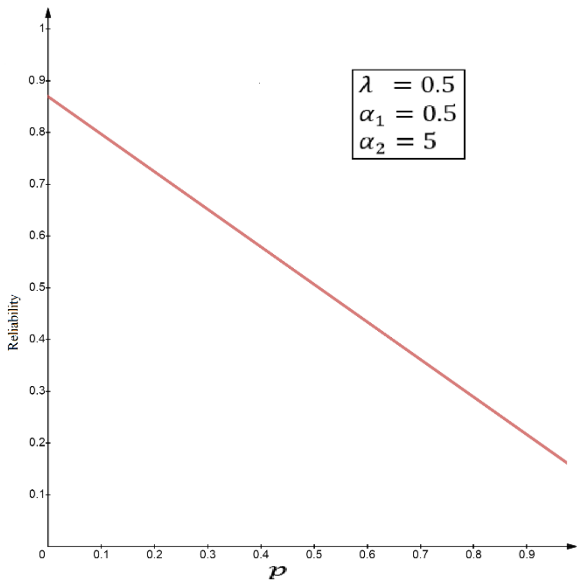

Figure 3

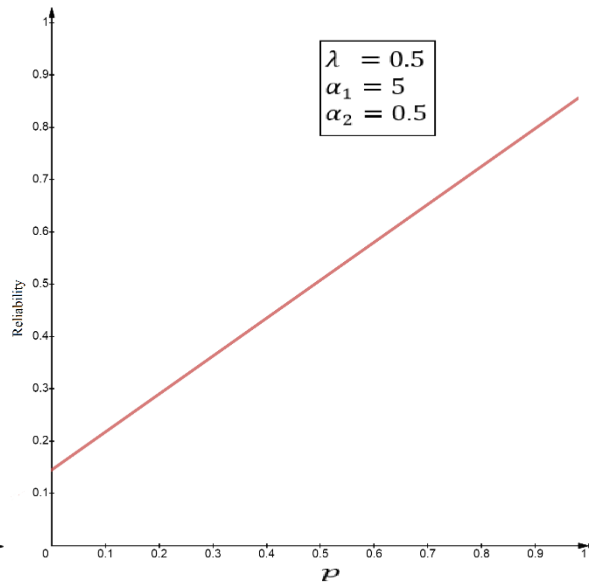

Figure 4

Case II: Stress follows Lindley distribution and strength follows finite mixture of M-Transformed Exponential distribution

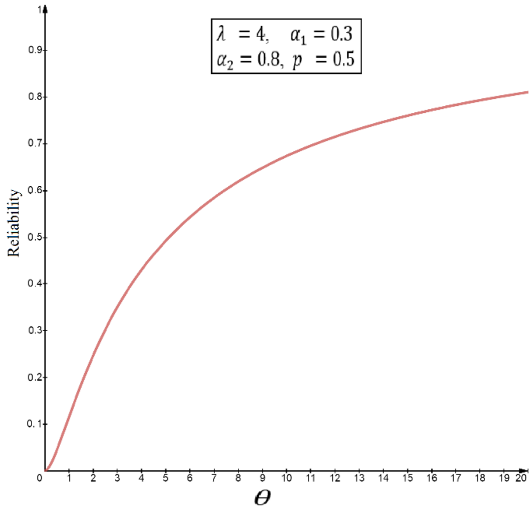

Figure 5

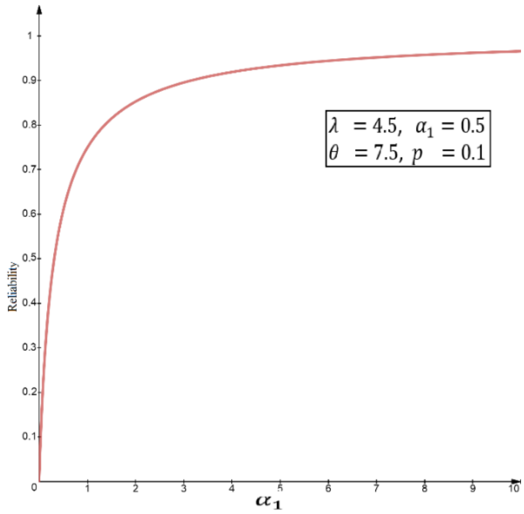

Figure 6

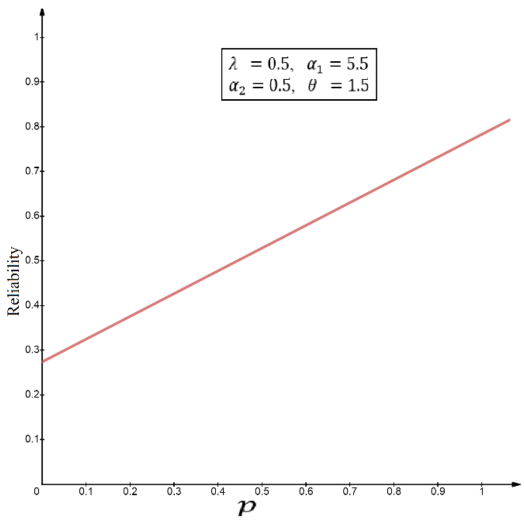

Figure 7

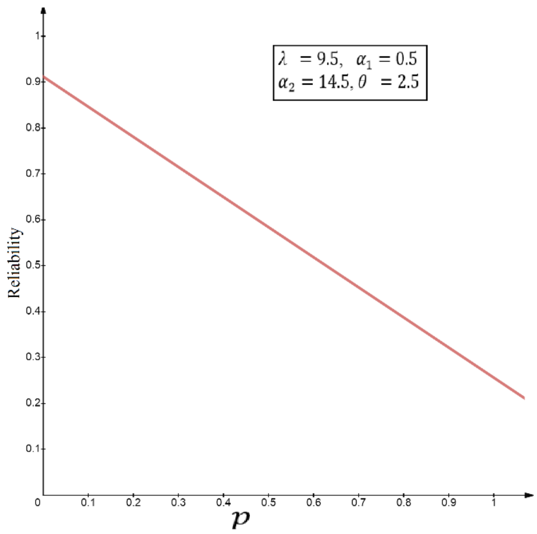

Figure 8

Case III: Stress follows finite mixture of Lindley distributions and strength follows finite mixture of M-Transformed Exponential distributions

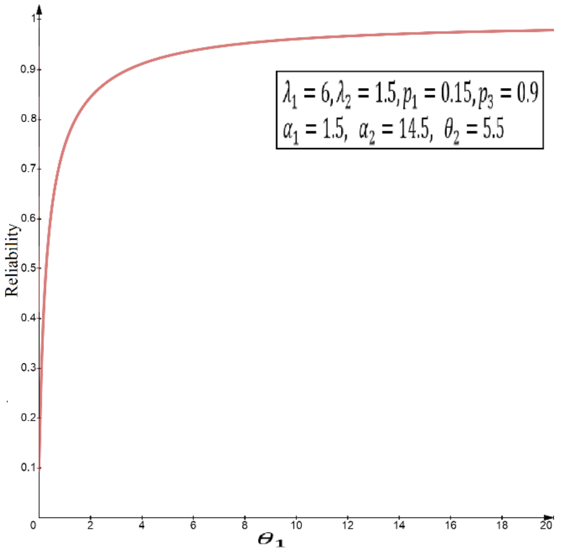

Figure 9

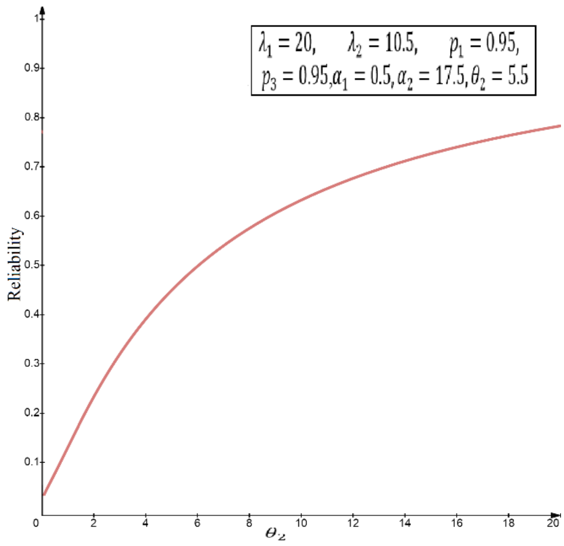

Figure 10

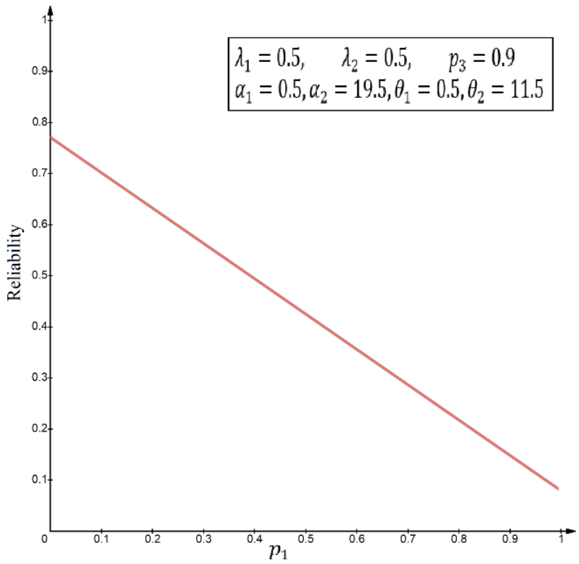

Figure 11

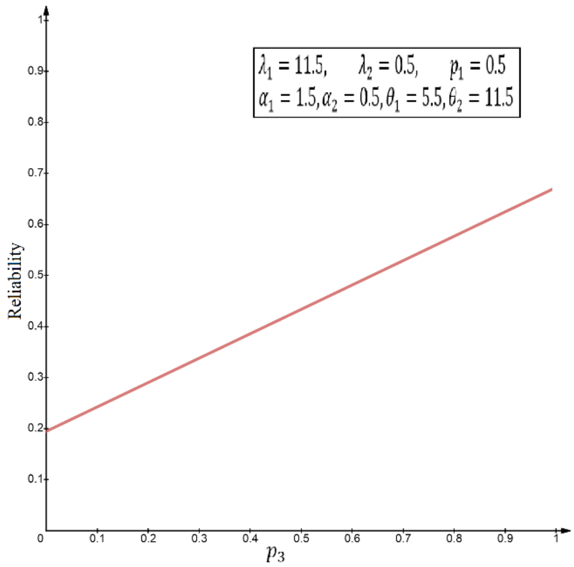

Figure 12

Case IV: Stress and strength both follows finite mixture of M-Transformed Exponential distributions

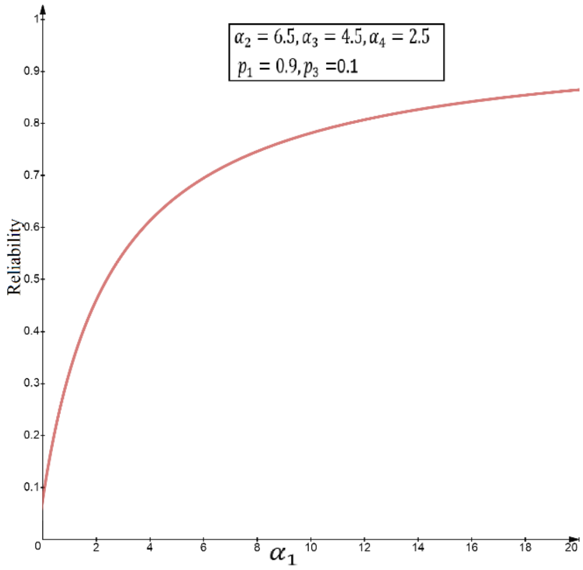

Figure 13

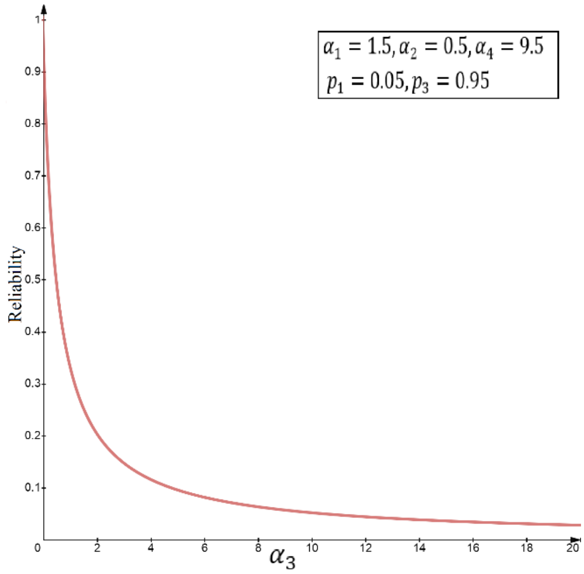

Figure 14

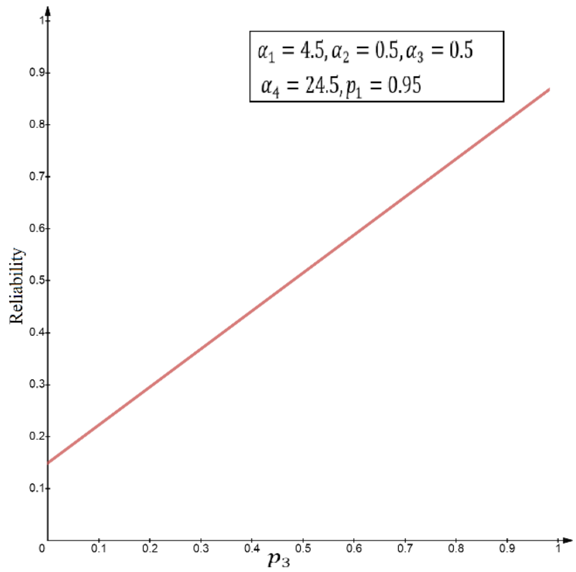

Figure 15

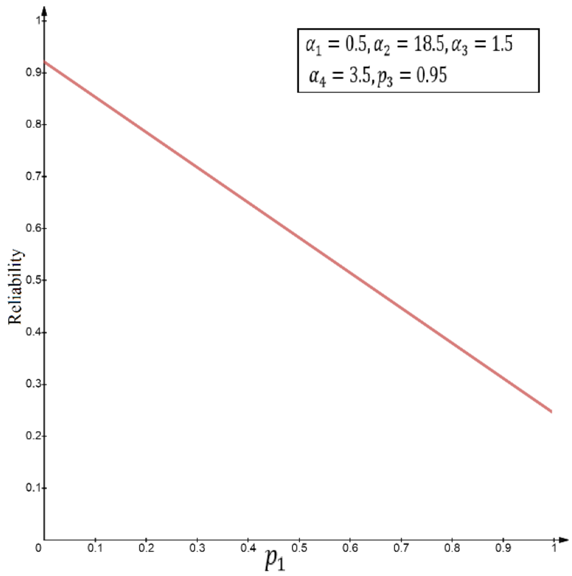

Figure 16

6. Real Data Analysis

In this sub section we analyze 33 leukaemia patients with two causes of death AG positive (presence of Auer rods and/or significant granulature of the leukaemic cells) or AG negative (both Auer rods and granulature are absent). The survival times in weeks are given

AG positive patients: 65, 156, 100, 134, 16, 108, 121, 4, 39, 143, 56, 26, 22, 1, 1, 5, 65

AG negative patients: 56, 65, 17, 7, 16, 22, 3, 4, 2, 3, 8, 4, 3, 30, 4, 43

We use the iterative procedure to obtain MLE of the parameters α_1 and α_2 using equation (4.5) and the final value of estimates are $\hat{\alpha}_1 = 90.65061, \hat{\alpha}_2 = 25.44067$ *and* $\hat{p}_1$ =0.5151.Based on these estimates the MLE of the reliability are given below

Stress and strength both follows finite mixture of M-Transformed Exponential distributions $\widehat{R}_2 = 0.68$

Table 1: Stress follows exponential distribution and strength follows finite mixture of M-Transformed Exponential distribution

R	0.9161	0.9547	0.9689	0.9763	0.9809	0.9840	0.9862	0.9880	0.9892	0.9903
λ	0.5	1	1.5	2	2.5	3	3.5	4	4.5	5

Table 2: Stress follows Lindley distribution and strength follows finite mixture of M-Transformed Exponential distribution

θ \ λ	0.5	1	1.5	2	2.5	3	3.5	4	4.5	5
0.5	0.9317	0.9264	0.9238	0.9223	0.9212	0.9205	0.9200	0.9195	0.9191	0.9189
1	0.9409	0.9340	0.9298	0.9270	0.9250	0.9235	0.9224	0.9215	0.9207	0.9200
1.5	0.9450	0.9375	0.9327	0.9292	0.9265	0.9245	0.9228	0.9215	0.9204	0.9195
2	0.9473	0.9399	0.9346	0.9306	0.9275	0.9250	0.9230	0.9214	0.9200	0.9187
2.5	0.9490	0.9415	0.9360	0.9317	0.9283	0.9255	0.9232	0.9212	0.9195	0.9181
3	0.9500	0.9427	0.9371	0.9326	0.9289	0.9258	0.9233	0.9211	0.9191	0.9174
3.5	0.9507	0.9436	0.9380	0.9333	0.9294	0.9262	0.9234	0.9209	0.9188	0.9169
4	0.9514	0.9444	0.9387	0.9340	0.9299	0.9264	0.9234	0.9208	0.9185	0.9164
4.5	0.9520	0.9450	0.9393	0.9344	0.9303	0.9266	0.9235	0.9207	0.9182	0.9160
5	0.9523	0.9455	0.9398	0.9348	0.9305	0.9268	0.9235	0.9206	0.9180	0.9156

Table 3: Stress follows finite mixture of Lindley distributions and strength follows finite mixture of M-Transformed Exponential distributions, $p_3 = 0.35, \lambda_2 = 6.5, \theta_2 = 5.5$

θ_1 \ λ_1	1	2	3	4	5	6	7	8	9	10
1	0.9680	0.9656	0.9644	0.9637	0.9632	0.9629	0.9626	0.9624	0.9622	0.9621
2	0.9784	0.9762	0.9748	0.9739	0.9733	0.9728	0.9725	0.9722	0.9719	0.9717
3	0.9818	0.9794	0.9780	0.9768	0.9760	0.9754	0.9749	0.9744	0.9741	0.9738
4	0.9833	0.9810	0.9792	0.9779	0.9770	0.9761	0.9754	0.9750	0.9744	0.9740
5	0.9842	0.9817	0.9798	0.9784	0.9771	0.9762	0.9754	0.9747	0.9741	0.9736
6	0.9848	0.9822	0.9801	0.9785	0.9772	0.9761	0.9751	0.9743	0.9736	0.9730
7	0.9852	0.9825	0.9803	0.9786	0.9771	0.9758	0.9747	0.9738	0.9730	0.9723
8	0.9855	0.9827	0.9804	0.9786	0.9769	0.9755	0.9743	0.9733	0.9723	0.9715
9	0.9858	0.9829	0.9805	0.9785	0.9768	0.9753	0.9740	0.9728	0.9717	0.9708
10	0.9860	0.9830	0.9806	0.9784	0.9766	0.9750	0.9735	0.9723	0.9711	0.9701

7. Conclusion

In this paper we have obtained stress strength reliability of a system with v-components using different distributions for stress variables viz. exponential distribution, M-Transformed exponential distribution and for strength variable we have used M-Transformed exponential distribution. In the last section we showed that reliability of 2-component system can be monotonically increasing and monotonically decreasing for specific values of parameter. Thus by proper choice of parameters leads to high reliability. The maximum likelihood estimates of parameters involved in the reliability function of v-components system are also obtained using iteration method.

