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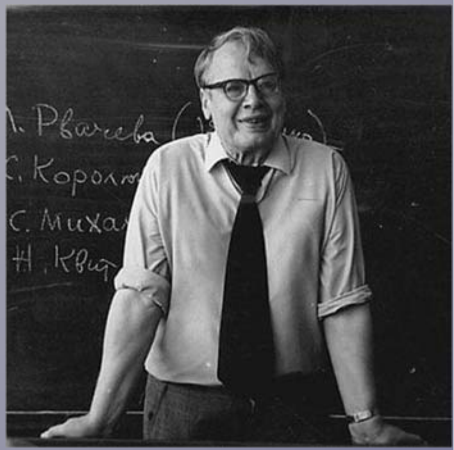

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ISSN 1932-2321

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VOL.15 NO.4 (59)
DECEMBER, 2020

RELIABILITY:
THEORY&APPLICATIONS

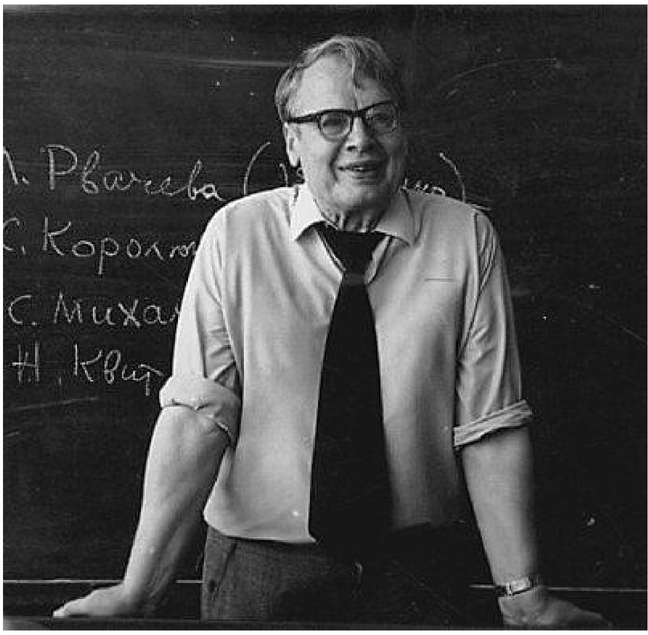
RELIABILITY
RISK ANALYSIS
MAINTENANCE
SAFETY

Vol.15
No.4 (59)
•
December
2020

San Diego

ISSN 1932-2321

Journal of International Group on Reliability



RELIABILITY:

THEORY & APPLICATIONS

Vol.15 No.4 (59),
December 2020

San Diego
2020

Editorial Board

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Gheorgy Konstantinovich Miscoy
(09.01.1944-12.09.2020)

On the12-th of September 2020 passed away Gheorghe Mishkoy, outstanding scientist, teacher, applied statistician, friend. His scientific degree was habilitated Doctor of Physics and mathematics, professor, academician at the Academy of Science of Moldova.

Illustrious scientist, mathematician, academician Gheorghe MISHKOY contributed to the development and strengthening of the applied mathematics. And he did this nationally and internationally during his 49 years of active, fruitful scientific, pedagogical and management remarkable work.

Gheorghe started his fundamental scientific activity at the Moscow State University named after M.V. Lomonosov. He worked as graduate student under the supervision of the famous scientist Gennadi Pavlovich Klimov, now living in the United States. At this time, research in the field of priority queueing systems was very popular and intensively developed at Moscow State University. Under the guidance of an outstanding scientist B.V. Gnedenko a group of young scientists and graduate students worked in this direction of research where G. Mishkoy was actively involved. The result of this activity was his Ph.D. thesis defended in 1974. His high achievement was acknowledged through a small monography issued by the University one year later. The continuation of these studies and numerous new results obtained by him were presented in his doctoral dissertation defended at the Institute of Cybernetics at the Academy of Sciences of Ukraine, in Kiev, 1989.

These results have left a deep mark in the scientific career of the Gheorghe. He remained faithful to this area of work throughout his subsequent scientific life. During his career Gheorghe stood firmly and resultative not only to the priority queues direction of research, but also continued to supervise and support the entire scientific, business, academic growth of the young students in Moldova, as well as the practical application of mathematical knowledge. Gheorghe lived and worked in his home country, but he used to keep all his friendly contacts with colleagues from Moscow, Kiev, Russia, and Ukraine and from universities of the whole world. For his outstanding results and contributions to the country Gheorghe Mishkoy was appointed as member-correspondent to the Moldovans Academy of Sciences in the year of 1995 and was elected for Academician in 2012. He established there the Department of Probability Theory and Applications for the needs of his country and for the perspectives of young scientists.

All subsequent professional activities of Academician Gheorghe MISHKOY were integrated into the Institute of Mathematics and Informatics of Academy of Sciences of Moldova. The Institution is named after Vladimir Andrunakievichas. Gheorghe worked at the same time for the International Independent University of Moldova. At the Institute of Mathematics and Informatics Gheorghe passed through the stages from an engineer to the head of a department and ultimately to the highest rank of academician of Moldova. In 2004-2008 working as academic coordinator of the Department of Precision and Economic Sciences of the Moldova Academy of Sciences, he made a significant contribution to the implementation of mathematical modeling in economy and other areas. At the Institute of Mathematics and Informatics for the first time in the Republic of Moldova, the scientific direction "Theory of Probabilities and its applications" has been founded by Gheorghe Mishkoy.

The fruitful scientific activity of this remarkable scientist is innovative and is marked by a number of significant results: the development of mathematical theory of queueing systems with priorities and semi-Markov transitions between states; construction of multi-dimensional versions of Kendall and Pollachek-Khinchin functional equations for polling systems; development of algorithms for modeling and numerical calculation of non-stationary characteristics and many others. In 2014 Gheorghe was invited guest speaker for the Flint International Conference on Statistics, organized in favor of The World Year of Statistics. Selected articles, one of which is his talk were published in world recognized journals.

Professionally, Academician Gheorghy MISHKOY harmoniously combined scientific research with teaching and educational activities. As a professor at the State University of Moldova, and the International Independent University of Moldova as well as in a number of other universities, he taught modern courses in mathematical statistics, econometrics, actuarial and financial mathematics, queueing theory.

Academician Gheorghe MISHKOY made a significant contribution to the training of highly qualified personnel, under his guidance and supervision, eight Doctors of Mathematical Sciences and habilitated doctors successfully got their doctors degree. Scientific results for many years are reflected in more than 300 scientific articles published in national and international journals, including 7 monographs, 4 textbooks. Numerous reports on various prestigious international scientific events in Austria, England, Holland, Spain, Canada, Russia, Romania. USA, France, Japan are tracing his scientific life.

The contributions of Academician Gheorghe MISHKOY to the development of science in the Republic of Moldova were highly appreciated and awarded. Gheorghe is granted by the title of Honorary Citizen and the Orden of "Gloria Muncii". He also is acknowledged with numerous prestigious academic awards: the silver medal of the ASM (2006), medals "Dimitri Cantemir" (2014), medals "Scientific merit", etc.

International recognition of his outstanding merits was confirmed by the electing Gheorghe MISHKOY as a member of the International Institute for Mathematical Research FIELDS (Toronto, Canada, 2010) and as member of the International Institute of Mathematical Statistics (USA, 1992).

The loss of Academician Gheorghe MISHKOY is huge for the entire scientific community and for the society in Republic of Moldova. In these difficult moments, we express our sincere condolences. Our thoughts and prays are with the staff of the Moldovan Academy, the Mathematical institute, his family and all the friends who knew Gheorghe.

May his soul rest in peace!

Gnedenko Forum Editorial

Received: November 10, 2020
Accepted: Desember 20, 2020

Fatigue of Unidirectional Fiber Composite and Static Strength of its Components. Simplified Daniels-Epsilon Sequence

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Abstract

A simple method is proposed for obtaining such a description of the average fatigue curve (and residual static strength) of a unidirectional fiber composite (UFC), which is directly related to the parameters of the static strength distribution of its components (SSDC), which are the longitudinal items bearing the main longitudinal load. This description makes it possible to predict changes in the fatigue life of the UFC when the SSDC changes. The method is based on a Daniels_epsilon_sequence (DeS), which is a modification of Daniels_sequence (DS) which takes into account the short-term damaging effects of one separate cycle of fatigue loading. Here we use a specific version of it in which the number of components of a critical link of UFC is equal to infinity. We call this version as simplified DeS. The concept of a DeS_fatigue equivalent distribution (DeS_FED) of local static strength of LI is introduced. The DS the calculations of fatigue life using the DeS_FED coinside with the test data. The simplified DeS model studied in this article should be used for preliminary analysis of the mean SN curve. For the more detailed analysis should be used the models considered by author some earlier which include the use of the theory of Markov processes and the Monte Carlo method, which allows modeling and statistical aspects of the problems under consideration, but require much more time-consuming calculations. At the end of the paper a numerical example of processing the fatigue test data and prediction a new fatigue life at some SSDC changes are given.

Keywords: Composite, Daniels'_epsilon_sequence, fatigue life, residual strength.

1. Introduction

In our previous publications [1, 2] it was shown that the use of the Daniels' sequence and Daniels'_epsilon_sequence (DeS) allows

1) to describe the process of step-by-step growth of local stresses in a weak link of theUFC. This process is similar to the well-known S-shaped change in some physical parameters during the fatigue tests (Fig. 1);

2) to relate directly the number of the DeS items (the calculated local stresses) at which the value DeS item tend to infinity with the composite DS-fatigue life (DS_FLf) which is a function of the parameters of the SSDC; this functon can be used for the regression analysys of the fatigue test data and the prediction of the composite fatigue life changes at some its component static strength changes; it is the main specific feature and advantage of the DeS models;

3) to explain the specific features of the residual strength: a long period of very gradual degradation of strength is suddenly replaced by a sharp drop to zero;

4) to explain the existence of an infinite calculated fatigue life and fatigue limit.

The main drawback of the considered previously DS model is this: if the cycle loads only slightly exceeds the fatigue limit, the predicted DS_FL is very small. In the work [1], this drawback is eliminated by the use the theory of Markov processes. In the work [2] the use of a Daniels'_epsilon_sequence (DeS) was proposed. Some additional parameter which is denoted by the symbol ε, was introduced in order makes it easy to control the value of the calculated DeS_FLf. This parameter takes into account that only a part of the components whose strength is less than the applied load is destroyed during one fatigue loading cycle. The value ε, apparently, depends on the maximum load in the current cycle, on the SSDC, on the frequency of loading, on the structure of the UFC and on the other circumstances.

In the paper [2], the initial number of components n in the considered critical link of UFC is assumed to be equal to some finite number. Assumption that the strength of each of these components is a random variable allows to model the statistical characteristics of the fatigue life of the UFC. But in this article, we make an additional simplification. We assume that this number is equal to infinity. It is clear that this assumption is acceptable only if we are interested in analyzing the average values of the fatigue life, but just this case is considered in this article. This assumption greatly simplifies calculations and is often appropriate in the applications. As it was told already, a detailed analysis involving the theory of Markov processes and Monte-Carlo method is described in our previous work [1].

Short history of the investigation of the fatigue of composite is considered in large number of the publications. Here we mention only most significant papers. The first scientific publication devoted to this problem appears to be the Peirce's work [3]. Peirce gives an approximate formula for the average strength of a bundle of LIs (fibers, bundles, strands) forming the UFC. The normal approximation of the strength distribution law of a LI parallel system was shown by Daniels [4, 5]. His result was refined by Smith [6] already with a reference to the series-parallel system (SPS), definition of which was earlier proposed in [7]. A detail review of the residual strength a given in [8, 9].

The term "Daniels Sequence" first appeared in work [10], but the first calculation of such a sequence took place much earlier [11, 12]. Later, some specific versions of DS, the Daniels_epsilon_Sequences (DeS) appears and was discussed in [1, 2]. But the models studiet in these papers take into account too many factors and required preliminary estimations of too large number of parameters. The main purpose of this paper is to offer the model which is much easier to use for the construction and the description of the SN fatigue curve and residual strength.

In section 2 of the paper the general mathematical definition of the DeS, in sections 3 and 4 its application to the the processing of the fatigue test data is studied. In the following section the prediction of the fatigue life after some changes of the statical strength of the components of the UFC and the following conclusions are discussed.

