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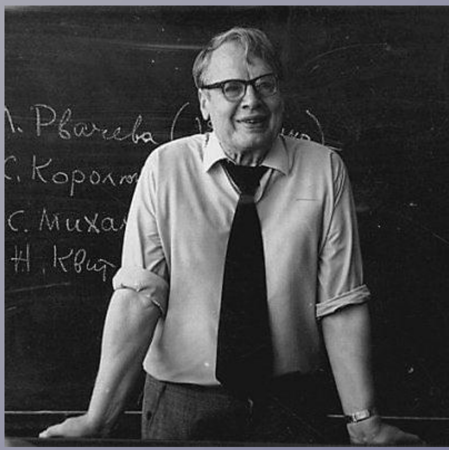

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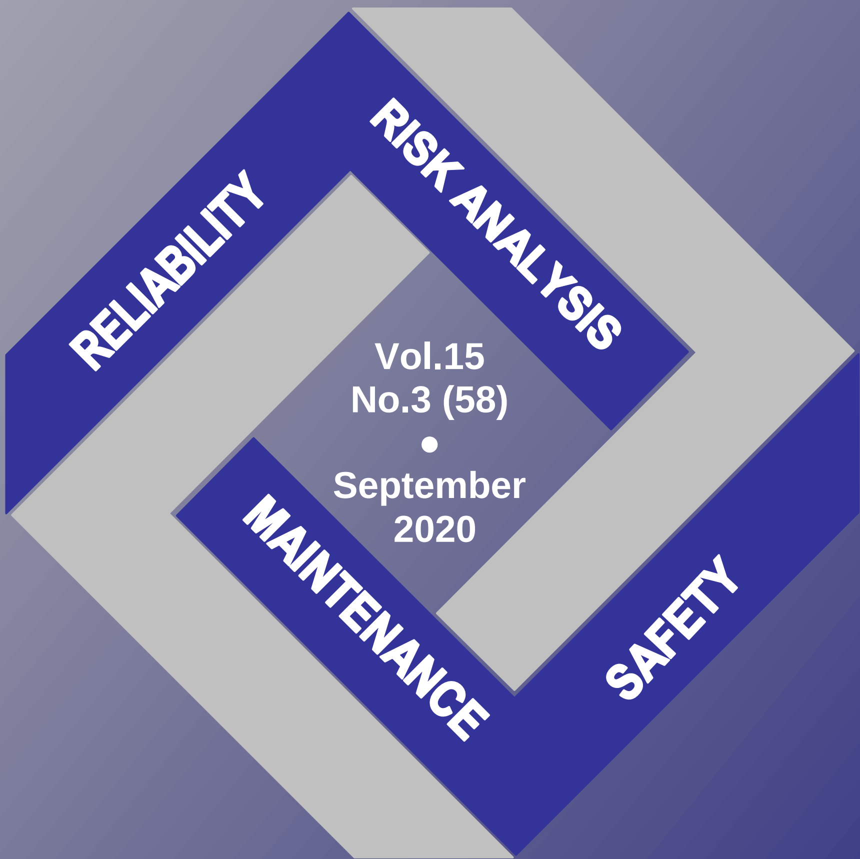

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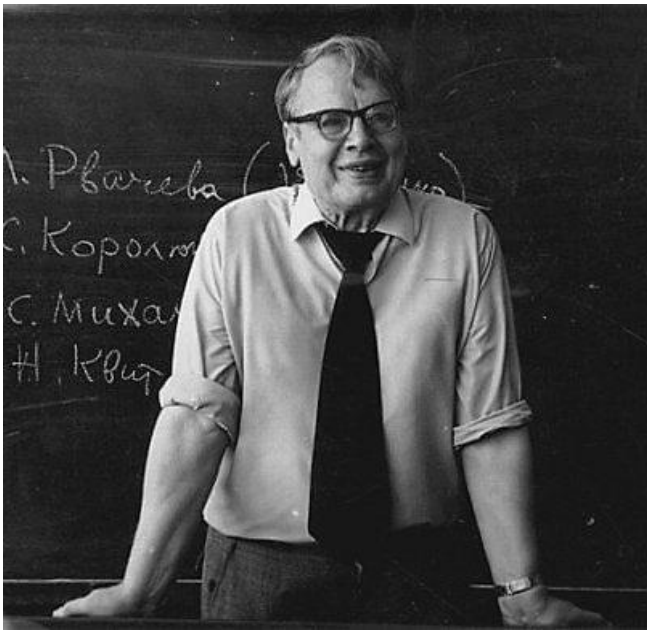

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In Good Memory of the Academician of the Academy of Sciences of Ukraine, Professor, Doctor of Physics and Mathematics Vladimir Semenovich Korolyuk

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Vladimir Semenovich Korolyuk, an outstanding Ukrainian mathematician, whose main work is related to probability theory, mathematical statistics, approximate and numerical methods, died on June 4, 2020 at the age of 95.

V.S.Korolyuk published about 350 scientific works, including about 20 monographs. Vladimir Semenovich Korolyuk was the Honored Worker of Science and Technology of Ukraine (1998), the laureate of the State Prize of Ukraine in the field of science and technology (1978, 2003), Prizes of the Academy of Sciences of Ukraine named after N. M. Krylov (1976), V. M. Glushkov (1988), N. N. Bogolyubov (1995), M. V. Ostrogradskii (2002).

In 1963, Korolyuk defended his doctoral dissertation becoming Doctor of Physical and Mathematical Sciences in 1964, and then Professor (1965) and corresponding member of the Academy of Sciences of the Ukraine (1967). Korolyuk became the academician of the Academy of Sciences of the Ukraine in 1976, where he served, as a chief research fellow from 1993 to 1999 and from 1999 became the Advisor to the Directorate of the Institute of Mathematics. After defending his candidate dissertation, Vladimir Semyonovich Korolyuk has forever connected his life with the Institute of Mathematics of the Academy of Sciences of the Ukraine, where he first worked as a junior and, since 1956, as a senior researcher. Since 1960, Korolyuk served as a Head of the Department of Probability Theory and Mathematical Statistics, organized by Korolyuk's mentor and teacher B.V. Gnedenko, who has left the Institute of Mathematics in the summer of 1960 to work at Moscow State University, and so Korolyuk has led this department until 1993. In 1966 - 1988, Korolyuk served also as the Deputy Director of the Institute for Scientific Work.

Besides his work at the Institute of Mathematics, Korolyuk was Professor in the Department of Probability Theory and Mathematical Statistics in the Faculty of Mechanics and Mathematics at Kiev State University named after T.G. Shevchenko (1965 - 1993) lecturing and supervising the work of undergraduate and graduate students.

More than 40 of his graduate students defended their Candidate (PhD) theses and 10 received their doctoral degree (equivalent to Full Professor in the US).

Studies and collaboration with Boris V. Gnedenko

Vladimir Semenovich was the first Kiev student of academician Boris Vladimirovich Gnedenko. In September 1949, the Presidium of the Academy of Sciences of the Ukraine decided to transfer B.V. Gnedenko from the city of Lviv to the city of Kiev. Gnedenko was asked to lead the department of probability theory at the Institute of Mathematics of the Academy of Sciences of the Ukraine and to become a Head of the Department of Probability Theory and Algebra in the Faculty of Mechanics and Mathematics at Kiev University. In the spring semester of 1950, Gnedenko organized a special seminar on probability theory and began reading a special course on limit theorems in the theory of probability. At this time, Vladimir Semyovich Korolyuk was a graduate student at Kiev University (in the USSR educational system, studying in his fifth year), and many years later in his memoirs about his mentor Gnedenko, he wrote that the first time (in the fall of 1949) he saw Gnedenko "at a meeting of the academic council of the faculty, delivering a bright, energetic speech, I decided without hesitation to become a student of Boris Vladimirovich Gnedenko." To begin, V.S. Korolyuk became a participant in Gnedenko's special seminar and enrolled in Gnedenko's special course. Boris Vladimirovich Gnedenko offered him to examine the conditions of attraction to stable laws in terms of characteristic functions as a perspective topic for a master thesis. Korolyuk wrote: "Of course, I understood that B.V. could develop this innovative approach on his own. However, he suggested me to work on this problem because he believed in my creative abilities. "Some remarks on the theory of regions of attraction of stable laws" became my first scientific work (co-authored with my teacher B.V. Gnedenko), which was published in the journal **«Reports of the Academy of Sciences of the Ukrainian SSR»** *(«Dopovidi AN URSR» 1950, number 4, 275 - 278)* in 1950.

After graduating from the university with the equivalent of the Master degree in 1950, Korolyuk worked one year at the Artyomovsk Teachers Institute (Artyomovsk - a city in the Donetsk region of Ukraine). Moreover, he tried, whenever possible, to come to Kiev to attend Gnedenko's seminar. At this time, Gnedenko was fascinated by the problem associated with the problem of equipment debugging. This means the following: a debugged machine starts to process certain parts, and for some time these parts are produced without defects, i.e. their sizes do not go beyond the permissible limits. Over time, equipment debugging begins (say, the cutter is blunted) and there is a danger of the parameter of interest to us exceeding the permissible limits (the dimensions of the parts fall into a certain range of values preceding the unacceptable). It is necessary, without stopping the production process itself, to determine whether the parameter of interest falls into such an acceptable interval.

In order to do this, the probability distributions of the two samples (the dimensions of the parts measured at the initial stage of the machine's operation and measured after some time passed) would be compared to each other and before 1951 the solution would typically involve two steps: 1. An assumption is made about the specific distribution of the parameter of interest to us and this hypothesis is checked against the sample obtained at the initial stage of the work; 2. After some time, the second sample is collected and a comparison is made with the first. Gnedenko was interested in the question of whether it is possible to solve this problem using methods of nonparametric statistics, i.e., without testing the hypothesis about a specific type of distribution, thereby collapsing the solution to only one-step procedure.

This task enthralled Korolyuk and already in a year, in 1951, collaborative work with Gnedenko has led to the publication of the article describing the new *trajectories method* for a given number of trials (Gnedenko & Korolyuk. 1951. "About the maximum divergence of the two empirical distributions" published in «**Reports of the Academy of Sciences of the *USSR*»** t. 80, number 4, 525 - 528).

In the fall of 1951, Korolyuk starts PhD program continuing working with his Major Advisor Professor Gnedenko on the same problem. In his memoirs, Korolyuk wrote: "I was carried away by the task proposed by B.V., and as a result, the criteria of Kolmogorov and Smirnov became the subject of my PhD thesis, defended at the Institute of Mathematics in 1954." Throughout his life, Korolyuk have organized many scientific and research conferences including the 2002 International Gnedenko Conference in Kiev, dedicated to the memory of his teacher and mentor, B.V. Gnedenko.

Near the building of Kiev University. From left to right: V.S. Korolyuk, V.S. Mikhalevich, V.S. Rodionova, B.V. Gnedenko, A.V. Skorohod (1953)

In November 1953, the USSR Ministry of Higher Education assigned Gnedenko to go for one year to Berlin, Germany, to restore university education destroyed during the WWII there. Anticipating this trip, Gnedenko asked his mentor, Professor Andrei Nikolaevich Kolmogorov to accept three Gnedenko's graduate students (Korolyuk, Mikhalevich and Skorokhod) at Kolmogorov's Department at Moscow State University. Korolyuk has arrived to Moscow in September, while V.S. Mikhalevich and A.V. Skorokhod joined him there in December.

In May 1954, Korolyuk returned from Moscow to Kiev, where he worked under the leadership of Kolmogorov who soon arrived to Kiev to attend the Korolyuk's dissertation defense, which took place on June 29.

Four friends. Top row, from left to right: Vladimir Sergeevich Mikhalevich, Anatoliy Vladimirovich Skorokhod. Bottom row, from left to right: Anatoly Gordeevich Kostyuchenko, Vladimir Semenovich Korolyuk (Moscow University. Spring 1954).

As Doctor of Philosophy, Vladimir Semyonovich Korolyuk became an employee of the department of probability theory, led by Gnedenko. At the end of 1954, Gnedenko returns from Germany and proposes to open a laboratory for special modeling and computer technology at the Kiev Institute of Mathematics, which would be in turn separated into an independent institution - the Computing Center of the Academy of Sciences of Ukraine. The laboratory was the first in continental Europe to use an electronic computer (MESM), created by the laboratory team under the leadership of its previous Head, Lebedev. Various affiliated organizations were utilizing that computer to perform all kinds of tasks.

Well before his trip to Germany in 1953, Gnedenko and his students, V.S. Korolyuk and V.S. Mikhalevich, were formulating and solving various tasks at MESM. Now, Gnedenko decides to attract his former graduate students to the development of the new, wide profile computer "for solving systems of linear algebraic equations with a number of unknowns exceeding seven hundred, to meet the growing demand from customers in geodetic, construction, physical fields that we came across when working with numerous practical applications. Of my students, V.S. Korolyuk and E.L. Rvacheva were the most active participants [of the project]."

As part of the new computer project, V.S. Korolyuk begins to teach a computer programming course to 5th year students (equivalent to master students in the US system of education), and, together with Gnedenko and Rvacheva-Yushchenko, begins working on a programming textbook, the first chapters of which were ready by the end of 1955, and it was entirely written in 1960, becoming the first in the USSR textbook on programming, "Programming elements". It was released by publishing house Fizmatgiz in 1961 (Moscow. Fizmatgiz. 1961, 3-348), with the second edition following in 1963. In 1964, the textbook was published in Germany ("Elementen der programmirung", Teubner, Leipzig: 1964, 1 - 327), Hungary ("Bevezetes a programozasba." Budapest, 1964, vol.1, 1 - 228 and vol.2, 1 - 204) and, in 1969, in France («Elements de programmation sur ordinateurs». Dunod. Paris. 1969. 1 - 362).

Korolyuk's last joint work with Gnedenko was the preparation of the report on *Asymptotic expansions in probability theory* for the IV Berkeley Symposium in the USA in 1960, in which Skorohod also participated. It was published in the «*Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability*», 1961, vol. II, 153 -170, University of California Press. From Korolyuk's memoirs: "Although my subsequent research activities have been taking place with my students without the direct participation of Gnedenko, I continued to discuss new problems and new results with Gnedenko, always receiving attention and sympathy from him." Inherently, the scientific interests of the teacher and student continued to directly intersect. This happened, for example, when Gnedenko began to study the reliability of duplicated systems with recovery, when both operating time and recovery time are assumed to have arbitrary distributions, and received the first results, Vladimir Semenovich Korolyuk saw that semi-Markov processes could be used to solve such problems. In addition, as Korolyuk wrote afterwards, "so under the influence of B.V. [Gnedenko], a new direction in the theory of reliability has arisen - an asymptotic analysis of semi-Markov processes in the phase enlargement scheme," to which both, the student and the teacher, have contributed significantly.

Recollections of M. Yastrebenetsky

I dealt with the reliability problems of control systems for various technological processes. For building reliability models in some situations, Markov processes were not enough for me. I drew attention to the new then term – semi-Markov and Markov recovery processes. These processes could be models of the reliability for a number of industrial control systems. The first works that I saw on this subject were the works of V.S. Korolyuk.

Shortly thereafter, a series of portraits of prominent Russian mathematicians — Chebyshev, Kovalevskaya, and others, including Markov — was published. I began with the fact that cut the portrait of Markov lengthwise into two parts and hung a half of portrait of Markov in my office, telling everyone visiting me that this is the Semi - Markov.

Vladimir Semenovich Korolyuk attended one of my research presentations in Kiev. After that, he invited me to give a talk on Markov recovery process rarefaction at a seminar in his department at the Institute Mathematics of the Academy of Sciences of Ukraine. Since then, a warm welcome and help from Vladimir Semenovich Korolyuk have always greeted me in my work on semi-Markov processes. I became a frequent visitor at cozy home of Institute of Mathematics, located on the street, then bearing the name of the Russian painter Repin. No wonder he wrote on his book with A.F. Turbin "Mikhail Yastrebenetsky, a victim of semi-Markov processes."

A lot of attention in my doctoral dissertation devoted to reliability of industrial control systems was paid to semi-Markov processes. V.S. Korolyuk, then a corresponding member of the Academy of Sciences of Ukraine, signed a review from the Institute of Mathematics for my dissertation. On the advice of B.V. Gnedenko, opponents in defense of my doctoral dissertation were A.D. Soloviev and I.A. Ushakov (and Prof. A.A.Larin). The defense went flawlessly, but the newly reformed USSR Government Higher Committee on Dissertations (so-called VAK, Higher Attestation Commission) sent me negative reference of their reviewer. He wrote that my work is not original and merely repeats the results of the article of corresponding member of the Academy of Sciences of Ukraine Anatoly Vladimirovich Skorokhod and Professor I. I. Yezhov in the "Journal of Probability Theory and its Application". The parallel between my results and the results of Skorohod and Yezhov was a great honor for me, because I have always considered my mathematical level (that of an engineer) and the level of Skorokhod's math incommensurate. However, I wanted to show that my results were by no means a simple replica of the previous researches To prove my point, I decided that it is best decision is to ask Skorokhod himself to write a letter to the Higher Attestation Commission and confirm that there is nothing in common between our works. A.V. Skorohod was a colleague of Korolyuk at the Institute of Mathematics and, as I learned later, his close friend. Thus, Vladimir Semenovich Korolyuk introduced me to Skorohod and explained the situation. Skorohod and Yezhov have seen my work and immediately wrote a letter to VAK, on the letterhead of the Institute of Mathematics. I still have a copy of this letter: "The process introduced in our article does not contain the model of regenerating semi-Markov process proposed in the dissertation of M. A. Yastrebenetsky as a special case." Further, in this letter, the difference between the article by Skorokhod and Yezhov and my work was described. That letter has become a significant contributor for the final approval of my dissertation. I am forever grateful to Vladimir Semenovich Korolyuk for help in such an important for me situation.

International conference «Mathematic methods in Reliability» (MMR 2009), Organized by Vladimir Rykov in Moscow. From left to right: Michael Yastrebenetsky, Vladimir Semenovich Korolyuk, Dmitry Borisovich Gnedenko.

Korolyuk's interest to the issues of the theory of reliability persisted throughout his lifetime, including his publication of the "Reference Book on Theory of Probability and Mathematical Statistics."

Always being interested in reliability issues, Korolyuk became one of the first presidents of the international association of specialists in reliability *Gnedenko-Forum*, created by Igor Ushakov and Alexander Bochkov and named after B.V. Gnedenko.

Speaking of Korolyuk, it is important to mention his wife - Nina Ivanovna Korolyuk-Andros. I remember her since my childhood in Kharkov– we have been of the same age and her female school was neighboring with my male school, so that our relationship was always friendly. Then it turned out that Nina became the wife of Vladimir Semenovich Korolyuk. Indeed, the world is small! Nina is - a servant of two muses. One of them - music. Nina is a musicologist, professor at the Kiev Conservatory, the author of scientific papers and textbooks in musical education. Her second muse is literature. Nina is the - author of children's books published in Kiev, Moscow, Riga, as well as the winner of a number of prestigious literary prizes. She has played a very important role in Korolyuk's life.

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Estimations of "Smart" Engineering Systems Operation by Probabilistic Measures of Correctness And Reliability

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Abstract

The estimations of critical operation quality deterioration for "Smart" Engineering Systems (SES) are analyzed considering impacts on SES operation correctness and reliability. Basic probabilistic models for the predictive analysis of SES operation are proposed. The effects are illustrated by examples that cover the comparisons of SES, acting as medium-level and skilled operators, the estimation of SES operation without and with considering correctness and reliability, rationale requirements to admissible level for the errors of analysis, predictions of the risks of critical SES operation quality deterioration, sensitivity estimation of predicted risks, the efficiency in practice applications. The novelty is formed from (1) the described methodology for risks of critical operation quality deterioration for SES, considering complexity and uncertainties for SES operation correctness and reliability, (2) the comparable impacts on SES operation quality from failures (reliability) and errors of the 1st and 2nd types (correctness), proved by probabilistic modeling, (3) the achievable levels about 0.999-0.9997 and more for probability of SES operation correctness and reliability and about 0.01 – 0.001 and less for probability of critical SES operation quality deterioration during 1 year, which characterizes risk for identical consequences.

Keywords: analysis, model, *estimation*, operation, prediction, probability, quality, system

I. Introduction

Operation of modern ad future SES in various areas of their applications is accompanied by a set of uncertainties. For general case the term "smart" is a mnemonic acronym, giving criteria of guiding on the set objectives. SES are defined as systems that incorporate functions of data sensing, information processing and transfer, monitoring of conditions, actuation and control of devices and other systems, improving their operation quality and/or safety. According to system engineering standards (see, for example, International standards ISO/IEC/IEEE 15288, ISO/IEC 16085, IEC 61508, etc.) the analysis of uncertainties should be carried out at all stages of system life cycle. Uncertainties are principal factors of risks. Today practice has shown: in the most cases from all properties of SES quality the main attention is paid to reliability of various systems. For an estimation of reliability there are created and used sets of standards, mathematical and methodical approaches [1-12] which can be applied for SES. At the same time, SES are differed from simple engineering systems that they are capable to carry out intellectual information processing based on formal modeling. Necessity of considering SES possibilities of monitoring parameters and making control actions, and also increasing SES complexity prove an actuality of models development for risks predictions. The scientific probabilistic approaches for an estimation of SES information correctness are at the beginning stage yet, some approaches – see for example [1-2, 5-12 etc.].

The novelty of the manuscript is formed from (1) the described methodology for risks of critical operation quality deterioration for SES, considering complexity and uncertainties for SES operation correctness and reliability, (2) the comparable impacts on SES operation quality from failures (reliability) and errors of the 1st and 2nd types (correctness), proved by probabilistic modeling, (3) the achievable levels for probability of SES operation correctness and reliability for probability of critical SES operation quality deterioration which characterizes risk for identical consequences. Considering an importance of researches for all-round maintenance and improvement of SES operation quality here are proposed: the basic ideas for the risks prediction of critical operation quality deterioration and for the further uses of predictions; the improved models to estimate information correctness and operation reliability (for "black box»); the consecutive algorithm to predict risks, considering SES operation correctness and reliability (for "black box»); the integrated model to predict the risks of critical operation quality deterioration for SES, composed as complex structures; the statements of optimization problems in SES life cycle; the practical examples with interpretation of probabilistic modeling SES operation quality.

Note. The used definitions are: Risk – 1) effect of uncertainty on objectives (ISO Guide 73); 2) the combination of the probability of an event and its consequence (ISO/IEC 16085, IEC 61508 etc.); Reliability of SES operation - the property of SES to perform its required functions of data sensing, information processing, transfer, monitoring of conditions and actuation of devices and other systems under stated conditions for a specified period of time; Correctness of SES operation – the property of SES to provide the right results or the coordinated effects of information processing and control.

II. Basic Ideas

The next ideas are intended to explain the approaches for the risks prediction and the further uses of predictions.

The idea 1 is: to focus on operation correctness and reliability as the main critical properties characterizing SES quality in use for conditions of uncertainties.

Note. SES quality in use also may be characterized additionally by other properties, for example – by the timeliness of information producing, the completeness, actuality, faultlessness, confidentiality of information etc. [1, 2, 12].

The idea 2 is: to improve created earlier probabilistic models for the correctness analysis and reliability prediction [1-2, 8-11] at mathematical SES description by a "black box»; to create the consecutive algorithm for prediction the risks of critical SES operation quality deterioration by the consecutive input of the results of the correctness modeling into the improved model for reliability prediction.

The idea 3 is: to create integrated probabilistic model for prediction the risks of critical operation quality deterioration of SES, composed as complex in the form of parallel-serial structures.

The idea 4 is: to use the risks of critical SES operation quality deterioration for rationale preventing measures by solving the problems of optimization in SES life cycle.

The idea 5 is: on the basis of application of the proposed models to demonstrate results of ideas 1-4 implementation by the examples devoted to:

-a comparison of possibilities of SES, acting as operators of medium-level and skilled operators;

- an estimation of SES operation reliability (without considering correctness of information processing) and SES operation considering correctness and reliability;

- rationale requirements to admissible level for the errors of analysis (of 1st and 2nd types) during SES information processing;
- predictions of the risks of critical SES operation quality deterioration during 1 month and 1 year;
- sensitivity estimation of predicted risks in dependence on changing initial data of probabilistic modeling of SES operation in diapason -50%+100%;
- reference on efficiency in practice applications.

III. The Proposed Probabilistic Models as "Black Box" for probabilistic estimations

I. The Improved Model of Items Content Analysis

Problems of item (text, forms, events) content analysis are inherent to SES operation. It may be identification of items, recognition of images, manuscripts, speech or signals, nondestructive control, hardware or software testing, information analysis for making decision etc. In any case there exist latent or suspicious objects for revealing and their following analysis by SES. Item content analysis depends on SES specificity and methods of content analysis. In general case methods of content analysis and SES decision-making may contain elements of guessing. Nonetheless, any analysis is based on logical positions. Logic implies argumentation based on essential items use. The way of logical approach is an algorithm of given items analysis by SES. Such algorithm is implemented by an operator (operator may be a SES-device or a SES, combined by a man, or their combination). See the core of formalization on Figure 1 (for information checking). The formalization core of item analysis is illustrated by Figure 1.

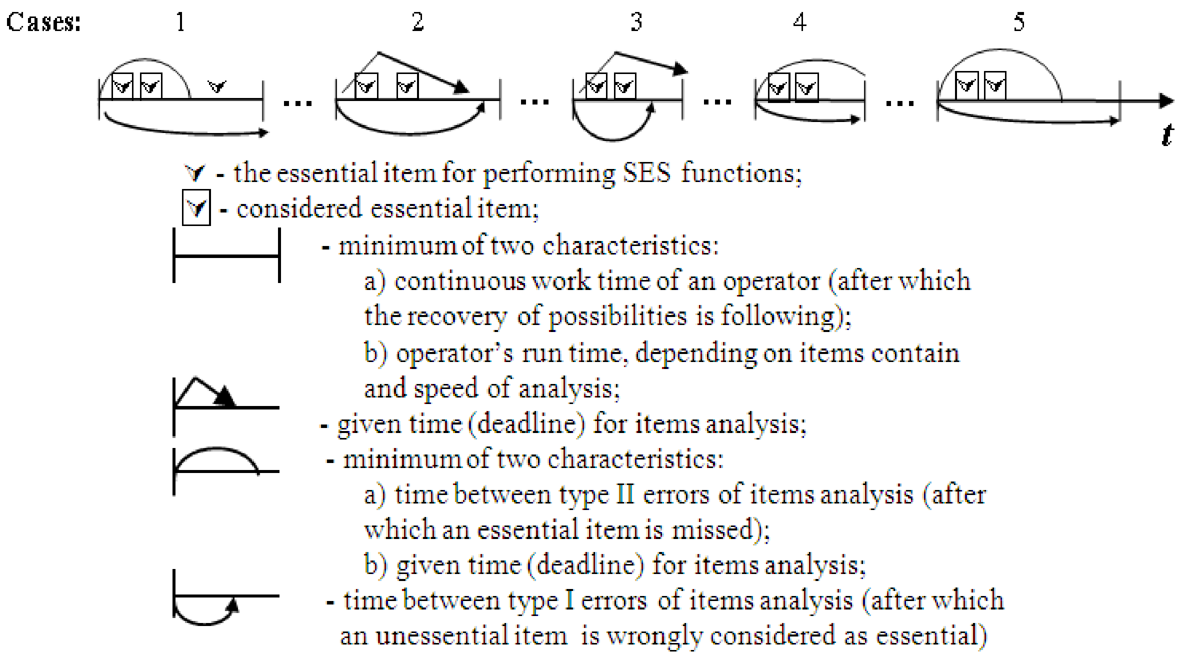

Figure 1: *Some possible cases in items analysis*

The case 1 illustrates error of the 2nd type (essential item is missed), the case 2 illustrates error of the 1st type after deadline, the case 3 illustrates error of the 1st type before deadline (unessential item is wrongly considered as essential), the cases 4 and 5 characterize correctness of items analysis during SES operation.

Because of uncertainties now and in future it may not always possible to distinguish a formal bound between essential and unessential items for performing some SES functions. A problem of a balance between input items content and quality of their analysis by SES within the given time should be solved. If input items content is big the number of errors increases what may cause a control action time waste. It is necessary to optimize the items content and to develop more rational technologies of SES operation.

Definition. SES operation is considered as correct during given time if all essential items are analyzed rightly and no an unessential item is wrongly considered as essential. Because of uncertainties SES operation correctness may be estimated by probabilistic modeling.

There are possible the four variants of correlations between the input characteristics for modeling.

Variant 1. The given time for items content analysis is no less than the real analysis time ($T_{real} \le T_{req.}$) and the content of analyzed items is such small that it is required only one continuous operator's work period ($T_{real} \le T_{cont.}$).

The next Statements 1-4 are used.

Statement 1. Under the condition of independence for considered input characteristics the probability of SES operation correctness during the given time is equal to:

$$P_{after(1)}(V, \mu, \nu, \eta, T_{MTBF}, T_{cont.}, T_{req.}) = [1 - N(V/\nu)]\left\{\int_0^{V/\nu} dA(t)[1 - M(V/\nu - t)] + \int_{V/\nu}^{\infty} dA(t)\right\}, \quad (1)$$

where V is an analyzed content of items;

μ is a relative fraction of essential items in items content, dimensionless ($\mu \cdot 100\%$ characterizes a relative fraction of essential items in percentage);

ν is a speed of items analysis;

N(t) is a probability distribution function (PDF) of time between errors of 1st type (when unessential item is wrongly considered as essential), η^{-1} is mean time, $N(t) = 1 - exp(-t \cdot \eta)$;

$M(t)$ is a PDF of time between real neighboring essential items in analyzed content for the given speed of items analysis and relative fraction of essential items in items content, $M(t) = 1 - exp(-t \cdot \mu \cdot \nu)$;

$A(t)$ is a PDF of time between errors of 2nd type (when essential item is missed), T_{MTBF} is mean time;

$T_{cont.}$ is a time of continuous operator's work;

$T_{req.}$ is a given time for items content analysis.

V, ν, $T_{cont.}$ and $T_{req.}$ are assigned as deterministic values.

The probability of operation correctness without items content analysis is $P_{no}(V) = e^{-\mu V}$. If μ=0%, $P_{no}(V) = 1$ (no comments for no essential items).

The final clear analytical formula for modeling is received by Lebesque-integration of expression (1).

Variant 2. The given time for items content analysis is no less than the real analysis time (i.e. $T_{real} \le T_{req.}$). But the content of analyzed items is comparatively large, i.e. $T_{real} > T_{cont.}$.

Statement 2. Under the condition of independence for considered characteristics the probability of SES operation correctness during the given time is equal to:

$$P_{after(2)} = \{P_{after(1)}(V_{part(2)}, \mu, \nu, \eta, T_{MTBF}, T_{cont.}, \tau_{part(2)})\}^L, \quad (2)$$

where $L=V/(\nu T_{cont.})$, $V_{part(2)}=V/L$, $\tau_{part(2)} = T_{req.}/L$.

Variant 3. The given time for items content analysis is less than the real analysis time ($T_{real}>T_{req.}$) and the content of analyzed items is such small that it is required only one continuous operator's work period ($T_{real} \le T_{cont.}$).

Statement 3. Under the condition of independence for considered characteristics the probability of SES operation correctness during the given time is equal to:

$$P_{after(3)} = (V_{part(3)}/V) \cdot P_{after(1)}\ (V_{part(3)}\ ,\mu,\ \nu,\ \eta,\ T_{MTBF},\ T_{cont.}, T_{req.}) + +[(V-V_{part(3)})/V] \cdot P_{without}, \qquad (3)$$

where $V_{part(3)} = \nu T_{req.}.$, $P_{without} = e^{-\mu (V-V_{part(3)})}$.

Variant 4. A given time for items content analysis is no less than the real analysis time (i.e. $T_{real}>T_{req.}$). But the content of analyzed items is comparatively large, i.e. $T_{real}>T_{cont.}$.

Statement 4. Under the condition of independence for considered characteristics the probability of SES operation correctness during the given time is equal to:

$$P_{after} = \begin{cases} [V_{part(4)}/V] \cdot {}_{after(1)}\ (V_{part(4)}\ ,\ \mu,\ \nu,\ \eta,\ T_{MTBF},\ T_{cont.},\ T_{req.}) + \\ +\ [(V-V_{part(4)})/V] \cdot e^{-\mu (V-V\,part(4))},\ \text{if } T_{req.} \le T_{cont.}; \\ [V_{part(4)}/V] \cdot \{P_{after(1)}\ (V_{part(4.2)}\ ,\ \mu,\ \nu,\ \eta,\ T_{MTBF},\ T_{cont.},\ \tau_{part(4.2)})\}^N + \\ +[(V-V_{part(4)})/V] \cdot e^{-\mu (V-V\,part(4))},\ \text{if } T_{req.} > T_{cont.}, \end{cases} \qquad (4)$$

where $V_{part(4)} = \nu T_{req.}$, $V_{part(4.2)}=V_{part(4)}/L$, $\tau_{part(4.2)}=\tau/L$, $L= V_{part(4)}/(\nu T_{cont.})$.

The fraction of no used essential items after analysis equals to $\mu_{after(1)}= \mu\cdot(1-P_{after(1)})$.

The final clear analytical formulas for modeling are received by integration (1) and using (2)-(4). The proofs of Statements are similar to the proofs for the probabilistic modeling [1, 3, 5].

II. The Improved Model for Operation Reliability Prediction

Nowadays in development and utilization an essential part of funds is spent on providing SES operation reliability. Various dangerous impacts on system integrity (these may be failures, defects events, "human factors" events etc.) deteriorate SES operation reliability. The improved model for operation reliability prediction especially uses the elementary events "correct operation" and alternatively "a loss of integrity" to link this model with the previous model and with the following consecutive algorithm for prediction the risks of critical SES operation quality deterioration.

There are examined two general technologies of providing SES operation reliability: periodical diagnostics of system integrity (technology 1, without monitoring between diagnostics) and additionally monitoring between diagnostics (technology 2).

Both Technologies 1 and 2 are focused on devices and systems, for which the SES functions are performed, including monitoring of conditions, actuation and control. Let a set of operating SES and these devices and systems, for which the SES functions are performed, is named a system and described by "black box".

Technology 1 is based on periodical diagnostics of such system integrity. Technology 1 is carried out to detect penetrated sources of potential unreliability or consequences of negative impacts (see Figure 2). The lost system integrity can be detected only as a result of diagnostics, after which recovery of system reliability is started. Dangerous impact on system is acted step-by step: at first a danger source penetrates and then after its activation begins to impact on reliability. System integrity can't be lost before a penetrated danger source is activated. A danger is considered to be realized only after a source of potential unreliability has impacted on a system.

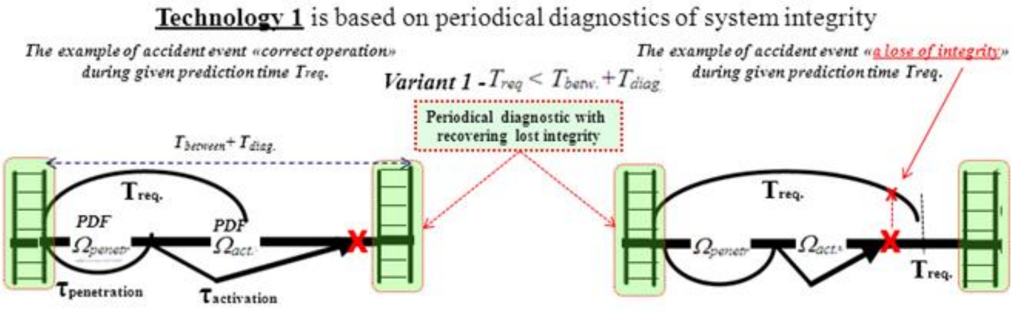

Figure 2: *Some explanation for probabilistic modeling Technology 1*

Technology 2, unlike the previous one, implies that an operator performs the functions of monitoring system integrity between diagnostics (operator may be a SES-device or a SES, combined by a man, or their combination). In case of detecting a danger source of potential unreliability an operator recovers system integrity. The ways of integrity recovering are analogous to the ways of technology 1. Correct operator's actions provide a neutralization of a source of potential unreliability. A penetration of a danger source of potential unreliability is possible only if an operator makes an error but an impact on reliability (with the lost integrity) occurs if the source is activated before the next diagnostic. Otherwise the source will be detected and neutralized during the next diagnostic (see Figure 3).

It is supposed for technologies 1 and 2 that the used SES diagnostic possibilities allow to provide required system integrity recovery after revealing penetrated sources or consequences of impacts. Assumption: for all time input characteristic the PDF exist. Thus the probability of correct system operation within the given prognostic period (i.e. probability of success) may be estimated as a result of use the next models. Risk to lose integrity is an addition to 1 for probability of correct system operation ("probability of success"). R=1-P.

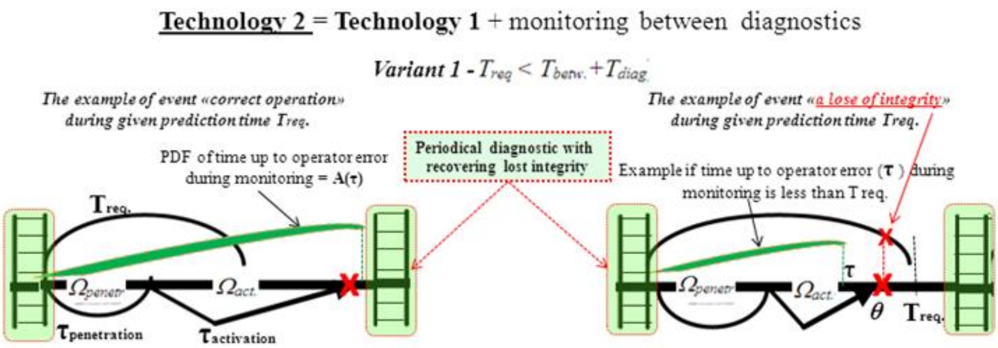

Figure 3: *Some explanation for probabilistic modeling Technology 2*

There are possible the next variants for technology 1 and 2: variant 1 – the given prognostic period T_{req} is less than established period between neighboring diagnostics ($T_{req} < T_{betw.}+T_{diag}$); variant 2 – the assigned period T_{req} is more than or equals to established period between neighboring diagnostics ($T_{req} \geq T_{betw.}+T_{diag}$). Here $T_{betw.}$ – is the time between the end of diagnostic and the beginning of the next diagnostic, T_{diag} – is the diagnostic time.

For the given period for prediction ($T_{req.}$) the next statements 5-8 are used [1, 3, 5, 11].

<u>Statement 5 (for technology 1).</u> Under the condition of independence of considered characteristics the probability of reliable system operation for variant 1 is equal to

$$P_{(1)}(T_{req}) = 1 - \Omega_{penetr} * \Omega_{activ}(T_{req}), \tag{5}$$

where $\Omega_{penetr}(t)$ – is the PDF of time between neighboring impacts for penetrating a danger source; $\Omega_{activ}(t)$ – is the PDF of activation time of a source of potential unreliability.

Statement 6 (for technology 1). Under the condition of independence for considered characteristics the probability of reliable system operation for variant 2 may be equal to:

measure a)

$$P_{(2)}(T_{req})=N((T_{betw}+T_{diag})/T_{req})P_{(1)}^N(T_{betw}+T_{diag})+(T_{rmn}/T_{req})\ P_{(1)}(T_{rmn}), \tag{6}$$

where $N=[\ T_{req.}/(T_{betw.}+\ T_{diag.})]$ – is the integer part,
$T_{rmn} = T_{req} - N(T_{betw} + T_{diag})$;

measure b)

$$P_{(2)}(T_{req})=P_{(1)}^N(T_{betw}+T_{diag})P_{(1)}(T_{rmn}). \tag{7}$$

The probability of success within the given time $P_{(1)}(T_{given})$ is defined by (5).

Statement 7 (for Technology 2). Under the condition of independence for considered characteristics the probability of reliable system operation for variant 1 is equal to

$$P_{(1)}(T_{req}) = 1 - \int_0^{T_{req.}} dA(\tau) \int_{\tau}^{T_{req.}} d\,\Omega_{penetr} * .\Omega_{act..}(\theta) \tag{8}$$

Here A(t) is the PDF of time from the last finish of diagnostic time up to the first operator error (is similar to PDF A(t) from subsection 3.1). T_{MTBF} is the mean time between errors.

Statement 8 (for Technology 2). Under the condition of independence of considered characteristics the probability of reliable system operation for variant 2 may be equal to:

measure a)

$$P_{(2)}(T_{req}) = N((T_{betw} + T_{diag})/T_{req})\ P_{(1)}^N(T_{betw} + T_{diag}) + (T_{rmn}/T_{req})\ P_{(1)}(T_{rmn}), \tag{9}$$

measure b)

$$P_{(2)}(T_{req}) = P_{(1)}^N(T_{betw} + T_{diag})\ P_{(1)}(T_{rmn}),\ \ \text{(see (7))},$$

where the probability of success within the given time $P_{(1)}(T_{req})$ is defined by (8).

The final clear analytical formulas for modelling are received by Lebesque-integration of expression (8) with due regard to Statements 5-8. The models are supported by software tools [1-2, 5, 11].

Comments: the measure a) allows to perform latent knowledge mining in the possibilities and impacts of every control because N is integer part. The measure b) allows to mine latent knowledge by average value of probability on the level of classical PDF.

III. The Algorithm of Prediction the Risks of Critical SES Operation Quality Deterioration

The critical deterioration of SES operation quality means such deterioration of operation reliability or correctness when the performance of destined functions (of data sensing, information processing or transfer, monitoring of conditions, actuation or control of devices or other systems) doesn't meet the defined requirements during given time.

Considering uncertainties the requirements to operation quality in SES life cycle (on stages of consept, development, production, utilization or support) may be defined in terms of admissible risks, for example: "the quality of SES operation is estimated as "acceptable" if the risk of operation reliability or correctness deterioration during given time of expected hard scenarios is less than established admissible risk".

The proposed consecutive algorithm for prediction the risks of SES operation quality deterioration is the next.

1. The acceptable level for the probability of reliable system operation during given time is defined. It should be achievable level considering system goals, possible conditions and dangers of uncertainties, resources, time and possible damages from unreliability. For estimations the improved model for operation reliability prediction may be used (from subsection 3.2). The used value T_{MTBF} is related with mean time between failures for SES.

Note. Here a set of operating SES and devices and systems, for which the SES functions are performed, is named a system.

2. The acceptable level for the probability of SES operation correctness during the given time is defined. It should be the level, possible near to acceptable level for the reliability, achievable considering SES goals, possible conditions of uncertainties, resources, time and possible damages from incorrectness. For estimations the improved model of items content analysis may be used (from subsection 3.1).

3. The maximal mean time between errors of the 1st type for which the calculated probability of SES operation correctness during the given time is equal to acceptable level, is defined for the given another input characteristics. For calculations the improved model of items content analysis may be used (from subsection 3.1).

4. The maximal mean time between errors of the 2nd type for which the calculated probability of SES operation correctness during the given time is equal to acceptable level, is defined for the given another input characteristics. For calculations the improved model of items content analysis may be used (from subsection 3.1).

5. Among the values of the mean time between errors of the 1st type and the mean time between errors of the 2nd type the minimum (min) and maximum (max) are defined.

6. Considering the specificity of SES, mean time between errors is defined (it is the weighed value inside of diapason of (min, max), considering frequency of errors of the 1st and 2nd types). Also in special case this may be or the mean time between errors of the 1st type or the mean time between errors of the 2nd type. This value (which characterizes SES operation correctness) is used instead of T_{MTBF} in modeling by using the improved model for operation reliability prediction may be used (from subsection 3.2). But now there are calculations in terms of risks. And results of modeling characterize the predicted risks of critical operation quality deterioration considering reliability and correctness for SES that are described by "black box".

IV. The Integrated Model to Predict Risks for SES, Composed as Complex Structures

For a complex system with parallel or serial structure existing models can be developed by usual methods of probability theory. Let's consider the elementary structure from two independent parallel or series elements.

Let's PDF of time between neighboring losses of i-th element integrity is $B_i(t) = P(\tau_i \le t)$, then:

1) time between losses of integrity for system combined from series connected independent elements is equal to a minimum from two times τ_i: failure of 1st or 2nd elements (i.e. the system goes into a state of lost integrity when either 1st, or 2nd element integrity will be lost). For this case the

PDF of time between the losses of system integrity is defined by expression

$$B(t) = P(\min(\tau_1,\tau_2)\le t)=1-P(\min(\tau_1,\tau_2)>t) = 1-P(\tau_1>t)P(\tau_2> t)= 1 - [1-B_1(t)]\,[1- B_2(t)], \quad (10)$$

2) time between losses of integrity for system combined from parallel connected independent elements (hot reservation) is equal to a maximum from two times τ_i: failure of 1st or 2nd elements (i.e. the system goes into a state of lost integrity when both 1st and 2nd element integrity will be lost). For this case the PDF of time between the losses of system integrity is defined by expression

$$B(t)=P(\max(\tau_1,\tau_2)\le t)=P(\tau_1 \le t)P(\tau_2 \le t)=B_1(t)B_2(t). \quad (11)$$

Note. The same approach is studied also by Prof. E.Ventcel (Russia) in 80th who has formulated the trying tasks for students.

Thus an adequacy of probabilistic models is reached by the consideration of real processes of control, monitoring, element recovery for complex structure. Applying recurrently expressions (10) – (11), it is possible to create PDF of time between the losses of integrity for any complex system with parallel and/or series structure.

The known kind of the more adequate PDF allows to define accordingly mean time between neighboring losses of system integrity $T_{exp.}$ (may be calculated from this PDF as mathematical expectation), and a frequency λ of system integrity losses, $\lambda=1/T_{exp.}$.

Risk to lose integrity (safety, quality or separate property, for example – reliability) is an addition to 1 for probability of providing system integrity (correct system operation or "probability of success") R=1-P. The formulas for probabilistic modeling technologies 1 and 2 and the proofs of them are proposed in [1-2, 5]

All these ideas are implemented by the software technologies of risk prediction for complex systems, for example, the "Mathematical modeling of system life cycle processes" – "know how" (registered by Rospatent №2004610858), "Complex for evaluating quality of production processes" (registered by Rospatent №2010614145) [1-2, 5 - 11].

V. Some Statements for System Optimization

Here a set of operating SES and devices and systems, for which the SES functions are performed, is named also a system.

The results of modeling processes can and should be used for optimization of systems operation on the base of risk prediction. For example, there are applicable the next general formal statements of problems for system optimization [1-2, 10-11]:

1) on the stages of system concept, development, production and support:

system parameters, software, technical and management measures (Q) are the most rational for the given period if on them the minimum of expenses ($Z_{dev.}$) for creation of system is reached:

$$Z_{dev.}(Q_{rational}) = \min_{Q} Z_{dev.}(Q),$$

at limitations on probability of an admissible level of risks $R(Q) \le R_{adm.}$ and/or admissible level of quality $P_{quality}(Q) \ge P_{adm.}$ and expenses for operation $C_{oper.}(Q) \le C_{adm.}$ and under other development, operation or maintenance conditions;

2) on operation stage:

system parameters, software, technical and management measures (Q) are the most rational for the given period of operation if on them the minimum of risks is reached:

$$R\,(Q_{\text{rational}}) = \min_{Q} R\,(Q),$$

at limitations on probability of an admissible level of quality $P_{quality}(Q) \geq P_{adm.}$ and expenses for operation $C_{oper.}(Q) \leq C_{adm.}$ and under other operation or maintenance conditions.

Of course these statements may be transformed into problems of minimization of expenses or mathematical expectation of damages in different limitations. System parameters, software, technical and management measures (Q) is a rule a vector of input – see sections 2 and 3. There may be combination of these formal statements in system life cycle.

VI. Examples

The examples cover:

- comparison of possibilities of SES, acting as operators of medium-level (Examples 1 and 2) and skilled operators (Example 3);
- estimation of SES operation reliability without considering correctness of information processing (Example 4) and SES operation considering correctness and reliability (Example 5);
- rationale requirements to admissible level for the errors of analysis (of 1st and 2nd types) during SES information processing and predictions of the risks of critical SES operation quality deterioration during 1 month and 1 year (Examples 6 and 7);
- sensitivity estimation of predicted risks in dependence on changing initial data of probabilistic modeling of SES operation in diapason -50%+100% (Examples 1-7);
- reference on the efficiency in practice applications (Examples 8 and 9).

Example 1 (estimation of items analysis correctness when functions of SES are performed by a man-operator).

Let's consider at first a situation when the items are analyzed by a man-operator of medium-level qualification. Similar systems are peculiar to many operating systems. Thus owing to purely human restricted possibilities operator covers only that volume of the items content which is capable to process for the given time. Let this analyzed content includes 20 items (in conditional units). Speed of the analysis is 20 items in a minute. Frequency of the 1st type errors (when unessential item is wrongly considered as essential) is equal to 1 error in a day. Mean time between errors of the 2nd type (when essential item is missed) is equal to 1 month. Operators replace each other through 2 hours for keeping attention concentration (it is a time of continuous operator's work). The given time for items content analysis is equal to 1 minute. It is required to estimate a correctness of items content analysis.

Solution. The analysis of modeling results by the improved model of items content analysis (see subsection 3.1) has shown the following. The probability of a man operation correctness will make 0.9993, that means absolutely correct processing with reference to small content volume 20 items. At increase items content volume twice the probability of operation correctness decreases to 0.5 – see Figure 4a). It is connected by that at the defined speed of analysis (20 items in a minute) for 1 minute half from 40 items will be analyzed only. The same effect is observed at researches of dependence of probability of operation correctness on speed of the analysis – see Figure 4b).

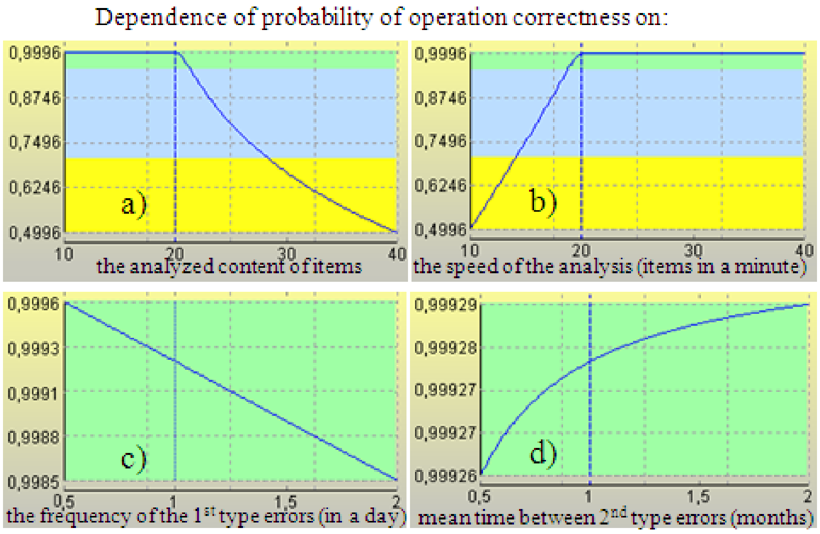

Figure 4. Dependence of probability of operation correctness on:
a) the frequency of the 1st type errors (errors in a day); b) the speed of the analysis (items in a minute);
c) the frequency of the 1st type errors (errors in a day); d) the mean time between errors of the 2nd type (months)

At the same time, change in a diapason -50 % + 100 % of frequency of the 1st type errors (Figure 4c)) and mean time between errors of the 2nd type (Figure 4d)) insignificantly impact probability of operation correctness.

The analysis of Example 1 has shown, that without SES use for a man-operator of medium-level qualification the content volume and speed of the analysis are critical only. The small volume analyzed items content on a man in practice is caused by the limited human possibilities. Thus big volumes of data which are expedient for considering at decision-making, are missed from consideration.

Example 2 (estimation of SES operation correctness).