References

1. Awad, A.M., Gharraf, M.K., 1986, Estimation of (P(Y < X) in the Burr case: A comparative study. Commun. Statist. Simul. Comp., 15(2), 389-403.
2. Beg, M.A., Singh, N., 1978, Estimation of P(Y < X) for the pareto distribution. IEEE Trans. Reliab., 28(5), 411-414.
3. Church, J.D., Harris, B., 1970, The estimation of reliability from stress strength relationships. Technometrics, 12, 49-54.
4. Khan, A.H., Jan, T.R., 2014, Estimation of Multi Component Systems Reliability in Stress-Strength Models. Journal of Modern Applied Statistical Methods, 13 (2), 389-398.
5. Khan, A.H., Jan, T.R., 2014, Reliability Estimates of Generalized Poisson Distribution and Generalized Geometric Series Distribution. Journal of Modern Applied Statistical Methods, 13 (2), 379-388.
6. Kotz, S., Lumelskii, Y., Pensky, M., 2003, The Stress-Strength Model and its Generalizations: Theory and Applications, World Scientific Publishing, Singapore.
7. Kumar, D., Singh, U., Singh, S.K, Mukherjee, S., 2017, The New Probability Distribution: An Aspect to Lifetime Distribution. Math. Sci. Lett. 6(1), 35-42.
8. Mokhlis, N.A., 2005, Reliability of a stress-strength model with Burr Type III distributions. Commun. Statist.Theory Meth. 34(7). 1643-1657.
9. Raqab, M. Z., Kunda, D., 2005, Comparison of Different Estimates of P(Y < X) for a Scaled Burr Type X distribution, Communication in Statistics-Computations and Simulations, 34(2), 465-483.
10. Saraçoğlu, B., Kınacı, İ., Kundu, D., 2011, On estimation of (P(Y < X) for exponential distribution under progressive type-II censoring. J. Stat. Comput. Simul. 82(5), 729-744.
11. Surles, J. G., Pudgett, W. J., 1998, Inference for P(Y < X) in the Burr X Model, Journal of Applied Statistical Sciences, 225-238.
12. Surles, J. G., Pudgett, W. J., 2001, Inference for Reliability and Stress-Strength for Scald Burr X distribution, Lifetime Data Analysis, 7, 187-200.
13. Woodward, W.A., Kelley, G.D., 1977, Minimum variance unbiased estimation of P(Y < X) in the normal case. Technometrics, 19, 95-98.

Analysis of MAP/PH/1 Queueing model with Setup, Closedown, Multiple Vacations, Standby Server, Breakdown, Repair and Reneging

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Abstract

We have analyzed a single server queueing model in which the arrival of customers according to the Markovian arrival process, the service process according to phase type distributions and the standby server who is serving at a lower rate also follows the phase type distribution. If any of the customers present in the system when the server completes a vacation who starts the setup process to initiate service to the customers. After service completion, the main server begins the closedown process. The total number of customers are present in the system under the steady-state probability vector has been investigated by using the Matrix-Analytic method. We have examined the stability condition, the analysis of the busy period and derived some important performance measures of our model. Numerical results and graphical representation are discussed for the proposed model.

Keywords: Markovian Arrival Process, Phase type Distributions, Standby Server, Setup, Closedown, Multiple Vacation.

AMS Subject Classification (2010): **60K25, 68M20, 90B22.**

1. Introduction

On the basis of the study, the concept of the Markovian Arrival Process (MAP) has been introduced by Neuts (1981), the PH-representation is a Markov renewal process in service times and in general, MAP is the non-renewal process and it is commensurate to the Versatile modeling tool of Markovian Point Process (VMPP). This point process is fairly extensively described and well developed by the MAP is a specific type of semi-Markov process with a transition probability matrix (TPM). Later, it was realized that the VMPP and Batch Markovian Arrival Process (BMAP) are equivalent processes.

The Matrix-analytic methods(MAM) had been first introduced and examined by Neuts(1981). Qi-Ming He (2004), has analyzed the fundamentals of Matrix-Analytic methods such that the concept of arrival and the service process. Chakravarthy(2010) has described the various types of arrivals in which the customer's arrival follows the Markovian Arrival Process (MAP) with representation (D0, D1) of a square matrix whose order is m. The representation of the service times is (α, T) which follows phase type distributions whose matrix order is n. Let the generator Q is defined by Q = D0 + D1, is an irreducible stochastic matrix. The matrix D0 governs for transitions corresponding to no arrival, it has non-negative off-diagonal elements and non-singular with negative diagonal elements. The matrix D1 governs for transitions corresponding to arrival such that both diagonal and off-diagonal elements are non-negative.

If $\pi 1$ is the unique probability vector of the Markov process described by the irreducible generator Q satisfying $\pi 1 Q = 0$ and $\pi 1 e = 1$. The constant $\lambda = \pi 1 D1e$ makes reference to the fundamental arrival rate, it will give us the expected number of customers arrive per unit time under the stationary version of the Markovian Arrival Process. The Marked Markovian Point Process (MMPP) is a special type of a doubly stochastic Poisson process whose arrival rate is modified by the states of an irreducible finite-state Continuous-Time Markov Chain (CTMC).

Attahiru Sule Alfa (1995) examined a discrete MAP/PH/1 in which the server offers service for a limited period of time and then the server goes on to another queue, in such a case it may consider server proceeds on a vacation. Jinbiao Wu et al. (2009) investigated the two types of arrivals such that positive and negative arrivals on the batch Markovian arrival process and the customer may go for G-queue with the second optional service. When the system empty, the server allows to take multiple vacations and they developed queue size distribution using the supplementary variable technique. Chesoong Kim et al. (2017) described unreliable BMAP/PH/N queueing type with breakdown occurrence moments are considered by Markovian arrival process and if the server fails immediately the repair period starts in which the duration of repair follows PH-distribution. Ayyappan and Shyamala (2014) have examined the concepts of setup time, breakdown and repair in a coherent way of batch arrival of customers with two heterogeneous servers.

Chakravarthy and Neuts (2014) have analyzed the queueing model of multi servers with two types of arrivals in which one type of customer is regular customers whose arrival follows the Markovian arrival process and another type is special customers whose arrival follows phase type renewal process. The regular customer requires only one server's attention but the special customer requires attention to all the servers. Furthermore, special customers possibly pre-empting the services of regular customers. Qi-Ming He and Attahiru Sule Alfa (2015) have studied the MAP/PH/K queueing model of the construction of Markov chains. Among these Markov chains, the first one is introduced through tracking service phases for the server which is construction of transition probability matrix in a straightforward manner and the second one is introduced through counting servers for phases which are using an algorithm for construction transition probability matrix. Zenios (1999) analyzed the queueing model with reneging such that the transplant waiting due to fear of organ transplantation that may lead to death. He has taken a survey at the end of 1996, the registered candidates for transplantation are 34,550 candidates but among these only 7,833 transplantations had performed and the remaining candidates were reneged due to impatience.

Arumuganathan and Jayakumar (2005) have analyzed the bulk queueing model with setup and closedown times. After completion of vacation if the queue size has reached N the server starts the setup process and the server starts the closedown process when the queue size less than N. Subsequently, they developed the cost model for their model. Wei Sun et al. (2012) developed Markovian queueing systems with three types of setup/closedown policies in which types are interruptible, skippable and insusceptible. Among these insusceptible explains if the customer comes during closedown times the service will start to the customer. The second type explains if a customer comes during closedown, they would be served after the closedown finishes and skipped the setup time and the third type tell the customers to arrive during the closedown time they could get service and they have to wait until the setup time finishes.

Tsung - Yin Wang (2012) has studied the Geo/G/1 queueing model with startup under N-policy. This N-policy is applicable for the server temporarily unavailable to the waiting customers. Service completed to all the customers, the server is being shut down in a closedown time. Zhisheng Niu (2003) examined the single server batch Markovian arrival processes of setup and closedown under single and multiple vacation queue in which they described the potential applications that the first one is switched virtual connection-based Internet protocols over Automated Teller Machine networks and the second one is multiple protocol label switched networks.

Many researchers incorporating their model with standby support. Sreekanth Kolledath et al. (2017) have analyzed a survey on standby support queueing models. They described different types of standby's are cold standby, warm standby, hot standby, mixed standby and standby switching. Among these the cold standby tells about the standby with zero failure rate, the warm standby tells about the standby with a lower failure rate than compared to the primary components, the hot standby tells about the standby has the same failure rate as the primary components, mixed standby is the new concept of the combination of cold and warm standby and the standby switching explains when switching the standby in place of the main one which may be unsuccessful that is standby switching failure has happened. In this situation, until switching is successful all the standby's are try to switch over one by one. Furthermore, we have referred to the concepts of standby in Subramanian and Sarm (1987) and Khalaf et al. (2011).

In real-life situations, pupils would like to do any of the work they have to do some of the preparatory action known as the setup process. After completion of that work, they do some shutdown action known as the closedown process. For example, the grocery store, supermarkets, Industries, computer systems, laptops, communication systems and hypermarkets. These examples are suitable for our model as setup time, closedown time, vacation, breakdown and standby server. Among these examples, the hypermarkets are an interesting concept of the combination of supermarkets and departmental stores. It has a wide range of shopping facilities such as including general merchandise and all kinds of grocery lines on one floor itself and it needs a large landscape to locate this one. In one trip itself, hypermarkets offer to the customers for buying whatever things they need in the routine shopping. These kinds of big-box stores need some amount of time to make the setup process and some amount of time to take for closing the hypermarkets. During the vacation times of hypermarkets, the reneging might happen due to impatience.

The remaining part of the article is organized as follows. We describe our mathematical model description in section 2. In section 3, we are generating our matrix formulation and notations of our model also included. In section 4 we discuss the stability condition and steady-state probability vector. In section 5, we have analyzed the busy period analysis and in section 6, measures of system performance are discussed. In section 7, presents some of the illustrated numerical and graphical representations. The main server and standby server service rates are compared in section 8. The conclusion of our model has been given in the last section 9.

2. The Mathematical model Description

In this model, we consider the arrival of customers follows the Markovian arrival process which represents (D0, D1), where D0 and D1 are square matrices of order is m′, the service process follows phase type distribution which represents (α, T) of order is n′ and the standby server service process also follows the phase type distribution which represents ($\alpha 1$, θT) of matrix order is n′ with T0 + Te = 1 such that T0 = -Te and we are taking α and $\alpha 1$ are the same for the purpose of differentiating the main server and standby server such that $\alpha = \alpha 1$. While the main server giving service to the customers, the server may breakdown at any time. When the breakdown has occurred, the main server goes for the repair process, immediately standby server switch over instead of the main server, then the standby server carry over the service process but at the lower rate compared to the main server service rate with representation θT where $0<\theta<1$. When the main server returns to the service station after rejuvenating from the repair, at that moment if the standby server is in the idle state. Obviously, the main server would be in the idle state until the customer's arrival to the system otherwise if the standby server is busy when the main server return to the system from the repair process then the main server would interrupt the standby server and carry over the service process. The breakdown times and repair times are exponentially distributed. After completing service to the customers, if there is no one in the system the main server starts the closedown process, afterward closedown the main server goes for vacation. When the main server return from vacation if there is no customer in the system then the server will go for vacation repeatedly until if the server finds at least one customer in the queue. The duration of vacation times follows exponential distribution at the rate η. After completion of vacation, if there is a customer in the queue then the main server does the setup process and then starts giving service to customers who are standing in the queue. Due to impatience, the customers who have an aspiration to get service may have reneged from the system that is they leave the system during the main server vacation period and the renege rate is ζ. The parameters of setup rate and closedown rate are σ and γ respectively.

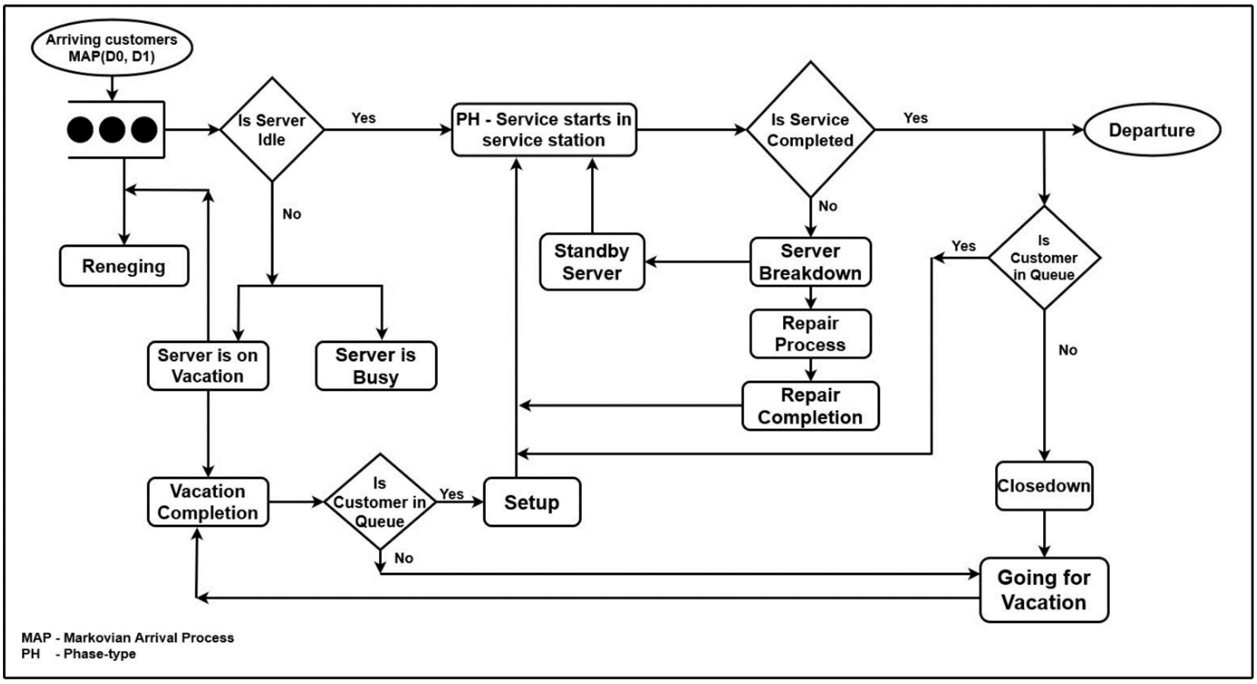

Figure 1: Pictorial Representation of our proposed model.

3. The Matrix Generation - QBD process

In this section, we have described the notation of our model as follows for the purpose of generating the QBD Process.

Notations for Matrix Generation

- $\otimes$ - Kronecker product of any two of the different order of matrices by using this symbol.
- $\oplus$ - Kronecker sum of any two of the different order of matrices by using this symbol.

- In - It denotes an n-dimensional Identity matrix.
- Im - It denotes an m-dimensional Identity matrix.
- Inm - It denotes an nm-dimensional Identity matrix.

- e - Column vector of suitable dimension each of its entry is 1.
- en - Column vector whose dimension is n and each of its entries are 1.
- enm - Column vector whose dimension is nm and each of its entries are 1.
- Let us denote λ be the arrival rate and is defined as λ = π1D1em, where π1 is the Probability vector of the generator matrix D = D0 + D1.
- The normal service rate of the main server and the standby server is denoted by δ and $\theta\delta$ where $\delta = [\alpha(-T)^{-1}en]^{-1}$.
- Define N(t) indicates the number of customers in the system.

- Define V(t) indicates the status of server at time t,

$$V(t) = \begin{cases} 1, & \text{ifthemainserverisinbusy,} \\ 2, & \text{ifthemainserverisinthebreakdown,} \\ 3, & \text{ifthemainserverisinthesetupprocess,} \\ 4, & \text{ifthemainserverisintheclosedown,} \\ 5, & \text{ifthemainserverisonvacation.} \end{cases}$$

- J(t) is the service process considered by phases.
- M(t) is the arrival process considered by phases.

Let {(N(t) , V(t) , J(t) , M(t) : t … 0 } be the Continuous-Time Markov Chain (CTMC) with state level independent Quasi-Birth-and-Death process whose state space is as follows,

Φ = l(0) ∪ l(p).

where,
l(0) = {(0,2,s) : 1 ≤ s ≤ m} ∪ {(0,4,s) : 1 ≤ s ≤ m} ∪ (0,5,s) : 1 ≤ s ≤ m}.
for p ≥ 1,
l(p) = {(p,1,r,s):1 ≤ r ≤ n; 1 ≤ s ≤ m} ∪ {(p,2,r,s): 1 ≤ r ≤ n; 1 ≤ s ≤ m}
∪ {(p,3,s) : 1 ≤ s ≤ m} ∪ {(p,4,s) : 1 ≤ s ≤ m} ∪ {(p,5,s) : 1 ≤ s ≤ m}.
The infinitesimal matrix generation of the QBD Process is given by,

$$Q = \begin{bmatrix} B_{00} & B_{01} & 0 & 0 & 0 & 0 & \cdots & \cdots \\ B_{10} & A_1 & A_0 & 0 & 0 & 0 & \cdots & \cdots \\ 0 & A_2 & A_1 & A_0 & 0 & 0 & \cdots & \cdots \\ 0 & 0 & A_2 & A_1 & A_0 & 0 & \cdots & \cdots \\ 0 & 0 & 0 & A_2 & A_1 & A_0 & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \\ \vdots & \vdots & \vdots & \vdots & & \ddots & \ddots & \ddots \end{bmatrix}$$

The entries in the block matrices of Q are defined as follows,

$$B_{00} = \begin{bmatrix} D0 - \Psi Im & \Psi Im & 0 \\ 0 & D0 - \gamma Im & \gamma Im \\ 0 & 0 & D0 \end{bmatrix},$$

$$B_{01} = \begin{bmatrix} 0 & \alpha 1 \otimes D1 & 0 & 0 & 0 \\ 0 & 0 & 0 & D1 & 0 \\ 0 & 0 & 0 & 0 & D1 \end{bmatrix},$$

$$B_{10} = \begin{bmatrix} 0 & T0 \otimes Im & 0 \\ \theta T0 \otimes Im & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \zeta Im & 0 \\ 0 & 0 & \zeta Im \end{bmatrix},$$

$$A_1 = \begin{bmatrix} T \oplus D0 - \tau Inm & \tau Inm & 0 & 0 & 0 \\ \Psi Inm & \theta T \oplus D0 - \Psi Inm & 0 & 0 & 0 \\ \alpha \otimes \sigma Im & 0 & D0 - \sigma Im & 0 & 0 \\ 0 & 0 & 0 & D0 - \gamma Im - \zeta Im & \gamma Im \\ 0 & 0 & \eta Im & 0 & D0 - \eta Im - \zeta Im \end{bmatrix},$$

$$A_0 = \begin{bmatrix} In \otimes D1 & 0 & 0 & 0 & 0 \\ 0 & In \otimes D1 & 0 & 0 & 0 \\ 0 & 0 & D1 & 0 & 0 \\ 0 & 0 & 0 & D1 & 0 \\ 0 & 0 & 0 & 0 & D1 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} T0\alpha \otimes Im & 0 & 0 & 0 & 0 \\ 0 & \theta T0\alpha 1 \otimes Im & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \zeta Im & 0 \\ 0 & 0 & 0 & 0 & \zeta Im \end{bmatrix}$$

4. Stability Condition

We have analyzed our model under some condition that whether the system is stable.

4.1. Analysis of Stability condition

Let us define the matrix A as A = A0 + A1 + A2. It has clearly shown that the arrangement of the square matrix A is 2nm+3m and this matrix is an irreducible infinitesimal generator matrix.

The vector ξ is denoted by ξ = (ξ1, ξ2, ξ3, ξ4, ξ5). Let ξ be the steady-state probability vector of A satisfying ξA = 0 and ξe = 1, where ξ1 and ξ2 are of dimension nm and ξ3, ξ4, ξ5 are of dimension m. The Markov process has the quasi-birth-and-death structure, there exits stability of our model should satisfy ξA0e < ξA2e, is the necessary and sufficient condition of a QBD process. The vector ξ is calculated by solving the following equations,

$$\xi 1[(T + T0\alpha) \oplus D - \tau Inm] + \xi 2[\Psi Inm] + \xi 3[\alpha \otimes \sigma Im] = 0.$$

$$\xi 1[\tau Inm] + \xi 2[(\theta T + \theta T0\alpha 1) \oplus D - \Psi Inm] = 0.$$

$$\xi 3[D - \sigma Im] + \xi 5[\eta Im] = 0.$$

$$\xi 4[D - \gamma Im] = 0.$$

$$\xi 4[\gamma Im] + \xi 5[D - \eta Im] = 0.$$

subject to the normalizing condition

$$\xi 1enm + \xi 2enm + \xi 3em + \xi 4em + \xi 5em = 1.$$

After some algebraical manipulation, the stability condition ξA0e < ξA2e which is turned to be as follows,

$$(\xi 1 + \xi 2)[en \otimes D1em] + (\xi 3 + \xi 4 + \xi 5)[D1em] < \xi 1[T0 \otimes em] + \xi 2[\theta T0 \otimes em] + (\xi 4 + \xi 5)[\zeta Im].$$

4.2. Analysis of Steady-state Probability vector

Let us take the variable x is the probability vector and is partitioned as $x = (x0, x1, x2, \ldots\ldots)$. Hence, x be the steady-state probability vector of Q. Here we are mentioning that $x0$ is of dimension 3m and $x1, x2, \ldots$ are of dimension 2nm+3m. Then x satisfies the condition $xQ = 0$ and $xe = 1$.