2. The Daniel's epsilon sequence

2.1. The Daniel's sequence

The UFC can be considered as the series-parallel system (SPS) with series of n_L links and n_C parallel items in every link. But in this section, we consider only one link of SPS that has n items. The strength of such link was studied by Daniels [4,5]. I t is called now the "classical model of bundle of n parallel fibers stretched between two clamps". In general case, strands or some set of strands can be considered instead of fibers. Here for all structural items of these types we'll use more general terms : „longitudional item" (LI) or just „component". The connection of the fatigue characteristic of one link and the SPS as a whole is described in [11,12]. We assume that the total load on the link in question is uniformly distributed among all functional LIs both at the beginning of the test and after the destruction of some of them. The state of the UFC is defined by the number of undisturbed workable LIs.

First, we will recall the general definition of DS with some corrections. The DS as some random process is defined by two components: a vector $X_{L,1:n}=(X_{L,1},X_{L,2},...,X_{L,n})$, whose components are mutually independent random variables with the same cumulative distribution function (cdf), and an infinite sequence $S^+_{0:\infty}=\{S^+_0,S^+_1,S^+_2,...\}$, whose components are random variables with a known law of joint probability distribution. In the description of the fatigue test $X_{L,1},X_{L,2},...,X_{L,n}$ are the strengths of the components of UFC, $S^+_0,S^+_1,S^+_2,...$ are the values of the maximum stress in the sequence of fatigue loading cycles. Specific realization of the *the Daniels sequence (DS)* for specific loading sequence, $s^+_{0:\infty}=\{s^+_0,s^+_1,s^+_2,...\}$, $s^+_0\le s^+_1\le s^+_2,...$ is defined by equation

$$s_0=s^+_0\ ,\quad s_{i+1}=s^+_{i+1}/(1-\nu(s_i)/n_c)\quad i=0,1,2,... \tag{1}$$

where $s^+_{0:\infty}=\{s^+_0,s^+_1,s^+_2,...\}$ is realization of the $S^+_{0:\infty}=\{S^+_0,S^+_1,S^+_2,...\}$; $\nu(s_i)$ is the number of of elements of vector $x_{L,1:n}=(x_{L,1},x_{L,2},...,x_{L,n})$, realization of $X_{L,1:n}=(X_{L,1},X_{L,2},...,X_{L,n})$, which are lower than s^+_i. The transition from s_i to s_{i+1} is what we call *the step of DS.*

We call the *Daniels_epsilon_Sequence (DeS)* the introdused in [2] modification of DS, which is defined by the equation

$$s_0=s^+_0\ ,\quad s_{i+1}=s^0_{i+1}\varepsilon+s_i(1-\varepsilon),\ \ i=0,1,2,... \tag{2}$$

where $s^0_{i+1}=s^+_{i+1}/(1-\nu(s_i)/n)\quad i=0,1,2,...\ ;\ \ 0<\varepsilon\le 1$,

As it was told already, the parameter ε defines the rate of accumulation of the fatigue damages into one cycle and takes into account the fact that the destruction of all $\nu(s_i)$ LIs requires both time and sufficient energy supply. The part of them the fracture of which takes place in one cycle depends on the frequency of loading and the other factors.

The ratio $\nu(s_i)/n_c$ is the empirical estimate, $\hat{F}(x)$, of the cdf, $F(x)$. For considered here simplified version of the DeS we assume that the dimension of the vector $X_{i:n}$ (number of LIs in the considered link) is so much that instead of $\hat{F}(x)$ in (1) can be used $F(x)$.

In the fatigue test to study the SN curve we are interested only in the case when $s^+_1=s^+_2=...=s^+$ where s^+ is some specific level of load. Then the equation (2) can be rewriten in this way

$$s_0=s^+_0\ ,\quad s_{i+1}=s^0_{i+1}\varepsilon+s_i(1-\varepsilon),\ \ i=0,1,2,... \tag{3}$$

where $s^0_{i+1}=s^+/(1-F(s_i))\quad i=0,1,2,...\ ;\ \ 0<\varepsilon\le 1$,

There are two types of the DeS (see Fig.1). The **first type** of it takes place if the parameter s^+ is large enough and items of DS has tendency to grow up to infinity.

Let for the fixed s^+ and some ε the integer function $n_{DeS}(s;s^+,\varepsilon),\ n=1,2,...$, $s=s_1,s_2,...$ determines the number of steps at which the sequence DeS reaches the value s. The event when s_i reaches some large enough critical value s_C we consider as fatigue failure and define the *DeS Fatigue Life (DeS_FLf)* to be equal to

$$n_{DeS.s_C}=n_{DeS}(s_C;s^+,\varepsilon),\quad n=1,2,... \tag{4}$$

Then for $s=s_C$ and $s=\infty$ the DeS_FLfs, the corresponding $n_{DeS,sC}$ and $n_{DeS,\infty}$, are equal to the values of this function.

We also introduce the concept of *the Daniels residual function* as a function that determines the residual strength after the DeS reaches the value s. We define it by the equation

$$r_{DeS}(s)=\max_{x>s} x(1-F(x))\ . \tag{5}$$

Now it can be introduced another definition of fatigue life, $n_{DeS,R}$, which is determined by the moment when the residual strength of the tested specimens decreases below a certain critical level r_C, $r_C \geq 0$. We define it by equation

$$n_{DeS,R}=1+\max(i: r_{DeS}(s_i)>r_C,\ i=0,1,2,... \tag{6}$$

The **second type** of DeS takes place if the stress s^+ is small enough. Then after some number of steps the increasing of the value of DeS items almost ceases in spite of increasing the number of steps up to infinity. The maximal value of cycle parameter s^+ for which this fenomenon takes place we considered as *the Daniels fatigue strength (DFSt)* or *the Daniels fatigue limit (DFLt),* s_D. The fatigue limit is determined by the maximum load value s^+ at which equality $s_{i^*+1}=s_{i^*}$ can occur for some i^*.

In [2] it is shown that regardless of the value of ε for load level, s^+, more than $\max x(1-F(x))$ the event $s_{i^*+1}=s_{i^*}$ can not take place but at every $i=1,2,...$, the inequality $s_{i+1}>s_i$ takes place; the items of the DeS grow up to infinity, all LIs will be destroyed, the value of the DeS_FLf is final. So the value

$$s_D=\max x(1-F(x))\ , \tag{7}$$

can be used as the definition of *the Daniels fatigue limit* **(***DFLt***).**

It is necessary to note the obvious. The values of s_D in (7) and $r_{DeS}(0)$ in (5) coincide with the value of the static strength of the "classical model of bundle of n parallel fibers stretched between two clamps" predicted by Daniels [4,5]. The difference between these values takes place only if there is a difference in the distribution functions of the strength of LIs, $F(x)$.

The mathematics of the above analysis is valid for any parallel system for which equations (1) and (2) are valid. The connection of this mathematics with the fatigue phenomenon of an UFC occurs if instead of the distribution of static strength, $F(x)$, we use some other cdf $F_L(x)$ of the local strength of the element working as part a the" weak segment " of the UFC. The length of the LI in the "weak segment" (the link in the considered here model) does not match the length of LI for laboratory static strength tests, the structure of the "weak segment" has a special support conditions, … So, in the following we will usually use the $F_L(x)$ instead of $F(x)$.

Later on it will be shown that processing the test data we can find the function $\varepsilon=\varepsilon(s^+)$ in such a way that the the DeS_FLfs, $n_{DeS,sC}$ or $n_{DeS,\infty}$, wil be equal to the corresponding test fatigue lifes. This function together with the equations (3) and (7) gives the description of fatigue curve which is directly related to the parameters of the static strength distribution of its components (SSDC).

Next, we will consider two examples of using the considered mathematical apparatus (as we will say, using the DeS approach) to process the results of fatigue tests in order to establish a connection between the description of the fatigue curve of a composite and the static strength of its components. In the second example, the analysis of residual strength will also be considered.

3. Application to the fatigue curve analysis. Numerical example 1

In this part of the article, we will review the analysis of data on fatigue tests of composite samples and on the static strength of its components presented in (13).

3.1. The cdf of the local tensile strength

Two simple assumptions help to explain (and to model) why during fatigue tests the composite collapses at the load significantly lower than its static strength:
A) The local strength in the considered specificweak link k_L time lower than in other links, $k_L \geq 1$;
B) in the considered specific link the local stress k_C time greater than in other links , $k_C \geq 1$.

Let's consider the difference between the influence of coefficients k_C and k_L on the results of analysis of fatigue test data. For simplicity we assume that ε is equal to 1. And instead of the symbol DeS, we'll use the DS symbol. Here we study the case of the lognormal distribution with the cdf $F(x) = \Phi\left((\log(x) - \theta_0)/\theta_1\right)$, where $\Phi(.)$ is cdf of the standard normal distribution, but all the following will be true in more general case (for example, for Weibull distributions with cdf $F(x) = 1 - \exp\left(-\exp\left((\log(x) - \theta_0)/\theta_1\right)\right)$). Note that if the explanation B of the local stress concentration is accepted, then instead of equation (3), we should use its corrected version - equation (3 b)

$$s_0 = s_0^+, \quad s_{i+1} = s_{i+1}^0 \varepsilon + s_i(1-\varepsilon), \ i = 0,1,2,..., \text{ where } s_{i+1}^0 = k_C s^+ / (1 - F_L(s_i)) \quad i = 0,1,2,...; \ 0 < \varepsilon \leq 1, \tag{3b}$$

It is clear that the use of $k_C s^+$ instead of s^+ increases the values of DeS curve by k_C times and reduces DeS_FLs, $n_{DeS;s_C}$. But it can be shown that fatigue limit s_D remains unchanged (see [1]). Now, if we use simultaneously k_C and k_L then $F_L(k_C x) = P(X/k_L < k_C x) = \Phi\left((\log(k_C k_L x) - \theta_0)/\theta_1\right)$ and in equation (7) we use $F_L(x) = F(k_C x)$ instead of $F(x)$ then the fatigue limit is determined by the product of $k_L k_C$. And if $k_L = 1$ then instead of equation (7), we should use its corrected version - equation (7a)

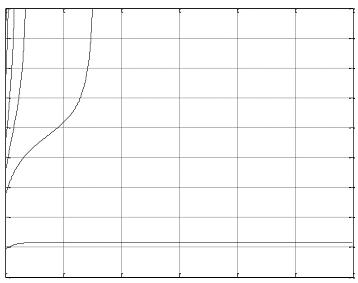
$$s_D = \max x(1 - F_L(x))/k_C \ , \tag{7a}$$

It is also clear that the values of the residual strength determined by the equation (5) do not depend on the k_C, but depend on the k_L and determine the **local residual strength** in the critical link under consideration. And equation (5) defines the residual static strength of the whole composite only in the case when $k_L = 1$. For this reason, some later in section 4 we use the explanation B for processing the data of residual strength.

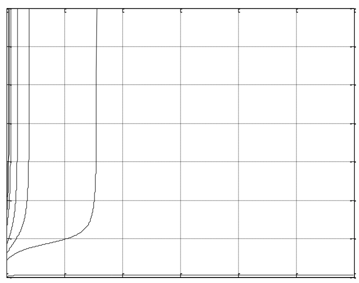
In order to see the difference between the influence of k_C and k_L under different values ε the calculation of DeS curves and the corresponding DeS_FLfs for two pairs: ($k_C = 1.75$, $k_L = 1.0$ **)** and **(**

$k_C = 1.0$, $k_L = 1.75$ **)** with two epsilon values: $\varepsilon = \mathbf{0}.\mathbf{00001}$ and $\varepsilon = \mathbf{0}.\mathbf{000001}$ were made. In the work [13] in the Table 2.1 the results of the carbon fiber strand specimen tensile test, in the table 2.11 of the results of the fatigue test at an approximately pulsed ($s_{\min} / s_{\max} = 0.1$) load on CFRP specimens are presented. The results of processing the data from static strength tests of carbon fiber strand specimens, including testing hypotheses about the type of distribution law using OSPPT and ρ criteria [11,14,15] show that the hypothesis about the lognormal distribution ("normal" on a logarithmic scale) was more plausible than the Weibull distribution. For the cdf $F(x) = \Phi((\log(x) - \theta_0)/\theta_1)$ the following parameter estimates are received $\theta_0 = 6.48$, θ_1=0.168 (where $\Phi(.)$ is the cdf of the standard normal distribution).

