RT&A, No 4 (59)
Volume 15, December 2020

Correlated Reneging in an Optional Service Markovian Queue With Working Vacations

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Abstract

This paper studies an $M/M/1$ queueing system with second optional service, correlated reneging and working vacations. All arriving customers require the first essential service whereas only a portion of them require a second optional service. The matrix geometric method is used to compute the stationary probability distribution of the system size. Further, various system performance measures are obtained and a cost optimization problem is considered using bat algorithm (BA). A variety of numerical illustrations are summarized in tables and graphs to provide an insight into the performance characteristics of the studied model.

Keywords: first essential service, second optional service, single working vacation, multiple working vacations, correlated reneging

I. Introduction

Vacation queues have been one of the intensive research topics for long time. There has been a considerable attention paid to the queueing models with server vacations, see Doshi [4]. During the vacation period, the server can be utilized for ancillary work, for example, in web services, file transfer services, manufacturing systems, etc. Such queueing model was first introduced by Servi and Finn [17] in an $M/M/1$ queueing system with working vacations and applied those results to analyze the performance of gate way router in communication networks. Later, Selvaraju and Goswami [16] have considered a single server impatient customers Markovian queueing system with single working vacation (SWV) and multiple working vacations (MWV). Rajadurai et al. [14] gave an analysis of a single server feedback retrial queueing system with subject to server breakdown and repair under MWV policy using probability generating function technique.

As for optional service, Madan [10] first investigated an $M/G/1$ queueing system with second optional service (SOS), in which some of the customers may require a SOS immediately after completion of the first essential service (FES). Using matrix geometric method, Jain and Chauhan [5] was able to approximate working vacation (WV) queueing system with SOS and unreliable server. Batch arrival bulk service queue with unreliable server, SOS and two different types of vacations has been investigated by Ayyappan and Supraja [2]. Manoharan and Sasi [11] discussed an $M/G/1$ queueing system with SOS and second optional vacation. A retrial queueing system with SOS under Erlang services has been investigated by Sekar et al. [15] using matrix geometric method.

Naturally, server vacations increase the waiting time of the customers. Due to longer wait in the queue, some customers may get discouraged and may decide to leave the queue without receiving the service (reneging). Such type of situations occur in real life like customers waiting in call centers, hospital emergency rooms, web services, programs waiting to be processed on a computer, etc. Queueing models with reneging have been investigated by many authors like Ancker and Gafarian [1], Baruah et al. [3], etc. The transient and steady-state behavior of the $M/M/1$ queue having customers' impatience with threshold has been discussed in Sharma et al. [18]. Mohan [12] introduced the concept of correlation in gambler's ruin problem. In Kim and Kim [7] , waiting time distribution of an $M/M/1$ queue was investigated where the inter arrival time between the n^{th} and $(n+1)^{th}$ customers and the service time of the n^{th} customer are correlated random variables with Downton's bivariate exponential distribution. A catastrophic queueing model with correlated input for the cell traffic generated by new broadband services has been studied by Jain and Kumar [6]. Kumar [8] studied a catastrophic-cum-restorative queueing problem with correlated input and impatient customer. There is another concept of correlated reneging wherein a customer's reneging at any time instant depends solely on the previous time instants' reneging or non-reneging. Transient numerical analysis of a single server queueing model with correlated reneging, balking and feedback has been carried out by Kumar and Soodan [9].

Existing literature shows frequent research topics related with WVs and SOS. However, a research gap observes no previous work on SOS, WVs in a queue with correlated reneging. As these topics are important in the real life situations, we consider an infinite capacity single server queueing system with SOS, WVs and correlated reneging. We have used matrix geometric method to obtain the steady state system length distributions. Some performance measures have been discussed. We employ the recently developed bat algorithm which was introduced by Yang [19] to achieve the optimal values of decision parameters and the expected cost. Particular cases of the model have been given. Later, a variety of numerical illustrations have been presented through tables and graphs.

The remainder of the model is structured as follows: Model description and practical justification of the model are presented in Section 2. In Section 3, the mathematical formulation of the model is given. Matrix geometric solution is given in Section 4. Section 5 is devoted to some performance measures, cost model and special cases of the model. Numerical investigations are given in the form of tables and graphs in Section 6. Finally, Section 7 concludes our paper.

II. Model Description

Consider a single server queueing system with SOS, WVs and correlated reneging. The model under consideration is schematically represented in Figure 1.

The queueing model is based on the following assumptions.

1. Customers arrive according to a Poisson process with rate λ.
2. The FES is provided to all customers. Immediately after completion of FES, a customer may demand SOS with probability r or he may leave the system with the complementary probability $(1-r)$. The service times of both FES and SOS are exponentially distributed with parameters μ_1 and μ_2, respectively.
3. At the end of a service, if there is no customer in the system, the server begins a WV of random length which is exponentially distributed with parameter θ. During WV, service is provided according to a Poisson distribution with parameter η. In SWV, when the server returns from WV period and finds no customer in the system, it does not take another WV but remains idle until the next arrival. But MWV policy requires the server to keep taking vacations until it finds at least one customer waiting in the system at a vacation completion instant. When the server returns from its vacation and finds at least one customer in the

system, it switches its service rate from η to μ_1 and a busy period starts; otherwise, it immediately leaves for another WV.

4. During WV, customers become impatient and they may renege from the queue. The reneging of customers can take place only at the transition marks t_0, t_1, t_2,... where $K_m = t_m - t_{m-1}$, $m = 1,2,3,..$, are random variables with $P[K_m \leq x] = 1 - exp(-\alpha x)$; $\alpha > 0, m = 1,2,3,\ldots$ $i.e.$, the distribution of inter-transition marks is negative exponential with parameter α. The average reneging rate of a customer is given by $\alpha_n = n\alpha$, $n \geq 1$.
5. The reneging at two consecutive transition marks is governed by the following transition probability matrix:

$$\begin{array}{ccc} & & \text{To } t_r \\ & & \begin{array}{cc} 0 & 1 \end{array} \\ \text{From } t_{r-1} & \begin{array}{c} 0 \\ 1 \end{array} & \begin{Vmatrix} q_{00} & q_{01} \\ q_{10} & q_{11} \end{Vmatrix} \end{array}$$

where $q_{00}+q_{01} = 1$ and $q_{10} + q_{11} = 1$.
Here, 0 refers to no reneging and 1 refers to the occurrence of reneging.
Thus, the reneging at two consecutive transition marks is correlated.