Moreover, when the stability condition has been satisfied with the subvectors of x except for $x0$ and $x1$, commensurate to the different level states are given by the equation as,

$$xj = x1\, R^{j-1}, \quad j \quad 2.$$

where the rate matrix R denotes the minimal non-negative solution of the matrix quadratic equation as $R^2A2 + RA1 + A0 = 0$, which is referred by the author Neuts(1981).

Since our system is stable, and if adding the square matrices of A0, A1 and A2 whose row sums are equal to zero, then the rate matrix R is a square matrix of order is 2nm+3m, it is obtained from the above quadratic equation and satisfies the relation RA2e = A0e.

The sub vectors of $x0$ and $x1$ have obtained by solving the following equations,

$$x0B00 + x1B10 = 0.$$
$$x0B01 + x1(A1 + RA2) = 0.$$

subject to the normalizing condition is

$$x0e3m + x1(I - R)^{-1}e2nm + 3m = 1.$$

Thus, the R matrix could be calculated mathematically using essential steps in the Logarithmic reduction algorithm of R are given below we have referred the author's Latouche and Ramaswami(1999).

Theorem: The structure of the rate matrix R is

$$R = \begin{bmatrix} R11 & R12 & 0 & 0 & 0 \\ R21 & R22 & 0 & 0 & 0 \\ R31 & R32 & R33 & 0 & 0 \\ R41 & R42 & R43 & R44 & R45 \\ R51 & R52 & R53 & 0 & R55 \end{bmatrix} \quad (1)$$

Proof: The computation of the matrix R, it is clearly shown that R must have the structure for our model as given in (1). The main server may be struck with breakdown while giving service leads to main server can go for repair, in this situation the standby server giving service at lower rate and the main server is being in service after return from the repair completion. Furthermore, here we will give proof of the construction of R. We can rewrite the matrix quadratic equation $R^2A2 + RA1 + A0 = 0$ is given by,

$$R = (R^2A2 + A0)(-A1)^{-1}$$

It can easily verify that the structure of the matrix $(-A1)^{-1}$ as follows,

$$(-A1)^{-1} = \frac{1}{V}\begin{bmatrix} f11 & f12 & 0 & 0 & 0 \\ f21 & f22 & 0 & 0 & 0 \\ f31 & f32 & f33 & 0 & 0 \\ f41 & f42 & f43 & f44 & f45 \\ f51 & f52 & f53 & 0 & f55 \end{bmatrix} \quad (2)$$

where the elements of $(-A1)^{-1}$ as follows,

$$V = [[(T \oplus D0) - \tau Inm][(\theta T \oplus D0) - \Psi Inm] - [\Psi Inm][\tau Inm]] \times [[D0 - \sigma Im] \times [D0 - \gamma Im - \zeta Im][D0 - \eta Im - \zeta Im]],$$

$$f11 = [[(\theta T \oplus D0) - \Psi Inm][\sigma Im - D0][D0 - \gamma Im - \zeta Im][D0 - \eta Im - \zeta Im]],$$

$$f12 = [[\tau Inm][D0 - \sigma Im][D0 - \gamma Im - \zeta Im][D0 - \eta Im - \zeta Im]],$$

$$f21 = [[\Psi Inm][D0 - \sigma Im][D0 - \gamma Im - \zeta Im][D0 - \eta Im - \zeta Im]],$$

$$f22 = [[\tau Inm - (T \oplus D0)][D0 - \sigma Im][D0 - \gamma Im - \zeta Im][D0 - \eta Im - \zeta Im]],$$

$$f31 = [[\alpha \otimes \sigma Im][(\theta T \oplus D0) - \Psi Inm][D0 - \gamma Im - \zeta Im][D0 - \eta Im - \zeta Im]],$$

$$f32 = [[\alpha \otimes \sigma Im][\tau Inm][\gamma Im + \zeta Im - D0][D0 - \eta Im - \zeta Im]],$$

$$f33 = [[\Psi Inm][\tau Inm] - [(T \oplus D0) - \tau Inm][(\theta T \oplus D0) - \Psi Inm]] \times [[D0 - \gamma Im - \zeta Im] \times [D0 - \eta Im - \zeta Im]],$$

$$f41 = [[\alpha \otimes \sigma Im][(\theta T \oplus D0) - \Psi Inm][\gamma Im \times \eta Im]],$$

$$f42 = [[-(\alpha \otimes \sigma Im)][\tau Inm][\gamma Im \times \eta Im]],$$

$$f43 = [[\Psi Inm][\tau Inm] - [(T \oplus D0) - \tau Inm][(\theta T \oplus D0) - \Psi Inm]] \times [\gamma Im \times \eta Im],$$

$$f44 = [[\Psi Inm][\tau Inm] - [(T \oplus D0) - \tau Inm][(\theta T \oplus D0) - \Psi Inm]] \times [[D0 - \sigma Im] \times [D0 - \eta Im - \zeta Im]],$$

$$f45 = [[(T \oplus D0) - \tau Inm][(\theta T \oplus D0) - \Psi Inm] - [\Psi Inm][\tau Inm]] \times [[D0 - \sigma Im][\gamma Im]],$$

$$f51 = [[\alpha \otimes \sigma Im][\Psi Inm - (\theta T \oplus D0)][\eta Im][D0 - \gamma Im - \zeta Im]],$$

$$f52 = [[\alpha \otimes \sigma Im][\tau Inm][D0 - \gamma Im - \zeta Im][\eta Im]],$$

$$f53 = [[(T \oplus D0) - \tau Inm][(\theta T \oplus D0) - \Psi Inm] - [\Psi Inm][\tau Inm]] \times [[\eta Im][D0 - \gamma Im - \zeta Im]],$$

$$f55 = [[\Psi Inm][\tau Inm] - [(T \oplus D0) - \tau Inm][(\theta T \oplus D0) - \Psi Inm]] \times [[D0 - \sigma Im][D0 - \gamma Im - \zeta Im]].$$

In the same way, pre-multiplying a diagonal block matrix with $(-A1)^{-1}$ matrix, it won't change the structure as seen in (2). Hence, the structure of matrix $A0(-A1)^{-1}$ is given by,

$$A0(-A1)^{-1} = \frac{1}{V}\begin{bmatrix} g11 & g12 & 0 & 0 & 0 \\ g21 & g22 & 0 & 0 & 0 \\ g31 & g32 & g33 & 0 & 0 \\ g41 & g42 & g43 & g44 & g45 \\ g51 & g52 & g53 & 0 & g55 \end{bmatrix} \quad (3)$$

where the elements of $A0(-A1)^{-1}$ as follows,

$g11 = [In \otimes D1]f11, \quad g12 = [In \otimes D1]f12, \quad g21 = [In \otimes D1]f21, \quad g22 = [In \otimes D1]f22$

$g31 = [D1]f31, \quad g32 = [D1]f32, \quad g33 = [D1]f33, \quad g41 = [D1]f41, \quad g42 = [D1]f42,$

$g43 = [D1]f43, \quad g44 = [D1]f44, \quad g45 = [D1]f45, \quad g51 = [D1]f51, \quad g52 = [D1]f52,$

$g53 = [D1]f53, \quad g55 = [D1]f55.$

Here, pre-multiplying a block matrix $A2$ with $(-A1)^{-1}$ matrix. Therefore, the structure of matrix $A2(-A1)^{-1}$ is given by,

$$A2(-A1)^{-1} = \frac{1}{V}\begin{bmatrix} h11 & h12 & 0 & 0 & 0 \\ h21 & h22 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ h41 & h42 & h43 & h44 & h45 \\ h51 & h52 & h53 & 0 & h55 \end{bmatrix} \quad (4)$$

where the elements of $A2(-A1)^{-1}$ as follows,

$h11 = [T0\alpha \otimes Im]f11, \quad h12 = [T0\alpha \otimes Im]f12, \quad h21 = [\theta T0\alpha \otimes Im]f21,$

$h22 = [\theta T0\alpha \otimes Im]f22, \quad h41 = [\zeta Im]f41, \quad h42 = [\zeta Im]f42, \quad h43 = [\zeta Im]f43,$

$h44 = [\zeta Im]f44, \quad h45 = [\zeta Im]f45, \quad h51 = [\zeta Im]f51, \quad h52 = [\zeta Im]f52,$

$h53 = [\zeta Im]f53, \quad h55 = [\zeta Im]f55.$

The sequence of $\{R^{(n)}, n = 0,1,2,3, \ldots.\}$ is defined by,

$$R^{(n+1)} = [(R^{(n)})^2 A2 + A0](-A1)^{-1}, \quad n = 0,1,2,3, \ldots.$$

The matrix quadratic equation $R^2A2 + RA1 + A0 = 0$ which has the minimal non negative solution as converges monotonically with $R^{(0)} = 0$. Hence, the structure of $\{A0(-A1)^{-1}\}$ and $\{(R^{(n)})^2 A2(-A1)^{-1}$, where $n = 1,2,3, \ldots.\}$ will remains the same as that of $(-A1)^{-1}$. Using $R^{(0)} = 0$, we can compute first iteration of R matrix i.e., $R^{(1)}$ then using first iteration of R matrix we can compute second iteration of R matrix i.e., $R^{(2)}$. Similarly, we can compute the further iterations of R matrix. Therefore, the each iteration of R matrix i.e., $R^{(n)}$ retains the same structure.

Logarithmic Reduction Algorithm of R

Step 0:
H ← $(-A_1)^{-1}A_0$, L ← $(-A_1)^{-1}A_2$, G = L, and T = H.
Step 1:
U = HL + LH
$M = H^2$
H ← $(I-U)^{-1}M$
M ← L^2
L ← $(I-U)^{-1}M$
G ← G + TL
T ← TH
Continue Step 1 Until $||e - Ge||_{\infty}\varepsilon$
Step 2:
$R = -A_0(A_1 + A_0G)^{-1}$.

5. Analysis of the Busy Period

• A Busy period is nothing but the interval between the customers arrives into the empty system and afterward the first interval once again the system becomes empty. So, it is the first passage from level 1 to 0. The busy cycle has described the first return time to level 0 with at least one visit to a state at any other level.

• Prior to examining the busy period, we have introduced an overview of the fundamental period. Under consideration of the QBD Process, it is the first passage time from level j to level $j-1, j$ 2.

• The cases $j = 0,1$ commensurate the boundary states have to be discussed individually. Note that for each and every level j, j 1 there corresponds (2nm+3m) states. Thus by the state (j,k) of level j we mention that the k^{th} state of level j when the states are arranged in alphabetical order.

• Let us denote $Gkk'(u,x)$ be the conditional probability that it started in the state (j,k) at time t = 0, the QBD process visits the level $j-1$ but not later than time x, we could make u transitions to the left and also entering the state (j,k').

Let us introduce the concept of the joint transform

$$\overline{G}kk'(z,s) = \sum_{u=1}^{\infty} z^u \int_0^{\infty} e^{-sx} dG\text{kk}'(u,x) \quad ; |z| \le 1, Re(s) \ge 0$$

and the matrix is denoted as follows

$$\overline{G}(z,s) = \overline{G}kk'(z,s)$$

then the above-defined matrix $\overline{G}(z,s)$ satisfies the equation

$$\overline{G}(z,s) = z(SI - A1)^{-1}A2 + (SI - A1)^{-1}A0\overline{G}^2(z,s)$$

• The matrix of $G = Gkk' = \overline{G}(1,0)$ would be taken for the first passage times, exclude for the boundary states. If we already know the matrix R then we could find the matrix G using the result

$$G = -(A1 + RA2)^{-1}A2$$

Otherwise, we may use the concept of a logarithmic reduction algorithm method to find the values of G matrix.

Notations of Boundary level states for Busy Period

- $Gkk'^{(1,0)}(u,x)$ denotes the conditional probability have been discussed for the first passage times from level 1 to the level 0 at time t = 0.
- $Gkk'^{(0,0)}(u,x)$ denotes the conditional probability have been discussed for the return time to the level 0.
- $\mathcal{F}1j$ denotes the mean first passage time from the level j to level $j-1$, given that the process is in the state (j,k) at time t = 0.
- $\overline{\mathcal{F}}1$ denotes the column vector with entries $\mathcal{F}1j$.
- $\mathcal{F}2j$ denotes the mean number of customers to be served during the first passage time from level j to level $j-1$, given that the first passage time has started in the state (j,k).
- $\overline{\mathcal{F}}2$ denotes the column vector with entries $\mathcal{F}2j$.
- $\overline{\mathcal{F}}1^{(1,0)}$ denotes the mean first passage time from level 1 to the level 0.
- $\overline{\mathcal{F}}2^{(1,0)}$ denotes the mean number of service completed during the first passage time from the level 1 to the level 0.
- $\overline{\mathcal{F}}1^{(0,0)}$ denotes the first return time to the level 0.
- $\overline{\mathcal{F}}2^{(0,0)}$ denotes the mean number of service completion in between first return time to the level 0.

For the boundary levels 1 and 0 we get,

$$\overline{G}^{(1,0)}(z,s) = z(SI - A1)^{-1}B10 + (SI - A1)^{-1}A0\overline{G}(z,s)\overline{G}^{(1,0)}(z,s)$$
$$0.3cm \quad \overline{G}^{(0,0)}(z,s) = (SI - B00)^{-1}B01\overline{G}^{(1,0)}(z,s)$$

Thus, the following instances are calculated using the matrices as $G(z,s)$, $\overline{G}^{(0,0)}(z,s)$ and $\overline{G}^{(1,0)}(z,s)$ are stochastic in nature. We can compute the instants as follows,

$$\overline{\mathcal{F}}1 = -\frac{\partial}{\partial s}\overline{G}(z,s)\Big|_{z=1,s=0} e = -[A1 + A0(I+G)]^{-1}e \tag{5}$$

$$\overline{\mathcal{F}}2 = \frac{\partial}{\partial z}\overline{G}(z,s)\Big|_{z=1,s=0} e = -[A1 + A0(I+G)]^{-1}A2e \tag{6}$$

$$\overline{\mathcal{F}}1^{(1,0)} = -\frac{\partial}{\partial s}\overline{G}^{(1,0)}(z,s)\Big|_{z=1,s=0} e = -[A1 + A0G]^{-1}(A0\overline{\mathcal{F}}1 + e) \tag{7}$$

$$\overline{\mathcal{F}}2^{(1,0)} = \frac{\partial}{\partial z}\overline{G}^{(1,0)}(z,s)\Big|_{z=1,s=0} e = -[A1 + A0G]^{-1}(A0\overline{\mathcal{F}}2 + B10e) \tag{8}$$

$$\overline{\mathcal{F}}1^{(0,0)} = -\frac{\partial}{\partial s}\overline{G}^{0,0}(z,s)\Big|_{z=1,s=0} e = -B00^{-1}[B01\overline{\mathcal{F}}1^{(1,0)} + e] \tag{9}$$

$$\overline{\mathcal{F}}2^{(0,0)} = \frac{\partial}{\partial z}\overline{G}^{(0,0)}(z,s)\Big|_{z=1,s=0} e = -B00^{-1}[B01\overline{\mathcal{F}}2^{(1,0)}]. \tag{10}$$

6. Measures of System Performance

The system performance measures are listed in this section for computation as follows,

1. Probability that the Main server is Busy in the system:
$P_{MSB} = \sum_{p=1}^{\infty} \sum_{r=1}^{n} \sum_{s=1}^{m} x_{p1rs}$

2. Probability that the Main server is the breakdown:
$P_{MSBD} = \sum_{s=1}^{m} x_{02s} + \sum_{p=1}^{\infty} \sum_{s=1}^{m} x_{p2rs}$

3. Probability of the Main server is in the setup process:
$P_{MSS} = \sum_{p=1}^{\infty} \sum_{s=1}^{m} x_{p3s}$

4. Probability of the Main server is in closedown period:
$P_{MSCD} = \sum_{p=0}^{\infty} \sum_{s=1}^{m} x_{p4s}$

5. Probability of the Main server is on vacation:
$P_{MSVAC} = \sum_{p=1}^{\infty} \sum_{s=1}^{m} x_{p5s}$

6. Expected system size:
$\mu_{NS} = \sum_{p=1}^{\infty} p x_p e_{2nm+3m} = x_1 (I - R)^{-2} e_{2nm+3m}$

7. Expected Queue size during the Main server is in Busy Period:
$\mu_{QMSB} = \sum_{p=1}^{\infty} \sum_{r=1}^{n} \sum_{s=1}^{m} (p-1) x_{p1rs} e_{nm}$

8. Expected Queue size during the Main server is in the breakdown:
$\mu_{MSBD} = \sum_{p=1}^{\infty} \sum_{r=1}^{n} \sum_{s=1}^{m} (p-1) x_{p2rs} e_{nm}$

9. Expected Queue size during the Main server is in the setup process:
$\mu_{MSS} = \sum_{p=1}^{\infty} \sum_{s=1}^{m} p x_{p3s} e_m$

10. Expected Queue size during the Main server is in the closedown period:
$\mu_{MSCD} = \sum_{p=0}^{\infty} \sum_{s=1}^{m} p x_{p4s} e_m$

11. Expected Queue size during the Main server is in Vacation:
$\mu_{MSV} = \sum_{p=0}^{\infty} \sum_{s=1}^{m} p x_{p5s} e_m$

12. Average Queue size:
$\mu_{QS} = \mu_{QMSB} + \mu_{MSBD} + \mu_{MSS} + \mu_{MSCD} + \mu_{MSV}$

7. Numerical Results

In this part, we are analyzing the model behavior in the form of numerical and graphical illustrations. The following five different MAP representations has mean value is same that is, 1 for all the different arrival process. These five sets of arrival values has taken as input data in published works in the literature, see Chakravarthy (2010).

Arrival in Erlang(ERLA) :

$$D0 = \begin{bmatrix} -3 & 3 & 0 \\ 0 & -3 & 3 \\ 0 & 0 & -3 \end{bmatrix}, D1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix}$$

Arrival in Exponential(EXPA) :

$$D0 = \begin{bmatrix} -1 \end{bmatrix}, D1 = \begin{bmatrix} 1 \end{bmatrix}$$

Arrival in Hyperexponential(HEXA) :

$$D0 = \begin{bmatrix} -1.90 & 0 \\ 0 & -0.19 \end{bmatrix}, D1 = \begin{bmatrix} 1.710 & 0.190 \\ 0.171 & 0.019 \end{bmatrix}$$

Arrival in MAP-Negative Correlation(MNCA) :

$$D0 = \begin{bmatrix} -1.00243 & 1.00243 & 0 \\ 0 & -1.00243 & 0 \\ 0 & 0 & -225.797 \end{bmatrix}, D1 = \begin{bmatrix} 0 & 0 & 0 \\ 0.01002 & 0 & 0.99241 \\ 223.539 & 0 & 2.258 \end{bmatrix}$$

Arrival in MAP-Positive Correlation(MPCA) :

$$D0 = \begin{bmatrix} -1.00243 & 1.00243 & 0 \\ 0 & -1.00243 & 0 \\ 0 & 0 & -225.797 \end{bmatrix}, D1 = \begin{bmatrix} 0 & 0 & 0 \\ 0.99241 & 0 & 0.01002 \\ 2.258 & 0 & 223.539 \end{bmatrix}$$

Let us consider three phase type distributions for the service process. Normalization of these three representations has done to get service rate δ. These sets of service values has taken as input data in published works in the literature, see Chakravarthy (2010).