For real SES the volume of the analyzed items content increases in hundred times. Let SES under the characteristics of a correctness of actions is similar to a man-operator of medium-level qualification (for example, it is neurosystem, trained on level of the specialist of medium-level qualification). Unlike conditions of the Example 1 the content in volume of 2000 items is analyzed at speed of the analysis about 2000 items in a minute (i.e. SES is 100 times more productivity). Unlike man-operator SES does not requires to keep attention concentration (it performs automatically). Therefore for SES the period of continuous work is defined at 2000 hours, that are equivalent approximately to 3 months and is comparable with the period between technical maintenance regulations at the enterprises. The given time for items content analysis is equal to 1 minute still. It is required to estimate operation correctness of items content analysis for such SES.

Solution**.** The probability of correct SES operation will make the same 0.9993. At increase items content volume twice the probability of operation correctness decreases to 0.5 also, as well as in the Example 1 (Figure 5a)). The results of the calculated dependence of operation correctness probability on the speed of the analysis are similar also (Figure 5b)).

At the same time, change in a diapason -50 % + 100 % of frequency of the 1st type errors (Figure 5c)) and mean time between errors of the 2nd type (Figure 5d)) also insignificantly impacts probability of operation correctness.

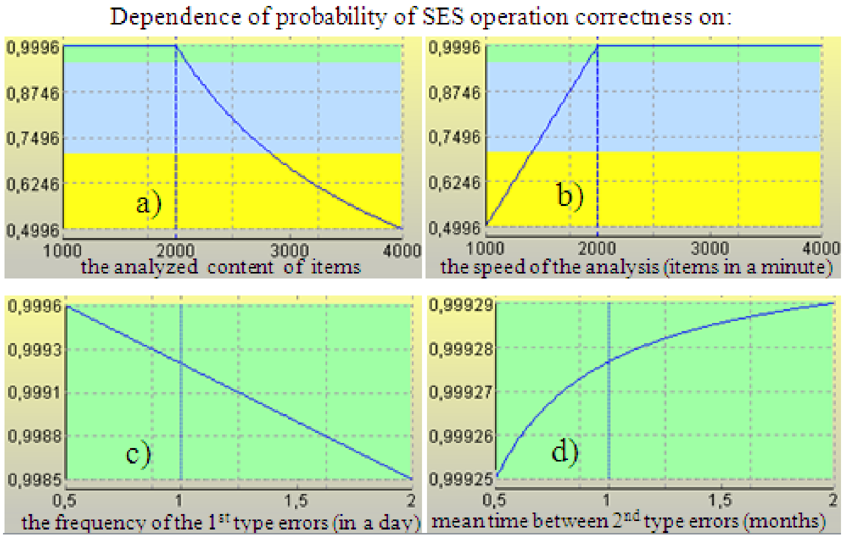

Figure 5. Dependence of probability of SES operation correctness on:
a) the analyzed content of items; b) the speed of the analysis (items in a minute);
c) the frequency of the 1st type errors (errors in a day); d) the mean time between errors of the 2nd type (months)

However, at all similarity of dependences on Figures 4 and 5 conclusions by the Example 2 are essentially others. Preliminary explanatories are the following – multiplying the volume 2000 items on probability of correct SES operation 0.9993, we receive, that in average there are correctly analyzed 1998.6 items. It means, that for 1.4 analyzed items (in average) a correctness is not provided! For example, incorrect interpretation in real time of one-two factors of threats in dangerous manufacture can lead for few minutes to occurrence of an explosive situation. If to provide quality and efficiency of SES application on a scientific basis, it is necessary to raise the correctness of SES operation for volume 2000 items to the level about 0.9998 so that the average quantity of items for which the correctness is not provided, was less than 0.5 (2000·0.9998=1999.6, i.e. only 0.4 items will be analyzed incorrectly).

Example 3 (estimation of operation correctness for SES with the improved characteristics).

Let SES is similar to the specialist of a high skill level (for example, well trained neurosystem). It is expressed that unlike conditions of the Example 2 the frequency of 1st type errors decreases twice, i.e. makes 0.5 errors a day. And mean time before error of 2nd type let will make 6 months (i.e. in 6 times more). Creation of such SES will demand increase in expenses essentially. Other conditions are the same, that in the Example 2. It is required to estimate changes in estimations concerning results of the Example 2 and to optimize requirements to such SES characteristics for admissible level of operation correctness.

Solution. For solving the proposed methods are applied (see section 5).The probability of correct SES operation will make the same 0.9993. At changes volume of the analyzed items and speed of analysis the dependencies of the probabilities of correct SES operation are similar resulted on Figure 5 a) and 5b).

Input changes in diapason -50 % + 100% led to following effects - see Figure 6a), b):

decrease in frequency of 1st type errors on the average more often 1 errors for three days provide probability of correct SES operation more low 0.9998;

keeping mean time between errors of the 2nd type up to 12 months can't increase the probability of correct SES operation more than 0.9997.

Researches have shown, that any efforts for increasing mean time between errors of the 2nd type in 2 times, demanding in practice essential expenses, will not lead to a desirable SES operation correctness (more 0.9998). Therefore optimization by criterion «quality - cost» the choice of requirements "mean time between errors of the 2nd type should not be less than 6 months" is made, reaching the effects at the expense of decrease in frequency of 1st type errors.

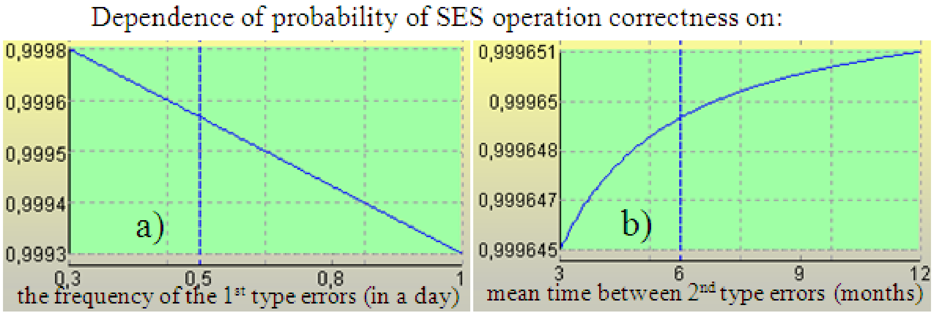

Figure 6. Dependence of probability of SES operation correctness on:
a) the frequency of the 1st type errors (errors in a day); b) the mean time between errors of the 2nd type (months)

Scientifically proved requirements by results of example 3 look as follows. For providing admissible high level of operation correctness (0.9998 and more) frequency of 1st type of errors should not exceed 1 error for three days, and mean time between errors of the 2nd type should not be less than 6 months. Let's remember this result.

Further we pass to application of the improved model for operation reliability prediction and the consecutive algorithm for prediction the risks of critical SES operation quality deterioration (subsections 3.2 and 3.3).

Example 4 (estimation of SES operation reliability without considering operation correctness).

Let's frequency of occurrence of the impacts for penetrating a danger source is equal to 300 times a year, and the mean activation time of a source of potential unreliability = 24 hours. The time between diagnostics of integrity traced by SES is equal to 1 hour. Duration of diagnostics and, if needs, a restoration of the lost integrity is equal in average to 20 minutes. The mean time before failures for SES is estimated about 2 years. It is required to predict SES reliability within prognostic period from 1 till 2 years.

Solution. The modeling results have shown - at change of input in a diapason -50 % + 100% the probability of reliable SES operation within 1 year is from 0.9993 to 0.9999. And even at increase prognostic period till 2th years the required probability for initial input data does not fall less 0.9995. Such usual estimation picture without considering correctness essentially varies if to consider not only reliability, but also SES operation correctness.

Example 5 (consideration of items analysis correctness and operation reliability when functions of SES are performed by a man-operator).

We take for a basis from the Example 4 initial given frequency of occurrence of the impacts for penetrating a danger source (300 times a year) and duration of diagnostics (20 minutes), and from the Example 1 - time between integrity diagnostics (8 hours instead of 1 hour from the Example 4), and the mean time before operator errors at monitoring is equal to 1 days (instead of 2th years from the Example 4). It is required to predict for 1 month and 1 year risks of critical operation quality deterioration taking into account impact of uncertainties on correctness and reliability (consequences in practice may be expressed in occurrence of emergencies, harm to health, damages) and estimate sensitivity of predicted risks.

Solution. The risks of critical operation quality deterioration during 1 month will make about 0.33 (considered consequences in practice may be expressed in occurrence of emergencies, harm to health, damages). The modeling has shown - at change of input in a diapason -50 % + 100% the risk during 1 month is from 0.11 to 0.70 (see Figure 7, demonstrating also sensitivity of predicted risks). A virtual interpretation of these figures may be the following: if to compare a lot of months of a similar mode of operation (for example, 100 months) in 17 % to 70 % of these months emergencies because of operator critical errors in the analyzed information will take place.

It is expected the risk to lose integrity increases in depending on increasing time between integrity diagnostics, duration of diagnostics, prognostic period (the logic explanation see [2,5].

Results for prognostic period 1 year (the risk is near 0.99) confirm inevitability of emergencies for this long period.

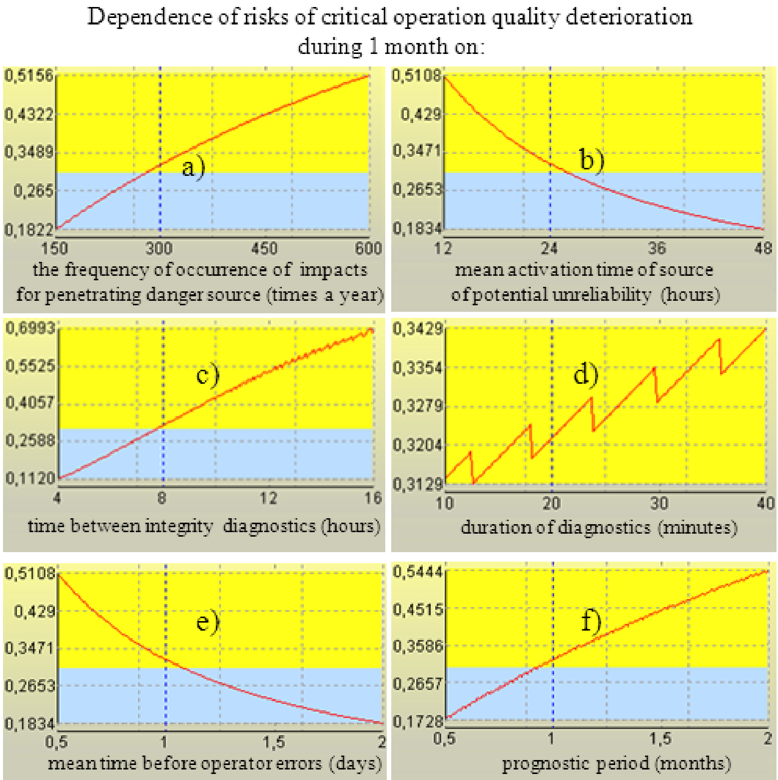

Figure 7 Dependence of risks of critical operation quality deterioration during 1 month on: a) the frequency of occurrence of impacts for penetrating danger source (times a year); b) mean activation time of source of potential unreliability (hours); c) time between integrity diagnostics (hours); d) duration of diagnostics (minutes); e) mean time before operator errors (days); f) prognostic period (months)

<u>Example 6</u> (the primary consideration of 1st type of errors during SES operation, when unessential item is wrongly considered as essential).

The applications area for such SES is characterized by a weak degree of scrutiny, for example, there may be prospecting works on shelves of Arctic regions or space research, innovative researches. Critical deterioration of SES operation in practice can be expressed in wrong primary conclusions, wasted expenses, unforeseen emergencies and damages etc. For similar SES the errors of 1st type are much more often, rather than errors of 2nd type. To form input we use the results of researches of example 3 (for providing SES operation correctness - a frequency errors of 1st type should not exceed one error for three days) are used. Other input - from the Example 5. The differences are only: time between integrity diagnostics = 1 hour, and the mean time before operator errors at monitoring = 3 days. It is required to predict for 1 month and 1 year risks of critical operation quality deterioration taking into account impact of uncertainties on correctness and reliability.

<u>Solution</u>**.** The risks of SES critical operation quality deterioration during 1 month will make about 0.004 and during 1 year - about 0.05 (considering consequences).

The modeling results have shown - at change of input in a diapason -50 % + 100% the risk during 1 month is in diapason from 0.0016 to 0.0124 and during 1 year - from 0.002 to 0.139 (see Fig. 8, demonstrating also sensitivity of predicted risks).

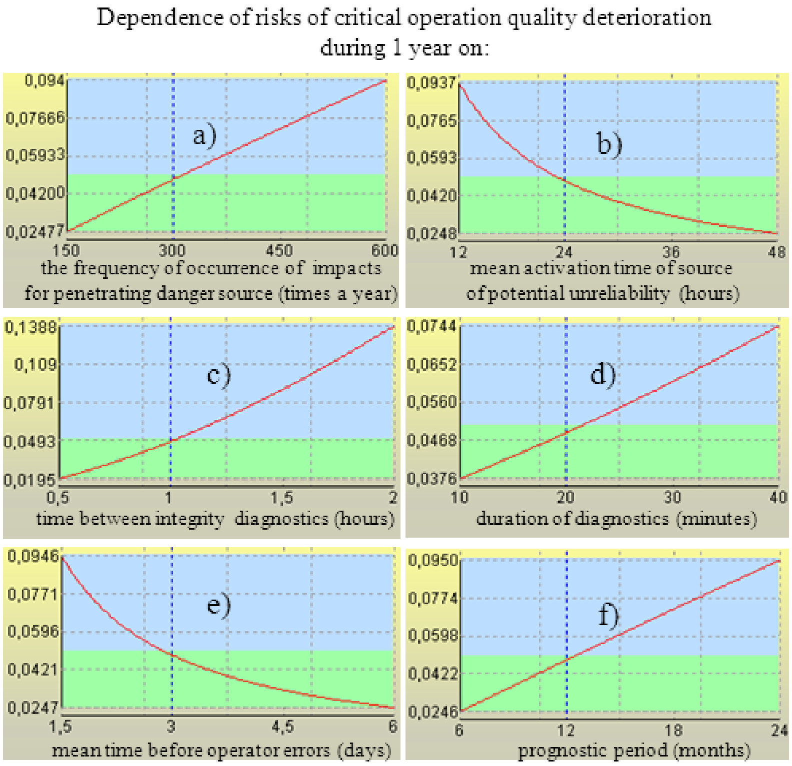

Figure 8 Dependence of risks of critical operation quality deterioration during 1 year on: a) frequency of occurrence of impacts for penetrating danger source (times a year); b) mean activation time of source of potential unreliability (hours); c) time between integrity diagnostics (hours); d) duration of diagnostics (minutes); e) mean time before operator errors (days); f) prognostic period (months)

Example 7 (the primary consideration of 2nd type of errors during SES operation, when essential item is missed).

The applications area for such SES is characterized by a high degree of scrutiny, for example, there may be monitoring of a current condition of the equipment and comparisons with admissible norms (for example, temperatures, pressure and so on – see requirements of standards ISO 13379, ISO 13381, ISO 17359, IEC 61508, etc.). Traced conditions of parameters are data about a condition before and on the current moment of time. Critical deterioration of SES operation in practice can be expressed in negative events after parameters abnormalities - failures, accidents, damages and-or the missed benefit because of equipment time out, etc. For similar SES the errors of 2nd type are much more often, rather than errors of 1st type. To form input we use the results of researches of example 3 (for providing SES operation correctness - "mean time between errors of the 2nd type should not be less than 6 months" instead of 3 days from Example 6) are used. Other input - from the Example 6. It is required to predict for 1 month and 1 year risks of critical operation quality deterioration taking into account impact of uncertainties on correctness and reliability.

Solution. The risks of SES critical operation quality deterioration during 1 month will make about 0.00007 and during 1 year - about 0.00086 (considering consequences). The modeling results have shown - at change of input in a diapason -50 % + 100% the risk during 1 month is in diapason from 0.00003 to 0.00021 and during 1 year - from 0.00032 to 0.00247 (see Fig. 9, demonstrating also sensitivity of predicted risks).

A virtual interpretation of these figures for 1 year SES operation may be the following: if to compare a lot of years of a similar mode of SES operation (for example, 1000 years) at worst within one or two years from 1000 years emergencies may be happen because of possible errors of 2nd type during items analysis by SES.

Example 8 (about pragmatic effects). Many examples demonstrating applications of the integrated model to predict the risks of critical operation quality deterioration for SES, composed as complex structures and optimization solutions cover oil&gas systems, systems of coal branch, robotic and automated systems [1-2, 5-11]. So, the Complex of risks predictions for technogenic safety support on the objects of oil&gas distribution has been awarded by Award of the Government of the Russian Federation in the field of a science and technics for 2014. The created peripheral posts are equipped additionally by SES of Complex to feel vibration, a fire, the flooding, unauthorized access, hurricane, and also intellectual SES of the reaction, capable to recognize, identify and predict a development of extreme situations. For 200 objects in several regions of Russia the applications of Complex during the period 2009-2014 have already provided economy about 8,5 Billions of Roubles. The economy is reached at the expense of effective implementation of the functions of risks prediction and processes optimization [6].

Conclusion

The methodology for risks of critical operation quality deterioration for SES is proposed. Complexity and uncertainties, impacting on SES operation correctness and reliability, are considered. Methodology covers improved models to estimate information correctness and operation reliability, consecutive algorithm to predict risks (for "black box»), integrated model to predict the risks for SES, composed as complex structures. Statements for optimization problems in SES life cycle are formulated. Researches have shown comparable impacts on SES operation quality from failures (reliability) and errors (correctness). The levels about 0.999-0.9997 and more for probability of SES operation correctness and reliability and about 0.01 – 0.001 and less for risks of critical SES operation quality deterioration during 1 year (against consequences) are achievable. The benefit from SES implementations in a system life cycle may be commensurable with expenses for a creation of this system.

References

[1] Kostogryzov, A. and Nistratov, G. Standardization, mathematical modelling, rational management and certification in the field of system and software engineering. Armament.Policy.Conversion, Moscow, 2004.

[2] Zio, En. An Introduction to the Basics of Reliability and Risk Analysis, World Scientific Publishing Co.Pte.Ltd., 2006.

[3] Kostogryzov, A.I. and Stepanov, P.V. Innovative management of quality and risks in systems life cycle. Moscow, Armament.Policy.Conversion, Moscow, 2008.

[4] Kolowrocki, K. and Soszynska-Budny, J. Reliability and Safety of Complex Technical Systems and Processes, Springer-Verlag London Ltd., 2011.

[5] Kostogryzov, A., Nistratov, G. and Nistratov, A. Some Applicable Methods to Analyze and Optimize System Processes in Quality Management. Total Quality Management and Six Sigma, InTech, 2012: 127-196.

[6] Abrosimov, N., at al. Security of Russia. Legal, Manufacturing&Economic and Scientific&Engineering Aspects. The Scientific Foundations of Technogenic Safety. Under the editorship of Mahutov N.A. Znanie, Moscow, 2015.

[7] Artemyev, V., Rudenko, Ju., Nistratov, G. Probabilistic modeling in system engineering. Probabilistic methods and technologies of risks prediction and rationale of preventive measures

by using "smart systems". Applications to coal branch for increasing Industrial safety of enterprises, IntechOpen, 2018: 23-51.

[8] Kershenbaum, V., Grigoriev, L., P. Kanygin, Nistratov, A. Probabilistic modeling in system engineering. Probabilistic modeling processes for oil and gas systems, IntechOpen, 2018: 55-79.

[9] Abrosimov, N., at al. Security of Russia. Legal, Social&Economic and Scientific&Engineering Aspects. The Scientific Foundations of Technogenic, Technologic and Technospheric Safety / Edited by N.Machutov – Moscow, «Znanie», 2018

[10] Lepikhin, A., Moskvichev, V., Machutov, N. Probabilistic Modelling in Solving Analytical Problems of System Engineering, Probabilistic modeling in system engineering. InTech, 2018, 3-22

[11] Kostogryzov, A., Korolev, V. Probabilistic Methods for Cognitive Solving of Some Problems in Artificial Intelligence Systems, IntechOpen, 2019. Doi: http://dx.doi.org/10.5772/intechopen.89168

[12] Kostogryzov, A., Kanygin, P., Nistratov, A. Probabilistic comparisons of systems operation quality for uncertainty conditions. RTA&A No1(56), 2020, 15:63-73.

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Evaluation of Customer Orientated Indices and Reliability Study of Electrical Feeder System

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Abstract

Reliability evaluation of a system or component or element is very important in order to predict its availability and other relevant indices. Reliability is the parameter which tells about the availability of the system under proper working conditions for a given period of time. The study of different reliability indices are very important considering the complex and uncertain nature of the power system. In this paper reliability evaluation of the electrical feeder system is presented. This paper also evaluates basic indices such as average failure rate, average outage time and average annual outage time. Along with basic indices, customer orientated indices such as system average interruption frequency index, system average interruption duration index and customer average interruption duration index of an electrical feeder system is also evaluated.

Keywords: Reliability, Availability, Electrical feeder system, average interruption frequency index, system average interruption duration index.

I. Introduction

Reliability evaluation of a system or component or element is very important in order to predict its availability and other relevant indices. Reliability is the parameter which tells about the availability of the system under proper working conditions for a given period of time. A Markov cut-set composite approach to the reliability evaluation of transmission and distribution systems involving dependent failures was proposed by Singh et al. [1]. The reliability indices have been determined at any point of composite system by conditional probability approach by Billinton et al. [2]. Wojczynski et al. [3] discussed distribution system simulation studies which investigate the effect of interruption duration distributions and cost curve shapes on interruption cost estimates. New indices to reflect the integration of probabilistic models and fuzzy concepts was proposed by Verma et al. [4].

Zheng et al. [5] developed a model for a single unit and derived expression for availability of a component accounting tolerable repair time. Distributions of reliability indices resulting from two sampling techniques are presented and analyzed along with those from MCS by Jirutitijaroen and Singh [6]. Dzobe et al. [7] investigated the use of probability distribution function in reliability worth analysis of electric power system. Bae and Kim [8] presented an analytical technique to evaluate the reliability of customers in a microgrid including distribution generations.

Reliability network equivalent approach to distribution system reliability assessment is proposed by Billinton and Wang [9].

Evaluation of Reliability indices accounting omission of random repair time for distribution systems using Monte Carlo simulation [10]. Determination of Optimum period between Inspections for Distribution system based on Availability Accounting Uncertainties in Inspection Time and Repair Time, Tiwary et al. [11]. Jirutitijaroen et al. [12] developed a comparison of simulation methods for power system reliability indexes and their distribution. Determination of reliability indices for distribution system using a state transition sampling technique accounting random down time omission Tiwary et al. [13].Tiwary et al. [14] proposed a methodology based on inspection repair based availability optimization of distribution systems using Teaching Learning based Optimization. Bootstrapping based technique for evaluating reliability indices of RBTS distribution system neglecting random down time was evaluated [15].

Volkanavski et al. [16] proposed application of fault tree analysis for assessment of the power system reliability. Li et al. [17] studies the impact of covered overhead conductors on distribution reliability and safety. Reliability enhancement of distribution system using Teaching Learning based optimization considering customer and energy based indices was obtained in Tiwary et al. [18]. Self-Adaptive Multi-Population Jaya Algorithm based Reactive Power Reserve Optimization Considering Voltage Stability Margin Constraints was obtained in Tiwary et al. [19]. A smooth bootstrapping based technique for evaluating distribution system reliability indices neglecting random interruption duration is developed [20]. The impact of covered overhead conductors on distribution reliability and safety is discussed [21]. Sarantakos et al. [22] introduced a method to include component condition and substation reliability into distribution system reconfiguration. Battu et al. [23] discussed a method for reliability compliant distribution system planning using Monte Carlo simulation.

Tiwary et al. [24] has discussed a methodology for reliability evaluation of an electrical power distribution system, which is radial in nature. Uspensky et al. [25] has developed a method for reliability assessment of the digital relay protection system. The paper established the load peak valley partition model and demand response model corresponding to every period based on load peak-valley characteristics [26]. A probabilistic method to assess the reliability impact of EVs penetration is discussed [27]. A BN-based unified modeling method for performance and reliability is proposed [28]. A framework for dynamic prediction of reliability weaknesses in power transmission systems based on imbalanced data is discussed [29]. Pham et al. [30] proposed a method for reliability evaluation of an aggregate battery energy storage system in microgrids under dynamic operation. Min et al. [31] proposes a research framework to evaluate the new policy in South Korea from various aspects using three simulation models in a series.

Ding et al. [32] presents a technique to evaluate reliability of a restructured power system with a bilateral market. Oh et al. [33] proposes a new methodology for a probabilistic power system reliability evaluation using a Monte Carlo simulation in case of multi-energy storage system installed at wind farms. Shrestha et al. [34] proposed the development of an operational adequacy evaluation framework for operational planning of bulk electric power systems. Gautam et al. [35] proposed the development and integration of momentary event models in active distribution system reliability assessment. Adinolfi et al. [36] proposed a multiobjective optimal design of photovoltaic synchronous boost converters assessing efficiency, reliability, and cost savings. Tiwary et al. [37] has discussed a methodology for evaluation of customer orientated indices and reliability of a meshed power distribution system.

Evaluation of different reliability indices are important for proper working of the distribution system. This paper provides a detailed study of different reliability indices of importance. Reliability of each and every distribution system and load point is been obtained. This paper also provides the value of the three basic reliability index along with the customer oriented reliability indices.

II. Reliability evaluation of series system

Physically a system configuration will be a series reliability network if system fails even if a single component fails or system survives if all the components are working successfully.

The system is having a constant failure rate and therefore the reliability of the system having constant failure rate is evaluated by using the following relation.

$$R(t) = e^{-\lambda t} \tag{1}$$

Where R(t) represents the reliability of each distribution section. λ represents the failure rate per year and t represents time period which is taken as one year.

If one assumes reliability of each component as r1,r2…rn, then reliability of series system (Rs) is given as

$$Rs = \prod ri \tag{2}$$

Where ri represents the reliability of components from i=1….n.

III. Basic reliability indices evaluation of series system

When series system with respect to reliability studies are concerns, three basic reliability parameters are average failure rate, average outage time and average annual outage time, which are mentioned as follows.

$$\lambda s=\sum \lambda i \tag{3}$$

$$Us=\sum [\![\lambda i.ri]\!] \tag{4}$$

$$rs=(\sum \lambda i ri)/(\sum \lambda i) \tag{5}$$

Where λi is failure rate per year
ri is average repair time, hours

IV. Customer orientated indices evaluation of series system

Customer orientated indices related to reliability studies are system average interruption frequency index, system average interruption duration index and customer average interruption duration index, which are mentioned as follows.

System average interruption frequency index (SAIFI)

$$SAIFI = \frac{total\ number\ of\ customer\ interruptions}{total\ number\ of\ customers\ served} \tag{6}$$

System average interruption duration index (SAIDI)

$$SAIDI = \frac{sum\ of\ customer\ interruption\ durations}{total\ number\ of\ customers} \tag{7}$$

Customer average interruption duration index (CAIDI)

$$CAIDI = \frac{sum\ of\ customer\ interruption\ durations}{total\ number\ of\ customer\ interruptions} \tag{8}$$

V. Results and Discussions

Eight node radial distribution system consists of 7 distributor segments and 7 load points from LP-2 to LP-8 Fig. 1,[4].

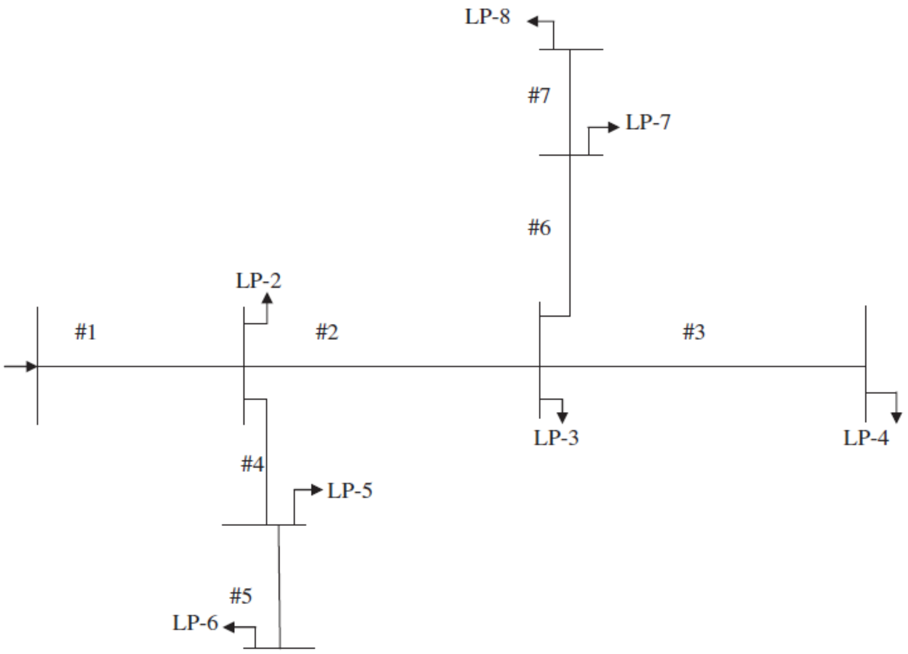

Fig. 1. Eightnode distribution system.

Table 1 [4] provides the initial data for the radial distribution system. Table 1 consists of failure rate per year and repair time in hours of each distribution section from 1 to 7 of the radial distribution system. Table 2 consists of number of customers at each load point LP-2 to LP-8.

Table 1. Initial data for the radial distribution system [4]

Distribution Section	1	2	3	4	5	6	7
λ, failure/year	0.4	0.2	0.3	0.5	0.2	0.1	0.1
r, Average repair time, hours	10	9	12	20	15	8	12

Table 2. Initial data for the load points

Load Point	2	3	4	5	6	7	8
Number of customers	1000	800	600	800	500	400	300

Evaluated Reliability at each distribution section is provided in Table 3. The reliability at each distribution section is evaluated by using equation 1. Fig.2 provides magnitude of reliability at different distribution sections.

Table 3 Evaluated Reliability at each distribution section

Distribution Section	Reliability
1	0.6703
2	0.8187
3	0.7408
4	0.6065
5	0.8187
6	0.9048
7	0.9048

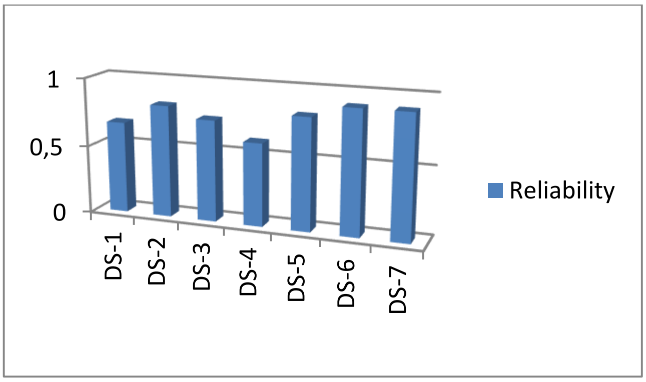

Fig. 2 Magnitude of Reliability at different distribution sections.

Table 4 gives the calculated value of the reliability at each and every load point of the radial distribution system. The reliability at each and every load point is obtained by using the equation 2. Fig.3 provides magnitude of reliability at different load points of the distribution system.

Table 4 Evaluated Reliability at each load points

Load point	Reliability
2	0.6703
3	0.5488
4	0.4065
5	0.4065
6	0.3328
7	0.4965
8	0.4492

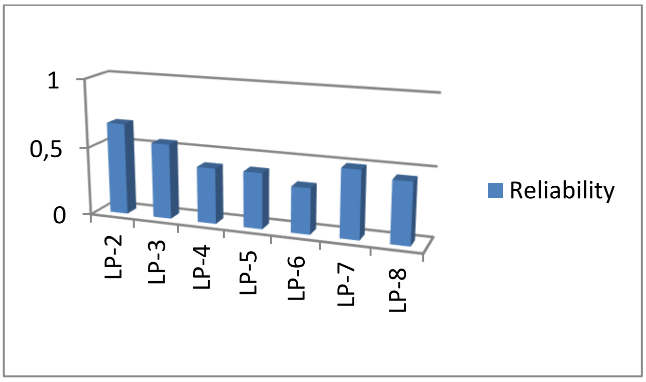

Fig. 3 Magnitude of Reliability at different Load Points.

Basic reliability indices at each load point i.eaverage failure rate, average outage time and average annual outage time are evaluated are presented in Table 5.

Customer orientated indices such as system average interruption frequency index (SAIFI) is evaluated as 0.7295, system average interruption duration index (SAIDI) is obtained as 8.8545 and customer average interruption duration index (CAIDI) is evaluated as 12.1371 for the radial distribution system.

Table 5 Evaluated Basic Reliability indices at each load points

Load point	2	3	4	5	6	7	8
average failure rate	0.4	0.6	0.9	0.9	1.1	0.7	0.8
average outage time	10	9.67	10.4	15.56	15.45	9.42	9.75
average annual outage time	4	5.8	9.4	14	17	6.6	7.8

VI. Conclusion

Evaluation of reliability and other indices related to it are very important for a power distribution system. In this paper a radial distribution system is taken in consideration for the study of different reliability parameters. Reliability of each and every distribution section is obtained for the radial distribution system. For each and every load point considered the value of the reliability is evaluated for the distribution system considered. The three basic reliability parameters of importance average failure rate, average outage time and average annual outage time are also obtained for the load points considered. Important customer orientated indices such as system average interruption frequency index, system average interruption duration index and customer average interruption duration index are also evaluated for the radial distribution system.

References

[1] C. Singh. (1981). Markov cut-set approach for the reliability evaluation of transmission and distribution systems. IEEE Trans. on Power Apparatus and Systems, 100: 2719-2725.

[2] R. Billinton. (1969). Composite system reliability evaluation. IEEE Trans. on Power Apparatus and Systems, 88:276-281.

[3] E. Wojczynski, R. Billinton. (1985). Effects of distribution system reliability index distributions upon interruption cost/reliability worth estimates. IEEE Trans. on Power Apparatus and Systems, 11:3229-3235.

[4] A. K. Verma, A. Srividya, H. M. R. Kumar. (2002). A framework using uncertainties in the composite power system reliability evaluation. Electric Power Components and Systems, 30:679-691.

[5] Z. Zheng, L. Cui, Alan G. Hawkes. (2006). A study on a single-unit Markov repairable system with repair time omission. IEEE Trans. on Reliability, 55:182-188.

[6] P. Jirutitijaroen, C. singh. (2008). Comparison of simulation methods for power system reliability indexes and their distributions. IEEE Trans. on Power Systems, 23:486-493.

[7] O. Dzobe, C. T. Gaunt, R. Herman. (2012). Investigating the use of probability distribution functions in reliability-worth analysis of electric power systems. Int. J. of Electrical Power and Energy Systems, 37:110-116.

[8] I. S. Bae, J. O. Kim. (2008). Reliability evaluation of customers in a microgrid.IEEE Trans. on Power Systems, 23:1416-1422.

[9] R. Billinton, P. Wang. (1998). Reliability-network-equivalent approach to distribution-system-reliability evaluation. IEE Proc. generation, transmission and distribution, 145:149-153.

[10] L. D. Arya, S. C. Choube, R. Arya and AdityaTiwary. (2012). Evaluation of Reliability indices accounting omission of random repair time for distribution systems using Monte Carlo simulation. Int. J. of Electrical Power and Energy System, (ELSEVIER), 42:533-541.

[11] AdityaTiwary, R. Arya, S. C. Choube and L. D. Arya.(2012). Determination of Optimum period between Inspections for Distribution system based on Availability Accounting Uncertainties in Inspection Time and Repair Time. Journal of The Institution of Engineers (India): series B (Springer), 93:67-72.

[12] Jirutitijaroen P, singh C. (2008).Comparison of simulation methods for power system reliability indexes and their distribution. IEEE Trans Power Syst, 23:486–92.

[13] AdityaTiwary, R. Arya, S. C. Choube and L. D. Arya.(2013). Determination of reliability indices for distribution system using a state transition sampling technique accounting random down time omission. Journal of The Institution of Engineers (India): series B (Springer), 94:71-83.

[14] AdityaTiwary, L. D. Arya, R. Arya and S. C. Choube. (2016). Inspection repair based availability optimization of distribution systems using Teaching Learning based Optimization. Journal of The Institution of Engineers (India): series B (Springer), 97:355-365.

[15] AdityaTiwary, R. Arya, L. D. Arya and S. C. Choube. (2017). Bootstrapping based technique for evaluating reliability indices of RBTS distribution system neglecting random down time. The IUP Journal of Electrical and Electronics Engineering, X:48-57.

[16] Volkanavski, Cepin M, Mavko B. (2009).Application of fault tree analysis for assessment of the power system reliability.ReliabEngSyst Safety, 94:1116–27.

[17] Li BM, Su CT, Shen CL. (2010).The impact of covered overhead conductors on distribution reliability and safety.Int J Electr Power Energy Syst, 32:281–9.

[18] AdityaTiwary. (2017). Reliability enhancement of distribution system using Teaching Learning based optimization considering customer and energy based indices. International Journal on Future Revolution in Computer Science & Communication Engineering, 3:58-62.

[19] AdityaTiwary. (2018). Self-Adaptive Multi-Population Jaya Algorithm based Reactive Power Reserve Optimization Considering Voltage Stability Margin Constraints. International Journal on Future Revolution in Computer Science & Communication Engineering, 4:341-345.

[20] R. Arya ,AdityaTiwary , S. C. Choube and L. D. Arya. (2013). A smooth bootstrapping based technique for evaluating distribution system reliability indices neglecting random interruption duration. Int. J. of Electrical Power and Energy System, (ELSEVIER), 51:307-310.

[21] M. BinLi, C. TzongSu, C. LungShen. (2010). The impact of covered overhead conductors on distribution reliability and safety. Int. J. of Electrical Power and Energy System, (ELSEVIER), 32:281-289.

[22] I. Sarantakos, D. M.Greenwood, J.Yi, S. R. Blake, P. C. Taylor. (2019). A method to include component condition and substation reliability into distribution system reconfiguration. Int. J. of Electrical Power and Energy System, (ELSEVIER), 109:122-138.

[23] N. R. Battu, A. R. Abhyankar, N. Senroy. (2019). Reliability Compliant Distribution System Planning Using Monte Carlo Simulation. Electric power components and systems, 47:985-997.

[24] AdityaTiwary. (2019). Reliability evaluation of radial distribution system – A case study. Int. J. of Reliability: Theory and Applications, 14,4(55):9-13.

[25] Michael Uspensky. (2019). Reliabilityassessmentof the digital relay protection system. Int. J. of Reliability: Theory and Applications, 14:10-17.

[26] N. Li, X. Wang, Z. Zhu, Y. Wang. (2019). The reliability evaluation research of distribution system considering demand response. The 6th International Conference on Power and Energy Systems Engineering (CPESE 2019). Okinawa. Japan.

[27] M. P. Anand, B. Bagen, A. Rajapakse. (2020). Probabilistic reliability evaluation of distribution systems considering the spatial and temporal distribution of electric vehicles. Int. J. of Electrical Power & Energy Systems. 117.

[28] Yi Ren, B. Cui, Q. A. Feng, D. Yang, D. Fan, Bo Sun, M. Li. (2020). A reliability evaluation method for radial multi-microgrid systems considering distribution network transmission capacity. Computers & Industrial Engineering.139.

[29] C. Sun, X. Wang, Y. Zheng, F. Zhang. (2020). A framework for dynamic prediction of reliability weaknesses in power transmission systems based on imbalanced data. Int. J. of Electrical Power & Energy Systems. 117.

[30] T. T. Pham, T. C. Kuo, D. M. Bui. (2020). Reliability evaluation of an aggregate battery energy storage system in microgrids under dynamic operation. Int. J. of Electrical Power & Energy Systems. 118.

[31] D. Min, J. H. Ryu, D. G. Choi. (2020). Effects of the move towards renewables on the power system reliability and flexibility in South Korea. Energy Reports. 6:406-417.

[32] Y. Ding, P. Wang, L. Goel, R. Billinton, R. Karki. (2007). Reliability assessment of restructured power systems using reliability network equivalent and pseudo-sequential simulation techniques. Electric Power Systems Research. 77:1665-1671.

[33] U. Oh, Y. Lee, J. Choi, R. Karki. (2019). Reliability evaluation of power system considering wind generators coordinated with multi-energy storage systems. IET Generation, Transmission & Distribution. 14.

[34] T. K. Shrestha, R. Karki, P. Piya. (2020). Development of an operational adequacy evaluation framework for operational planning of bulk electric power systems. International Journal of Reliability, Quality and Safety Engineering.

[35] P. Gautam, P. Piya, R. Karki. (2019). Development and integration of momentary event models in active distribution system reliability assessment. IEEE Transactions on Power Systems.

[36] G. Adinolfi, G. Graditi, P. Siano, A. Piccolo. (2015). Multiobjective optimal design of photovoltaic synchronous boost converters assessing efficiency, reliability, and cost savings. IEEE Transactions on Industrial Informatics. 11.

[37] Aditya Tiwary. (2020). Customer orientated indices and reliability evaluation of meshed power distribution system. Int. J. of Reliability: Theory and Applications, 15,1(56):10-19.

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Synergetic Effects in Applied Poisson Flow Models

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Abstract

In this paper, we consider applied models of Poisson flow, in which synergetic effects are identified. The synergistic effect here is understood as a significant increase/decrease of the efficiency of the simulated system, while increasing the flow rate and a certain additional parameter of the model. Three applied Poissom flow models are considered. First of them is application of Boolean model in powder metallurgy. Second describes process of $3D$ printer and xerocopying. Third deals with non-stationary model of queuing acyclic network with assembly and disassambley of customers and with deterministic service times. In all these models sufficiently strong synergetic effects of considered systems objective functions are obtained.

Keywords: synergetic effect, Poisson flow, Boolean model, acyclic directed graph.

1 Introduction

In this paper, we consider applied models of Poisson flow, in which synergetic effects are identified. The synergistic effect here is understood as a significant increase/decrease in the efficiency of the simulated system, while increasing the flow rate and a certain parameter of the model. The paper considers the following applied models based on Poisson point flow.

The first model is a Boolean model of a random set based on a Poisson flow of points in three-dimensional space. Balls of a fixed radius are attached to each of the flow points by their centres. The set of balls is projected onto a flat unit square. This model may characterize the process of spraying a certain surface with a metal powder.

Using well-known formulas for the Boolean model [1], we study the average square area uncovered by balls projections. It is shown that with a proportional increase in the intensity of the Poisson flow by k times and the same decrease in the volume of each ball, this efficiency indicator tends to zero and the rate of such convergence has the form $\exp(-bk^a)$, for some $a, b > 0$, i.e. the convergence is fast enough. The selected efficiency indicator characterizes the degree of protection of the surface to be sprayed with the powder.

The second model is used for quality analysis in modern printing technologies: xerocopying, 3D printing. This model is based on the Poisson flow model of points of variable intensity $\lambda(x, y)$ (intensity $\lambda(x, y, z)$) of a fixed mass m in two-dimensional space (in three-dimensional space). We investigate the proximity of the resulting random set and the intensity of λ, while simultaneously reducing the mass of m: $m \to m/k$ by a factor of k and increasing the flow intensity of $\lambda \to k\lambda$. A special probability metric is selected for this purpose and $\rho \to \frac{\rho}{\sqrt{k}}$. The choice of this metric is based on the Poisson point flow model and the Boolean model and plays an important role in evaluating the quality of the $3D$ printer and the xerocopying procedure.

The third model is being built for research in the non-stationary model of operation of a conveyor with the assembly and disassembly of customers. In this model, it is assumed that the system input receives a non-stationary Poisson flow of customers, that pass through the acyclic service network. The customers are assembled and/or disassembled at each node of the network. Moreover, the customer begins to be served only when all its parts arrive at the node, which means that these parts are located on the edges of the network. The time spent in the node is deterministic, without any delays in the queue. It is proved that the distribution of the number of customers in the network nodes has a Poisson distribution and its parameter is calculated at any time. This allows us to see that the input stream intensity is being smoothed in this system.

A special feature of this model is the absence of a queue in nodes, which is caused by the fact that each node is a multi-channel system. In this system, due to the synergetic effect known for queuing systems [2], when the intensity of the input Poisson flow and the number of channels increase by a factor of k, the random number of customers in the queue tends to zero. In other words, such a multi-channel system is approximated by a system with an infinite number of channels and, consequently, with no queue. It should be noted that the product theorem for stationary distributions in open queuing networks with division and merging of customers was obtained only recently [3], [4]. Algorithms for numerical calculations of non-stationary model in open queuing networks were constructed in the works [5], [6]. However, exact formulas for distributions of the numbers of customers in the network nodes in non-stationary model were not received earlier even for partial service time distributions.

2 Synergetic effects in the Boolean model

Consider the problem of the quality of the protective layer, formed by spraying the powder on a certain surface. We will consider that the flow of powder particles – balls of volume v is applied to a flat surface that has the shape of a square with sides of unit length. The area s of the projection of the ball on the plane is equal to $s = \pi^{1/3}\left(\frac{3v}{4}\right)^{2/3}$.

Let's assume that the centres of the balls in the projection on the sprayed surface form a Poisson flow of intensity λ. Then from the known properties of the Boolean model [1, pp. 295, 335] the average area q of the set of points, not covered by circles of radius s on a unit square, satisfies the equality

$$q = \exp(-\lambda s). \tag{1}$$

Let us now assume that the volume v of the ball decreases by a factor of k, and the intensity of the Poisson flow of the centres of the balls in the projection on the unit square increases by a factor of k. Then the area s of the projection of the ball on the square is reduced by $k^{2/3}$ times. Let's denote $q(k)$ the average area of the set of uncovered projections of balls of reduced volume of points on the unit square. Using the formula (1), we get the equality

$$q(k) = \exp(-\lambda s k^{1/3}), \tag{2}$$

the average thickness of the protective layer is preserved.

If the intensity of the Poisson flow of the projection centres of the balls increases by $k^{1-\gamma}$, $0 < \gamma < 1/3$ times, when the volume of the ball decreases by k times, then equality (2) is replaced with equality

$$q(k) = \exp(-\lambda s k^{1/3-\gamma}), \tag{3}$$

and the average thickness of the protective layer decreases by k^{γ} times.

3 Quality of the $3D$ printer

Let's first consider the problem of evaluating the quality of the $3D$ printer, assuming that there is a design distribution density the mass of the part specified by the Lebesgue-measurable function $\lambda(x,y,z) \geq 0,\ (x,y,z) \in A,\ A = \{0 \leq x \leq a,\ 0 \leq y \leq b,\ 0 \leq z \leq c\}$, and

$$\Lambda = \int_0^a \int_0^b \int_0^c \lambda(x,y,z)dx\ dy\ dz < \infty.$$

We will assume that a copy of this image is generated by Poisson flow of particles of dye powder with intensity $\lambda(x,y,z),\ (x,y,z) \in A$. The mass of each particle is equal to m. Find out, how the image quality changes if the mass of each particle is reduced by a factor of k, and the intensity of the Poisson flow of powder particles increases by a factor of k.

To evaluate the quality of $3D$ printing, the following criteria are introduced. Let A^* is some partition of the rectangle A by $n = n(A^*)$ disjoint and measurable by Lebesgue subsets $A_1, \dots, A_n$. Denote $\eta_i^{(k)}$ a random number of Poisson flux particles with intensity $k\lambda(x,y,z)$, falling into the set A_i, and put

$$\Lambda(A_i) = \int_{A_i} \int \lambda(x,y,z)dx\ dy\ dz.$$

Since the Poisson flow has an intensity $k\lambda(x,y,z)$, and each particle has a mass $\frac{m}{k}$, then the total mass of these particles is $\frac{m\eta_i^{(k)}}{k}$ and satisfies the following equalities

$$M\frac{m\eta_i^{(k)}}{k} = m\Lambda_i, D\frac{m\eta_i^{(k)}}{k} = m^2\frac{\Lambda_i}{k}$$

this means that the following equality is fulfilled

$$M\left(\frac{m\eta_i^{(k)}}{k} - m\Lambda_i\right)^2 = \frac{m^2\Lambda_i}{k},\ M\sum_{i=1}^{n(A^*)}\left(\frac{m\eta_i^{(k)}}{k} - m\Lambda_i\right)^2 = \frac{m^2\Lambda}{k}.$$

The last equality follows from the formula $\Lambda = \sum_{i=1}^{n(A^*)} \Lambda_i$.

Let $\mathcal{A}$ be a collection of all possible partitions A^* of the set A into disjoint and Lebesgue-measurable subsets $A_1, \dots, A_n,\ n = n(A^*)$. Now let's define the quality of the function $\rho(k)$ of the $3D$ printer with k – multiple powder grinding equal to

$$\rho(k) = \left(\sup_{A^* \in \mathcal{A}} M\sum_{i=1}^{n(A^*)}\left(\frac{m\eta_i^{(k)}}{k} - m\Lambda_i\right)^2\right)^{1/2} = m\sqrt{\frac{\Lambda}{k}}.$$

Therefore, the value $\rho(k)$, which characterizes the quality of the coating with crushed particles, decreases as $k^{-1/2}$. Note that the image quality in the resulting xerocopying process can be studied in the same way.

4 Non-stationary model of the queuing system with the assembly and disassembly of customers and deterministic service times

Consider an acyclic directed graph (digraph) G with a finite set of vertices U and a set of edges V. Assume that in the digraph G there is a single (input) vertex u_0, from which a path can be drawn to any vertex. Denote $S(u)$ the set of all paths $s(u)$ from the vertex u_0 to the vertex u in the digraph G. Let's put $l(u)$ the maximum number of edges in paths $s(u) \in S(u)$, let $L = \max_{u \in U} l(u)$. The set of vertices U is divided into subsets $U(k),\ 0 \leq k \leq L$, of the form

$$U(k) = \{u \in U\colon\ l(u) = k\},\ k = 0, \dots, L.$$

It is proved [7], that in the digraph G any edge $(u_1,u_2) \in V;\ u_1 \in U(k_1),\ u_2 \in U(k_2)$ satisfies the inequality $k_1 < k_2$. In the cited article, the following analog of the Floyd-Warshell algorithm [8] is developed for finding the matrix of maximum lengths of all paths between the vertices of an acyclic digraph and hence for calculating $l(u),\ u \in U$.

Let an acyclic digraph contains n vertices, denoted by the indexes $1, \dots, n$. Let's put $D^k = ||d_{i,j}^k||_{i,j=1}^n,\ k = 0, \dots, n$, where the value $d_{i,j}^k$ is equal to infinity, if and only if the graph G has no path, connecting the vertices i, j, which can only pass through the vertices $1, \dots, k$. If such paths exist, the value $D_{i,j}^{k}$ is equal to the maximum length of such paths. It is not difficult to prove that the following theorem holds.

Statement 1. *The matrices* $D^k = ||d_{i,j}^k||_{i,j=1}^n$, $k = 2, \dots, n$, *satisfy the recurrent relations*

$$d_{i,j}^k = \max(d_{i,j}^{k-1},\ d_{i,k}^{k-1} + d_{k,j}^{k-1}),\ \ if\ \ \max(d_{i,j}^{k-1},\ d_{i,k}^{k-1} + d_{k,j}^{k-1}) < \infty, \tag{4}$$

$$else\ \ d_{i,j}^k = \min(d_{i,j}^{k-1},\ d_{i,k}^{k-1} + d_{k,j}^{k-1}). \tag{5}$$

The matrix $D^n = ||d_{i,j}^n||_{i,j=1}^n$ defines the maximum length of paths between vertices of the graph G, if such paths exist. If there are no such paths, the corresponding elements of the matrix are equal to infinity.

Assume that $(u_1, u_2) \in V$, $l(u_1) = k_1$, $l(u_2) = k_2$, $k_2 > k_1 + 1$. Enter into the (u_1, u_2) edge of the graph G dummy vertices $u_1', \dots, u_{k_2-k_1-1}'$ and thus turn it into a path, consisting of sequentially connected edges (u_1, u_1'), $(u_1', u_2'), \dots, (u_{k_2-k_1-1}', u_2)$ (see Fig. 1).