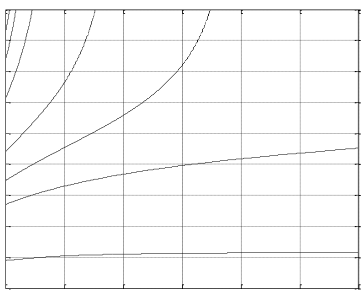
The results of calculation DeS curves and $n_{DeS}(s_C; s^+, \varepsilon)$ for $s_C = 600$ are shown on Fig. 1. It turned out that at high stress loads there is some small difference in the values of DeSFLf for two pairs: ($k_C = 1.75$, $k_L = 1.0$ **)** and **(** $k_C = 1.0$, $k_L = 1.75$ **)**. But this difference disappears and for $\varepsilon = 0.00001$ and for $\varepsilon = 0.000001$ if the load values are small**.** So if we need to know only the fatigue curve, there is no big difference : to use the k_C or k_L parameter, but if the residual strength data is also analyzed at the same time, it is more convenient to use parameter k_C assuming parameter k_L to be 1 because , as it was told already the equation (5) defines local residual stress corresponding to cdf $F_L(.)$.



$\varepsilon = 0.00001$**;** $k_C = 1.75$, $k_L = 1.0$ DeS_DFLs : 8000 19000 47000 154000 349000 1507000 >6000000



$\varepsilon = 0.00001$**;** $k_C = 1.0$, $k_L = 1.75$ DeS_DFLs : 37000 50000 82000 195000 396000 1558000 >6000000



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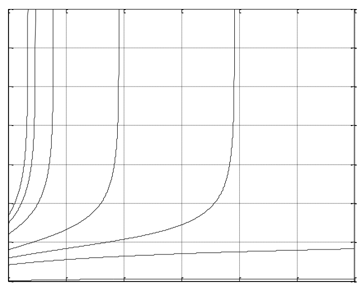


$\varepsilon = 0.000001$, $k_C = 1.0$, $k_L = 1.75$ DeS_DFLs : 343000 473000 785000 1919000 3920000 >6000000 >6000000

Figure 1: DeS and $n_{DeS}(s_C; s^+, \varepsilon)$ for load levels: 333.5 323.7 309.7 290.1 279.6 270.8 250.2 MPa

There is a need to investigate the influence of the parameter ε more closely. In first and second lines of Table 1 the results of the fatigue test [13] are shown. The results of calculating the corresponding $n_{DeS}(600; s^+, \varepsilon)$ for $k_C = 1.75$ and different ε are presented.

Table 1: Comparison of test data and calculations of DeS_FLfs

S	333.5	323.7	309.7	290.1
Test (cycles)	4928	115733	373199	1004800
ε	DeS_FLf			
0.1	1	3	5	16
0.01	8	19	47	154
0.00001	7038	18129	46031	153012

The analysis of the Table shows that for relatively small value of ε, $\varepsilon < 0.01$, DeS_FLf is approximately proportional to the $1/\varepsilon$ and to some specific value $N_{DeS}(s^+, \varepsilon_{DeS})$ of the DeS_FLf which is defined by some function $N_{DeS}(s^+, \varepsilon)$ for some specific value of $\varepsilon = \varepsilon_{DeS}$ .The values of DeS_FLf for some other ε, $\varepsilon < 0.01$, is defined by equation

$$N_{DeS}(s^+, \varepsilon) = (\varepsilon_{DeS} / \varepsilon) N_{DeS}(s^+, \varepsilon_{DeS}) . \tag{8}$$

Now we see that in order to get reasinoble fitting of the test SN curve by the use of The equation (8) we should fine the function $\varepsilon(s)$ corresponding to equation

$$N_{DeS}(s^+, \varepsilon(s^+)) = N_T(s^+) , \tag{9}$$

where $N_T(s^+)$ mean test fatigue life as function of s^+ which define SN curve.

By the way, let us note, the tedious calculations with a very small ε value in some cases can be replaced by faster calculations with a larger ε value. However, this rule does not take place for $\varepsilon > 0.01$.

3.2. The DeS fatigue equivalent distribution of the local strength

Using the corresponding value of the parameter ε we can obtain arbitrarily large calculated fatigue life . But the natural question appears: is there such a cdf, we denote it by $F_{DeS}(.)$, that the calculations using the equation (3) and a pair (ε , $F_L(.)$) give the same results as when using the same equation but pair (1 , $F_{DeS}(.)$) (using value $\varepsilon = 1$). The function $F_{DeS}(.)$ should correspond to the equation

$$(1-\varepsilon)s + \varepsilon s^+ / (1 - F_L(s)) = s^+ / (1 - F_{DeS}(s)) , \qquad s^+ \le s < \infty .$$

It is easy to get the following solution for $F_{DeS}(.)$

$$F_{DeS}(s)) = 1 + s^+ (1 - F_L(s)) / ((1 - F_L(s))(1-\varepsilon)s + \varepsilon s^+) , \qquad s^+ \le s < \infty . \tag{10}$$

The function $F_{DeS}(.)$ we will call *the DeS fatigue equivalent distribution (DeS_FD) of local strength*. The fatigue life calculated using this function and DS approach (instead of DeS approach) would coincide with the data of fatigue tests.

3.3 Approximation

The way to calculate the DeS_FLf is the use of requrent formule (3) and its modifications. It is easy if the DeS_FLf is not too large. In other case it is very exhausting work. The following approximation can be used.

Let us consider i as continuous variable. If i is very large then the difference $(s_{i+1} - s_i)$ is very small and in subsequent calculations it can be used as a derivative $ds/di = \varepsilon(-s + k_C s^+ / (1 - F_L(s_i))$. Then the value of step corresponding to increasing s_i from s_0 to s

$$i(s_0, s, \varepsilon) = (1/\varepsilon) K(s_o, s), \tag{11}$$

where

$$K(s_0, s) = \int_{s_0}^{s} (1/(-x + k_C s^+ / (1 - F_L(x)))dx . \tag{12}$$

If $s_0 = k_C s^+$, $s = s_C$ then we can get the approximate estimate of $n_{DeS}(s_C; s^+, \varepsilon)$.

For example, in Table 1 we see the values $n_{DeS}(600; s^+, 0.00001)$: 7038; 18129; 46031; 153012 for s^+ =333.5, 323.7, 309.7, 290.1. Corresponding calculations using equation (9) give the very similar results : 7108 ; 18312 ; 46503 ; 154498 .

Let us note that the integral $K(s_0, s)$ can be used for approximate calculation of the $n_{DeS}(s_C; s^+, \varepsilon)$ and then for calculation of the approximate function $\varepsilon(s)$ which is necessary for fitting real test fatigue lives.

4. Numerical example 2. Residual strength

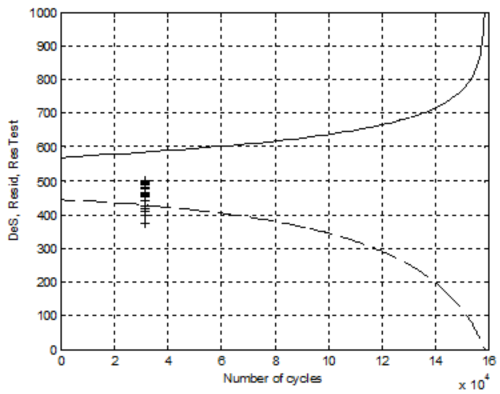
Now we consider the processing of the result of the fatigue test in which the data not only about the fatigue life but about the residual strength was obtained. The test data was taken from Tables 1-3 in Ref. [16] concerned T300/934 graphite/epoxy laminates with $[0/45/90-45_2/90/45/0]_2$ lay-up. In Table 1 of this paper the static strength of 25 specimens, in Table 2 numers of cycles to failure at three different stress levels (namely: for $\sigma_{\max}$ = 400, 380 and 290 MPa, $R = \sigma_{\min}/\sigma_{\max} = 0$) and in Table 3 two sets of residual strength data are reported for 15 and 18 specimens subjected to cyclic loading up to 3,640,000 and 31,400 cycles at a maximum stress, $\sigma_{\max}$ = 290 and 345 MPa respectively.

The tested specimens are not the UFC. But we suppose that the failure of this composite takes the place after the failure of some weak microvolume (WMV) which is the bundle of the n parallel LIs. We make (enough rough) assumption that this WMV is a *UFC - equivalent* which has the same distribution of fatigue strength. In work [16], there is no information about the static strength of the composite components. We will use the data which we have used already in the previous example in which carbon fiber longitudinal elements were also used for the test specimens. This, of course, means that the following should be considered as an example of the application of the technique in question, and not as a study of a specific experiment. We accept also the lognormal distribution $F(x) = \Phi((\log(x) - \theta_0)/\theta_1)$ of the static strength of LIs with the same values of parameters $\theta_0 = 6.475$, $\theta_1 = 0.168$ as in the section 3.1.

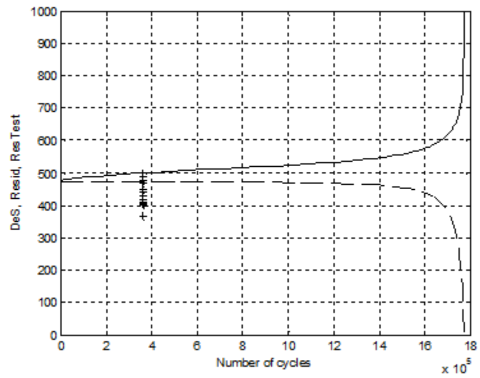
After transformation of equation (11) we can calculate the value of ε

$$\varepsilon = K(s_o, s^+) / N_T(s^+), \tag{13}$$

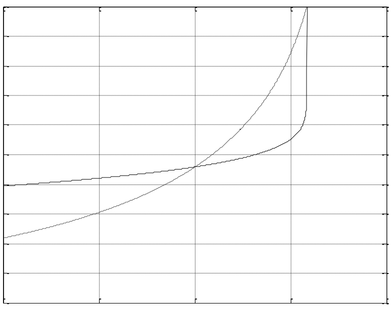
which should ensure the equality of the calculated DeS_FL and the corresponding test value for specific load level. For two load levels, 345 and 290 MPa, we have got the corresponding values ε: 0.00000242 and 0.00000464. Then using these values further in equations (3 b), (5) and (8) , we get the results shown in Fig. 2. In the left part of this figure, for the load level of 345 MPa, the following is shown : a1) DeS curves (-), calculated (--) and test residual strengths (+); a2) cdf $F_L(.)$ and cdf $F_{DeS}(.)$; a3) "the derivatives" of these functions (df=$(F_L(s_{i+1})-F_L(s_i))/(s_{i+1}-s_i)$; def= $(F_{DeS}(s_{i+1})-F_{DeS}(s_i))/(s_{i+1}-s_i)$, $i=1,2,...$) .



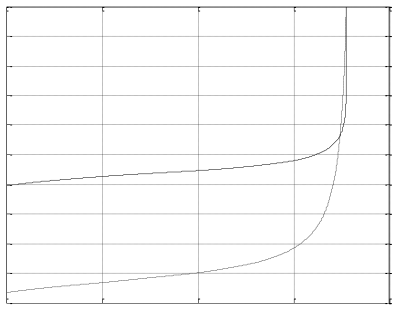
a1



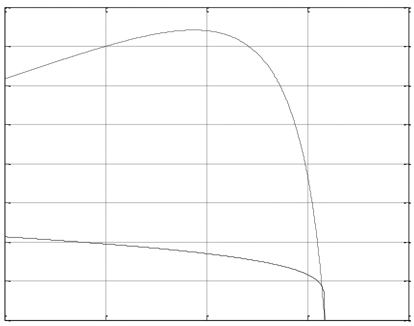
b1



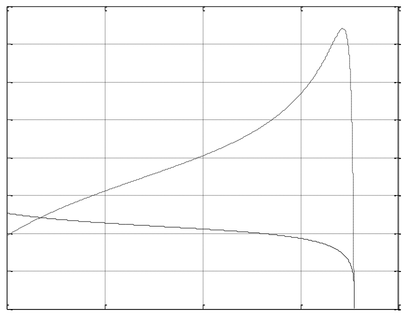
a 2



b 2



a 3



a 4

Figure.2. Daniels' sequences (DeS) , calculated (--) and test residual strength (+) (a1, b1); cdf $F_L(.)$ and Daniels'- equivalent cdf $F_{DeS}(.)$ (a2, b2) and the "derivatives" of these functions (a3, b3) for two load levels 345(a) and 290(b) MPa.

The right part of the figure shows similar results for the load level of 290 MPa. Let us note that the Daniels fatigue equivalent distribution, $F_{DeS}(.)$, provides another measurement of differences in the static strength distributions of separate longitudinal components, $F(.)$, and inside the structure of the composite.

