A Matrix Method for Transient Solution of an M/M/2/N Queuing System with Heterogeneous Servers and Retention of Reneging Customers

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Abstract

In this paper, a finite capacity two heterogeneous servers' queuing system with retention of reneging customers is studied. The explicit transient probabilities of system size are obtained using matrix method. Further, the time-dependent mean and variance are presented. Finally, a numerical example is provided to show the behavior of the system.

Keywords: Retention of reneging customers, Heterogeneous servers, matrix method, Transient solution

1 Introduction

Queuing theory emerges as proficient instrument in solving the difficulties of clogging in telecommunication systems, computer-communication systems, service systems and traffic systems. Most of the work done in multi-server queuing, researchers have assumed equal service rate for all the servers. This assumption is validated only in mechanically or electronically controlled systems. But, when the servers are human they will perform with different rates as per their abilities. For example, counters of a library where different library assistants work on different rates can be accurately demonstrated using heterogeneous multi-server queuing systems. Kumar et al. [14] state that it is quite difficult to obtain the analytical results for queuing systems multi-heterogeneous servers. Morse [20] was the first to incorporate the idea of heterogeneity in service. Gumbel [11] derived the expressions for steady-state system size probabilities and the expected queue length for non-homogeneous multiple server queuing model. Saaty [21] obtained time-independent probabilities for heterogeneous server queuing system. He extended the Morse's model with two different service rates. The same model with two types of queue disciplines was studied by Krishnamoorthy [13]. Godini [10] considered a heterogeneous server $M/M/S$ queuing system. Singh [29] analyzed three heterogeneous servers' $M/M/3$ queue where the first server is faster than other two and second server is faster than third server. He also obtained steady-state results and compared them with the existing three server homogeneous system. Cooper [5] studied $M/M/S$ queuing system with different service rates. He also obtained the steady-state to analyze the performance measures like number of customers in the system and server utilization. Queuing system with two types of processors was studied by Trivedi [32] to get the steady-state results. Sharma and Dass [23] did the busy period analysis of $M/M/2/N$ queuing system with heterogeneous servers and also acquired the expression for customers' number in the system and density function. For the same model Sharma and Dass [24] achieved the steady-state results for number of customers using Laplace transform and matrix method. Dharmaraja [6] obtained the transient solutions for the model already studied by Trivedi's [32].

In our daily life, a customer may not be allowed to be served at time instance that he joins the queue, and he has to wait some time duration until his service process is started. During waiting time period, he may become impatient when the waiting time is higher than his expected service time duration and may leave the system before getting service. Telephone switchboard customers, perishable goods, inventory systems, and hospital emergency rooms' handling of critical patients are the most prominent examples for above mentioned situations. Taking an example of a call centre where a customer who is told to hold on for some time to contact customer care officer may renege if he becomes impatient before his connection with customer care officer is established if his waiting time more than his patience level. This behavior can be observed in train ticket booking also. A customer in queue may renege after waiting for some time. Both balking and reneging influence the performance of the queuing system. Thus, many researchers have shown their keen interest in these two concepts. Singh [28] has analyzed an $M/M/2$ queuing system with heterogeneity and balking, and furthermore, the results with corresponding two-server homogeneous system are compared. About-El-Ata [1] also studied an $M/M/2$ queuing system with balking and heterogeneity. Al-seedy [4] attempted to obtain the transient solutions for system size probabilities of an $M/M/2$ queue with balking, heterogeneous servers, and an additional server is set up for longer queues. El-Paoumy [7] has analyzed a finite capacity queuing system what has batch arrival, balking, reneging and two heterogeneous servers. Yue and Yue [34] have studied a two heterogeneous servers queuing system with balking, single vacation, and under Bernoulli schedules. A two heterogeneous servers queuing system was discussed by Yue et al. [33] by adding the feature of balking. They implemented the condition that first server is reliable and second server is subject to breakdown by extending the model of Singh [28]. Matrix-geometric method was used to derive the steady-state results for the system size probabilities. Furthermore, they have presented the performance measures such as the mean system size, and the average balking rate. El-Sherbiny [9] has studied a finite capacity two heterogeneous server queuing system with two general different balk functions to derive the steady-state results, and he probability generating function technique along with hypergeometric function. Kumar and Sharma [16] obtained the steady-state probabilities of number of customers in the system and some performance measures for an $M/M/2/N$ queue with discouraged arrivals, two-heterogeneous servers, reneging and retention of reneged customers. This was an extending of Kumar and Sharma [17] model. Impatient behavior on a two heterogeneous servers queuing system was studied by Ammar [2] to obtain the transient and steady-state results along with some performance measures. a two heterogeneous servers $M/M/2/N$ queue subject to reverse balking and reneging has been studied by Som and Kumar [30], and they presented the steady-state expressions and some performance measures for that model. Kumar and Sharma [18] have recently obtained the transient and steady-state system size probabilities for a heterogeneous servers' $M/M/2$ queuing system with retention of reneged customers. Furthermore, they have presented mean and variance as the performance measures and numerical illustrations also are provided.