Service in Erlang(ERLS) :

$$\alpha = \begin{pmatrix} 1,0 \end{pmatrix}, T = \begin{bmatrix} -2 & 2 \\ 0 & -2 \end{bmatrix}$$

Service in Exponential(EXPS) :

$$\alpha = \begin{pmatrix} 1 \end{pmatrix}, T = \begin{bmatrix} -1 \end{bmatrix}$$

Service in Hyperexponential(HEXS) :

$$\alpha = \begin{pmatrix} 0.8,0.2 \end{pmatrix}, T = \begin{bmatrix} -2.80 & 0 \\ 0 & -0.28 \end{bmatrix}$$

Illustrated Example 1:

We have examined the consequence of the renege rate ζ against the expected system size in the following table. We fix $\lambda = 1$; $\theta = 0.7$; $\Psi = 3$; $\gamma = 6$; $\sigma = 8$; $\delta = 4$; $\eta = 5$; $\tau = 2$.

Table 1: Expected System size ʻ

Erlang service					
ζ	**ERLA**	**EXPA**	**HEXA**	**MNCA**	**MPCA**
8	0.2551100830	0.3191208819	0.4237650475	0.4033760196	19.5652579340
13	0.1876854481	0.2403341600	0.3314934478	0.3076333066	19.3768425434
18	0.1485579728	0.1928559259	0.2726619048	0.2483769578	19.2023395965
23	0.1229681989	0.1610892244	0.2317527947	0.2081541358	19.0357254576
28	0.1049145893	0.1383294179	0.2016045028	0.1790782187	18.8745536749
33	0.0914907791	0.1212157553	0.1784409219	0.1570846970	18.7176231039
38	0.0811162436	0.1078763507	0.1600754865	0.1398694196	18.5642607563
43	0.0728570202	0.0971852548	0.1451515493	0.1260294489	18.4140497046
48	0.0661255200	0.0884242695	0.1327815261	0.1146619073	18.2667099490

Table 2: Expected System size

Exponential service					
ζ	**ERLA**	**EXPA**	**HEXA**	**MNCA**	**MPCA**
8	0.2660071394	0.3337868611	0.4545621888	0.4165792561	19.5853519487
13	0.1959931546	0.2514044546	0.3554004824	0.3180855132	19.3960615220
18	0.1552672619	0.2017442209	0.2921818413	0.2570158696	19.2210068115
23	0.1285938228	0.1685137353	0.2482416632	0.2155124581	19.0539913023
28	0.1097574927	0.1447040698	0.2158754166	0.1854852167	18.8925006470
33	0.0957419592	0.1268006869	0.1910190187	0.1627575449	18.7353016465
38	0.0849044456	0.1128456776	0.1713193512	0.1449587238	18.5817042082
43	0.0762731206	0.1016612272	0.1553168399	0.1306438676	18.4312814390
48	0.0692360437	0.0924960096	0.1420568962	0.1188823783	18.2837471323

Table 3: Expected System size

Hyperexponential service					
ζ	**ERLA**	**EXPA**	**HEXA**	**MNCA**	**MPCA**
8	0.3475067705	0.4289873057	0.6229687643	0.5140562334	19.7220351977
13	0.2572426019	0.3236600000	0.4859931583	0.3946837704	19.5250809913
18	0.2043399957	0.2599631914	0.3987735746	0.3200301148	19.3452627410
23	0.1695337242	0.2172643308	0.3382751167	0.2690145569	19.1748608531
28	0.1448792919	0.1866366331	0.2938010054	0.2319627085	19.0107486316
33	0.1264946416	0.1635891620	0.2597065716	0.2038376098	18.8513991043
38	0.1122553475	0.1456147747	0.2327267733	0.1817628664	18.6959635987
43	0.1009001983	0.1312030431	0.2108393303	0.1639773995	18.5439220356
48	0.0916328750	0.1193894349	0.1927236849	0.1493429129	18.3949298110

From the above tables 1, 2 and 3, we conclude that while increasing the reneging rate, the expected system size decreases in case of the variety of arrangements of services and arrivals. Nevertheless, ERLA slowly decreases than the EXPA, HEXA rapidly. Among these, MPCA decreases much faster than the other arrivals.

Illustrated Example 2:

We have examined the consequence of the service rate δ of main server against the expected system size in the following table. We fix $\lambda = 1$; $\Psi = 5$; $\sigma = 8$; $\gamma = 6$; ζ=9; $\theta = 0.7$; $\eta = 5$; $\tau = 2$.

Table 4: Expected System size

Erlang service					
δ	**ERLA**	**EXPA**	**HEXA**	**MNCA**	**MPCA**
4	0.2379764747	0.2994591234	0.4013151438	0.3798100776	19.5258931834
5	0.2065039525	0.2487301273	0.3063421739	0.3207713481	14.4614145429
6	0.1880284636	0.2199680438	0.2582097489	0.2861007651	11.4819360299
7	0.1758546173	0.2015441054	0.2295481094	0.2632472972	9.5197772389
8	0.1672205738	0.1887802063	0.2106828814	0.2470346455	8.1299142452
9	0.1607759003	0.1794400127	0.1973928640	0.2349313955	7.0938468136
10	0.1557808065	0.1723229710	0.1875600607	0.2255502697	6.2917434540
11	0.1517955387	0.1667282839	0.1800107682	0.2180662304	5.6523974928
12	0.1485420686	0.1622200781	0.1740444599	0.2119573947	5.1308356413

Table 5: Expected System size

Exponential service					
δ	**ERLA**	**EXPA**	**HEXA**	**MNCA**	**MPCA**
4	0.2482351638	0.3132315037	0.4304393361	0.3923562659	19.5457697200
5	0.2120552776	0.2567327335	0.3227377870	0.3276776653	14.4728125522
6	0.1914637163	0.2252509825	0.2687014614	0.2904550358	11.4893149129
7	0.1781755609	0.2053235695	0.2368546405	0.2662474615	9.5249442035
8	0.1688892137	0.1916365013	0.2160806861	0.2492348851	8.1337368759
9	0.1620323328	0.1816856929	0.2015562395	0.2366207794	7.0967923028
10	0.1567611666	0.1741418179	0.1908778569	0.2268934813	6.2940852915
11	0.1525823509	0.1682358002	0.1827228795	0.2191636915	5.6543062914
12	0.1491880874	0.1634927441	0.1763070028	0.2128737519	5.1324233062

Table 6: Expected System size

Hyperexponential service					
δ	**ERLA**	**EXPA**	**HEXA**	**MNCA**	**MPCA**
4	0.3246765587	0.4027499695	0.5896485286	0.4848163361	19.6805443062
5	0.2546633015	0.3080706837	0.4156634604	0.3790445701	14.5503737715
6	0.2183704443	0.2586826177	0.3294321844	0.3228206429	11.5395557565
7	0.1965817028	0.2289316791	0.2796591165	0.2883845215	9.5600778968
8	0.1822049566	0.2092691192	0.2479171397	0.2652822023	8.1596645385
9	0.1720738738	0.1954065551	0.2262029037	0.2487705926	7.1167066591
10	0.1645808739	0.1851571755	0.2105570033	0.2364084960	6.3098612364
11	0.1588301261	0.1772982721	0.1988249493	0.2268191719	5.6671156423
12	0.1542858532	0.1710966489	0.1897458792	0.2191703336	5.1430353189

From the above tables 4, 5 and 6, we conclude that while increasing the main server service rate, the expected system size decreases in case of the variety of arrangements of services and arrivals. Eventhough, ERLA and EXPA decreases slowly, HEXA and MNCA decreases gradually and the MPCA decreases rapidly than compared to the other arrivals.

Illustrated Example 3:

We have examined the consequence of the repair rate Ψ against the expected system size in the following table. We fix $\lambda = 1$; $\theta = 0.7$; $\sigma = 8$; $\gamma = 6$; $\zeta=9$; $\delta = 4$; $\eta = 5$; $\tau = 2$.

Table 7: Expected System size

Erlang service					
Ψ	**ERLA**	**EXPA**	**HEXA**	**MNCA**	**MPCA**
4	0.2351597114	0.2946564762	0.3905212078	0.3743593312	18.9274045812
6	0.2319256796	0.2890301295	0.3781560286	0.3679926754	18.2284868245
8	0.2300699236	0.2857783885	0.3712174710	0.3643031902	17.8330653966
10	0.2288467240	0.2836400981	0.3667572229	0.3618672995	17.5787028633
12	0.2279733379	0.2821209855	0.3636427225	0.3601306660	17.4013390100
14	0.2273159732	0.2809839024	0.3613424418	0.3588270709	17.2706052797
16	0.2268022053	0.2800998845	0.3595729794	0.3578113063	17.1702481848
18	0.2263890870	0.2793924588	0.3581691038	0.3569969784	17.0907820211
20	0.2260494077	0.2788132852	0.3570278602	0.3563293036	17.0262983354

Table 8: Expected System size

Exponential service					
Ψ	**ERLA**	**EXPA**	**HEXA**	**MNCA**	**MPCA**
4	0.2447096506	0.3075250014	0.4177427601	0.3860232879	18.9461545159
6	0.2406602054	0.3008823356	0.4032632476	0.3786534491	18.2459354054
8	0.2383541519	0.2970792602	0.3951810279	0.3744184322	17.8497897169
10	0.2368484292	0.2945978474	0.3900064894	0.3716437228	17.5949679582
12	0.2357825582	0.2928457732	0.3864044486	0.3696776380	17.4172877193
14	0.2349862035	0.2915406696	0.3837506133	0.3682089168	17.2863230878
16	0.2343676345	0.2905299637	0.3817132155	0.3670688570	17.1857901956
18	0.2338728056	0.2897237091	0.3800993764	0.3661576879	17.1061857821
20	0.2334677012	0.2890653405	0.3787892125	0.3654124806	17.0415905583

Table 9: Expected System size

Hyperexponential service					
Ψ	**ERLA**	**EXPA**	**HEXA**	**MNCA**	**MPCA**
4	0.3161898145	0.3913958479	0.5673779642	0.4725395327	19.0733802339
6	0.3066305618	0.3784862880	0.5422524487	0.4585297312	18.3645535023
8	0.3013461895	0.3712989978	0.5283974514	0.4506946122	17.9636497210
10	0.2979805242	0.3667051229	0.5196056510	0.4456686565	17.7058140995
12	0.2956447368	0.3635109574	0.5135264742	0.4421640141	17.5260552766
14	0.2939269314	0.3611595202	0.5090707869	0.4395779898	17.3935710289
16	0.2926095607	0.3593553104	0.5056641815	0.4375899624	17.2918789671
18	0.2915667102	0.3579267684	0.5029748399	0.4360133363	17.2113612827
20	0.2907203721	0.3567673791	0.5007976263	0.4347320116	17.1460279908

From the above tables 7, 8 and 9, we conclude that while increasing the repair rate, the expected system size decreases in case of the variety of arrangements of services and arrivals. Though, ERLA and EXPA decreases slowly, HEXA and MNCA decreases than ERLA, EXPA and the MPCA decreases fastly than compared to the other arrivals.

Illustrated Example 4:

We have examined the consequence of the vacation rate η against the expected system size in the following table. We fix $\lambda = 1$; $\Psi = 3$; $\sigma = 8$; $\theta = 0.7$; ζ=9; $\delta = 4$; $\gamma = 6$; $\tau = 2$.

Table 10: Expected System size

Erlang service					
η	**ERLA**	**EXPA**	**HEXA**	**MNCA**	**MPCA**
3	0.2048797964	0.2580248808	0.3496636085	0.3295547779	19.3974064241
5	0.2379764747	0.2994591234	0.4013151438	0.3798100776	19.5258931834
7	0.2609374022	0.3264662539	0.4321131261	0.4126041589	19.5849169838
9	0.2778495302	0.3454994764	0.4525827108	0.4356964425	19.6195291979
11	0.2908481194	0.3596511214	0.4671791623	0.4528335256	19.6425405482
13	0.3011626554	0.3705930064	0.4781152395	0.4660511754	19.6590576277
15	0.3095532427	0.3793097174	0.4866153454	0.4765521438	19.6715425632
17	0.3165160745	0.3864195048	0.4934122466	0.4850930542	19.6813385223
19	0.3223893119	0.3923304733	0.4989714475	0.4921737642	19.6892448290

Table 11: Expected System size

Exponential service					
η	**ERLA**	**EXPA**	**HEXA**	**MNCA**	**MPCA**
3	0.2125347615	0.2684064944	0.3724672735	0.3392800892	19.4162589083
5	0.2482351638	0.3132315037	0.4304393361	0.3923562659	19.5457697200
7	0.2729429633	0.3424982484	0.4652043037	0.4269134345	19.6054778723
9	0.2911060532	0.3631501692	0.4884068653	0.4512048870	19.6405877132
11	0.3050432170	0.3785206714	0.5050054121	0.4692067998	19.6639788918
13	0.3160871004	0.3904145455	0.5174736353	0.4830756755	19.6807958537
15	0.3250602101	0.3998959660	0.5271852411	0.4940835793	19.6935236950
17	0.3324986711	0.4076338334	0.5349648003	0.5030296401	19.7035204476
19	0.3387673636	0.4140700670	0.5413374657	0.5104411633	19.7115955170

Table 12: Expected System size

Hyperexponential service					
η	**ERLA**	**EXPA**	**HEXA**	**MNCA**	**MPCA**
3	0.2689766678	0.3360384794	0.4968181848	0.4105364262	19.5428638563
5	0.3246765587	0.4027499695	0.5896485286	0.4848163361	19.6805443062
7	0.3631292339	0.4465468202	0.6463852161	0.5328031754	19.7458368472
9	0.3913353241	0.4775803982	0.6847672480	0.5663216126	19.7850697502
11	0.4129378486	0.5007531426	0.7125078950	0.5910302936	19.8116452740
13	0.4300271643	0.5187319646	0.7335154646	0.6099806636	19.8310007263
15	0.4438915101	0.5330955374	0.7499870410	0.6249636134	19.8458016026
17	0.4553693955	0.5448394330	0.7632546739	0.6370991280	19.8575241033
19	0.4650307329	0.5546232529	0.7741738593	0.6471232193	19.8670586028

From the above tables 10, 11 and 12, we conclude that when we are increasing the vacation rate then the expected system size is also increases in the variety of arrangements of arrivals and services. Nonetheless, ERLA and EXPA increases slowly, HEXA and MNCA increases rapidly but in the case of MPCA increases gradually than compared to the other arrivals.

Illustrated Example 5:

We have examined the consequence of the setup rate σ against the expected system size in the following table. We fix $\lambda = 1$; $\theta = 0.7$; $\Psi = 3$; $\gamma = 6$; ζ=9; $\delta = 4$; $\eta = 5$; $\tau = 2$.

Table 13: Expected System size

Erlang service					
σ	**ERLA**	**EXPA**	**HEXA**	**MNCA**	**MPCA**
7	0.2459777032	0.3089365521	0.4130365944	0.3898273658	19.5394272035
11	0.2230858302	0.2816829834	0.3792769140	0.3609173026	19.5001558027
15	0.2128018626	0.2693028343	0.3638927871	0.3476665782	19.4819360808
19	0.2069616016	0.2622371767	0.3551019328	0.3400646246	19.4714200228
23	0.2031965713	0.2576700983	0.3494163091	0.3351341033	19.4645746841
27	0.2005674189	0.2544758358	0.3454383758	0.3316773097	19.4597637942
31	0.1986274089	0.2521164492	0.3424995291	0.3291194092	19.4561977552
35	0.1971369360	0.2503025272	0.3402398004	0.3271501234	19.4534487560
39	0.1959559466	0.2488645462	0.3384482366	0.3255872405	19.4512648638

Table 14: Expected System size

Exponential service					
σ	**ERLA**	**EXPA**	**HEXA**	**MNCA**	**MPCA**
7	0.2563481281	0.3228302711	0.4424163840	0.4024462732	19.5593192999
11	0.2331297801	0.2952183454	0.4078875333	0.3733239025	19.5200048733
15	0.2226919076	0.2826652387	0.3921160745	0.3599734036	19.5017673293
19	0.2167619277	0.2754977271	0.3830919110	0.3523134799	19.4912413486
23	0.2129380461	0.2708635304	0.3772503625	0.3473450301	19.4843894587
27	0.2102672745	0.2676217056	0.3731608641	0.3438614526	19.4795737454
31	0.2082962544	0.2652268528	0.3701382388	0.3412836023	19.4760038850
35	0.2067817721	0.2633854671	0.3678132866	0.3392988602	19.4732517000
39	0.2055816379	0.2619255910	0.3659695047	0.3377236335	19.4710650544

Table 15: Expected System size

Hyperexponential service					
σ	**ERLA**	**EXPA**	**HEXA**	**MNCA**	**MPCA**
7	0.3333551730	0.4130708338	0.6030453013	0.4952896493	19.6941862430
11	0.3085394278	0.3833293074	0.5642591570	0.4650633254	19.6546155835
15	0.2974150057	0.3697515263	0.5463606703	0.4512114996	19.6362724604
19	0.2911082897	0.3619815009	0.5360605806	0.4432663296	19.6256896147
23	0.2870474039	0.3569506282	0.5293684884	0.4381139280	19.6188021143
27	0.2842140195	0.3534278749	0.5246714808	0.4345018467	19.6139618281
31	0.2821245619	0.3508236306	0.5211932306	0.4318290984	19.6103738246
35	0.2805199810	0.3488201520	0.5185139054	0.4297713517	19.6076075622
39	0.2792490027	0.3472310819	0.5163866252	0.4281381748	19.6054095731

From the above tables 13, 14 and 15, we examined that when we are increasing the setup rate, the expected system size decreases in the variety of arrangements of services and arrivals. Though, ERLA and EXPA decreases slowly, HEXA and MNCA decreases than ERLA, EXPA but in the case of MPCA decreases gradually than compared to the other arrivals.

Illustrated Example 6:

We have examined the consequence of the breakdown rate τ against the expected system size in the following table. We fix $\lambda = 1$; $\theta = 0.7$; $\Psi = 3$; $\gamma = 6$; ζ=9; $\delta = 4$; $\eta = 5$; $\sigma = 8$.

Table 16: Expected System size

Erlang service					
τ	**ERLA**	**EXPA**	**HEXA**	**MNCA**	**MPCA**
1.2	0.2325503701	0.2902695261	0.3821097387	0.3693639979	18.5182290797
1.4	0.2339881967	0.2927164860	0.3872259225	0.3721512928	18.7936229079
1.6	0.2353697421	0.2950595706	0.3921235659	0.3748163882	19.0524217592
1.8	0.2366981857	0.2973051588	0.3968158982	0.3773669628	19.2960816647
2.0	0.2379764747	0.2994591234	0.4013151438	0.3798100776	19.5258931834
2.2	0.2392073439	0.3015268799	0.4056326106	0.3821522351	19.7430042545
2.4	0.2403933354	0.3035134296	0.4097787711	0.3843994331	19.9484393654
2.6	0.2415368142	0.3054233983	0.4137633345	0.3865572118	20.1431157099
2.8	0.2426399843	0.3072610704	0.4175953123	0.3886306959	20.3278568743

Table 17: Expected System size

Exponential service					
τ	**ERLA**	**EXPA**	**HEXA**	**MNCA**	**MPCA**
1.2	0.2416670895	0.3026331558	0.4081984191	0.3805408844	18.5362367193
1.4	0.2434137950	0.3054613010	0.4141292614	0.3837002241	18.8121367719
1.6	0.2450878289	0.3081652249	0.4198026751	0.3867164611	19.0714145056
1.8	0.2466935983	0.3107528650	0.4252344938	0.3895989516	19.3155280116
2.0	0.2482351638	0.3132315037	0.4304393361	0.3923562659	19.5457697200
2.2	0.2497162727	0.3156078338	0.4354307133	0.3949962680	19.7632892724
2.4	0.2511403881	0.3178880161	0.4402211261	0.3975261857	19.9691127063
2.6	0.2525107158	0.3200777309	0.4448221522	0.3999526729	20.1641586293
2.8	0.2538302269	0.3221822235	0.4492445268	0.4022818646	20.3492519179

Table 18: Expected System size

Hyperexponential service					
τ	**ERLA**	**EXPA**	**HEXA**	**MNCA**	**MPCA**
1.2	0.3102292500	0.3832114374	0.5520675036	0.4635680590	18.6585235243
1.4	0.3140934994	0.3884433265	0.5621199692	0.4692715155	18.9377955214
1.6	0.3177817606	0.3934329312	0.5717146896	0.4747016151	19.2002701798
1.8	0.3213058493	0.3981967858	0.5808816904	0.4798774333	19.4474184089
2.0	0.3246765587	0.4027499695	0.5896485286	0.4848163361	19.6805443062
2.2	0.3279037649	0.4071062588	0.5980405239	0.4895341627	19.9008081433
2.4	0.3309965207	0.4112782628	0.6060809685	0.4940453865	20.1092456548
2.6	0.3339631373	0.4152775420	0.6137913152	0.4983632567	20.3067843045
2.8	0.3368112576	0.4191147133	0.6211913473	0.5024999226	20.4942570689

From the above tables 16 ,17 and 18, we conclude that maximizing the breakdown rate then the expected system size is also maximizes in different arrangements of services and arrivals of ERLA, EXPA, HEXA, MNCA and MPCA. Nevertheless, Erlang arrival and exponential arrival increases slowly, hyperexponential arrival and negative correlation arrival increases rapidly but in the case of positive correlation arrival increases gradually than compared to the other arrivals.