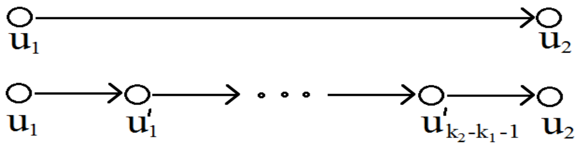

Fig 1. Converting the edge (u_1, u_2) (above) to the sequence of edges (below).

Then we can determine the length of the maximum path from the vertex u_0 to the dummy vertex u_j' by the equality $l(u_j') = l(u_1) + j$, $1 \le j \le k_2 - k_1 - 1$. Denote G_* an acyclic digraph obtained by introducing dummy vertices to the digraph G. Let U_* be the set of vertices in the digraph G_*, and V_* - the set of edges in it.

We assume that the input vertex u_0 receives a non-stationary Poisson flow of intensity $\lambda(t.)$ Let $R(u) = \{u' \colon (u', u) \in V_*\}$ and means $l(u') + 1 = l(u)$, $u' \in R(u)$. In each vertex $u \in U_*$, the customer is being served for a deterministic time $t(u)$, and in dummy vertices this time is zero. If the vertex $u \in U_*$ contains several edges, the customer service begins when all input customers are collected at this vertex, forming a single customer. If several edges come out of the vertex $u \in U$, this means that after servicing, the customer/customers received at this vertex are disassembled. Moreover, as a result of waiting for all input customers, going to the vertex u, on the edges (u', u), these customers can be collected.

Let's construct a recurrent algorithm with the value $l(u)$ for determining the time $T(u)$ of the exit from the vertex u of the customer, received at the zero moment at the vertex u_0. Then equality is true

$$T(u) = \max_{s(u) \in S(u)} \sum_{u' \in s(u)} t(u').$$

It is obvious that $T(u)$ satisfies the relations

$$T(u) = \underline{T}(u) + t(u),\ \underline{T}(u) = \max_{u' \in R(u)} T(u').$$

Using this construction, we can calculate the parameter $\lambda(t, u)$ of the Poisson distribution of the number of customers, that are at the time t at the vertex u

$$\lambda(t, u) = \int_{t-T(u)}^{t-\underline{T}(u)} \lambda(\tau) d\tau.$$

In turn, the parameter $\lambda(t, (u', u))$ of the Poisson distribution of the number of customers located in the edge (u', u), $u' \in R(u)$ at moment t, satisfies the equality

$$\lambda(t, (u', u)) = \int_{t-\underline{T}(u)}^{t-T(u')} \lambda(\tau) d\tau.$$

References

[1] Ambartsumian, R. V., Mecke, Y., Stoyan, D. Introduction to stochastic geometry. Nauka, Moscow, 1989. (In Russian).

[2] Borovkov, A. A. Asymptotic methods in queuing theory. Nauka, Moscow, 1980. (In Russian).

[3] Mitrofanov, Yu. I., Rogachko, E. S., Stankevich, E. P. (2015) Analysis of queueing networks with batch movements of customers and control of flows among clusters. Automatic Control and Computer Sciences, 49 (4): 221–230.

[4] Osipov, O. A. (2018) Analysis of fork/join queueing networks with retrials. (In Russian). Tomsk state university journal of control and computer science, 43: 49–55.

[5] Ivnitsky, V. A. Theory of queuing networks. Fizmatlit, Moscow, 2004. (In Russian).

[6] Katrakhov, V. V., Ryzhkov D. E., Introduction to the functional-analytical method in the dynamic theory of queuing. DVGU publishing house, Vladivostok, 2004. (In Russian).

[7] Tsitsiashvili, G. Sh., Osipova, M A. (2016) Stationary flows in acyclic queuing networks. (In Russian). Far Eastern Mathematical Journal, 16 (2): 223–228.

[8] Cormen, T. H., Leiserson, C. E., Rivest, R. L. Introduction to Algorithms (1st ed.). MIT Press and McGraw-Hill. 1990.

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Exponential-Gamma $(3, \theta)$ Distribution and its Applications

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Abstract

Lifetime distributions for many components usually have a bathtub shape for its failure rate function in practice. However, there are a very few distribution have bathtub shaped failure rate function. Models with bathtub-shaped failure rate functions are useful in reliability analysis, particularly in reliability related decision making, cost analysis and burn-in analysis. When considering a failure mechanism, the failure of units in system may be due to random failure occurred by change in temperature, voltage, jurking etc or due to ageing. This paper study on a distribution, which is a mixture of Exponential and Gamma (3) distribution, which have bathtub shaped failure rate function. Moments, skewness, kurtosis, moment generating function, characteristic function are derived. Renyi entroy, Lorenz curve and Gini index are obtained. Reliability of stress-strength model is derived. Distribution of maximum and minimum order statistics are obtained. We have obtained maximum likelihood estimators. A simulation study is conducted to illustrate the performance of the accuracy of the estimation method used. Application is illustrated using real data.

Keywords: Reliability, Bathtub shaped failure rate, Moments, Entropy, Maximum Likelihood estimator.

I. Introduction

Modeling and analysis of lifetime data has a prominent role in many applied sciences such as medicine, engineering and finance. Various lifetime data have been modeled using distributions such as Exponential, Weibull, Gamma, Rayleigh distributions and their generalizations. It is proved that Exponential distribution (ED) have constant failure function and Rayleigh distribution have monotone increasing failure functions. Two parameter generalized Exponential distribution is introduced by Gupta and Kundu [6] and proved that it has monotone failure functions, depending on its shape parameter. Generalized Rayleigh distribution has an increasing or bathtub shaped failure function, see Surles and Padgett [13]. A new distribution with probability density function

$$f(x,\theta)=\frac{\theta^2}{1+\theta}(1+x)e^{-\theta x}, x>0,\ \theta>0.$$

is proposed by Lindley [8] in the context of Bayesian statistics. Ghitany et al. [5] studied the properties and application of the Lindley distribution. They highlighted that the Lindley distribution is a better model than one based on the exponential distribution. Ghitany et al. [3] showed that the Lindley distribution can be written as a mixture of a Exponential distribution and a Gamma distribution with shape parameter 2. Sankaran [12] proposed the discrete Poisson-Lindley distribution as a combination of the Poisson and Lindley distributions. An Upside-down Bathtub Shaped failure rate model using DUS Transformation of Lomax Distribution is discussed by Deepthi and Chacko (2020). When considering a failure mechanism, the failure of units in system may be due to random failure occurred by change in temperature, voltage, jurking etc or due to ageing. In such situations, we need to use Exponential distribution for random failures and other lifetime distributions for failure due to aging. Mixture of Exponential distribution and a Gamma distribution with shape parameter 2 is not appropriate in some real life situations. So here we examine the mixture of Exponential distribution and a Gamma distribution with shape parameter 3.

The rest of the paper is organized as follows. Section II discussed Exponential-Gamma$(3,\theta)$ distribution. In Section III, the statistical properties are given. Section IV deals with computation of reliability. Section V described the distribution of maximum and minimum. In Section VI, the maximum likelihood method to estimate the unknown parameter is given and two real data sets are analysed. In Section VII, detailed simulation study is given. The comparison of Exponential-Gamma(3) distribution with Exponential and Exponentiated Exponential distribution (EED) for examples from reliability and survival analysis is discussed in Section VIII. Conclusions are given in section IX.

II. Exponential-Gamma $(3,\theta)$ Distribution

A mixture of Exponential (θ) and Gamma $(3,\theta)$ distribution is considered. It is denoted as $EGD(\theta)$. Probability density function (pdf) of mixture of the Exponential $(\theta)$and Gamma $(3,\theta)$ distribution is as follows:

$$f(x,\theta) = pf_1(x,\theta) + (1-p)f_2(x,3,\theta),$$

where $p = \dfrac{\theta}{1+\theta},\ f_1 = \theta e^{-\theta x}\ and\ f_2 = \theta^3 \dfrac{x^2}{2} e^{-\theta x}.$

$$f(x,\theta) = \frac{\theta^2}{(1+\theta)}\left[1+\frac{\theta}{2}x^2\right]e^{-\theta x},\ x>0,\ \theta>0. \tag{2.1}$$

The corresponding cumulative distribution function (cdf) of $EGD(\theta)$ distribution is

$$F(x;\theta) = 1 - \frac{(\theta(x(\theta x+2)+2)+2)e^{-\theta x}}{2(1+\theta)},\ x>0,\ \theta>0. \tag{2.2}$$

The Survival function associated with (2.2) is

$$\overline{F}(x,\theta) = 1 - F(x,\theta) = \frac{(\theta(x(\theta x+2)+2)+2)e^{-\theta x}}{2(1+\theta)},\ x>0, \theta>0. \tag{2.3}$$

The first derivative of the pdf is

$$f'(x) = \frac{\theta^3 e^{-\theta x}}{1+\theta}\left(x-1-\frac{\theta x^2}{2}\right).$$

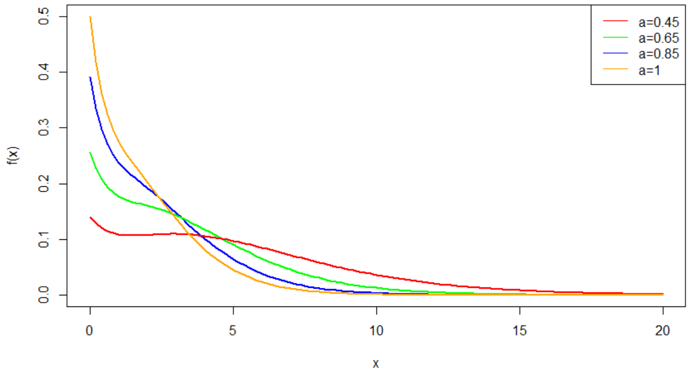
The second derivative of the pdf is

$$f''(x) = \frac{\theta^3 e^{-\theta x}}{1+\theta}\left(1-2\theta x+\theta+\frac{\theta^2 x^2}{2}\right).$$

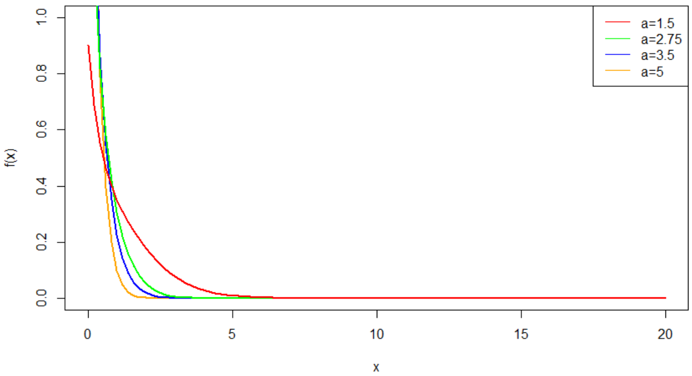
The mode of $f(x)$ is the point $x = x_0$ satisfying $f'(x_0)=0$. Here $f'(x_0)=0$ at the

$x_0 = \dfrac{1 \pm \sqrt{1-\dfrac{\theta}{2}}}{\theta}$. $f''(x) < 0$ for $0 < x < 1$ and $f''(x) > 0$ for $1 \le x \le 2$.

Shape of the probability density function is given in figure 1 below.



(a)



(b)

Figure 1 (a) & (b): pdf of EGD (θ) for θ =0.45, 0.65, 0.85, 1 and θ = 1.5, 2.75, 3.5, 5.

From the above figures it is obvious that the pdf can be decreasing or unimodal.
The failure rate function of EGD (θ) is given in (2.4) below.

$$h(x)=\frac{f(x,\theta)}{\overline{F}(x,\theta)}=\frac{2(1+\theta)\theta^2\left(1+\theta x^2/2\right)}{\left(\theta(x(\theta x+2)+2)+2\right)}, x>0,\, \theta>0 \tag{2.4}$$

The first derivative of failure rate function is

$$h'(x)=2(1+\theta)\theta^2\frac{d}{dx}\left[\frac{1+\theta\frac{x^2}{2}}{\theta(x(\theta x+2)+2)+2}\right]$$

$$=2(1+\theta)\theta^2\frac{\theta x(\theta(x(\theta x+2)+2)+2)-\theta(2\theta x+2)\left(1+\theta\frac{x^2}{2}\right)}{(\theta(x(\theta x+2)+2)+2)^2}$$

$$=\frac{2\theta^2\left(\theta^2x^2+2\theta x-2\theta\right)(1+\theta)}{(\theta(x(\theta x+2)+2)+2)^2}.$$

The second derivative of the failure rate function is given by

$$h''(x) = 2(1+\theta)\theta^2 \frac{d}{dx}\left[\frac{\theta^2 x^2 + 2\theta x - 2\theta}{(\theta(x(\theta x+2)+2)+2)^2}\right] = \frac{4\theta^3(\theta x+1)(-\theta^2 x^2 + 6\theta - 2\theta x + 2)(1+\theta)}{(\theta(x(\theta x+2)+2)+2)^3}$$

The extremum of h(x) is the point x = x_0 satisfying $h'(x) = 0$ and these points correspond to a maximum or a minimum or a point of inflection according as $h''(x) < 0$, $h''(x) > 0$ and $h''(x) = 0$ respectively. Here $h'(x) = 0$ at the point $x_0 = \frac{-1+\sqrt{1+2\theta}}{\theta}$ and $h''(x) > 0$ for $\theta > 0$. So h(x) must attain a unique minimum at x = x_0. Initially, plot of h(x) decreases monotonically and then increases giving a bathtub shape.

Figure 2 provide the failure rate functions of EGD(θ) for different parameter values.

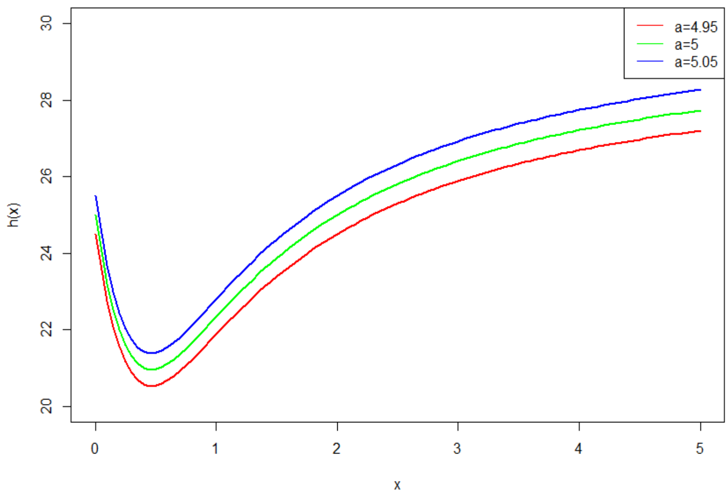


Figure 2: Failure rate function of EGD(θ) for θ = 4.95, 5, 5.15.

III. Statistical Properties

Here, we discuss the statistical measures for the EGD (θ) distribution, such as moments, skewness, kurtosis, moment generating function, characteristic function, quantile function, median, entropy, Lorenz curve and Gini index.

I. Moments

The concept of moment is important in statistical literature. We can measure the central tendency of a population by using moments. Moments also help in measuring the scatteredness, asymmetry and peakedness of a curve for a particular distribution.

The r^{th}raw moment (about origin) of EGD (θ) is

$$\mu_r' = p\frac{r!}{\theta^r} + (1-p)\frac{\Gamma(r+3)}{2\theta^r} = \frac{2\theta r! + \Gamma(r+3)}{2(1+\theta)\theta^r}.$$

Therefore, the mean and variance of EGD(θ) are

$$\mu = \frac{\theta+3}{\theta(1+\theta)} \text{ and } \sigma^2 = \frac{\theta^2+8\theta+3}{\theta^2(1+\theta)^2}.$$

The skewness and kurtosis can be obtained using these raw moments as

$$\text{Skewness} = \frac{2\theta^3+30\theta^2-63\theta+16}{\theta^2+8\theta+3} \text{ and Kurtosis} = \frac{9\theta^4+192\theta^3+306\theta^2+216\theta+45}{(\theta^2+8\theta+3)^2}$$

II. Moment Generating Function and Characteristic Function

Let X has $EGD(\theta)$ distribution, then the moment generating function of X, $M_X(t) = E[\exp(tX)]$, is

$$M_X(t) = \frac{\theta^2}{1+\theta}\left[-\frac{(t-\theta)^2+\theta}{(t-\theta)^3}\right]$$

for $t > \theta$. Similarly, the characteristic function of X becomes $\phi(t) = M_X(it)$,

$$\phi(t) = \frac{\theta^2}{1+\theta}\left[-\frac{(it-\theta)^2+\theta}{(it-\theta)^3}\right]$$

where $i = \sqrt{-1}$.

III. Quantile and Median

Here, we determine the formulas of the quantile and the median of $EGD(\theta)$ distribution. The quantile x_p of the $EGD(\theta)$ is given from

$$F(x_p) = p, 0 < p < 1.$$

We obtain the 100 p^{th} percentile,

$$(\theta(x(\theta x+2)+2)+2)e^{-\theta x} = 2(1-p)(1+\theta) \tag{3.1}$$

Setting $p = 0.5$ in Eq. (3.1), we get the median of $EGD(\theta)$ from

$$(\theta(x(\theta x+2)+2)+2)e^{-\theta x} = 1+\theta.$$

$x_{0.5}$ is the solution of above monotone increasing function. Using different statistical softwares we can obtain the quantiles or percentiles.

IV. Entropy

An important entropy measure is Rènyi entropy [11]. If X has the $EGD(\theta)$, then Rènyi entropy is defined by

$$\Im_R(\gamma)=\frac{1}{1-\gamma}\log\left\{\int f^{\gamma}(x)dx\right\},$$

where $\gamma>0$ and $\gamma\neq 1$. Then, we can calculate, for $EGD(\theta)$,

$$\int f^{\gamma}(x)dx=\int_0^{\infty}\left\{\frac{\theta^2}{1+\theta}e^{-\theta x}\left(1+\frac{\theta}{2}x^2\right)\right\}^{\gamma}dx=\left(\frac{\theta^2}{1+\theta}\right)^{\gamma}\int_0^{\infty}\left\{1+\frac{\theta}{2}x^2\right\}^{\gamma}e^{-\gamma\theta x}dx$$

$$=\left(\frac{\theta^2}{1+\theta}\right)^{\gamma}\sum_{k=0}^{\infty}\binom{\gamma}{k}(-1)^k\left(\frac{\theta}{2}\right)^k\int_0^{\infty}x^{2k}e^{-\gamma\theta x}dx=\left(\frac{\theta^2}{1+\theta}\right)^{\gamma}\sum_{k=0}^{\infty}\binom{\gamma}{k}(-1)^k\left(\frac{\theta}{2}\right)^k\frac{\Gamma(2k+1)}{(\gamma\theta)^{2k+1}}.$$

Therefore, Rènyi entropy is given by

$$\Im_R(\gamma)=\frac{1}{1-\gamma}\log\left\{\left(\frac{\theta^2}{1+\theta}\right)^{\gamma}\sum_{k=0}^{\infty}\binom{\gamma}{k}(-1)^k\left(\frac{\theta}{2}\right)^k\frac{\Gamma(2k+1)}{(\gamma\theta)^{2k+1}}\right\}$$

$$=\frac{\gamma}{1-\gamma}\log\left(\frac{\theta}{1+\theta}\right)+\frac{1}{1-\gamma}\log\left\{\sum_{k=0}^{\infty}\binom{\gamma}{k}(-1)^k\left(\frac{\theta}{2}\right)^k\frac{\Gamma(2k+1)}{(\gamma\theta)^{2k+1}}\right\}$$

V. Lorenz Curve and Gini Index

The Lorenz curve and Gini index have applications not only in economics but also in reliability. The Lorenz curve is defined by

$$L(p)=\frac{1}{\mu}\int_0^q xf(x)dx$$

or equivalently

$$L(p)=\frac{1}{\mu}\int_0^p F^{-1}(x)dx$$

where $\mu=E(X)$ and $q=F^{-1}(p)$. Gini index is defined by

$$G = 1 - 2\int_0^1 L(p)dp$$

If X has $EGD(\theta)$ then

$$L(p) = \frac{1}{\mu}\left[\frac{\theta+3}{\theta(\theta+1)} - \frac{(\theta(q(\theta(q(\theta q+3)+2)+6)+2)+6)e^{-\theta q}}{2\theta(1+\theta)}\right].$$

Gini index is

$$G = 1 - \frac{2}{\mu\theta(1+\theta)}\left[\theta+3-\frac{(\theta(q(\theta(q(\theta q+3)+2)+6)+2)+6)e^{-\theta q}}{2}\right], \theta > 0.$$

IV. Reliability

Suppose that X and Y are two independent strength and stress random variables. We derive the reliability $R = P(Y < X)$ when X and Y are independent random variables distributed according to EGD distribution with parameters θ_1 and θ_2 , respectively. Then system reliability is

$$R = \int_0^\infty\int_0^x f(x)f(y)dydx = \int_0^\infty\int_0^x \frac{\theta_1^2}{1+\theta_1}(1+\frac{\theta_1}{2}x^2)e^{-\theta_1 x}\frac{\theta_2^2}{1+\theta_2}(1+\frac{\theta_2}{2}y^2)e^{-\theta_2 y}dydx$$

$$= \int_0^\infty \frac{\theta_1^2}{1+\theta_1}(1+\frac{\theta_1}{2}x^2)e^{-\theta_1 x}\left\{\int_0^x \frac{\theta_2^2}{1+\theta_2}(1+\frac{\theta_2}{2}y^2)e^{-\theta_2 y}dy\right\}dx$$

$$= \int_0^\infty \frac{\theta_1^2}{1+\theta_1}(1+\frac{\theta_1}{2}x^2)e^{-\theta_1 x}\left\{\frac{\theta_2}{1+\theta_2}\left[\left(1-e^{-\theta_2 x}\right)\frac{(1+\theta_2)}{\theta_2} - xe^{-\theta_2 x}(1+\theta_2 x)\right]\right\}dx$$

$$= \frac{\theta_1^2}{1+\theta_1}\int_0^\infty\left\{(1+\frac{\theta_1}{2}x^2)e^{-\theta_1 x} - (1+\frac{\theta_1}{2}x^2)e^{-(\theta_1+\theta_2)x} - x\frac{\theta_2}{1+\theta_2}e^{-(\theta_1+\theta_2)x}(1+\theta_2 x)\left(1+\frac{\theta_1}{2}x^2\right)\right\}dx$$

$$= \frac{\theta_1^2}{1+\theta_1}\left\{\frac{1+\theta_1}{\theta_1^2} - \frac{1}{\theta_1+\theta_2} - \frac{\theta_1}{(\theta_1+\theta_2)^3} - \frac{\theta_2}{(1+\theta_2)(\theta_1+\theta_2)^5}\left[(\theta_1+\theta_2)^3 + 2\theta_2(\theta_1+\theta_2)^2 + 3\theta_1(\theta_1+\theta_2) + 12\theta_1\theta_2\right]\right\}.$$

V.Distribution of Maximum and Minimum

Let $X_1, X_2, \ldots, X_n$ be a simple random sample from $EGD(\theta)$. Let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ denote the order statistics obtained from this sample. The pdf of $X_{(r)}$ is given by,

$$f_{r:n}(x) = \frac{1}{B(r, n-r+1)}[F(x;\theta)]^{r-1}[1-F(x;\theta)]^{n-r} f(x;\theta)$$

where $F(x;\lambda)$, $f(x;\lambda)$ are the cdf and pdf given by (2.1) and (2.2), respectively.

$$f_{r:n}(x)=\frac{1}{B(r,n-r+1)}\left[1-\frac{\left(\theta\left(x(\theta x+2)+2\right)+2\right)e^{-\theta x}}{2(1+\theta)}\right]^{r-1}\left[\frac{\left(\theta\left(x(\theta x+2)+2\right)+2\right)e^{-\theta x}}{2(1+\theta)}\right]^{n-r}\frac{\theta^2\left(1+\frac{\theta}{2}x^2\right)e^{-\theta x}}{1+\theta}. \tag{5.1}$$

Then the pdf of the smallest and largest order statistics, $X_{(1)}$ and $X_{(n)}$, respectively, are

$$f_1(x)=\frac{1}{B(1,n)}\left[\frac{\left(\theta\left(x(\theta x+2)+2\right)+2\right)}{2(1+\theta)}\right]^{n-1}\left[\frac{\theta^2\left(1+\frac{\theta}{2}x^2\right)e^{-\theta x}}{1+\theta}\right]$$

and

$$f_n(x)=\frac{1}{B(n,1)}\left[1-\frac{\left(\theta\left(x(\theta x+2)+2\right)+2\right)}{2(1+\theta)}\right]^{n-1}\left[\frac{\theta^2\left(1+\frac{\theta}{2}x^2\right)e^{-\theta x}}{1+\theta}\right]$$

The cdf of $X_{(r)}$ is

$$F_{r:n}(x)=\sum_{j=r}^{n}\binom{n}{j}F^{j}(x)[1-F(x)]^{n-j}$$

$$F_{r:n}(x)=\sum_{j=r}^{n}\binom{n}{j}\left[1-\frac{\left(\theta\left(x(\theta x+2)+2\right)+2\right)e^{-\theta x}}{2(1+\theta)}\right]^{j}\left[\frac{\left(\theta\left(x(\theta x+2)+2\right)+2\right)e^{-\theta x}}{2(1+\theta)}\right]^{n-j} \tag{5.2}$$

Then the cdf of the smallest and largest order statistics $X_{(1)}$ and $X_{(n)}$, respectively, are

$$F_1(x)=1-\left[\frac{\left(\theta\left(x(\theta x+2)+2\right)+2\right)e^{-\theta x}}{2(1+\theta)}\right]^{n},\ \theta>0$$

and

$$F_n(x)=\left[1-\frac{\left(\theta\left(x(\theta x+2)+2\right)+2\right)e^{-\theta x}}{2(1+\theta)}\right]^{n},\ \theta>0.$$

These distributions can be used in reliability operations.

VI. Parametric Estimation

In this section, point estimation of the unknown parameter of the $EGD(\theta)$ is described by using the method of maximum likelihood for a complete sample data, as given below.

The likelihood function of $EGD(\theta)$ distribution is

$$L=\prod_{i=1}^{n}f(x_i;\theta)=\prod_{i=1}^{n}\frac{\theta^2\left(1+\frac{\theta}{2}x_i^2\right)e^{-\theta x_i}}{1+\theta}.$$

The log-likelihood function is,

$$\log L(x_i,\theta) = 2n\log\theta - n\log(1+\theta) + \sum_{i=1}^{n}\left[\log\left(1+\frac{\theta}{2}x_i^2\right) - \theta x_i\right].$$

The first partial derivatives of the log-likelihood function with respect to θ is

$$\frac{\partial L}{\partial\theta} = \frac{2n}{\theta} - \frac{n}{1+\theta} + \sum_{i=1}^{n}\left(\frac{x_i^2}{2\left(1+\frac{\theta}{2}x_i^2\right)} - x_i\right).$$

Setting the left side of the above equation to zero, we get the likelihood equation as a system of nonlinear equation in θ. Solving this system in θ gives the MLE of θ. It is easy to obtain numerically by using statistical software package like *nlm* package in R programming with arbitrary initial values.

The Fisher information about θ, I (θ), is

$$I(\theta) = E\left\{-\frac{\partial^2}{\partial\theta^2}\ln f(X;\theta)\right\} = E\left\{\frac{2}{\theta^2} - \frac{1}{(1+\theta)^2} + \frac{x^4}{4}\frac{1}{\left(1+\frac{\theta}{2}x^2\right)^2}\right\}$$

$$= \frac{2}{\theta^2} - \frac{1}{(1+\theta)^2} + E\left\{\frac{x^4}{4}\frac{1}{\left(1+\frac{\theta}{2}x^2\right)^2}\right\}.$$

Then the asymptotic 100(1-α) % confidence interval for θ is given by

$$\hat{\theta} \mp z_{\alpha/2}\frac{I^{-1/2}(\hat{\theta})}{\sqrt{n}}$$

VII. Simulation

A simulation study is conducted to illustrate the performance of the accuracy of the estimation method. The following scheme is used:

(i) Specify the value of the parameter θ.

(ii) Specify the sample size n.

(iii) Generate a random sample with size n from $EGD(\theta)$.

(iv) Using the estimation method used in this paper, calculate the point estimate of the parameter θ.

(v) Repeat steps 3-4, N times.

(vi) Calculate the bias and the mean squared error (MSE).

The simulation study is performed at different sample sizes and different parameter values, θ= 1, 1.5, 1.85 and bias and MSEs for the parameter θ is given in table 1. MSE decreases as sample size increases.

Table.1. Simulation Results

Θ	N	Bias	MSE
1	50	-0.000854222	$3.648476e^{-05}$
	100	0.00039463	$1.557328e^{-05}$
	500	$1.3114e^{-05}$	$8.59885e^{-08}$
	1000	$3.6889e^{-05}$	$1.490841e^{-09}$
1.5	50	-0.00072618	$2.636687e^{-05}$
	100	-0.00058251	$3.393179e^{-05}$
	500	$-3.906e^{-06}$	$7.628418e^{-09}$
	1000	$-3.8229e^{-05}$	$1.461456e^{-06}$
1.85	50	0.00174578	0.000152387
	100	0.00092697	$8.592734e^{-05}$
	500	0.00016791	$1.409688e^{-05}$
	1000	$3.2956e^{-05}$	$1.086098e^{-06}$

VIII. Data Analysis

Applications of the $EGD(\theta)$ distribution is illustrated in two examples.

Data set 1:- We provide a data analysis to see how the new model works. The data set is taken from Klein and Berger [9]. It shows the survival data on the death times of 26 Psychiatric inpatients admitted to the University of Iowa hospitals during the years 1935-1948.

Table 2: The survival data on the death times of Psychiatric inpatients

1	1	2	22	30	28	32	11	14	36	31	33	33
37	35	25	31	22	26	24	35	34	30	35	40	39

We have used different distributions namely, *ED, EED* and $EGD(\theta)$ to analyse the data. The estimate(s) of the unknown parameter(s), corresponding Kolmogorov-Smirnov (K-S) test statistic and Log L values for three different models are given in table 3.

Table 3: The estimates, K-S test statistic and log-likelihood for the dataset 1

Model	Estimates	K-S	LogL
ED	$\hat{\theta} = 0.03784579$	0.3728	-111.1302
EED	$\hat{a} = 1.79724674, \hat{b} = 0.05254319$	0.3146	-108.9871
$EGD(\theta)$	$\hat{\theta} = 0.10500099$	0.2613	-104.5856

We present the p-values, corresponding Akaikes Information Criterion (AIC) (see [1]) and Bayesian Information Criterion (BIC) in the following table 4.

Table 4: The p-value, AIC and BIC of the models based on the dataset 1

Model	p- value	AIC	BIC
ED	0.001455	224.2604	225.5185
EED	0.01162	221.9741	224.4903
$EGD(\theta)$	0.0574	211.1713	212.4294

The table 3 shows the parameter MLEs and log likelihood values of the fitted distributions and table 4 show the values of AIC, BIC and the p-value. The values in tables 3 and 4, indicate that the $EGD(\theta)$ distribution is a strong competitor to other distribution used here for fitting the dataset.

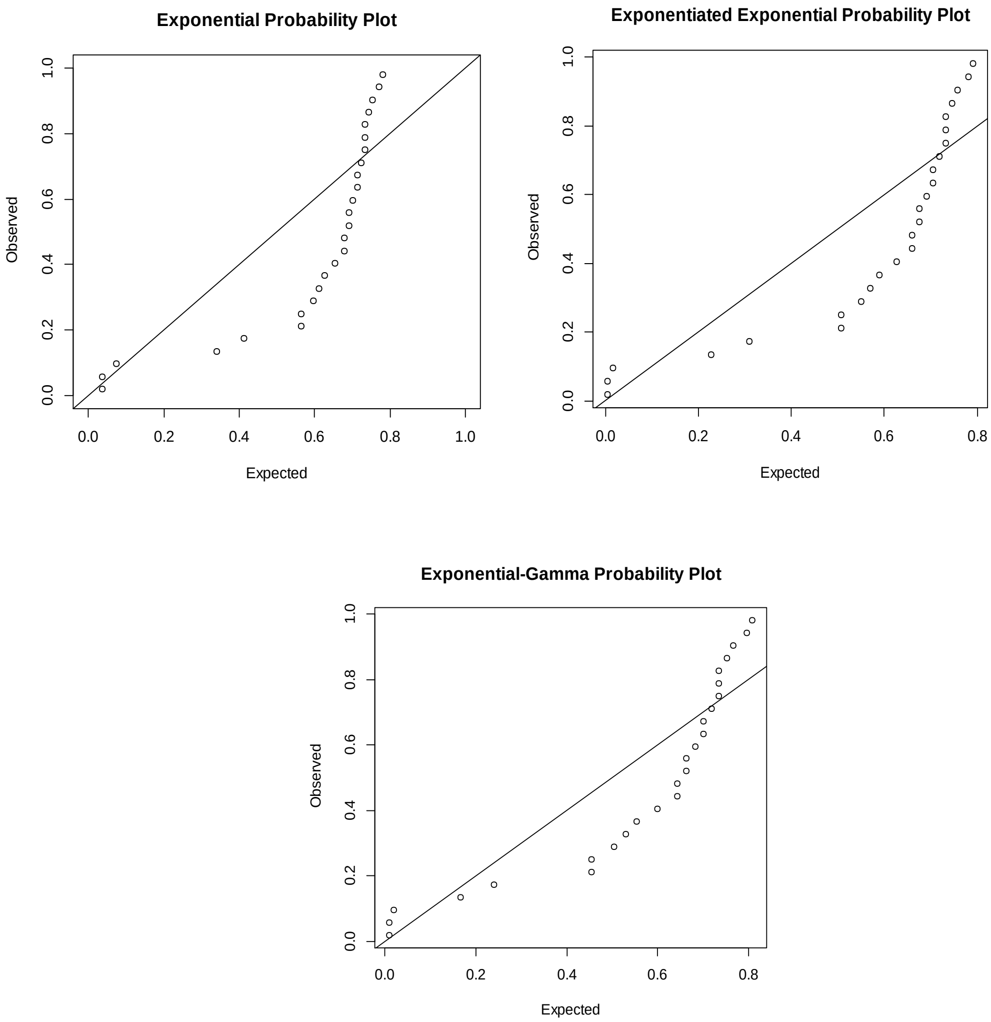


Figure 3: P-P plots for fitted ED, EED and EGD

P-P plot for *ED, EED* and $EGD(\theta)$ are given in Fig.3 which shows that $EGD(\theta)$ model is more plausible than *ED* and *EED* models.

Data set 2:- Chen [6] presented a type-II censoring data of samples, in which there was complete unit failures: 0.29, 1.44, 8.38, 8.66, 10.20, 11.04, 13.44, 14.37, 17.05, 17.13, and 18.35. The estimate(s) of the unknown parameter(s), corresponding Kolmogorov-Smirnov (K-S) test statistic and Log L values for three different models are given in table 5.

Table 5: The estimates, K-S test statistic and log-likelihood for the dataset 2

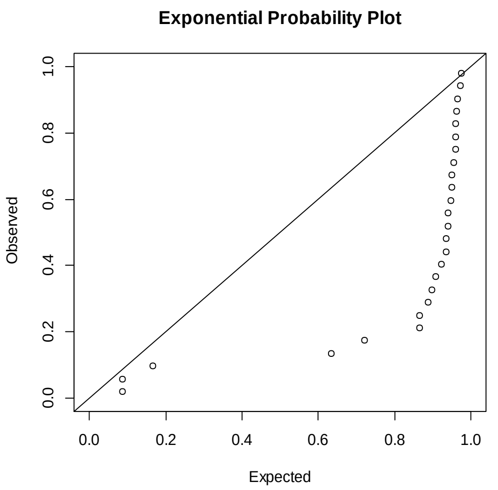
Model	Estimates	K-S	LogL
ED	$\hat{\theta}=0.0913995$	0.3533	-37.3176
EED	$\hat{a}=1.3514168, \hat{b}=0.1090155$	0.3183	-37.04664
EGD	$\hat{\theta}=0.2375122$	0.243	-35.25229

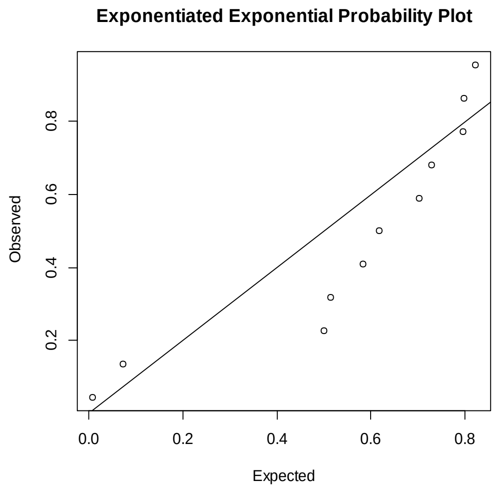
We present the p-values, corresponding Akaikes Information Criterion (AIC) and Bayesian Information Criterion (BIC) for the dataset 2 in the following table 6.

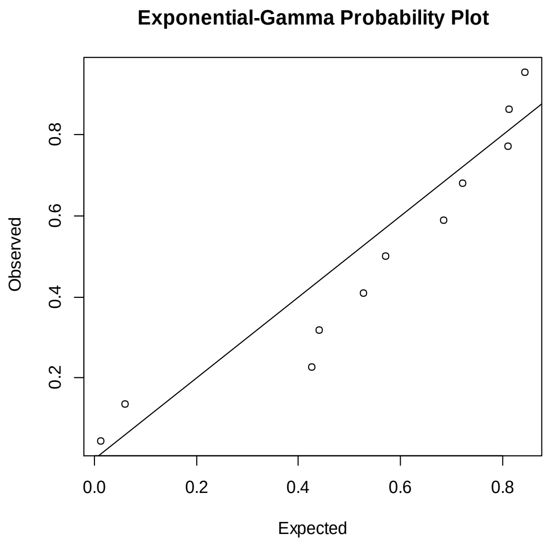
Table 6: The p-value, AIC and BIC of the models based on the dataset 2

Model	P value	AIC	BIC
ED	0.09856	76.6352	77.03309
EED	0.1722	78.09328	78.88907
EGD	0.4625	72.50459	72.90248

The table 5 shows the parameter MLEs and log likelihood values of the fitted distributions and table 6 show the values of AIC, BIC and the p-value. The values in tables 5 and 6, indicate that the $EGD(\theta)$ distribution is a strong competitor to other distribution used here for fitting the dataset.







IX. Conclusion

A bathtub shaped failure rate model, Exponential-Gamma$(3, \theta)$ distribution is considered and its properties are studied. Moments, skewness, kurtosis, moment generating function, characteristic function, etc are derived. Renyi entropy, Lorenz curve and Gini index are obtained. Reliability of stress-strength model is derived. Distribution of maximum and minimum are obtained. We have obtained maximum likelihood estimators. A simulation study is conducted to illustrate the performance of the accuracy of the estimation method used. Applications of $EGD(\theta)$ to real data show that Exponential-Gamma$(3, \theta)$ distribution is effective in providing better fits than the Exponential and Exponentiated Exponential distribution.

References

[1] Akaike, H. (1974). A new look at the statistical model identification, IEEE Transactions on Automatic Control, AC-19, pp. 716-723.
[2] Z. Chen. (2000). A new two-parameter lifetime distribution with bathtub shape or increasing failure rate function, Statistics & Probability Letters, Vol. 49, No. 2, pp. 155–161.
[3] Ghitany M. E., Al-Mutairi D. K., Balakrishnan N. and Al-Enezi L. J. (2013). Power Lindley distribution and associated inference, Computational Statistics and Data Analysis, Vol.64, pp. 20-33.
[4] Deepthi, K S and Chacko, V M. (2020). An Upside-down Bathtub Shaped failure rate model using DUS Transformation of Lomax Distribution, Chapter 6, Stochastic Models in Reliability Engineering, CRC Press Taylor & Francis Group, London.
[5] Ghitany M. E., Ateih B. and Nadarajah S. (2008). Lindley distribution and its application, Mathematics and Computers in Simulation, Vol. 78, pp. 493-506.
[6] Gupta R. D. and Kundu D. (1999). Generalized Exponential distribution. Australian and New Zealand Journal of Statistics, Vol. 41, no.2, pp.173-188.
[7] Gupta R. D. and Kundu D.(2001). Exponentiated Exponential family; An alternative to Gamma and Weibull, Biometrical Journal, Vol.43, pp.117-130.
[8] Lindley D. V. (1958). Fiducial distributions and Bayes' theorem, Journal of the Royal Statistical Society, A, Vol.20, pp.102-107.
[9] Klein J. P. and Moesch Berger M. L.(1997). Survival Analysis Techniques for Censored and Truncated Data. Springer Verlag, New York.
[10] Lawless J. F. (2003). Statistical models and methods for lifetime data. John Wiley and Sons, New York.
[11] Rènyi A. (1961). On Measures of Entropy and Information. In Proceedings of the 4th Berkeley Symposium on Mathematical Statistics and Probability, Vol.1,pp.547-561. Berkeley: University Press.
[12] Sankaran M. (1970). The discrete Poisson-Lindley distribution. Biometrics, pp.145-149.
[13] Surles J. G. and Padgett W. J. (2005). Some properties of a scaled Burr type X distribution. Journal of Statistical Planning and Inference, Vol. 128, pp.271-280.
[14] Zakerzadah H. and Dolati A. (2010). Generalized Lindley distribution. Journal of Mathematical Extension, Vol. 3(2), pp. 13-25.

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Application of Non-Parametric Bootstrap Technique for evaluating MTTF and Reliability of a Complex Network with Non-Identical Component Failure Laws

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Abstract

This paper describes a non-parametric bootstrap technique and Monte Carlo simulation technique to evaluate MTTF and reliability for a complex network, where components have different failure density functions. Algorithm for complete reliability analysis with boot strapping and Monte Carlo simulation (MCS) technique for the complex network has been developed. The algorithm has been implemented on a bridge network. The result obtained has been compared with those obtained using MCS.

Keywords: Monte Carlo simulation, bootstrap, failure density function, mean time to failure, reliability.

I. Introduction

Reliability evaluation of a system is an important issue. Various analytical methods for reliability evaluation of network have been presented in the literature [1, 2, 3]. Average reliability indices are evaluated using analytical technique where as simulation techniques are used to generate probability distribution of these indices. Billinton et al. [4] developed a technique for reliability evaluation of distribution system. Volkanavski et al. [5] used fault tree analysis for power system reliability evaluation. Various MCS based methodologies have been developed for reliability evaluation [6, 7]. Several variations of MCS methods have been developed to probabilistically evaluate long term reliability of power system [8]. Billinton et al. [9] have developed a new MCS technique which is based on a system state transition sampling approach for composite power system reliability. Evaluation of Tsai and Lu [10] utilized the non-parametric bootstrap to estimate available transfer capacity (ATC). Othman et al. [11] developed a novel approach to determine transmission reliability margin using bootstrap technique. Reliability indices accounting omission of random repair time for distribution systems usingMonte Carlo simulation [14]. Jirutitijaroen et al. [15] developed a comparison of simulation methods for power system reliability indexes and their distribution. Determination of reliability indices for distribution system using a state transition sampling technique accounting random down time omission was discussed by Tiwary et al. [16]. Bootstrapping based technique for evaluating reliability indices of RBTS distribution system neglecting random down time was evaluated [17]. Volkanavski et al. [18] proposed application of fault tree analysis for assessment of the power system reliability. Li et al. [19] studies the impact of covered overhead conductors on distribution reliability and safety.

Reliability enhancement of distribution system using Teaching Learning based optimization considering customer and energy based indices was obtained in Tiwary et al. [20]. A smooth bootstrapping based technique for evaluating distribution system reliability indices neglecting random interruption duration is developed [21]. Sarantakos et al. [22] introduced a method to include component condition and substation reliability into distribution system reconfiguration. Battu et al. [23] discussed a method for reliability compliant distribution system planning using Monte Carlo simulation. Tiwary et al. [24] has discussed a methodology for reliability evaluation of an electrical power distribution system, which is radial in nature.Uspensky et al. [25] has developed a method for reliability assessment of the digital relay protection system. A probabilistic method to assess the reliability impact of EVs penetration is discussed [26]. A BN-based unified modeling method for performance and reliability is proposed [27]. A framework for dynamic prediction of reliability weaknesses in power transmission systems based on imbalanced data is discussed [28]. Shrestha et al. [29] proposed the development of an operational adequacy evaluation framework for operational planning of bulk electric power systems. Tiwary et al. [30] has discussed a methodology for evaluation of customer orientated indices and reliability of a meshed power distribution system.

Evaluation of reliability and MTTF of a network become tedious analytically if the failure density functions of different components are non-exponential and different. In such situation simulation approach provides a simple way to obtain these indices. MCS procedure requires large number of samples to obtain desired accuracy results. Hence re-sampling techniques (boot-strap) have been used in various studies. Boot strapping requires lesser number of samples and thus considerable reduction in CPU time may be achieved. In view of this in this paper boot strapping has been employed along with MCS to evaluate reliability and MTTF of a complex reliability network.

II. Monte Carlo Simulation

The reliability indices of a system can be evaluated using one of two basic approaches, direct analytical technique or stochastic simulation. Simulation is a very valuable method which is widely used in the solution of real engineering problems. Analytical technique represents the system by a mathematical model, which is often simplified and evaluates the reliability indices from this model using direct mathematical solutions. Simulation technique, on the other hand, estimates the reliability indices by simulating the actual process and random behavior of the system [1]. Computational algorithm for evaluating mean time to failure (MTTF) and reliability by using MCS technique is as follows:

Step-1 Obtain minimal cut-set for the given network.

Step-2 Obtain random variates for time to failure for each of the components by considering their respective failure density functions. Based on the cut sets, obtain time to failure of the system.

Step-3 Repeat step-2 for large number of times. e.g. NS=10000.

Step-4 Estimate mean time to failure $(M\hat{T}TF)$ by using following relation:

$$M\hat{T}TF = \frac{1}{NS}\sum_{i=1}^{NS} t_i \tag{1}$$

t_i is time to failure for i^{th} sample

Step-5 Obtain coefficient of variations for convergence

$$\beta = s / M\hat{T}TF \tag{2}$$

β is coefficient of variation for MTTF

Where $s = \hat{\sigma} / \sqrt{NS}$

(3)

$$\hat{\sigma}^2 = \frac{1}{NS} \sum_{i=1}^{NS} (t_i - M\hat{T}TF)^2 \tag{4}$$

Step-6 If

$\beta < \xi$

Then solution has converged.

ξ is tolerance specified e.g. 0.0010.

If not converged, increase sample "NS" and repeat from step-2.

Step-7 Obtain the function I_i as follows:

$I_i = 0$ if $t_i \le t_r$

$= 1$ if $t_i > t_r$

t_r is the duration for which reliability is required.

Step-8 Estimate reliability by using following relation:

$$\hat{R}(t_r) = \frac{1}{NS} \sum_{i=1}^{NS} I_i$$

Thus reliability may be calculated for various values of time t_r.

III. Non-parametric Bootstrap technique

The non-parametric bootstrap technique, randomly replace the position ofactual data yielding to several new samples of data [10, 11, 12, 13]. Theprocedure of non-parametric bootstrap technique commences with anindependent random sample. A smaller size of time to failure is obtainedusing MCS and assume that it is given as

$$[t_{f1}, t_{f2}, \ldots, t_{fNS'}] \tag{5}$$

Re-sampling with replacement (boot strapping) is adopted from (5) and 'NB' samples are obtained as follows.

$$\{t_{fB1}^{(i)}, t_{fB2}^{(i)}, \ldots, t_{fB,NS'}^{(i)}\} \tag{6}$$

$i = 1, \ldots\ldots\ldots NB$

(6) is set of time to failure as obtained from (5) after re-sampling and may have repeated time to failure from (5).Now for each re-sampled set of data (6) sample mean is calculated

$$\bar{t}_{fB}^{(i)} = \frac{1}{NS} \sum_{k=1}^{NS'} t_{fBK}^{(i)} \tag{7}$$

$i = 1, \ldots\ldots\ldots NB$

Over all estimated mean time to failure is calculated as follows

$$M\hat{T}TF_B = \frac{1}{NB}\sum_{i=1}^{NB} \bar{t}_{fB}^{(i)} \tag{8}$$

$M\hat{T}TF_B$ is the estimated mean time to failure obtained using boot strapping. Convergence may be checked using coefficient of variation.

Computational algorithm for evaluating mean time to failure (MTTF) using non-parametric bootstrap technique is given as follows:

Step-1 Obtain data set of time to failures using MCS as explained inprevious section. This is given by relation (5). Note that $NS' << NS$.

Step-2 Obtain re-sampled data set as given by relation (6).

Step-3 Calculate mean time to failure using relation (7) and (8).

Step-4 Obtain coefficient of variations for convergence

$$\beta = s / M\hat{T}TF_B$$

Where $s = \hat{\sigma} / \sqrt{NB}$

$$\hat{\sigma}^2 = \frac{1}{NB-1}\sum_{i=1}^{NB} (\bar{t}_{fBi}^{(i)} - M\hat{T}TF_B)^2$$

Step-5 If

$$\beta < \xi$$

Then solution has converged.

ξ is tolerance specified e.g. 0.0010

Step-6 In each resample calculate reliability for time

$$r_B^{(i)}(t_r) = \frac{n^{(i)}(t_r)}{NS}$$

$n^{(i)}(t_r)$ number of data is for which time to failure is greater than t_r

$i = 1,.......NB$

Step-7 Calculate average reliability

$$\bar{r}(t_r) = \frac{1}{NB}\sum_{i=1}^{NB} r_B^{(i)}(t_r)$$

Thus reliability $\bar{r}(t_r)$ is obtained for various time values.

IV. Result and discussion

The two techniques non-parametric bootstrap technique and MCS technique are used to evaluate MTTF and reliability for bridge network as shown in Fig.-1.

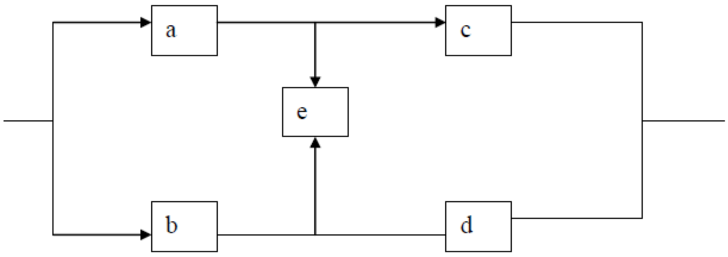

Fig-1 Complex network with five components a, b, c, d and e respectively.

Developed algorithms have been implemented by considering failure density function of components 'a' and 'b' exponentially, component 'c' and 'd' uniformly distributed and component 'e' Normally distributed. The failure rate for components 'a' and 'b' have been taken as 0.1 and 0.2 per year respectively. The range for the uniformly distributed components c and d are $0 \le t \le 5$ and $0 \le t \le 8$ respectively. For the component 'e' which is Normally distributed the mean value is 5 and the standard deviation is 1. Table-1 and Table-2 shows the values of MTTF obtained by MCS and nonparametric bootstrap technique respectively.

Table 1 MTTF as obtained using MCS and CPU time

Number of Samples, NS	MTTF, yrs.	CPU Time in seconds
101000	3.7955	0.117450

Table 2 MTTF as obtained using Boot strap technique and CPU time

Number of Samples, NS'	$MTTF_B$, yrs.	CPU Time in seconds
100	3.8390	0.047849

The error is 1.133 % using nonparametric bootstrap technique as compared to MCS. The CPU time required for convergence on Intel core 2 duo processor, 2.10 GHZ by using bootstrap technique have reduced by 59.26 %. The plot for coefficient of variation with respect to number of samples by using MCS and nonparametric bootstrap technique is shown in Fig.-2 and Fig.-3 respectively.