Even though researchers usually consider infinite capacity queuing systems, it may not appear in real life situations. For example, an e-mail server system has to limit its waiting line for mails considering available limit of memory. It never becomes infinite and should be limited to some finite capacity. The transient solution of a finite capacity $M/M/1$ queue was obtained by Takacs [?] by making use of a technique involving eigenvectors and eigenvalues. Sharma and Gupta [25] have used Chebyshev polynomials to analyze an $M/M/1/N$ queuing system in transient state. Sharma and Maheswar [27] have applied the matrix geometric method to derive the time dependent results for $M/M/1/N$ queue. A finite capacity correlated two server Markovian queuing system was analyzed by Sharma and Maheswar [22] by using matrix method. Ammar et al. [3] obtained the transient solutions for an $M/M/1/N$ queuing system with discouraged arrivals and reneging by using computable matrix technique. Yue and yue [35] derived the steady state expressions for a finite capacity multi-server queuing system with simultaneous balking, reneging, and synchronous vacations of servers. An $M/M/c/N$ queuing system with balking and retention of reneged customers was analyzed by Kumar [15] and he used probability generating function technique to derive steady-state solutions with some performance measures for that model. Sharma [22] obtained the transient solution for two-heterogeneous servers' queuing system in general form by making use of computable matrix technique. A finite capacity two heterogeneous servers queuing system with general balk function, reneging was studied by El-Paoumy and Nabwey [8] to obtain the steady-state expressions.Recently, Isguder and Kocer [12] have studied finite capacity queueing system with recurrent input and two heterogeneous servers and they derived the steady-state expressions for system size.

The applicability of this model can be seen in communication and computer systems. Messages arrive to communication device as data packets are transferred through one of the communication channels which are working with different rates. The model we are considering has two heterogeneous processors and one of them is faster. If the data packets takes too much time to transmit, then the sender may recall the message sent. This is known as reneging in queuing theory. Customer retention strategies can be applied to reduce the dropping of the packets. Capacity of memory of both processors are limited to finite value. Therefore, this system has to limit their waiting room capacity for some finite number of messages.

The application as discussed above motivates us to analyze the behavior of a finite capacity two-heterogeneous servers' queuing system with retention of reneging customers. From the literature servey, it has been noticed that transient solution of the queuing model considered in this paper has not been obtained by using matrix method. Hence, we study and $M/M/2/N$ queuing system with two-heterogeneous servers and retention of reneging customers, and obtained its transient solution by employing matrix method.

Rest of the paper has been arranged as follows; In section 3, the model is described. Section 4 provides the transient solution. In section 5, time-dependent mean and variance are presented. Section 6 deals numerical illustrations. Finally, the paper is concluded in section 7.

2 Model Description

A finite capacity two-heterogeneous queueing system with impatient customers is considered. Arrivals occur to the system in accordance with Poisson process with rate λ. The system has two severs and they have different exponentially distributed service rates μ_1 (server-1) and μ_2 (server-2) such that $\mu_1 < \mu_2$. In this model, we consider modified queue discipline i.e. an arriving customer goes to the server-1 if there is no customers in the system. Otherwise, it joins the server who is free. After joining the queue, the arrivals activates an individual timer, exponentially distributed with parameter ξ. If the customer's service has not been started before the customer's timer expires, he abandons the system with probability p or may remain in the queue for his service with probability $q(= 1 - p)$. The reneging rate when there are n customers in the queue is given by $(n - 2)\xi p$. The number of customers in the system is limited to N. It is assumed that inter-arrival times, service times are mutually independent and the service discipline is First-In, First-Out(FIFO).