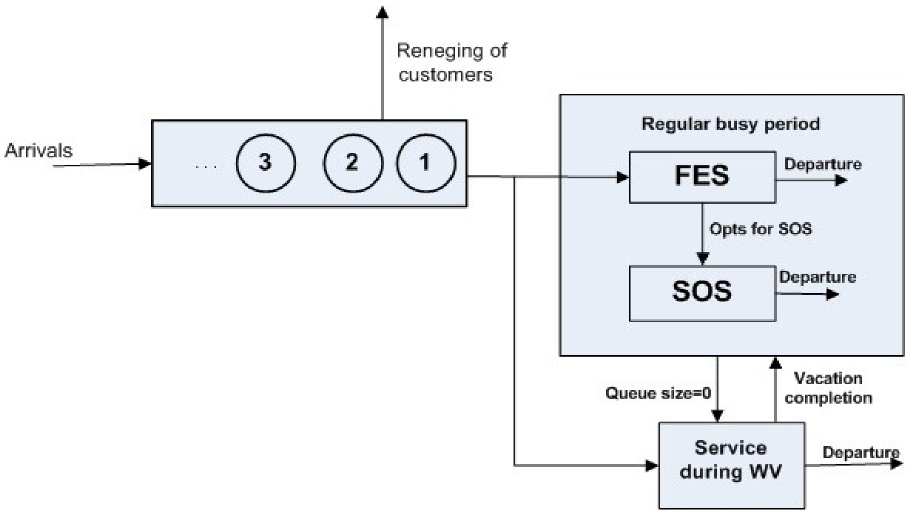

Figure 1: *General structure of the model.*

2.1 Practical Justification of the Model

The above discussed model has real time applications in electronic commerce (also known as E-commerce) which is a process of buying and selling of products, making money transfers and transferring data over an electronic medium. The whole E-commerce process can be divided into three main components, viz. receiving orders, processing order information and shipping. Cross-selling is a sales technique to increase sales by suggesting additional items to customers. When the sales are at their lowest, E-commerce merchant carries out the tasks like contacting suppliers and important clients, managing accounts up to date, etc. The speed of a service process will take a hit during this time and the customers may cancel the orders as they anticipate longer wait. If an order is canceled (not canceled) at any time instant, then there is a chance that an order may or may not be canceled at next time instant. Here, orders, selling of products, cross-selling, maintaining accounts, canceling of orders at time instants can be represented by the arrivals, FES, SOS, WV, correlated reneging, respectively in basic queueing situations.

III. Mathematical Formulation of the Model

At time t, let $N(t)$ be the number of customers in the queue, $J(t)$ be the state of the server, which is defined as

$$J(t)=\begin{cases} 0, \textit{the server is on } WV, \\ 1, \textit{the server is idle or busy } (SWV) \& \\ \quad \textit{the server is busy} (MWV) \end{cases}$$

and $S(t)$ be the state of the customer which is given as

$$S(t)=\begin{cases} 0, \; \textit{no reneging}, \\ 1, \; \textit{occurrence of reneging}. \end{cases}$$

The process $\{L(t), J(t), S(t), t \geq 0\}$ defines a continuous-time Markov process with state space χ $=\{(n,j,s): n \geq 0, j = 0,1, s = 0,1\}$. For mathematical formulation purpose, we define the following steady-state probabilities:

$E_{0,0,0}(E_{0,0,1})$ = Probability that the queue is empty, the server is idle, server is on WV, and a customer has not reneged (reneged) at the previous transition mark.

$E_{0,1,0}(E_{0,1,1})$ = Probability that the queue is empty, the server is idle, server is in busy state, and a customer has not reneged (reneged) at the previous transition mark.

$P_{n,0,0}(P_{n,0,1})$ = Probability of n customers in the queue, the server is not idle, server is on WV, and a customer has not reneged (reneged) at the previous transition mark.

$P_{n,1,0}(P_{n,1,1})$ = Probability of n customers in the queue, the server is not idle, server is rendering FES, and a customer has not reneged (reneged) at the previous transition mark.

$Q_{n,1,0}(Q_{n,1,1})$ = Probability of n customers in the queue, the server is not idle, server is rendering SOS, and a customer has not reneged (reneged) at the previous transition mark.

Steady-state equations:

$$(\lambda + \omega\theta)E_{0,0,0} = \eta P_{0,0,0} + (1-r)\mu_1 P_{0,1,0} + \mu_2 Q_{0,1,0}, \tag{1}$$
$$(\lambda + \omega\theta)E_{0,0,1} = \eta P_{0,0,1} + (1-r)\mu_1 P_{0,1,1} + \mu_2 Q_{0,1,1}, \tag{2}$$
$$(\lambda + \theta + \eta)P_{0,0,0} = \eta P_{1,0,0} + \lambda E_{0,0,0}, \tag{3}$$
$$(\lambda + \theta + \eta)P_{0,0,1} = \eta P_{1,0,1} + \lambda E_{0,0,1} + \alpha[q_{11}P_{1,0,1} + q_{01}P_{1,0,0}], \tag{4}$$
$$(\lambda + \theta + \eta + n\alpha)P_{n,0,0} = \lambda P_{n-1,0,0} + \eta P_{n+1,0,0} + n\alpha[q_{00}P_{n,0,0} + q_{10}P_{n,0,1}], n \geq 1, \tag{5}$$
$$(\lambda + \theta + \eta + n\alpha)P_{n,0,1} = \lambda P_{n-1,0,1} + \eta P_{n+1,0,1} + (n+1)\alpha[q_{01}P_{n+1,0,0} + q_{11}P_{n+1,0,1}], n \geq 1, \tag{6}$$
$$\lambda E_{0,1,0} = \omega\theta E_{0,0,0}, \tag{7}$$
$$\lambda E_{0,1,1} = \omega\theta E_{0,0,1}, \tag{8}$$
$$(\lambda + r\mu_1 + (1-r)\mu_1)P_{0,1,0} = \theta P_{0,0,0} + (1-r)\mu_1 P_{1,1,0} + \mu_2 Q_{1,1,0} + \omega\lambda E_{0,1,0}, \tag{9}$$
$$(\lambda + r\mu_1 + (1-r)\mu_1)P_{0,1,1} = \theta P_{0,0,1} + (1-r)\mu_1 P_{1,1,1} + \mu_2 Q_{1,1,1} + \omega\lambda E_{0,1,1}, \tag{10}$$
$$(\lambda + r\mu_1 + (1-r)\mu_1)P_{n,1,0} = \lambda P_{n-1,1,0} + (1-r)\mu_1 P_{n+1,1,0} + \mu_2 Q_{n+1,1,0} + \theta P_{n,0,0}, n \geq 1, \tag{11}$$
$$(\lambda + r\mu_1 + (1-r)\mu_1)P_{n,1,1} = \lambda P_{n-1,1,1} + (1-r)\mu_1 P_{n+1,1,1} + \mu_2 Q_{n+1,1,1} + \theta P_{n,0,1}, n \geq 1, \tag{12}$$
$$(\lambda + \mu_2)Q_{0,1,0} = r\mu_1 P_{0,1,0}, \tag{13}$$
$$(\lambda + \mu_2)Q_{n,1,0} = r\mu_1 P_{n,1,0} + \lambda Q_{n-1,1,0}, n \geq 1, \tag{14}$$
$$(\lambda + \mu_2)Q_{0,1,1} = r\mu_1 P_{0,1,1} \tag{15}$$
$$(\lambda + \mu_2)Q_{n,1,1} = r\mu_1 P_{n,1,1} + \lambda Q_{n-1,1,1}, n \geq 1. \tag{16}$$

Here, $\omega = 1$ or 0 correspond to the steady-state equations for SWV or MWV.