Illustrated Example 7:

We have examined the consequence of the standby server service rate $\theta\delta$ against the expected system size in the following table. We fix λ = 1 ; Ψ = 3; σ = 8; γ = 6; ζ=9; δ = 4 ; η = 5; τ = 2.

Table 19: Expected System size

Erlang service					
$\theta\delta$	**ERLA**	**EXPA**	**HEXA**	**MNCA**	**MPCA**
2.2	0.2493345337	0.3189650962	0.4442998622	0.4016843196	21.5840044974
2.4	0.2451019008	0.3117401217	0.4282485811	0.3936201806	20.8508674817
2.6	0.2413398016	0.3052737730	0.4140070993	0.3863659406	20.1663913544
2.8	0.2379764747	0.2994591234	0.4013151438	0.3798100776	19.5258931834
3.0	0.2349534663	0.2942073951	0.3899556725	0.3738598100	18.9252717370
3.2	0.2322228221	0.2894444617	0.3797470180	0.3684374863	18.3609199326
3.4	0.2297449241	0.2851080872	0.3705365387	0.3634777414	17.8296526306
3.6	0.2274868176	0.2811457369	0.3621954802	0.3589252447	17.3286467252
3.8	0.2254209087	0.2775128340	0.3546148072	0.3547329083	16.8553911568

Table 20: Expected System size

Exponential service					
$\theta\delta$	**ERLA**	**EXPA**	**HEXA**	**MNCA**	**MPCA**
2.2	0.2620613084	0.3355792516	0.4791805431	0.4171768202	21.6079764679
2.4	0.2569279619	0.3273268675	0.4610597903	0.4080428441	20.8733500039
2.6	0.2523461596	0.3199161494	0.4449036704	0.3998098925	20.1875135419
2.8	0.2482351638	0.3132315037	0.4304393361	0.3923562659	19.5457697200
3.0	0.2445286090	0.3071765721	0.4174381012	0.3855803680	18.9440047937
3.2	0.2411715731	0.3016705653	0.4057075188	0.3793968794	18.3786007973
3.4	0.2381183070	0.2966453703	0.3950849869	0.3737337400	17.8463630966
3.6	0.2353304623	0.2920432536	0.3854325696	0.3685297555	17.3444602783
3.8	0.2327757010	0.2878150282	0.3766327917	0.3637326873	16.8703739928

Table 21: Expected System size

Hyperexponential service					
$\theta\delta$	**ERLA**	**EXPA**	**HEXA**	**MNCA**	**MPCA**
2.2	0.3547810935	0.4431870322	0.6674822570	0.5295597332	21.7696904127
2.4	0.3437154609	0.4283603500	0.6389033334	0.5131722013	21.0252665947
2.6	0.3337276417	0.4149413304	0.6130742909	0.4983227530	20.3304822178
2.8	0.3246765587	0.4027499695	0.5896485286	0.4848163361	19.6805443062
3.0	0.3164438165	0.3916345369	0.5683321841	0.4724883079	19.0712565623
3.2	0.3089294302	0.3814663829	0.5488748637	0.4611988887	18.4989294453
3.4	0.3020484686	0.3721358261	0.5310622021	0.4508287634	17.9603060193
3.6	0.2957283944	0.3635488745	0.5147098562	0.4412755675	17.4525004402
3.8	0.2899069400	0.3556245899	0.4996586252	0.4324510585	16.9729466378

From the above tables 19, 20, and 21, we conclude that increasing the standby service rate, the expected system size decreases in case of the variety of arrangements of services and arrivals. Eventhough, ERLA and EXPA decreses slowly, HEXA and MNCA decreases their values than ERLA and the MPCA decreases more rapidly than compared to the other arrivals.

Illustrated Example 8:

We have examined the consequence of the closedown rate γ against the expected system size in the following table. We fix $\lambda = 1$; $\Psi = 3$; $\sigma = 8$; $\theta = 0.7$; ζ=9; $\delta = 4$; $\eta = 5$; $\tau = 2$.

Table 22: Expected System size

Erlang service					
γ	**ERLA**	**EXPA**	**HEXA**	**MNCA**	**MPCA**
6	0.2379764747	0.2994591234	0.4013151438	0.3798100776	19.5258931834
9	0.2395334980	0.3014656569	0.4047552133	0.3816022397	19.5279632231
12	0.2402164837	0.3023627617	0.4062973564	0.3823623790	19.5286980831
15	0.2405789359	0.3028461514	0.4071288880	0.3827566641	19.5290420827
18	0.2407949971	0.3031378197	0.4076307286	0.3829876942	19.5292306987
21	0.2409343973	0.3033278535	0.4079577337	0.3831347288	19.5293453188
24	0.2410296700	0.3034587825	0.4081830537	0.3832340971	19.5294202030
27	0.2410977093	0.3035529176	0.4083450702	0.3833043942	19.5294718322
30	0.2411480148	0.3036229156	0.4084655580	0.3833559509	19.5295089408

Table 23: Expected System size

Exponential service					
γ	**ERLA**	**EXPA**	**HEXA**	**MNCA**	**MPCA**
6	0.2482351638	0.3132315037	0.4304393361	0.3923562659	19.5457697200
9	0.2498484226	0.3154037298	0.4343315782	0.3941987613	19.5478091696
12	0.2505494261	0.3163754300	0.4360797056	0.3949768093	19.5485318940
15	0.2509192056	0.3168992191	0.4370235337	0.3953792754	19.5488698925
18	0.2511387310	0.3172153589	0.4375937005	0.3956146569	19.5490551173
21	0.2512799490	0.3174213865	0.4379655094	0.3957642577	19.5491676409
24	0.2513762514	0.3175633626	0.4382218574	0.3958652568	19.5492411422
27	0.2514449089	0.3176654569	0.4384062772	0.3959366509	19.5492918133
30	0.2514956029	0.3177413838	0.4385434836	0.3959889793	19.5493282322

Table 24: Expected System size

Hyperexponential service					
γ	**ERLA**	**EXPA**	**HEXA**	**MNCA**	**MPCA**
6	0.3246765587	0.4027499695	0.5896485286	0.4848163361	19.6805443062
9	0.3269565720	0.4059890577	0.5960709245	0.4871727892	19.6825830695
12	0.3279220141	0.4074410391	0.5989721822	0.4881523092	19.6833068224
15	0.3284224851	0.4082249413	0.6005447623	0.4886538794	19.6836459575
18	0.3287159437	0.4086986463	0.6014975096	0.4889451390	19.6838321413
21	0.3289030001	0.4090076556	0.6021201914	0.4891292753	19.6839454308
24	0.3290296693	0.4092207649	0.6025502744	0.4892530810	19.6840195391
27	0.3291194782	0.4093741107	0.6028601346	0.4893403108	19.6840706941
30	0.3291854945	0.4094882160	0.6030909498	0.4894040754	19.6841075030

From the above tables 22, 23 and 24, we conclude that increasing the closedown rate then the expected system size is also increases in the variety of arrangements of services and arrivals. Eventhough, ERLA and EXPA increases slowly, HEXA and MNCA increases their values than EXPA but in the case of MPCA increases gradually than compared to the other arrivals.

Illustrated Example 9:

We fix $\lambda = 1$; $\Psi = 3$; $\theta = 0.7$; $\zeta = 9$; $\eta = 5$; $\tau = 2$; $\gamma = 6$.

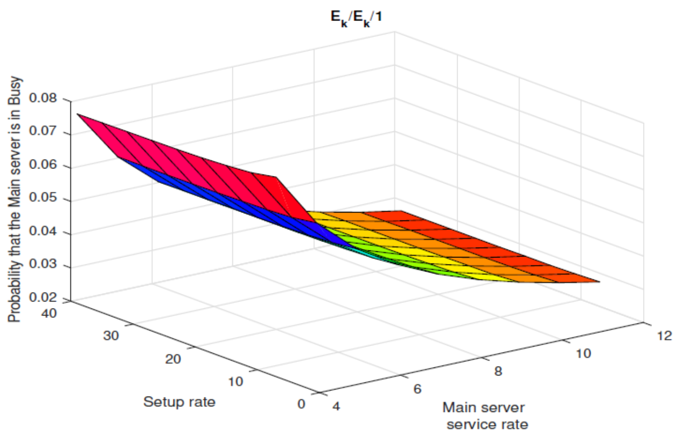

Figure 2: The graph of $E_k/E_k/1$ - setup rate(σ) and main server service rate(δ) versus probability that the main server is in busy

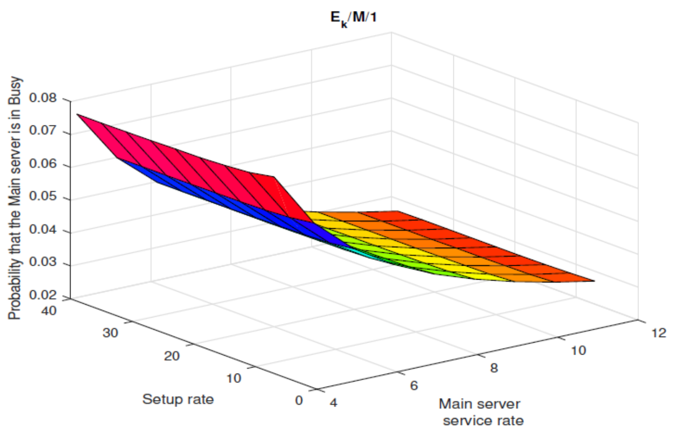

Figure 3: The graph of $E_k/M/1$ - setup rate(σ) and main server service rate(δ) versus probability that the main server is in busy

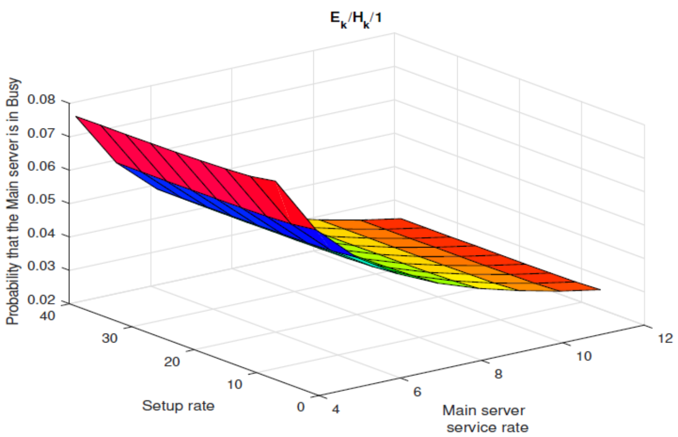

Figure 4: The graph of $E_k/H_k/1$ - setup rate(σ) and main server service rate(δ) versus probability that the main server is in busy

We observe from figures 2, 3 and 4 that the impact of setup rate and main server service rate on the probability of the main server service is in the busy mode. We have examined the probability of the main server is in the busy mode decreases while we are increasing both the setup rate and main server service rate for the arrangement of Erlang arrival with ERLS, EXPS and HEXS. However, the Erlang arrival decreases slowly in hyperexponential service.

We fix $\lambda = 1$; $\Psi = 3$; $\theta = 0.7$; $\zeta = 9$; $\eta = 5$; $\tau = 2$; $\gamma = 6$.

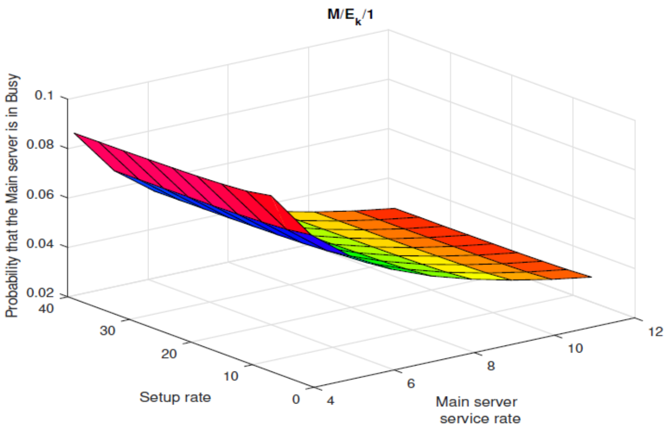

Figure 5: The graph of $M/E_k/1$ - setup rate(σ) and main server service rate(δ) versus probability that the main server is in busy

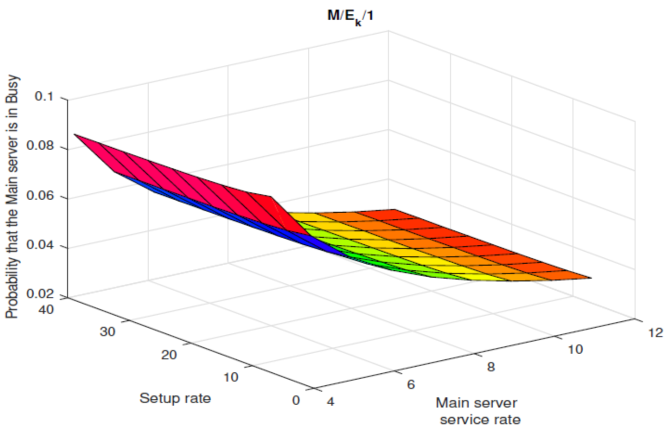

Figure 6: The graph of $M/M/1$ - setup rate(σ) and main server service rate(δ) versus probability that the main server is in busy

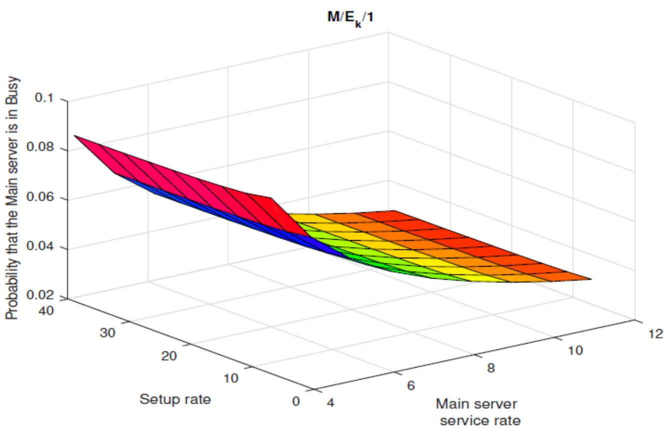

Figure 7: The graph of $M/H_k/1$ - setup rate(σ) and main server service rate(δ) versus probability that the main server is in busy

We observe from figures 5, 6 and 7 that the impact of setup rate and main server service rate on the probability of the main server service is in the busy mode. We have examined the probability of the main server is in the busy mode decreases while we are increasing both the setup rate and main server service rate for the arrangement of exponential arrival with ERLS, EXPS and HEXS. Nevertheless, the Erlang service times decreases than the hyperexponential service times.

We fix $\lambda = 1$; $\Psi = 3$; $\theta = 0.7$; $\zeta = 9$; $\eta = 5$; $\tau = 2$; $\gamma = 6$.

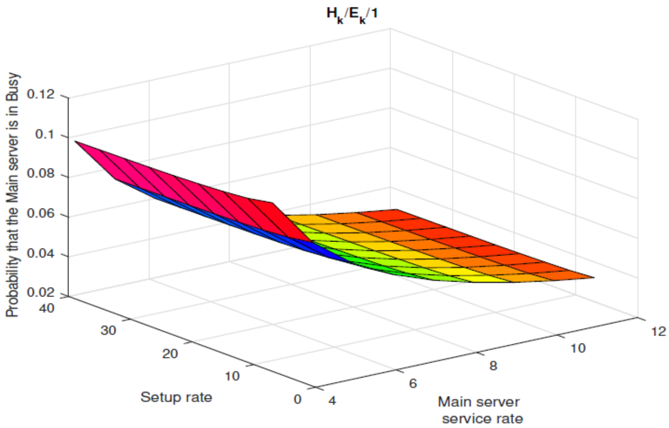

Figure 8: The graph of $H_k/E_k/1$ - setup rate(σ) and main server service rate(δ) versus probability that the main server is in busy

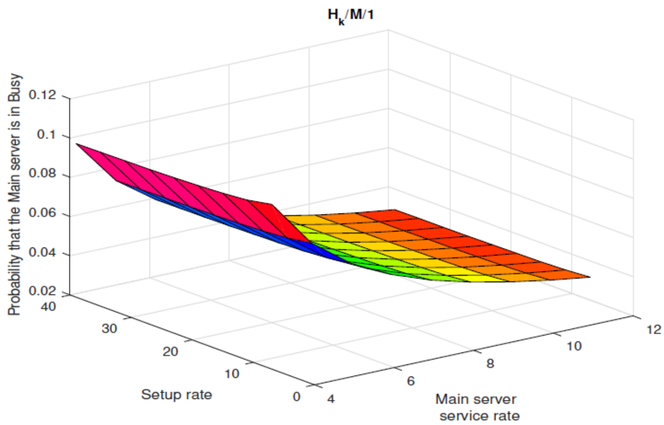

Figure 9: The graph of $H_k/M/1$ - setup rate(σ) and main server service rate(δ) versus probability that the main server is in busy

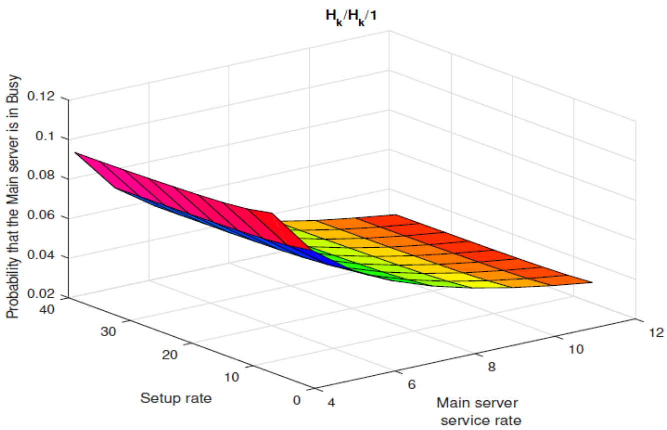

Figure 10: The graph of $H_k/H_k/1$ - setup rate(σ) and main server service rate(δ) versus probability that the main server is in busy

We observe from figures 8, 9 and 10 that the consequence of setup rate and main server service rate on the probability of the main server is in the busy mode. We have examined the probability of the main server is in the busy mode decreases while we are increasing both the setup rate and main server service rate for the arrangement of hyperexponential arrival with ERLS, EXPS and HEXS. Meanwhile, the hyperexponential arrival decreases slowly in hyperexponential service times.

We fix $\lambda = 1$; $\Psi = 3$; $\theta = 0.7$; $\zeta = 9$; $\eta = 5$; $\tau = 2$; $\gamma = 6$.