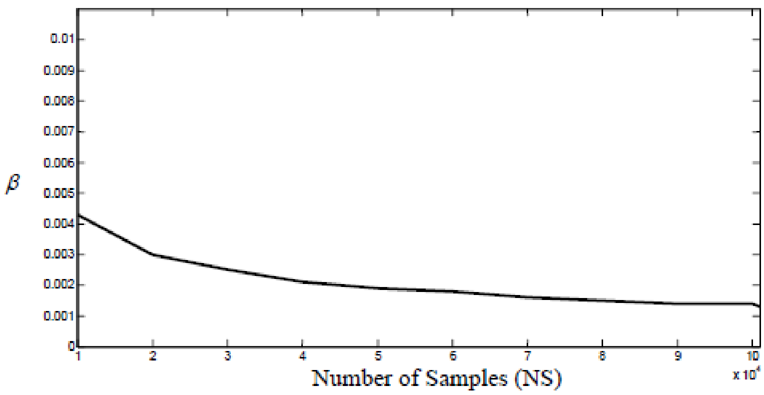

Fig-2 Variation of coefficient of variation with respect to number of samples in MCS

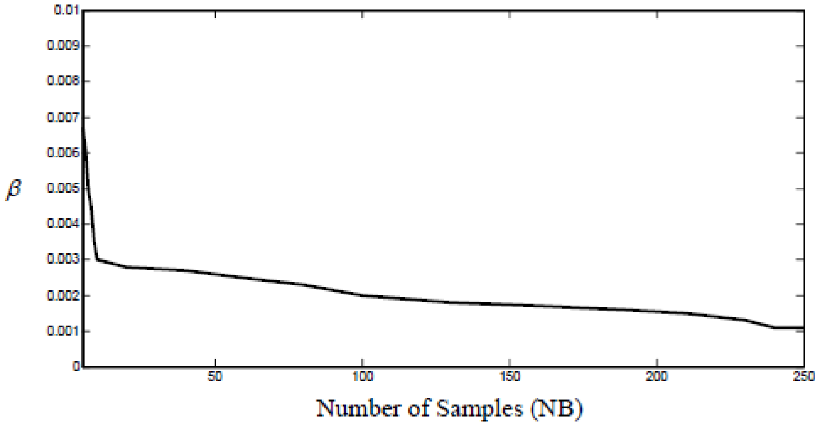

Fig-3 Variation of coefficient of variation with boot strap samples

The plot of reliability with respect to time by using MCS and bootstrap technique is shown in Fig.-4. Fig.-5 gives relative frequency distribution of mean time to failure $(\bar{t}_{fB}^{(i)})$ as obtained using boot strapping. It is observed that the shape is that of Normal distribution as is the case with boot strapping.

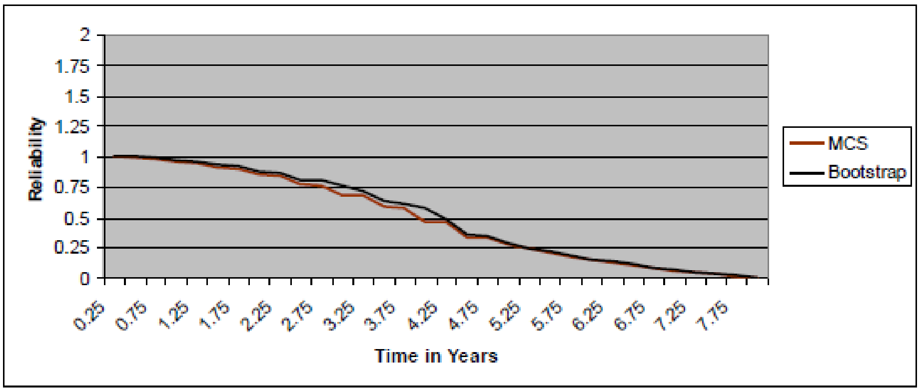

Fig-4 Plots showing variation of reliability with respect to time of operation using MCS and boot strap technique

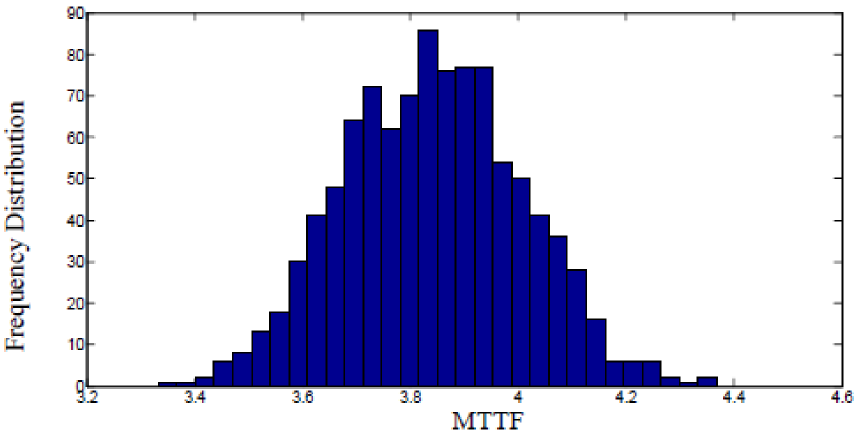

Fig-5 Frequency distribution of MTTF as obtained from non-parametric bootstrap technique

V. Conclusion

The paper has provided a comparison between MCS and enhanced sampling using bootstrap technique for calculating MTTF and reliability for the complex network by considering different distributions. The result obtained by two methods is in close agreement. The error introduced in MTTF using boot strapping is 1.133 % with respect to MTTF obtained using general MCS. Considerable reduction in CPU time is observed using bootstrap technique. The convergence has been visualized by the coefficient of variation.

References

1. R. Billinton, R. N. Allan. (2007). Reliability Evaluation of Engineering systems: Concepts and techniques. Springer International edition.
2. R. Billinton, R. N. Allan. (1996). Reliability evaluation of power system. Springer International Edition.
3 R. N. Allan, R. Billinton. (2000). Probabilistic assessment of power systems. Proc. IEEE, 88(2):140-162.
4 R. Billinton, M. S. Grover. (1975). Reliability evaluation in distribution and transmission system. Proc. IEEE, 122(5):517-524.
5 Volkanavski, M. Cepin, B. Mavko. (2009). Application of fault tree analysis for assessment of the power system reliability. Reliability engineering and system safety, 94(6):1116-1127.
6 E. Zio, L. Podofillini, V. Zille. (2006). A combination of Monte Carlo simulation and cellular automata for computing the availability of complex network system. Reliability Engineering and system safety, 91: 181-190.
7 P. Jirutitijaroen, C. singh. (2008). Comparison of simulation methods for power system reliability indexes and their distribution. IEEE Trans. power systems, 23(2):486-492.
8 R. Billinton, W. Wangdee. (2006). Delivery point reliability indices of a bulk electric system using sequential Monte Carlo simulation. IEEE Trans. on power Delivery, 21(1): 345-352.
9 R. Billinton, W. Li. (1993). A system state transition sampling method for composite system reliability evaluation. IEEE Trans. on power system, 8(3):761-770.
10 C. Y. Tsai, C. N. Lu. (2001). Bootstrap application in ATC estimation. IEEE Power Eng Rev, 21: 40-2.
11 M. M. Othaman, I. Musirin. (2001). A novel approach to determine transmission reliability margin using parametric bootstrap technique. Int. J. of Electrical Power and Energy System, 33:1666-1674.
12 H. C. Frey, D. E. Burmaster. (1999). Methods for characterizing variability and uncertainty: comparision of bootstrap simulation and likelihood based approaches. Int J Risk Anal, 19:109-30.
13 R. Wehrens, H. Putter, L. M. C. Buydens. (2000). The bootstrap: a tutorial. Chemom Intell Lab Syst, 54:35-52.
14 L. D. Arya, S. C. Choube, R. Arya, Aditya Tiwary. (2012). Evaluation of Reliability indices accounting omission of random repair time for distribution systems using Monte Carlo simulation. Int. J. of Electrical Power and Energy System, 42:533-541.
15 Jirutitijaroen P, singh C. (2008).Comparison of simulation methods for power system reliability indexes and their distribution. IEEE Trans Power Syst, 23:486–92.

16 Aditya Tiwary, R. Arya, S. C. Choube and L. D. Arya. (2013).Determination of reliability indices for distribution system using a state transition sampling technique accounting random down time omission. Journal of The Institution of Engineers (India): series B (Springer), 94:71-83.
17 Aditya Tiwary, R. Arya, L. D. Arya and S. C. Choube. (2017). Bootstrapping based technique for evaluating reliability indices of RBTS distribution system neglecting random down time. The IUP Journal of Electrical and Electronics Engineering, X: 48-57.
18 Volkanavski, Cepin M, Mavko B. (2009).Application of fault tree analysis for assessment of the power system reliability. Reliab Eng Syst Safety, 94:1116–27.
19 Li BM, Su CT, Shen CL. (2010).The impact of covered overhead conductors on distribution reliability and safety. Int J Electr Power Energy Syst, 32:281–9.
20 Aditya Tiwary. (2017). Reliability enhancement of distribution system using Teaching Learning based optimization considering customer and energy based indices. International Journal on Future Revolution in Computer Science & Communication Engineering, 3:58-62.
21 R. Arya, Aditya Tiwary, S. C. Choube and L. D. Arya. (2013). A smooth bootstrapping based technique for evaluating distribution system reliability indices neglecting random interruption duration. Int. J. of Electrical Power and Energy System, 51:307-310.
22 I. Sarantakos, D. M. Greenwood, J. Yi, S. R. Blake, P. C. Taylor. (2019). A method to include component condition and substation reliability into distribution system reconfiguration. Int. J. of Electrical Power and Energy System, 109:122-138.
23 N. R. Battu, A. R. Abhyankar, N. Senroy. (2019). Reliability Compliant Distribution System Planning Using Monte Carlo Simulation. Electric power components and systems, 47:985-997.
24 Aditya Tiwary. (2019). Reliability evaluation of radial distribution system – A case study. Int. J. of Reliability: Theory and Applications, 14, 4(55):9-13.
25 Michael Uspensky. (2019). Reliability assessment of the digital relay protection system. Int. J. of Reliability: Theory and Applications, 14:10-17.
26 M. P. Anand, B. Bagen, A. Rajapakse. (2020). Probabilistic reliability evaluation of distribution systems considering the spatial and temporal distribution of electric vehicles. Int. J. of Electrical Power & Energy Systems.117.
27 Yi Ren, B. Cui, Q. A. Feng, D. Yang, D. Fan, Bo Sun, M. Li. (2020).A reliability evaluation method for radial multi-microgrid systems considering distribution network transmission capacity. Computers & Industrial Engineering.139.
28 C. Sun, X. Wang, Y. Zheng, F. Zhang. (2020). A framework for dynamic prediction of reliability weaknesses in power transmission systems based on imbalanced data. Int. J. of Electrical Power & Energy Systems. 117.
29 T. K. Shrestha, R. Karki, P. Piya. (2020). Development of an operational adequacy evaluation framework for operational planning of bulk electric power systems. International Journal of Reliability, Quality and Safety Engineering.
30 Aditya Tiwary. (2020).Customer orientated indices and reliability evaluation of meshed power distribution system. Int. J. of Reliability: Theory and Applications, 15, 1(56):10-19.

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Contribution of Hardware, Software, and Traffic to the Wams Communication Network Availability

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Abstract

Modes of the power system can now be controlled by use of a Wide Area Measurement Systems (WAMS). It is based on the Phasor Measurement Unit (PMU), connected by an information network covering a significant territory. The hardware reliability of such a network is determined to a big extend by the reliability of storage media (optical fiber, radio waves, etc.) and by the devices that ensure their operation - Phasor Data Concentrator (PDC). The paper proposes an approach to determining the parameters of reliability on the example of a 10-bus power system. The ways to improve the hardware reliability of the information network are considered.

Key words: reliability, availability, WAMS communication network.

I Introduction

The need for a correct assessment of the power system state has led to the creation of a Wide Area Measurement Systems (WAMS). It is based on the measuring technology of phasors (phase vectors), on the Phasor Measurement Unit (PMU) by the signal of global navigation systems, which ensures simultaneous measurement of phasors [1]. WAMS includes measuring transformers, PMUs, Phasor Data Concentrators (PDCs) and equipment of the local information network. It allows controlling the behavior of power system by continuously observing system events. The reliability of WAMS the operation is determined by the monitoring system reliability of each element.

The paper considers the WAMS network structure, proposes the assessment of its reliability based on the network links. The network reliability includes four components, as follows:

1) Hardware or technical reliability associated with the failure (destruction) of the transmission channel elements or the integrity of communication lines;
2) Traffic reliability, determined by time loss or data corruption without element failure of channel transmissions;
3) Software reliability due to errors in the development of exchange execution programs; and
4) Resistance to external targeted influence on the transmitted information.

This paper considers first three components of the network reliability. The paper includes the example of an application for assessing the network availability of a 10-bus power system for the positions under consideration is given. Optical fiber or high frequency channels on power lines adopted in the example as carriers.

II The structure of the WAMS communication network

An important part of the WAMS are communication networks combining PMUs with PDCs. Concentrated PDC information then goes to the upper level of the WAMS to determine the actions of the automation or the dispatcher, depending on the modes and processes in the power system. Communication networks are divided:

• By hierarchy: the upper level is monitoring and control centers of the system, regional dispatch organizations; middle level is data concentrators and means of their delivery to the upper level; lower level is synchronous data collection in the buses of the system (fig. 1);

• According to the remoteness of the information sources and receivers: the lower level is the connection of the PMU with the PDC usually lies within hundreds of meters to kilometer, the upper one is the connection of the PDC with dispatch centers – from units to hundreds of kilometers;

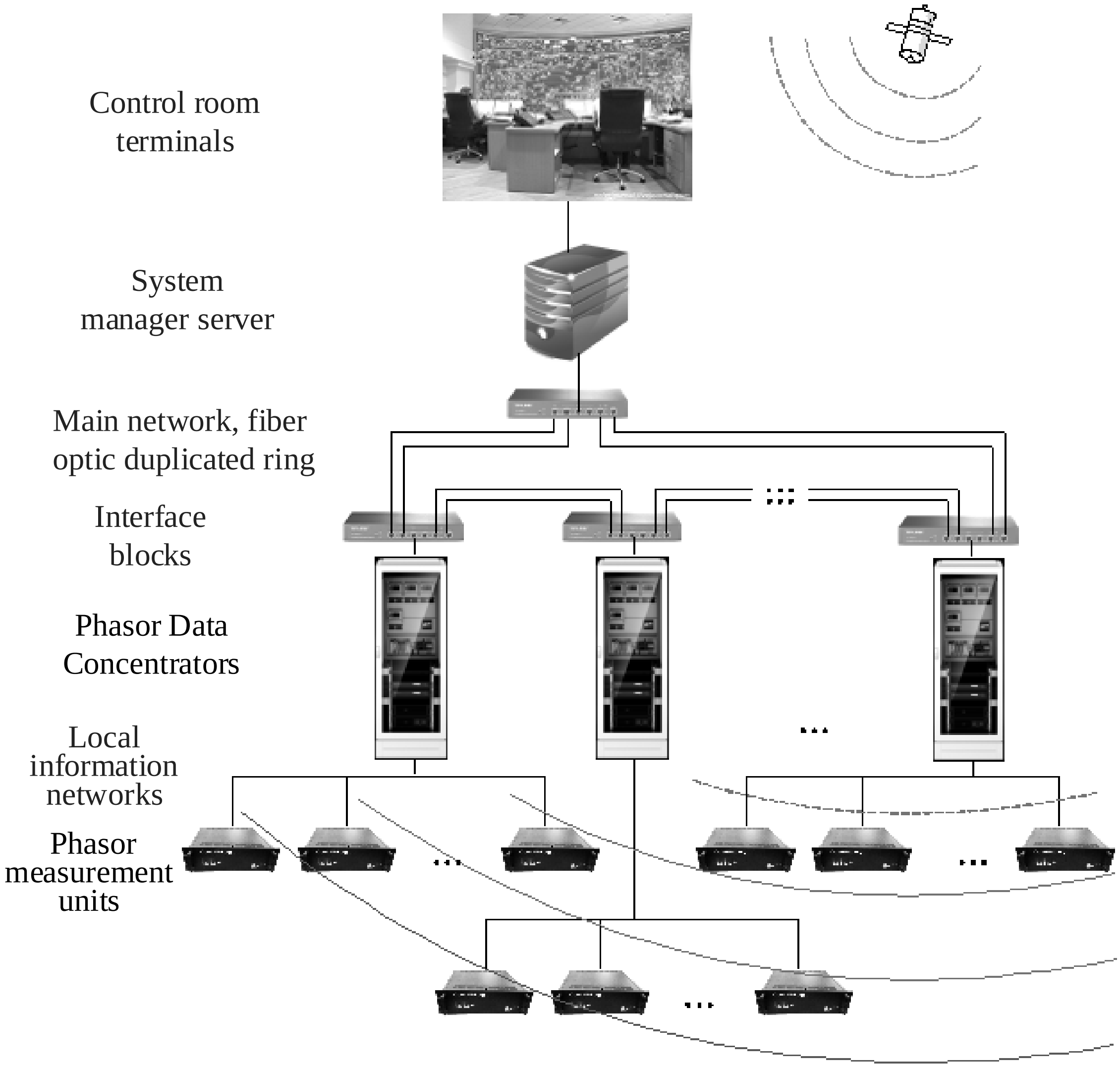


Fig. 1. Example of a hierarchical WAMS system.

• According to the network diagram: a star, a star with backup, a ring, a ring with backup, the first type of scheme being applied to the lower level of the hierarchy for linking PMU with PDC, since the distances and volumes of information here are relatively small. PMUs are installed with redundancy in such a way that when one of them comes out, its information can be restored using PMUs remaining in the work. If this is not possible, then the second or third scheme is used. A

backup ring scheme is usually used to communicate with the upper level;
• By type of information carrier: optical fiber, wire pair, microwave or high frequency radio waves. The choice of one or another carrier is determined by its distance, cost, and noise immunity;
• By affiliation: own and third party (commercial). An example of third-party media is the Internet.

From the point of view on the operation reliability of the WAMS network, four aspects should be considered:

1) The element reliability of electronic devices and the reliability of information carriers (wires, optical fiber, ether) in the sense of physical failure (change in the parameters of the medium under the influence of external factors, for example measures, open circuit) namely hardware reliability;

2) Software reliability due to errors in the development of exchange execution programs;

3) Traffic reliability, determined by the temporary loss or distortion of data without failure of the transmission channel element;

4) Opposition to an external targeted effect on the transmitted information.

III Determination of the network WAMS hardware reliability

The WAMS network hardware e is made up by the PDC electronics. Since the operation of the central processor units and the PDC communication interface during backup is similar to the operation of these elements in the PMU, we use the reliability estimate of these units in [2], which obtained from the system of Markov equations of state probabilities, taking into account different lengths of the main and backup communicate channels. Then the network channel availability, A_{ch}, consisting of a duplicated information source (PMU, PDC or, if necessary, an intermediate amplifier) and communication channel can be defined as

$$A_{ch} = A_{PDC} \cdot A_{com}, \tag{1}$$

where

$$A_{PDC} = \frac{\mu_{PDC}^2}{(\mu_{PDC}+\lambda_{PDC})^2}, \tag{2}$$

since PDCs are the same type, and

$$A_{com} = \frac{\mu_{lm} \cdot \mu_{lb}}{(\mu_{lm}+\lambda_{lm})(\mu_{lb}+\lambda_{lb})}. \tag{3}$$

Here A_{PDC} is the availability of the duplicated information source; λ_{PDC} and μ_{PDC} are the failure and recovery rates of the source, respectively. The physical availability of information carriers (twisted pair, optical fiber, high-frequency channel), each element of which is characterized by the length l_i, specific failure rate λ_{lm} for main and λ_{lb} for backup line and average recovery time r_{lm} for main and r_{lb} for backup line i per unit length. Since the reliability indicators of communication lines λ_l and r_l approximately linearly depend on the their length, and $\mu_l = 1/r_l$, then the working state probability of the information carrier element (availability) is easy to evaluate as

$$A_{ln,i} = \frac{\mu_{ln,i}}{(\mu_{ln,i}+\lambda_{ln,i})} = \frac{1/(r_{ln,i} \cdot l_i)}{1/(r_{ln,i} \cdot l_i)+\lambda_{ln,i} \cdot l_i} = \frac{1}{1+\lambda_{ln,i} \cdot r_{ln,i} \cdot l_i^2}. \tag{4}$$

It should be noted that $r_{ln,i}$ includes two components: the distance-searching violation variable, and the recovery-related constant. Since the second component has small values, we neglect it. Consequently, the availability of a communication line is inversely proportional to the square of its length. Unlike duplication in electronics, where the backup device usually repeats the basic one, duplication of storage media is most often provided by elements of various reliability indicators. This is due to the fact that in normal mode, communication is provided via the shortest line in the communication network, and in the case of the backup mode, information goes through the remaining in the communication network, which can be significantly longer than the main one. Moreover, the approach in solving such a problem is the same as when duplicating electronic units (2), only taking into account the different values of λ_j and μ_j for the j-th connection (3).

The algorithm for the calculation is as follows. After setting the initial data on the known link lengths of the information channels and the necessary reliability parameters, a table of the link participation in the formation of the main and backup channels is compiled (see the example of table 1 below). Further, the reliability characteristics of the links (λ_j, μ_j and A_j for the j-th link), the same parameters are determined for the main and backup channels of information exchange and the availability and channel characteristics with redundancy are calculated. At the same time, the initial data and the link participation table determine all changes in the network configuration. The estimated part remains unchanged.

IV An example of calculating the reliability of a WAMS network

Let us consider the described approach using the example of a 10-bus system considered in [3], fig. 2 and fig. 3. Without dwelling on the optimal composition of the PMU, assign them to each network bus and select the locations for PDC in nodes 4 and 9. We will determine the main and backup communication channel from the PMU of each node to its PDC, table 1 and fig. 3. In fig. 3a, such connections are without reservation, and in 3b, with back up. For communication lines on fiber, the specific indicators from the table 12.4 in [4] and the data from [5, 6] λ_l = 0.01752 failure /(km·year); r_l = 0.2088 hour/(km·recovery). Reliability indicators of electronic devices with their reservation are as follows: λ_{PMU} = 1.539·10^{-3} failure/year, μ_{PMU} = 5.922 recovery/year, A_{PMU} = 0.999740 [2], λ_{PDC} = 2.673·10^{-6} failure/year and μ_{PDC} = 740 recovery/year, A_{PDC} = 0.999999996 [2].

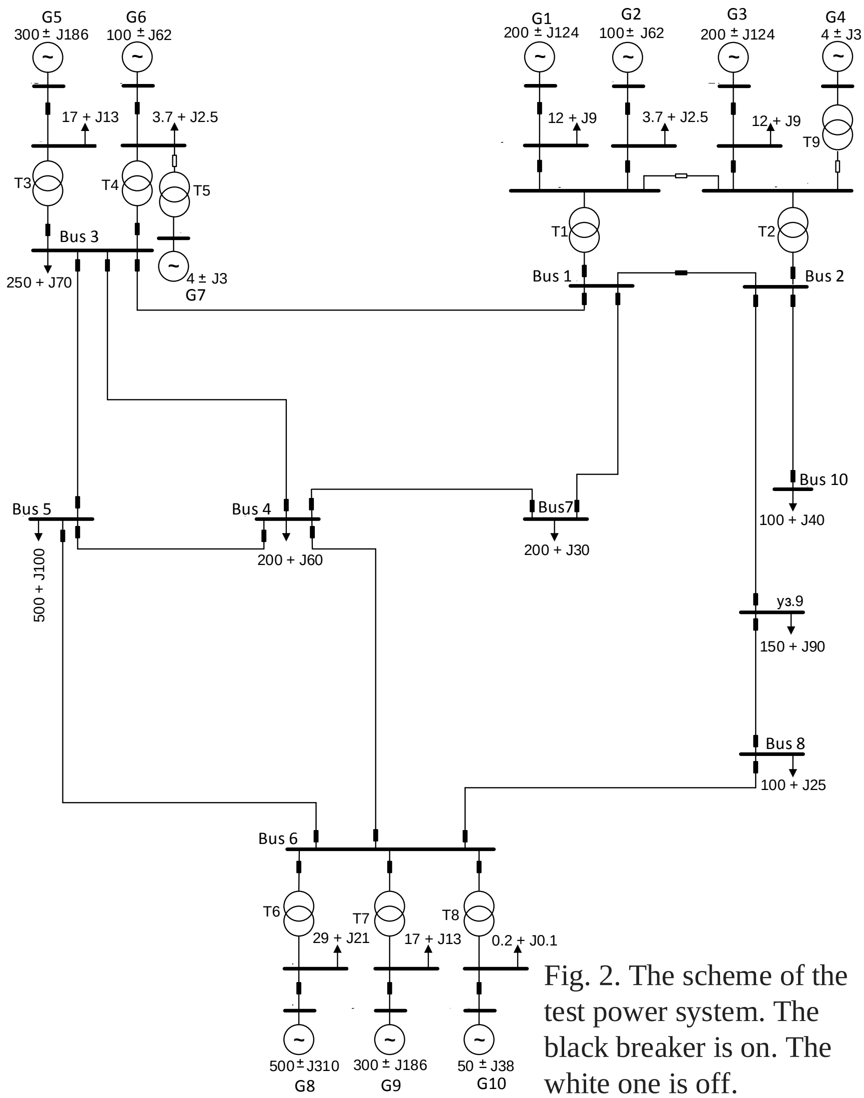


Fig. 2. The scheme of the test power system. The black breaker is on. The white one is off.

In the table 2, the link availabilities of the information exchange channel are determined, each of which includes an information source (PMU or PDC) and the actual fiber-optic connection, taking into account the fact $\mu_{con} = \frac{8760}{r_{con}}$ recovery/year. Then the availability of individual information channel is defined as the availability product of sequential links, which corresponds to a nonreserved channel, and the availability of the i-th channel with redundancy is defined as $A_{ch,i} = 1 - (1 - A_{m\_ch,i}) \cdot (1 - A_{b\_ch,i})$, (5)

where $A_{m\_ch,i}$ is i-th main channel connection availability, $A_{b\_ch,i}$ is i-th backup channel connection availability. The individual channel availabilities are given in the table 3 according to the connection table 1 and according of the channel availabilities with back up.

Let us agree to call a separate information line as "connection", a line with the source of information (PMU or PDC) - as "link", and a set of connections from the source node to the dispatch server - as "channel".

The table 3 shows that when using a single communication channel, its availability spread lies in the range from one to three nines after the decimal point. With reservation, the communication availability is maintained at the level of three nines after the point, even at a source sufficiently far from the control room, such as nodes 1 and 2.

Table 1. **Routes for the main and backup channels of information exchange**

Source node	Main channel	Backup channel
1	1-7-4	1-9-8-6-4
2	2-7-4	2-9-7-4
3	3-4	3-5-4
5	5-4	5-6-4
6	6-4	6-5-4
7	7-4	7-6-4
8	8-6-4	8-9-7-4
9	9-7-4	9-8-6-4
10	10-2-7-4	-

For communication lines using a high-frequency signal on power lines, specific indicators from the table 12.3 [4] λ_l = 0.0196 failure/(km·year); r_l = 0.19 hour/(km·recovery), the rest of the data is the same. Then the channel link availability and information exchange channels are given in table 4 and table 5, respectively.

As a comparison of tables 2, 3 and 4, 5 shows, the difference in availability between fiber and high frequency transmission is negligible. In the last table, as for optical fibers, only the main channel determines the availability of node 10.

Considering the sequential link inclusion of the main or backup information channel, as well as the parallel operation of these channels on the server, we determine the failure rate λ_Σ and the recovery rate μ_Σ for information exchange channels with optical fiber. Then, $\lambda_{i,\Sigma} = \sum_j \lambda_{i,j}$, where i is the main or backup information channel, j is the link element of this channel. Further, we determine $\mu_{i,\Sigma} = \frac{\lambda_{i,\Sigma} \cdot A_{ch,i}}{1 - A_{ch,i}}$ from the relation $A = \frac{\mu}{\mu + \lambda}$ and find $\mu_\Sigma = \sum_i \mu_{i,\Sigma}$ and $\lambda_\Sigma = \frac{\mu_\Sigma (1 - A_{ch,i})}{A_{ch,i}}$. The resulting λ_Σ and μ_Σ for fiber and power lines are summarized in table 6.

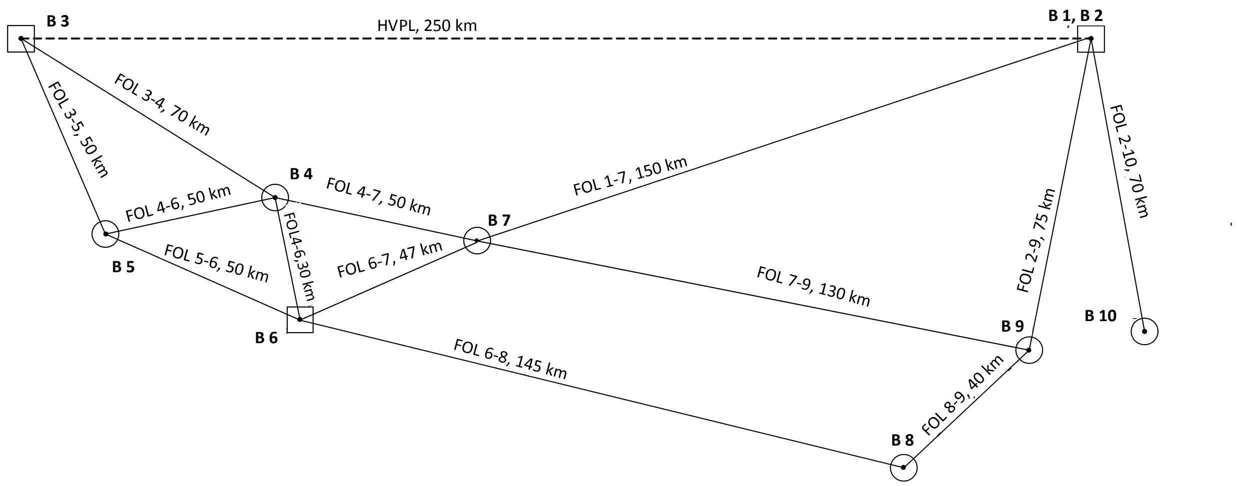
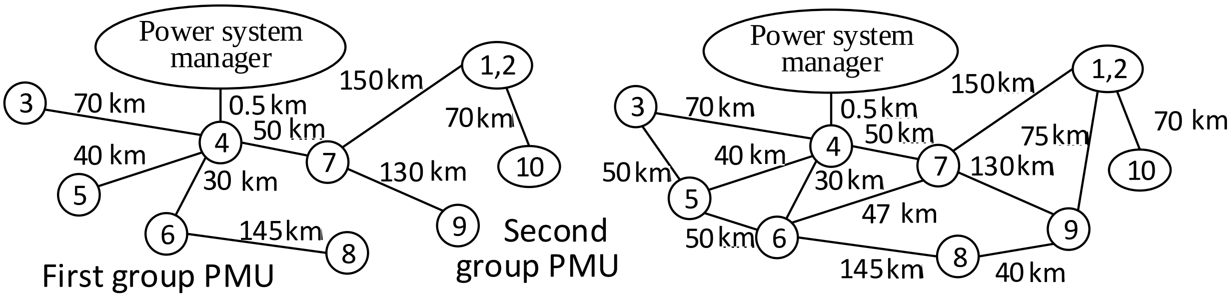


Fig. 3. The geographical location of the test power system. Scale 1 cm = 17 km. Rectangle nodes have generation. Circular nodes have only a load.



a) b)

Fig. 4. Communication channels: a) without redundancy, b) with redundancy.

Table 2. **Fiber optic link availability of the information exchange channel**

Link	l, km	$\lambda_{св.}$, failure/year	$r_{св.}$, h/recovery	A_{link}	Link	l, km	$\lambda_{св.}$, failure/year	$r_{св.}$, h/recovery	A_{link}
1-7	150.0	2.628	31.32	0.990433883	4-6	30.0	0.5256	6.264	0.999364399
1-9	75.0	1.314	15.66	0.997397114	4-7	50.0	0.876	10.44	0.99869736
2-7	150.0	2.628	31.32	0.990433883	5-6	50.0	0.876	10.44	0.99869736
2-9	75.0	1.314	15.66	0.997397114	6-7	47.0	0.82344	9.8136	0.998818611
2-10	70.0	1.2264	14.616	0.997698469	6-8	145.0	2.5404	30.276	0.991038641
3-4	70.0	1.2264	14.616	0.997698469	7-9	130.0	2.2776	27.144	0.99273384
3-5	50.0	0.876	10.44	0.99869736	8-9	40.0	0.7008	8.352	0.99907246
4-5	40.0	0.7008	8.352	0.99907246					

Table 3. **Fiber optic channel availability**

Bus #	$A_{main\ channel}$	$A_{backup\ channel}$	$A_{channel\ with\ backup}$	Bus #	$A_{main\ channel}$	$A_{backup\ channel}$	$A_{channel\ with\ backup}$
1	0.989400945	0.987684744	0.99986947	7	0.990433883	0.998443352	0.999985109
2	0.989400945	0.989374459	0.999887379	8	0.990666305	0.991036328	0.999916336
3	0.997698469	0.998030512	0.999995467	9	0.991698503	0.99000482	0.999917025
5	0.99907246	0.998322147	0.999998444	10	0.987380523	0	0.987380523
6	0.999364399	0.998030512	0.999998748				

Table 4. **Link availability of the information exchange channel for power lines**

Link	l, km	$\lambda_{св.}$, failure/year	$r_{св.}$, h/recovery	A_{link}	Link	l, km	$\lambda_{св.}$, failure/year	$r_{св.}$, h/recovery	A_{link}
1-7	150.0	2.94	28.5	0.990268019	4-6	30.0	0.588	5.7	0.999357643
1-9	75.0	1.47	14.25	0.997355058	4-7	50.0	0.98	9.5	0.998678619
2-7	150.0	2.94	28.5	0.990268019	5-6	50.0	0.98	9.5	0.998678619
2-9	75.0	1.47	14.25	0.997355058	6-7	47.0	0.9212	8.93	0.998802048
2-10	70.0	1.372	13.3	0.997661811	6-8	145.0	2.842	27.55	0.990883459
3-4	70.0	1.372	13.3	0.997661811	7-9	130.0	2.548	24.7	0.992608673

3-5	50.0	0.98	9.5	0.998678619	8-9	40.0	0.784	7.6	0.999060456
4-5	40.0	0.784	7.6	0.999060456					

Table 5. **Channel availability on power line**

Bus #	$A_{main\ channel}$	$A_{backup\ channel}$	A channel with backup	Bus #	$A_{main\ channel}$	$A_{backup\ channel}$	A channel with backup
1	0.98921669	0.987469907	0.999864884	7	0.99026802	0.998420046	0.999984624
2	0.98921669	0.989189439	0.999883426	8	0.99050449	0.990880875	0.999913409
3	0.99766181	0.997999793	0.999995323	9	0.99155486	0.989831215	0.999914123
5	0.99906046	0.998296664	0.9999984	10	0.98716037	0	0.98716037
6	0.99935764	0.997999793	0.999998715				

With a complex network of information connections, it can find a backup connection from the server node to the node with the failed connection, excluding the latter. To do this, we use the search algorithm first in depth and then in broadwise, as proposed in [7]. It allows taking to find a backup path into account the failed connections, if one exists, or to warn about its absence. When searching, the column "Reserve channel" is built in the table 1 and then the hardware reliability is evaluated for the found path.

These backup routes are stored in table 1 in order of decreasing a availability. A similar operation is performed in the process of network building. In real mode, if necessary, a backup channel with operational connections and highest availability is used.

Table 6. **Resulting values λ_Σ and μ_Σ of communication channels**

Bus source	Fiber optic channel		Power line channel	
	λ_Σ	μ_Σ	λ_Σ	μ_Σ
1	0.09589628	734.572164	0.109127394	807.5486398
2	0.083695895	743.08248	0.095243883	816.9324165
3	0.006031754	1330.67373	0.006854215	1465.551612
5	0.002472612	1588.79738	0.00280486	1752.642178
6	0.00203475	1625.44545	0.002306221	1794.935514
7	0.016936974	1137.37422	0.019264457	1252.864972
8	0.062884682	751.569245	0.071559069	826.33357
9	0.062221822	749.824533	0.070804518	824.418488
10	0.077536602	642.210362	0.088232547	706.0262268

V Traffic reliability

Traffic reliability is information transfer in a timely manner, without losses and associated with the exchange channel loading the distortions. Losses due to traffic are induced by an unacceptable delay or loss of some information due to an overload of the information channel, but are not associated with the failure of the device elements of this channel, which is taken into account in hardware reliability. Therefore, the traffic reliability is determined by the choice of bandwidth, taking into account the delay in the transmitted information.

The information frame from the generation unit or power line, formed by each PMU, combines 9 vector measurements: 3 currents and 3 voltages (magnitude and phase), 3 power (active and reactive components); 2 analog values: generator current and voltage; the state of the PMU device and the state of the switching elements. In addition, the transmission package includes the frequency and speed of its change, the time stamp and the binding for interaction with the information network in the standard C.37.118-2011. The structure of the data frame is given in table. 7.

Table 7 **C37.118-2011 frame structure**

Field	**Size**
Sync byte (SYNC)	2 bytes
Byte number of frame (FRAMESIZE)	2 bytes
Identifier PMU (IDCODE)	2 bytes
Second counting (SOC)	4 bytes
Fraction of a second/quality flag (FRACSEC)	4 bytes
Status flag (STAT)	2 bytes
Vectors (PHASORS)	$8 \cdot n$ bytes (floating point)
Frequency (FREQ)	4 bytes (floating point)
Frequency change rate (DFREQ)	4 bytes (floating point)
Analog data (ANALOG)	$8 \cdot m$ bytes (floating point)
Digital data (DIGITAL)	$2 \cdot l$ bytes (discrete values)
Cyclic Redundancy Check (CHK)	2 bytes

l – number of discrete information sources; m – number of analog information sources; n – synchronized vectors (magnitude and phase).

Then the amount of information from one PMU takes $b_{in} = 8 \cdot 9 + 2 \cdot 8 + 2 + 2 = 92$ bytes. The amount of information per frame of one node is $b_{fr} = 6 + 8 + 8 + 2 = 24$ bytes. Depending on the number of PMU - sources of measurement information, and transmitted measurements per second, the packet volume often have the range of 100 - 400 bytes. The approximate channel bandwidth in kbit/s, depending on the number of source devices and the sampling rate, taking into account a margin of 10%, is given in table 8, [8]. In this case, 1kbit = 1024 bits. Information delay is caused both with the type of the exchange channel and with the time of unloading its receive buffer. Packet delivery to a receiver requires time consuming T_d, which, in the general case, is determined by the propagation time of the signal T_{pc}, the time of transmission of the packet over the communication line T_{tp} and the waiting time of the packet in the queue in the communication unit T_{wp}

Table 8. **Required channel bandwidth, kbit/s**

Samples per second	Number of PMU's			
	2	10	40	100
25	50	249	997	2392
50	100	499	1994	4984
100	200	997	3988	9969

$$T_d = T_{pc} + T_{tp} + T_{wp}. \tag{6}$$

The propagation time, T_{pc}, of the signal in most communication systems is determined by the propagation time of the electric (electromagnetic field) or optical signal. The pulse delay in the optical fiber is (3.5–5) $\cdot\ l$ (ns) [6], and in the copper wire 5 $\cdot\ l$ (μs) [9], where l is the channel length in km.

The packet transmission time, T_{tp}, depends on the data transfer rate on the communication line ν_{tr} (kbit/s) and the volume or length of the packet L_p (kbit)

$$T_{tp} = L_p/\nu_{tr}. \tag{7}$$

Obviously, the propagation speed depends only on the channel material; therefore, the propagation time along the channel is constant. Transmission time depends only on the packet length.

The main task in designing a data transmission network is to ensure a balance between traffic (the flow of requests λ, in our case, the measurement frequency), the amount of network resources (bandwidth) and the quality of the service (service flow μ, parameter of request processing). In solving this problem, two levels of the open system interaction model (OSI) are considered: network and channel.

Network level. Traffic transit routes on the network are considered at the network level. For this, it is convenient to describe the communication network as the graph model [10] (in this case, non-oriented), in which the network nodes (routers) correspond to the graph vertices, and the communication lines to the graph arcs. The transmission time to the receiving node is the time that the packet spends on the network line. This time is to some extent random.

The load intensity on the arcs of the network graph ρ_{ij}, determined by the ratio of the request flow intensity the from the source node information i to the intensity of the service flow by the destination node j (λ_i/μ_j), depends on the number of devices and the amount of information from each device. In our case, the request flow rate is determined by the frequency of parameter measurement in the nodes of the power system: $\lambda = f_{msr.} = \frac{1}{T_{msr.}}$, service flow intensity – by the revers of the packet delivery time: $\mu = \frac{1}{T_d} = \frac{1}{L_p/v_{tr}+T_{pc}}$, and since this time in our case is shorter than the request period, $T_{wp} = 0$. On the other hand, the receiver electronics creates an additional delay, T_{re}, of about 5 μs on average. Then

$$\rho_{ij} = \frac{L_p/v_{tr}+T_{pc}+T_{re}}{T_{msr.}}. \tag{8}$$

Channel level. At this level, it is required to evaluate the necessary bandwidth of communication lines between network nodes. In the general case, an approximate formula can be used to estimate the probability of losses [11]

$$q_{ij} = \frac{1-\rho_{ij}}{1-\rho_{ij}^{N_j+1}}\rho_{ij}^{N_j}, \tag{9}$$

where N_j is the section number of the receiver accumulator j; ρ_{ij} is the load intensity of line ij.
The loss absence is defined as

$$p_{ij} = 1 - q_{ij}. \tag{10}$$

It is clear that such an assessment corresponds to one information line connecting two nodes. Taking into account the sequence of switching on the communication lines of two nodes passing through the intermediate nodes, the overall assessment of the probability of information loss is defined as

$$P_\Sigma = \prod_{ij} p_{ij}. \tag{11}$$

We estimate the WAMS information channels for the electric power system fig. 2, cited in detail in [3]. The scheme of information links with the distance scale is shown in fig. 3. Define the conditions and characteristics of the network. All information connections are made with fiber optic with propagation delay T_{pc} = 5 ns. Electronics delay is T_{re} = 5μs. Transmission rate is v_{tr} = 1Mbit/s = 1048576 bit/s [12]. The measurement transmission frequency is 10 Hz or $T_{msr.}$ = 0.1 s. Since the dispatch center is defined in four node of the system, the information routes in the normal and emergence mode for several information transmission line is given in table 1. The last column shows the connection of the source node with node 4 in case of failure for one of the link components bypass routes. Note that 10-2 communication failure leads to a complete loss of information exchange with node 10. The initial data for the calculations are summarized in table 9. b_{in} and b_{fr} are directly related to the corresponding line in the third and fourth columns, and Σb^{nr} and Σb^{em} are byte batches, including intermediate packets for the communication, both in normal mode and in emergency one, associated with the failure of one from the lines. N is determined by the maximum frames in normal mode and equals five.

Table 9. **Input data on the information network**

Link	l, km	b_{in}	b_{fr}	$\sum b_{in}^{nr}$	$\sum b_{fr}^{nr}$	$\sum b_{in}^{em}$	$\sum b_{fr}^{em}$
1-7	150	2	1	2	1	5	3
2-9	75	2	1	3	2	5	3
10-2	70	1	1	1	1	1	1
3-4	70	6	2	6	2	6	2
3-5	50	0	0	0	0	6	2
9-7	130	1	1	4	3	6	4
9-8	40	0	0	0	0	6	4
8-6	145	1	1	1	1	7	5
7-4	50	1	1	7	5	7	5
6-5	50	1	1	0	0	7	3

The simulation results are given in

6-4	30	6	2	7	3	13	7
5-4	40	1	1	1	1	8	4
7-6	50	6	2	0	0	7	5
2-7	150	0	0	0	0	5	3
1-9	75	0	0	0	0	5	3

tables 10 and 11 from which it can be seen that with the calculated loads, the probability of information loss is very low.

Let us consider the dependence of the information loss probability on the load intensity ρ by the example of a 7–4 connection under the remaining conditions. In the same example, we consider the influence of the number of sections drive *N*, tab. 12.

Table 10. **Loads and Probabilities of information loss for a separate link**

Link	$\rho_{ij}^{\text{н}}$	$q_{ij}^{\text{н}}$	ρ_{ij}^{a}	q_{ij}^{a}
1-7	0.01593	1.008E-09	0.04065	1.0643E-07
2-9	0.02477	9.099E-09	0.04064	1.0638E-07
10-2	0.00890	5.545E-11	0.00890	5.5455E-11
3-4	0.04583	1.929E-07	0.04580	1.9296E-07
3-5	–	–	0.04583	1.9289E-07
9-7	0.03363	4.154E-08	0.04949	2.8233E-07
9-8	–	–	0,04949	2.8221E-07
8-6	0.00891	5.557E-11	0.05835	6.3671E-07
7-4	0.05834	6.364E-07	0.05834	6.3645E-07
6-5	–	–	0,05468	4.6204E-07
6-4	0.05468	4.619E-07	0.10412	1.0961E-05
5-4	0.00890	5.541E-11	0.06353	9.6904E-07
7-6	–	–	0,05834	6.3644E-07
2-7	–	–	0,04065	1.0649E-07
1-9	–	–	0,04064	1.0638E-07

Table 11. **Probabilities of no loss for route information**

Route	$Q_{\Sigma\text{м}}^{nr}$	Route	$Q_{\Sigma\text{м}}^{em}$
1-7-4	6,37E-07	1-9-8-6-4	1,199E-05
2-9-7-4	6,87E-07	2-7-4	7,429E-07
3-4	1,93E-07	3-5-4	1,162E-06
5-4	5,54E-11	5-6-4	1,142E-05
6-4	4,62E-07	6-5-4	1,431E-06
7-4	6,36E-07	7-6-4	1,16E-05
8-6-4	4,62E-07	8-9-7-4	1,201E-06
9-7-4	6,78E-07	9-8-6-4	1,188E-05
10-2-7-4	6,37E-07	–	–

Table 12. **Influence of load intensity and number of sections on the probability of information loss q and error-free operation p for communication 7-4**

#	ρ	N	p	q	#	ρ	N	p	q
1		0	0	1	5		7	0.9998469	0.0001531
2		1	0.99009901	0.00990099	6	0.3	10	0.999995867	4.13344E-06
3		3	0.99999901	9.9E-07	7		100	1	0
4	0.01	5	1	9.9E-11	1		0	0	1
5		7	1	9.88098E-15	2		1	0.666666667	0.333333333
6		10	1	0	3		3	0.933333333	0.066666667
7		100	1	0	4		5	0.984126984	0.015873016
1		0	0	1	5	0.5	7	0.996078431	0.003921569
2		1	0.944874979	0.055125021	6		10	0.99951148	0.00048852
3		3	0.999813008	0.000186992	7		100	1	0
4	0.05834	5	0.999999364	6.36452E-07	1		0	0	1
5		7	0.999999998	2.16628E-09	2		1	0.588235294	0.411764706
6		10	1	4.30211E-13	3		3	0.864587446	0.135412554
7		100	1	0	4		5	0.942856074	0.057143926
1		0	0.000000000	1.000000000	5	0.7	7	0.973782312	0.026217688
2		1	0.909090909	0.090909090	6		10	0.991354799	0.008645201
3		3	0.999099909	0.000900090	7		100	1	1.11022E-16
4	0.1	5	0.999990999	0.000009000	1		0	0	1
5		7	0.999999909	0.000000090	2		1	0.500000025	0.499999975
6		10	0.999999999	9.00007E-11	3		3	0.750000038	0.249999962
7		100	1.000000000	0.000000000	4	0.9999999	5	0.833333375	0.166666625
1		0	0	1	5		7	0.875000044	0.124999956
2	0.3	1	0.769230769	0.230769231	6		10	0.909090955	0.090909045
3		3	0.98094566	0.01905434	7		100	0.990099059	0.009900941
4		5	0.998297759	0.001702241					

It is clear that at $N = 0$ the probability of information loss equals 1, because there is simply nowhere to take it. With increasing N, the q value drops rather steeply, turning almost to zero already at $N = 10$. It is also obvious that the greater is the load intensity ρ, the greater is the probability of information loss q, and the increase is quite fast, requiring an increase in the number of sections of the receiver drive N.

VI The network software availability

The failure of the software (SW) is associated with its inconsistency with the set tasks. There are many definitions of a software error. The most acceptable definition seems to be [13]: *Software reliability is probability that the program will work without failures for a certain period, taking into account the degree of their influence on the output results.*

The frequency occurrence of errors from the statistical data, reduced to 100% errors, is given in table 13, with the position "Incomplete or erroneous task" disclosed in more detail.

The software is not the subject to wear and tear and its reliability is determined only by development errors. Thus, over time, this indicator should increase if the correction of the detected errors does not introduce new errors.

For critical applications, which should include the WAMS SW, by the time the system is delivered to the client, it may contain from 4 to 15 errors per 100,000 lines of program code [4]. For clarity, we note that the code line number of WINDOWS XP more than 45 million, NASA - 40 million, Linux 4.11 kernel more than 18 million. When evaluating the WAMS program of 10 million lines of code, the number of errors at the beginning of operation of the program E = (V/100000) · 4 = 400 errors. Then, using the formula for the mean time between failures of the software, we get

$\lambda_{SW} = \beta \frac{E}{V} = 0.01 \frac{400}{10^7} = 4 \cdot 10^{-7}$ or $t_{SW} = \frac{1}{SW} = \frac{10^7}{4 \cdot 8760} \approx 285$ years,

Table 13. **Frequency occurrence of errors for certain types** [14]

Cause of error	**Fraction**, %
Deviation from the task	12
Neglecting programming rules	10
Incorrect data sampling	10
Erroneous logic or sequence of operations	12
Erroneous arithmetic operations	9
Lack of time to resolve	4
Incorrect interrupt handling	4
Invalid constants or source data	3
Inaccurate recording	8
Incomplete or erroneous task	28
⇓	
Errors in numeric values	12
Insufficient accuracy requirements	4
Erroneous characters or signs	2
Registration errors	15
Incorrect hardware description	2
Incomplete or inaccurate development basics	52
Ambiguity of requirements	13

where E is the number of errors per accepted program for operation, V is the program volume in lines of code, β is the program complexity coefficient, usually in the range 0.001 ... 0.01, λ_{SW} is the failure rate and t_{SW} is the mean time between failures of the software, 8760 is the number of hours per year. With a value of one error per 1000 code lines, accepted for application software after testing with the same amount of lines E = 10 000 errors

$$\lambda_{SW} = \beta \frac{E}{V} = 0.01 \frac{10000}{10^7} = 10^{-5} \text{ or } t_{SW} = \frac{1}{\lambda_{SW}} = \frac{10^5}{8760} \approx 11.4 \text{ years}$$

or about one failure in 12 years.

VII Conclusions

The correct functioning of the local information network in the WAMS is ensured by four components of operation reliability: hardware or technical reliability associated with the failure of transmission channel elements or the integrity of information transmission lines; software reliability due to errors in the development of exchange execution programs; and traffic reliability determined by temporary loss or distortion of data without failure of the elements of the transmission channel, and opposition to the external targeted influence on the transmitted information. The influence of the latter component is devoted to a number of works, for example, [15, 16], and is not considered by this paper.

Hardware reliability of such the network is largely determined by the reliability of the information carriers (optical fiber, radio waves, etc.) and the devices that ensure their work - concentrators of vector measurement data. The paper proposes an approach to determining the parameters of such reliability using the example of the 10-bus power system. So, with the proper organization of the backup, the hardware availability of the network, including information sources (PMU), exceeds three nines after the decimal point for optical fiber and is slightly less when exchanging via power lines. Ways of increasing the hardware reliability of the information network are considered.