IV. Matrix Geometric Solution

Matrix geometric method is used for the analysis of quasi-birth-death (QBD) process with continuous time Markov chains whose transition rate matrices have a repetitive block structure. The method was developed by Neuts [13]. The transition rate matrix Q of the Markov chain corresponding to the coefficients of equations (1) to (16) has the block tridiagonal form given by:

$$\mathbf{Q}=\begin{pmatrix} \mathbf{A}_0 & \mathbf{C}_0 & & & & & & & \\ \mathbf{B}_1 & \mathbf{A}_1 & \mathbf{C}_1 & & & & & & \\ & \mathbf{B}_2 & \mathbf{A}_2 & \mathbf{C}_1 & & & & & \\ & & \mathbf{B}_3 & \mathbf{A}_3 & \mathbf{C}_1 & & & & \\ & & & \vdots & \vdots & \vdots & & & \\ & & & & \vdots & \vdots & \vdots & & \\ & & & & & \mathbf{B}_{N-1} & \mathbf{A}_{N-1} & \mathbf{C}_1 & \\ & & & & & & \mathbf{B}_N & \mathbf{A}_N & \mathbf{C}_1 \\ & & & & & & & \vdots & \vdots & \vdots \end{pmatrix}$$

The transition rate matrix Q of the QBD process has the sub-matrices given as:

$$\mathbf{A}_0=\begin{cases}\begin{pmatrix} -(\lambda+\theta) & 0 & \theta & 0 \\ 0 & -(\lambda+\theta) & 0 & \theta \\ 0 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & -\lambda \end{pmatrix} (for\ SWV), & , \\ \begin{pmatrix} -\lambda & 0 \\ 0 & -\lambda \end{pmatrix} (for\ MWV), \end{cases}$$

$$\mathbf{C}_0=\begin{cases}\begin{pmatrix} \lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 & 0 \end{pmatrix} (for\ SWV), \\ \begin{pmatrix} \lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 & 0 \end{pmatrix} (for\ MWV), \end{cases}$$

$$\mathbf{C}_1=\begin{pmatrix} \lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda \end{pmatrix}, A_1=\begin{pmatrix} \delta_1 & 0 & \theta & 0 & 0 & 0 \\ 0 & \delta_1 & 0 & \theta & 0 & 0 \\ 0 & 0 & \delta_3 & 0 & r\mu_1 & 0 \\ 0 & 0 & 0 & \delta_3 & 0 & r\mu_1 \\ 0 & 0 & 0 & 0 & \delta_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & \delta_4 \end{pmatrix},$$

$$\mathbf{B}_1=\begin{cases}\begin{pmatrix} \eta & 0 & 0 & 0 \\ 0 & \eta & 0 & 0 \\ (1-r)\mu_1 & 0 & 0 & 0 \\ 0 & (1-r)\mu_1 & 0 & 0 \\ \mu_2 & 0 & 0 & 0 \\ 0 & \mu_2 & 0 & 0 \end{pmatrix} (for\ SWV), \\ \begin{pmatrix} \eta & 0 \\ 0 & \eta \\ (1-r)\mu_1 & 0 \\ 0 & (1-r)\mu_1 \\ \mu_2 & 0 \\ 0 & \mu_2 \end{pmatrix} (for\ MWV), \end{cases}$$

$$\mathbf{B}_i = \begin{pmatrix} \eta & (i-1)\alpha q_{01} & 0 & 0 & 0 & 0 \\ 0 & \eta+(i-1)\alpha q_{11} & 0 & 0 & 0 & 0 \\ 0 & 0 & (1-r)\mu_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & (1-r)\mu_1 & 0 & 0 \\ 0 & 0 & \mu_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu_2 & 0 & 0 \end{pmatrix}, 2 \le i \le N-1,$$

$$\mathbf{B}_i = \begin{pmatrix} \eta & (N-1)\alpha q_{01} & 0 & 0 & 0 & 0 \\ 0 & \eta+(N-1)\alpha q_{11} & 0 & 0 & 0 & 0 \\ 0 & 0 & (1-r)\mu_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & (1-r)\mu_1 & 0 & 0 \\ 0 & 0 & \mu_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu_2 & 0 & 0 \end{pmatrix}, i \ge N,$$

$$\mathbf{A}_i = \begin{pmatrix} \delta_2+(i-1)\alpha q_{00} & 0 & \theta & 0 & 0 & 0 \\ (i-1)\alpha q_{10} & \delta_2 & 0 & \theta & 0 & 0 \\ 0 & 0 & \delta_3 & 0 & r\mu_1 & 0 \\ 0 & 0 & 0 & \delta_3 & 0 & r\mu_1 \\ 0 & 0 & 0 & 0 & \delta_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & \delta_4 \end{pmatrix}, 2 \le i \le N-1,$$

$$\mathbf{A}_i = \begin{pmatrix} \delta_5+(N-1)\alpha q_{00} & 0 & \theta & 0 & 0 & 0 \\ (N-1)\alpha q_{10} & \delta_2 & 0 & \theta & 0 & 0 \\ 0 & 0 & \delta_3 & 0 & r\mu_1 & 0 \\ 0 & 0 & 0 & \delta_3 & 0 & r\mu_1 \\ 0 & 0 & 0 & 0 & \delta_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & \delta_4 \end{pmatrix}, i \ge N.$$

where $\delta_1 = -(\lambda+\theta+\eta)$, $\delta_2 = -(\lambda+\theta+\eta+(i-1)\alpha)$, $\delta_3 = -(\lambda+\mu_1)$, $\delta_4 = -(\lambda+\mu_2)$ and $\delta_5 = -(\lambda+\theta+\eta+(N-1)\alpha)$.