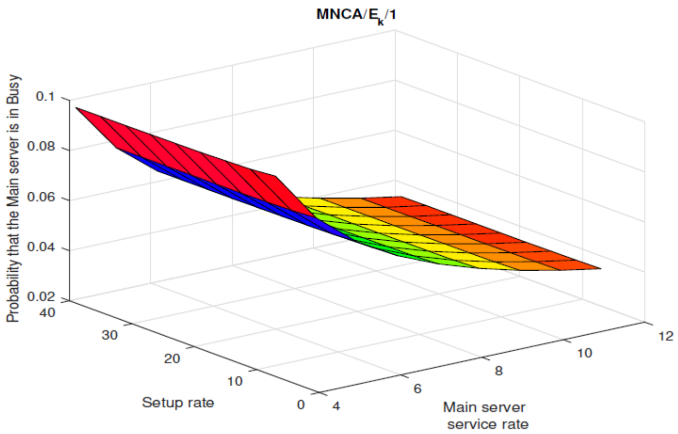

Figure 11: The graph of the $MAP-NC/E_k/1$ - setup rate(σ) and main server service rate(δ) versus probability that the main server is in busy

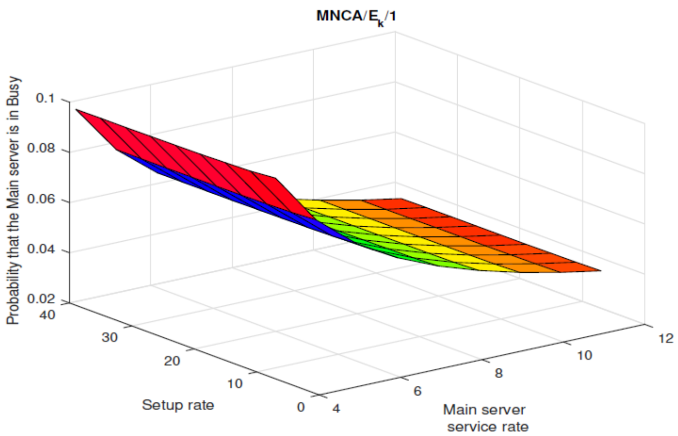

Figure 12: The graph of the $MAP-NC/M/1$ - setup rate(σ) and main server service rate(δ) versus probability that the main server is in busy

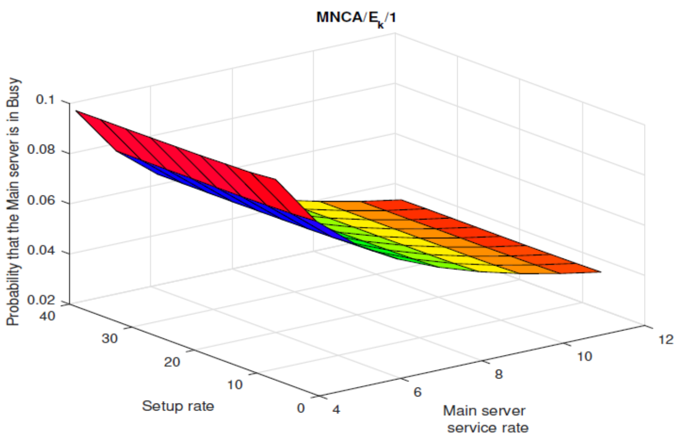

Figure 13: The graph of the $MAP-NC/H_k/1$ - setup rate(σ) and main server service rate(δ) versus probability that the main server is in busy

We observe from figures 11, 12 and 13 that the impact of setup rate and main server service rate on the probability of the main server service is in the busy mode. We have examined the probability of the main server is in the busy mode decreases while we are increasing both the setup rate and main server service rate for the arrangement of MAP-Negative correlation arrival (MNCA) with ERLS, EXPS and HEXS. Nevertheless, MAP-Negative correlation arrival decreases slowly in HEXS than ERLS service times.

We fix $\lambda = 1$; $\Psi = 3$; $\theta = 0.7$; $\zeta = 9$; $\eta = 5$; $\tau = 2$; $\gamma = 6$.

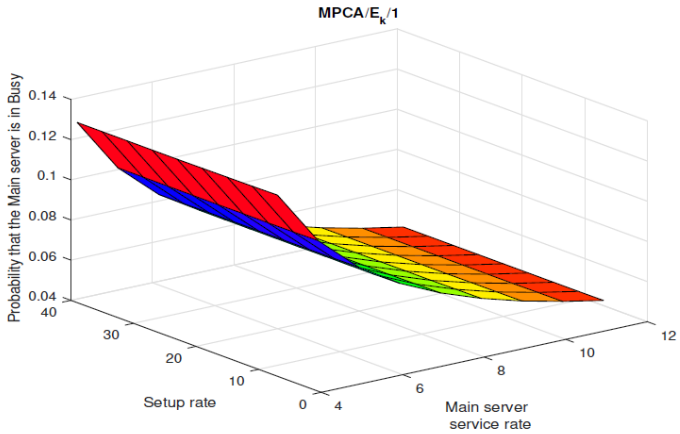

Figure 14: The graph of the $MAP-PC/E_k/1$ - setup rate(σ) and main server service rate(δ) versus probability that the main server is in busy

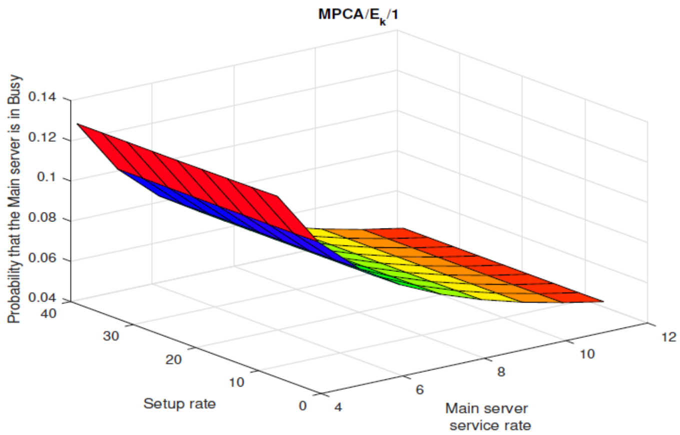

Figure 15: The graph of the $MAP-PC/M/1$ - setup rate(σ) and main server service rate(δ) versus probability that the main server is in busy

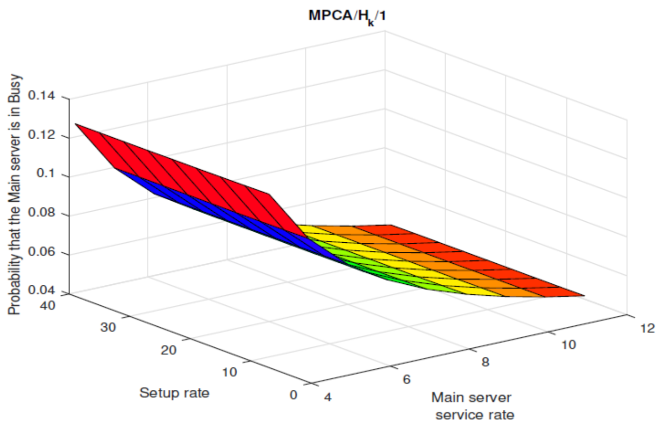

Figure 16: The graph of the $MAP-PC/H_k/1$ - setup rate(σ) and main server service rate(δ) versus probability that the main server is in busy

We observe from figures 14, 15 and 16 that the impact of setup rate and main server service rate on the probability of the main server service is in the busy mode. We have examined the probability of the main server is in the busy mode decreases while we are increasing both the setup rate and main server service rate for the arrangement of MAP-Positive correlation arrival(MPCA) with ERLS, EXPS and HEXS. Nevertheless, MAP-Positive correlation arrival decreases slowly in the hyperexponential service times.

Illustrated Example 10:

We fix $\lambda = 1$; $\delta = 4$; $\Psi = 3$; $\theta = 0.7$; $\eta = 5$; $\tau = 2$; $\sigma = 8$.

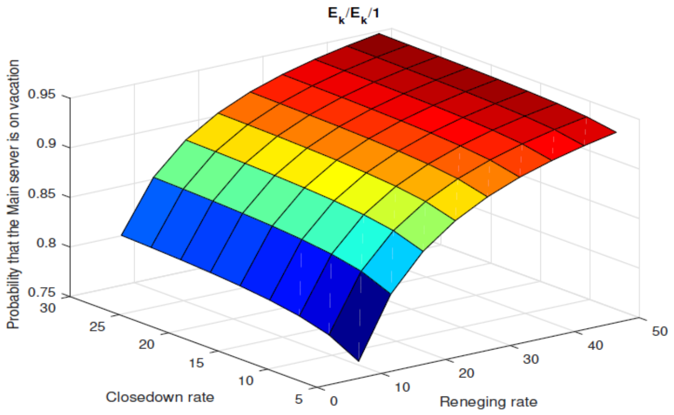

Figure 17: The graph of the $E_k/E_k/1$ - closedown rate(γ) and reneging rate(ζ) versus probability that the main server is on vacation

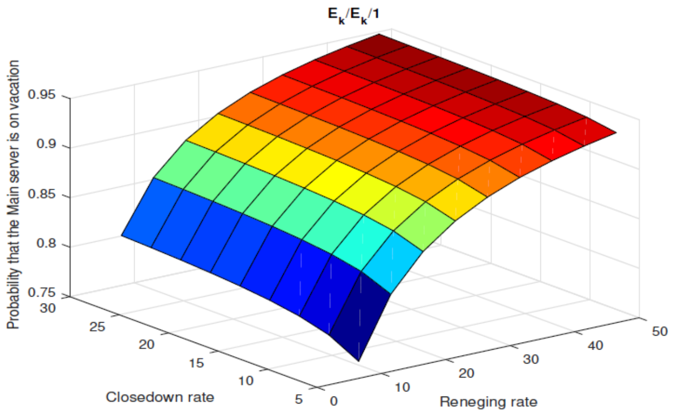

Figure 18: The graph of the $E_k/M/1$ - closedown rate(γ) and reneging rate(ζ) versus probability that the main server is on vacation

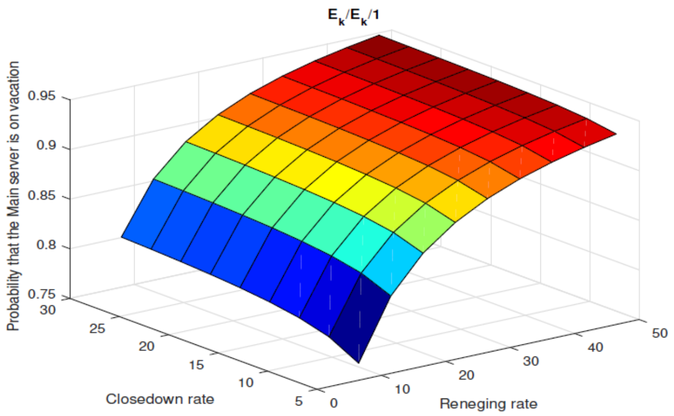

Figure 19: The graph of the $E_k/H_k/1$ - closedown rate(γ) and reneging rate(ζ) versus probability that the main server is on vacation

We observe from figures 17,18 and 19 that the impact of closedown rate and reneging rate on the probability of the main server service is on vacation. We have examined the probability of the main server is in the vacation increases while we are increasing both the closedown rate and reneging rate for the arrangement of Erlang arrival with ERLS, EXPS and HEXS. However, the Erlang arrival increases fastly in hyperexponential service times.

We fix $\lambda = 1$; $\delta = 4$; $\Psi = 3$; $\theta = 0.7$; $\eta = 5$; $\tau = 2$; $\sigma = 8$.

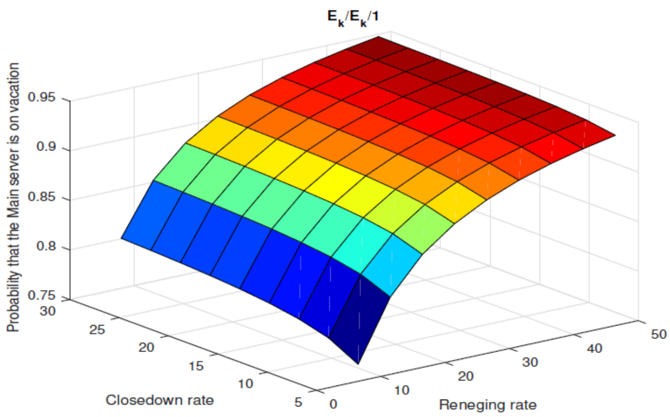

Figure 20: The graph of the $M/E_k/1$ - closedown rate(γ) and reneging rate(ζ) versus probability that the main server is on vacation

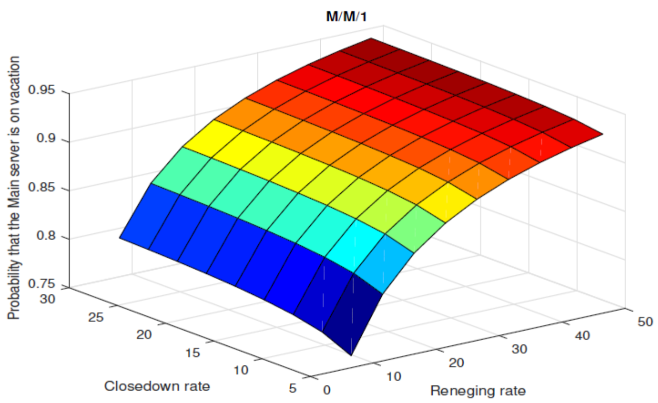

Figure 21: The graph of the $M/M/1$ - closedown rate(γ) and reneging rate(ζ) versus probability that the main server is on vacation

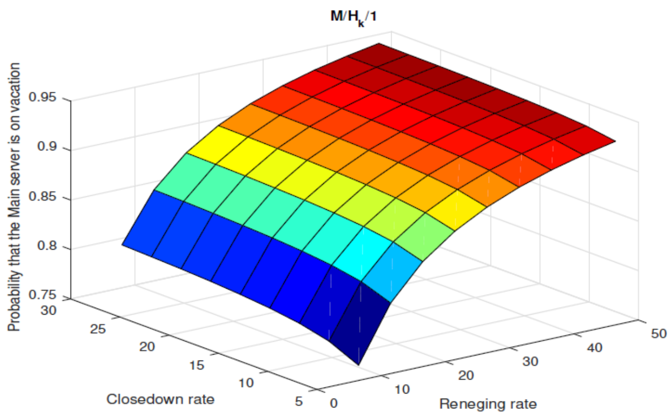

Figure 22: The graph of the $M/H_k/1$ - closedown rate(γ) and reneging rate(ζ) versus probability that the main server is on vacation

We observe from the figures 20, 21 and 22 that it shows the consequence of closedown rate and reneging rate on the probability of the main server service is on vacation. We have examined that the probability of the main server is in the vacation increases while we are increasing both the closedown rate and reneging rate for the arrangement of exponential arrival with ERLS, EXPS and HEXS. Therefore, the exponential arrival highly increases in hyperexponential service times.

We fix $\lambda = 1$; $\delta = 4$; $\Psi = 3$; $\theta = 0.7$; $\eta = 5$; $\tau = 2$; $\sigma = 8$.

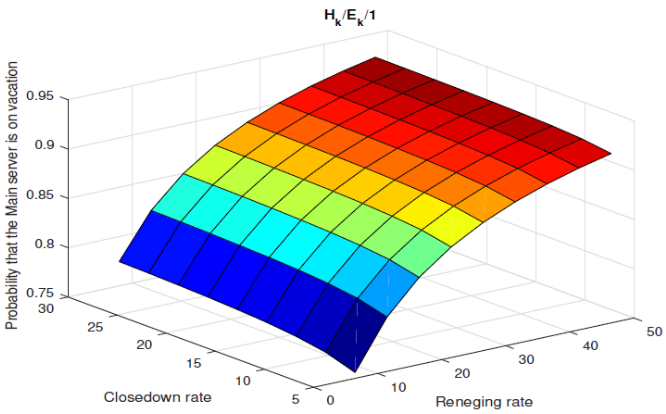

Figure 23: The graph of the $H_k/E_k/1$ - closedown rate(γ) and reneging rate(ζ) versus probability that the main server is on vacation

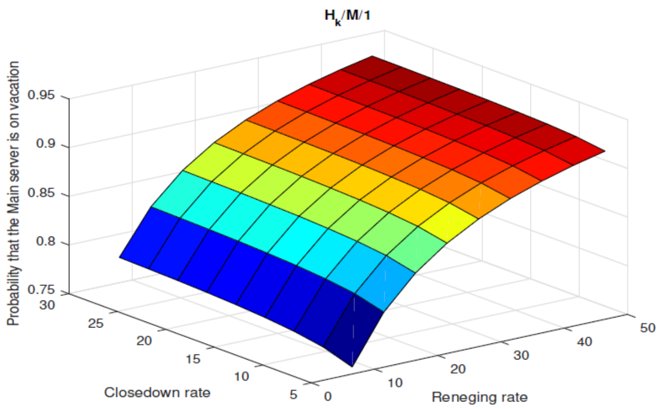

Figure 24: The graph of the $H_k/M/1$ - closedown rate(γ) and reneging rate(ζ) versus probability that the main server is on vacation

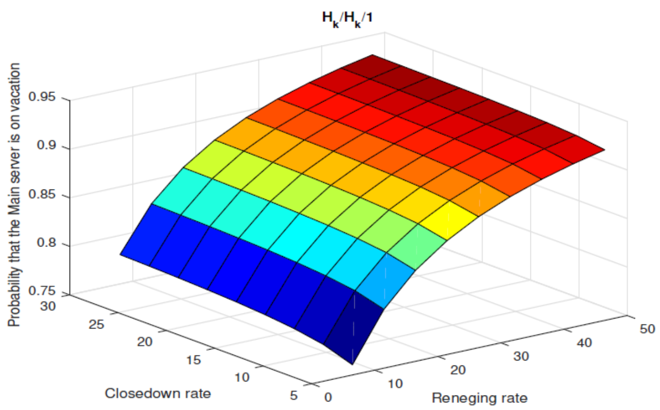

Figure 25: The graph of the $H_k/H_k/1$ - closedown rate(γ) and reneging rate(ζ) versus probability that the main server is on vacation

We observe from figures 23, 24 and 25 that it shows the consequence of closedown rate and reneging rate on the probability of the main server is on vacation. We have examined the probability of the main server is in the vacation increases while we are increasing both the closedown rate and reneging rate for the arrangement of hyperexponential arrival with ERLS, EXPS and HEXS. Nonetheless, the hyperexponential arrival times increases slowly in the case of Erlang service times.

We fix $\lambda = 1$; $\delta = 4$; $\Psi = 3$; $\theta = 0.7$; $\eta = 5$; $\tau = 2$; $\sigma = 8$.

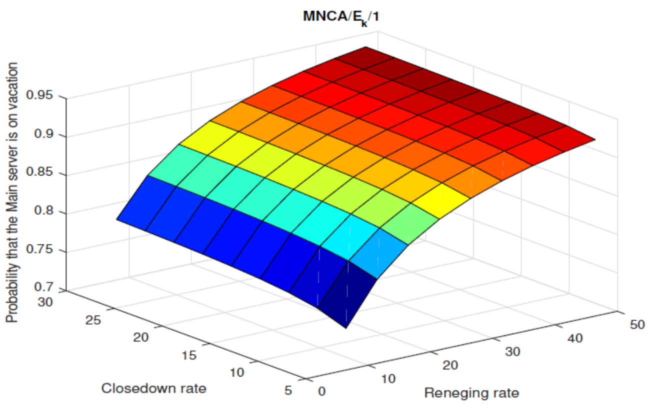

Figure 26: The graph of the $MAP - NC/E_k/1$ - closedown rate(γ) and reneging rate(ζ) versus probability that the main server is on vacation

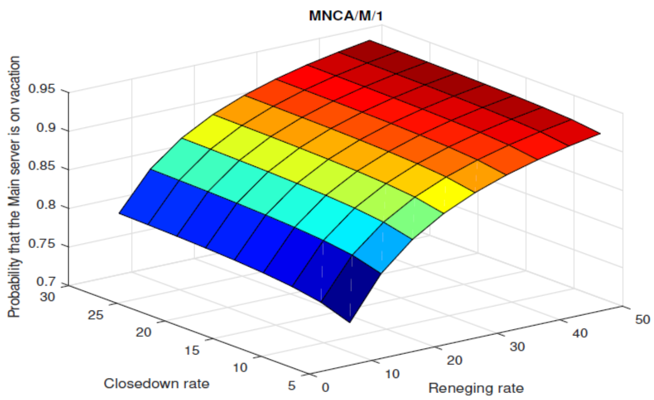

Figure 27: The graph of the $MAP - NC/M/1$ - closedown rate(γ) and reneging rate(ζ) versus probability that the main server is on vacation

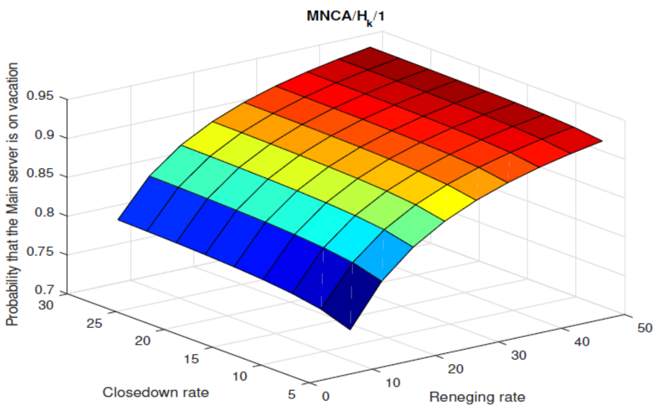

Figure 28: The graph of the $MAP - NC/H_k/1$ - closedown rate(γ) and reneging rate(ζ) versus probability that the main server is on vacation

We observe from figures 26, 27 and 28 that it shows the consequence of closedown rate and reneging rate on the probability of the main server service is on vacation. We have examined the probability of the main server is in the vacation increases while we are increasing both the closedown rate and reneging rate for the arrangement of MAP-Negative correlation arrival(MNCA) with ERLS, EXPS and HEXS. Nevertheless, MAP-Negative correlation arrival increases fastly in case of hyperexponential service times.