Traffic reliability component is determined by the load intensity of each connection and the capabilities of receiving information, determined by the volume of the receiver's storage. It should be noted that the probability of information loss on the number of sections in the receiver's drive is rather strong. Their increase in a certain range makes it possible to compensate for the growth of this probability with increasing load intensity. The availability of the test network for traffic also exceeded three nines.

In the software plan, the effect of the code line volume on the value of this parameter is noted and its estimate is shown depending on the number of commands. An important property of this indicator is its improvement with increasing operating time. However, it can be adjusted due to the introduction of new errors in the correction of those identified in operation. So for the example of the WAMS program of 10 million code lines the mean time between failures should be 285 years.

References

1. Phadke A. G. and Thorp J. S. Synchronized Phasor Measurements and Their Applications. New York: Springer, 2008, 260 p.
2. Uspensky M.I. Functional reliability assessment of the phasor measurement unit // Relay protection and automation. 2017, No. 3, pp. 39-44. (In Russian).
3. Uspensky M.I., Smirnov S.O. The Occurrence Reasons and Countermeasures to Power System Blackouts // The International Journal of Energy Engineering, 2014, No. 1, pp. 1- 8.
4. Li W. Risk Assessment of Power Systems: Models, Methods, and Applications. New York: Wiley-IEEE Press, 2005, 325 p.
5. Reliability assessment of the designed FOCL (*fiber optic communication line*). [Electronic resource]. URL: http://www.icete.ru/pegibs-567-1.html. (Application date: 10.03.2019). (In Russian).
6. Tsukanov V.N., Yakovlev M.Ya. Fiber optic technology. Practical Guide. M.: Infra-Engineering, 2014, 304 p. (In Russian).
7. Uspensky M.I., Kyzrodev I.V. Combined Method of a Distribution Network Reconfiguration for Power Supply Restoration // Proceedings of the IEEE PowerTech, St. Petersburg, Russia, 27-30 June, ref. 33, 2005.
8. Real-Time Application of Synchrophasors for Improving Reliability 10/18/2010. [Electronic resource]. URL: http://www. naspi.org. (application date: 09.12.2019).
9. Davydov A.E., Smirnov P.I., Paramonov A.I. Design of telecommunication systems and networks. Calculation of communication network parameters and traffic analysis. St. Petersburg: University of IT-MO, 2016, 47 p. (In Russian).
10. Asanov M.O., Baransky V.A., Rasin V.V. Discrete mathematics: graphs, matroids, algorithms. M. PXD, 2001, 288 p. (In Russian).
11. N.B. Zeliger, O.S. Chugreev, G.G. Yanovsky. Designing networks and systems for transmitting discrete messages. M.: Radio and communications, 1984, 177 p. (In Russian).
12. Gordienko V. N. et al. Optical telecommunication systems. Textbook for high schools/ V. N. Gordienko, V. V. Krukhmalev, A. D. Mochenov, R. M. Sharafutdinov. M: Hotline – Telecom, 2011, 368 p. (In Russian).
13. Morozov Yu.M. Reliability of hardware and software systems. St. Petersburg, 2011, 136 p. (In Russian).
14. Shklyar V.N. Reliability of control systems. Tomsk: Publishing House of Tomsk Polytechnic University, 2009, 126 p. (In Russian).
15. Zhang Y. etc. Cyber Physical Security Analytics for Transactive Energy Systems / Y. Zhang, V. V. G. Krishnan, J. Pi, K. Kaur, A. Srivastava, A. Hahn and S. Suresh // IEEE Trans. on Smart Grid, vol. 11, no. 2, March, 2020, pp. 931-941.

16. Martel E., Kariger R., Graf P.-A. Cyber Resilience in the Electricity Ecosystem: Principles and Guidance for Boards / Center for Cybersecurity and Electricity Industry Community. January, 2019, 29 p. [Electronic resource]. URL: http://www3.weforum.org. (Application date: 03.03.2020).

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Length Biased Exponential Distribution as a Reliability Model: a Bayesian Approach

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Abstract

In this paper, mathematical properties of Length biased Exponential distribution via Bayesian approach are derived under various loss functions. These properties include Bayes estimators and posterior risks for the simulation study. The comparison was made based on the performance of the Bayes estimate for the parameter under different loss functions with respect to the posterior risk. Also, obtained the reliability characteristics of this distribution.

Keywords: Bayes estimator,Failure rate function, Length biased exponential distribution,Posterior risk, Reliability, Survival Analysis.

1 Introduction

Weighted distributions take into report the method of ascertainment, by modifying the probabilities of the actual occurrence of events to reach at a specification of the probabilities of those events as observed and recorded [4]. To introduce the concept of a weighted distribution, suppose that X is a random variable(r.v.) with its natural probability density function (PDF) $g(x|\theta)$, where the natural parameter $\theta \in \Theta$(Θ is the parameter space). A weighted distribution with kernel $g(x|\theta)$ and weight function $w(x,\beta)$ is defined as

$$f(x|\theta,\beta) = w(x,\beta).\, g(x|\theta)/C(\theta,\beta) \tag{1}$$

where, $w(x,\beta) > 0$ and $C(\theta,\beta) = E_f[\omega(X,\beta)]$. When X is a non-negative random variable and $w(x,\beta)$ = x, the resultant weighted distribution is known as length-biased distribution.

The study is developed as follows.In section 2 we consider the derivation of posterior distribution using non informative and informative priors.In Section 3 we derived Bayes estimators and respective posterior risks under various loss functions using different priors . A simulation study of Bayes estimators and their posterior risk is performed in Section 4. In section 5, we derived the characteristic property of LBE distribution. The reliability characteristics of the distribution is obtained in Section 6 and Finally, Section 7 deals with the conclusion.

2 Length biased Exponential distribution

A random variable X is said to possess a Length biased Exponential distribution if it has the following probability density function,

$$f(x|\theta) = \frac{x}{\theta^2} e^{-\frac{x}{\theta}}, x, \theta > 0 \tag{2}$$

and the cumulative distribution function(CDF) is,

$$F(x|\theta) = 1 - (1 + \frac{x}{\theta})e^{-\frac{x}{\theta}}, x, \theta > 0 \quad (3)$$

The likelihood function for a random sample $x_1, x_2, \ldots x_n$ which is taken from Length biased Exponential distribution is:

$$L(x, \theta) = \frac{\prod_{i=1}^{n} x_i}{\theta^{2n}} e^{-\frac{\sum_{i=1}^{n} x_i}{\theta}}, x, \theta > 0 \quad (4)$$

The posterior distribution consists of the probabilistic information about the parameters in the form of prior distribution and the sample information involved in the likelihood function. The likelihood principle propose that the information about the parameter will depend only on its posterior distribution. In this section, we will use the Length biased Exponential model as sampling distribution with Jeffreys Prior as a non informative prior and inverse gamma distribution as a conjugate prior for the derivation of the posterior distribution.

2.1 Posterior Distribution Using the Jeffreys Prior

The posterior distribution based on Jeffreys prior(JP) may be used as a standard or a reference for the class of posterior distributions which may be obtained from other priors. [3] proposed a formal rule for obtaining a non-informative prior as: If θ is a k-vector valued parameter, then JP of θ is: $p(\theta) \propto \sqrt{detI(\theta)}$ where $I(\theta)$ is a $k \times k$ Fishers (information) matrix whose $(i,j)^{th}$element is $-nE[\frac{\partial^2 log f(x)}{\partial\theta_i \partial\theta_j}]$ i, j = 1, 2,..., k. Fishers information matrix is not exactly related to the notation of lack of information. The relation comes from the role of Fishers matrix in asymptotic theory. Jeffreys non-informative priors based on Fishers information matrix often point to a family of improper priors. The Jeffreys prior of the parameter θ is:

$$I(\theta) = \frac{2n}{\theta^2}$$
$$p(\theta) = \frac{\sqrt{2n}}{\theta}$$
$$p(\theta) \propto \frac{1}{\theta}, \theta > 0$$

and the posterior distribution using Jeffreys Prior is

$$p(\theta|x) = \frac{(\sum_{i=1}^{n} x_i)^{2n}}{\Gamma(2n)} \frac{e^{\frac{-1}{\theta}\sum_{i=1}^{n} x_i}}{\theta^{2n+1}}, \theta > 0 \quad (5)$$

2.2 Posterior distribution using conjugate prior

If the selection of the prior is such that prior and posterior belong to the same family of distributions then prior is called conjugate prior(CP). For more discussion about conjugate priors see detail in [1]. Assuming the natural conjugate prior for θ to be inverse gamma distribution defined by

$$p(\theta) = \frac{a^b}{\Gamma(b)} \frac{e^{-\frac{a}{\theta}}}{\theta^{b+1}}, \theta > 0 \quad (6)$$

Posterior distribution using Inverse Gamma prior is

$$p(\theta|x) = \frac{(a+\sum_{i=1}^{n} x_i)^{b+2n}}{\Gamma(b+2n)} \frac{e^{\frac{-1}{\theta}(a+\sum_{i=1}^{n} x_i)}}{\theta^{b+2n+1}}, \theta > 0 \tag{7}$$

Lemma: For the given posterior 7, we have

1. $E(\theta|x) = \frac{(a+\sum_{i=1}^{n} x_i)}{b+2n-1}$
2. $E(\theta^2|x) = \frac{(a+\sum_{i=1}^{n} x_i)^2}{(b+2n-1)(b+2n-2)}$
3. $E(\theta^{-1}|x) = \frac{(b+2n)}{(a+\sum_{i=1}^{n} x_i)}$
4. $E(\theta^{-2}|x) = \frac{(b+2n)(b+2n+1)}{(a+\sum_{i=1}^{n} x_i)^2}$

Proof: The results can be simply derived from the definition

$$E(\theta^k|x) = \int_0^\infty \theta^k p(\theta|x) d\theta \tag{8}$$

On using Lemma, Bayes estimators of the parameter θ can be simply obtained.

3 Bayes estimators and Posterior risk under different loss functions

This section gives the derivation of the Bayes Estimator under different loss functions and their respective Posterior Risk. The results are compared for Jeffreys prior and conjugate prior. The Bayes estimators are determined under squared error loss function (SELF), weighted squared error loss function (WSELF), modified squared error loss function (MSELF), precautionary loss function (PLF) and K-Loss function. Table 1 will show the Bayes estimators and their posterior risks under various loss functions and Table 2 and Table 3 gives Bayes estimators of θ under different loss functions along with their Posterior risk using Jeffreys and conjugate prior respectively.

Table 1: Bayes Estimator and Posterior Risk under different Loss Functions.

Loss Function	Bayes Estimator	Posterior Risk
L_1=SELF= $(\theta - d)^2$	$E(\theta|x)$	$Var(\theta|x)$
L_2=WSELF= $\frac{(\theta-d)^2}{\theta}$	$(E(\theta^{-1}|x))^{-1}$	$E(\theta|x) - (E(\theta^{-1}|x))^{-1}$
L_3=MSELF= $(1-\frac{d}{\theta})^2$	$\frac{E(\theta^{-1}|x)}{E(\theta^{-2}|x)}$	$1 - \frac{E(\theta^{-1}|x)^2}{E(\theta^{-2}|x)}$
L_4=PLF= $\frac{(\theta-d)^2}{d}$	$\sqrt{E(\theta^2|x)}$	$2\left(\sqrt{E(\theta^2|x)} - E(\theta|x)\right)$
L_5=KLF= $(\sqrt{\frac{d}{\theta}} - \sqrt{\frac{\theta}{d}})^2$	$\sqrt{\frac{E(\theta|x)}{E(\theta^{-1}|x)}}$	$2\left(\sqrt{E(\theta|x)E(\theta^{-1}|x)} - 1\right)$

Table 2: Bayes Estimator and Posterior Risk under different Loss Functions using Jeffreys prior.

Loss Function	Bayes Estimator	Posterior Risk
L_1	$\frac{\sum_{i=1}^{n} x_i}{2n-1}$	$\frac{(\sum_{i=1}^{n} x_i)^2}{(2n-1)^2(2n-2)}$
L_2	$\frac{\sum_{i=1}^{n} x_i}{2n}$	$\frac{\sum_{i=1}^{n} x_i}{(2n-1)2n}$
L_3	$\frac{\sum_{i=1}^{n} x_i}{2n+1}$	$\frac{(\sum_{i=1}^{n} x_i+2n)(\sum_{i=1}^{n} x_i-2n)}{2n(2n+1)}$
L_4	$\frac{\sum_{i=1}^{n} x_i}{\sqrt{(2n-1)(2n-2)}}$	$2\sum_{i=1}^{n} x_i \left(\frac{\sqrt{2n-1}-\sqrt{2n-2}}{(2n-1)\sqrt{(2n-2)}}\right)$
L_5	$\frac{\sum_{i=1}^{n} x_i}{\sqrt{(2n-1)2n}}$	$2\left(\sqrt{\frac{2n}{2n-1}} - 1\right)$

Table 3: Bayes Estimator and Posterior Risk under different Loss Functions using Conjugate prior.

Loss Function	Bayes Estimator	Posterior Risk
L_1	$\frac{(a+\sum_{i=1}^{n} x_i)}{b+2n-1}$	$\frac{(a+\sum_{i=1}^{n} x_i)^2}{(b+2n-1)^2(b+2n-2)}$
L_2	$\frac{(a+\sum_{i=1}^{n} x_i)}{b+2n}$	$\frac{(a+\sum_{i=1}^{n} x_i)}{(b+2n-1)(b+2n)}$
L_3	$\frac{(a+\sum_{i=1}^{n} x_i)}{b+2n+1}$	$\frac{((a+\sum_{i=1}^{n} x_i)+(b+2n))((a+\sum_{i=1}^{n} x_i)-b-2n)}{(b+2n)(b+2n+1)}$
L_4	$\frac{(a+\sum_{i=1}^{n} x_i)}{\sqrt{(b+2n-1)(b+2n-2)}}$	$2\left(a+\sum_{i=1}^{n} x_i\right)\left(\frac{\sqrt{b+2n-1}-\sqrt{b+2n-2}}{(b+2n-1)\sqrt{(b+2n-2)}}\right)$
L_5	$\frac{a+\sum_{i=1}^{n} x_i}{\sqrt{(b+2n-1)(b+2n)}}$	$2\left(\sqrt{\frac{b+2n}{b+2n-1}}-1\right)$

4 Simulation of Bayes Estimates and Posterior Risk

The inverse cumulative distribution function (CDF) is commonly used for generating random variates . For arbitrary CDF, define $F^{-1}(u) = min\{x; F(x) \geq u\}$. The inverse CDF method can not be directly applied for Length biased Exponential distribution because closed form expression for its quantile function is not available. Here, we consider Newtons method for the calculation of the quantile function numerically. The following algorithm from [6] considered for this purpose:

Algorithm

1. Set n, θ and initial value x^0.
2. Generate U ~ Uniform(0, 1).
3. Update x^0 by using the Newtonυs formula, $x^{å}$=x^0 - R(x^0,θ) where R(x^0,θ) = $\frac{F(x^0;\theta)-U}{f(x^0;\theta)}$,

where, f(.) and F(.) are PDF and CDF of the Length biased Exponential respectively.

4. If $|x^0$ - $x^{å}| \leq \varepsilon$, (very small, $\varepsilon > 0$ tolerance limit), store x = $x^{å}$ as a sample from Length biased Exponential distribution.
5. If $|x^0 - x^{å}| > \varepsilon$ then, set $x^0 = x^{å}$ and go to step 3.
6. Repeat steps 2-5, n times for $x_1, x_2, \ldots, x_n$ respectively.

We use R-codes given in [5] for the algorithm. On the basis of simulated samples, we study the behaviour of Bayes estimators of θ are compared in terms of their average risks. We made such comparisons on the basis of ten thousand samples drawn from Length biased Exponential distributions. The performances of the Length biased Exponential parameter are studied for different sets of values of θ and n. Under conjugate prior, we consider a = 3 and b = 2 such that prior mean is 1. From the simulation results (Tables 4,45,456,4567,45678), we reach the following conclusions.

As we increase sample size posterior risk decreases and also with the increase of parameters values posterior risk also increases. Prior selection concern, the CP has smaller posterior risk than JP. The choice of loss function as concerned, one can easily observe that symmetric loss function has smaller posterior risk as compared to the asymmetric loss function. That is PLF has greater posterior risk as compared to other loss functions. Bayes estimator obtained under SELF over estimate the parameter while using Jeffreyυs prior for the parameter θ. Bayes estimator obtained under WSELF shows under estimation while assuming conjugate prior for the parameter θ. Bayes estimator obtained under MSELF under estimate the parameter in case of both Jeffreys and conjugate priors. Bayes estimator obtained under PLF over estimate the parameter while using Jeffreys prior for the parameter θ. Bayes estimator obtained under KLF over estimate the parameter in case of Jeffreys prior but under estimates in case of conjugate prior.

Table 4: Bayes Estimator and Posterior Risk under SELF.

θ	1	1.5	2	2.5	3
n	JP				
10	1.050839(0.057050)	1.577861(0.128166)	2.105983(0.240905)	2.621651(0.361315)	3.151374 (0.523698)
20	1.025913(0.026958)	1.537422(0.0617144)	2.047911(0.108105)	2.565001(0.168103)	3.075444 (0.245407)
30	1.016123(0.017494)	1.523956(0.038925)	2.035946(0.073026)	2.546169(0.112006)	3.046986(0.159117)
40	1.013658(0.012998)	1.516555(0.029366)	2.026467(0.052290)	2.531208(0.081310)	3.040588(0.118794)
50	1.009951(0.010188)	1.516402(0.023346)	2.021150(0.040501)	2.521196(0.064712)	3.037411(0.094524)
n	CP				
10	1.001784(0.041110)	1.452043(0.093017)	1.910218(0.172565)	2.364622(0.277175)	2.805224(0.411331)
20	0.9992899(0.022447)	1.476889(0.051386)	1.955403(0.091665)	2.424122(0.146191)	2.909636(0.212392)
30	1.000618(0.015602)	1.484401(0.036184)	1.970960(0.063898)	2.447151(0.099446)	2.937984(0.145412)
40	1.002039(0.012052)	1.491129(0.027227)	1.971900(0.047652)	2.462013(0.075992)	2.950605(0.108099)
50	0.9996826(0.009717)	1.489947(0.021964)	1.978285(0.038388)	2.472500(0.060986)	2.956754(0.088349)

Table 5: Bayes Estimator and Posterior Risk under WSELF.

θ	1	1.5	2	2.5	3
n	JP				
10	0.998860(0.050155)	1.500322(0.110952)	2.000823 (0.201593)	2.492451(0.304005)	2.996755(0.448970)
20	1.001486(0.025353)	1.499948(0.056375)	2.006655(0.101651)	2.49867(0.154002)	2.996174(0.225370)
30	1.000853(0.016561)	1.497054(0.036835)	1.998304(0.067076)	2.496211(0.104501)	2.999777(0.151519)
40	1.000404(0.012403)	1.49859(0.027182)	1.998168(0.050678)	2.504209(0.080532)	2.99885(0.115215)
50	1.001162(0.009948)	1.50144(0.022596)	1.996856(0.039692)	2.503607(0.063678)	3.000517(0.090971)
n	CP				
10	0.957114(0.039384)	1.396833(0.096399)	1.834240(0.181456)	2.263538(0.293588)	2.692223(0.439493)
20	0.974198(0.022048)	1.442250(0.051543)	1.908784(0.095777)	2.368487(0.153005)	2.831670(0.223166)
30	0.984746(0.015216)	1.458896(0.035855)	1.938499(0.065078)	2.409828(0.101718)	2.884782(0.151596)
40	0.988603(0.011672)	1.470956(0.027086)	1.94954(0.048403)	2.429858(0.076752)	2.911890(0.110910)
50	0.987665(0.011788)	1.475537(0.022217)	1.960522(0.038718)	2.446095(0.062097)	2.933175(0.091132)

Table 6: Bayes Estimator and Posterior Risk under MSELF.

θ	1	1.5	2	2.5	3
n	JP				
10	0.9538054(0.047443)	1.427201(0.106207)	1.912552(0.190068)	2.384794(0.298815)	2.852106(0.429349)
20	0.975606(0.024253)	1.45857(0.055761)	1.952824(0.096199)	2.441506(0.152615)	2.932368(0.221503)
30	0.983787(0.016673)	1.473163(0.036916)	1.969054(0.066145)	2.45936(0.102367)	2.953521(0.144153)
40	0.9877532(0.012546)	1.480203(0.027746)	1.971907(0.049682)	2.474515(0.076571)	2.967487(0.113820)
50	0.9905735(0.010151)	1.485611(0.022328)	1.981082(0.038633)	2.475351(0.061634)	2.973353(0.090230)
n	CP				
10	0.915634(0.042617)	1.336096(0.105570)	1.752156(0.198349)	2.165991(0.326690)	2.583195(0.484528)
20	0.952708(0.022514)	1.408673(0.054173)	1.862613(0.099694)	2.313020(0.1618012)	2.770039(0.237690)
30	0.970615(0.016079)	1.439304(0.036717)	1.906157(0.067063)	2.379814(0.105828)	2.835735(0.157739)
40	0.976914(0.011966)	1.434702(0.036722)	1.925851(0.050425)	2.398621(0.083745)	2.839411(0.156844)
50	0.981895(0.009633)	1.460042(0.022305)	1.943517(0.040534)	2.422901(0.063676)	2.905176(0.090337)

Table 7: Bayes Estimator and Posterior Risk under PLF.

θ	1	1.5	2	2.5	3
n	**JP**				
10	1.080143(0.064980)	1.626697(0.153575)	2.155806(0.254323)	2.700976(0.405576)	3.238162(0.066076)
20	1.040659(0.028565)	1.555354(0.063628)	2.083347(0.115160)	2.597052(0.174577)	3.116272(0.254988)
30	1.024338(0.018080)	1.541264(0.041020)	2.051707(0.071975)	2.562563(0.114311)	3.079783(0.162872)
40	1.019863(0.013347)	1.528745(0.030176)	2.038116(0.054966)	2.547378(0.083676)	3.052076(0.119659)
50	1.014274(0.010459)	1.524923(0.024347)	2.031945(0.043019)	2.537732(0.066076)	3.043194(0.094411)
n	**CP**				
10	1.026494(0.044506)	1.460923(0.094313)	1.913013(0.176025)	2.366208(0.274394)	2.825168(0.403514)
20	1.013772(0.024099)	1.478809(0.051262)	1.958041(0.093682)	2.432739(0.143970)	2.908967(0.212538)
30	1.008354(0.016062)	1.481696(0.034607)	1.966843(0.062606)	2.450746(0.098866)	2.936851(0.146527)
40	1.005892(0.011971)	1.489377(0.026776)	1.974448(0.048959)	2.46400(0.076449)	2.954996 (0.108458)
50	1.005354(0.009836)	1.491817(0.021753)	1.980016(0.038384)	2.470013(0.060978)	2.952238(0.089848)

Table 8: Bayes Estimator and Posterior Risk under KLF.

θ	1	1.5	2	2.5	3
n	**JP**				
10	1.025750(0.052648)	1.542011(0.122813)	2.051425(0.215322)	2.560207(0.333654)	3.088071(0.496273)
20	1.013489(0.025740)	1.518582(0.058053)	2.030651(0.104347)	2.535404(0.164987)	3.037618(0.232088)
30	1.010831(0.017366)	1.510478(0.037983)	2.018387(0.068729)	2.518961(0.103003)	3.023398(0.152086)
40	1.007564(0.012665)	1.51111(0.028126)	2.011791(0.049675)	2.511053(0.080001)	3.011605(0.113605)
50	1.005230(0.010113)	1.505956(0.023395)	2.012489(0.040545)	2.514406(0.062844)	3.014673(0.092787)
n	**CP**				
10	0.979370(0.038938)	1.417620(0.093463)	1.868669(0.177487)	2.309896(0.284355)	2.753182(0.415612)
20	0.987159(0.021805)	1.460179(0.051668)	1.938010(0.092811)	2.393906(0.148467)	2.870227(0.214058)
30	0.989702(0.015561)	1.474116(0.035223)	1.954692(0.062533)	2.433272(0.098115)	2.910851 (0.145990)
40	0.992784(0.011793)	1.480878(0.026994)	1.960131(0.047240)	2.450976(0.076274)	2.933232 (0.109091)
50	0.993924(0.009522)	1.483697(0.021450)	1.971755(0.038569)	2.454816(0.061290)	2.944691(0.088662)

5 Characterization Property

Result: The length biased exponential distribution and the distribution of $X_1 + X_2$ are the same if and only if X_1 and X_2 are independent and identically distributed exponential random variables with parameter θ.

Proof: *Necessary Part*

Suppose $X_1, X_2 \sim$ iid exponential (θ) with g(x | θ) = $\frac{1}{\theta} e^{\frac{-x}{\theta}}$, then $f(x|\theta) = \frac{xf(x|\theta)}{E(x)} = \frac{x}{\theta^2} e^{\frac{x}{\theta}}$, which is the pdf of Y = $X_1 + X_2$.

Sufficiency Part

Suppose that the Length biased distribution of X and Y= $X_1 + X_2$ are the same. Now the characteristic function of the length biased distribution can be obtained as follows. Suppose $\psi(t)$ is the characteristic function of X.

$$\psi(t) = \int_{-\infty}^{\infty} e^{itx} g(x) dx$$
$$\Rightarrow \psi^{`}(t) = i \int_{-\infty}^{\infty} e^{itx} x g(x) dx$$
$$\Rightarrow \frac{\psi^{`}(t)}{iE_g(x)} = \int_{-\infty}^{\infty} e^{itx} \frac{xg(x)}{E_g(x)} dx$$
$$\Rightarrow \psi(t) = \frac{\psi^{`}(t)}{i\mu} = \int_{-\infty}^{\infty} e^{itx} f(x) dx$$

provided g(x) is a density with support (0,∞).

$\Rightarrow \psi(t) = \frac{\psi^{`}(t)}{i\mu}$ is the characteristic function of the LBE distribution. Now under the assumption of the sufficiency part,

$$\psi(t) = [\psi(t)]^2$$
$$\Rightarrow \psi^{`}(t) = (i\mu)[\psi(t)]^2$$
$$\Rightarrow y^{`} - (i\mu)y^2 = 0, y = \psi(t)$$

On solving this differential equation , we get the solution as y = $(1 - i\mu t)^{-1}$. Hence the proof.

6 Reliability characteristics of Length Biased Exponential distribution

In this section, we consider Length Biased Exponential distribution as a lifetime model and study some reliability characteristics. The reliability function of the Length Biased Exponential distribution is given by,

$$R(t) = \bar{F}(t) = P(X > t) = 1 - F(t) = \int_t^{\infty} \frac{x}{\theta^2} e^{-\frac{x}{\theta}} dx = (1 + \frac{t}{\theta}) e^{\frac{-t}{\theta}}, \theta > 0 \quad (9)$$

The mean residual life function (MRLF) is given by,

$$\delta(x) = \frac{2\theta^2 + x\theta}{\theta + x}, x > 0 \quad (10)$$

The hazard rate function is given by,

$$h(x) = (\theta + \frac{\theta^2}{x})^{-1}, \theta > 0 \quad (11)$$

The cumulative hazard function is given by,

$$H(x) = -log(\bar{F}(x)) = -log(R(x)) = -log\left((1 + \frac{x}{\theta})e^{\frac{-x}{\theta}}\right), \theta, x > 0 \quad (12)$$

The conditional survival of t is given by,

$$R(x|t) = \frac{R(x+t)}{R(t)} = \left(1 + \frac{x}{\theta + t}\right) e^{\frac{-x}{\theta}}, \theta, x, t, R(.) > 0 \quad (13)$$

The failure rate average (FRA) is given by,

$$FRA(x) = \frac{H(x)}{x} = \frac{-log\left((1+\frac{x}{\theta})e^{\frac{-x}{\theta}}\right)}{x}, x > 0 \quad (14)$$

In this case on solving numerically, R(x|t) < R(t). Then by [2], we can conclude that the distribution of X belongs to the new better than used (NBU) classes. The reversed hazard function of LBE distribution is given by,

$$r(t) = \frac{te^{\frac{-t}{\theta}}}{\theta^2\left(1 - (1+\frac{t}{\theta})e^{\frac{-t}{\theta}}\right)}, \theta > 0 \quad (15)$$

It has been described that reversed hazard function or hazard rate function uniquely determines the corresponding probability density function. Figure 1 shows that reversed hazard function of LBE distribution with various values of parameter. This shape of reversed hazard function shows that h(t) of LBE distribution is increasing failure rate (IFR).

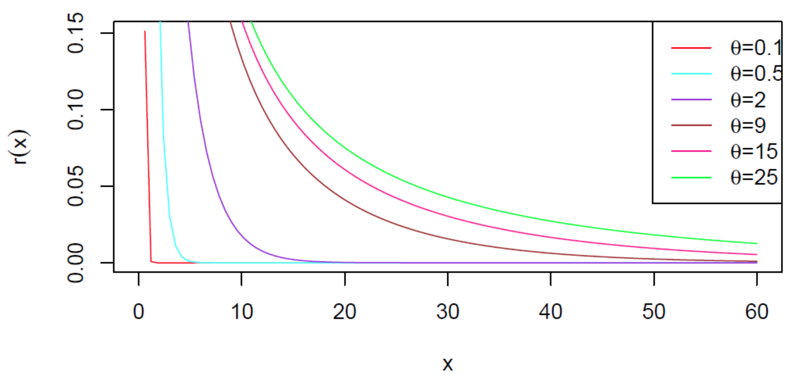

Figure 1: Reversed Hazard Function

7 Conclusion

We consider the Bayesian analysis of the Length biased Exponential model via informative and non informative prior under different loss functions. Based on posterior distribution, we conclude that informative prior (CP) has smaller posterior risk as compared to non informative prior (JP). The selection of loss function as concerned, one can easily observed that KLF is suitable than other asymmetrical loss functions. Also, we can conclude that when the sample size increases posterior risk decreases.

References

[1] Bansal, Ashok K. Bayesian parametric inference. Alpha Science International Limited, 2007.

[2] R.C. Gupta, R.D. Gupta(2007), Proportional reversed hazard model and its applications, *J. Stat. Plan. Infer.*, 137: 3525-3536.

[3] Jeffreys H. Theory of Probability, third ed., Oxford University Press, 1964.

[4] Patil, G.P. (2002). Weighted distributions. *Encyclopedia of Environmetrics* 4: 2369- 2377.

[5] Sharma, V.K., Singh, S.K., Singh, U., Merovci, F., (2016b). The generalized inverse Lindley distribution: A new inverse statistical model for the study of upside-down bathtub data. *Communications in Statistics-Theory and Methods* ,45:5709-5729.

[6] Sharma, V.K., Sanku D., Singh, S.K. and Manzoor U.(2018). On Length and Area biased Maxwell distributions. *Communications in Statistics - Simulation and Computation*,47:1506-1528

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Composite Disc Brake Design and Analysis

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Abstract

Brakes are the major components of the automobiles which are used to reduce the speed of the vehicle or to hold in the desired position by ceasing the motion of the vehicle. During braking, due to vehicle kinetic energy of the vehicle and the mechanical forces applied on the Disc leads to a temperature rise of the contact pairs of the discs results in heat dissipation. This project aims to select the best design of a Disc brake using the TOPSIS method and to develop a design of composite Disc and develop a methodology to check the durability under thermo-mechanical cyclic stresses. Initially, 5 models of the discs having various types of vents are selected and their durability was analyzed and the best Disc geometry was selected by using the ranking method, TOPSIS. The best-selected Disc design was analyzed with cast iron (Fe) and Titanium - cast iron metal matrix composite (Ti – Fe MMC) to sustain harsh working conditions. The Ti – Fe composites were prepared numerically by using MSC Digimat. The designed composite Disc brake is analyzed with existing material (Fe) and proposed Ti – MMC with Ti particle reinforcements in 5%, 10%, 15%, and 20% volume, for its sustainability using FEA and the results are compared and validated. The design of the composite brake is carried out in Solidworks Part Design and the thermo-mechanical analysis is carried out in ANSYS. From the results of thermal coupled with mechanical FE analysis (thermo-mechanical FEA) it is found that the Ti – Fe MMC 20% composite disc brake can work at temperatures up to 1050 K whereas the grey cast iron disc brake can only function up to 548 K.

Keywords: FEA, Disc brake, Composite, Ti- Fe MMC, Thermo-mechanical

I. Introduction

Among the automobile safety components, the braking system plays a vital role. The braking system deaccelerates the vehicle providing control over the speed of an automobile. It is also used to bring the driving speeds to the desired value. The kinetic energy of the wheel is converted into thermal energy through friction by using a friction braking system in which employs friction pads and rotating discs [1] Under the same operating conditions, Disc brakes have less wear compared to drum brakes. The linear relationship between the brake pad coefficient of friction and braking torque is the major advantage in the Disc brakes [2].

Disc brakes show prominent merits compared to drum brakes which made them a common braking system in light trucks and passenger cars. Brake friction materials employ high friction coefficient materials and are used in friction linings and brake pads. Braking materials should absorb a high amount of heat without showing any adverse braking performance [3]. Hence, the selection of braking material depends on the braking application. Thermal effects play a predominant role in the wear and tear of the braking systems. So, the thermo-structural analysis helps in finding out the thermo-mechanical behavior of the braking system.

The energy dissipated results in the temperature rise ranging from 200° C to 850° C which results in the micro – deformations due to frictional forces in the contact regions. As a result, when the vehicle travel at high speed the wear and tear of the rotating parts especially the brake rotors will be high resulting in a decrease in its life. This study aims at reducing the damage to Disc brake due to thermal and mechanical loads by using the Ti – Fe composite Disc brake.

II. Methodology

A. Disc Working Parameters

The four-wheeler vehicle weighing 855 kgs with weight transfer of 60:40 percentage on front and rear axle during braking is assumed. Table 1 represents the working parameters and their associated values used in this study which are imparted from the work of D Karan et al.

Tab.1-Disc Working Parameters

Parameter	Unit	Value
Average braking power	Watt	16531.96
Final breaking power	Nm/sec	1053.78
Heat flux	W/m^2	57217.8
Braking force	MPa	2.454
Braking torque	Nm	101.76
Convective heat transfer coefficient	W/m^2K	106.18

B. Ti-Fe MMC Properties

The mechanical properties for Ti – Fe MMC are obtained with the help of MSC Digimat. The analysis type used is Thermomechanical. Fe is in the matrix phase and Ti is in the inclusion phase (reinforcement). The mechanical properties are obtained for different volume fraction after simulations and are listed in Table 2.

Tab.2- Ti-Fe MMC Properties

Ti - Volume fraction (ϕ)	**Ti- Fe MMC Youngs Modulus (GPa)**	**Ti- Fe MMC Poisons Ratio**	**Ti- Fe MMC Shear Modulus (GPa)**	**Ti- Fe MMC Density (kg/m^3)**	**Ti- Fe MMC Yield Stress (Mpa) (σ_Y)**
0.05	81.506	0.25929	32.362	6850	154.45
0.1	83.037	0.26355	32.859	6700	157.9
0.15	84.594	0.26779	33.363	6550	161.35
0.20	86.179	0.27202	33.875	6400	164.8

The thermal properties for Ti – Fe MMC are obtained by using linear rule of mixtures prosed by Pac and Cho in the year 1981. Table 3 represents the thermal properties of the materials considered in thhis study. The linear rule of mixtures equations are as follows:

$$\text{Property}_{\text{Composite}} = (\phi)^* (\text{Property}_{\text{Reinforcement}}) + (1-\phi)^*(\text{Property}_{\text{Matrix}})$$

Thus,

$$k_{\text{Ti- Fe MMC}} = (\phi)^* (k_{\text{Ti}}) + (1-\phi)^*(k_{\text{Fe}}) \quad (1)$$

$$\text{CTE}_{\text{Ti – Fe MMC}} = (\phi)^* (\text{CTE}_{\text{Ti}}) + (1-\phi)^*(\text{CTE}_{\text{Fe}}) \quad (2)$$

Where, k = thermal conducticvity (W/mK)
CTE = coefficeint of thermal expansion (10^{-6}/K)
ϕ = volume fraction of reinforcement

Tab.3-Ti-Fe Properties

S. No	Material	ϕ	1-ϕ	k (W/mK)	CTE (10^{-6}/K)
1	Ti-Fe MMC 5%	0.05	0.95	38.85	10.895
2	Ti-Fe MMC 10%	0.1	0.9	37.7	10.79
3	Ti-Fe MMC 15%	0.15	0.85	36.55	10.68
4	Ti-Fe MMC 20%	0.20	0.80	35.4	10.58

C. Disc Brake Design

The design of brake discs was carried out in Solidworks part design. Five discs with various geometric features on the swept area were designed. The model I Disc is a solid Disc with a pitch circle diameter 100 mm and swept area of Disc pad is located at a distance of 58 mm from the Disc center. The Disc sept area is 18400 mm^2. Model II. III, IV, and V are made from Model I geometry. The five models are shown in fig 1 (a) to 1 (e).

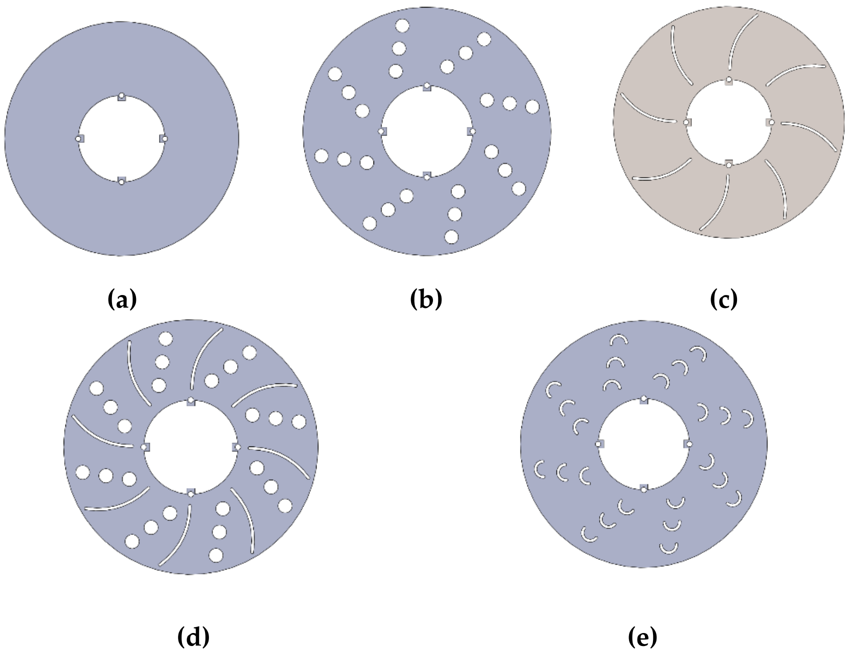


Fig 1: Schematic model of (a) Disc Model I (b) Disc Model II (c) Disc Model III (d) Disc Model IV (e) Disc Model V

D. FEA Analysis

To select the best geometric model from the five-disc models a thermo-mechanical analysis is carried out using the ANSYS workbench module under the working parameters represented in Table 1. Steady-state thermal analysis and thermo-mechanical fatigue analysis are done and the obtained results are taken to select the best by using TOPSIS methodology. The boundary conditions applied are as follows:

- Ambient temperature: 295.01 K
- The moment on the disc: 101.01 N-m
- Heat flux: 57216.621 W/m²
- Convection film coefficient: 106.181 W/m²K
- The cutoff limit for infinite cycles: 10^7 cycles

The thermo-mechanical analysis is a coupled steady-state thermal and static – structural analysis. These two interfaces must be coupled in ANSYS workbench as represented in Fig 2 for the selection of best geometry and represented in Fig 3 for the selection of best geometry. The mesh details are tabulated in Table 4. The analysis is carried out using the specified boundary conditions and with structural steel as a material.

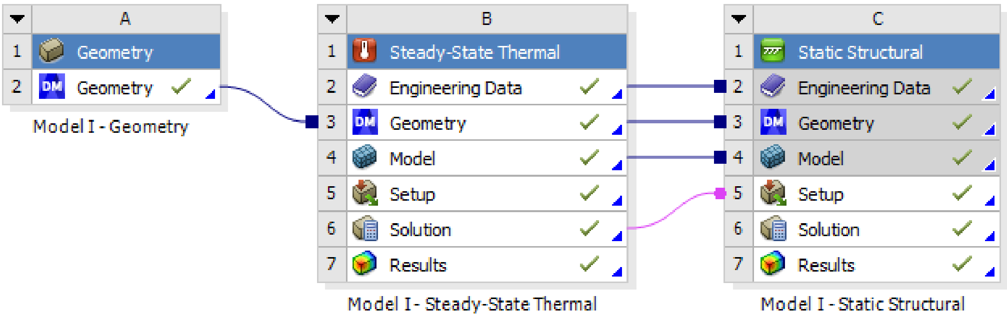


Fig 2: ANSYS Thermo-mechanical coupled FEA for selection of best geometry

Tab.4 – FEA Analysis

Parameter	**Value**
Elements	54564
Nodes	275785
Element size	2

E. TOPSIS Methodology

The best geometric model from the five-disc models and best Ti – Fe MMC composition is selected by using TOPSIS (Technique of Order Preference Similarity to the Ideal Solution) which is a straightforward MCDA (Multiple-criteria decision analysis) method. The method is based on finding an ideal and an anti-ideal solution and comparing the distance of each one of the alternatives to those. The method as follows:

Step-1: Calculate Normalised Matrix

$$\overline{X}_{ij} = \frac{X_{ij}}{\sqrt{\sum_{i=1}^{n} X_{ij}^2}}$$

Step-2: Calculate weighted Normalised Matrix

$$V_{ij} = \bar{X}_{ij} \times W_j$$

Step-3: Calculate the ideal best and ideal worst value

Step-4: Calculate the Euclidean distance from the ideal best

$$S_i^+ = \left[\sum_{j=1}^{m} \left(V_{ij} - V_j^+ \right)^2 \right]^{0.5}$$

Step-5: Calculate the Euclidean distance from the ideal worst

$$S_i^- = \left[\sum_{j=1}^{m} \left(V_{ij} - V_j^- \right)^2 \right]^{0.5}$$

Step-6: Calculate Performance Score

$$P_i = \frac{S_i^-}{S_i^+ + S_i^-}$$

III. Results

A. Geometry influence on the disc life

The thermo-mechanical FE analysis is conducted under the boundary conditions discussed in section II-D on the five models and the results obtained are used for the best geometric model by using the TOPSIS method. The results of thermomechanical FE analysis are tabulated in Table 5.

Tab.5 – Thermo-mechanical FE analysis

Model	**Temperature (k)**	**Total Deformation (mm)**	**Factor of Safety (NA)**	**Von Mises Stress (Mpa)**
I	581.69	0.26635	0.5918	222.9
II	599.7	0.27926	0.6757	221.2
III	574.64	0.23864	1.2781	144.6
IV	573	0.21745	1.2939	145.8
V	581.49	0.26448	0.9475	199.6

The steps involved in the TOPSIS are discussed in section II-E and the results are tabulated in Tables 6, 7, and 8. Table 6 shows the weightage and Table 7 shows the normalized matrix. Table 8 shows the remaining steps involved in the TOPSIS method and ranks.

Tab.6 - Weightage

Weightage	**Non-Beneficial**	**Beneficial**	**Beneficial**	**Beneficial**
	0.35	0.25	0.25	0.15
Parameter	Temperature	Total Deformation	Factor of Safety	Von Mises Stress

Tab.7 – Normalized matrix results

Model	**Temperature (k)**	**Total Deformation (mm)**	**Factor of Safety (NA)**	**Von Mises Stress (Mpa)**
I	0.446836357	0.468572033	0.26435	0.5244
II	0.460671085	0.491283747	0.30182	0.5206
III	0.441420764	0.419823653	0.57088	0.3403
IV	0.440160966	0.38254548	0.57794	0.3431
V	0.446682723	0.465282265	0.42322	0.4696

Tab.8- TOPSIS Results

Model	Temperature (k)	Total Deformation (mm)	Factor of Safety (NA)	Von Mises Stress (Mpa)	Si+	Si-	Pi	Rank
I	0.156392725	0.117143008	0.06609	0.0787	0.0786	0.0353	0.31	5
II	0.16123488	0.122820937	0.07546	0.0781	0.0694	0.0395	0.3625	4
III	0.154497267	0.104955913	0.14272	0.051	0.0329	0.0775	0.7017	1
IV	0.154056338	0.09563637	0.14448	0.0515	0.0385	0.0787	0.6719	2
V	0.156338953	0.116320566	0.1058	0.0704	0.0401	0.049	0.5499	3

By following the TOPSIS method the Model III has the Pi value of 0.7017 which is greater than all the remaining models. So, Model III disc geometry is considered in the design of the Ti- Fe MMC composite disc.

B. Influence of volume fraction of reinforcement on the disc life

The thermo-mechanical FE analysis is conducted on the brake disc Model III under the boundary conditions discussed in section II-D and by using the metal matrix composites discussed in Table 3 of section II-B and the results obtained are tabulated in Table 9.

Tab.9 – TOPSIS Results

S.No	Material	T (K)	Displacement (mm)	Strain (mm/mm) (ϵ)	Von Mises Stress (Mpa) (σ_V)
1	Grey Cast Iron	576.64	0.20321	$7.0959e^{-004}$	77.386
2	Ti-Fe MMC 5%	576.64	0.19952	$7.0129e^{-004}$	56.671
3	Ti-Fe MMC 10%	576.64	0.20424	$7.0978e^{-004}$	58.422
4	Ti-Fe MMC 15%	576.64	0.19712	$6.9097e^{-004}$	57.951
5	Ti-Fe MMC 20%	576.64	0.19531	$6.8381e^{-004}$	58.425

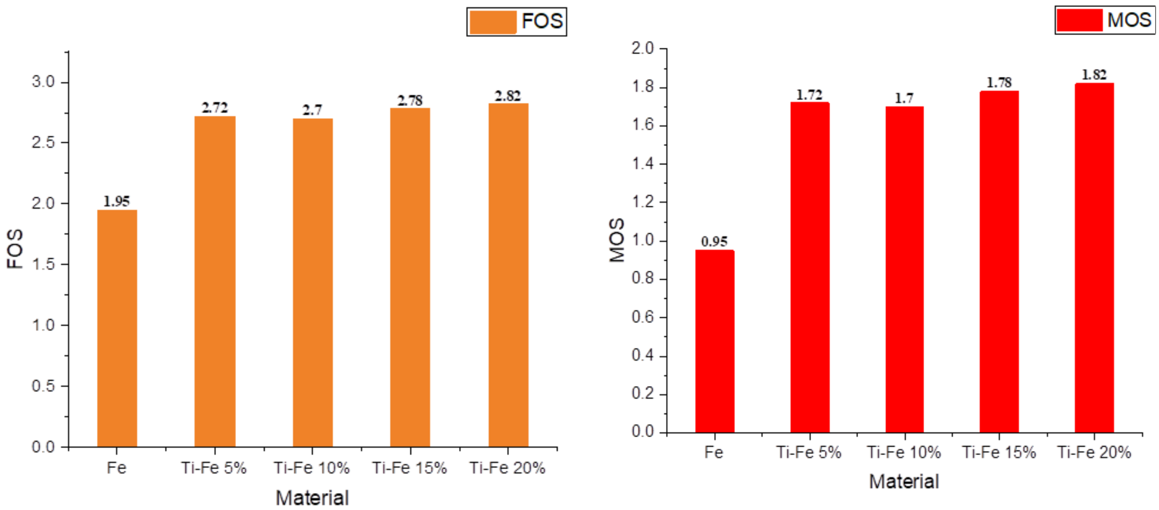
The factor of safety is calculated by using "Distortion energy theory with Von Mises Stress." The factor of safety and margin of safety is calculated by using the following equations and are tabulated in Table 10. The plots are shown in fig 4 and fig 5.

$$FOS = (\sigma_Y / \sigma_V)$$
$$MOS = FOS - 1$$

Where FOS = Factor of Safety
MOS = Margin of Safety
σ_Y = Material Yield Stress
σ_V =Von Mises Stress

Tab.10- Results of FOS & MOS

S.No	**Material**	**Yield Stress (Mpa) (σ_Y)**	**Factor of Safety FOS = (σ_Y/ σ_V)**	**Margin of safety MOS = FOS – 1**
1	Grey Cast Iron	151.1	1.95	0.95
2	Ti-Fe MMC 5%	154.45	2.72	1.72
3	Ti-Fe MMC 10%	157.9	2.70	1.70
4	Ti-Fe MMC 15%	161.35	2.78	1.78
5	Ti-Fe MMC 20%	164.8	2.82	1.82



(5) (6)

Fig: (5) FOS and **(6)** MOS results plot for materials specified in Table 10

The TOPSIS results are tabulated in Tables 10, 11, and 12. Table 11 shows the weightage and Table 12 shows the normalized matrix. Table 13 shows the remaining steps involved in the TOPSIS method and ranks.

Tabl.11- Weightage

Weightage	**Non-Beneficial**	**Beneficial**	**Beneficial**	**Beneficial**
	0.35	0.25	0.25	0.15
Parameter	Temperature	Margin of Safety	Factor of Safety	Displacement

Tab.12- TOPSIS Results

Material	**Temperature (k)**	**Margin of Safety (NA)**	**Factor of Safety (NA)**	**Displacement (mm)**
Grey Cast Iron	0.447213595	0.26116549	0.333581	0.45459735
Ti-Fe MMC 5%	0.447213595	0.47284698	0.465303	0.446342519
Ti-Fe MMC 10%	0.447213595	0.46734876	0.461881	0.456901544
Ti-Fe MMC 15%	0.447213595	0.48934165	0.475567	0.440973523
Ti-Fe MMC 20%	0.447213595	0.50033809	0.482409	0.436924405

Tab.13 – Ranking of TOPSIS

Material	Temperature (k)	MOS (NA)	FOS (NA)	Displacement (mm)	Si+	Si-	Pi	Rank
Grey Cast Iron	0.156524758	0.06529137	0.083395	0.068189603	0.07	0.003	0.036	**5**
Ti-Fe MMC 5%	0.156524758	0.11821175	0.116326	0.066951378	0.008	0.062	0.883	**3**
Ti-Fe MMC 10%	0.156524758	0.11683719	0.11547	0.068535232	0.01	0.061	0.862	**4**
Ti-Fe MMC 15%	0.156524758	0.12233541	0.118892	0.066146028	0.004	0.067	0.943	**2**
Ti-Fe MMC 20%	0.156524758	0.12508452	0.120602	0.065538661	0.003	0.07	0.959	**1**

Tab.14- Maximum Operating Temperatures for Grey Cast Iron and Ti-Fe MMC models

S.NO	Materials	MOS	Initial Temperature (Ti)	MOS*Initial Temperature
1	Grey Cast Iron	0.95	576.64	547.808
2	Ti-Fe MMC 5%	1.72	576.64	991.820
3	Ti-Fe MMC 10%	1.70	576.64	980.288
4	Ti-Fe MMC 15%	1.78	576.64	1026.419
5	Ti-Fe MMC 20%	1.82	576.64	1049.484

Tab.15- Temperature difference based on Margin Of Safety

S.NO	Materials	MOS	Initial Temperature	Grey Cast Iron Temperature	ΔT= (MOS* Initial Temperature)- Grey Cast Iron Temperature
1	Grey Cast Iron	0.95	576.64	576.64	-28.83
2	Ti-Fe MMC 5%	1.72	576.64	576.64	415.18
3	Ti-Fe MMC 10%	1.70	576.64	576.64	403.64
4	Ti-Fe MMC 15%	1.78	576.64	576.64	449.77
5	Ti-Fe MMC 20%	1.82	576.64	576.64	472.84

By following the TOPSIS method the Ti-Fe MMC 20% has Pi value of 0.959 which is greater than all the remaining composites. From this, we can state that the Ti-Fe MMC 20% can be used in the preparation of composite disc brakes. The post-processing images of the Model III disk with Ti-Fe MMC 20% composite are shown in fig 6, fig 7 and fig 8.