Let $\mathbf{P}$ be the corresponding steady state probability vector of $\mathbf{Q}$. By partitioning the vector $\mathbf{P}$ as $\mathbf{P} = \{\mathbf{P}_0, \mathbf{P}_1, \mathbf{P}_2, \dots\}$ where
$\mathbf{P}_0$=$[E_{0,0,0}, E_{0,0,1}, E_{0,1,0}, E_{0,1,1}]$, (for SWV) and $\mathbf{P}_0$=$[E_{0,0,0}, E_{0,0,1}]$, (for MWV),
$\mathbf{P}_{i+1}$=$[P_{i,0,0}, P_{i,0,1}, P_{i,1,0}, P_{i,1,1}, Q_{i,1,0}, Q_{i,1,1}]$, $i \ge 0$.

According to Neuts [13] , the system is stable and the steady state probability vector exists if and only if $\mathbf{YC}_1\mathbf{e}_6 < \mathbf{YB}_N\mathbf{e}_6$ where $\mathbf{Y}$ is an invariant probability of the matrix $\mathbf{M}$=$\mathbf{A}_N + \mathbf{B}_N + \mathbf{C}_1$. $\mathbf{e}_n$ denotes a column vector with size n, and all elements equal to 1. $\mathbf{Y}$ satisfies the equations $\mathbf{YM} = 0$ and $\mathbf{Ye}_6 = 1$.

Apparently, when the stability condition is satisfied, the sub-vectors of $\mathbf{P}$, corresponding to different levels satisfy

$$\mathbf{P}_n = \mathbf{P}_N\mathbf{R}^{n-N}, n \ge N, \tag{17}$$

where the matrix R is the minimal non-negative solution of the matrix quadratic equation

$$\mathbf{C}_1 + \mathbf{RA}_N + \mathbf{R}^2\mathbf{B}_N = \mathbf{0}, \tag{18}$$

which can be obtained by using the following iterative procedure.

Computational algorithm for R:

Step 1: Set $k = 1$.
Step 2: Set $\mathbf{U} = \mathbf{A}_N$ and calculate $\mathbf{G} = (\mathbf{I}-\mathbf{U})^{-1}\mathbf{B}_N$.
Step 3: Increment k by 1.
Step 4: Replace $\mathbf{U} = \mathbf{A}_N + \mathbf{C}_1\mathbf{G}$ and $\mathbf{G} = (\mathbf{I}-\mathbf{U})^{-1}\mathbf{B}_N$.
Step 5: Repeat Steps 3 and 4 until $\| \mathbf{e}_n - \mathbf{Ge}_n \|_\infty < \epsilon$, where ϵ is a stopping tolerance.
Step 6: Calculate $\mathbf{R} = \mathbf{C}_1(\mathbf{I}-\mathbf{U})^{-1}$.

From the equation $\mathbf{PQ} = \mathbf{0}$, the governing system of difference equations can be given as

$$\mathbf{P}_0\mathbf{A}_0 + \mathbf{P}_1\mathbf{B}_1 = \mathbf{0}, \tag{19}$$

$$\mathbf{P}_0\mathbf{C}_0 + \mathbf{P}_1\mathbf{A}_1 + \mathbf{P}_2\mathbf{B}_2 = \mathbf{0}, \tag{20}$$

$$\mathbf{P}_{n-1}\mathbf{C}_1 + \mathbf{P_n}\mathbf{A}_n + \mathbf{P}_{n+1}\mathbf{B}_{n+1} = \mathbf{0}, 2 \le n \le N-1, \tag{21}$$

$$\mathbf{P}_{n-1}\mathbf{C}_1 + \mathbf{P}_n\mathbf{A}_N + \mathbf{P}_{n+1}\mathbf{B}_N = \mathbf{0}, n \ge N, \tag{22}$$

and the normalizing condition

$$\sum_{n=0}^{\infty} \mathbf{P}_n\mathbf{e}_n = 1. \tag{23}$$

From equations (19) to (22), after some mathematical manipulations, we get

$$\mathbf{P}_{n-1} = \mathbf{P}_n\boldsymbol{\phi}_n, 1 \le n \le N, \tag{24}$$

$$\mathbf{P}_N[\boldsymbol{\phi}_N\mathbf{C}_1 + \mathbf{A}_N + \mathbf{R}\mathbf{B}_N] = \mathbf{0}. \tag{25}$$

where

$$\boldsymbol{\phi}_1 = -\mathbf{B}_1(\mathbf{A}_0^{-1}), \boldsymbol{\phi}_2 = -\mathbf{B}_2[\boldsymbol{\phi}_1\mathbf{C}_0 + \mathbf{A}_1]^{-1}, \boldsymbol{\phi}_n = -\mathbf{B_n}(\mathbf{A_{n-1}} + \boldsymbol{\phi}_{\mathbf{n-1}}\mathbf{C_1})^{-\mathbf{1}}, 3 \le n \le N.$$

solving equations (23) and (24), we get

$$\mathbf{P}_N[\sum_{j=1}^{N} \prod_{i=N}^{m} \boldsymbol{\phi}_i + (\mathbf{I} - \mathbf{R})^{-1}]\mathbf{e}_n = 1. \tag{26}$$

Solving equations (25) and (26), we obtain $\mathbf{P}_N$. We use equations (17) and (24) to get $\mathbf{P}_n$ for $n \ge 0$.

V. Performance Measures

- Expected number of customers in the queue, when the server is busy in FES and SOS, respectively are

$$E[QF] = \sum_{n=1}^{\infty} nP_{n,1,0} + \sum_{n=1}^{\infty} nP_{n,1,1}; E[QS] = \sum_{n=1}^{\infty} nQ_{n,1,0} + \sum_{n=1}^{\infty} nQ_{n,1,1}.$$

- Expected number in the queue, when the server is in WV is given as

$$E[Q_{WV}] = \sum_{n=1}^{\infty} nP_{n,0,0} + \sum_{n=1}^{\infty} nP_{n,0,1}.$$

- Expected number of customers in the system is

$$E[L] = \sum_{n=0}^{\infty} (n+1)[P_{n,1,0} + P_{n,1,1}] + \sum_{n=0}^{\infty} (n+1)[P_{n,0,0} + P_{n,0,1}] + \sum_{n=0}^{\infty} (n+1)[Q_{n,1,0} + Q_{n,1,1}].$$

- Expected reneging rate of the customer is

$$E[RC] = \sum_{n=1}^{\infty} n\alpha P_{n,0,0} + \sum_{n=1}^{\infty} n\alpha P_{n,0,1}.$$