We fix $\lambda = 1$; $\delta = 4$; $\Psi = 3$; $\theta = 0.7$; $\eta = 5$; $\tau = 2$; $\sigma = 8$.

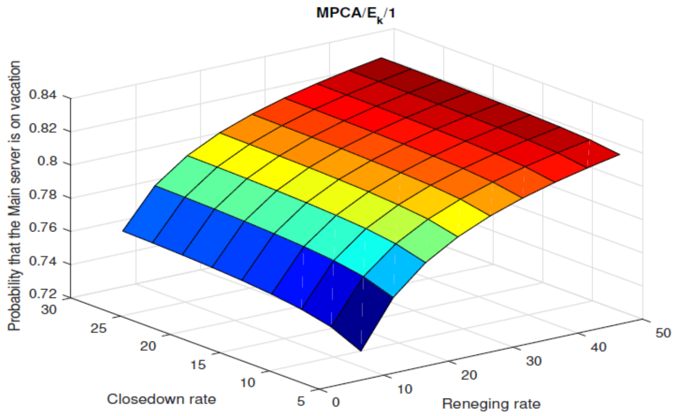

Figure 29: The graph of the $MAP-PC/E_k/1$ - closedown rate(γ) and reneging rate(ζ) versus probability that the main server is on vacation

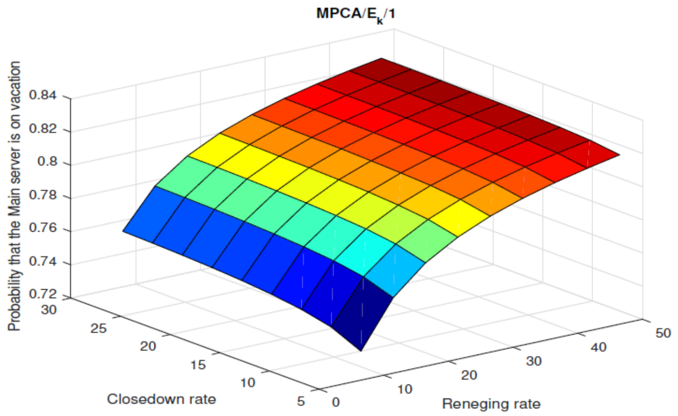

Figure 30: The graph of the $MAP-PC/M/1$ - closedown rate(γ) and reneging rate(ζ) versus probability that the main server is on vacation

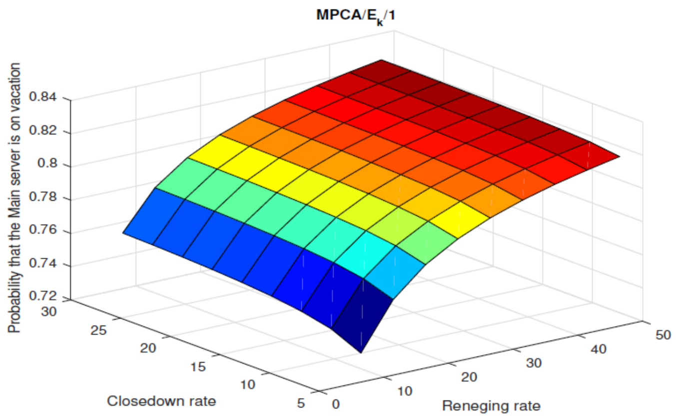

Figure 31: The graph of the $MAP-PC/H_k/1$ - closedown rate(ζ) and reneging rate(γ) versus probability that the main server is on vacation

We observe from figures 29, 30 and 31 that it shows the consequence of closedown rate and reneging rate on the probability of the main server service is on vacation. We have examined the probability of the main server is in the vacation increases while we are increasing both the closedown rate and reneging rate for the arrangement of MAP-Positive correlation arrival(MPCA) with ERLS, EXPS and HEXS. Moreover, the hyperexponential service fastly increases in MAP-Positive correlation arrival.

Illustrated Example 11:

We fix $\lambda = 1$; $\delta = 4$; $\eta = 5$; $\tau = 2$; $\sigma = 8$; $\gamma = 6$; $\zeta = 9$.

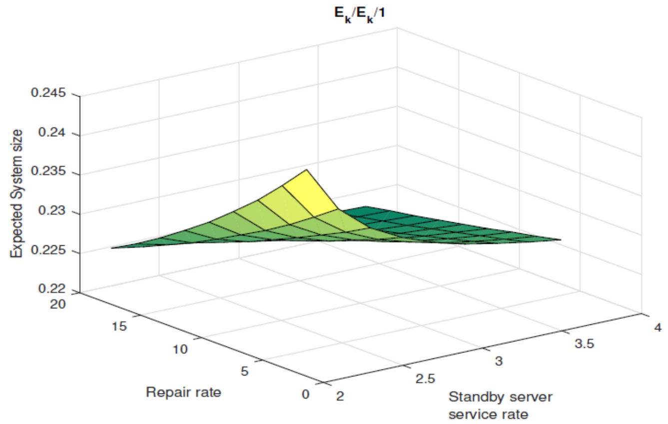

Figure 32: The graph of $E_k/E_k/1$ - repair rate(Ψ) and standby server service rate($\theta\delta$) versus the expected system size

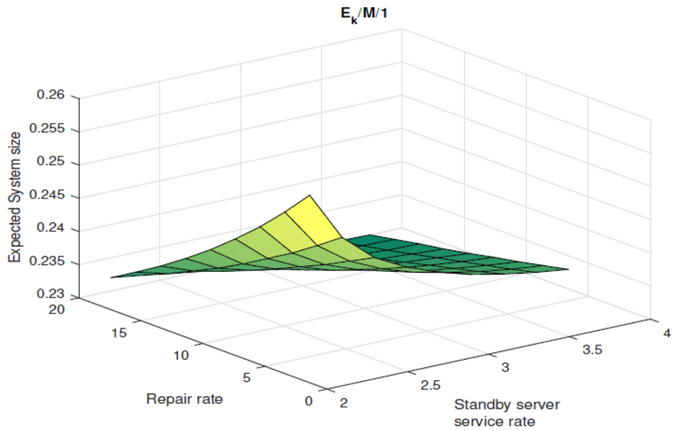

Figure 33: The graph of $E_k/M/1$ - repair rate(Ψ) and standby server service rate($\theta\delta$) versus the expected system size

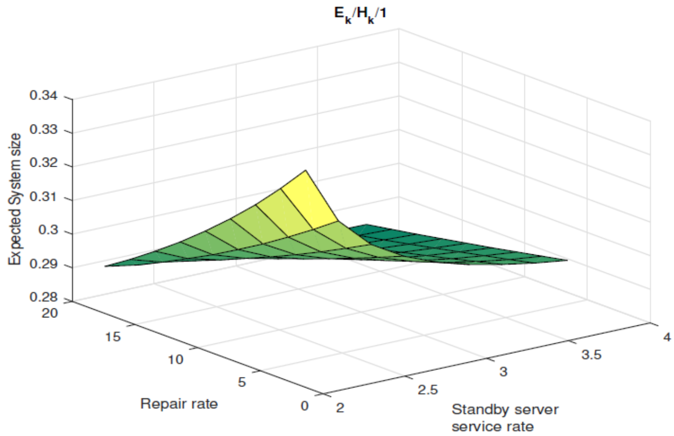

Figure 34: The graph of $E_k/H_k/1$ - repair rate(Ψ) and standby server service rate($\theta\delta$) versus the expected system size

We observe from the figures 32, 33 and 34 that it shows the consequence of standby server service rate and repair rate on the expected system size. We have examined that the expected system size decreases while we are increasing both the repair rate and standby server service rate for the arrangement of Erlang arrival with ERLS, EXPS and HEXS. However, the Erlang arrival fastly decreases with hyperexponential service.

We fix $\lambda = 1$; $\delta = 4$; $\eta = 5$; $\tau = 2$; $\sigma = 8$; $\gamma = 6$; $\zeta = 9$.

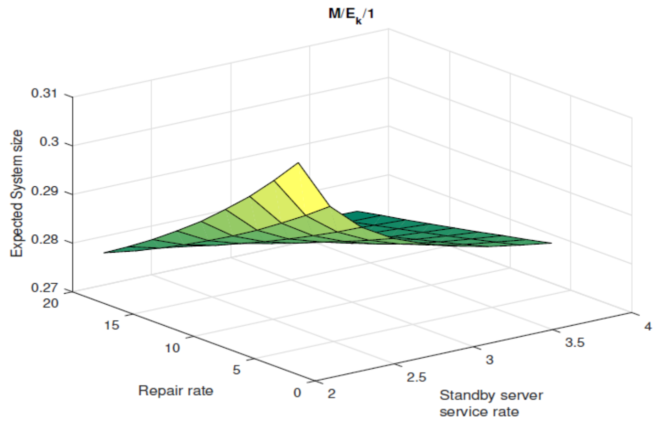

Figure 35: The graph of $M/E_k/1$ - repair rate(Ψ) and standby server service rate($\theta\delta$) versus the expected system size

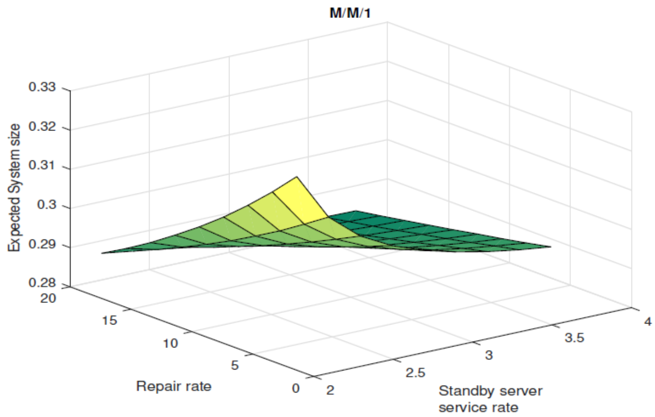

Figure 36: The graph of $M/M/1$ - repair rate(Ψ) and standby server service rate($\theta\delta$) versus the expected system size

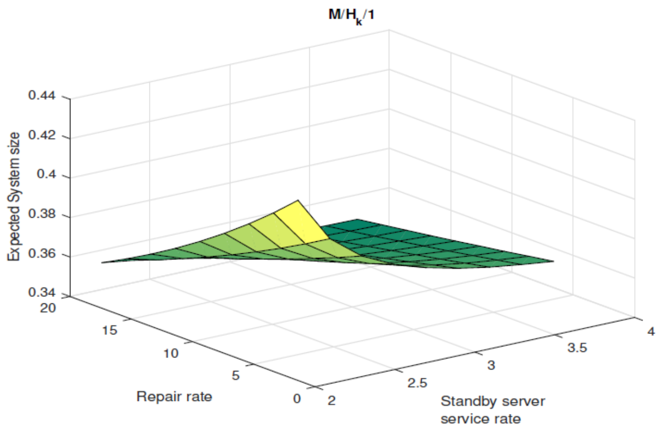

Figure 37: The graph of $M/H_k/1$ - repair rate(Ψ) and standby server service rate($\theta\delta$) versus the expected system size

We observe from the figures 35, 36 and 37 that it shows the consequence of standby server service rate and repair rate on the expected system size. We have examined that the expected system size decreases while we are increasing both the repair rate and standby server service rate for the arrangement of exponential arrival with services of ERLS, EXPS and HEXS. Nevertheless, exponential arrival decreases fastly in hyperexponential service, gradually in exponential service and slowly in Erlang service times.

We fix $\lambda = 1$; $\delta = 4$; $\eta = 5$; $\tau = 2$; $\sigma = 8$; $\gamma = 6$; $\zeta = 9$.

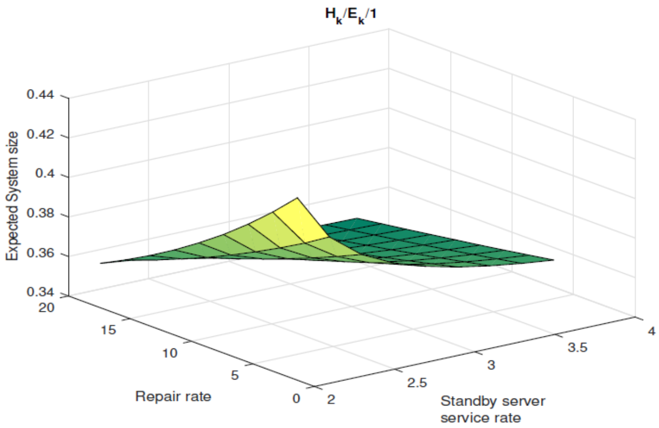

Figure 38: The graph of $H_k/E_k/1$ - repair rate(Ψ) and standby server service rate($\theta\delta$) versus the expected system size

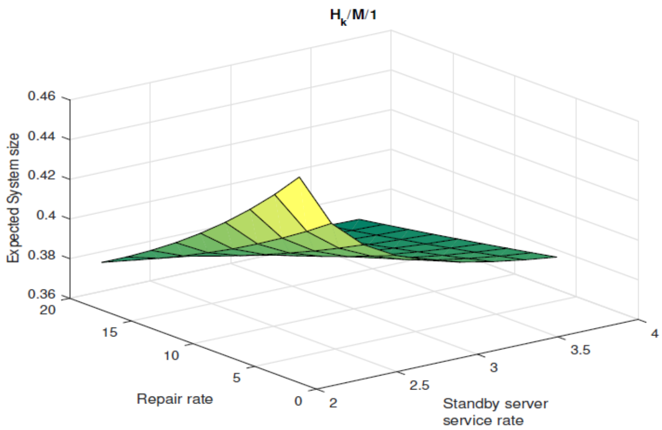

Figure 39: The graph of $H_k/M/1$ - repair rate(Ψ) and standby server service rate($\theta\delta$) versus the expected system size

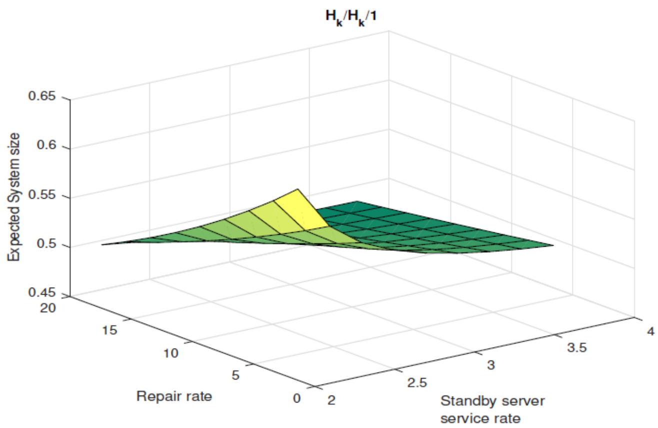

Figure 40: The graph of $H_k/H_k/1$ - repair rate(Ψ) and standby server service rate($\theta\delta$) versus the expected system size

We observe from the figures 38, 39 and 40 that it shows the consequence of standby server service rate and repair rate on the expected system size. We have examined that the expected system size decreases slowly while we are increasing both the repair rate and standby server service rate for the arrangement of hyperexponential arrival with services of ERLS, EXPS and HEXS. Moreover, the hyperexponential service times decreases than the Erlang service times with the hyperexponential arrival times.

We fix $\lambda = 1$; $\delta = 4$; $\eta = 5$; $\tau = 2$; $\sigma = 8$; $\gamma = 6$; $\zeta = 9$.

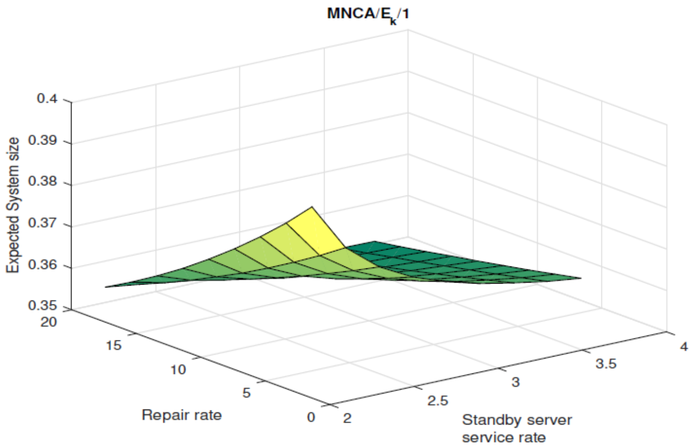

Figure 41: The graph of the $MAP - NC/E_k/1$ - standby server service rate($\theta\delta$) and repair rate(Ψ) versus the expected system size

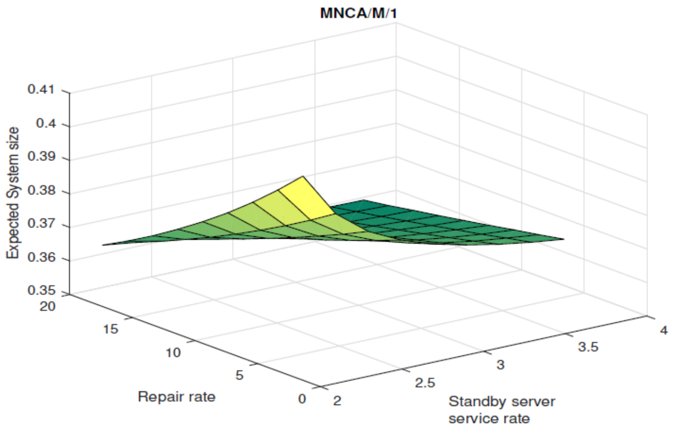

Figure 42: The graph of the $MAP - NC/M/1$ - standby server service rate($\theta\delta$) and repair rate(Ψ) versus the expected system size

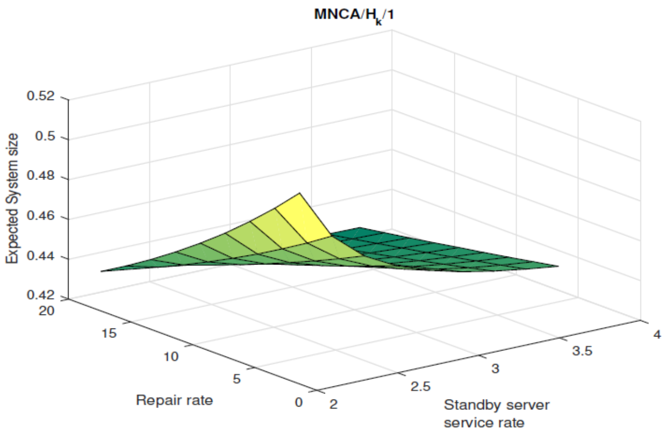

Figure 43: The graph of the $MAP - NC/H_k/1$ - standby server service rate($\theta\delta$) and repair rate(Ψ) versus the expected system size

We observe from the figures 41, 42 and 43 that it shows the consequence of standby server service rate and repair rate on the expected system size. We have examined that the expected system size decreases randomly while we are increasing both the repair rate and standby server service rate for the arrangement of MAP-Negative correlation arrival(MNCA) with services of ERLS, EXPS and HEXS. Nonetheless, the MAP-Negative correlation arrival decreases slowly in Erlang service times and fastly in hyperexponential service times.

