Synergetic Effects in Applied Poisson Flow Models

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Abstract

In this paper, we consider applied models of Poisson flow, in which synergetic effects are identified. The synergistic effect here is understood as a significant increase/decrease of the efficiency of the simulated system, while increasing the flow rate and a certain additional parameter of the model. Three applied Poissom flow models are considered. First of them is application of Boolean model in powder metallurgy. Second describes process of $3D$ printer and xerocopying. Third deals with non-stationary model of queuing acyclic network with assembly and disassambley of customers and with deterministic service times. In all these models sufficiently strong synergetic effects of considered systems objective functions are obtained.

Keywords: synergetic effect, Poisson flow, Boolean model, acyclic directed graph.

1 Introduction

In this paper, we consider applied models of Poisson flow, in which synergetic effects are identified. The synergistic effect here is understood as a significant increase/decrease in the efficiency of the simulated system, while increasing the flow rate and a certain parameter of the model. The paper considers the following applied models based on Poisson point flow.

The first model is a Boolean model of a random set based on a Poisson flow of points in three-dimensional space. Balls of a fixed radius are attached to each of the flow points by their centres. The set of balls is projected onto a flat unit square. This model may characterize the process of spraying a certain surface with a metal powder.

Using well-known formulas for the Boolean model [1], we study the average square area uncovered by balls projections. It is shown that with a proportional increase in the intensity of the Poisson flow by k times and the same decrease in the volume of each ball, this efficiency indicator tends to zero and the rate of such convergence has the form $\exp(-bk^a)$, for some $a, b > 0$, i.e. the convergence is fast enough. The selected efficiency indicator characterizes the degree of protection of the surface to be sprayed with the powder.

The second model is used for quality analysis in modern printing technologies: xerocopying, 3D printing. This model is based on the Poisson flow model of points of variable intensity $\lambda(x, y)$ (intensity $\lambda(x, y, z)$) of a fixed mass m in two-dimensional space (in three-dimensional space). We investigate the proximity of the resulting random set and the intensity of λ, while simultaneously reducing the mass of m: $m \to m/k$ by a factor of k and increasing the flow intensity of $\lambda \to k\lambda$. A special probability metric is selected for this purpose and $\rho \to \frac{\rho}{\sqrt{k}}$. The choice of this metric is based on the Poisson point flow model and the Boolean model and plays an important role in evaluating the quality of the $3D$ printer and the xerocopying procedure.

The third model is being built for research in the non-stationary model of operation of a conveyor with the assembly and disassembly of customers. In this model, it is assumed that the system input receives a non-stationary Poisson flow of customers, that pass through the acyclic service network. The customers are assembled and/or disassembled at each node of the network. Moreover, the customer begins to be served only when all its parts arrive at the node, which means that these parts are located on the edges of the network. The time spent in the node is deterministic, without any delays in the queue. It is proved that the distribution of the number of customers in the network nodes has a Poisson distribution and its parameter is calculated at any time. This allows us to see that the input stream intensity is being smoothed in this system.

A special feature of this model is the absence of a queue in nodes, which is caused by the fact that each node is a multi-channel system. In this system, due to the synergetic effect known for queuing systems [2], when the intensity of the input Poisson flow and the number of channels increase by a factor of k, the random number of customers in the queue tends to zero. In other words, such a multi-channel system is approximated by a system with an infinite number of channels and, consequently, with no queue. It should be noted that the product theorem for stationary distributions in open queuing networks with division and merging of customers was obtained only recently [3], [4]. Algorithms for numerical calculations of non-stationary model in open queuing networks were constructed in the works [5], [6]. However, exact formulas for distributions of the numbers of customers in the network nodes in non-stationary model were not received earlier even for partial service time distributions.

2 Synergetic effects in the Boolean model

Consider the problem of the quality of the protective layer, formed by spraying the powder on a certain surface. We will consider that the flow of powder particles – balls of volume v is applied to a flat surface that has the shape of a square with sides of unit length. The area s of the projection of the ball on the plane is equal to $s = \pi^{1/3}\left(\frac{3v}{4}\right)^{2/3}$.

Let's assume that the centres of the balls in the projection on the sprayed surface form a Poisson flow of intensity λ. Then from the known properties of the Boolean model [1, pp. 295, 335] the average area q of the set of points, not covered by circles of radius s on a unit square, satisfies the equality

$$q = \exp(-\lambda s). \tag{1}$$

Let us now assume that the volume v of the ball decreases by a factor of k, and the intensity of the Poisson flow of the centres of the balls in the projection on the unit square increases by a factor of k. Then the area s of the projection of the ball on the square is reduced by $k^{2/3}$ times. Let's denote $q(k)$ the average area of the set of uncovered projections of balls of reduced volume of points on the unit square. Using the formula (1), we get the equality

$$q(k) = \exp(-\lambda s k^{1/3}), \tag{2}$$

the average thickness of the protective layer is preserved.