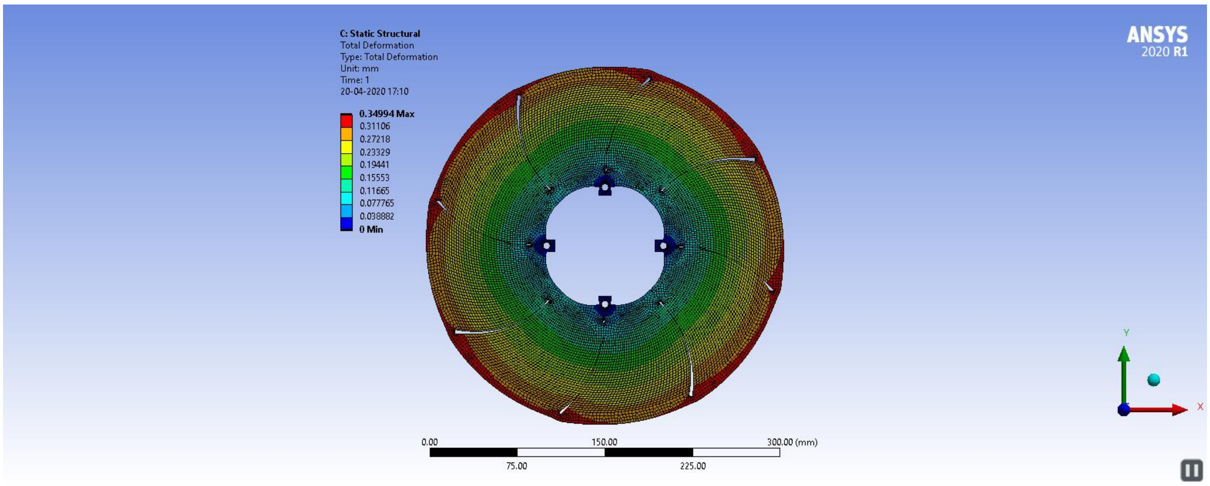


Fig 6: ANSYS Thermo-mechanical total deformation contour of Ti-Fe MMC 20% composite disc

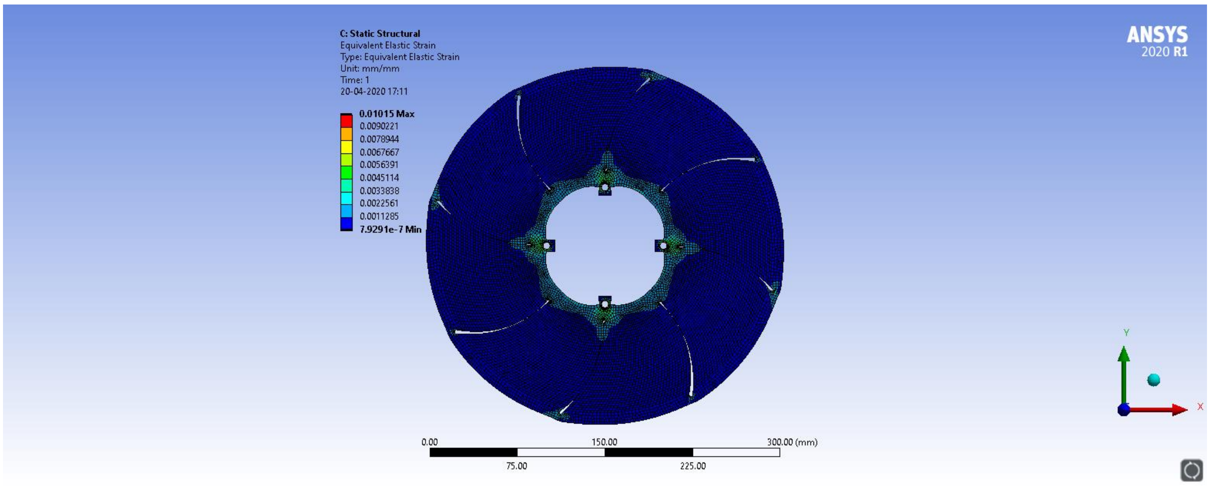


Fig 7: ANSYS Thermo-mechanical strain contour of Ti-Fe MMC 20% composite disc

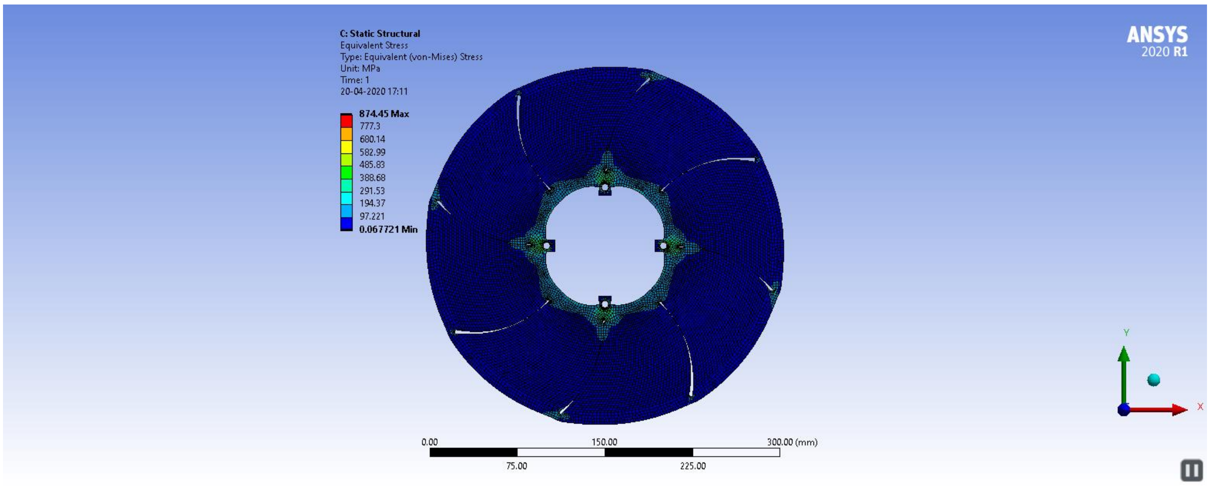


Fig 8: ANSYS Thermo-mechanical Von Mises stress contour of Ti-Fe MMC 20% composite disc

IV. Conclusions

From the studies, we can say the geometric design of the disc plays a major role in the thermomechanical characteristics. The disc with vents supports the rapid cooling of the brake rotor and thereby facilitating faster cooling and reducing the weight which makes it usable for racing applications. From the study following can be concluded:

1. From the thermo-mechanical analysis, it can be seen that by using grey cast iron in the disk fabrication resulted in FOS of 1.95. Though the factor of safety is greater than 1 which implies the disk support the design load, the MOS is 0.95 which is a measure of requirement verification states that it may fail at the sudden rise in loads.
2. While the results of the thermo-mechanical analysis on the Ti – Fe MMC discs shows that the composite disc with Ti-Fe MMC 20% resulted in FOS of 2.82 and MOS of 1.82 which states that the design can sustain even there is a 1.82 times sudden rise in both thermal and mechanical loads than the designed loads.

From this, it can be seen that the Ti – Fe MMC 20% composite disc brake can work at temperatures up to 1050 K whereas the grey cast iron disc brake can only function up to 548 K. Therefore the aim of the study to design a composite disc rotor which can sustain high thermal loads is met by using Ti – Fe MMC composite discs.

Future Scope: The percentage of Ti reinforcement in Ti – Fe MMC composite discs should be based on the operating temperatures of the brakes. And before utilizing the Ti – Fe MMC composite discs in practical applications for the specific model of disc experimentations should be carried out before its application which is left as future scope.

References

[1] Aleksander Yevtushkendo, Michal Kuciej, “Temperature and Thermal Stresses in a pad/disc during braking”, Elsevier, Applied Thermal Engineering 30 (2010) 354-359,2011, doi: 10.1016/j.applthermaleng.2009.09.015
[2] Ali Belhocine, Mostefa Bouchetara, “Thermal analysis of solid brake disc”, Elsevier, Applied Thermal Engineering 32 (2012)59-67,2012, doi: 10.1016/j.applthermaleng.2011.08.029
[3] Fred Puhn, “Brake Handbook” (HP Books, U.S.A., 1985), ISBN: 0-89586-232-8.
[4] F. Talati, S.Jalalifar, “Investigation of heat transfer phenomenon in a ventilated disc brake rotor with straight radial rounded vanes”, Journal of Applied Sciences, 8:3583-35922008, doi: 10.3923/jas.2008.3583.3592
[5] Guru Murthy Nathi, T N Charyulu, K. Gowtham, P. Satish Reddy, “Coupled Structural/ Thermal Analysis of Disc Brake”, IJERT: 539-553, 2012, doi.org/10.15623/ijret.2012.0104004
[6] Giancarlo Genta, Lorenzo Morello, “The Automotive Chassis (Volume 1), Springer, 2009”, ISBN: 978-1-4020-8674-8.
[7] Rudolf Limpert, “Brake Design and Safety” (S.A.E.Internationl, U.S.A., 1999), ISBN: 1-56091-915-9
[8] ANSYS (Version 15.0), ANSYS Meshing User’s Guide, Computer Software, ANSYS Inc., Houston, USA, 2015.
[9] M. Boniardi, F.D’Errico, C.Tagliabue, G.Gotti, G.Perricone, “Failure analysis of motorcycle brake disc”, Engineering Failure Analysis, 13 (2006) 933–945, doi:10.1016/j.engfailanal.2005.04.01.
[10] Daanvir Karan Dhir, Thermo-mechanical performance of automotive disc brakes, Materials Today: Proceedings, Volume 5, Issue 1, Part 1, 2018, Pages 1864-1871, ISSN 2214-7853, https://doi.org/10.1016/j.matpr.2017.11.287.

[11] Mahle Metal Leve S/A, Mahle International GmbH, "Cylinder Line and Cast-Iron Alloy", U.S. Patent 14/239,240, Oct 30, 2014.

[12] Julie A. Bannantine, Jess J. Corner, James L. Handrock, "Fundamentals of Metal Fatigue Analysis" (Prentince Hall, 1990), ISBN: 013340191X

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A Truncated Two Parameter Pranav Distribution and its Applications

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Abstract

In this paper, two parameter truncated Pranav distribution has been proposed. Some statistical properties including moments, coefficient of variation, skewness and index of dispersion have been derived and presented graphically. Survival and Hazard functions are derived and its behaviors are presented graphically. Maximum likelihood method of estimation has been used to estimate the parameters of proposed model. Simulation study has also been carried out. A proposed distribution has been applied on two data sets and compares its superiority over two parameter and one parameter classical distributions.

Keywords: Truncated, Pranav distribution, Lindley distribution, Skewness, Kurtosis

I. Introduction

In the recent past decades, life time modeling has been becoming popular in distribution theory, where many statisticians are involved in introducing new models. Some of the life time models are very popular and applied in biological, engineering and agricultural areas, such as Lindley distribution; Lindley (1958), weighted Lindley distribution; Gitany et al. (2008), Akash distribution; Shanker (2015), Ishita distribution; Shanker and Shukla (2017), Pranav distribution; Shukla (2018) etc and extension of above mentioned distribution has also been becoming popular in different areas of statistics.

Shukla (2018) has introduced a Pranav distribution which is mixture of exponential distribution having scale parameter θ and gamma distribution having shape parameter 4 and scale parameter θ, is defined by its pdf and cdf :

$$f_1(y;\theta)=\frac{\theta^4}{\theta^4+6}\left(\theta+y^3\right)e^{-\theta y}\quad ;y>0,\,\theta>0 \tag{1.1}$$

$$F_2(y;\theta)=1-\left[1+\frac{\theta y\left(\theta^2y^2+3\theta y+6\right)}{\theta^4+6}\right]e^{-\theta y}\quad ;y>0,\theta>0 \tag{1.2}$$

Shukla (2018) has discussed in details about its mathematical and statistical properties. Estimation of parameter using both the method of moment and the maximum likelihood estimation have mentioned in his paper and its application to model lifetime data from engineering and biomedical sciences.
Recently, Umeh and Ibenegbu (2019) have introduced extension of Pranav distribution and named as two parameter Pranav (TPPD) distribution and its pdf and cdf are defined as follows:

$$f_2(y;\theta)=\frac{\theta^4}{\alpha\theta^4+6}\left(\alpha\theta+y^3\right)e^{-\theta y}\quad ;y>0,\ \theta>0,\alpha\ge 0 \tag{1.3}$$

$$F_2(y;\theta,\alpha)=1-\left[1+\frac{\theta y\left(\theta^2y^2+3\theta y+6\right)}{(\alpha\theta^4+6)}\right]e^{-\theta y}\quad ;y>0,\theta>0,\alpha\ge 0 \tag{1.4}$$

Umeh and Ibenegbu (2019) have discussed in details about its mathematical and statistical properties. Estimation of parameter using both the method of moment and the maximum likelihood estimation including its application have mentioned in their paper.

Truncated type of distribution are more effective in application to modeling life time data because its limits used as bound either upper or lower or both according to given data. Truncated normal distribution is proposed by Johnson et al (1994). It has wide application in economics and statistics. Many researchers have been proposed truncated type of distribution and applied in different areas of statistics, especially in censor data such as truncated Wiebull distribution; Zange and Xie (2010) , truncated Lomax distribution; Aryuyuen and Bodhisuwan (2018), truncated Pareto distribution; Janinetti and Ferraro (2008), truncated Lindley distribution; Singh et al (2014). Truncated version of distribution can be defined as:

Definition1. Let X be a random variable that is distributed according to some pdf $g(x;\Theta..)$ and cdf $G(x;\Theta..)$, where Θ is a parameter vector of X.

Let X lies within the interval $[a,b]$ where $-\infty<a\le x\le b<\infty$ then the conditional on $a\le x\le b$ is distributed. We have the pdf of truncated distribution as reported by Singh et al (2014) defined by:

$$f(x;\Theta..)=g(x/a\le x\le b;\Theta..)=\frac{g(x;\Theta..)}{G(b;\Theta..)-G(a;\Theta..)}\frac{-b\pm\sqrt{b^2-4ac}}{2a} \tag{1.5}$$

where $f(x;\Theta..)=g(x;\Theta..)$ for all $a\le x\le b$ and $f(x;\Theta)=0$ elsewhere.
Notice that $f(x;\theta)$ in fact $f(x;\theta)$ is a pdf of X on interval $[a,b]$.

$$f(x;\Theta..)=\int_a^b f(x;\Theta..)dx=\frac{1}{G(b;\Theta..)-G(a;\Theta..)}\int_a^b g(x;\Theta..)dx$$

$$=\frac{1}{G(b;\Theta..)-G(a;\Theta..)}G(b;\Theta..)-G(a;\Theta..)=1 \tag{1.6}$$

The cdf of truncated distribution is given by

$$F(x;\Theta..)=\int_a^x f(x;\Theta..)dx=\frac{G(x;\Theta..)-G(a;\Theta..)}{G(b;\Theta..)-G(a;\Theta..)} \tag{1.7}$$

The main objectives of this paper are (i) to propose new truncated distribution using two parameter Pranav distribution, which is called as Truncated Two Parameter Pranav distribution (TTPPD) (ii) to know statistical mathematical properties and its suitability, it has been compared with classical distributions of two parameter as well as one parameter using two lifetime datasets. The study has been divided in eight sections. Introduction about the paper is described in the first section. In the second section, TTPPD has been derived. Mathematical and statistical properties including its moment have been discussed in third section. Behavior of hazards rate has been presented mathematically as well as graphically in fourth section. Moments and its related expression have been discussed in fifth section. Simulation study of the presented distribution has been discussed to check estimation parameters using Bias and Mean square error in sixth section.

Estimation of parameter of proposed distribution has been discussed in seven section where its applications and comparative study of other classical two parameter life time distributions as well as one parameter distributions have been illustrated using life time data. In the last, conclusions have been drawn according to studied of behavior and properties of TTPPD.

II. Truncated Two Parameter Pranav distribution

In this section, pdf and cdf of new truncated distribution is proposed and named Truncated two parameter Pranav distribution(TTPPD) , using (1.5) & (1.7) of definition1 and from (1.3) & (1.4) , which is defined as :

Definition 2: Let X be random variable which is distributed as TTPPD with scale parameter θ, location parameter a & b ,and shape parameter α , will be denoted by TTPPD (a,b,θ,α). The pdf and cdf of X are respectively:

$$f(x;\theta,\alpha)=\frac{\theta^4(x^3+\theta\alpha)e^{-\theta x}}{\left((\theta^3a^3+\alpha\theta^4+3a^2\theta^2+6a\theta+6)e^{-\theta a}-(\theta^3b^3+\alpha\theta^4+3b^2\theta^2+6b\theta+6)e^{-\theta b}\right)}$$

(1.8)

$$F(x;\theta,\alpha)=\frac{\left((\theta^3a^3+\alpha\theta^4+3a^2\theta^2+6a\theta+6)e^{-\theta a}-(\theta^3x^3+\alpha\theta^4+3x^2\theta^2+6x\theta+6)e^{-\theta x}\right)}{\left((\theta^3a^3+\alpha\theta^4+3a^2\theta^2+6a\theta+6)e^{-\theta a}-(\theta^3b^3+\alpha\theta^4+3b^2\theta^2+6b\theta+6)e^{-\theta b}\right)}$$

(1.9)

Where $-\infty<a\le x\le b<\infty$, and $\theta>0,\alpha>0$

Following conditions can be categories:

(i) When $a=0$ and $b=\infty$ it reduces to two parameter Pranav distribution.
(i) When $a=0$ it is known as right truncated two parameter Pranav distribution.
(ii) When $b=\infty$ it is known as left truncated two parameter Pranav distribution.
(iii) When and it reduces to two parameter Pranav distribution.
(iv) When $a=0$, $b=\infty$ and $\alpha=1$ it reduces to Pranav distribution.

Performance of pdf of TTPPD for varying values of parameter has been illustrated in the fig.1. From the figure1, it is clearly indicates that parameters a & b are the location parameter, θ scale parameter and α is the shape parameter, and value of pdf is decreasing as increased value of θ when ($\theta<1$) at fixed values of a, b and α , whereas pdf is increasing as increased value of θ when ($\theta>1$) at fixed values of a, b and α .

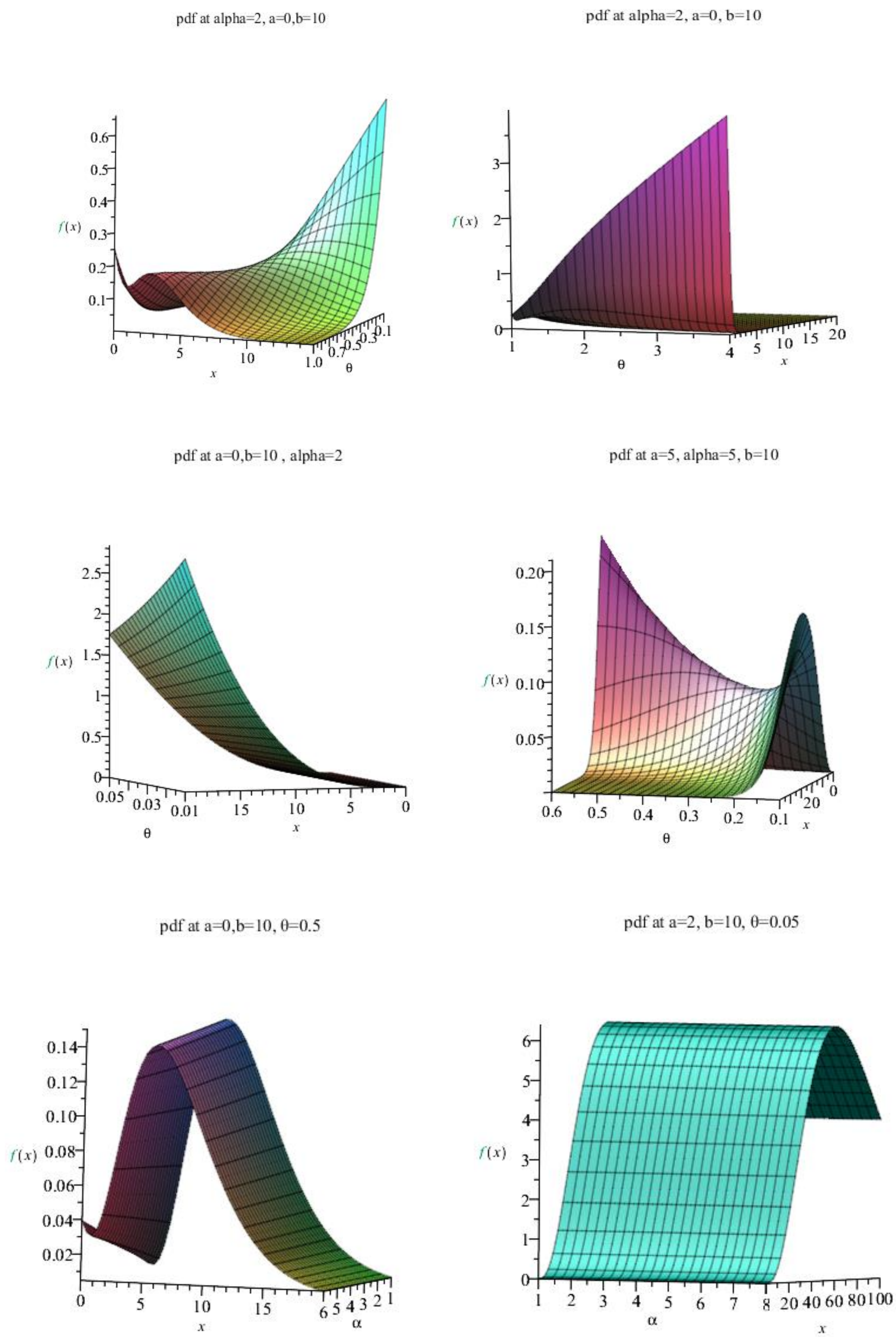

Figure 1. *pdf plots of TTPPD for varying values of parameters*

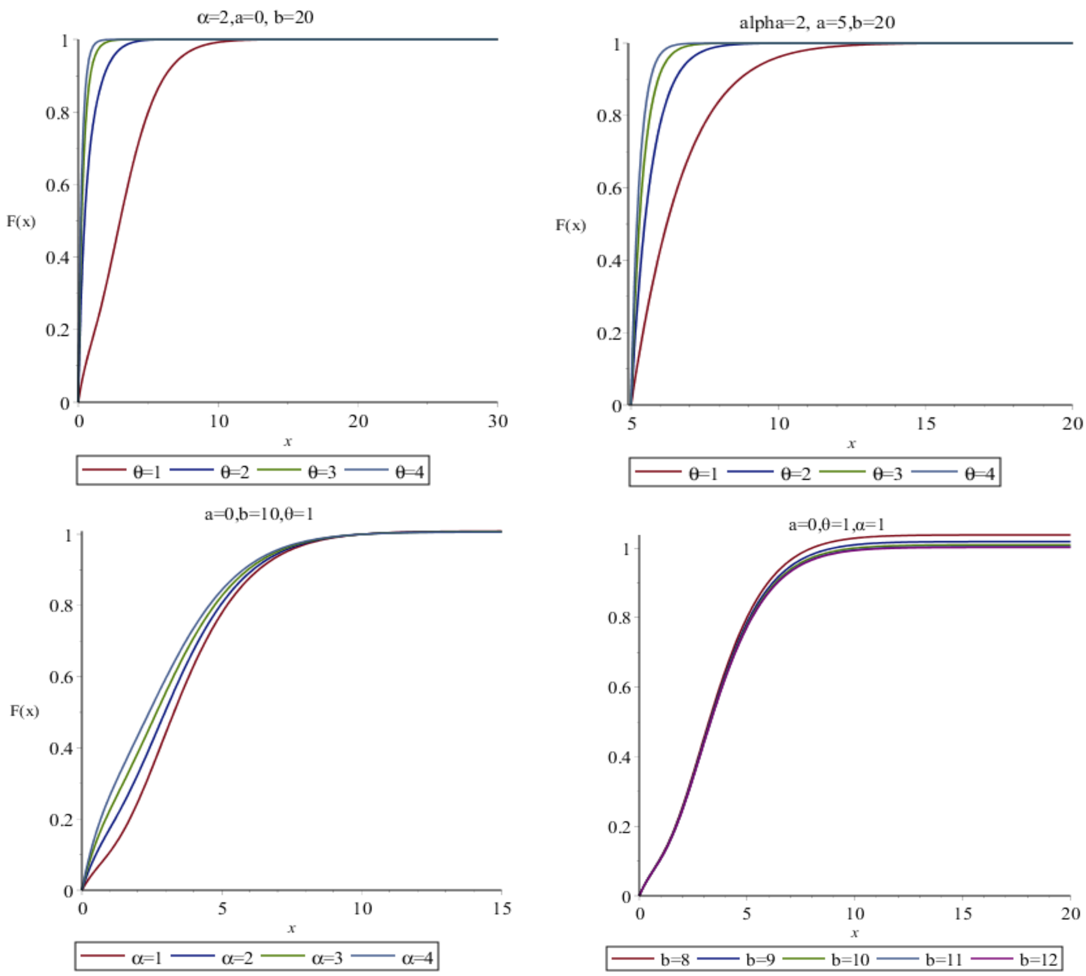

Figure 2. *cdf plots of TTPPD for varying values of parameter*

III. Survival and Hazard function

$S(x)$ and $h(x)$ are the survival function and hazard function respectively, which are defined as:

$$S(x) = 1 - F(x)$$

$$S(x;\theta,\alpha) = \frac{\left((\theta^3 x^3 + \alpha\theta^4 + 3x^2\theta^2 + 6x\theta + 6)e^{-\theta x} - (\theta^3 b^3 + \alpha\theta^4 + 3b^2\theta^2 + 6b\theta + 6)e^{-\theta b}\right)}{\left((\theta^3 a^3 + \alpha\theta^4 + 3a^2\theta^2 + 6a\theta + 6)e^{-\theta a} - (\theta^3 b^3 + \alpha\theta^4 + 3b^2\theta^2 + 6b\theta + 6)e^{-\theta b}\right)}$$

$$h(x) = \frac{f(x)}{S(x)}$$

$$h(x) = \frac{\theta^4 (x^3 + \theta\alpha)e^{-\theta x}}{\left((\theta^3 x^3 + \alpha\theta^4 + 3x^2\theta^2 + 6x\theta + 6)e^{-\theta x} - (\theta^3 b^3 + \alpha\theta^4 + 3b^2\theta^2 + 6b\theta + 6)e^{-\theta b}\right)}$$

From the equation, It is notice that $h(x)$ is independent of parameter 'a'

Behavior of hazard function of TTPPD for varying values of parameter is presented in figure3:

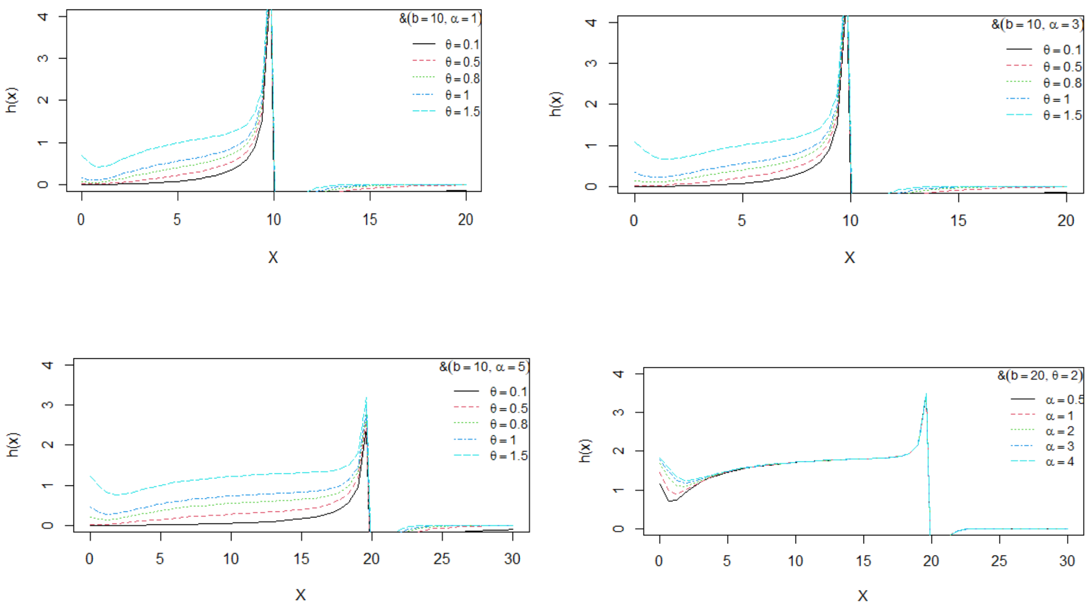

Figure 3. *h(x) plots of TTPPD for varying values of parameter*

IV. Moments and Mathematical Properties

Theorem: Suppose X follows doubly TTPPD(θ,α,a,b). Then the r th moment about origin μ_r' of TTPPD is

$$\mu_r' = \frac{\theta^4\alpha\{\gamma(r+1,\theta b)-\gamma(r+1,\theta a)\}+\{\gamma(r+4,\theta b)-\gamma(r+4,\theta a)\}}{\theta^r\begin{pmatrix}(a^3\theta^3+3a^2\theta^2+6a\theta+\alpha\theta^4+6)e^{-\theta a}-\\(b^3\theta^3+3b^2\theta^2+6b\theta+\alpha\theta^4+6)e^{-\theta b}\end{pmatrix}};r=1,2,3,...$$

Proof: Considering $K=\begin{Bmatrix}(a^3\theta^3+3a^2\theta^2+6a\theta+\alpha\theta^4+6)e^{-\theta a}-\\(b^3\theta^3+3b^2\theta^2+6b\theta+\alpha\theta^4+6)e^{-\theta b}\end{Bmatrix}$

in (2.1), we have

$$\mu_r' = \frac{\theta^4}{K}\int_a^b x^r\left(\theta\alpha+x^3\right)e^{-\theta x}dx$$

$$=\frac{\theta^4}{K}\left[\int_a^b \alpha\theta e^{-\theta x}x^r dx+\int_a^b e^{-\theta x}x^{r+3}dx\right]$$

Taking $u=\theta x, x=\dfrac{u}{\theta}$

$$= \frac{\theta^4}{K}\left[\begin{array}{l} \frac{\theta\alpha}{\theta^{r+1}}\left\{\int_0^{\theta b} e^{-u}x^r du - \int_0^{\theta a} e^{-u}x^r du\right\} + \\ \frac{1}{\theta^{r+4}}\left\{\int_0^{\theta b} e^{-u}u^{r+3}du - \int_0^{\theta a} e^{-u}x^{r+3}du\right\} \end{array}\right]$$

Where $\gamma(\alpha,z)=\int_0^z e^{-x}x^{\alpha-1}dx, \alpha>0, x>0$ is the lower incomplete gamma function.

$$=\frac{\theta^4}{K}\left[\frac{\alpha\left\{\gamma(r+1,\theta b)-\gamma(r+1,\theta a)\right\}}{\theta^r}+\frac{\gamma(r+4,\theta b)-\gamma(r+4,\theta a)}{\theta^{r+4}}\right]$$

$$=\frac{1}{K}\left[\frac{\theta^4\alpha\left\{\gamma(r+1,\theta b)-\gamma(r+1,\theta a)\right\}+\left\{\gamma(r+4,\theta b)-\gamma(r+4,\theta a)\right\}}{\theta^r}\right]$$

$$=\frac{\theta^4\alpha\left\{\gamma(r+1,\theta b)-\gamma(r+1,\theta a)\right\}+\left\{\gamma(r+4,\theta b)-\gamma(r+4,\theta a)\right\}}{\theta^r\begin{pmatrix}(a^3\theta^3+3a^2\theta^2+6a\theta+\alpha\theta^4+6)e^{-\theta a}-\\(b^3\theta^3+3b^2\theta^2+6b\theta+\alpha\theta^4+6)e^{-\theta b}\end{pmatrix}}$$

(4.1)

Now taking $r=1,2$, mean and variance can be obtained as

$$\mu_1'=\frac{\theta^4\alpha\left\{\gamma(2,\theta b)-\gamma(2,\theta a)\right\}+\left\{\gamma(5,\theta b)-\gamma(5,\theta a)\right\}}{\theta\begin{pmatrix}(a^3\theta^3+3a^2\theta^2+6a\theta+\alpha\theta^4+6)e^{-\theta a}-\\(b^3\theta^3+3b^2\theta^2+6b\theta+\alpha\theta^4+6)e^{-\theta b}\end{pmatrix}}$$

$$\mu_2'=\frac{\theta^4\alpha\left\{\gamma(3,\theta b)-\gamma(3,\theta a)\right\}+\left\{\gamma(6,\theta b)-\gamma(6,\theta a)\right\}}{\theta^2\begin{pmatrix}(a^3\theta^3+3a^2\theta^2+6a\theta+\alpha\theta^4+6)e^{-\theta a}-\\(b^3\theta^3+3b^2\theta^2+6b\theta+\alpha\theta^4+6)e^{-\theta b}\end{pmatrix}}$$

Variance $\mu_2=\mu_2'-(\mu_1')^2$

Similarly rest two moment of origin as well as coefficient of variation, coefficient of skewness, coefficient of kurtosis and Index of dispersion can be obtained, substituting $r=3,4$ in the equation (4.1), which are as follows:

$$\mu_3'=\frac{\theta^4\alpha\left\{\gamma(4,\theta b)-\gamma(4,\theta a)\right\}+\left\{\gamma(7,\theta b)-\gamma(7,\theta a)\right\}}{\theta^3\begin{pmatrix}(a^3\theta^3+3a^2\theta^2+6a\theta+\alpha\theta^4+6)e^{-\theta a}-\\(b^3\theta^3+3b^2\theta^2+6b\theta+\alpha\theta^4+6)e^{-\theta b}\end{pmatrix}}$$

$$\mu_4'=\frac{\theta^4\alpha\left\{\gamma(5,\theta b)-\gamma(5,\theta a)\right\}+\left\{\gamma(8,\theta b)-\gamma(8,\theta a)\right\}}{\theta^4\begin{pmatrix}(a^3\theta^3+3a^2\theta^2+6a\theta+\alpha\theta^4+6)e^{-\theta a}-\\(b^3\theta^3+3b^2\theta^2+6b\theta+\alpha\theta^4+6)e^{-\theta b}\end{pmatrix}}$$

Coefficient of Variation= $\frac{\left(\mu_2'-(\mu_1')^2\right)^{1/2}}{\mu_1'}$, Coefficient of Skweness= $\frac{\left(\mu_3'+3\mu_2'\mu_1'-(\mu_1')^2\right)}{(\mu_2'-(\mu_1')^2)^{3/2}}$, Coefficient of Kurtosis= $\dfrac{\left(\mu_4^{'}-4\mu_3^{'}\mu_1^{'}+6\mu_2^{'}(\mu_1^{'})^2-3(\mu_1^{'})^4\right)}{(\mu_2^{'}-(\mu_1^{'})^2)^2}$,

Index of dispersion= $\dfrac{\left(\mu_2^{'}-(\mu_1^{'})^2\right)}{\mu_1^{'}}$, graph of above measures are presented in figure 4,5,6,7,8 &9.

Expressions of other central moments are not being given here because they have lengthy expressions. However, they can be easily obtained.

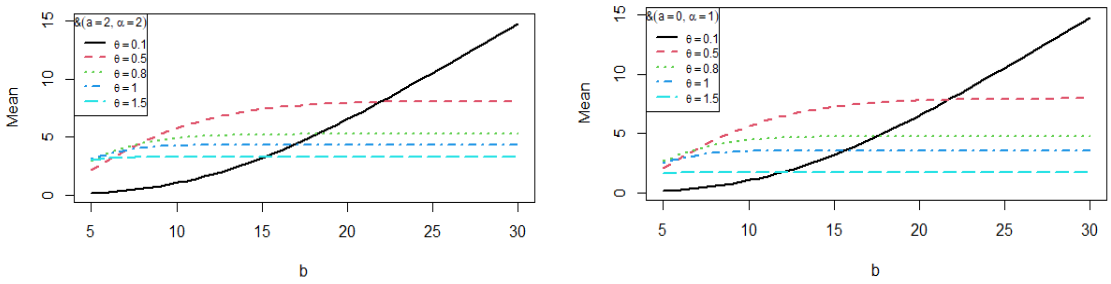

Figure4. *Mean of TTPPD on varying value of parameters*

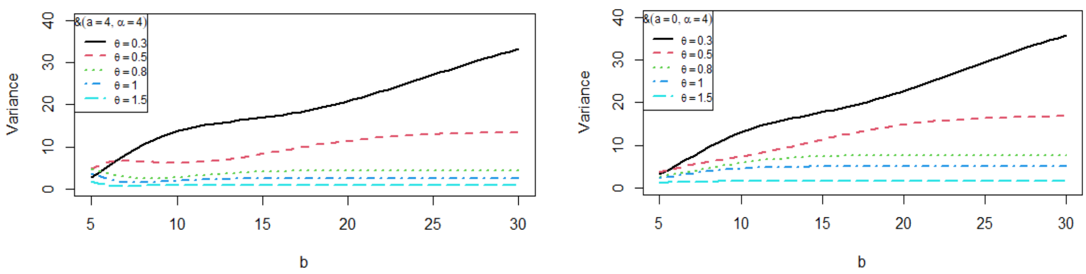

Figure5.*Variance of TTPPD on varying value of parameters*

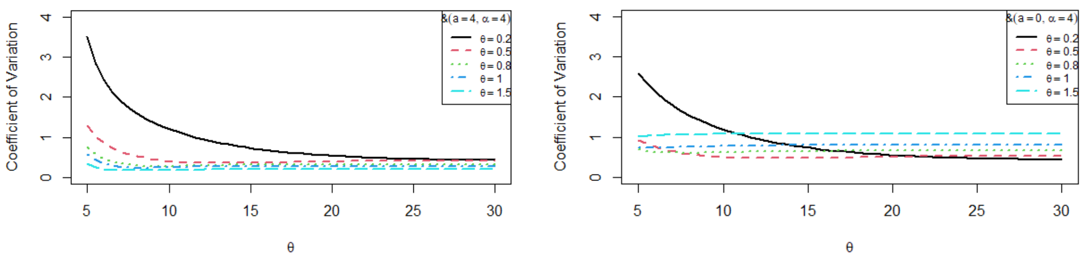

Figure 6.*Variance of TTPPD on varying value of parameters*

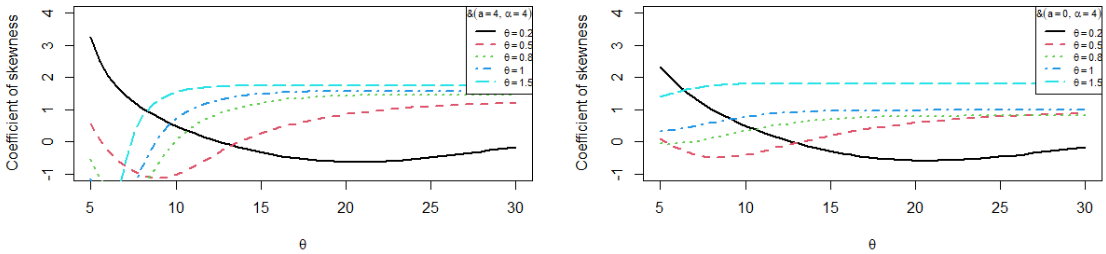

Figure7.*Variance of TTPPD on varying value of parameters*

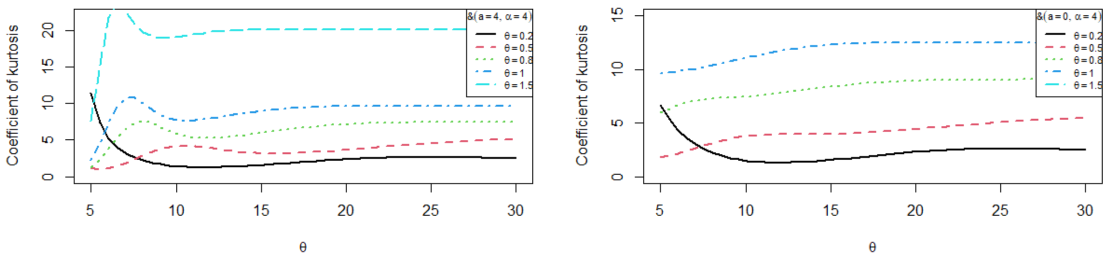

Figure8.*Variance of TTPPD on varying value of parameters*

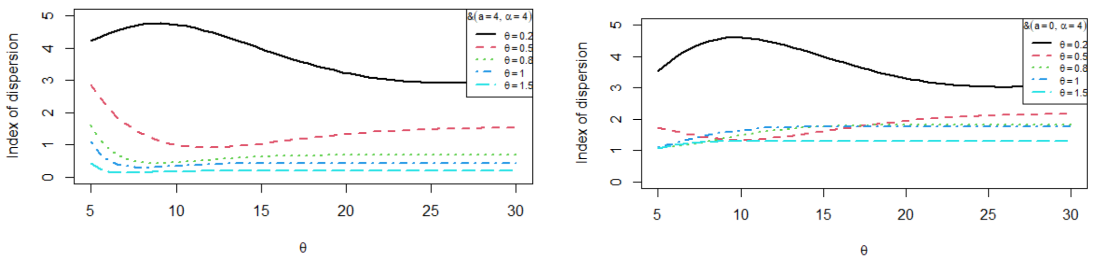

Figure9.*Variance of TTPPD on varying value of parameters*

V. Maximum likelihood Method of Estimation

Let $\left(x_1, x_2, x_3, \dots, x_n\right)$ be a random sample of size n from (1.1). The likelihood function, L of Pranav distribution is given by

$$L=\left(\frac{\theta^4}{\left((\theta^3a^3+\alpha\theta^4+3a^2\theta^2+6a\theta+6)e^{-\theta a}-(\theta^3b^3+\alpha\theta^4+3b^2\theta^2+6b\theta+6)e^{-\theta b}\right)}\right)^n\prod_{i=1}^{n}\left(\alpha\theta+x_i^3\right)e^{-n\theta\bar{x}}$$

and its log likelihood function is thus obtained as

$$\ln L=n\ln\left(\frac{\theta^4}{\left((\theta^3a^3+\alpha\theta^4+3a^2\theta^2+6a\theta+6)e^{-\theta a}-(\theta^3b^3+\alpha\theta^4+3b^2\theta^2+6b\theta+6)e^{-\theta b}\right)}\right)+$$

$$\sum_{i=1}^{n}\ln\left(\alpha\theta+x_i^3\right)-n\theta\bar{x}$$

Taking $\hat{a}=\min\left(x_1, x_2, x_3, \dots, x_n\right)$, $\hat{b}=\max\left(x_1, x_2, x_3, \dots, x_n\right)$, the maximum likelihood estimate $\hat{\theta}$ of parameter θ is the solution of the log-likelihood equation $\frac{\partial \log L}{\partial \theta} = 0$. It is obvious that $\frac{\partial \log L}{\partial \theta} = 0$ will not be in closed form and hence some numerical optimization technique can be used e the equation for θ. In this paper the nonlinear method available in R software has been used to find the MLE of the parameter θ.

VI. Simulation Study

In this section, simulation of study of (2.1) has been carried out. Acceptance and Rejection method has been used to generate random number. Bias Error and Mean square Error have been calculated for varying values parameters θ and α whereas parameter a and b kept constant.

Table 1. *Simulation of TSD at a=10, b=100,* $\theta = 0.1$ *and* $\alpha = 1$

Sample Size (n)	θ	α	Bias Error(θ)	MSE(θ)	Bias Error(α)	MSE(α)
20	0.1	1.0	0.035225	0.024817	1.121145	25.13931
	0.5	2.0	0.015225	0.004636	1.071145	22.94702
	1.0	3.0	-0.00977	0.001911	1.021145	20.85473
	1.5	4.0	-0.03477	0.024186	0.971145	18.86244
40	0.1	1.0	0.016613	0.01104	1.228047	60.32398
	0.5	2.0	0.006613	0.001749	1.203047	57.89288
	1.0	3.0	-0.00589	0.001386	1.178047	55.51179
	1.5	4.0	-0.01839	0.013523	1.153047	53.18069
60	0.1	1.0	0.011075	0.00736	0.818698	40.21598
	0.5	2.0	0.004409	0.001166	0.802031	38.59525
	1.0	3.0	-0.00392	0.000924	0.785365	37.00786
	1.5	4.0	-0.01226	0.009016	0.768698	35.4538
80	0.1	1.0	0.013449	0.014469	0.010876	0.009463
	0.5	2.0	0.008449	0.00571	-0.00162	0.000211
	1.0	3.0	0.002199	0.000387	-0.01412	0.01596
	1.5	4.0	-0.00405	0.001313	-0.02662	0.056708

Table 2. *Simulation of TSD at a=10, b=100,* $\theta = 0.5$ *and* $\alpha = 2$

Sample Size (n)	θ	α	Bias Error(θ)	MSE(θ)	Bias Error(α)	MSE(α)
20	0.1	1.0	0.031624	0.020001	1.096027	24.0255
	0.5	2.0	0.011624	0.002702	1.046027	21.88345
	1.0	3.0	-0.01338	0.003578	0.996027	19.84139
	1.5	4.0	-0.03838	0.029454	0.946027	17.89934
40	0.1	1.0	0.016796	0.011285	0.764069	23.35205
	0.5	2.0	0.006796	0.001848	0.739069	21.84891
	1.0	3.0	-0.0057	0.001301	0.714069	20.39577
	1.5	4.0	-0.0182	0.013255	0.689069	18.99264
60	0.1	1.0	0.015861	0.015095	0.059787	0.214469
	0.5	2.0	0.009195	0.005073	0.04312	0.111562
	1.0	3.0	0.000861	4.45E-05	0.026454	0.041988
	1.5	4.0	-0.00747	0.00335	0.009787	0.005747
80	0.1	1.0	0.012453	0.012405	0.027456	0.060306
	0.5	2.0	0.007453	0.004443	0.014956	0.017894
	1.0	3.0	0.001203	0.000116	0.002456	0.000483
	1.5	4.0	-0.00505	0.002038	-0.01004	0.008071

Table 3. *Simulation of TSD at a=10, b=100,* $\theta = 1$ *and* $\alpha = 3$

Sample Size (n)	θ	α	Bias Error(θ)	MSE(θ)	Bias Error(α)	MSE(α)
20	0.1	1.0	0.03261	0.021269	0.989088	19.56588
	0.5	2.0	0.01261	0.00318	0.939088	17.63771
	1.0	3.0	-0.01239	0.00307	0.889088	15.80953
	1.5	4.0	-0.03739	0.02796	0.839088	14.08136
40	0.1	1.0	0.01753	0.012292	0.631985	15.97622
	0.5	2.0	0.00753	0.002268	0.606985	14.73725
	1.0	3.0	-0.00497	0.000988	0.581985	13.54828
	1.5	4.0	-0.01747	0.012208	0.556985	12.40931
60	0.1	1.0	0.015636	0.014669	0.071319	0.305184
	0.5	2.0	0.008969	0.004827	0.054652	0.179212
	1.0	3.0	0.000636	2.43E-05	0.037986	0.086574
	1.5	4.0	-0.0077	0.003555	0.021319	0.02727
80	0.1	1.0	0.012021	0.011561	0.044151	0.155947
	0.5	2.0	0.007021	0.003944	0.031651	0.080144
	1.0	3.0	0.000771	4.76E-05	0.019151	0.029342
	1.5	4.0	-0.00548	0.002401	0.006651	0.003539

Table 4. *Simulation of TSD at a=10, b=100 ,* $\theta = 1.5$ *and* $\alpha = 4$

Sample Size (n)	θ	α	Bias Error(θ)	MSE(θ)	Bias Error(α)	MSE(α)
20	0.1	1.0	0.052018	0.054117	0.279555	1.563022
	0.5	2.0	0.032018	0.020503	0.229555	1.053912
	1.0	3.0	0.007018	0.000985	0.179555	0.644801
	1.5	4.0	-0.01798	0.006467	0.129555	0.335691
40	0.1	1.0	0.035047	0.04913	0.055135	0.121594
	0.5	2.0	0.025047	0.025093	0.030135	0.036325
	1.0	3.0	0.012547	0.006297	0.005135	0.001055
	1.5	4.0	4.66E-05	8.68E-08	-0.01987	0.015785
60	0.1	1.0	0.02594	0.040373	0.017838	0.019092
	0.5	2.0	0.019273	0.022288	0.001172	8.24E-05
	1.0	3.0	0.01094	0.007181	-0.01549	0.014406
	1.5	4.0	0.002607	0.000408	-0.03216	0.062062
80	0.1	1.0	0.016822	0.022637	0.059504	0.283254
	0.5	2.0	0.011822	0.01118	0.047004	0.176747
	1.0	3.0	0.005572	0.002483	0.034504	0.09524
	1.5	4.0	-0.00068	3.68E-05	0.022004	0.038733

VII. Applications on Life time data

In this section, TTPPD has been applied on following three data sets, where maximum likelihood method of estimation has been used for estimation of its parameter. Parameter θ is estimated whereas another parameters a, and b are considered as lowest and highest values of data. i. e. $a = \min(x)$ and $b = \max(x)$, where x is data set. Goodness of fit has been decided using Akaike information criteria (AIC), Bayesian Information criteria (BIC) and Kolmogorov Simonov test (KS) values respectively, which are calculated for each distribution and also compared with p-value. As we know that best goodness of fit of the distribution can be decide on the basis minimum value of

KS, AIC and BIC and maximum p-value for K.S.

Data Set 1: The data is given by Birnbaum and Saunders (1969) on the fatigue life of 6061 – T6 aluminum coupons cut parallel to the direction of rolling and oscillated at 18 cycles per second. The data set consists of 100 observations with maximum stress per cycle 31,000 psi. The data ($\times 10^{-3}$) are presented below (after subtracting 65**).**

5 25 31 32 34 35 38 39 39 40 42 43
43 43 44 44 47 47 48 49 49 49 51 54
55 55 55 56 56 56 58 59 59 59 59 59
63 63 64 64 65 65 65 66 66 66 66 66
67 67 67 68 69 69 69 69 71 71 72 73
73 73 74 74 76 76 77 77 77 77 77 77
79 79 80 81 83 83 84 86 86 87 90 91
92 92 92 92 93 94 97 98 98 99 101 103
105 109 136 147

Data Set 2: This data set is the strength data of glass of the aircraft window reported by Fuller *et al* (1994):

18.83 20.8 21.657 23.03 23.23 24.05 24.321 25.5 25.52 25.8 26.69 26.77
26.78 27.05 27.67 29.9 31.11 33.2 33.73 33.76 33.89 34.76 35.75 35.91
36.98 37.08 37.09 39.58 44.045 45.29 45.381

Table 5: *MLE's, - 2ln L, AIC, K-S and p-values of the fitted distributions for data set*-1

Distributions	ML Estimates	$-2\ln L$	AIC	BIC	K-S	p-value
TTPPD	$\theta = 0.05527$ $\alpha = 2.02011$	**927.37**	**931.37**	**930.14**	**0.136**	0.056
TTPLD	$\theta = 0.02238$ $\alpha = 15.80197$	957.94	961.94	967.15	0.191	**0.001**
TPPD	$\theta = 0.05853$ $\alpha = 2.13206$	934.06	938.06	940.93	0.173	**0.005**
TPWD	$\theta = 0.00272$ $\theta = 1.39558$	989.35	993.35	998.56	0.294	**0.000**
TAD	$\theta = 0.03917$	939.13	941.13	942.05	0.153	**0.017**
TLD	$\theta = 0.02199$	958.88	960.88	962.31	0.186	**0.001**
Pranav	$\theta = 0.05853$	934.06	936.06	937.49	0.167	**0.007**
Lindley	$\theta = 0.02886$	983.10	985.10	986.54	0.252	**0.000**
Exponential	$\theta = 0.01463$	**1044.87**	**1046.87**	**1048.30**	**0.336**	**0.000**

Table 6: *MLE's, - 2ln L, AIC, K-S and p-values of the fitted distributions for data set-2*

Distributions	ML Estimates	$-2\ln L$	AIC	BIC	K-S	p-value
TTPPD	$\theta = 0.12066$ $\alpha = 0.5999$	**201.80**	**205.80**	**204.57**	**0.107**	0.829
TTPLD	$\theta = 0.12981$ $\alpha = 2.17254$	232.77	236.77	239.64	0.282	**0.011**
TPPD	$\theta = 0.12981$ $\alpha = 1.9946$	232.77	236.77	239.64	0.282	**0.011**
TPWD	$\theta = 0.00203$ $\theta = 1.80566$	241.61	245.61	247.61	0.353	**0.000**
TAD	$\theta = 0.08776$	201.96	203.96	205.58	0.112	**0.786**
TLD	$\theta = 0.05392$	202.18	204.18	205.61	0.117	**0.738**
Pranav	$\theta = 0.12981$	232.77	234.77	236.20	0.267	**0.019**
Lindley	$\theta = 0.06299$	253.98	255.98	256.98	0.365	**0.000**
Exponential	$\theta = 0.032452$	**274.52**	**276.52**	**277.52**	**0.458**	**0.000**

VIII. Conclusions

In this paper, Truncated Two Parameter Distribution (TTPPD) has been proposed. Its mathematical and statistical properties including coefficient of variation, skewness, kurtosis and Index of dispersion have been derived and presented graphically. Maximum likelihood method has been used for estimation of its parameters. Goodness of fit of TTPPD has been discussed with two life time data sets and compared with truncated two parameter Lindley distribution (TTPLD), two parameter Weibull distribution (TPWD), two parameter Pranav distribution (TPPD), truncated Akash distribution (TAD), truncated Lindley distribution (TLD), Pranav, Lindley and Exponential distribution. It has been observed from above results, TTPPD gives good fit over above mentioned distributions on both the data sets.