On the sub-plots a1 and b1, we see that equation (5) for the assumed distribution function $F_L(.)$ gives a plausible description of the residual strength. It is useful to note: the DeS curve rushes to infinity and the calculated residual strength rushes to zero at the same number of cycles. Since equation (10) is suitable only for relatively low load levels, for levels 400 and 380 MPa, the values of ε were selected from the equation of the coincidence of the calculated and tested values of durability: 0.0000181 and 0.00000166. The final calculated fatigue curve is shown in Fig. 3.

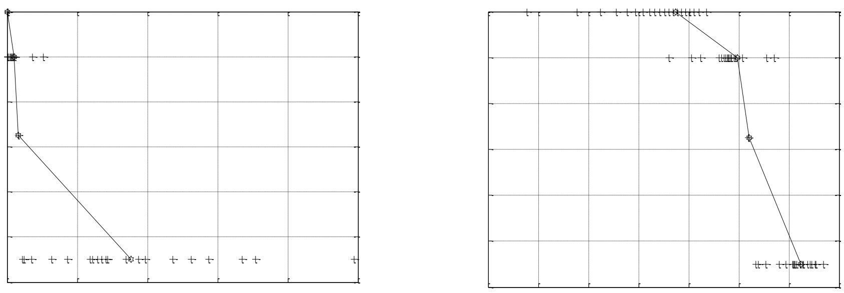


Figure 3. DeS_SN, log10DeS_SN (-) and tast data

Let us clarify that the description of the fatigue curve is obtained by selecting the coefficient $k_C = 1.6$, using equation (3b) and preliminary calculation values of ε: 0.00000242 and 0.00000464 using equation (10) for loading levels 345 and 290 MPa and by direct selection 0.0000181 and 0.00000166 for levels 400 and 380 MPa.

5. Prediction

The study of the possibility and accuracy of predicting changes in the fatigue life of a composite with changes in the static strength of its components using the DeS approach requires a volumetric experiment. And this is the task of subsequent research. Here we will limit ourselves to analyzing the effect of reducing the spread of static strength, more precisely, the parameter θ_1 on the calculated average fatigue life at the level of loads considered in the last example. A comparison of the results of the the calculations DeS_FLf for the two different values θ_1 but the same all the other parameters is shown in Table 2.

Table 2. ***Comparison of the calculations of DeS_FLfs for two values*** θ_1

s^+	480	380	345	290
Test (cycles)	5313	95760	158489	1772812
θ_1	DeS_FLf			
0.168	5500	95900	158600	1772800
0.015	4800	88100	164300	>4000000

We see a significant increase in DeS_FLf when the load is close to the fatigue limit. But with a relatively high load, the decrease led to a slight decrease in DeS_FLf. Recall that the same effect applies to calculating the average static strength. For the studied here lognormal distribution the average static strength is equal to $\exp(\theta_0 + (\theta_1)^2/2)$. These conclusions should be taken into account in the desine of a new composite which is similar to the studied here.

Conclusions

It is shown that a simplified version of DeS can be used for such description of the average fatigue curve (and residual static strength) of an UFC, which is directly related to the parameters of the static strength distribution of its components (SSDC). This description is very desirable because it makes it possible to predict changes in the fatigue characteristics of the UFC when the static strength characteristics of its components change. A numerical example of processing the test data on fatigue life and residual strength of a carbon-fiber reinforced composite confirms the reasonable coincidence of the calculation result and test data. The offered type of description of fatigue curve

The concept of a DeS fatigue equivalent distribution (DeS_FED) of local static strength of LI is introduced. The fatigue life calculated using this distribution and the basic Daniels sequence (DS) would coincide with the data of fatigue tests. Comparison of the DeS_FED with the real SSDC shows the specific behavior of the UFC components under short-term loading during a single fatigue loading cycle, as opposed to loading during static tests.

The application of the proposed method for processing fatigue tests data of composite material samples that differ in structure from the UFC will allow us to judge the influence of its features on the fatigue curve.

Randomization of the considered version of the DeS, using the Monte Carlo method allows to analyze the scatter of the fatigue life. But this time the search of the parameters of the corresponding nonlinear regression is a difficult task an example of this analysis is given in [2]. Against all the odds, we think that, in due course, the structure of the models suggested will be of the interest not only for the graduation theses of the students but also for the engineering applications, in particular, for the prediction of the variations fatigue life of the UFC after the changes in the parameters of their components.

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Received: November 23, 2020
Accepted: Desember 20, 2020

Sensitivity Analysis of a Complex Engineering System with the Application of Wait in Line Theory

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Abstract

In this bloodthirsty scenario of competition, rapid and cost-effective production is a key obligation for endurance. To attain this objective the thought of production area is fetching people now days. In this sector, non- identical equipment is arranged rationally to execute desired procedure to convert unprocessed materials into the processed material. Sometimes during the manufacturing, the problem of waiting line arises because of some unpredictable reasons, so here this paper reveals the same problem with its effects on system reliability and availability, also system sensitive nature is analyzed with respect to unexpected failures. Supplementary variable technique and copula method is used to solve the system.

Keywords: Markov processes, steady state behavior, supplementary variable technique, reliability, wait in line etc.

1. Introduction

Wait in line theory has an important role in almost all analysis of repair services. Wait in line models are very helpful tool for calculating the performance of various repair systems like automated systems, production systems, computer systems, telecommunication systems, networking systems like computer networking or communication networking, and flexible manufacturing systems. Conventional wait in models forecast the system performance on the basis of the assumption that all service facilities offer failure-free service

With reference to the above facts, here we have discussed the behavior analysis of an engineering system having three-subsystems 1, 2 and 3, connected in series, having waiting line for repair. First subsystem has two units one is key unit and another is active superfluous. Second subsystem has one key Unit and another is cold superfluous while Third subsystem consists of two units' connected in parallel arrangement. The system can completely fail due to failure of any of the subsystems [3]. In the beginning when the system starts working, the key units of subsystem 1 and 2 and both units of the subsystem 3 are fully operational. When the main units of the subsystems 1 and 2 fail, the supporting units are switched on automatically and failed units are sent for repair to repairing section [1, 2]. Here, a realistic situation is taken into the consideration that when main units and supporting units of both subsystems 1 and 2 are failed and they are sent for repair to repairing section, they are in waiting line because repairmen is busy somewhere else. In this case a wait in line is there at repair section. So the main concern of the study is that all the four units are waiting for repair [4]. Transition state diagram is shown by Figures 1. Table 1 shows the state specification of the system.

Assumptions

- In the beginning the system is in good operating state.
- All Subsystems are connected in series.
- System has two states only good and failed not degraded.
- Catastrophic failure is also responsible for system failure in the study also they require constant and exponential repair. So, copula technique is used for finding probability distribution [5].
- Repair facility which follows general time distribution is there for the service of both the subsystems of unit 3 and also failure are exponential in both cases.
- For the subsystems 1 and 2 failure and repairs both are exponential.

Table 1: *State specification of the system*

States	Description	System State
S0	The system is in good working state	G
S1	The system is in working state when key unit is failed.	G
S2	The system is in failed state because of failure of superfluous unit.	F
S3	When all four units are in waiting at repair section, system is in failed state.	F
S4	The system is in working state when superfluous unit of subsystem 1 is failed.	G
S5	The system is in failed state due to the failure of key unit of subsystem 1.	F
S6	The system is in working condition when key unit subsystem 2 is failed.	G
S7	The system is in failed state when superfluous unit of subsystem 2 is failed.	F
S8	The system is in operable condition when key unit of subsystem 3 failed.	G
S9	The system is in failed state from the state S_8 due to failure of superfluous unit of subsystem 3.	FR
S10	The system is in operable condition when superfluous unit of subsystem 3 is failed.	G
S11	The system is in failed state from the state S_{10} due to failure of key unit of subsystem 3.	FR
S12	The system is in failed state from the state S_1 due to failure of subsystem 3.	FR
S13	The system is in failed state from the state S_6 due to failure of subsystem 3.	FR
S14	System is failed state because of catastrophic failure.	FR

G: Good state; F: Failed State; FR= Failed state and under repair.

2. Notations

Pr	Probability
$P_0(t)$	Pr (at time t system is in good state S0)
$P_i(t)$	Pr {the system is in failed state due to the failure of the ith subsystem at time t}, where i=2, 5, 7, 14.
λ_i	Failure rates of subsystems, where i=a_1, a_2, b_1, b_2, c_1, c_2, CSF.
ψ	Arrival rate of all four units of subsystems 1 and 2 to the repair section named as a_1, a_2, b_1, b_2.
μ	Repair rate of unit's a_1, a_2, b_1, b_2.
$\phi_i(k)$	General repair rate of ith system in the time interval (k, k+Δ), where i= c_1, c_2, (names for

	the units of subsystem 3) CSF and k=v, g, r, l.
$P_3(t)$	Pr (at time t there is a queue (a_1, a_2, b_1, b_2) in the maintenance section due to servicing of some other unit and all four machines are waiting for repair.
$P_i(j,k,t)$	Pr (at time t system is in failed state due to the failure of j[th] unit when k[th] unit has been already failed, where i=9, 11. j=g, v. and k=v, g.
K1, K2	Profit cost and service cost per unit time respectively.

Let $u_1 = e^l$ and $u_2 = \phi_{CSF}(l)$ then the expression for joint probability according to Gumbel-Hougaard family of copula is given as $\phi_{CSF}(l) = \exp[l^\theta + (\log\phi_{CSF}(l))^\theta)^{1/\theta}]$

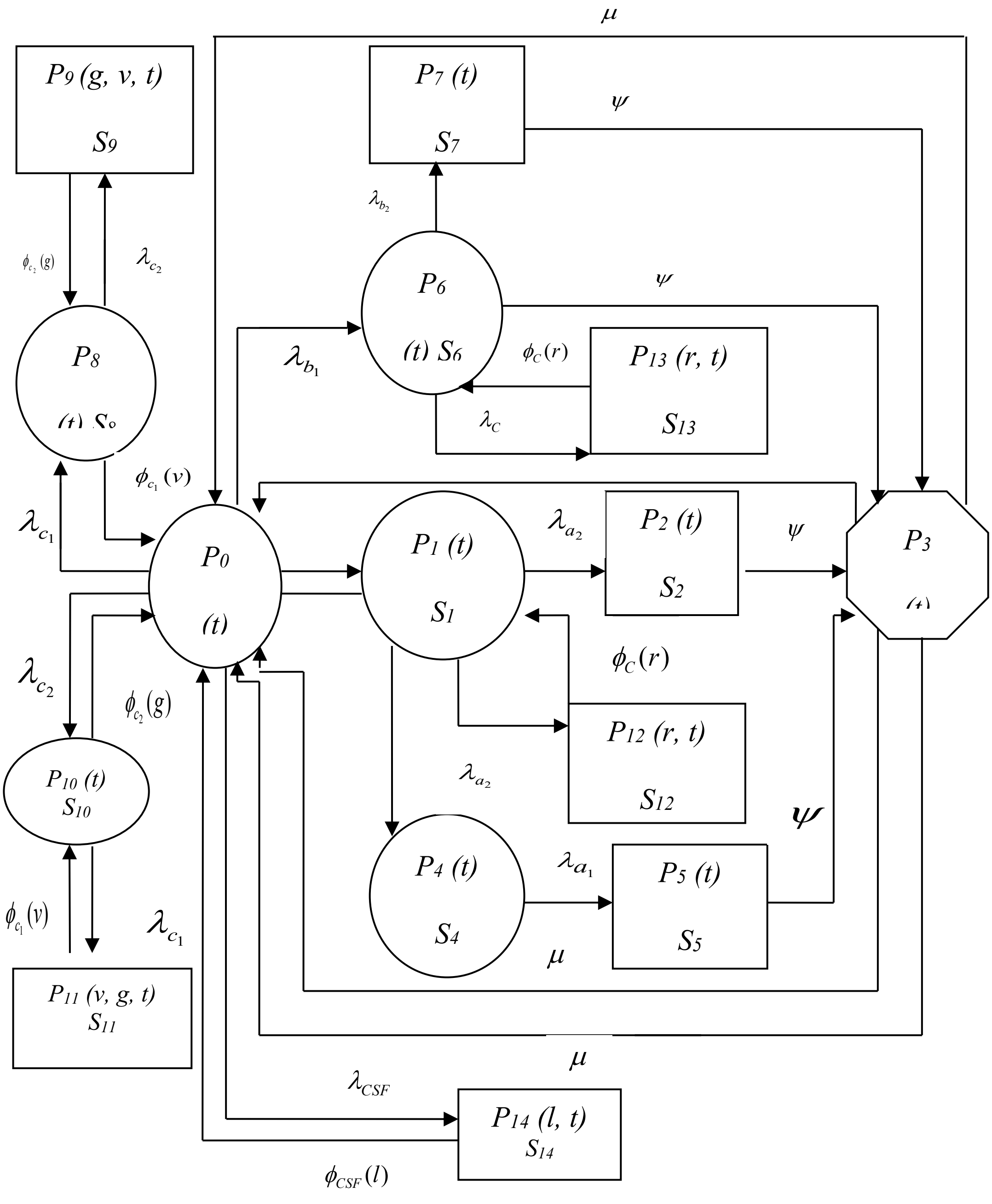

Figure 1: *Transition state diagram*

3. Formulation of the mathematical model

The following differential equations have been obtained by considering limiting procedures and different probability constraints which satisfying the model:

$$\left[\frac{d}{dt}+\lambda_{a_1}+\lambda_{a_2}+\lambda_{b_1}+\lambda_{b_2}+\lambda_{c_1}+\lambda_{c_2}+\lambda_{CSF}\right]P_0(t)=\int_0^\infty \mu\ (i)P_3(t)di+\phi_{c_1}P_8(t)+\phi_{c_2}P_{10}(t)+\int_0^\infty \phi_{CSF}(l)P_{14}(l,t)dl \quad \ldots (1)$$

$$\left[\frac{\partial}{\partial t}+\lambda_{a_2}+\lambda_c\right]P_1(t)=\lambda_{a_1}P_0(t)+\int_0^\infty \phi_C(r)P_{12}(r,t)dr \quad \ldots (2)$$

$$\left[\frac{\partial}{\partial t}+\psi\right]P_2(t)=\lambda_{a_2}P_1(t) \quad \ldots (3)$$

$$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial i}+(\mu+\psi)\right]P_3(t)=\psi[P_2(t)+P_5(t)+P_6(t)+P_7(t)]+\frac{(\psi t)^3e^{-\psi t}}{6} \quad \ldots (4)$$

$$\left[\frac{\partial}{\partial t}+\lambda_{a_1}\right]P_4(t)=\lambda_{a_2}P_0(t) \quad \ldots (5)$$

$$\left[\frac{\partial}{\partial t}+\psi\right]P_5(t)=\lambda_{a_1}P_4(t) \quad \ldots (6)$$

$$\left[\frac{\partial}{\partial t}+\lambda_{b_2}+\psi\right]P_6(t)=\lambda_{b_1}P_0(t)+\int_0^\infty \phi_C(r)P_{13}(r,t)dr \quad \ldots (7)$$

$$\left[\frac{\partial}{\partial t}+\psi\right]P_7(t)=\lambda_{b_2}P_6(t) \quad \ldots (8)$$

$$\left[\frac{\partial}{\partial t}+\phi_{c_1}(v)+\lambda_{c_2}\right]P_8(t)=\lambda_{c_1}P_0(t)+\int_0^\infty \phi_{c_2}(g)P_{11}(g,v,t)dg \quad \ldots (9)$$

$$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial g}+\phi_{c_2}(g)\right]P_9(g,v,t)=0 \quad \ldots (10)$$

$$\left[\frac{\partial}{\partial t}+\phi_{c_2}(g)+\lambda_{c_1}\right]P_{10}(t)=\lambda_{c_2}P_0(t)+\int_0^\infty \phi_{c_1}(v)P_{11}(v,g,t)dv \quad \ldots (11)$$

$$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial v}+\phi_{c_1}(v)\right]P_{11}(v,g,t)=0 \quad \ldots (12)$$

$$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial r}+\phi_C(r)\right]P_{12}(r,t)=0 \quad \ldots (13)$$

$$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial r}+\phi_C(r)\right]P_{13}(r,t)=0 \quad \ldots (14)$$

$$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial l}+\phi_{CSF}(l)\right]P_{14}(l,t)=0 \quad \ldots (15)$$

Boundary Conditions:

$$P_3(i=0,t)=\psi[P_2(t)+P_3(t)+P_6(t)+P_7(t)] \quad \ldots (16)$$

$$P_8(0,t)=\lambda_{c_1}P_0(t) \quad \ldots (17)$$

$$P_9(0,v,t)=\lambda_{c_2}P_8(t) \quad \dots (18)$$

$$P_{10}(0,t)=\lambda_{c_2}P_0(t) \quad \dots (19)$$

$$P_{11}(0,g,t)=\lambda_{c_1}P_{10}(t) \quad \dots (20)$$

$$P_{12}(0,t)=\lambda_C P_1(t) \quad \dots (21)$$

$$P_{13}(0,t)=\lambda_C P_6(t) \quad \dots (22)$$

$$P_{14}(0,t)=\lambda_{CSF} P_0(t) \quad \dots (23)$$

Initial condition:

$P_0(0)=1$, otherwise zero.

Solving equations (1) through (15) by taking Laplace transform and by using initial and boundary conditions we obtained following probabilities of system is in up) and down states at time t,

$$\overline{P}up=\overline{P}_0(s)+\overline{P}_1(S)+\overline{P}_4(s)+\overline{P}_6(s)+\overline{P}_8(s)+\overline{P}_{10}(s)$$

$$=\frac{1}{K(s)}\Big[\,1+\frac{\lambda_{a_1}}{[s+\lambda_{a_2}+\lambda_C-\lambda_C\overline{S}_{\phi_C}(s)]}+\frac{\lambda_{a_2}}{[s+\lambda_{a_1}]}+\frac{\lambda_{b_1}}{[s+\lambda_{b_2}+\psi-\lambda_C\overline{S}_{\phi_C}(s)]}+$$
$$\frac{\lambda_{c_1}}{[s+\lambda_{c_2}+\phi_{c_1}(v)-\lambda_{c_2}\overline{S}_{\phi_{C_2}}(s)]}+\frac{\lambda_{c_2}}{[s+\lambda_{c_1}+\phi_{c_2}(g)-\lambda_{c_1}\overline{S}_{\phi_{c1}}(s)]}\Big] \quad \dots (24)$$

$$\overline{P}down=\overline{P}2(s)+\overline{P}5(s)+\overline{P}7(s)+\overline{P}9(s)+\overline{P}11(s)+\overline{P}12(s)+\overline{P}13(s)+\overline{P}14(s)$$

$$=\frac{\lambda_{a_1}\lambda_{a_2}}{[s+\psi][s+\lambda_{a_2}+\lambda_C-\lambda_C\overline{S}_{\phi_C}(s)]}\frac{1}{K(s)}+\frac{\lambda_{a_1}\lambda_{a_2}}{[s+\psi][s+\lambda_{a_1}]}\frac{1}{K(s)}+$$
$$\frac{\lambda_{b_1}\lambda_{b_2}}{[s+\psi][s+\lambda_{b_2}+\psi-\lambda_C\overline{S}_{\phi_C}(s)]}\frac{1}{K(s)}+$$
$$\frac{\lambda_{c_1}\lambda_{c_2}D_{\phi_{c2}}(s)}{[s+\lambda_{c_2}+\phi_{c_1}(v)-\lambda_{c_2}\overline{S}_{\phi_{c2}}(s)]}\frac{1}{K(s)}+\frac{\lambda_{c_1}\lambda_{c_2}D_{\phi_{c1}}(s)}{[s+\lambda_{c_1}+\phi_{c_2}(g)-\lambda_{c_1}\overline{S}_{\phi_{c1}}(s)]}\frac{1}{K(s)}+$$
$$\frac{\lambda_c\lambda_{a_1}D_{\phi_c}(s)}{[s+\lambda_{a_2}+\lambda_C-\lambda_C\overline{S}_{\phi_C}(s)]}\frac{1}{K(s)}+\frac{\lambda_c\lambda_{b_1}D_{\phi_c}(s)}{[s+\lambda_{b_2}+\psi-\lambda_C\overline{S}_{\phi_C}(s)]}\frac{1}{K(s)}+$$
$$\frac{\lambda_{CSF}D_{\phi_{CSF}}(s)}{K(s)} \quad \dots (25)$$

where,

$$K(s)=s+\lambda_{a_1}+\lambda_{a_2}+\lambda_{b_1}+\lambda_{b_2}+\lambda_{c_1}+\lambda_{c_2}+\lambda_{CSF}-\psi\{[\frac{\lambda_{a_1}\lambda_{a_2}}{[s+\psi][s+\lambda_{a_2}+\lambda_C-\lambda_C\overline{S}_{\phi_C}(s)]}$$
$$+\frac{\lambda_{a_1}\lambda_{a_2}}{[s+\psi][s+\lambda_{a_1}]}+\frac{\lambda_{b_1}}{[s+\lambda_{b_2}+\psi-\lambda_C\overline{S}_{\phi_C}(s)]}+\frac{\lambda_{b_1}\lambda_{b_2}}{[s+\psi][s+\lambda_{b_2}+\psi-\lambda_C\overline{S}_{\phi_C}(s)]}]D_\mu(s)$$
$$+\frac{\psi^3}{(s+\psi)^4}\}-\frac{\lambda_{c_1}\phi_{c_1}(v)}{[s+\lambda_{c_2}+\phi_{c_1}(v)-\lambda_{c_2}\overline{S}_{\phi_{C_2}}(s)]}-\frac{\lambda_{c_2}\phi_{c_2}(g)}{[s+\lambda_{c_1}+\phi_{c_2}(g)-\lambda_{c_1}\overline{S}_{\phi_{c1}}(s)]}$$
$$-\lambda_{CSF}\overline{S}_{\phi_{CSF}}(s)$$

$$M(s)=\psi\{[\frac{\lambda_{a_1}\lambda_{a_2}}{[s+\psi][s+\lambda_{a_2}+\lambda_C-\lambda_C\overline{S}_{\phi_C}(s)]}+\frac{\lambda_{a_1}\lambda_{a_2}}{[s+\psi][s+\lambda_{a_1}]}+\frac{\lambda_{b_1}}{[s+\lambda_{b_2}+\psi-\lambda_C\overline{S}_{\phi_C}(s)]}$$

$$+\frac{\lambda_{b_1}\lambda_{b_2}}{[s+\psi][s+\lambda_{b_2}+\psi-\lambda_C\overline{S}_{\phi_C}(s)]}]D_\mu(s)++\frac{\psi^3}{(s+\psi)^4}\}$$

... (26)

$$D_\mu(s)=\frac{1-\overline{S}_\mu(s)}{s+\psi} \quad \text{... (27)}$$

$$\phi_{CSF}(l)=\exp[l^\theta+(\log\phi_{CSF}(l))^\theta)^{1/\theta}] \quad \text{... (28)}$$

Also,

$$\overline{P}_{up}(s)+\overline{P}_{down}(s)=\frac{1}{s}$$

... (29)

Steady state behavior of the system By Abel's lemma we have,

$$\lim_{s\to 0}\{s\overline{F}(s)\}=\lim_{t\to\infty}F(t)$$

In equations (24) and (25) we get,

$$\overline{P}_{up}(s)=\frac{1}{K(0)}[1+\frac{\lambda_{a_1}}{\psi\lambda_{a_2}}+\frac{\lambda_{a_2}}{\lambda_{a_1}}+\frac{\lambda_{b_1}}{\lambda_{b_2}+\psi-\lambda_c}+\frac{\lambda_{c_1}}{\phi_{c_1}(v)}+\frac{\lambda_{c_2}}{\phi_{c_2}(g)}] \quad \text{... (30)}$$

$$\overline{P}_{down}=\frac{1}{K(0)}[\frac{\lambda_{a_1}}{\psi}+M(0)+\frac{\lambda_{a_2}}{\psi}+\frac{\lambda_{b_1}\lambda_{b_2}}{\psi(\lambda_{b_2}+\psi-\lambda_C)}+\frac{\lambda_{c_1}\lambda_{c_2}M_{\phi_{c2}}}{\phi_{c_1}(v)}+\frac{\lambda_{c_1}\lambda_{c_2}M_{\phi_{c1}}}{\phi_{c_2}(g)}+$$

$$\frac{\lambda_c\lambda_{a_1}M_{\phi_C}}{\lambda_{a_2}}+\frac{\lambda_C\lambda_{b_1}M_{\phi_C}}{\lambda_{b_2}+\psi-\lambda_C}+\lambda_{CSF}M_{\phi_{CSF}}] \quad \text{... (31)}$$

where,

$$M(0)=\lim_{s\to 0}M(s)$$

... (32)

$$M_{\phi_i}=\lim_{s\to 0}\frac{1-\overline{S}_{\phi_i}(s)}{s}$$

... (33)

$$\mathrm{S}_{\phi_i}(s)=\frac{\phi_1}{\mathrm{s}+\phi_1} \quad \text{... (34)}$$

Sensitivity analysis:

Here we have done sensitivity analysis of the system for catastrophic failure rate and key unit of subsystem 1.