If the intensity of the Poisson flow of the projection centres of the balls increases by $k^{1-\gamma}$, $0 < \gamma < 1/3$ times, when the volume of the ball decreases by k times, then equality (2) is replaced with equality

$$q(k) = \exp(-\lambda s k^{1/3-\gamma}), \tag{3}$$

and the average thickness of the protective layer decreases by k^{γ} times.

3 Quality of the $3D$ printer

Let's first consider the problem of evaluating the quality of the $3D$ printer, assuming that there is a design distribution density the mass of the part specified by the Lebesgue-measurable function $\lambda(x,y,z) \geq 0,\ (x,y,z) \in A,\ A = \{0 \leq x \leq a,\ 0 \leq y \leq b,\ 0 \leq z \leq c\}$, and

$$\Lambda = \int_0^a \int_0^b \int_0^c \lambda(x,y,z)dx\ dy\ dz < \infty.$$

We will assume that a copy of this image is generated by Poisson flow of particles of dye powder with intensity $\lambda(x,y,z),\ (x,y,z) \in A$. The mass of each particle is equal to m. Find out, how the image quality changes if the mass of each particle is reduced by a factor of k, and the intensity of the Poisson flow of powder particles increases by a factor of k.

To evaluate the quality of $3D$ printing, the following criteria are introduced. Let A^* is some partition of the rectangle A by $n = n(A^*)$ disjoint and measurable by Lebesgue subsets $A_1, \dots, A_n$. Denote $\eta_i^{(k)}$ a random number of Poisson flux particles with intensity $k\lambda(x,y,z)$, falling into the set A_i, and put

$$\Lambda(A_i) = \int_{A_i} \int \lambda(x,y,z)dx\ dy\ dz.$$

Since the Poisson flow has an intensity $k\lambda(x,y,z)$, and each particle has a mass $\frac{m}{k}$, then the total mass of these particles is $\frac{m\eta_i^{(k)}}{k}$ and satisfies the following equalities

$$M\frac{m\eta_i^{(k)}}{k} = m\Lambda_i, D\frac{m\eta_i^{(k)}}{k} = m^2\frac{\Lambda_i}{k}$$

this means that the following equality is fulfilled

$$M\left(\frac{m\eta_i^{(k)}}{k} - m\Lambda_i\right)^2 = \frac{m^2\Lambda_i}{k},\ M\sum_{i=1}^{n(A^*)}\left(\frac{m\eta_i^{(k)}}{k} - m\Lambda_i\right)^2 = \frac{m^2\Lambda}{k}.$$

The last equality follows from the formula $\Lambda = \sum_{i=1}^{n(A^*)} \Lambda_i$.

Let $\mathcal{A}$ be a collection of all possible partitions A^* of the set A into disjoint and Lebesgue-measurable subsets $A_1, \dots, A_n,\ n = n(A^*)$. Now let's define the quality of the function $\rho(k)$ of the $3D$ printer with k – multiple powder grinding equal to

$$\rho(k) = \left(\sup_{A^* \in \mathcal{A}} M\sum_{i=1}^{n(A^*)}\left(\frac{m\eta_i^{(k)}}{k} - m\Lambda_i\right)^2\right)^{1/2} = m\sqrt{\frac{\Lambda}{k}}.$$

Therefore, the value $\rho(k)$, which characterizes the quality of the coating with crushed particles, decreases as $k^{-1/2}$. Note that the image quality in the resulting xerocopying process can be studied in the same way.

4 Non-stationary model of the queuing system with the assembly and disassembly of customers and deterministic service times

Consider an acyclic directed graph (digraph) G with a finite set of vertices U and a set of edges V. Assume that in the digraph G there is a single (input) vertex u_0, from which a path can be drawn to any vertex. Denote $S(u)$ the set of all paths $s(u)$ from the vertex u_0 to the vertex u in the digraph G. Let's put $l(u)$ the maximum number of edges in paths $s(u) \in S(u)$, let $L = \max_{u \in U} l(u)$. The set of vertices U is divided into subsets $U(k),\ 0 \leq k \leq L$, of the form

$$U(k) = \{u \in U:\ l(u) = k\},\ k = 0, \dots, L.$$

It is proved [7], that in the digraph G any edge $(u_1, u_2) \in V;\ u_1 \in U(k_1),\ u_2 \in U(k_2)$ satisfies the inequality $k_1 < k_2$. In the cited article, the following analog of the Floyd-Warshell algorithm [8] is developed for finding the matrix of maximum lengths of all paths between the vertices of an acyclic digraph and hence for calculating $l(u),\ u \in U$.