References

[1] Aryuyuen, S.(2018), Truncated Two parameter Lindley distribution and application, *The Journal of Applied Science,* 17(1),19-32

[2] Aryuyuen,S. and Bodhisuwan, W. (2018). The truncated power Lomax distribution: Properties and applications, *Walailak Journal of Science and Technology*, 16 , 655-668.

[3] Birnbaum, Z.W., Saunders, S.C. (1969), Estimation for a family of life distribution with applications to fatigue. *Journal of applied probability*, 6(2), 328-347.

[4] Fuller, E.J., Frieman, S., Quinn, J., Quinn, G., and Carter, W.(1994): Fracture mechanics approach to the design of glass aircraft windows: A case study, SPIE Proc 2286, 419-430.

[5] Ghitany ME, Atieh B, Nadarajah S. Lindley distribution and its Applications. *Mathematics Computing and Simulation.* 2008;78:493– 506.

[6] Johnson, N. L., Kotz,S. and Balakrishnan,N. (1994), Continuous Univariate Distributions, Wiley, New York (1994).

[7] Lindley, D.V. (1958): Fiducial distributions and Bayes' Theorem, *Journal of the Royal Statistical Society*, Series B, 20, 102 – 107.

[8] Shukla, K. K. and Shanker, R., The Truncated Akash distribution: properties and applications communicated.

[9] Shukla, K. K (2018), Pranav distribution with Properties and Applications, Biometrics and Biostatistics International Journal,7(3),244-254.

[10] Singh,S. K., Singh,U. and Sharma, V.K. (2014), The truncated Lindley distribution: Inference and application, *Journal of Statistics Applications & Probability*, 3, 219-228.

[11] Umeh,E. and Ibenegbu, A. (2019): A two parameter Pranav distribution with properties and its application, *J. Biostat. Epidemiol*. 5(1),74-90

[12] Zange,Z., and Xie, M., (2011),On the upper truncated Weibull distribution and its reliability implications, *Reliability Engineering & System Safety*, 96 , 194-200.

[13] Zaninetti,L. and Ferraro,M. (2008), On the truncated Pareto distribution with applications, *Central European Journal of Physics*, 6 , 1-6.

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Analogy between Agent Less Monitoring and Agent Based Monitoring

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Abstract

There has been an ongoing war among Agent-based vs. Agentless control and management technologies in the IT Service Management sector. Several vendors claim to be agentless, and do not need any special deployment of agents. Someone else include the use of proprietary agents. Both strategies have pros and cons. This study explores the agents versus agentless approach to management of enterprise processes, looking at emerging developments in both technological advances and the industry. We have compared Agent Based and Agent Less Monitoring for two different scenarios on following parameters - Installation and depth of monitoring, Cost, Maintainance, Network Overhead, Server Overhead, Knowledge, Security.In scenario 1 we conclude that Agent based collector has edge over Agent less and in scenario 2 we conclude that Agent Less collector has edge over Agent Based.

Keywords- Agent Based Collectors, Agent Less Collectors, Nagios, Prometheus and Telegraf

I. INTRODUCTION

During the first five years of the new millennium, the transition in the late 1990s' IT industry slowed to an evolutionary rate. This was not because of a lack of modern technologies and innovation but because of fiscal constraints imposed by cost-conscious management and guided by cautious shareholders. Ironically, as businesses continued to expand, IT spending flat rates allowed business IT workers to catch up in several ways. Instead of introducing new technologies at a breakneck pace, IT has focused on enhancing the current infrastructure's stability.
Agent-based systems management tools distribute server-wide control applications around the network. These agents gather data through some kind of variety of API, devicecalls, sometimes parsing log files and leveraging other personal data stores. Information from these agents will then be fed back to central tracking servers where the results will be tabulated, thresholds and alarms reviewed, and the results delivered to the end user.Control strategies for agentless environments do not distribute applications to specific servers.In the meanwhile, the central management server is surveying devices in the network, collecting usability and performance information through published API, framework.

As with its agent-based equivalent, the end-user is provided with real-time and historical data, warnings, and metrics through a system management dashboard.Enterprise Management Associates (EMA) launched a vendor-independent Internet survey to examine surveillance patterns, especially with regard to agents and agentless approaches. Such findings were followed by in-depth interviews with two Fortune 1000 firms that use management methods in both agent-based and agentless systems. Survey findings and interviews were well linked. Agentless monitoring solutions tackle the most important issues today for businesses: cost, ease of maintenance, faster delivery and less burden on the managed server. Agent-based solutions were required to provide knowledge that their agentless counterparts did not have available.

A. *AGENT* BASED COLLECTORS

Agent based monitoring has authorized the installation of an agent (small executable) on the system. To offer an example: Prometheus and Telegraf

Advantages of getting agent-based collectors:

• When an agent is mounted on a node, it can do in-depth monitoring and management.
• Data is stored on a server, and the data will be processed even though there is a network interruption.
• Because data resides on the node, data can be collected, deduced and aggregated before saving network bandwidth is transmitted.
• After collection of data a custom protection channel may also be used to transfer data over the network.

PROMETHEUS

An open-source monitoring framework with a model of dimensional data, versatile query language, effective database of regression analysis and modern alerting strategy.

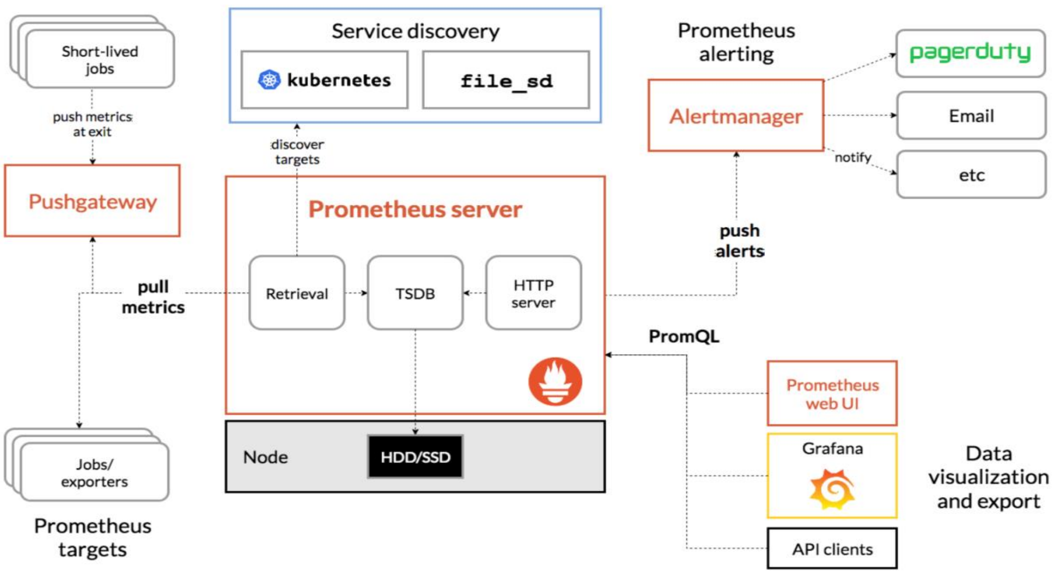

Figure:1 Prometheus Architecture

B. AGENT LESS COLLECTORS

Agentless monitoring is a type of network monitoring in which a monitor extracts performance measurements through devices while installing a software agent on the monitored devices or servers. Agentless monitoring is common as it helps companies to escape the complexities of installing software agents on servers, as they would have to do with a monitoring ideology based on agents.

Benefits of providing agent with fewer collectors:

- Easy to deploy, agentless control. Get up in a few minutes and continue playing.
- Low maintenance, since only a few data collectors need regular monitoring
- Maximum flexibility: identify the infrastructure levels to be monitored with agents and those to be monitored in an agent-free manner
- Integrated integration of agent indicators and agent-free monitors into a single glass screen.

NAGIOS

Nagios Core, is a free and open source and computer software monitoring systems, networks and infrastructure. Nagios offers servers, switches, software and facilities with monitoring and alerting systems. This warns users when things go wrong and warns them when the situation has been fixed a second time.The Nagios Log Server significantly simplifies the search process for your log info. Set up alarms to warn you when there are possible risks, or simply check your log data to inspect any device quickly. You get all of your log data in one location with Nagios Log Server, with high availability and built fail-over right in.

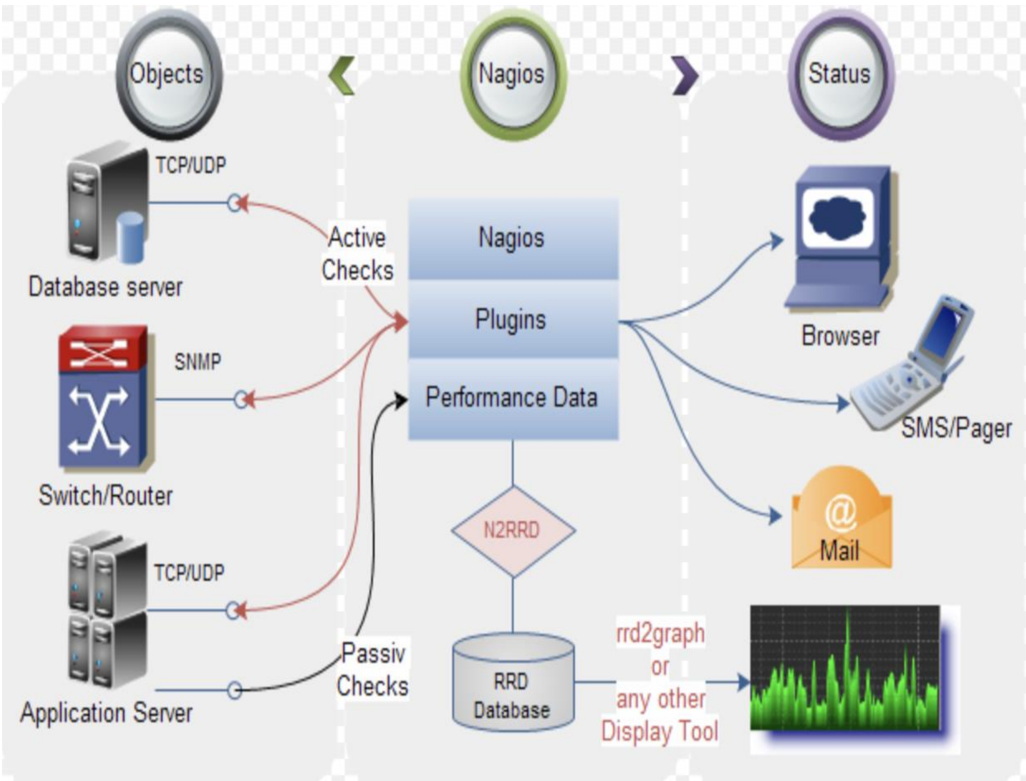

Figure 2: Nagios Architecture

We have compared Agent Based and Agent Less Monitoring for two different scenarios:

Scenario 1: Data Center has spread across private and public (Aws/Azure, etc.) cloud. Collect logs generated from the servers and also monitor status of all its servers.

Scenario 2: Data centers having various network devices, storage devices and embedded devices such as routers, printers, switches and firewalls. Collect logs generated from network devices, storage devices and embedded devices and also monitor status of all its servers.

II. RESEARCH WORK

SCENARIO 1:

Data Center has spread across private and public (Aws/Azure, etc.) cloud. PerformanceMonitoring ofservers across private and public cloud.

Above that the scenario can be tracked with fewer agent as well as collectors dependent on agent. But both can have pluses and minuses of their own.

1) INSTALLATION AND DEPTH OF MONITORING:

- Agent Based monitoring involves a server-installed agent with one central system, while Agent Less just requires central system deployment.
- Some data stored in local files or databases cannot be collected because Agent Less collectors track device from outside.
- Processing data correctly on the source network is not always feasible, thereby requiring extra bandwidth.

2) COST:

- Any additional server needed by Agent Lesscollector for monitoring against Agent Based Collector and additional costs for such servers. Thus costs in Agent Less collectors are higher.

3) MAINTENANCE:

- InAgent Based Collectors software needs to be installed and managed on all tracked servers, making the management of Agent Based collectors more difficult.

- A centralized data collector collects Performance Measurements with Agentless control. The central management tool uses standard protocols to use a remote API to access the monitoring data.

4) NETWORK OVERHEAD:

- Agentless monitoring needs an external network, as raw data information is transmitted to remote data collectors.

- Agent Based Control gathers data locally, aggregates the data and sends it over the network, thereby saving a lot of overhead network.

5) SERVER OVERHEAD:

- Agent Dependent Control agents are installed on the target server and consume both a CPU cycle and low memory. There would be more overhead if the collection frequency is low.
- Agent Less has no permanent overhead on the target server but some CPU cycle will be consumed upon receipt of the polling order.

6) KNOWLEDGE:

- Agent Less collector includes comprehensive network routing expertise and some custom configuration to capture an adequate traffic analysis amongst monitored devices.

7) SECURITY:

- The OS communications agent is managed internally to the server in Agent Based Monitoring. Therefore no need to install additional firewall rules.

- The activation of multiple methods for collecting data remotely offers additional attack vectors in Agent Less Monitoring.

1STCONCLUSION**:** We conclude that Agent based collector has edge over Agent less as:

- Agent Based collector collect more in-depth metrics,
- Since additional hardware is not required so low cost.
- Low network overhead (network transfer takes lot of time) makes more suitable.
- Highly secure as everything is behind firewall.
- Less understanding of network concepts is required.
- A little more Server Overhead but hardware is cheaper now so manageable.
- The only major drawback is maintenance as we have to maintain agents installed on all machines.

Scenario 2: Data centers having various network devices like routers, switches, hubs, etc., storage devices and embedded devices such as ATM machines, printers, switches and firewalls. Collect logs generated from network devices, storage devices and embedded devices and also performance monitoring all its servers.

Since Agent Based collectors have to installed on the devices to monitor it. All network, storage and embedded devices can't have agents installed on it.

List of devices and whether they can be monitored using Agent Less and Agent Based Collector-

Table 1: List of devices monitored using agent less and agent based collector

Device	Agent Less Collector	Agent Based Collector
Network Device	Yes	No
Storage Device	Yes	Yes
Embedded Device	Yes	No
Servers	Yes	Yes

2ND CONCLUSION: We conclude that not all the use cases can be handled by Agent Based Collector. So, either we can use:

1. Agent Less Collector – Since all the information can be fetched using Agent Less Collector, we can use Agent Less Collector.

2. A combination of Agent Based and Agent Less Collector -

   - Agent Less Collector can be used for monitoring of Network and embedded devices.
   - Agent Based Collector for monitoring Storage devices and Servers and get some benefit of collecting monitoring data in depth, Security, Network Overhead and cost.

COMPANIES USING PROMETHEUS:

COMPANIES USING NAGIOS:

From the above discussion we conclude that for Scenario one, Agent Based Collector is more preferred as it stands out for more number of parameters and for Scenario two, agent less collector is preferred as Agent Based single handily cant cater to all the scenarios.

References

1. R. Axtell, "Why agents? On the varied motivations for agent computing in the social sciences" in Working Paper 17 Center on Social and Economic Dynamics, Washington, D.C:Brookings Institution, 2000.
2. E. Bonabeau, "Agent-based modeling: methods and techniques for simulating human systems", *In Proceedings of National Academy of Sciences*, vol. 99, no. 3, pp. 7280-7287, 2001.
3. K. M. Carley, D. B. Fridsma, E. Casman, A. Yahja, N. Altman, L.-C. Chen, et al., "BioWar: scalable agent-based model of bioattacks", *IEEE Transactions on Systems Man and Cybernetics - Part A: Systems and Humans*, vol. 36, no. 2, pp. 252-265, 2006.
4. A.C. Charania, J. R. Olds and D. D ePasquale, "Sub-Orbital Space Tourism Market: Predictions of the Future Marketplace Using Agent-Based Modeling", 2006.
5. S. R. Conway, "An Agent-Based Model for Analyzing Control Policies and the Dynamic Service-Time Performance of a Capacity-Constrained Air Traffic Management Facility", *ICAS 2006–25th Congress of the International Council of the Aeronautical Sciences*, Sep. 2006.
6. V.A. Folcik, G.C. An and C.G. Orosz, "The Basic Immune Simulator: An Agent-Based Model to Study the Interactions between Innate and Adaptive Immunity", *Theoretical Biology and Medical Modelling*, vol. 4, no. 1, pp. 39-56, 2007.
7. A.F. Griffin and C. Stanish, "An Agent-Based Model of Prehistoric Settlement Patterns and Political Consolidation in the Lake Titicaca Basin of Peru and Bolivia", *Structure and Dynamics: eJournal of Anthropological and Related Sciences*, vol. 2, no. 2, 2007.
8. N. R. Jennings, "On agent-based software engineering", *Artificial Intelligence*, vol. 117, pp. 277-296, 2000.
9. C. M. Macal, "Agent-Based Modeling and Social Simulation with Mathematica and MATLAB", *in Proceedings of Agent 2004 Conference on Social Dynamics: Interaction Reflexivity and Emergence*, pp. 185-204, 2004, Oct. 7–9.
10. C. M. Macal and M. J. North, "Agent-based Modeling and Simulation: Desktop ABMS", *Proceedings of the 2007 Winter Simulation Conference*, pp. 95-106, 2007.
11. E. H. M. Mast, Kuik G.A.M. van and G.J.W. van Bussel, "Agent-Based Modelling for Scenario Development of Offshore Wind Energy" in Delft University of Technology, The Netherlands, 2007.
12. K. J. Mock and J. W. Testa, "An agent-based model of predator-prey relationships between transient killer whales and other marine mammals", May 2007.
13. C. Nikolai and G. Madey, "Tools of the Trade: A Survey of Various Agent Based Modeling Platforms", *Journal of Artificial Societies and Social Simulation*, vol. 12, no. 2, 2009.
14. M. J. North and C. M. Macal, "Managing business complexity: discovering strategic solutions with agent-based modeling and simulation" in , Oxford, U.K.:Oxford University Press, 2007.
15. M. North, N. Collier and J. Vos, "Experiences in creating three implementations of the repast agent modeling toolkit", *ACM Transactions on Modeling and Computer Simulation*, vol. 16, no. 1, pp. 1-25, 2006.
16. R. Tobias and C. Hofmapp, "Evaluation of free Java-libraries for social-scientific agent based simulation", *Journal of Artificial Societies and Social Simulation*, vol. 7, no. 1, Jan. 2004.
17. A.Troisi, V. Wong and M. A. Ratner, "An Agent-Based Approach for Modeling Molecular Self-Organization", *Proceedings of the National Academy of Sciences USA*, vol. 102, no. 2, pp. 255-260, 2005.
18. K. H. van Dam, Z. Lukszo, L. Ferreira and A. Sirikijpanichkul, *Planning the location of intermodal freight hubs: an agent based approach Proceedings of the 2007 IEEE International Conference on Networking Sensing and Control*, pp. 187-192, 2007, 15–17 April 2007.

19. S. Kumar. S and H. Pandey, Review Article: Necessity of User Friendly Sequence Analysis Tools & Servers and Different Analysis through Computational Biology Approaches in Modern era", International Journal of Research in Electronics and Computer Engineering, VOL. 7 ISSUE 2, PP. 2133-2139, ISSN: 2348-2281 (Online), April- June 2019.
20. S. Kumar. S and H. Pandey, Modelling of an offline software DNA & AA-Align Editor of biological sequence retrieval and phylogenetic analysis by Perl, Bioperl and PerlTk downloadable from webserver, International Journal of Information and Computing Science, Volume 6, Issue 5, PP. 462-470, ISSN NO: 0972-1347 May 2019.
21. GK Srivastava, S. Kumar, H. Pandey, Modelling of an offline and online software for normalization of microarray data of gene expression by Perl, Bioperl and PerlTk and Perl-CGI", Reliability: Theory & Applications 14 (2).
22. H. Pandey, M. Darbari, Coalescence of Evolutionary Multi-Objective Decision making approach and Genetic Programming for Selection of Software Quality Parameter, International Journal of Applied Information System (IJAIS), Foundation of Computer Science, New York, USA, Volume 7, No. 11, PP. ISSN: 2249-0868, Nov. 2014.
23. S. Bansal and H. Pandey, Develop Framework for selecting best Software Development Methodology, International Journal of Scientific and Engineering Research, Volume 5, Issue 4, PP. 1067-1070, ISSN: 2229-5518, Apr. 2014.
24. M. Srivastava and H. Pandey,A Literature Review of E- Learning Model Based on Semantic Web Technology, International Journal of Scientific and Engineering Research" Volume 5, Issue 10, PP. 174-178, ISSN: 2229-5518, Oct. 2014.
25. H. Pandey, A New NFA Reduction Algorithm for State Minimization Problem, International Journal of Applied Information Systems (IJAIS), Foundation of Computer Science FCS, New York, USA, Volume 8, No.3, PP. 27-30, ISSN: 2249-0868, Feb. 2015.
26. H.Pandey, LR Rotation rule for creating Minimal NFA, International Journal of Applied Information Systems (IJAIS), Foundation of Computer Science FCS, New York, USA, Volume 8, No.6, PP. 1-4, ISSN: 2249-0868, Apr. 2015.

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Optimal order quantity and pricing policies for EPL model with rework and quality inspection

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Abstract

This article develops a three-echelon supply chain coordination policy composing of a supplier, a manufacturer, and a wholesaler. The economic production lot-size (EPL) model comprises of perfect and defective items, quality inspection, return policy and reworking of defective items by the manufacturer. The defective items produced are reworked at a cost just after the regular production time. Here, return policy is considered between the outside supplier and the supplier, and the manufacturer and the wholesaler. Also, we considered the production cost as a function of production rate. We have formulated the profit functions of each member of the supply chain and optimized the total profit function of the whole supply chain system. Next, we have shown that the profit function of each member is concave. A numerical example with graphical representation presented to illustrate the proposed model.

Keywords: Supply chain, Rework, Inspection error, Return policy, quality inspection
AMS Subject Classification: 90B05, 90B30, 90B50

1 Introduction

Ordering and pricing decisions play an important role in optimizing the costs and profits of supply chain members. Recently, many researchers were executed in the area of the supply chain by applying ordering and pricing decisions. Also, the demand of customers is usually price-sensitive and they are looking for the lower price therefore firm attract customers by utilizing optimal pricing policies. In the study of supply chain network, the members of the network like supplier, manufacturer, wholesaler and retailer play an equal role and are interconnected for the proper functioning of the supply chain. To highlight such type of phenomenon, Sana (2011) developed an integrated production-inventory model for supplier, manufacturer and retailer consisting of perfect and imperfect quality items. He maximized total profit of supply chain in coordination with production and inventory decisions. Banerjee (1986) shows that the joint optimal ordering policy, together with an appropriate price adjustment, can be economically beneficial for both the purchaser and the vendor. Ben-Daya (1999) discussed a multi-stage lot sizing models for imperfect production processes.

Khan *et al.* (2014) presented a simple but integrated mathematical model that incorporates quality inspection errors at the buyer's end and learning in production at the vendor's end for determining an optimal vendor-buyer inventory policy. Khanna *et al.* (2017) developed a finite production model to inquire the optimal production quantity for maximizing the total expected profit per unit time. They constructed model in an imperfect production, imperfect inspection and imperfect rework environment. An inventory model with a money back agreement between the retailer and the supplier means defective products are returned to the supplier via the vehicle that brings new order, was considered by Vishkaei *et al.* (2014) instead of selling defective items at a lower price at the end of the period. Eroglu and Ozdemir (2007) addressed an EOQ model for defective items which are classified as scraps and imperfects, they are sold on a discounted selling price as a single lot. Authors assumed defective rate is a random variable with uniformly distributed.

Further, Priyan and Manivannan (2016); Noori-daryan and Taleizadeh (2018); Wang Chiu (2006) investigated the concept of inspection, random defective rate, return policy and rework process for different situations. In the classical economic production lot size (EPL) model, it is often assumed that the production cost is constant and predetermined, and items produced are of perfect quality. In reality, unit production cost depend on the production rate. The cost of the production process decreases with increased production rate. During production, both perfect and imperfect quality items are produced, and imperfect quality items are reworked or send back. Khouja and Mehrez (1994) developed an EPL model with unit production costs depending on production rate. They considered production rate as a decision variable. In this paper, result shows that optimal production rate may be different from the production rate that minimizes unit production cost.

Taleizadeh and Noori-daryan (2014) proposed a mathematical model which shows that by increasing the ordering cost, the number of shipments received by the supplier and the producer is decreased and the total cost of the supply chain network is increased. Khan *et al.* (2011) used the similar approach of Salameh and Jabeh (2000) to produce an optimal order production quantity with imperfect items. They incorporated the screening costs more accurately in the economic order sizing decisions. Huang *et al.* (2011) developed a model as a three-level dynamic non-cooperative game composed of multiple suppliers, a single manufacturer and multiple retailers. They used both analytical and computational methods for the derivative of the optimal decisions of all members of the supply chain.

In manufacturing industries, researchers and practitioners gives importance to develop and pricing policies and inventory control models in supply chain management. In this direction, some notable researches were addressed by Khedlekar and Singh (2019); Liu et al. (2009); Cárdenas-Barrón (2008,2009) and others. Pal *et al.* (2012) showed that the integrated profit function is more profitable compared to the profit of the whole chain. In this paper, production system may follows a probability density function. Mukherjee *et al.* (2019) investigated learning effect in production and investment process for quality improvement at the vendor's end, and lot-size dependent lead-time at the buyer's end. They observed that the expected annual total cost and investment required to improve the production process quality decrease, as the value of learning exponent increases. Barzegar Astanjin and Sajadieh (2017) shows that a coordination in supply chains is more beneficial if the defective rate is high. The investment in quality does not essentially leads to reduced costs, while the coordination of decision making strengthen such policies. In this paper, we extend EPL model by considering production cost as a function of production rate with reworkable items.

2 Model Assumptions and Notations

2.1 Notations

1. The level of positive inventory at time t of the supplier
2. The level of positive inventory at time t of the manufacturer
3. The level of positive inventory at time t of the wholesaler
4. The supplier's selling price per unit
5. The inventory holding cost per unit per unit time at supplier
6. The inventory holding cost per unit per unit time at manufacturer

7. The inventory holding cost per unit per unit time at wholesaler
8. The ordering cost of the supplier per order
9. The ordering cost of the manufacturer per order
10. The ordering cost of the wholesaler per order
11. The inspection cost per item of the supplier
12. The inspection cost per item of the manufacturer
13. The inspection cost per item of the wholesaler
14. The production rate of the manufacturer per unit time
15. The reworking rate of the defective items by the manufacturer per unit time, $P' = zP$, $z = 1$ (say)
16. The purchasing cost of raw material per item of the supplier
17. Buyback/repurchase price per returned items of the supplier per item
18. Reduced price of the defective items per unit, $p'_m = xp_m$
19. Buyback price of the returned items of the wholesaler per item, $p'_w = yp_w$
20. The proportion of defective items at the supplier, $0 \leq \alpha < 1$
21. The proportion of defective items at the manufacturer, $0 \leq \beta < 1$
22. The proportion of defective items at the wholesaler, $0 \leq \gamma < 1$
23. The cycle length of the supplier
24. The cycle length of the manufacturer
25. The cycle length of the wholesaler
26. The cycle length of the manufacturer for production and selling both
27. The cycle length of the manufacturer for reworking and selling both
28. The cycle length of the manufacturer for selling
29. The cycle length of the wholesaler for collecting and selling both
30. The cycle length of the wholesaler for selling
31. The manufacturer's suggested retail price (MSRP)
32. The manufacturer's demand rate per unit time, where $D_m = a - bp_s$
33. The wholesaler's demand rate per unit time, where $D_w = a - bp_m + \theta(M_p - p_m)$
34. The customer's demand rate per unit time, where $D_c = a - bp_w$
35. The market potential
36. The price sensitivity factor of demand
37. The sensitivity which determines the effect on demand when a wholesaler sells product below or above MSRP
38. The production cost per unit
39. The fixed cost like labour, energy and technology cost
40. The variation constant of tool/die costs
41. The total profit of the supplier per unit time, ($)
42. The total profit of the manufacturer per unit time, ($)
43. The total profit of the wholesaler per unit time, ($)
44. The total profit of the supply chain per unit time, ($)

Decision variables:

1. The ordering quantity(lot size) of the supplier per cycle
2. The manufacturer's selling price per unit
3. The wholesaler's selling price per unit

2.2 Assumptions

A single type of item is developed in the model. Shortage is not allowed. The model consists of a supplier, a manufacturer and a wholesaler. The demand is constant and price-sensitive. We considered a return policy between a supplier and a manufacturer; a manufacturer and a wholesaler. The manufacturer does rework after manufacturing with the same rate.

3 The Mathematical Model

Fig. 1 represents the outline of the proposed model. We considered a supplier, a manufacturer and a wholesaler. In this section we developed the profit function of each member of the supply chain. The total inventory system is depicted in Fig. 2.

3.1 The Supplier's Model

Firstly, supplier inspects all the raw material received from the outside supplier. Then, the supplier returns the defective items to the outside supplier and supplied good quality items to the manufacturer at a demand rate D_m. Suppose $I_s(t)$ is on hand inventory of the supplier at time t.

Initially, the inventory cycle begins with maximum stock-level $(1-\alpha)Q$ at $t=0$. During the time interval $[0,T_s]$, the stock-level decreases due to price-dependent demand. Finally, inventory level becomes zero at $t=T_s$. The differential equation satisfies in time interval $[0,T_s]$ is as follows

$$\frac{dI_s(t)}{dt} = -D_m, \; 0 \le t \le T_s \tag{1}$$

with boundary condition $I_s(0) = (1-\alpha)Q$ and $I_s(T_s) = 0$.

Condition: $(1-\alpha)Q \ge D_m T_s$

Solving Eq. (1), we obtain inventory level $I_s(t)$ as

$$I_s(t) = (1-\alpha)Q - D_m t; \;\; 0 \le t \le T_s \tag{2}$$

At $t = T_s$, above equation becomes

$$(1-\alpha)Q - D_m T_s = 0$$

$$T_s = \frac{(1-\alpha)Q}{D_m} \tag{3}$$

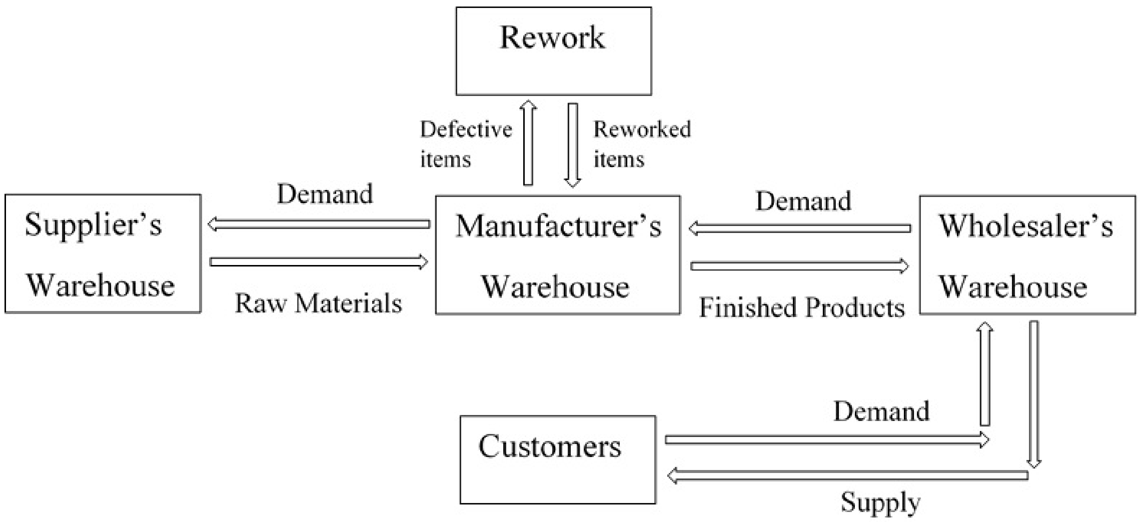

Figure 1: A three-level supply chain

Now, the total profit consists the following values:

Holding cost of the supplier

$$HC = \frac{1}{T_s} C_h^s \int_0^{T_s} I_s(t)\; dt = \frac{C_h^s(1-\alpha)Q}{2}$$

Ordering cost

$$OC = \frac{C_o^S}{T_s} = \frac{C_o^S D_m}{(1-\alpha)Q}$$

$$IC = \frac{C_i^S Q}{T_s} = \frac{C_i^S D_m}{(1-\alpha)}$$

Purchasing cost

$$PC = \frac{C_p Q}{T_s} = \frac{C_p D_m}{(1-\alpha)}$$

Hence, The total profit of the supplier is given by

TP_s = sales revenue - purchasing cost - inspection cost - holding cost - annual ordering cost

$$TP_s = \left(p_s + \frac{\alpha}{1-\alpha} C_p'\right) D_m - (C_p + C_i^s) \frac{D_m}{1-\alpha} - \frac{C_h^S (1-\alpha) Q}{2} - \frac{C_o^S D_m}{(1-\alpha)Q} \tag{4}$$

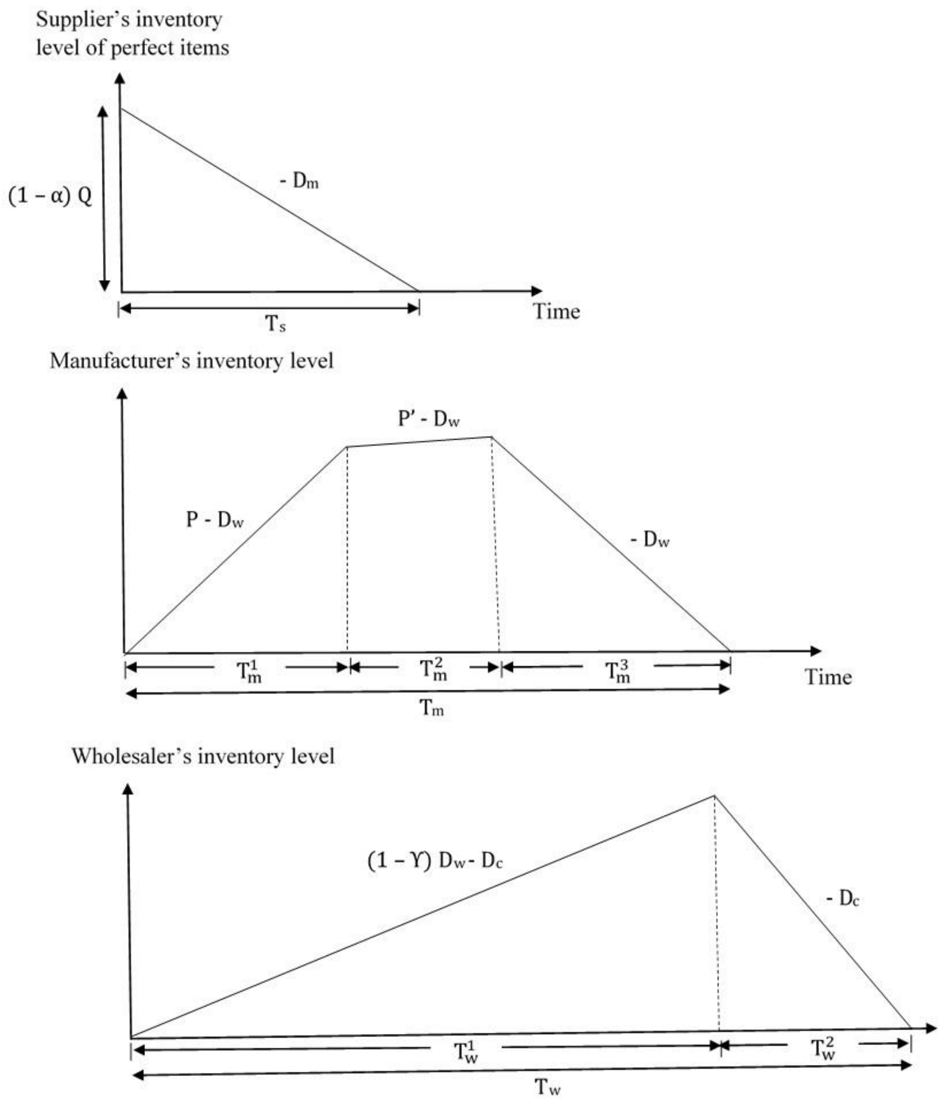

Figure 2: The inventory levels of the supply chain members

3.2 The Manufacturer's Model

The manufacturer produces the finished products from raw materials at a rate P during the time interval $[0, T_m^1]$with satisfying the wholesaler demand at the rate D_w. in the same interval, some defective products are manufactured which are reworked by manufacturer with the same rate P in the interval $[T_m^1, T_m^1 + T_m^2]$. In the interval $[T_m^1 + T_m^2, T_m]$, the inventory level decreases due to wholesaler's demand rate D_w and reaches zero at $t = T_m$. To avoid shortage, we consider $P \geq D_w$. Also, the unit production cost as $C(P) = p_s + \frac{L}{P} + \Gamma P$. Suppose $I_m(t)$ is on hand inventory of the manufacturer at time t. Thus, to represent the manufacturer's inventory system, the governing differential equations are

$$\frac{dI_m(t)}{dt} = (1-\beta)P - D_w, \; 0 \leq t \leq T_m^1 \tag{5}$$

$$\frac{dI_m(t)}{dt} = P - D_w, \; T_m^1 \leq t \leq T_m^1 + T_m^2 \tag{6}$$

$$\frac{dI_m(t)}{dt} = -D_w, \; T_m^1 + T_m^2 \leq t \leq T_m \tag{7}$$

with boundary conditions $I_m(0) = 0$ and $I_m(T_m) = 0$.

Solving Eqs. (5) - (7), we obtain the inventory level $I_m(t)$ as

$$I_m(t) = \{(1-\beta)P - D_w\}t; \quad 0 \leq t \leq T_m^1$$

$$I_m(t) = \{(1-\beta)P - D_w\}T_m^1 + (P - D_w)(t - T_m^1); \quad T_m^1 \leq t \leq T_m^1 + T_m^2$$

$$I_m(t) = D_w(T_m - t); \quad T_m^1 + T_m^2 \leq t \leq T_m$$

Also,

$$T_m^1 = \frac{(1-\alpha)Q}{P} \tag{8}$$

$$T_m^2 = \frac{\beta(1-\alpha)Q}{P} \tag{9}$$

$$T_m^3 = \frac{(1-\alpha)Q}{D_w}\left[1 - \frac{(1+\beta)D_w}{P}\right] \tag{10}$$

and

$$T_m = \frac{(1-\alpha)Q}{D_w} \tag{11}$$

Now, the total profit of the manufacturer consists the following terms:

Holding cost of the manufacturer

$$HC = \frac{1}{T_m} C_h^m \int_0^{T_m} I_m(t)\ dt = \frac{C_h^m(1-\alpha)Q}{2}\left[1 - (1+\beta+\beta^2)\frac{D_w}{P}\right]$$

Ordering cost

$$OC = \frac{C_o^m}{T_m} = \frac{C_o^m D_w}{(1-\alpha)Q}$$

Inspection cost

$$IC = \frac{C_i^m(PT_m^1 + P'T_m^2)}{T_m} = C_i^m D_w(1+\beta z)$$

Production cost

$$PC = \frac{1}{T_m}\left[P\int_0^{T_m^1} C(P)\ dt + P'\int_{T_m^1}^{T_m^1+T_m^2} C(P')\ dt\right]$$

$$= [C(P) + z\beta C(P')]D_w$$

Supplier cost

$$SC = \frac{p_s(1-\alpha)Q}{T_m} = p_s D_w$$

Hence, the total profit of the manufacturer is given by

TP_m = sales revenue - supplier cost - inspection cost - holding cost - annual ordering cost - production cost

$$\begin{aligned} TP_m &= (p_m - p_s - \gamma p'_m)D_w - C_i^m D_w(1+\beta z) - \frac{C_h^m(1-\alpha)Q}{2}\left[1 - (1+\beta+\beta^2)\frac{D_w}{P}\right] \\ &- \frac{C_o^m D_w}{(1-\alpha)Q} - [C(P) + z\beta C(P')]D_w \end{aligned} \tag{12}$$

3.3 The Wholesaler's Model

The wholesaler inspects all the products received from the manufacturer and return the defective products back to the manufacturer. In the interval $[0, T_w^1]$, the stock-level at the wholesaler is accumulates at the rate $(1-\gamma)D_w - D_c$. In the interval $[T_w^1, T_w]$, the accumulated inventory depletes with the rate D_c and reaches zero at $t = T_w$. Suppose I_w is on hand inventory at time t. The differential equations in time interval $[0, T_w]$

$$\frac{dI_w(t)}{dt} = (1-\gamma)D_w - D_c,\ \ 0 \le t \le T_w^1 \tag{13}$$

$$\frac{dI_w(t)}{dt} = -D_c,\ \ T_w^1 \le t \le T_w \tag{14}$$

with the boundary conditions $I_w(0) = 0$ and $I_w(T_w) = 0$.

From Eqs. (13) and (14), we have the inventory level $I_w(t)$ as

$$I_w(t) = \{(1-\gamma)D_w - D_c\}t; \quad 0 \le t \le T_w^1 \tag{15}$$

$$I_w(t) = D_c(T_w - t); \quad T_w^1 \le t \le T_w \tag{16}$$

We have the following equations

$$(1-\gamma)D_w T_w^1 \ = (1-\beta)r_p T_m^1 + \beta r_p T_m^2$$
$$(1-\gamma)D_w T_w^1 \ = (1-\alpha)Q$$

This implies

$$T_w^1 = \frac{(1-\alpha)Q}{(1-\gamma)D_w}$$

$$T_w^2 = \frac{(1-\alpha)Q}{D_c}\left[1 - \frac{D_c}{(1-\gamma)D_w}\right]$$

At $t = T_w^1$ in Equations (15) and (16)

$$\{(1-\gamma)D_w - D_c\}T_w^1 = D_c(T_w - T_w^1)$$

$$T_w = \frac{(1-\alpha)Q}{D_c}$$

Now, the total profit consists the following values:

Holding cost of the wholesaler

$$HC = \frac{1}{T_m} C_h^m \int_0^{T_w} I_w(t)\ dt = \frac{C_h^W(1-\alpha)Q}{2}\left[1 - \frac{D_c}{(1-\gamma)D_w}\right]$$

Ordering cost

$$OC = \frac{C_O^W}{T_w} = \frac{C_O^W D_c}{(1-\alpha)Q}$$

Inspection cost

$$IC = \frac{C_i^W D_w T_w^1}{T_w} = \frac{C_i^W D_c}{(1-\gamma)}$$

Hence, the total profit of the wholesaler is given by

TP_w= sales revenue - inspection cost - holding cost - ordering cost

$$TP_w = (p_w - p_m + \gamma p'_w)D_c - \frac{C_i^W D_c}{(1-\gamma)} - \frac{C_h^W(1-\alpha)Q}{2}\left[1 - \frac{D_c}{(1-\gamma)D_w}\right] - \frac{C_O^W D_c}{(1-\alpha)Q} \tag{17}$$

Finally, the total profit of chain is the summation of the total profits of the supplier, the manufacturer and the wholesaler shown in Equation 18,

$$TP = TP_s + TP_m + TP_w \tag{18}$$

so we have

$$\begin{aligned}TP &= \left(p_s + \frac{\alpha}{1-\alpha} C_p'\right)(a - bp_s) - (C_p + C_i^s)\frac{(a-bp_s)}{(1-\alpha)} - \frac{C_h^s(1-\alpha)Q}{2} - \frac{C_o^s(a-bp_s)}{(1-\alpha)Q} \\ &+\{(1-\gamma x)p_m - p_s\}\{a - bp_m + \theta(M_p - p_m)\} - C_i^m\{a - bp_m + \theta(M_p - p_m)\}(1+\beta) \\ &- \frac{C_h^m(1-\alpha)Q}{2}\left[1 - (1+\beta+\beta^2)\frac{\{a-bp_m+\theta(M_p-p_m)\}}{P}\right] \\ &- \frac{C_o^m\{a-bp_m+\theta(M_p-p_m)\}}{(1-\alpha)Q} - \left(p_s + \frac{L}{P} + \gamma P\right)\{a - bp_m + \theta(M_p - p_m)\}(1+\beta) \\ &+\{(1+\gamma y)p_w - p_m\}(a - bp_w) - \frac{C_i^w(a-bp_w)}{(1-\gamma)} - \frac{C_o^w(a-bp_w)}{(1-\alpha)Q} \\ &- \frac{C_h^w(1-\alpha)Q}{2}\left[1 - \frac{(a-bp_w)}{(1-\gamma)\{a-bp_m+\theta(M_p-p_m)\}}\right]\end{aligned} \tag{19}$$

Theorem 4.1 *The supplier's total profit function $TP_s(Q)$ is concave.*

Proof. Concavity of the suppler's total profit can be proved by taking the second-order derivative of $TP_s(Q)$ respect to Q which is strictly negative,

$$\frac{d^2TP_s}{dQ^2} = -\frac{2C_o^s}{(1-\alpha)Q^3}(a - bp_s) < 0.$$

The root of the first derivative of objective function Eq. (4) with respect to Q is global maximum of total profit function (Fig. 3). That means the optimal ordering quantity of supplier is

$$\frac{dTP_s}{dQ} = -\frac{(1-\alpha)C_h^s}{2} + \frac{C_o^s}{(1-\alpha)Q^2}(a - bp_s)$$

this implies,

$$Q^* = \frac{1}{(1-\alpha)}\sqrt{\frac{2C_o^s(a-bp_s)}{C_h^s}}. \tag{20}$$

Theorem 4.2 *The manufacturer's total profit function $TP_m(p_m)$ is concave.*

Proof. Differentiating Eq. (12) with respect to p_m

$$\begin{aligned}\frac{dTP_m}{dp_m} &= -\frac{(1-\alpha)(b+\theta)Q(1+\beta+\beta^2)C_h^m}{2P} + (1+z\beta)(b+\theta)C_i^m + \frac{(b+\theta)C_o^m}{(1-\alpha)Q} \\ &+(1-x\gamma)\{a - bp_m + \theta(M_p - p_m)\} - (b+\theta)\{(1-\gamma x)p_m - p_s\} \\ &+(b+\theta)\left\{\left(p_s + \frac{L}{P} + \gamma P\right) + z\beta\left(p_s + \frac{L}{zP} + \gamma zP\right)\right\}\end{aligned}$$

The necessary condition $\frac{dTP_m}{dp_m} = 0$, for existence of optimal solution, yields

$$\begin{aligned}p_m^* &= \frac{1}{2(x\gamma-1)(b+\theta)}\Big[a(x\gamma - 1) - (1+z\beta)(b+\theta)C_i^m + (x\gamma - 1)\theta M_p \\ &-(b+\theta)p_s + \frac{(b+\theta)C_o^m}{Q(\alpha-1)} - \frac{Q(\alpha-1)(1+\beta+\beta^2)(b+\theta)C_h^m}{2P} \\ &- \frac{(b+\theta)}{P}\{L(1+\beta) + P^2(1+z^2\beta)\gamma + P(1+z\beta)p_s\}\Big].\end{aligned} \tag{21}$$

Also,

$$\frac{d^2TP_m}{dp_m^2} = -2(b+\theta)(1-x\gamma) < 0.$$

Therefore, TP_s is a concave function of p_m and thus (12) provides global maximum profit of the manufacturer.

Theorem 4.3 *The wholesaler's total profit function $TP_w(p_w)$ is concave.*

Proof. Differentiate Eq. (17) with respect to p_w

$$\begin{aligned}\frac{dTP_w}{dp_w} &= \frac{bC_i^w}{(1-\gamma)} + \frac{bC_o^w}{Q(1-\alpha)} - \frac{bQ(1-\alpha)C_h^w}{2(1-\gamma)\{a-bp_m+\theta(M_p-p_m)\}} \\ &+(1+y\gamma)(a - bp_w) - b\{(1+\gamma y)p_w - p_m\}\end{aligned}$$

The necessary condition $\frac{dTP_w}{dp_w} = 0$, for existence of optimal solution, yields

$$p_w^* = \frac{1}{2b(1+y\gamma)}\left[a(1+y\gamma) + bp_m + \frac{bC_i^w}{(1-\gamma)} + \frac{bC_o^w}{Q(1-\alpha)} - \frac{bQ(1-\alpha)C_h^w}{2(1-\gamma)\{a-bp_m+\theta(M_p-p_m)\}}\right]. \tag{22}$$

Further, $\frac{d^2TP_w}{dp_w^2} = -2b(1+y\gamma) < 0$ ensures concavity condition. Substituting the optimal value of the wholesaler's selling price in (17), we have the global maximum profit of the wholesaler.

Theorem 4.4 *The unit production cost function $C(P)$ is convex.*

Proof Since

$$C(P) = p_s + \frac{L}{P} + \gamma P \tag{23}$$

$$\frac{dC(P)}{dP} = -\frac{L}{P^2} + \gamma \tag{24}$$

$$\frac{d^2C(P)}{dP^2} = \frac{2L}{P^3} > 0 \tag{25}$$

$$P = \sqrt{\frac{L}{\gamma}}$$

so, this production rate leads to minimum production cost.

4 Numerical Examples

Example 1. Suppose the input data is as follows: $a = 250$, $b = 0.6$, $p_s = 25$, $C_h^s = 3$, $C_h^m = 4$, $C_h^w = 5$, $C_o^s = 100$, $C_o^m = 250$, $C_o^w = 200$, $C_i^s = 3$, $C_i^m = 2$, $C_i^w = 3$, $P = 100$, $C_p = 10$, $C_p' = 6$, $\alpha = 0.2$, $\beta = 0.5$, $\gamma = 0.1$, $x = 0.5$, $y = 0.4$, $L = 1$, $\Gamma = 0.8$, $M_p = 50$ and $\theta = 0.5$. We check the condition

$$\frac{d^2TP_s}{dQ^2} = -\frac{2C_o^s}{(1-\alpha)Q^3}(a - bp_s) = -0.015339 < 0.$$

Hence the optimal solution is $Q^* = 156.46$, $TP_s^* = 2033.25$ (Fig. 3).