We fix $\lambda = 1$; $\delta = 4$; $\eta = 5$; $\tau = 2$; $\sigma = 8$; $\gamma = 6$; $\zeta = 9$.

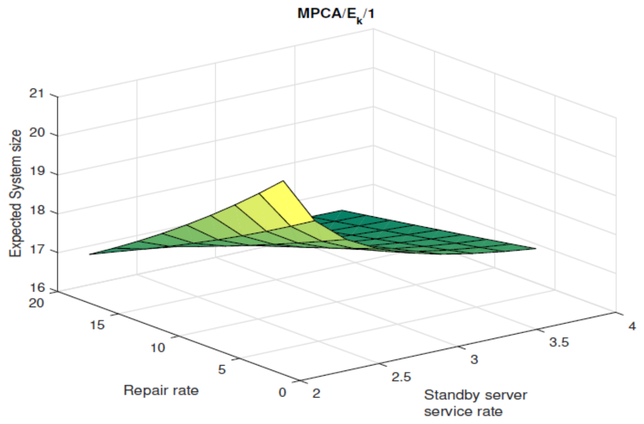

Figure 44: The graph of the $MAP - PC/E_k/1$ - standby server service rate($\theta\delta$) and repair rate(Ψ) versus the expected system size

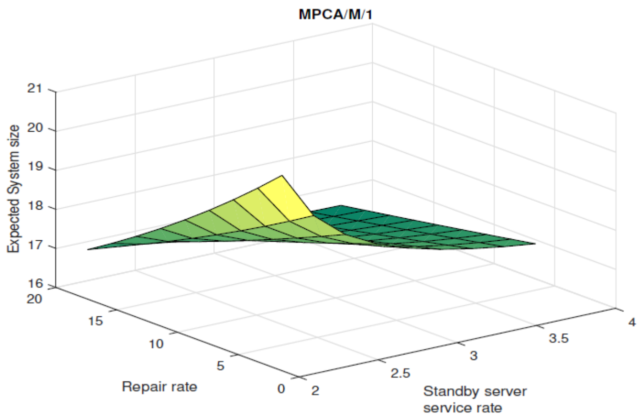

Figure 45: The graph of the $MAP - PC/M/1$ - standby server service rate($\theta\delta$) and repair rate(Ψ) versus the expected system size

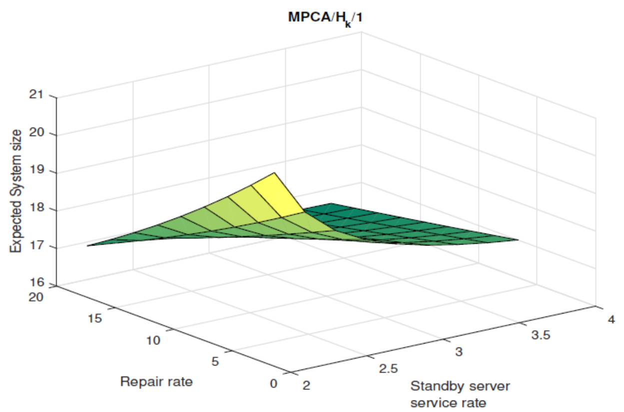

Figure 46: The graph of the $MAP - PC/H_k/1$ - standby server service rate($\theta\delta$) and repair rate(Ψ) versus the expected system size

We observe from the figures 44, 45 and 46 that it shows the consequence of standby server service rate and repair rate on the expected system size. We have examined that the expected system size decreases rapidly while we are increasing both the repair rate and standby server service rate for the arrangement of MAP-Positive correlation arrival(MPCA) with services of ERLS, EXPS and HEXS. Furthermore, the Erlang service times slowly decreases than the exponential and hyperexponential service times in case of MAP-Positive correlation arrival.

8. Comparing the service rate of the Main server and Standby server

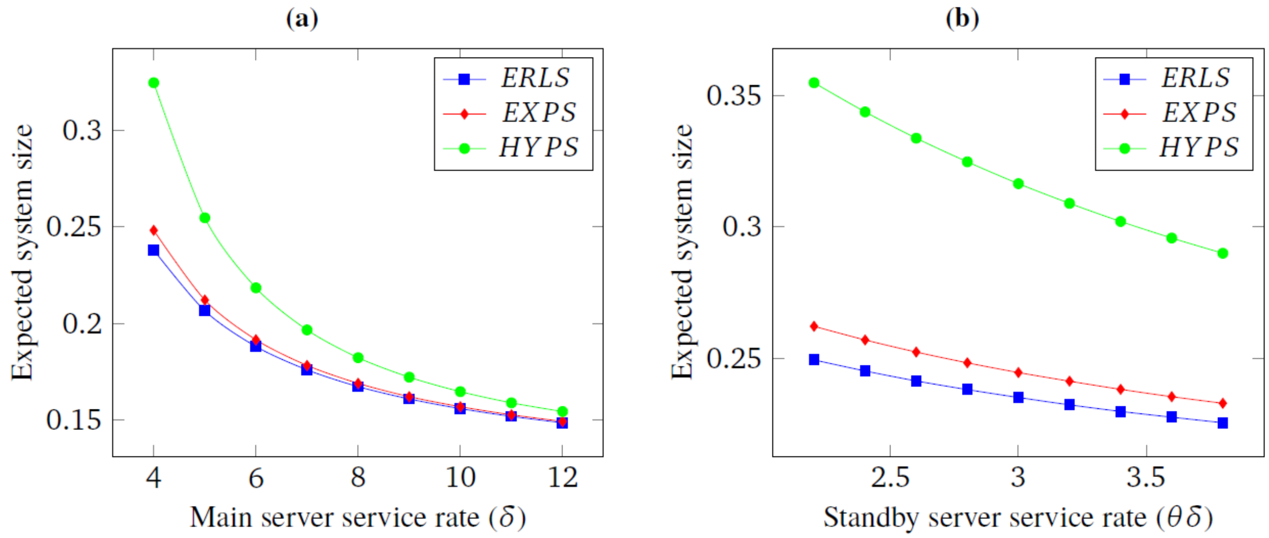

Figure 47: Expected system sizes Vs Main server and Standby server service rate of Erlang arrival

From figure 47, by comparing both the service rate of the main server and standby server contrast to the expected system size, it decreases rapidly in main server service rate and in the case of the standby server service rate decreases slowly.

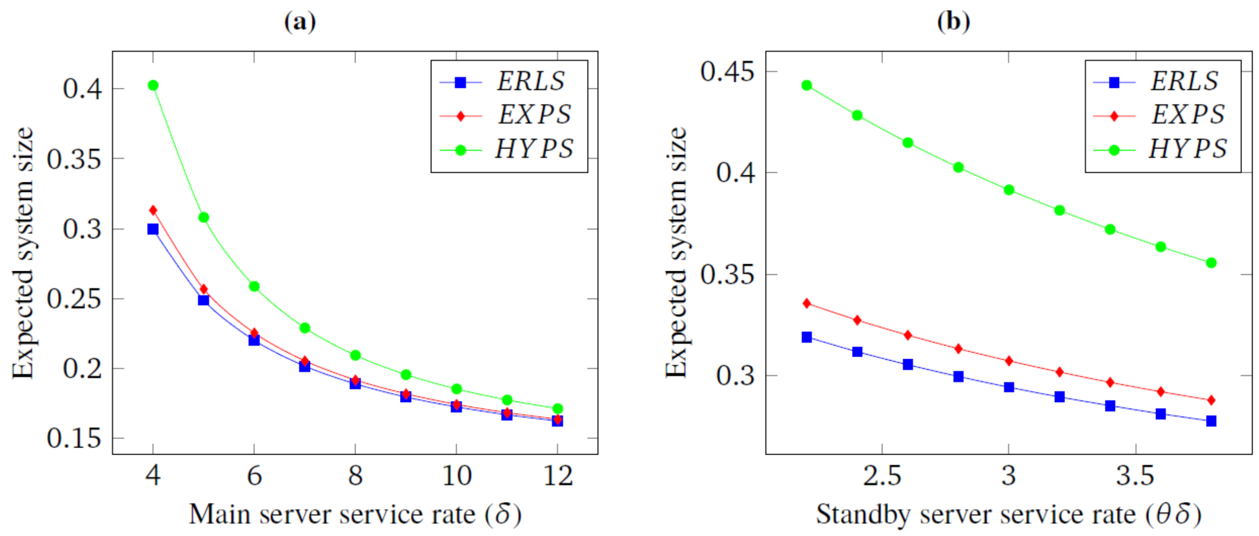

Figure 48: Expected system sizes Vs Main server and Standby server service rate of exponential arrival

From figure 48, by comparing both the service rate of the main server and standby server against to the expected system size, it decreases fastly in main server service rate and in the case of the standby server service rate decreases gradually.

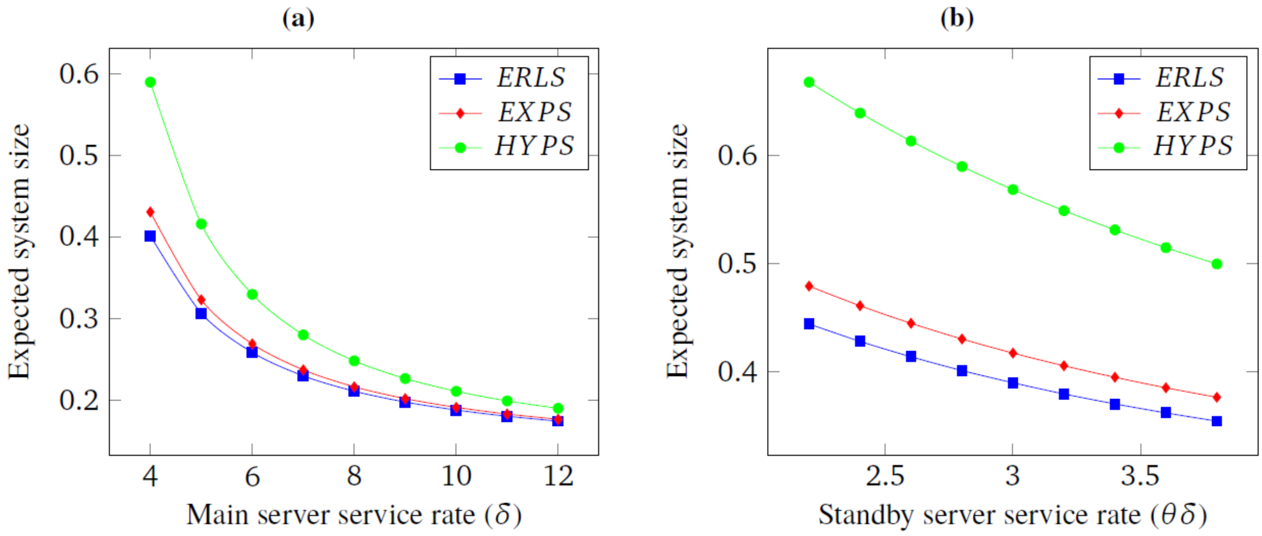

Figure 49: Expected system sizes Vs Main server and Standby server service rate of hyperexponential arrival

From figure 49, by comparing both the service rate of the main server and standby server contrast to the expected system size, it decreases more rapidly in main server service rate and in the case of the standby server service rate decreases gradually.

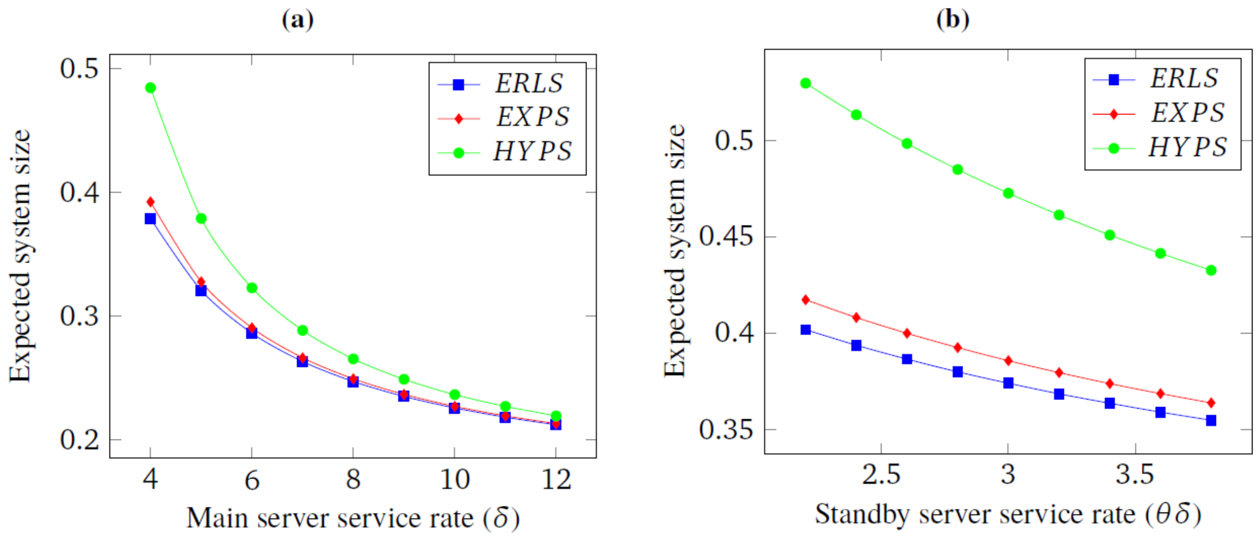

Figure 50: Expected system sizes Vs Main server and Standby server service rate of MAP-Negative correlation arrival

From figure 50, by comparing the service rate of the main server and standby server contrast to the expected system size, it decreases rapidly in main server service rate and in the case of the standby server service rate decreases slowly.

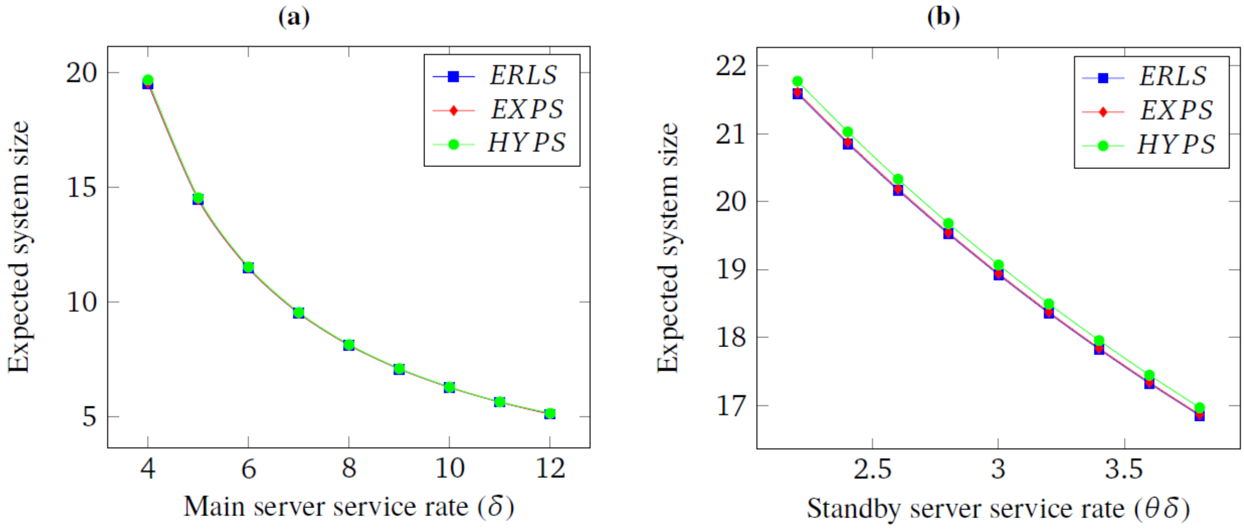

Figure 51: Expected system sizes Vs Main server and Standby server service rate of MAP-Positive correlation arrival

In figure 51, by comparing the service rate of the main server and standby server on the expected system size, main server service rate decreases such that all types of services converge and in the case of the standby server service rate decreases rapidly.

9. Conclusion

In our model, customers arrive in Markovian Arrival Process and the process of service in phase type distribution with server breakdown, multiple vacations, reneging, standby server, setup, closedown and repair. In our work, we also compute the busy period analysis. Using numerical arrivals and services we tabulated the expected system size values for the breakdown rate, repair rate, standby server service rate, setup rate, closedown rate, service rate and vacation rate. We have compared the both the setup rate and main server service rate contrast to the probability that the main server is in the busy mode, both the closedown rate and reneging rate contrast to the probability that the main server is on vacation and both the standby server service rate and repair rate contrast to expected system size showed through the graphical demonstrations. Furthermore, We have compared the service rate of the main server and standby server graphically.

References

1. Arumuganathan, R., Jeyakumar, S., (2005). Steady state analysis of a bulk queue with multiple vacations, setup times with N-policy and closedown times, *Applied Mathematical Modelling*, Vol. 29, pp. 972-986.
2. Attahiru Sule Alfa (1995). A discrete MAP/PH/1 queue with vacations and exhaustive time-

limited service, Operations Research Letters, Vol. 18, pp. 31-40.
3. Ayyappan and Shyamala (2014). $M^{[X]}/G_1, G_2/1$ with Setup Time, Bernoulli Vacation, Break Down, and Delayed Repair, International Journal of Stochastic Analysis, Article ID 892867, pp. 1-12.
4. Chakravarthy, S. R. (2010). Markovian Arrival Process, Wiley Encyclopaedia of Operation Research and Management Science.
5. Chakravarthy, S.R., Neuts, M.F. (2014). Analysis of a multi-server queueing model with MAP arrivals of regular customers and phase type arrivals of special customers, Simulation Modelling Practice and Theory, Vol.43, pp. 79-95.
6. Chesoong Kim, Klimenok, V.I., Dudin, A.N. (2017). Analysis of unreliable BMAP/PH/N type queue with Markovian flow of breakdowns, Applied Mathematics and Computation, Vol. 314, pp. 154-172.
7. Jinbiao Wu, Zaiming Liu, Yi Peng (2009). On the BMAP/G/1 G-queues with second optional service and multiple vacations, Applied Mathematical Modelling, Vol. 33, pp. 4314-4325.
8. Khalaf, R.F, Madan, K.C and Lukas, C.A. (2011). On a Batch Arrival Queuing System Equipped with a Stand-by Server during Vacation Periods or the Repairs Times of the Main Server, Journal of Probability and Statistics, Article ID 812726, pp. 1-11.
9. Latouche, G. and Ramaswami, V. (1999): Introduction of Matrix-Analytic Methods in Stochastic Modeling. Society for Industrial and Applied Mathematics, Philadelphia.
10. Neuts, M.F.(1981). Matrix geometric Solutions in Stochastic Models: an algorithmic approach. John Hopkins Series in Mathematical Sciences, John Hopkins University Press, Baltimore, Md, USA.
11. Qi-Ming He (2004). Fundamentals of Matrix-Analytic Methods, Springer, New York.
12. Qi-Ming He, Attahiru Sule Alfa (2015). Construction of Markov chains for discrete time MAP/PH/K queues, Performance Evaluation, Vol. 93. pp. 17-26.
13. Subramanian, R., Sarm, Y.V.S. (1987). Two-Unit Standby Systems with Non-Instantaneous Switchover and Imperfect Switch, Pergamon Journals, Vol. 27, No. 2, pp. 237-247.
14. Sreekanth Kolledath, Kamlesh Kumar, Sarita Pippal (2017). Survey on Queueing Models with Standbys Support, Yugoslav Journal of Operations Research, vol. 28, Issue. 1, pp. 3-20.
15. Stefanos A. Zenios (1999). Modeling the transplant waiting list: A queueing model with reneging, Queueing Systems, Vol. 31, pp. 239-251.
16. Tsung-Yin Wang (2012). Random N-Policy Geo/G/1 Queue with Startup and Closedown Times, Journal of Applied Mathematics, Article ID 793801, pp. 1-19.
17. Wei Sun, Yulan Wang, Naishuo Tian (2012). Pricing and setup/closedown policies in unobservable queues with strategic customers, Springer, Vol. 10, pp. 287-311.
18. Zhisheng Niu, Tao Shu, Yoshitaka Takahashi (2003). A vacation queue with setup and close-down times and batch Markovian arrival processes, Performance Evaluation, Vol. 54, pp. 225-248.

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