- Expected number of customers served is

$$ECS = \sum_{n=0}^{\infty} \eta(P_{n,0,0} + P_{n,0,1}) + \sum_{n=0}^{\infty} \mu_1(P_{n,1,0} + P_{n,1,1}) + \sum_{n=0}^{\infty} r\mu_2(Q_{n,1,0} + Q_{n,1,1}).$$

- Probability that the server is on WV is

$$P_{WV} = \sum_{n=0}^{\infty} P_{n,0,0} + \sum_{n=0}^{\infty} P_{n,0,1}.$$

- Probability that the server is idle is

$$P_0 = E_{0,0,0} + E_{0,0,1} + E_{0,1,0} + E_{0,1,1}(for\ SWV); P_0 = E_{0,0,0} + E_{0,0,1}(for\ MWV)$$

- Probability that the server is busy with FES and SOS is

$$PBF = \sum_{n=0}^{\infty} P_{n,1,0} + \sum_{n=0}^{\infty} P_{n,1,1}; PBS = \sum_{n=0}^{\infty} Q_{n,1,0} + \sum_{n=0}^{\infty} Q_{n,1,1}.$$

5.1 Special Cases of the Model

Case 1: Taking particular values of the parameters as $\alpha = 0$, $r = 0$, $\mu_2 = 0$ and $\omega = 0$, our model reduces to $M/M/1$ queueing model with MWV and results match with Servi and Finn [17].
Case 2: The present model reduces to an $M/M/1$ queueing model with SWV and MWV by taking values of the parameters as $\alpha = 0$, $r = 0$ and $\mu_2 = 0$. Results match with Selvaraju and Goswami [16] (by taking $\alpha = 0$ in their paper).

5.2 Cost Model

This section develops a cost model in order to carry out an economic analysis of the queueing system under consideration. We formulate an expected cost function per unit time, where the service rate in FES (μ_1) and that in SOS (μ_2) are decision variables.

Let us define
$c_1 \equiv$ Cost per unit time when the customer waits for the service,
$c_2 \equiv$ Cost per unit time when the server is on WV,
$c_3 \equiv$ Cost per unit time when the customer reneges,
$c_4 \equiv$ Cost per unit time when the server is busy with SOS.
Using the above cost parameters, the following cost optimization problem is designed as

$$minimize\ \tau_c[\mu_1, \mu_2] = c_1\mu_1 E[L] + c_2\eta P_{WV} + c_3 E[RC] + c_4\mu_2.$$

Let Rev be the revenue earned by providing service to a customer, τ_r be the total expected revenue per unit time of the system and τ_p be the total expected profit per unit time of the system. Thus,

$$\tau_r = Rev \times ECS, \tau_p = \tau_r - \tau_c.$$

5.3 Bat Algorithm

Bat algorithm is an innovative technique proving to give better solution than many popular traditional and heuristic algorithms for solving complex engineering problems. The bat algorithm is a meta-heuristic algorithm for global optimization. It was inspired by the echolocation behavior of micro bats, with varying pulse rates of emission and loudness. The bat algorithm was developed by Yang in 2010.

The bat algorithm works with the following three idealized rules

1. All bats use the echolocation to detect the distance from a food source and also have the knowledge to distinguish between foods/victims and background barriers.
2. Bats fly randomly in the surroundings with velocity $\mathbf{V_i}$ at position $\mathbf{x_i}$ with a frequency f_{min}, varying wavelength W and loudness L_0 in search for prey. They can automatically regulate the frequency(or wavelength) of their emitted pulses and change the rate of pulse emission (p) correspondingly in the range between 0 and 1, depending on the proximity of their target.
3. Though the loudness can vary in a variety of ways, we consider that the loudness varies from a large (positive) L_0 to a minimum constant value L_{min}.

In addition to these assumptions, for simplicity, the frequency f is taken in a range $[f_{min}, f_{max}]$ corresponds to a range of wavelengths $[W_{min}, W_{max}]$. We can either use wave lengths or frequencies for implementation, we use $f_{min} = 0$ and f_{max} depending on the domain size of the problem of interest. Therefore, with the help of the mentioned assumptions, the updated equations for frequency f_i, position $\mathbf{x_i}$ and velocity $\mathbf{V_i}$ are as follows

$$f_i = f_{min} + (f_{max} - f_{min})\boldsymbol{\vartheta},$$
$$\mathbf{V_i^{t+1}} = \mathbf{V_i^t} + (\mathbf{x}_i^t - \mathbf{x}^*)f_i,$$
$$\mathbf{x}_i^{t+1} = \mathbf{x}_i^t + \mathbf{V_i^{t+1}}.$$

where

- $\boldsymbol{\vartheta} \in [0,1]$ is a uniformly distributed random vector.
- f_i is the frequency that i^{th} bat emits and f_{min}, f_{max} are the lower and upper bounds of frequencies, respectively.
- V_i^t is the velocity of i^{th} bat after t generations.
- x_i^t is the position of i^{th} bat after t generations.
- x^* is the current best position (solution) of the fitness function among all the bats.

After selecting a solution among the current best solutions, for the local search we use the random walk for each bat. Hence, the new position updating formula is generated locally and is expressed as

$$\mathbf{x}_{new} = \mathbf{x}_{old} + \epsilon_1 L^{(t)}$$

where $\epsilon_1 \in [-1,1]$ is a random number and $L^{(t)} =< L_i^t >$ is the average loudness of all the bats at time instant t. Now to control the step size, the new position updating formula is rewritten as

$$\mathbf{x}_{new} = \mathbf{x}_{old} + \varsigma\epsilon_t L^{(t)}$$

where, the value of ϵ_t is taken from the Gaussian normal distribution N(0,1), and ς is a scaling factor having standard value 0.001.

VI. Numerical Investigations

This section is devoted to study numerically the performance measures and cost profit aspects associated with the model using Mathematica software. The parameters of the model are assumed to be $\lambda = 0.8$, $\mu_1 = 3.5$, $\mu_2 = 3.0$, $\eta = 2.5$, $\theta = 0.5$, $\alpha = 0.7$, $r = 0.6$, $q_{00} = 0.6$, $q_{11} = 0.5$. For the economic analysis of the system, we fix the different costs as $c_1 = 5$, $c_2 = 4$, $c_3 = 3$, and $c_4 = 2$, for bat algorithm, we assume $f_{min} = 0$, $f_{max} = 2$, $L = 0.5$, $p = 0.5$, lower and upper bounds of μ_1 and μ_2 are taken as [1.5, 4.5] and [1.0, 4.0], respectively.