Let $\{X(t), t \geq 0\}$ denotes the number of customers in the system at time t, and $P_n(t)$, $n = 0,1,2,3,4,\ldots.,N$ be the time-dependent probabilities for the number of customers at time t. Initially, it is assumed that there are i customers in the queuing system.

Then, the set of forward Kolmogorov differential difference equations governing the process are given by

$$P_0'(t) = -\lambda P_0(t) + \mu_1 P_1(t) \tag{1}$$

$$P_1'(t) = \lambda P_0(t) - (\lambda + \mu_1)P_1(t) + (\mu_1 + \mu_2)P_2(t) \tag{2}$$

$$P_2'(t) = \lambda P_1(t) - (\lambda + \mu_1 + \mu_2)P_2(t) + (\mu_1 + \mu_2 + \xi p)P_3(t) \tag{3}$$

$$P_n'(t) = \lambda P_{n-1}(t) - (\lambda + \mu_1 + \mu_2 + (n-2)\xi p)P_n(t)$$
$$+(\mu_1 + \mu_2 + (n-1)\xi p)P_{n+1}(t); n = 3,4,5. \dots, N-1 \tag{4}$$

$$P_N'(t) = \lambda P_{N-1}(t) - (\mu_1 + \mu_2 + (N-2)\xi p)P_N(t) \tag{5}$$

3 Transient solutions

In this section, the transient solution of the above described model is derived by employing matrix method.

4.1 Evaluation of $P_n(t)$

Taking Laplace transform of the equations (1)-(5), we have

$$AP(s) = P(0) \tag{6}$$

Where A is a tridiagonal matrix of order $(N+1) \times (N+1)$, and $P(s)$ and $P(0)$ are column vectors of order $N+1$. Matrix A is given by

$$\begin{pmatrix} s+\lambda & -\mu_1 & .. & .. & .. & .. & 0 \\ -\lambda & s+\lambda+\mu_1 & -(\mu_1+\mu_2) & .. & .. & .. & 0 \\ 0 & -\lambda & s+\lambda+\mu_1+\mu_2 & -(\mu_1+\mu_2+\xi p) & .. & .. & 0 \\ 0.75\ .. & .. & .. & .. & .. & .. & .. \\ .. & .. & .. & .. & .. & .. & .. \\ 0 & .. & .. & .. & .. & s+\lambda+\mu_1+\mu_2+(N-3)\xi p & -[\mu_1+\mu_2+(N-2)\xi p] \\ 0 & .. & .. & .. & .. & -\lambda & -[\mu_1+\mu_2+(N-2)\xi p] \end{pmatrix}$$

and

$$P(s) = \left[\hat{P}_0(s), \hat{P}_1(s), \dots\dots, \hat{P}_N(s)\right]^T,$$
$$P(0) = [P_0(0), P_1(0), \dots\dots, P_N(0)]^T,$$

where $\hat{P}_n(s)$ is the Laplace transform of $P_n(t)$.

The matrix A can be transformed into the symmetric tridiagonal form by the diagonal matrix

$$M = dg[d_0, d_1, d_2, \dots\dots, d_N]$$

with

$$d_0 = 1$$
$$d_n = \prod_{k=1}^{n} \sqrt{\frac{\mu_1 + (1-\delta_{1k})\mu_2 + (n-2+\delta_{1k})\xi p}{\lambda}}, 1 \le n \le N$$

Using the diagonal matrix M, a symmetric tridiagonal matrix, $sI + B = MAM^{-1}$, is obtained. Diagonal entries of this matrix are same as in matrix A and off diagonal entries in the nth row are represented by

$-\sqrt{\lambda(\mu_1 + (1-\delta_{1n})\mu_2 + (n-2+\delta_{1n})\xi p)}$ and $-\sqrt{\lambda(\mu_1 + \mu_2 + (n-1)\xi p)}$ respectively. This matrix and matrix **A** have same eigenvalues.