S = -t*exp (-(Lcsf + La1 + La2 + Lb1 + Lb2 + Lc1 + Lc2)*t)

5. Results and Discussion

Here reliability, availability and sensitivity analysis with respect to catastrophic failure and key unit of subsystem 1 is done for the considered system by employing Supplementary variables technique and Copula methodology. The Figure 2 shows the movement of reliability of the system against time for fixed values of failure and repair rates. From the graph we conclude that the reliability of the system reduces hastily with passage of time because of waiting line for repair in the repair section.

Figure 3 talks about the availability of the system which says that the availability reduces approximately in a constant manner as time increases.

The sensitivities of the system reliability R (t) with respect to system parameters like catastrophic failure and key unit of subsystem 1 are shown in figures-4 and 5. It can easily be observed that there is very negligible impact of both the parameters on system sensitivity.

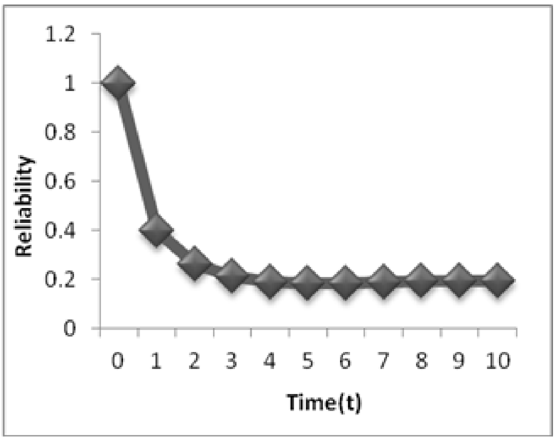

Figure 2: *Reliability against time*

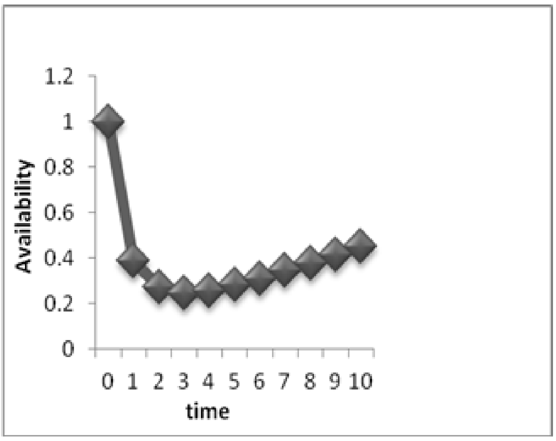

Figure 3: *Availability against time*

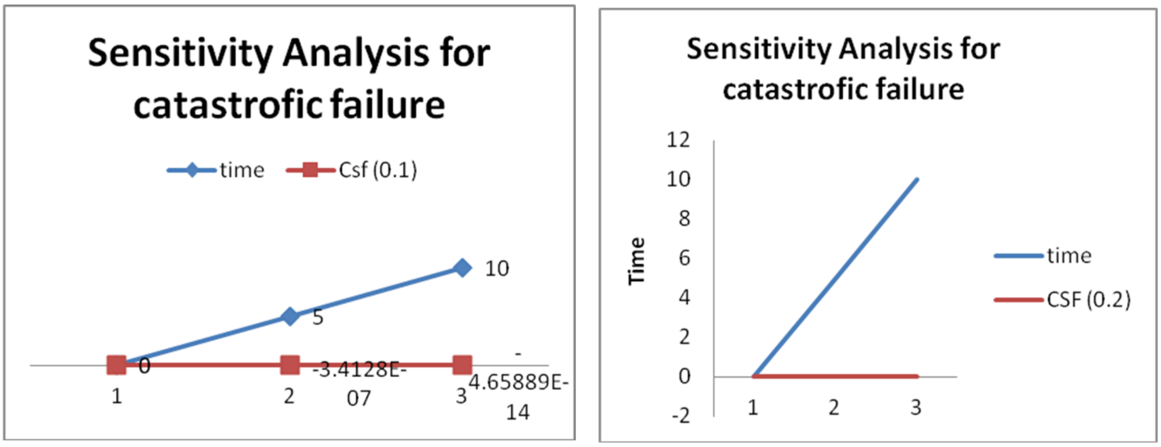

Figure 4: *Sensitivity Analysis for catastrophic failure*

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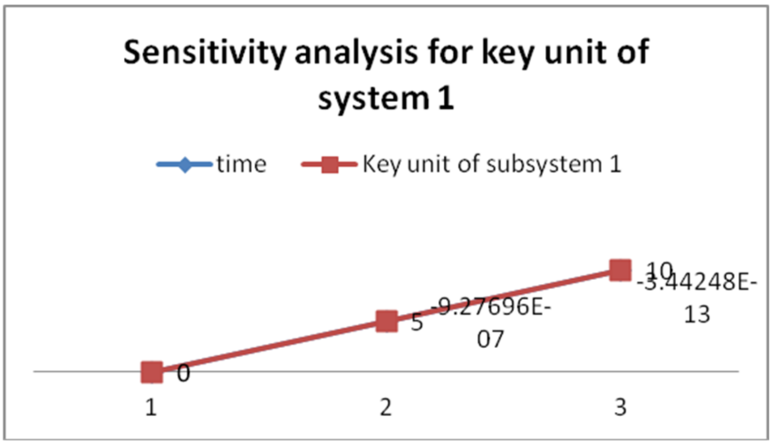

Figure 5: *Sensitivity Analysis for key unit of system1*

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Received: August 09, 2020
Accepted: Oktober 12, 2020

RT&A, No 4 (59)
Volume 15, December 2020

RAM Analysis of E_r/M/1/N Phase-Type Queueing System with Working and Working-Breakdown States

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Abstract

In this paper, the Reliability, Availability and Maintainability (RAM) analysis for the finite capacity Erlangian Phase-type Queueing model is studied with regard to failure and recovery rates. The arrival process of the machines to repair is assumed to follow Erlang distribution and the service process is exponentially distributed in FCFS discipline. Apart from the multi-phases in the queueing system two different environmental states such as the working and working-breakdown states were also taken into consideration. The transient state differential-difference equations for the general case and for the special case of N=5 is obtained. The results are presented numerically and graphically along with some special metrics such as MTBF and MTTR. The sensitivity analysis is also performed to find changes in different parametric values for the model.

Keywords: Availability, Erlang distribution, Multi-phase queueing system, Maintainability, Reliability, Sensitivity Analysis.

I. Introduction

Queueing theory was developed in order to provide models to predict the behaviour of systems that aims to provide service for randomly arising demands. Any system in which arrivals place demands upon a finite capacity resource may be determined as queueing system. In order to describe the queueing of systems more effectively it is necessary to understand Erlang theory. The main assumption of the Erlang in queueing model is that the calls arrive as a Poisson process and when there is more than one inter-related Poisson process occurring in phases it is considered to be Phase-Type distribution with continuous variable. Thus, in Phase-Type theory Erlang distribution is considered to be a special case. The phase-type distribution and phase-type renewal processes were introduced by Neuts [7], who formed the substrata for the definition of the N-process and the Markov-modulated Poisson process (MMPP). Binkowski and Carragher [3] employed an Er/Ek/1/N queuing system to model the operation of a stockyard mining. Baba [2] studied GI/M/1 queue with working vacations by using the matrix analytic method and subsequently, for the queue with working vacation and vacation interruption. Plumchitchom and Thomopoulos [8] made a study on a single-server queuing system with Erlang distributed inter-arrival and service times, Li et al. [5] studied the GI/M/1 queue such that the vacation time follows an exponential distribution.

Along with Queueing of the systems, it is also important to analyze the performance of the industrial systems by using the most important metric such as the Reliability. In order to receive effective results in the Industrial systems it is proved to analyze the Availability and Maintainability of the machines along with Reliability. Performance modelling and availability analysis are applied by many researchers on different industrial systems such as the paper plant, paint, and thermal power plant Industry etc., Singh, and Goyal, [11] developed a methodology to study the transient behaviour of repairable mechanical biscuit shaping system on a biscuit manufacturing plant for determining the availability of the system based on Markov modelling. Lin, et al. [6] made a study on reliability using both classical and Bayesian semi-parametric frame-works, they illustrated modelled a wheel- set's degradation data and analyzed to ease the calculation of system reliability during applying preventive maintenance. The differential equations have been solved using Laplace Transforms. These Laplace Transform are commonly used in the transient state to obtain the state probabilities. Aggarwal, et. al. [1] presented a model using Markov birth-death process with the concept of fuzzy reliability and availability assuming that the failure and repair rates of each subsystem as exponential distribution.

In this paper RAM analysis of the Er/M/1 finite space queueing model for different environmental states such as the Working state and the Working-Breakdown state is studied. The differential-difference equations for the model are formed and a special case of N=5 is considered. The transient equations are solved using Fourth-Order Runge Kutta numerical method. The results are shown numerically and graphically for reliability, availability and maintainability analysis for the queueing system. The Sensitivity Analysis is also carried out for the changes in different parametric values involved in the model.

II. Assumptions and Notations

The following are the assumptions that are used in this model:

- The arrival of machines for repair to the queueing system is independent according to the Erlang process with a constant parameter λ
- The service process is exponentially distributed with First Come First Service (FCFS) queue discipline
- When the system is in the working state (i.e., there should be at least one machine) failure occurs at the interarrival phase which is also exponentially distributed and once the failure occurs in the system the process is moved to the working-breakdown state where the it is performed at a low rate
- Whenever working-breakdown occurs in the system, it is immediately recovered in the recovery state which is also exponentially distributed. Once the system recovers it performs its activity at a normal arrival rate
- All inter-arrival times and the service times are independent of each other

The following are the notations that are used in this paper:

N(t) : Total no of machines in the system at any time t

Er : Erlang distribution with r identical phases

S(t) : The environmental state at any instant of time t which is given by

$$S(t) = \begin{cases} 0, & \text{if the server is in the working environment state for Phase 1 \& 2} \\ 1, & \text{if the server is in the working breakdown state for Phase 1 \& 2} \end{cases}$$

λ : Arrival rate

$\mu 1$: Service rate for working state

$\mu 2$: Service rate for working-breakdown state ($\mu 1 > \mu 2$).

α : Failure rate of the queueing system
β : Recovery rate of the queueing system

The transient state-probabilities that are used in this model:

P0,0,0(t): Probability of arrival of a machine (i.e., at least one machine) in the system

Pn, i, j(t): Probability that there are (n-1) machines in the system with i (i=0, 1, …, r) phases and j (j=1,2) states

III. Description of the model

The RAM analysis of an Erlang phase type arrival and single server queue with finite capacity queueing system is considered. The arrival of machines to repair follows Erlang distribution with the parameter rλ is used for this model. Two different service mechanisms are exponentially distributed with parameters $\mu 1$ and $\mu 2$ are considered for this model based on the environmental states namely, working and working break-down states respectively. When the system is not empty (i.e., at least one machine in the system) failure occurs in the arrival process of the system with the failure rate α. Therefore, whenever failure occurs it is immediately recovered in the recovery state with rate β. The failure and recovery rate are assumed to be exponentially distributed. The state-transition diagram for the RAM analysis of the phase type Erlang queuing model is presented in Figure 1:

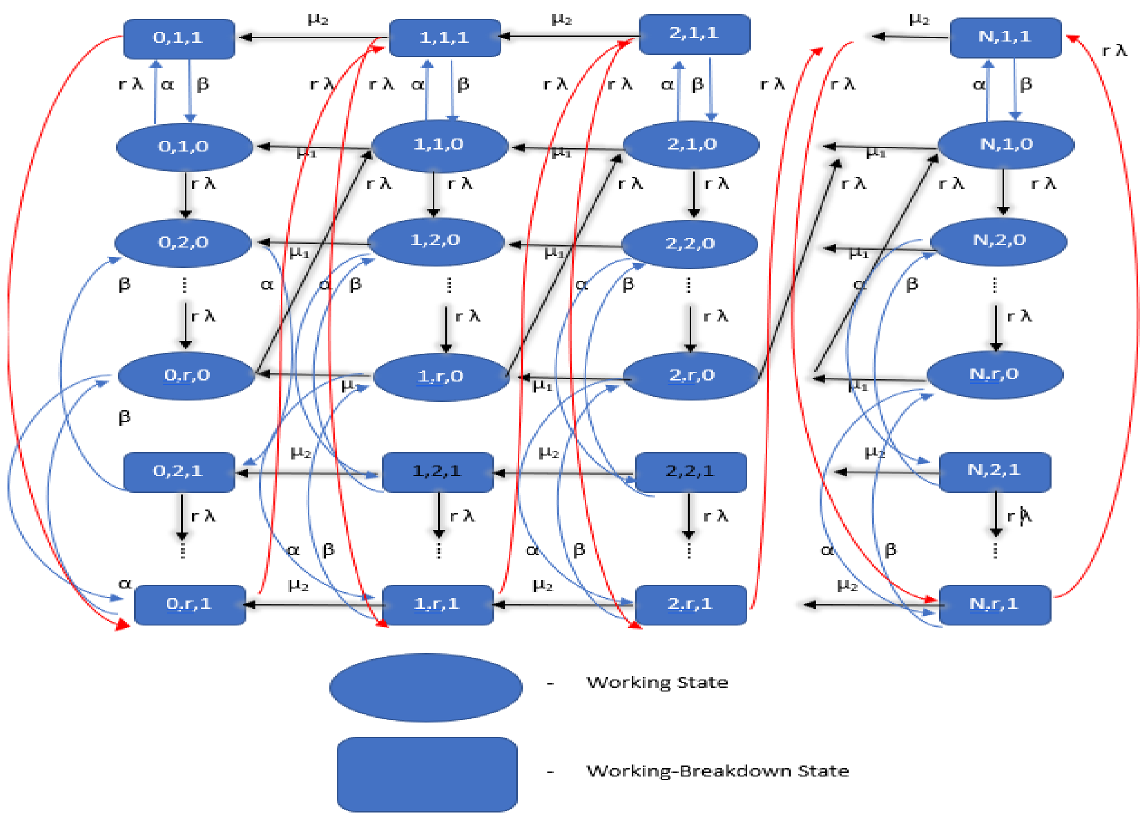

Figure 1: State-Transition Diagram of E_r/M/1

By using the state-transition diagram, the transient state differential-difference equations are formed for the Erlangian Phase-type queueing model with working and working-breakdown states.

WORKING STATE

$$\frac{dP_{0,1,0}(t)}{dt} = \mu_1 P_{n+1,1,0}(t) + \beta P_{0,1,0}(t) - (r\lambda + \alpha) P_{0,1,0}(t), \qquad n=0 \qquad (3.1)$$

$$\frac{dP_{n,1,0}(t)}{dt} = r\lambda P_{n-1,r,0}(t) + \mu_1 P_{n+1,1,0}(t) + \beta P_{n,1,1}(t) - (r\lambda + \mu_1 + \alpha) P_{n,1,0}(t), \qquad n \geq 1 \qquad (3.2)$$

$$\frac{dP_{N,1,0}(t)}{dt} = r\lambda P_{N-1,1,0}(t) + \beta P_{N,1,1}(t) - (\mu_1 + r\lambda + \alpha_1) P_{N,1,0}(t), \qquad n=N \qquad (3.3)$$

$$\frac{dP_{0,r,0}(t)}{dt} = \mu_1 P_{n+1,r,0}(t) + \beta P_{0,r,0}(t) + r\lambda P_{0,r-1,0}(t) - (r\lambda + \alpha) P_{0,1,0}(t), \qquad n=0 \qquad (3.4)$$

$$\frac{dP_{n,r,0}(t)}{dt} = r\lambda P_{n-1,r-1,0}(t) + \mu_1 P_{n+1,r,0}(t) + \beta P_{n,r,1}(t) - (r\lambda + \mu_1 + \alpha) P_{n,r,0}(t), \qquad n \geq 1 \qquad (3.5)$$

$$\frac{dP_{N,r,0}(t)}{dt} = r\lambda P_{N-1,r,0}(t) + \beta P_{N,r,1}(t) - (\mu_1 + \alpha) P_{N,r,0}(t), \qquad n=N \qquad (3.6)$$

WORKING - BREAKDOWN STATE

$$\frac{dP_{0,1,1}(t)}{dt} = \mu_2 P_{1,1,1}(t) + \alpha P_{0,1,0}(t) - (r\lambda + \beta) P_{0,1,1}(t), \qquad n=0 \qquad (3.7)$$

$$\frac{dP_{n,1,1}(t)}{dt} = r\lambda P_{n-1,r,1}(t) + \mu_2 P_{n+1,r-1,1}(t) + \alpha P_{n,1,0}(t) - (r\lambda + \mu_2 + \beta) P_{n,1,1}(t), \qquad n \geq 1 \qquad (3.8)$$

$$\frac{dP_{N,1,1}(t)}{dt} = r\lambda P_{N-1,r,1}(t) + \alpha P_{N,1,1}(t) - (\mu_2 + r\lambda + \beta) P_{N,1,1}(t), \qquad n=N \qquad (3.9)$$

$$\frac{dP_{0,r,1}(t)}{dt} = \mu_2 P_{n+1,r,1}(t) + \alpha P_{0,r,0}(t) + r\lambda P_{0,r-1,1}(t) - (r\lambda + \beta) P_{0,1,1}(t), \qquad n=0 \qquad (3.10)$$

$$\frac{dP_{n,r,1}(t)}{dt} = r\lambda P_{n-1,r,1}(t) + \mu_2 P_{n+1,r,1}(t) + \alpha P_{n,r,0}(t) - (r\lambda + \mu_2 + \beta) P_{n,r,1}(t), \qquad n \geq 1 \qquad (3.11)$$

$$\frac{dP_{N,r,1}(t)}{dt} = r\lambda P_{N-1,r,1}(t) + \alpha P_{N,r,0}(t) - (\mu_2 + \beta) P_{N,r,1}(t), \qquad n=N \qquad (3.12)$$

without loss of generality the initial state *conditions* are given by
$P_{0,0,0}(0) = 0, P_{n,i,j}(0) = 0, \forall n = 1,2,\ldots,N;$ i=1,2; j=0,1

The system reliability at time t is calculated as follows:

$$R(t) = \sum_{n=0}^{N} \sum_{i=1}^{2} \sum_{j=0}^{1} P_{n,i,j} \qquad (3.13)$$

The system Availability at time t is calculated by considering all the working states is as follows:

$$A(t) = \sum_{n=0}^{N} \sum_{i=1}^{2} \sum_{j=0} P_{n,i,j} \qquad (3.14)$$

The system Maintainability at time t is calculated by considering working-breakdown state which is calculated as follows:

$$M(t) = \sum_{n=0}^{N} \sum_{i=1}^{2} \sum_{j=1} P_{n,i,j} \qquad (3.15)$$

Apart from the RAM, the special metrics such as MTBF (Mean time between failures) and MTTR (Mean Time till Recovery) are also calculated as follows:

$$MTBF = \frac{1}{\alpha}$$

$$MTTR = \frac{1}{\beta}$$

IV. Special case

The differential-difference equations for N=5 is formed for the transient state of the Reliability model for the Erlangian Phase-type queueing system. The equations for the working and working-breakdown states are given below:

WORKING STATE

$$\frac{dP_{0,1,0}(t)}{dt} = \mu_1 P_{1,1,0}(t) + \beta P_{0,1,1}(t) - (2\lambda + \alpha) P_{0,1,0}(t) \tag{3.1.1}$$

$$\frac{dP_{1,1,0}(t)}{dt} = 2\lambda P_{0,2,0}(t) + \mu_1 P_{2,1,0}(t) + \beta P_{1,1,1}(t) - (2\lambda + \mu_1 + \alpha) P_{1,1,0}(t), \tag{3.1.2}$$

$$\frac{dP_{2,1,0}(t)}{dt} = 2\lambda P_{1,2,0}(t) + \mu_1 P_{3,1,0}(t) + \beta P_{2,1,1}(t) - (2\lambda + \mu_1 + \alpha) P_{2,1,0}(t), \tag{3.1.3}$$

$$\frac{dP_{3,1,0}(t)}{dt} = 2\lambda P_{2,2,0}(t) + \mu_1 P_{4,1,0}(t) + \beta P_{3,1,1}(t) - (2\lambda + \mu_1 + \alpha) P_{3,1,0}(t), \tag{3.1.4}$$

$$\frac{dP_{4,1,0}(t)}{dt} = 2\lambda P_{3,2,0}(t) + \beta P_{4,1,1}(t) - (2\lambda + \mu_1 + \alpha) P_{4,1,0}(t), \tag{3.1.5}$$

$$\frac{dP_{0,2,0}(t)}{dt} = \mu_1 P_{1,2,0}(t) + \beta P_{0,2,1}(t) + 2\lambda P_{0,1,0}(t) - (2\lambda + \alpha) P_{0,2,0}(t) \tag{3.1.6}$$

$$\frac{dP_{1,2,0}(t)}{dt} = \mu_1 P_{2,2,0}(t) + \beta P_{1,2,1}(t) + 2\lambda P_{1,1,0}(t) - (2\lambda + \mu_1 + \alpha) P_{1,2,0}(t), \tag{3.1.7}$$

$$\frac{dP_{2,2,0}(t)}{dt} = 2\lambda P_{2,1,0} + \mu_1 P_{3,2,0}(t) + \beta P_{2,2,1}(t) - (2\lambda + \mu_1 + \alpha) P_{2,2,0}(t), \tag{3.1.8}$$

$$\frac{dP_{3,2,0}(t)}{dt} = 2\lambda P_{3,1,0} + \mu_1 P_{4,2,0}(t) + \beta P_{3,2,1}(t) - (2\lambda + \mu_1 + \alpha) P_{3,2,0}(t), \tag{3.1.9}$$

$$\frac{dP_{4,2,0}(t)}{dt} = 2\lambda P_{4,1,0}(t) + \beta P_{4,2,1}(t) - (\mu_1 + \alpha) P_{4,2,0}(t), \tag{3.1.10}$$

WORKING - BREAKDOWN STATE

$$\frac{dP_{0,1,1}(t)}{dt} = \mu_2 P_{1,1,1}(t) + \alpha P_{0,1,0}(t) - (2\lambda + \beta) P_{0,1,1}(t), \tag{3.1.11}$$

$$\frac{dP_{1,1,1}(t)}{dt} = 2\lambda P_{0,2,1}(t) + \mu_2 P_{2,1,1}(t) + \alpha P_{1,1,0}(t) - (2\lambda + \mu_2 + \beta) P_{1,1,1}(t), \tag{3.1.12}$$

$$\frac{dP_{2,1,1}(t)}{dt} = 2\lambda P_{1,2,1}(t) + \mu_2 P_{3,1,1}(t) + \alpha P_{2,1,0}(t) - (2\lambda + \mu_2 + \beta) P_{2,1,1}(t), \tag{3.1.13}$$

$$\frac{dP_{3,1,1}(t)}{dt} = 2\lambda P_{2,2,1}(t) + \mu_2 P_{4,1,1}(t) + \alpha P_{3,1,0}(t) - (2\lambda + \mu_2 + \beta) P_{3,1,1}(t), \tag{3.1.14}$$

$$\frac{dP_{4,1,1}(t)}{dt} = 2\lambda P_{3,2,1}(t) + \alpha P_{4,1,0}(t) - (\mu_2 + 2\lambda + \beta) P_{4,1,1}(t), \tag{3.1.15}$$

$$\frac{dP_{0,2,1}(t)}{dt} = \mu_2 P_{1,2,1}(t) + \alpha P_{0,2,0}(t) + 2\lambda P_{0,1,1}(t) - (2\lambda + \beta) P_{0,2,1}(t), \tag{3.1.16}$$

$$\frac{dP_{1,2,1}(t)}{dt} = \mu_2 P_{2,2,1}(t) + \alpha P_{1,2,0}(t) + 2\lambda P_{0,1,1}(t) - (2\lambda + \mu_2 + \beta) P_{1,2,1}(t), \tag{3.1.17}$$

$$\frac{dP_{2,2,1}(t)}{dt} = 2\lambda P_{1,1,1}(t) + \mu_2 P_{3,2,1}(t) + \alpha P_{2,2,0}(t) - (2\lambda + \mu_2 + \beta) P_{2,2,1}(t), \tag{3.1.18}$$

$$\frac{dP_{3,2,1}(t)}{dt} = 2\lambda P_{2,1,1}(t) + \mu_2 P_{4,2,1}(t) + \alpha P_{3,2,0}(t) - (2\lambda + \mu_2 + \beta) P_{3,2,1}(t), \tag{3.1.19}$$

$$\frac{dP_{4,2,1}}{dt} = 2\lambda P_{3,1,1}(t) + \alpha P_{4,2,0}(t) - (\mu_2 + \beta) P_{4,2,1}(t), \tag{3.1.20}$$

V. Numerical illustration

The transient behaviour of the Reliability, Availability and Maintainability for the Erlangian Phase-Type queueing model of N=5, has been analyzed and are solved by using Fourth-Order Runge-Kutta numerical method. Assuming the time range from t=0 to t=200 (in hours) and the parametric values as λ=0.6, μ1=1.0, μ2=0.7, α=0.05, β=0.03, the values of Pn(t), the transient probabilities are obtained by solving the system of equations 3.1.1- 3.1.20.