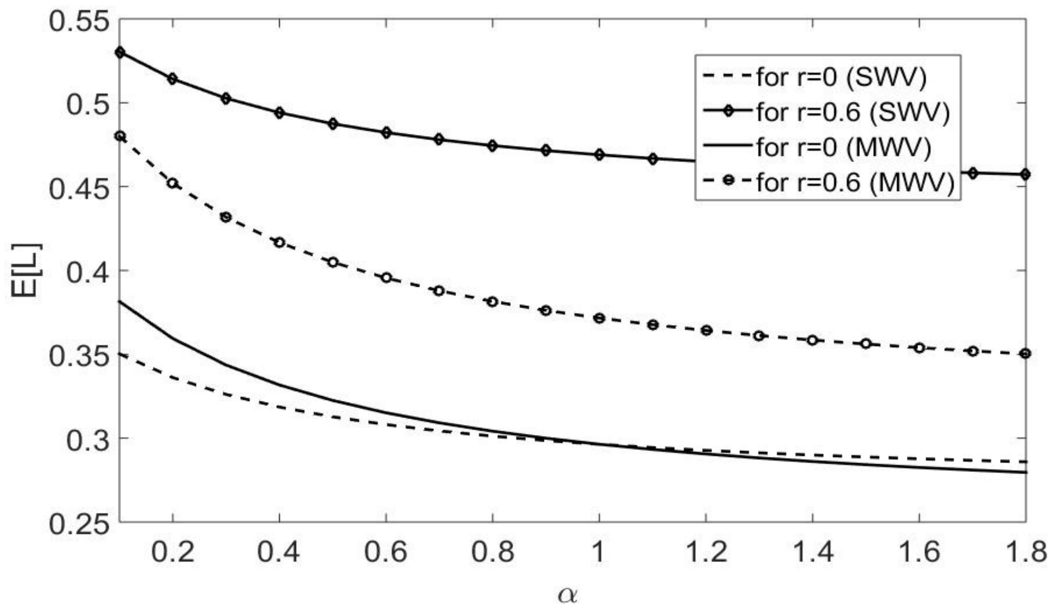

Figure 2: *Effect of α on $E[L]$ for different r.*

Table 1: *Effect of r on performance measures.*

		SWV		MWV	
	Cases	$E[L]$	P_0	$E[L]$	P_0
$r = 0$	Correlated reneging	0.30435	0.75028	0.30917	0.73764
	No reneging	0.37128	0.72819	0.41401	0.70354
$r = 0.3$	Correlated reneging	0.38191	0.70939	0.34312	0.72109
	No reneging	0.45345	0.68633	0.46087	0.68318
$r = 0.6$	Correlated reneging	0.47796	0.66375	0.38789	0.70126
	No reneging	0.55452	0.63993	0.52144	0.65913

Table 2: *Effect of r and μ_1 on performance measures.*

		SWV		MWV	
	Cases	$E[L]$	P_0	$E[L]$	P_0
$r = 0$	$\mu_1 = 3.6$	0.29911	0.75334	0.30689	0.73883
	$\mu_1 = 3.8$	0.29826	0.75892	0.30288	0.74099
	$\mu_1 = 4.0$	0.29757	0.76387	0.29944	0.74289
$r = 0.3$	$\mu_1 = 3.6$	0.35797	0.71279	0.34001	0.72252
	$\mu_1 = 3.8$	0.35550	0.71900	0.33453	0.72508
	$\mu_1 = 4.0$	0.35341	0.72449	0.32985	0.72734
$r = 0.6$	$\mu_1 = 3.6$	0.43122	0.66757	0.38355	0.70298
	$\mu_1 = 3.8$	0.42638	0.67451	0.37591	0.70607
	$\mu_1 = 4.0$	0.42227	0.68065	0.36941	0.70878

Table 3: *Effect of q_{11} and q_{00} on τ_c, τ_r and τ_p.*

	SWV			MWV		
	τ_c	τ_r	τ_p	τ_c	τ_r	τ_p
$q_{11} = 0.2$	15.7559	41.3303	25.5743	15.1813	37.6903	22.5090
$q_{11} = 0.4$	15.6551	41.1014	25.4463	15.0058	37.2814	22.2756
$q_{11} = 0.6$	15.5365	40.8314	25.2950	14.7985	36.7970	21.9986
$q_{00} = 0.3$	15.4533	40.6499	25.1966	14.6526	36.4701	21.8175
$q_{00} = 0.5$	15.5335	40.8292	25.2957	14.7933	36.7929	21.9997
$q_{00} = 0.7$	15.6963	41.1860	25.4897	15.0775	37.4326	22.3551

Table 4: *Effect of λ, q_{11} and r on optimum cost.*

	μ_1^*	μ_2^*	τ_c^*
$\lambda = 0.6$	2.2495	1.6517	9.8199
$\lambda = 0.8$	2.5264	2.1841	13.2489
$\lambda = 1.0$	2.9319	2.6845	17.2492
$q_{11} = 0.4$	2.5313	2.2132	13.6841
$q_{11} = 0.6$	2.5176	2.2015	13.5692
$q_{11} = 0.8$	2.4924	2.1524	13.4325
$r = 0.2$	1.8532	1.3842	10.7163
$r = 0.4$	2.4609	1.9185	11.9251
$r = 0.6$	2.7886	2.4301	12.1792