where matrix B is given by

$$\begin{pmatrix} \lambda & -\sqrt{\lambda\,\mu_1} & 0 & \cdot\cdot & \cdot\cdot & \cdot\cdot & 0 \\ -\sqrt{\lambda\,\mu_1} & \lambda+\mu_1 & -\sqrt{\lambda\,(\mu_1+\mu_2)} & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ 0 & -\sqrt{\lambda\,(\mu_1+\mu_2)} & \lambda+\mu_1+\mu_2 & -\sqrt{\lambda\,(\mu_1+\mu_2+\xi\,p)} & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ 0.68\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & s+\lambda+\mu_1+\mu_2+(N-3)\xi\,p & -\sqrt{\lambda\,(\mu_1+\mu_2+(N-2)\xi\,p)} \\ 0 & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & -\sqrt{\lambda\,(\mu_1+\mu_2+(N-2)\xi\,p)} & -[\mu_1+\mu_2+(N-2)\xi\,p\,] \end{pmatrix}$$

Two matrices $A_n(s)$ and $B_n(s)$ are defined with the determinants $T_n(s)$ and $U_n(s)$. They represent the bottom right and top left $(n \times n)$ square matrices of the matrix $sI + B$ respectively. The determinants $T_n(s)$ and $U_n(s)$ satisfy the following difference equations,

$$T_n(s) = [s + \lambda + \mu_1 + (1 - \delta_{Nn})\mu_2 + (N - n - 1 + \delta_{Nn})\xi p]T_{n-1}(s)$$
$$-[\lambda(\mu_1 + \mu_2 + (N - n)\xi p)]T_{n-2}(s)$$
$$U_n(s) = [s + \lambda + \mu_1 + (1 - \delta_{2n})\mu_2 + (n - 3 + \delta_{2n})\xi p]U_{n-1}(s)$$
$$-[\lambda(\mu_1 + (1 - \delta_{2n})\mu_2 + (n - 3 + \delta_{2n})\xi p)]U_{n-2}(s)$$

with the initial conditions

$$T_0(s) = 1 = U_0(s)$$

$$T_1(s) = s + \lambda + \mu_1 + \mu_2 + (N - 2)\xi p$$
$$U_1(s) = s + \lambda$$

Using the Lemma (1) and (2) of Lewis [19], we are able to derive the following results

$$(sI + B)^{-1} = \frac{C}{|sI+B|}, C = \left(C_{ij}(s)\right),$$

$$C_{ij}(s) = \sqrt{\prod_{r=j+1}^{i} \lambda[\mu_1 + (1 - \delta_{1r})\mu_2 + (r - 2 + \delta_{1r})\xi p]}\,U_j(s)T_{N-i}(s), i > j$$
$$= U_i(s)T_{N-j}(s), i = j$$
$$= \sqrt{\prod_{r=i+1}^{j} \lambda[\mu_1 + (1 - \delta_{1r})\mu_2 + (r - 2 + \delta_{1r})\xi p]}\,U_i(s)T_{N-j}(s), i < j$$

and

$$|A| = |sI + B| = T_N(s) = U_N(s)$$

The equation (6) is rearranged as follows,

$$P(s) = A^{-1}P(0)$$
$$= M^{-1}(sI + B)^{-1}MP(0)$$

$$P_n(s) = \frac{\sum_{j=0}^{N} d_n^{-1} C_{nj}(s) d_j P_j(0)}{|sI+B|}$$
$$= \frac{d_n^{-1} d_i C_{ni}(s)}{|sI+B|}$$

where

$$d_n^{-1}d_i = \prod_{k=n+1}^{i} \sqrt{\frac{\mu_1+(1-\delta_{1kr})\mu_2+(k-2+\delta_{1k})\xi p}{\lambda}}, i > n$$
$$= 1, i = n,$$
$$= \frac{1}{\prod_{k=i+1}^{n} \sqrt{\frac{\mu_1+(1-\delta_{1k})\mu_2+(k-2+\delta_{1k})\xi p}{\lambda}}}, n > i$$

Then, we can derive the following expression for $P_n(s)$,

$$\hat{P}_n(s) = \prod_{r=n+1}^{i} [\mu_1 + (1 - \delta_{1r})\mu_2 + (r - 2 + \delta_{1r})\xi p] \frac{U_n(s)T_{N-i}(s)}{|sI+B|}, n < i$$
$$= \frac{U_n(s)T_{N-n}(s)}{|sI+B|}, n = i$$
$$= \lambda^{n-i} \frac{U_n(s)T_{N-i}(s)}{|sI+B|}, n > i, 0 \le i, n \le N \tag{7}$$

Since symmetric tridiagonal matrix B is a diagonally dominant matrix, eigenvalues of its are real, positive and distinct.