Let an acyclic digraph contains n vertices, denoted by the indexes $1, \dots, n$. Let's put $D^k = ||d_{i,j}^k||_{i,j=1}^n,\ k = 0, \dots, n$, where the value $d_{i,j}^k$ is equal to infinity, if and only if the graph G has no path, connecting the vertices $i,\ j$, which can only pass through the vertices $1,\dots,k$. If such paths exist, the value $D_{i,j}^{k}$ is equal to the maximum length of such paths. It is not difficult to prove that the following theorem holds.

Statement 1. *The matrices* $D^k = ||d_{i,j}^k||_{i,j=1}^n,\ k = 2,\dots,n$, *satisfy the recurrent relations*

$$d_{i,j}^k = \max(d_{i,j}^{k-1},\ d_{i,k}^{k-1} + d_{k,j}^{k-1}),\ \ if\ \ \max(d_{i,j}^{k-1},\ d_{i,k}^{k-1} + d_{k,j}^{k-1}) < \infty, \tag{4}$$

$$else\ \ d_{i,j}^k = \min(d_{i,j}^{k-1},\ d_{i,k}^{k-1} + d_{k,j}^{k-1}). \tag{5}$$

The matrix $D^n = ||d_{i,j}^n||_{i,j=1}^n$ defines the maximum length of paths between vertices of the graph G, if such paths exist. If there are no such paths, the corresponding elements of the matrix are equal to infinity.

Assume that $(u_1,u_2) \in V,\ l(u_1) = k_1,\ l(u_2) = k_2,\ k_2 > k_1 + 1$. Enter into the (u_1,u_2) edge of the graph G dummy vertices $u_1',\dots,u_{k_2-k_1-1}'$ and thus turn it into a path, consisting of sequentially connected edges $(u_1,u_1'),\ (u_1',u_2'),\dots,(u_{k_2-k_1-1}',u_2)$ (see Fig. 1).

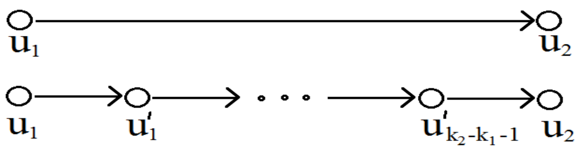

Fig 1. Converting the edge (u_1,u_2) (above) to the sequence of edges (below).

Then we can determine the length of the maximum path from the vertex u_0 to the dummy vertex u_j' by the equality $l(u_j') = l(u_1) + j,\ 1 \le j \le k_2 - k_1 - 1$. Denote G_* an acyclic digraph obtained by introducing dummy vertices to the digraph G. Let U_* be the set of vertices in the digraph G_*, and V_* - the set of edges in it.

We assume that the input vertex u_0 receives a non-stationary Poisson flow of intensity $\lambda(t.)$ Let $R(u) = \{u'\colon (u',u) \in V_*\}$ and means $l(u') + 1 = l(u),\ u' \in R(u)$. In each vertex $u \in U_*$, the customer is being served for a deterministic time $t(u)$, and in dummy vertices this time is zero. If the vertex $u \in U_*$ contains several edges, the customer service begins when all input customers are collected at this vertex, forming a single customer. If several edges come out of the vertex $u \in U$, this means that after servicing, the customer/customers received at this vertex are disassembled. Moreover, as a result of waiting for all input customers, going to the vertex u, on the edges (u',u), these customers can be collected.

Let's construct a recurrent algorithm with the value $l(u)$ for determining the time $T(u)$ of the exit from the vertex u of the customer, received at the zero moment at the vertex u_0. Then equality is true

$$T(u) = \max_{s(u)\in S(u)} \textstyle\sum_{u'\in s(u)} t(u').$$

It is obvious that $T(u)$ satisfies the relations

$$T(u) = \underline{T}(u) + t(u),\ \underline{T}(u) = \max_{u'\in R(u)} T(u').$$

Using this construction, we can calculate the parameter $\lambda(t,u)$ of the Poisson distribution of the number of customers, that are at the time t at the vertex u

$$\lambda(t,u) = \int_{t-T(u)}^{t-\underline{T}(u)} \lambda(\tau)d\tau.$$

In turn, the parameter $\lambda(t,(u',u))$ of the Poisson distribution of the number of customers located in the edge $(u',u),\ u' \in R(u)$ at moment t, satisfies the equality

$$\lambda(t,(u',u)) = \int_{t-\underline{T}(u)}^{t-T(u')} \lambda(\tau)d\tau.$$

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