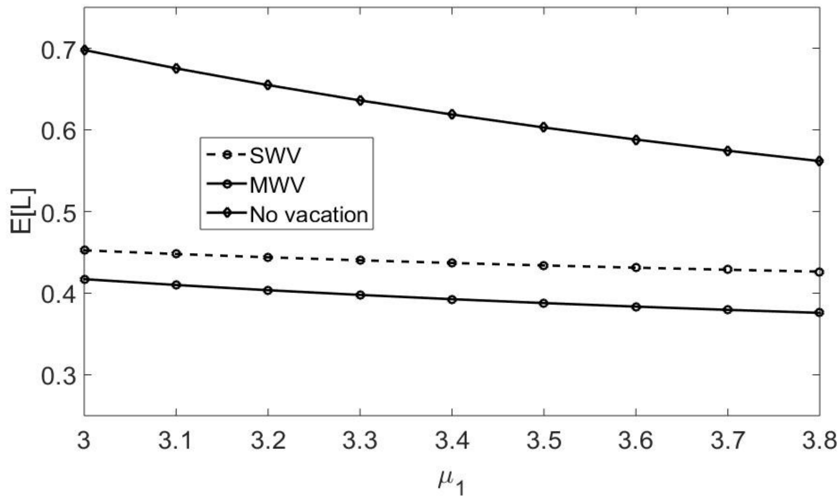

Figure 3: *Effect of μ_1 on $E[L]$.*

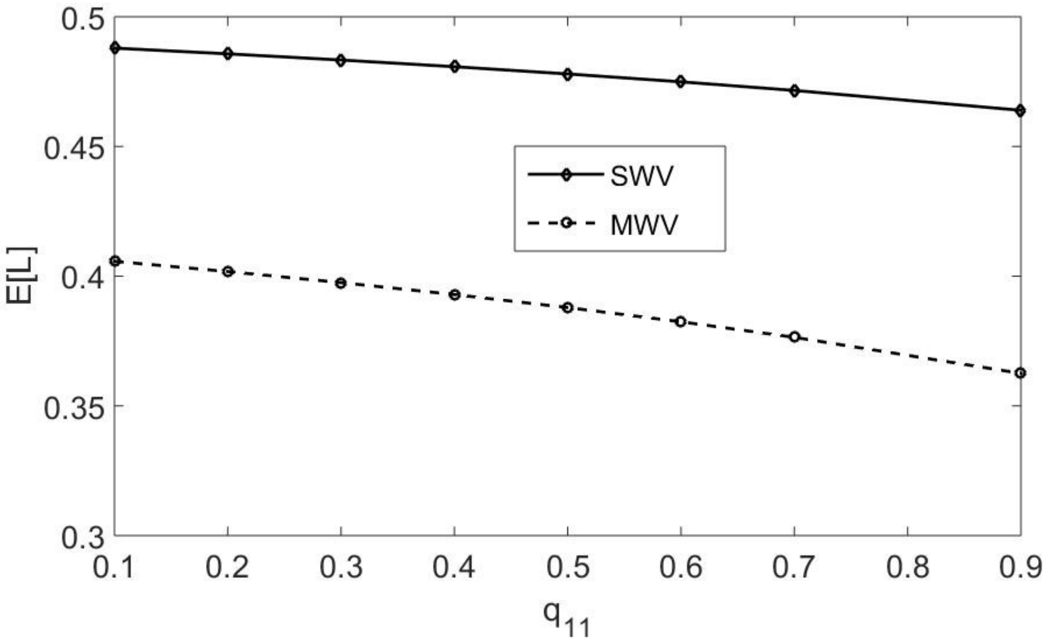

Figure 4: *Effect of q_{11} on $E[L]$.*

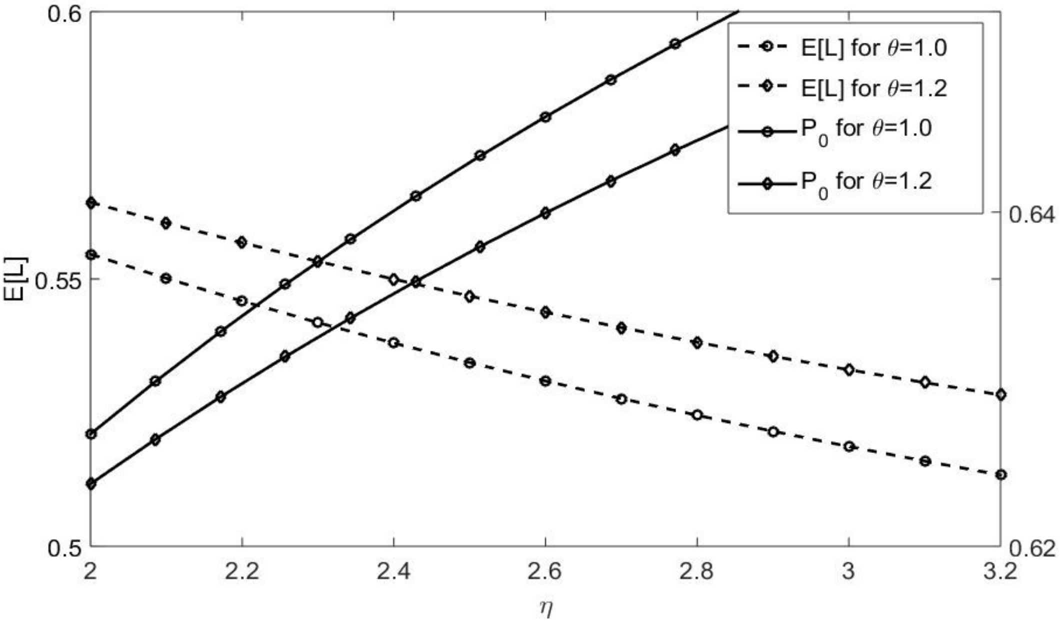

Figure 5: *Effect of η on $E[L]$ and P_0 for different values of θ.*

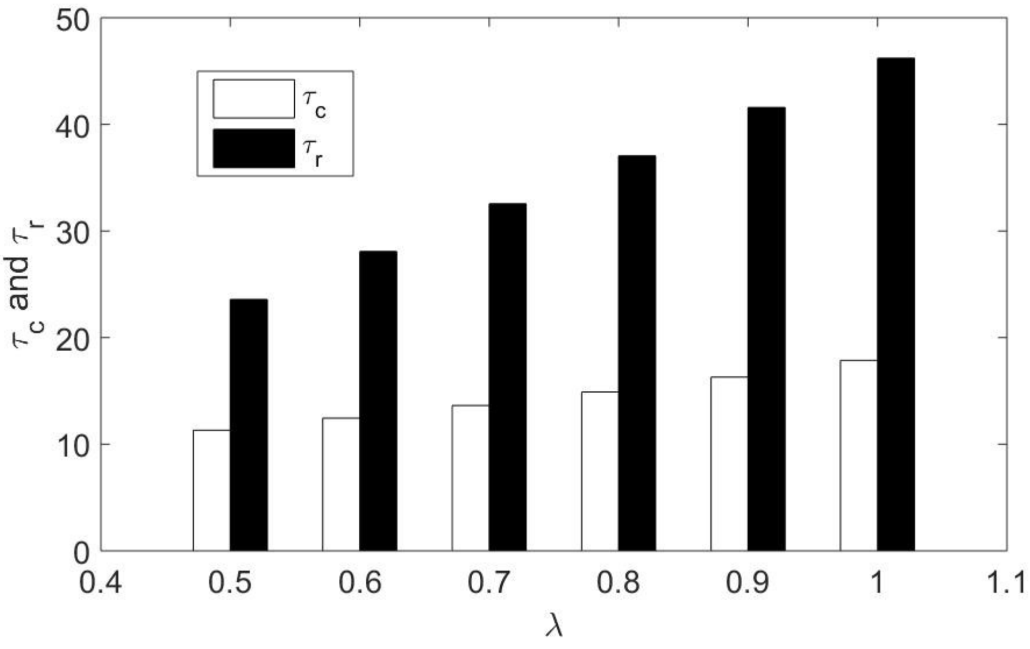

Figure 6: *Effect of λ on τ_c and τ_r.*

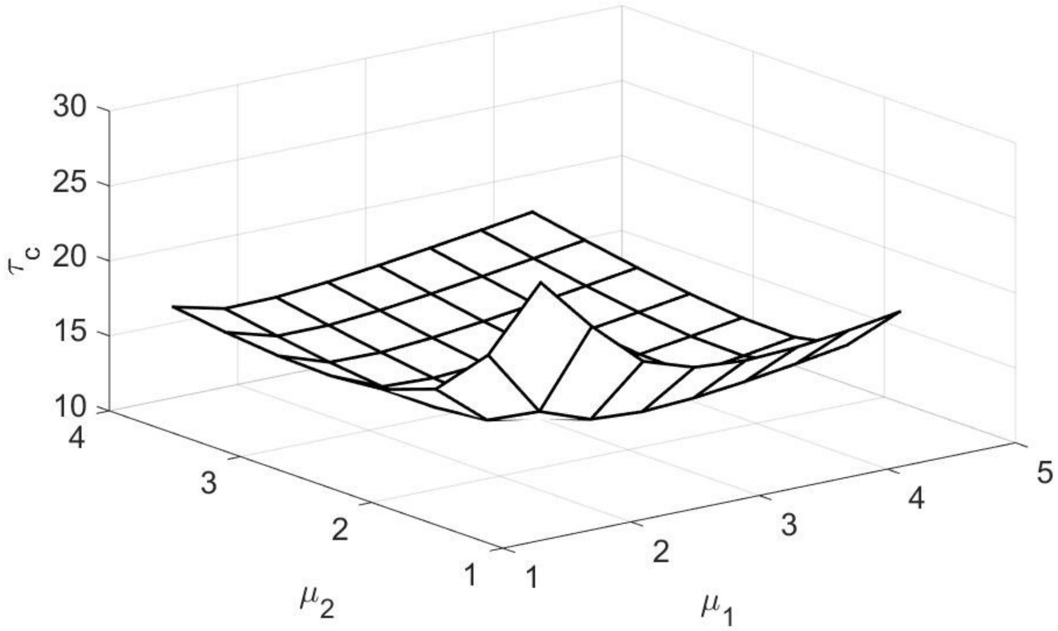

Figure 7: *Effect of* μ_1 *and* μ_2 *on* τ_c.

The effect of reneging rate (α) on average system length ($E[L]$) is shown in Figure 2. It is clear that as α increases, $E[L]$ decreases for both the models. In absence of SOS ($r = 0$) , we see an interesting behavior; for $\alpha < 1$ system length in SWV is smaller, for $\alpha > 1$ system length in MWV is smaller and at $\alpha = 1$ they coincide. In presence of an optional service, obviously MWV gives smaller system size because of the predominant effect of reneging.

In Tables 1 and 2, for fixed μ_1, as SOS probability (r) increases, the average system length ($E[L]$) increases and idle probability P_0 decreases. Further, for fixed r, as μ_1 increases a completely opposite trend is observed.

Table 3 illustrates the impact of reneging (non-reneging) probabilities of customers at both transition marks q_{11} (q_{00}) on total expected cost (τ_c), total expected revenue (τ_r) and total expected profit (τ_p) for both the models. As expected, an increase in q_{11}, decrease τ_c, τ_r and τ_p. This is because of the significant number of lost customers. On the other hand, opposite trend is observed for q_{00}. Therefore, it reveals the fact that q_{00} has positive effect on the economy of the system as it enforces the customers to be held in the system.

In table 4, using bat algorithm, the effect of λ, q_{11} and r on optimal service rates (μ_1^* and μ_2^*) and minimum expected cost (τ_c^*) is shown for MWV model. We observe that

- when arrival rate (λ) increases,μ_1^*, μ_2^* and τ_c^* increase as expected in the view of stability of the system.
- most importantly, increase in reneging probability (q_{11}), substantially reduce the optimal service rates and minimum cost due to lost customers.
- as r increases,μ_1^*, μ_2^* and τ_c^* increase. This agrees with our intuition.

In Figure 3, we show the effect of μ_1 on the system lengths for the model with SWV, MWV and no vacation. The graphs show the larger system lengths in the absence of vacation. This is explained by the fact that reneging occurs only during WV. When there is no WV, customers are remain in the system till they get served.