Let $\alpha_m (m = 0,1,2, \ldots, N)$ be an eigenvalue of matrix B, and $\alpha_0 = 0$, Then obviously, we have

$$|sI + B| = s\prod_{m=1}^{N}(s+\alpha_m) \quad (8)$$

Substituting the equation (8) in (??), and making use of partial fraction decomposition, we derive

$$\hat{P}_n(s) = \prod_{r=n+1}^{i}[\mu_1 + (1-\delta_{1r})\mu_2 + (r-2+\delta_{1r})\xi p]\left(\frac{\pi_n}{s} + \sum_{k=1}^{N}\frac{A_{nk}}{s+\alpha_k}\right),$$
$$0 \le n < i$$
$$= \frac{\pi_n}{s} + \sum_{k=1}^{N}\frac{B_{nk}}{s+\alpha_k}, n = i$$
$$= \lambda^{n-i}\left(\frac{\pi_n}{s} + \sum_{k=1}^{N}\frac{B_{nk}}{s+\alpha_k}\right), i < n \le N \quad (9)$$

where

$$\pi_n = \frac{U_n(0)T_{N-i}(0)}{\prod_{j=1}^{N}\alpha_j}, 0 \le n \le i$$
$$= \frac{U_i(0)T_{N-n}(0)}{\prod_{j=1}^{N}\alpha_j}, i \le n \le N$$
$$A_{nk} = \frac{U_n(-\alpha_k)T_{N-i}(-\alpha_k)}{(-\alpha_k)\prod_{j=1,j\neq k}^{N}(\alpha_j-\alpha_k)}$$
$$B_{nk} = \frac{U_i(-\alpha_k)T_{N-n}(-\alpha_k)}{(-\alpha_k)\prod_{j=1,j\neq k}^{N}(\alpha_j-\alpha_k)}$$

The inversion of the equation (??) yields

$$P_n(t) = \prod_{r=n+1}^{i}[\mu_1 + (1-\delta_{1r})\mu_2 + (r-2+\delta_{1r})\xi p](\pi_n + \sum_{k=1}^{N}A_{nk}e^{-\alpha_k t}),$$
$$0 \le n < i$$
$$= \pi_i + \sum_{k=1}^{N}B_{ik}e^{-\alpha_k t}, n = i$$
$$= \lambda^{n-i}(\pi_n + \sum_{k=1}^{N}B_{nk}e^{-\alpha_k t}), i < n \le N$$