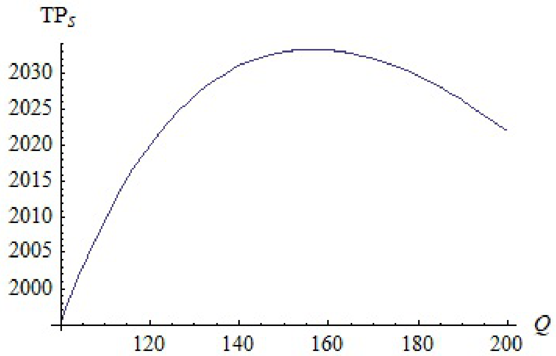

Figure 3: Supplier's total profit versus Q

Example 2. Let us take the values of Example 1 in appropriate units with $Q^* = 156.46$ and apply the solution procedure, to find the optimal solutions $p_m^* = 221.385$, $TP_m^* = 605.331$ (Fig. 4).

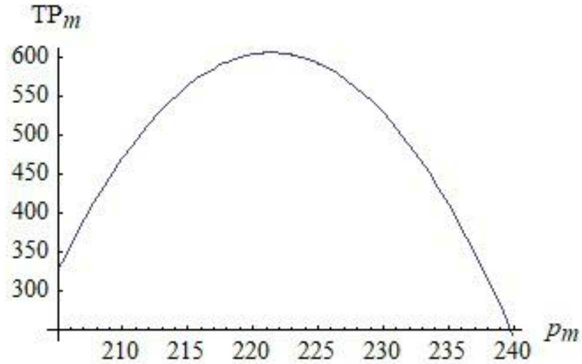

Figure 4: Manufacturer's total profit versus p_m

Example 3. By using above parameters $Q^* = 156.46$, $p_m^* = 221.385$. the optimal values of solution are calculated as: $p_w^* = 311.829$, $TP_w^* = 6545.47$. Fig. 5 illustrate the optimal price p_w^* and the optimal wholesaler's profit TP_w^*.

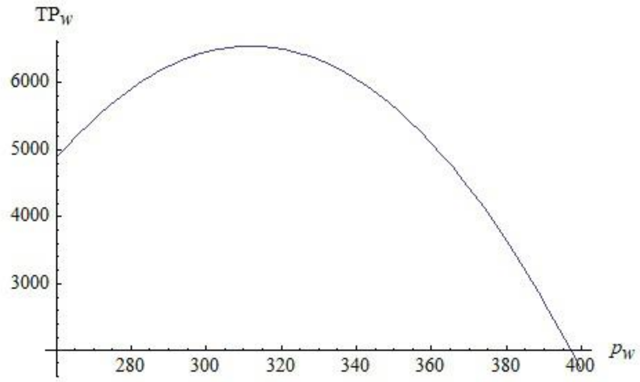

Figure 5: Wholesaler's total profit versus p_w

5 Conclusions

We developed a three-echelon supply chain including a supplier, a manufacturer and a wholesaler. The wholesaler order his demands to the manufacturer and the manufacturer order his demands to the supplier. The shortage is not permitted. The goal of the paper is to optimise the total profit of the supply chain members. The decision variables of the proposed model are the manufacturer's selling price, the wholesaler's selling price and the number of orders received by the supplier, respectively.

Here, demand is price-sensitive and rework at the manufacturer is allowed. However, a Nash-equilibrium method is considered among the members of the supply chain. A numerical example is presented to show the practicality of the proposed model. We considered the unit production cost as a function of production rate. We found that the profits of the chain members are concave with respect to their decision variables.

For future research, the present model can be extended under uncertain demand, the stock out situation, profit sharing, multiple competing wholesalers, delivery lead time may be incorporated.

References

[1] Banerjee, A. (1986). A joint economic-lot-size model for purchaser and vendor. *Decision Sciences*, 17(3), 292–311.

[2] Barzegar Astanjin, M. and Sajadieh, M. S. (2017). Integrated production-inventory model with price-dependent demand, imperfect quality, and investment in quality and inspection. *Aut Journal of Modeling and Simulation*, 49(1) , 43–56.

[3] Ben-Daya, M. and Rahim, A. (1999). Multi-stage lot sizing models with imperfect processes and inspection errors. *Production Planning & Control*, 10(2), 118–126.

[4] Cárdenas-Barrón, L.E. (2008). Optimal manufacturing batch size with rework in a single-stage production system вЂ" A simple derivation. *Computers & Industrial Engineering*, 55(4) , 758–765.

[5] Cárdenas-Barrón, L. E. (2009). Economic production quantity with rework process at a single-stage manufacturing system with planned backorders. *Computers & Industrial Engineering*, 57(3), 1105–1113.

[6] Eroglu, A. and Ozdemir, G. (2007). An economic order quantity model with defective items and shortages. *International Journal of Production Economics*, 106(2), 544–549.

[7] Huang, Y., Huang, G. Q. and Newman, S. T. (2011). Coordinating pricing and inventory decisions in a multi-level supply chain: A game-theoretic approach. *Transportation Research Part E: Logistics and Transportation Review*, 42(2), 115–129.

[8] Khan, M., Jaber, M. Y. and Bonney, M. (2011). An economic order quantity (EOQ) for items with imperfect quality and inspection errors. *International Journal of Production Economics*, 133(1), 113–118.

[9] Khan, M., Jaber, M. Y. and Ahmad, A.-R. (2014). An integrated supply chain model with errors in quality inspection and learning in production. *Omega*, 42(1), 16–24.

[10] Khanna, A., Kishore, A. and Jaggi C. K. (2017). Inventory modeling for imperfect production process with inspection errors, sales return, and imperfect rework process. *International Journal of Mathematical, Engineering and Management Sciences*, 42(2), 242–258.

[11] Khedlekar, U. K. and Singh, P. (2019). Three-layer supply chain policy under sharing recycling responsibility. *Journal of Advances in Management Research*, 16(5), 734–762.

[12] Khouja, M. and Mehrez, A. (1994). Economic production lot size model with variable production rate and imperfect quality. *Journal of the Operational Research Society*, 45(12), 1405–1417.

[13] Liu, N., Kim, Y. and Hwang, H. (2009). An optimal operating policy for the production system with rework. *Computers & Industrial Engineering*, 56(3), 874–887.

[14] Mukherjee, A., Dey, O. and Giri, B. C. (2019). An integrated vendorвЂ"buyer model with stochastic demand, lot-size dependent lead-time and learning in production. *Journal of Industrial Engineering International*, 15, 165–178.

[15] Noori-daryan, M. and Taleizadeh, A. A. (2018). Optimizing pricing and ordering strategies in a three-level supply chain under return policy. *Journal of Industrial Engineering International*, 15, 73–80.

[16] Pal, B., Sana, S. S. and Chaudhuri, K. (2012). Three-layer supply chain - A production-inventory model for reworkable items. *Applied Mathematics and Computation*, 219(2), 530–543.

[17] Priyan, S. and Manivannan, P. (2016). Optimal inventory modeling of supply chain system involving quality inspection errors and fuzzy defective rate. *OPSEARCH*, 54(1), 21–43.

[18] Salameh, M. K. and Jaber, M. Y. (2000). Economic production quantity model for items with imperfect quality. *International Journal of Production Economics*, 64(1), 59–64.

[19] Sana, S. S. (2011). A production-inventory model of imperfect quality products in a three-layer supply chain, *Decision Support Systems*, 50(2), 539–547.

[20] Taleizadeh, A. A. and Noori-daryan, M. (2014). Pricing, manufacturing and inventory policies for raw material in a three-level supply chain. *International Journal of Systems Science*, 47(4) , 919–931.

[21] Vishkaei, B. M., Niaki, S. T. A., Farhangi, M. and Rashti, M. E. M. (2014). Optimal lot sizing in screening processes with returnable defective items. *Journal of Industrial Engineering International*, 10(3), 1–9.

[22] Wang Chiu, S. (2006). Optimal replenishment policy for imperfect quality EMQ model with rework and backlogging. *Applied Stochastic Models in Business and Industry*, 23(2), 165–178.

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Performance Analysis of Network System Configured as Series-Parallel Subject to Different Repair Policies: Copula Approach to Joint Probability Distribution

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Abstract

A network system is designed in this paper. It has two labs connected in parallel with four units each; working under 3-out-of-4: G; policy, two servers connected in parallel working under 1-out-of-2: G; policy and a router. The labs, servers and the router are connected in series all together. By making use of a supplementary variable and Laplace transforms to varies measures of system reliability, the network system has been studied and evaluation of Sensitivity, Availability, Reliability, (Mean time to failure) MTTF, and cost analysis for particular values of the failure and repair rates is made. Conclusion has been done with computed results demonstrated by tables, figures and graphs.

Keywords: Network, System, Availability; Reliability; Gumbel Hougard, series-parallel.

1. Introduction

It is very vital that a network system or any industrial system should be defendable and consistent; such system is more desirable than any other. The availability and reliability of that system plays a big role in its production output, performance, industrial growth and expected profit. In trying to enhance the performance of such desirable network/industrial systems many researchers evaluate different systems. Kabiru and Singh [1] made a reliability assessment of complex system consisting two subsystems connected in series configuration using Gumbel-Hougaard family copula distribution. Yusuf and Ismail [6] focused on reliability analysis of communication network system with redundant Relay Station under Partial and Complete Failure.

Singh and Poonia [14] study the performance analysis of a complex repairable system with two subsystems in series configuration with an imperfect switch. Using transition diagrams and systems of first order differential difference equations Yusuf and Sani in [2] developed and solved a system recursively to obtain the steady-state availability, busy period of repair men and profit function. Kakkar and Chitkara [15] consider finding the reliability of two unit parallel industrial system. Lado and Singh [4] develop a model of a complex repairable system having two subsystems A and B which is connected in a series configuration. Kakkar and Chitkara [9] present reliability of two dissimilar parallel units in the present of preventive maintenance. Yusuf and Mahmud [3] use Markov model of a system derived through the system state transition table and differential difference equations which are further used to evaluate the system availability and mean time to system failure and profit.

Using Copula Singh and Gulati [13] studied the performance analysis of complex system in series configuration under different failure and repair discipline. Garg [18] use fuzzy Kolmogorov's differential equations to make an approach for analyzing the reliability of industrial system. Yusuf and Bala [21] made an analysis of reliability characteristics of a parallel system with external supporting devices for operation.

2. State Description, Notation, and Assumptions

2.1 State description

S_0: This state represents a fully functional system with three working units and one standby unit in both labs, one operational server and one standby server, and working router.
S_1: This represents operational state with one unit down in lab 1.
S_2: This represents failure state due to failure of two units in lab 1.
S_3: This represents operational state with one unit down in lab 2.
S_4: This represents failure state due to failure of two units in lab 2.
S_5: This represents operational state with one server down.
S_6: This represents failure state due to failure of both servers.
S_7: This represents failure state due to failure of the main router.
S_8: This represents operational state with one unit in lab 1 down and one server under repair.
S_9: This represents operational state with one unit in lab 2 down and one server under repair.
S_{10}: This represents operational with one server down and a unit in lab 2 under repair.
S_{11}: This represents operational with one server down and a unit in lab 1 under repair.

2.2 Notations

- t Stands for Time variable on a time scale.
- s Stands for Laplace transform variable for all expressions.
- β_1 Stands for Failure rate of any unit in lab 1.
- β_2 Stands for Failure rate of any unit in lab 2.
- β_3 Stands for Failure rate of any server.
- β_4 Stands for Failure rate of the router.
- $\phi(a)$ Stands for all Repair rates.
- $P_i(t)$ Stands for the probability that the system is in S_i state at instants for i = 0 to 9.
- $\bar{P}(t)$ Stands for Laplace transformation of the state transition probability P(t).
- $P_i(a,t)$ Stands for the probability that a system is in state S_i for $i = 1,2,\dots,11$, the system is running under repair and elapse repair time is (a, t) with repair variable a and time variable t.
- $E_p(t)$ Stands for Expected profit during the time interval [0, t).
- K_1, K_2 Stands for Revenue and service cost per unit time respectively.

2.3 Assumptions

It's assumed that at the beginning all the units in labs, servers and the router are working perfectly. At least three units from the two labs, one server and a router are needed for the system to operate. Failure rates are regarded as the same and may follow exponential distribution. Likewise repairs follow general distribution.

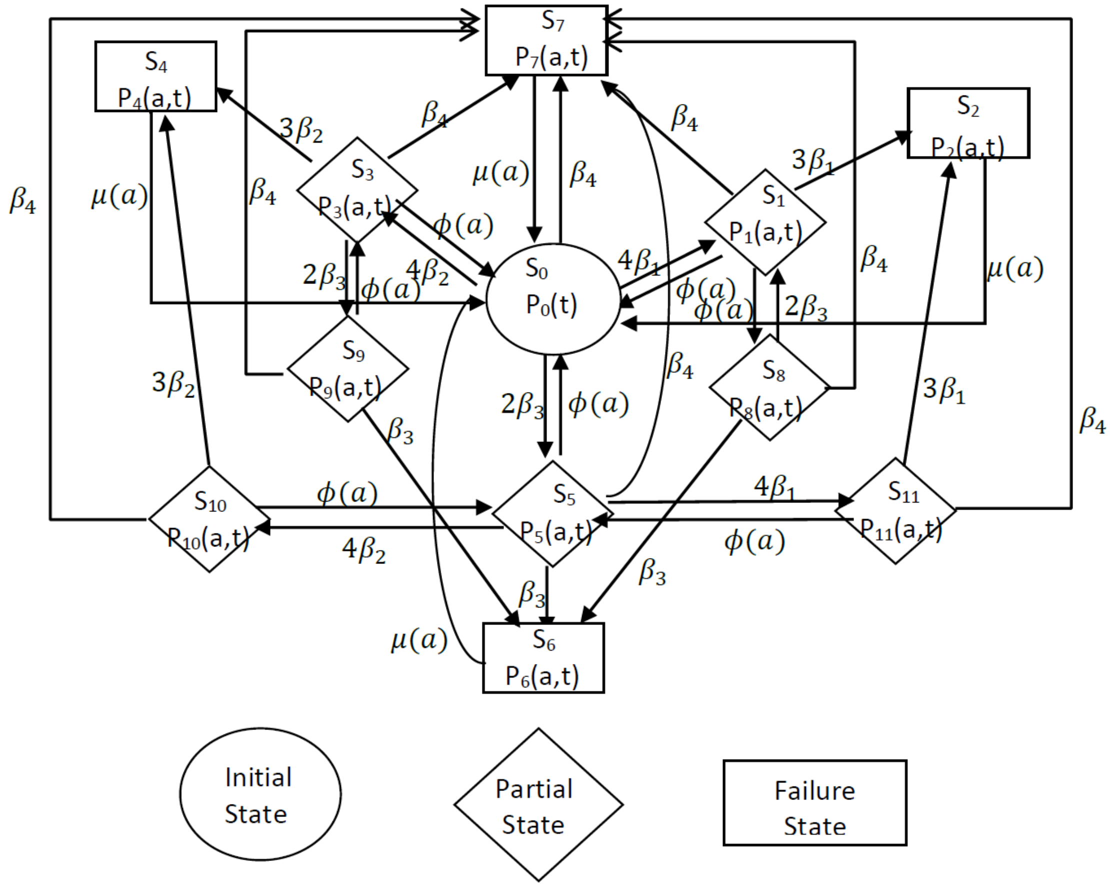

Figure 1: Transition Diagram

3. Mathematical Model Formulation

Bearing in mind the probability and continuity arguments, train set of difference differential equations and the present mathematical model are lump together as:

$$\left(\frac{\partial}{\partial t}+4\beta_1+4\beta_2+2\beta_3+\beta_4\right)p_0(t)=\int_0^\infty \emptyset(a)p_1(a,t)da+\int_0^\infty \mu(a)p_2(a,t)da+\int_0^\infty \emptyset(a)p_3(a,t)da+\int_0^\infty \mu(a)p_4(a,t)da+\int_0^\infty \emptyset(a)p_5(a,t)da+\int_0^\infty \mu(a)p_6(a,t)da+\int_0^\infty \mu(a)p_7(a,t)da \tag{1}$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}+3\beta_1+2\beta_3+\beta_4+\emptyset(a)\right)p_1(a,t)=0 \tag{2}$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}+\mu(a)\right)p_2(a,t)=0 \tag{3}$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}+3\beta_2+2\beta_3+\beta_4+\emptyset(a)\right)p_3(a,t)=0 \tag{4}$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}+\mu(a)\right)p_4(a,t)=0 \tag{5}$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}+4\beta_1+4\beta_2+\beta_3+\beta_4+\emptyset(a)\right)p_5(a,t)=0 \tag{6}$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}+\mu(a)\right)p_6(a,t)=0 \tag{7}$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}+\mu(a)\right)p_7(a,t)=0 \tag{8}$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}+\beta_3+\beta_4+\emptyset(a)\right)p_8(a,t)=0 \tag{9}$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}+\beta_3+\beta_4+\emptyset(a)\right)p_9(a,t)=0 \tag{10}$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}+3\beta_2+\beta_4+\emptyset(a)\right)p_{10}(a,t)=0 \tag{11}$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}+3\beta_1+\beta_4+\emptyset(a)\right)p_{11}(a,t)=0 \tag{12}$$

Boundary condition

$$p_1(a,t)=4\beta_1 p_0(t) \tag{13}$$
$$p_2(a,t)=3\beta_1 p_1(a,t) \tag{14}$$
$$p_3(a,t)=4\beta_2 p_0(t) \tag{15}$$
$$p_4(a,t)=3\beta_2 p_3(a,t) \tag{16}$$
$$p_5(a,t)=2\beta_3 p_0(t) \tag{17}$$
$$p_6(a,t)=\beta_3 p_5(a,t) \tag{18}$$
$$p_7(a,t)=\beta_4\big(p_0(t)+p_1(a,t)+p_3(a,t)+p_5(a,t)+p_8(a,t)+p_9(a,t)+p_{10}(a,t)+p_{11}(a,t)\big) \tag{19}$$
$$p_8(a,t)=2\beta_3 p_1(a,t) \tag{20}$$
$$p_9(a,t)=2\beta_3 p_3(a,t) \tag{21}$$
$$p_{10}(a,t)=4\beta_2 p_5(a,t) \tag{22}$$
$$p_{11}(a,t)=4\beta_1 p_5(a,t) \tag{23}$$

Solution of the Model:

Using initial condition, P_0 (0) =1and Laplace transformation of equations (1) to (23) we have the following:

$$(s+4\beta_1+4\beta_2+2\beta_3+\beta_4)\overline{p_0}(s)=\int_0^\infty \emptyset(a)\overline{p_1}(a,s)da+\int_0^\infty \mu(a)\overline{p_2}(a,s)da+\int_0^\infty \emptyset(a)\overline{p_3}(a,s)da+\int_0^\infty \mu(a)\overline{p_4}(a,s)da+\int_0^\infty \emptyset(a)\overline{p_5}(a,s)da+\int_0^\infty \mu(a)\overline{p_6}(a,s)da+\int_0^\infty \mu(a)\overline{p_7}(a,s)da \tag{24}$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}+3\beta_1+2\beta_3+\beta_4+\emptyset(a)\right)p_1(a,s)=0 \tag{25}$$
$$\left(s+\frac{\partial}{\partial a}+\mu(a)\right)\overline{p_2}(a,s)=0 \tag{26}$$
$$\left(s+\frac{\partial}{\partial a}+3\beta_2+2\beta_3+\beta_4+\emptyset(a)\right)\overline{p_3}(a,s)=0 \tag{27}$$
$$\left(s+\frac{\partial}{\partial a}+\mu(a)\right)\overline{p_4}(a,s)=0 \tag{28}$$
$$\left(s+\frac{\partial}{\partial a}+4\beta_1+4\beta_2+\beta_3+\beta_4+\emptyset(a)\right)\overline{p_5}(a,s)=0 \tag{29}$$
$$\left(s+\frac{\partial}{\partial a}+\mu(a)\right)\overline{p_6}(a,s)=0 \tag{30}$$
$$\left(s+\frac{\partial}{\partial a}+\mu(a)\right)\overline{p_7}(a,s)=0 \tag{31}$$
$$\left(s+\frac{\partial}{\partial a}+\beta_3+\beta_4+\emptyset(a)\right)\overline{p_8}(a,s)=0 \tag{32}$$
$$\left(s+\frac{\partial}{\partial a}+\beta_3+\beta_4+\emptyset(a)\right)\overline{p_9}(a,s)=0 \tag{33}$$
$$\left(s+\frac{\partial}{\partial a}+3\beta_2+\beta_4+\emptyset(a)\right)\overline{p_{10}}(a,s)=0 \tag{34}$$
$$\left(s+\frac{\partial}{\partial a}+3\beta_1+\beta_4+\emptyset(a)\right)\overline{p_{11}}(a,s)=0 \tag{35}$$

Laplace transform of boundary conditions

$$\overline{p_1}(0,s)=4\beta_1\overline{p_0}(s) \tag{36}$$
$$\overline{p_2}(0,s)=3\beta_1\overline{p_1}(0,s) \tag{37}$$
$$\overline{p_3}(0,s)=4\beta_2\overline{p_0}(s) \tag{38}$$
$$\overline{p_4}(0,s)=3\beta_2\overline{p_3}(0,s) \tag{39}$$
$$\overline{p_5}(0,s)=2\beta_3\overline{p_0}(s) \tag{40}$$
$$\overline{p_6}(0,s)=\beta_3\overline{p_5}(0,s) \tag{41}$$
$$\overline{p_7}(0,s)=\beta_4\big(\overline{p_0}(s)+\overline{p_1}(0,s)+\overline{p_3}(0,s)+\overline{p_5}(0,s)+\overline{p_8}(0,s)+\overline{p_9}(0,s)+\overline{p_{10}}(0,s)+\overline{p_{11}}(0,s)\big) \tag{42}$$
$$\overline{p_8}(0,s)=2\beta_3\overline{p_1}(0,s) \tag{43}$$

$$\overline{p_9}(0,s) = 2\beta_3\overline{p_3}(0,s) \tag{44}$$
$$\overline{p_{10}}(0,s) = 4\beta_2\overline{p_5}(0,s) \tag{45}$$
$$\overline{p_{11}}(0,s) = 4\beta_1\overline{p_5}(0,s) \tag{46}$$

The solution of (24) to (35) with (36) to (46) results as;

$$\overline{P_0}(s) = \frac{1}{D(s)} \tag{47}$$
$$\overline{P_1}(s) = \frac{4\beta_1}{D(s)}\left\{\frac{1-\bar{S}_\phi(s+3\beta_1+2\beta_3+\beta_4)}{s+3\beta_1+2\beta_3+\beta_4}\right\} \tag{48}$$
$$\overline{P_2}(s) = \frac{12\beta_1{}^2}{D(s)}\left\{\frac{1-\bar{S}_\mu(s)}{s}\right\} \tag{49}$$
$$\overline{P_3}(s) = \frac{4\beta_2}{D(s)}\left\{\frac{1-\bar{S}_\phi(s+3\beta_2+2\beta_3+\beta_4)}{s+3\beta_2+2\beta_3+\beta_4}\right\} \tag{50}$$
$$\overline{P_4}(s) = \frac{12\beta_2{}^2}{D(s)}\left\{\frac{1-\bar{S}_\mu(s)}{s}\right\} \tag{51}$$
$$\overline{P_5}(s) = \frac{2\beta_3}{D(s)}\left\{\frac{1-\bar{S}_\phi(s+4\beta_1+4\beta_2+\beta_3+\beta_4)}{s+4\beta_1+4\beta_2+\beta_3+\beta_4}\right\} \tag{52}$$
$$\overline{P_6}(s) = \frac{2\beta_3{}^2}{D(s)}\left\{\frac{1-\bar{S}_\mu(s)}{s}\right\} \tag{53}$$
$$\overline{P_7}(s) = \frac{\beta_4(1+4\beta_1+4\beta_2+2\beta_3+16\beta_1\beta_3+16\beta_2\beta_3)}{D(s)}\left\{\frac{1-\bar{S}_\mu(s)}{s}\right\} \tag{54}$$
$$\overline{P_8}(s) = \frac{8\beta_1\beta_3}{D(s)}\left\{\frac{1-\bar{S}_\phi(s+\beta_3+\beta_4)}{s+\beta_3+\beta_4}\right\} \tag{55}$$
$$\overline{P_9}(s) = \frac{8\beta_2\beta_3}{D(s)}\left\{\frac{1-\bar{S}_\phi(s+\beta_3+\beta_4)}{s+\beta_3+\beta_4}\right\} \tag{56}$$
$$\overline{P_{10}}(s) = \frac{8\beta_2\beta_3}{D(s)}\left\{\frac{1-\bar{S}_\phi(s+3\beta_2+\beta_4)}{s+\beta_3+\beta_4}\right\} \tag{57}$$
$$\overline{P_{11}}(s) = \frac{8\beta_1\beta_3}{D(s)}\left\{\frac{1-\bar{S}_\phi(s+3\beta_1+\beta_4)}{s+\beta_3+\beta_4}\right\} \tag{58}$$

Where $D(s) = s + 4\beta_1 + 4\beta_2 + 2\beta_3 + \beta_4 - \Big\{4\beta_1\{\bar{S}_\phi(s+3\beta_1+2\beta_3+\beta_4)\} + 12\beta_1{}^2\{\bar{S}_\mu(s)\} + 4\beta_2\{\bar{S}_\phi(s+3\beta_2+2\beta_3+\beta_4)\} + 12\beta_2{}^2\{\bar{S}_\mu(s)\} + 2\beta_3\{\bar{S}_\phi(s+4\beta_1+4\beta_2+\beta_3+\beta_4)\} + 2\beta_3{}^2\{\bar{S}_\mu(s)\} + \beta_4(1+4\beta_1+4\beta_2+2\beta_3+16\beta_1\beta_3+16\beta_2\beta_3)\{\bar{S}_\mu(s)\}\Big\}$ (59)

The followings are Laplace transformations of the state transition probabilities when the system is in initial, partial failure and failed condition at any time:

$$\overline{P_{up}}(s) = \overline{P_0}(s) + \overline{P_1}(s) + \overline{P_3}(s) + \overline{P_5}(s) + \overline{P_8}(s) + \overline{P_9}(s) + \overline{P_{10}}(s) + \overline{P_{11}}(s) \tag{60}$$

$$\overline{P_{up}}(s) = \overline{P_0}(s)\Big(1 + 4\beta_1\left\{\frac{1-\bar{S}_\phi(s+3\beta_1+2\beta_3+\beta_4)}{s+3\beta_1+2\beta_3+\beta_4}\right\} + 4\beta_2\left\{\frac{1-\bar{S}_\phi(s+3\beta_2+2\beta_3+\beta_4)}{s+3\beta_2+2\beta_3+\beta_4}\right\} + 2\beta_3\left\{\frac{1-\bar{S}_\phi(s+4\beta_1+4\beta_2+\beta_3+\beta_4)}{s+4\beta_1+4\beta_2+\beta_3+\beta_4}\right\} + 8\beta_1\beta_3\left\{\frac{1-\bar{S}_\phi(s+\beta_3+\beta_4)}{s+\beta_3+\beta_4}\right\} + 8\beta_2\beta_3\left\{\frac{1-\bar{S}_\phi(s+\beta_3+\beta_4)}{s+\beta_3+\beta_4}\right\} + 8\beta_2\beta_3\left\{\frac{1-\bar{S}_\phi(s+3\beta_2+\beta_4)}{s+\beta_3+\beta_4}\right\} + 8\beta_1\beta_3\left\{\frac{1-\bar{S}_\phi(s+3\beta_1+\beta_4)}{s+\beta_3+\beta_4}\right\}\Big) \tag{61}$$

$$\bar{P}_{down}(s) = 1 - \bar{P}_{up}(s)$$

4. Analytical Study of the Model for Particular Case

Availability Analysis

With$\bar{S}_\phi(s) = \frac{\phi}{s+\phi}$, $\bar{S}_\mu(s) = \frac{\mu}{s+\mu}$, $\frac{1-\bar{S}_\phi(s)}{s} = \frac{1}{s+\phi}$, $\frac{1-\bar{S}_\mu(s)}{s} = \frac{1}{s+\mu}$ and considering the values of different parameters as $\beta_1 = 0.03, \beta_2 = 0.02, \beta_3 = 0.05, \beta_4 = 0.06$, $\phi = \mu = 1$ in (61), the expression for availability by taking the inverse Laplace transform is obtained as:

$$\bar{P}_{up}(t) = 0.03892291366e^{-2.838301077t} - 0.06425625912e^{-1.489776405t} - 0.00241062060e^{-1.283327761t} + 0.0002345183193e^{-1.229731692t} + 1.036198863e^{-0.01716306460t} - 0.00204112976e^{-1.150000000t} - 0.006655302231e^{-1.110000000t} \tag{63}$$

Table 1 and Figure 2 below shows different values of $P_{up}(t)$ using (63) and time variable t= 0, 1, 2, 3, 4, 5, 6, 7, 8, 9,

Table 1: Variation of Availability with respect to time

Time(t)	Availability
0	1
2	0.9970
4	0.9672
6	0.9348
8	0.9033
10	0.8728
12	0.8433
14	0.8149
16	0.7874
18	0.7608

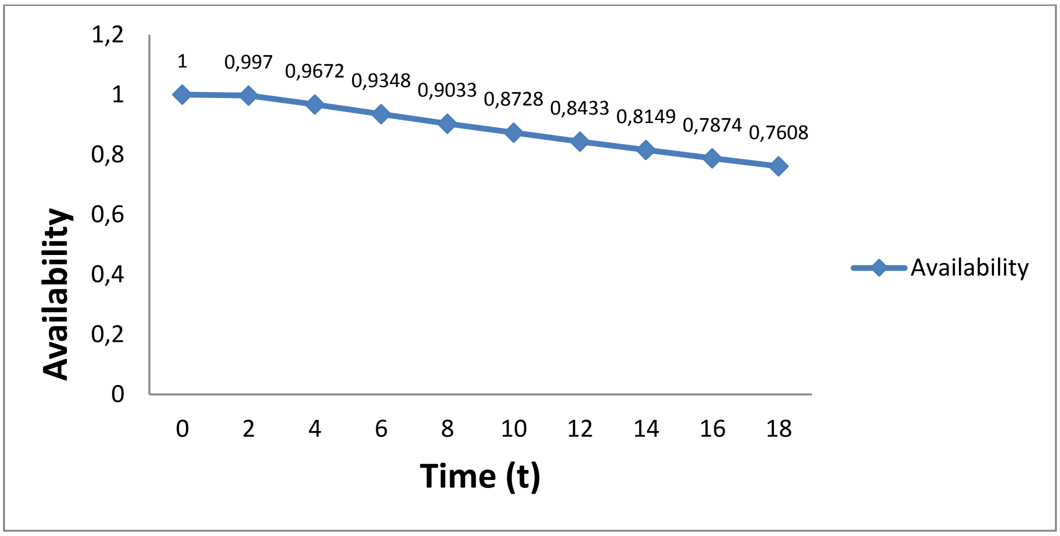

Figure 2: Availability as function of time

Reliability Analysis

All repair rates, ϕ, μ, set to zero in equation (61), the values of failure rates as $\beta_1 = 0.03, \beta_2 = 0.02, \beta_3 = 0.05, \beta_4 = 0.06$ and then computing inverse Laplace transform, express the reliability for the system as;

$$R(t) = -2.889523810e^{-0.3600000000t} + 0.03809523810e^{-0.1500000000t} + 0.08000000000e^{-0.1100000000t} + 0.5714285714e^{-0.2200000000t} + 1.2000000000e^{-0.2500000000t} + 2.\,e^{-0.3100000000t} \tag{64}$$

Table 2 and Figure 3 below shows different values of $P_{up}(t)$ using (63) and time variable t= 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.

Table 2: Computation of reliability for different values of time

Time(t)	Reliability
0	1
2	0.8577
4	0.6451
6	0.4553
8	0.3106
10	0.2081
12	0.1382
14	0.0917
16	0.0610
18	0.0409

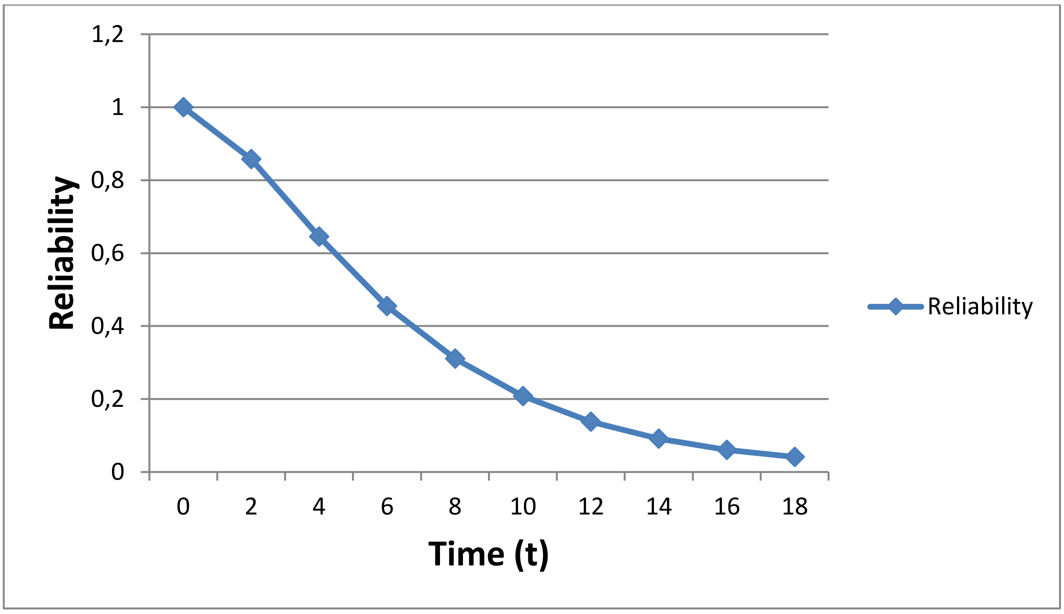

Figure 3: Reliability as function of time

Mean Time to Failure (MTTF) Analysis

All repairs are set to zero in equation (49), and then taking limit, as s tends to zero express MTTF as:

$$MTTF = \lim_{s\to 0} \overline{P_{up}}(s) = \frac{1}{4\beta_1+4\beta_2+2\beta_3+\beta_4}\Big(1+\frac{4\beta_1}{3\beta_1+2\beta_3+\beta_4}+\frac{4\beta_2}{3\beta_2+2\beta_3+\beta_4}+\frac{2\beta_3}{4\beta_1+4\beta_2+\beta_3+\beta_4}+\frac{8\beta_1\beta_3}{\beta_3+\beta_4}+\frac{8\beta_2\beta_3}{\beta_3+\beta_4}+\frac{8\beta_2\beta_3}{3\beta_1+\beta_4}+\frac{8\beta_1\beta_3}{3\beta_1+\beta_4}\Big) \quad (65)$$

Table 3 and figure 4 below express the variation of MTTF with respect to failure rates, by fixing$\beta_2 = 0.02, \beta_3 = 0.05, \beta_4 = 0.06$ and varying β_1 as $0.01, 0.02, 0.03, 0.04, 005, 0.06, 0.07, 0.08, 0.09$, fixing $\beta_1 = 0.03, \beta_3 = 0.05, \beta_4 = 0.06$ and varying β_2 as $0.01, 0.02, 0.03, 0.04, 005, 0.06, 0.07, 0.08, 0.09$, fixing$\beta_1 = 0.03, \beta_2 = 0.02, \beta_4 = 0.06$ and varying β_3 as $0.01, 0.02, 0.03, 0.04, 005, 0.06, 0.07, 0.08, 0.09$, fixing $\beta_1 = 0.03, \beta_2 = 0.02, \beta_3 = 0.05$, and varying β_4 as$0.01, 0.02, 0.03, 0.04, 005, 0.06, 0.07, 0.08, 0.09$, in (61).

Table 3: Computation of MTTF corresponding to the various values of failure rates

Failure rate	MTTF β_1	MTTF β_2	MTTF β_3	MTTF β_4
0.01	8.0406	7.2282	8.6971	9.6404
0.02	7.4263	6.8927	8.0926	8.9180
0.03	6.8927	6.5597	7.6128	8.3040
0.04	6.4307	6.2478	7.2208	7.7725
0.05	6.0297	5.9624	6.8927	7.3061
0.06	5.6798	5.7036	6.6126	6.8927
0.07	5.3726	5.4699	6.3694	6.5231
0.08	5.1012	5.2586	6.1555	6.1903
0.09	4.8600	5.0673	5.9650	5.8889

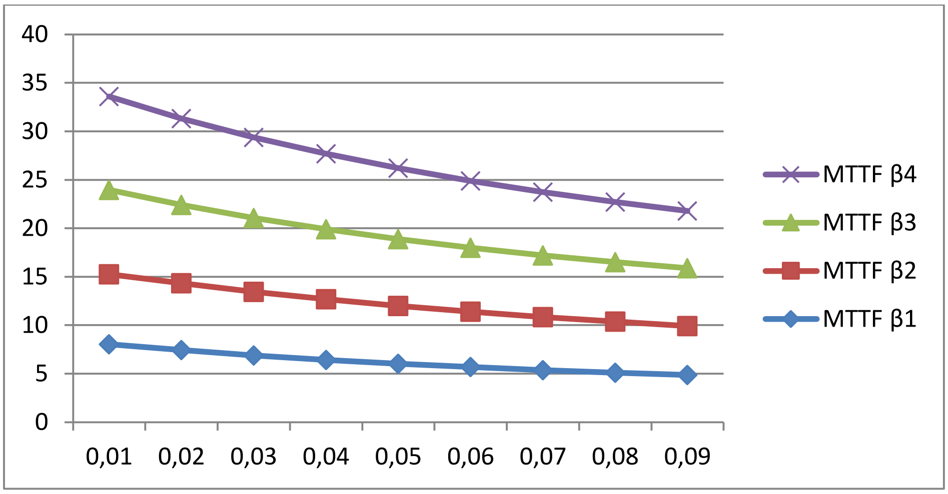

Figure 4: MTTF corresponding to the various values of failure rates

Sensitivity Analysis corresponding to (MTTF)

Table 4 and figure 5 below, express the sensitivity in MTTF of the system, using the partial differentiation of MTTF with respect to the failure rates of the system and setting parameters as $\beta_1 = 0.03, \beta_2 = 0.02, \beta_3 = 0.05, \beta_4 = 0.06$.

Table 4: MTTF sensitivity as function of time

Failure rate	∂(MTTF)/ β_1	∂(MTTF)/ β_2	∂(MTTF)/ β_3	∂(MTTF)/ β_4
0.01	-65.5677	-32.4004	-68.4440	-78.8498
0.02	-57.2902	-33.9081	-53.4467	-66.2544
0.03	-49.6020	-32.4031	-43.1042	-56.9357
0.04	-42.9726	-29.8976	-35.6883	-49.6544
0.05	-37.3926	-27.1880	-30.1944	-43.8144
0.06	-32.7291	-24.5889	-26.0096	-39.0221
0.07	-28.8269	-22.2133	-22.7442	-35.0194
0.08	-25.5471	-20.0894	-20.7423	-31.6289
0.09	-22.7743	-18.2092	-28.7236	-28.7236

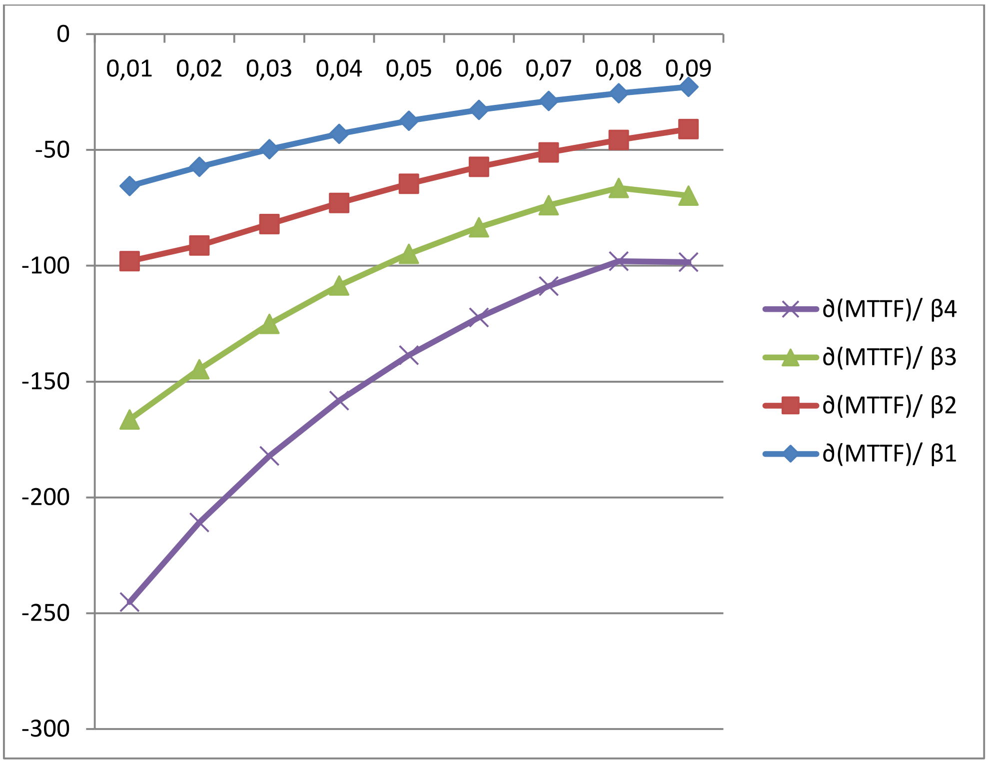

Figure 5: MTTF sensitivity as function of time

Cost Analysis

$$E_P(t) = K_1 \int_0^t P_{up}(t)dt - K_2 t \tag{65}$$

Equation (65) can be used to attain expected profit of the system in the interval [0,t), if service facilities is always available.

Using (61) and (65), we obtain:

$$E_P(t) = K_1(-0.01371345485e^{-2.838301077t} + 0.04313147859e^{-1.489776405t} + 0.001878413822e^{-1.2833327761t} - 0.0001907069004e^{-1.229731692t} - 60.37376699e^{-0.017163064600t} + 0.001768793892e^{-1.150000000t} + 0.005995767776e^{-1.110000000t} + 60.335) - K_2 t \tag{66}$$

Table 5 and Figure 6 express the expected profit by setting $K_1 = 1$and $K_2 =$ 0.6, 0.5, 0.4, 0.3, 0.2 and 0.1 respectively and varying time t = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 in (66).

Table 5: Expected profit as function of time

Time(t)	Ep(t);K2=0.6	Ep(t);K2=0.5	Ep(t);K2=0.4	Ep(t);K2=0.3	Ep(t);K2=0.2	Ep(t);K2=0.1
0	0	0	0	0	0	0
2	0.8016	0.0016	1.2016	1.4016	1.6016	1.8016
4	1.5672	1.9672	2.3672	2.7672	3.1672	3.5672
6	2.2690	2.8690	3.4690	4.0690	4.6690	5.2690
8	2.9069	3.7069	4.5069	5.3069	6.1069	6.9069
10	3.4828	4.4828	5.4828	6.4828	7.4828	8.4828
12	3.9987	5.1987	6.3987	7.5987	8.7987	9.9987
14	4.4567	5.8567	7.2567	8.6567	10.0567	11.4567
16	4.8588	6.4588	8.0588	9.6588	11.2588	12.8588
18	5.2069	7.0069	8.8069	10.6069	12.4068	14.2069

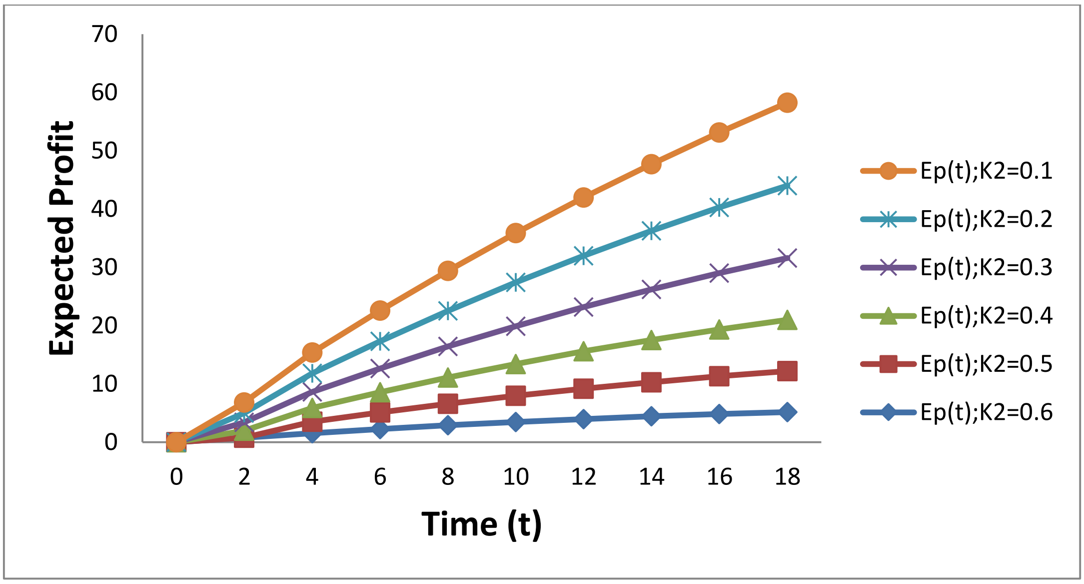

Figure 6: Expected profit as function of time

5. Conclusions

To make a conclusion on the performance of the system in this study, a study of the reliability measures on repair and failure rates with different values has been made. A provision of information on the availability of the system changing with time at fixed failure rates with different values has been made in table 1 and figure 2. It has been evidently found that gently and slowly the availability of system decreases and the probability of the failure increases at fixed failure rates $\beta_1 = 0.03, \beta_2 = 0.02, \beta_3 = 0.05, \beta_4 = 0.06$, and at a long run it will become steady to zero value. Therefore, at any chosen time and set of parametric values, one can easily say the future behavior of the system.

Setting repairs to zero, an analysis of the system reliability has been provided in table 2 and figure 3. By studying the availability and reliability of the system, one can conclude that providing repair to the system is better than replacement.

With other parameters considered as constant, mean-time-to-failure (MTTF) in variation of $\beta_1, \beta_2, \beta_3$ *and* β_4 respectively has been provided in figure 4. It also signifies that the variation of $\beta_1, \beta_2, \beta_3$ *and* β_4 are responsible for the better performance of the system.

Figure 5 shows the sensitivity analysis of system.

Figure 6 and table 5 shows the calculations of the profit at fixed revenue cost $K_1 = 1$ per unit time and service cost $K_2 = 0.6, 0.5, 0.4, 0.3, 0.2$ and 0.1. The result shows that when the service cost $K2$ fixed at minimum value 0.1, the expected profit increases with respect to the time. In the end, one can observe that profit decrease whenever service cost increase.

References

[1] Kabiru, H., Singh, V., & Lado, A. (2017). Reliability assessment of complex system consisting two subsystems connected in series configuration using Gumbel-Hougaard family copula distribution. *Journal of Applied Mathematics & Bioinformatics*, 7(2), 1-27.

[2] Yusuf, I., Sani, B., & Yusuf, B. (2019). Profit Analysis of a Serial-parallel System under Partial and Complete Failures. Journal of Applied Sciences ISSN 1812-5654, 565-574. DOI: 10.3923/jas.

[3] Yusuf, I., Mahmud, S., & Sulaiman, K. (2018). Performance analysis of a single host system with three types of heterogeneous software. Research Journal of Applied Sciences, 13(12), 736-741.

[4] Lado, A., Singh, V. V. & Yusuf, I. (2018). Performance and cost assessment of repairable complex system with two subsystems connected in series configuration. International journal of reliability and applications, 19(1), pp. 27-42.

[5] Singh, V.V. & Ayagi, H.I., (2018). Stochastic analysis of a complex system under preemptive resume repair policy using Gumbel-Hougaard family of copula. International Journal of Mathematics in Operational Research, 12(2). 273-292.

[6] Yusuf, I., Ismail, A. L. & Ali, U. A., (2020). Reliability Analysis of Communication Network System with redundant Relay Station under Partial and Complete Failure. Journal of Mathematical And Computational Science 10(4) 863-880,

[7] Gahlot, M., Singh V. V., Ayagi, H. I., & Goel, C. K., (2018). Performance assessment of repairable system in series configuration under different types of failure and repair policies using copula linguistics, Int. J. Reliab. Safety, 12(4), 348-363.

[8] Gulati, J., Singh, V. V. & Rawal, D. K., (2018). Performance assessment of two unit's redundant system under different failure and repair policies using copula. Int. J. Reliab. Appl. 19(2), 109-124.

[9] Kakkar, M., Chitkara, A. & Bhatti, J. (2016). Reliability analysis of two dissimilar parallel unit repairable systems with failure during preventive maintenance. Manage. Sci. Lett. 6(4), 285-296.

[10] Kumar, A., Pant, S. & Singh, S. B. (2016). Reliability optimization of complex system by using cuckoos search algorithm, in Ram, M. and Davim, J.P. (Eds), Mathematical Concepts and Applications in Mechanical Engineering and Mechatronics. IGI Global Publisher, Hershey, PA, 95-112.

[11] Sadeghi, M. & Roghanian, E. (2017). Reliability analysis of a warm standby repairable system with two cases of imperfect switching mechanis. Sci. Iran. E, 24(2), 808-822.

[12] Kumar, A., Pant, S., Singh, S. B., (2017). Availability and cost analysis of an engineering system involving subsystems in series configuration. Int. J. Qual. Reliab. Manage. 34(6), 879-894

[13] Singh, V. V., Gulati, J., Rawal, D. K, & Goel, C. K., (2016). Performance analysis of complex system in series configuration under different failure and repair discipline using copula. Int. J. Reliab. Qual. Safety Eng. 23(2), 1650010.

[14] Singh, V. V., Poonia, P. K. & Abdullahi, A. F. (2020). Performance analysis of a complex repairable system with two subsystems in series configuration with an imperfect switch. Journal of Mathematical and Computational science, Journal of Mathematics & Computational Science. (JMCS). Vol. 10(2).pp. 359-383

[15] Kakkar, M., Chitkara, A. and Bhatti, J. (2015a). Reliability analysis of two unit parallel repairable industrial system", Decision Sci. Lett., 4(4), pp. 525-536.

[16] Kumar, A. and Malik, S. (2015). Reliability measures of a computer system with priority to PM over the H/W repair activities subject to MOT and MRT", Manage. Sci. Lett., 5(1), pp. 29-38.

[17] Niwas, R. & Kadyan, M.S. (2015) 'Reliability modeling of a maintained system with warranty and degradation', *Journal of Reliability and Statistical Studies*, Vol. 8, No. 1, pp.63–75.

[18] Garg, H. (2015). An approach for analyzing the reliability of industrial system using fuzzy Kolmogorov's differential equations, Arabian Journal for Science and Engineering, 40(3), 975-987.

[19] Garg, H. (2014). Reliability, availability and maintainability analysis of industrial systems using PSO and fuzzy methodology, MAPAN, 29(2), 115-129.

[20] Pundir, P. S & Patawa, R. (2019). Stochastic behavior of dissimilar units cold standby system waiting for repair. *Life Cycle Reliability and Safety Engineering*, *8*(1), 43-53.

[21] Yusuf, I and Bala, S.I. (2013). Analysis of reliability characteristics of a parallel system with external supporting devices for operation, International Journal of Information and Management Sciences, 24, 297-306.

[22] Yang, D. Y & Tsao, C. L. (2019). Reliability and availability analysis of standby systems with working vacations and retrial of failed components. *Reliability Engineering & System Safety*, *182*, 46-55.

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