Figure 4 depicts that an increase in q_{11}, decreases the system length ($E[L]$), which is obviously true. Through Figure 5 we demonstrate the effect of the service rate in WV period (η) on $E[L]$ and P_0 for different values of vacation rate (θ) in SWV model. It is quite obvious that for a fixed θ, increase in η, decreases $E[L]$ and increases P_0. Moreover, upon increasing of θ, reverse trend is observed.

The impact of arrival rate λ on τ_c and τ_r for MWV policy is shown in Figure 6. We observe that, τ_c and τ_r increase with the increasing of λ. This is quite reasonable, the bigger the arrival rate, the greater the total expected cost and the total expected revenue. In Figure 7, we portray the three-dimensional surface plot generated through the joint variation of decision parameters μ_1 and μ_2 for MWV model. It prompts the convex nature of τ_c with respect to μ_1 and μ_2. As per the restriction of the system resources, the analyst can design parameters for the optimal service cost.

VII. Conclusion

In this paper, we have carried out an analysis of infinite buffer single server queueing system with SOS and correlated reneging under single and multiple working vacation policies. Using matrix geometric method, we derived the steady-state probabilities of the system. Some performance measures are developed. A cost model was established, and bat algorithm is applied to determine the optimal values of service rates in FES and SOS with the aim of minimizing the expected cost per unit time. The effects of various parameters on the system performance measures were explored by numerical experiments. Our study shows that

- increasing the service rates reduces the average system length.
- increase of the non-reneging probability of the customer at both transition marks, increases the expected system length and it shows the positive effect on the economy of the system as it increases the revenue.
- MWV model has lower system lengths for higher reneging rates due to the departures of customers by the way of reneging.

According to the analysis of expected system length by numerical examples, we find that our model represents some practical problems reasonably. The obtained results have potential applications in modeling computer and telecommunication systems, computer networks, manufacturing, and so on. So, the service companies may design the reasonable WV rate, service rates and correlated reneging rates to enable the companies to operate more flexibly and efficiently. To make the system modeling more closer to real world problems, we extend our model to consider general service times and server breakdowns.

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Received: September 11, 2020
Accepted: November 25, 2020