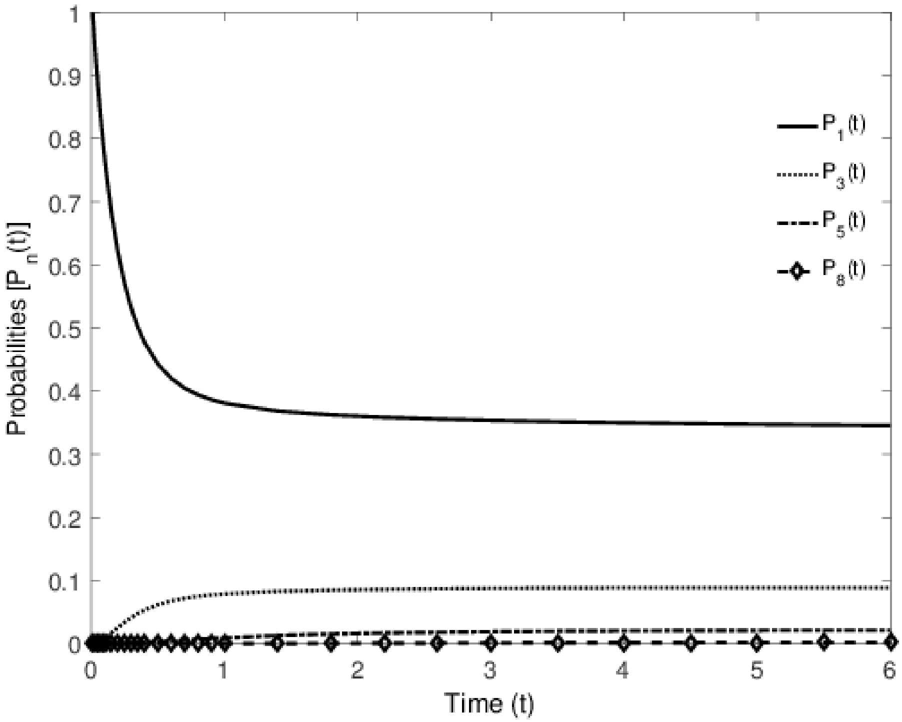

Figure 1: Variation in transient probabilities with time

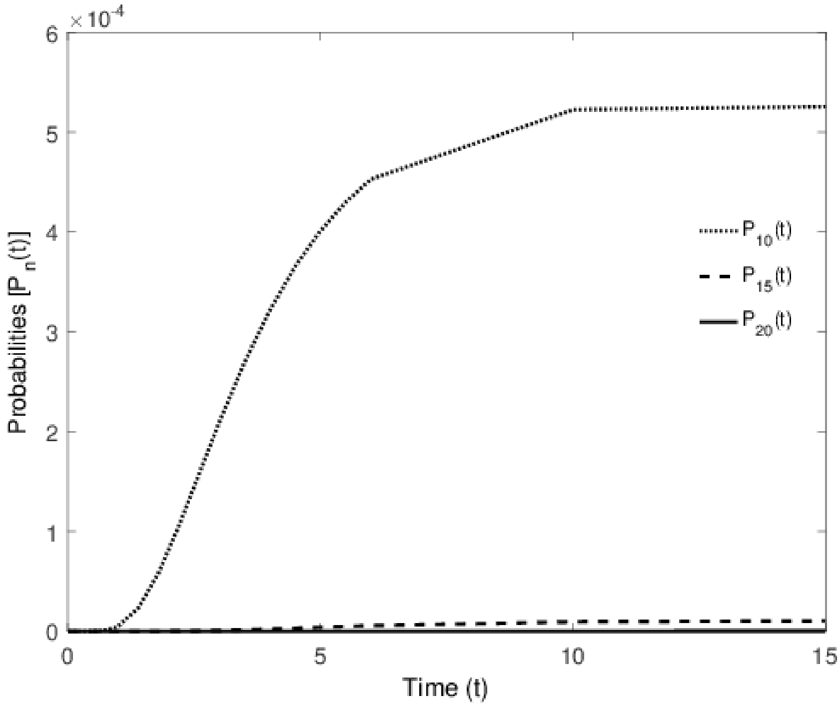

Figure 2: Variation in transient probabilities with time

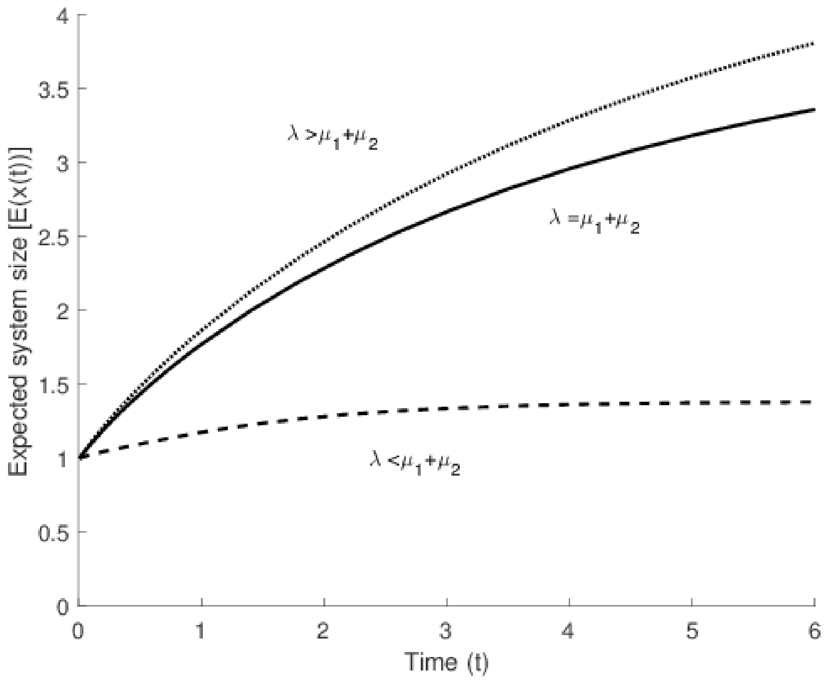

Figure 3: Comparison of expected system size $[E(X(t))]$ against time (t) for varying arrival rate (λ)

5 Time dependent mean and variance

In this section, time dependent expected value and variance of the system size distribution are derived.

5.1 Mean

Let $X(t)$ denotes the number of jobs in the system at time t. The average number of jobs in the system at time t is given by

$$E(X(t)) = \sum_{j=1}^{N} jP_n(t)$$
$$= \sum_{j=1}^{i} jP_j(t) + nP_n(t) + \sum_{j=n+1}^{N} jP_j(t)$$
$$= \sum_{j=1}^{i} j \prod_{r=j+1}^{i} [\mu_1 + (1-\delta_{1r})\mu_2 + (r-2+\delta_{1r})\xi p]$$
$$\times \left(\pi_j + \sum_{k=1}^{N} A_{jk} e^{-\alpha_k t}\right)$$
$$+ n\pi_n + \sum_{k=1}^{N} B_{nk} e^{-\alpha_k t}$$
$$+ \sum_{j=n+1}^{N} j\lambda^{j-i}\left(\pi_j + \sum_{k=1}^{N} B_{jk} e^{-\alpha_k t}\right)$$

5.2 Variance

Let $X(t)$ denotes the number of jobs in the system at time t. The variance of jobs in the system at time t is given by

$$Var(X(t)) = \sum_{j=1}^{N} j^2 P_n(t) - [E(X(t))]^2$$
$$= \sum_{j=1}^{i} j^2 P_j(t) + n^2 P_n(t) + \sum_{j=n+1}^{N} j^2 P_j(t) - [E(X(t))]^2$$
$$= \sum_{j=1}^{i} j^2 \prod_{r=j+1}^{i} [\mu_1 + (1-\delta_{1r})\mu_2 + (r-2+\delta_{1r})\xi p]$$
$$\times \left(\pi_j + \sum_{k=1}^{N} A_{jk} e^{-\alpha_k t}\right)$$
$$+ n^2\pi_n + \sum_{k=1}^{N} B_{nk} e^{-\alpha_k t}$$
$$+ \sum_{j=n+1}^{N} j^2\lambda^{j-i}\left(\pi_j + \sum_{k=1}^{N} B_{jk} e^{-\alpha_k t}\right) - [E(X(t))]^2$$

6 Numerical illustrations

The numerical examples which illustrate the functioning of concerned model in transient state are presented in this section.

Figures 1 and 2 presents the behaviour of the probabilities $P_n(t)$ against time t for varying values of n with parameters $\lambda = 1.8$, $\mu_1 = 1.5$, $\mu_2 = 2$, $\xi = 0.1$, $p = 0.4$ and initial value $i = 1$. It can be noticed that all the probabilities tend to settle at steady-state when time progresses.

Figure 3 is plotted to describe the comparison of the expected system sizes $E(X(t))$ with same parameter values and three types of arrivals. Here, if $\lambda < \mu_1 + \mu_2$, it can be seen that expected system size of the queue reaches its steady state with time t. But, for other two cases, it rapidly increases expected number of customers in the system when time progresses.

7 Conclusion

A finite capacity two-heterogeneous servers' queuing system with retention of reneging customers is studied. The matrix method is used to derive the transient solution. Additionally, mean and variance of the system size are presented as the performance measures. Finally, numerical analysis is added to express the behaviour of system size probabilities and expected system size against time

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Declaration of Conflicting Interests

The Authors declare that there is no conflict of interest

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