Figure 2 shows the probability distribution, Pn(t), time-dependent total system size for the queueing system. The probability curves are displayed to understand the distribution trend of the system probabilities over the specified time interval.

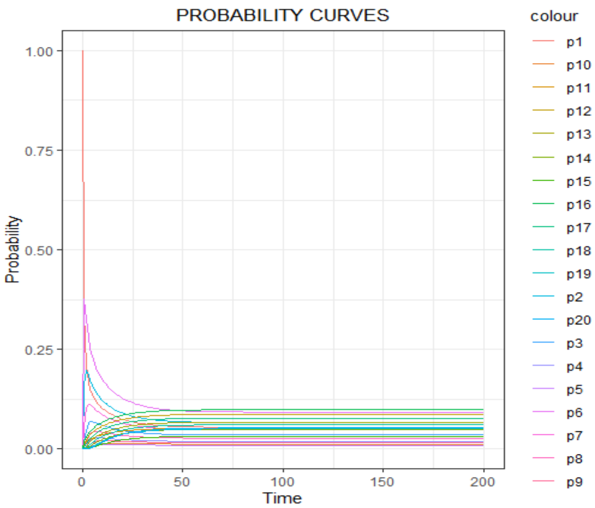

Figure 2

Figure 3, represents the Reliability of the system of the Erlangian Phase-Type queueing model. It is found out that as time increases the reliability of the system decreases. The reliability of the system is found out to be 38% after 200 hours. Figure 4, shows the Availability of the system and it is found out that as time increases the availability of the system decreases. Figure 5, depicts the maintainability of the system of the Erlangian Phase-Type queueing model. It is seen that as time increases the maintainability of the system increases. It is found out that the Maintainability of the system is 62% after 200 hours. The values of MTBF (Mean Time Between Failures) and MTTR (Mean Time till Repair) for the Erlangian Phase-Type Queueing system are found to be 20 hours/failure and 33 hours/recovery.

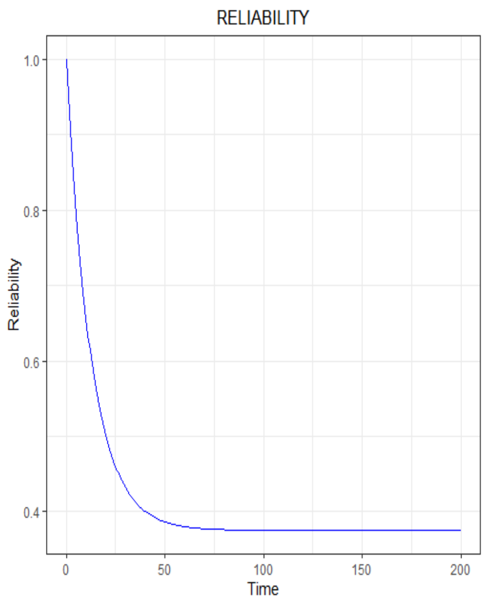

Figure 3

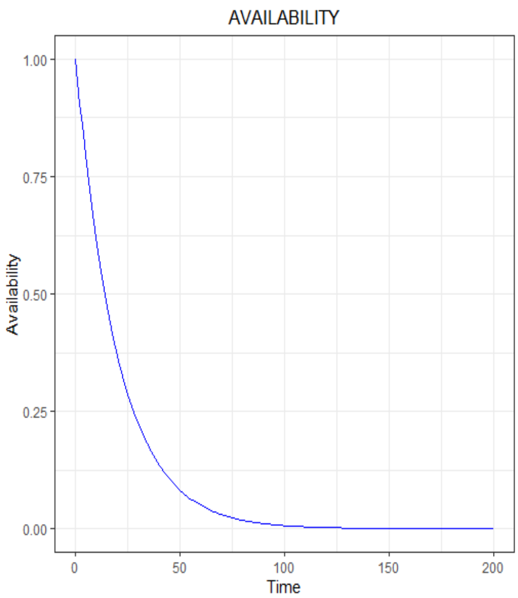

Figure 4

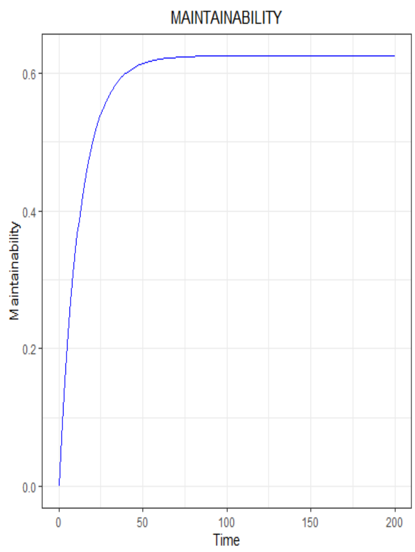

Figure 5

VI. Sensitivity analysis

For different parametric values sensitivity analysis has been carried out for RAM model for the Erlangian Phase-Type queueing model. Figures 6,7 and 8 shows the Reliability, Availability and Maintainability for different sets of Failure rates (0.05,0.06,0.07). By keeping other parameters constant, it is observed that as the failure rate value increases Reliability and Availability of the system decreases, whereas Maintainability of the system increases.

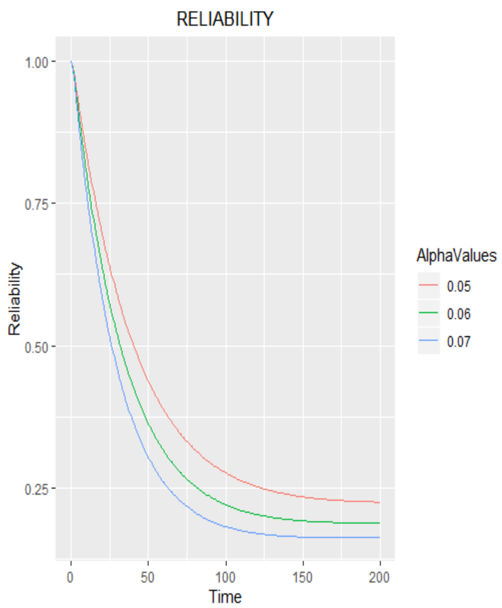

Figure 6

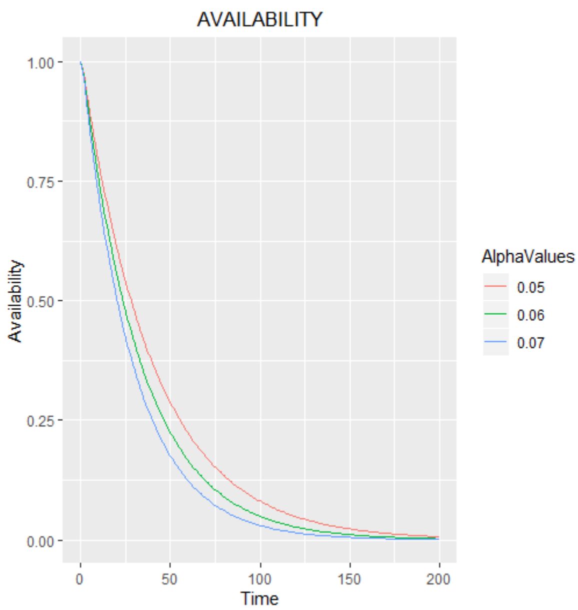

Figure 7

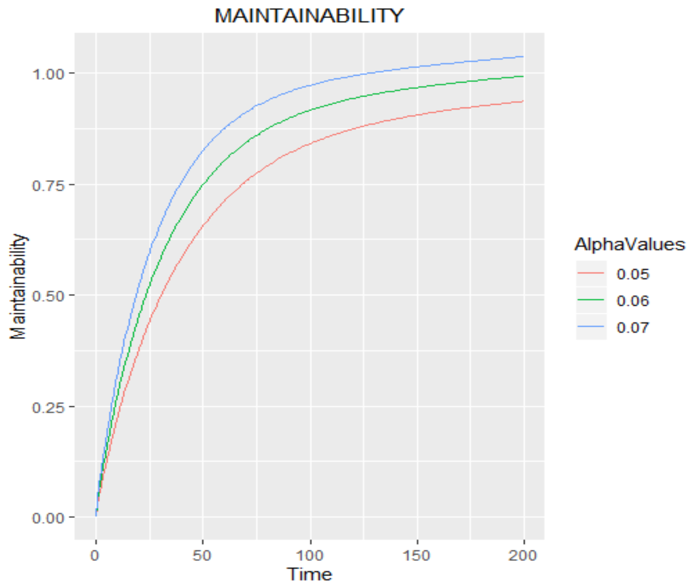

Figure 8

Figures 9 and 10 illustrates Reliability and Maintainability of the system for different Recovery rates (0.03,0.04,0.05) by keeping the other parameters constant. It can be seen from the graph that as the recovery rate value increases the Reliability of the system increases whereas Maintainability decreases.

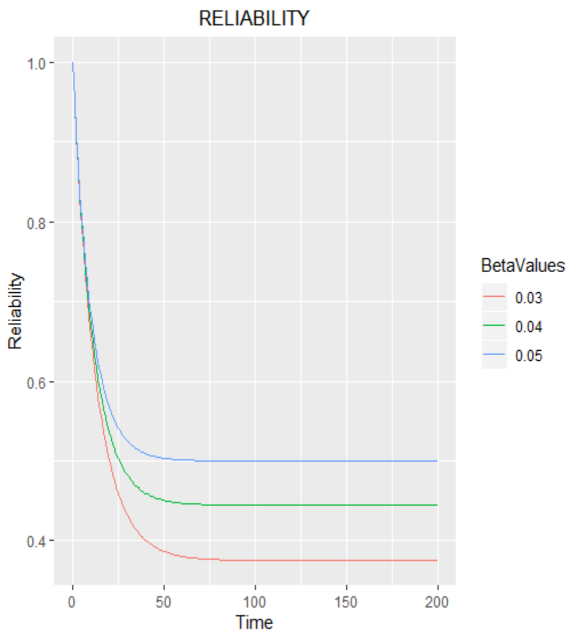

Figure 9

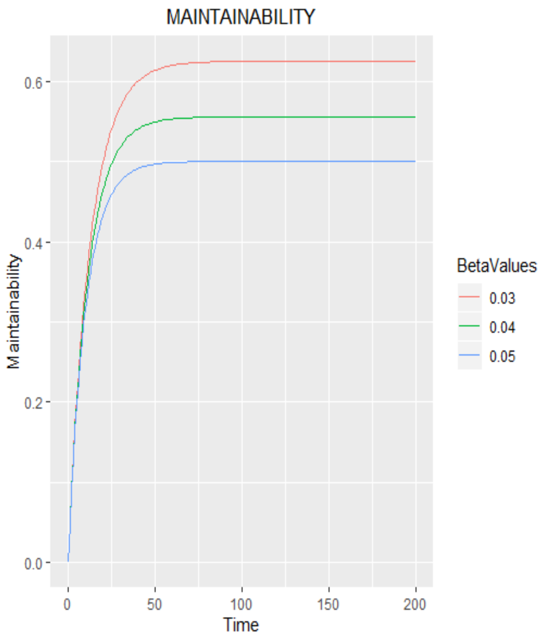

Figure 10

Table 1, represents the changes in the Reliability, Availability and Maintainability of the system for different values of arrival rates and failure rates by keeping the other parameters constant. It is found out that as the failure rate value increases keeping the arrival rate constant Reliability and Availability of the system decreases whereas the Maintainability of the system increases. It is also found that after 100 hours the Reliability, Availability and Maintainability of the system becomes constant.

Table 1: Sensitivity Analysis for the change of Arrival (0.1,0.2,0.3) and Failure (0.03,0.04,0.05) rate values

ARRIVAL RATE Vs FAILURE RATE										
TIME		**λ=0.1**			**λ=0.2**			**λ=0.3**		
		R(t)	**M(t)**	**A(t)**	**R(t)**	**M(t)**	**A(t)**	**R(t)**	**M(t)**	**A(t)**
	$\alpha1$=0.03	0.4014	0.5986	0.3012	0.4116	0.6004	0.3112	0.4218	0.6115	0.3213
40	$\alpha1$=0.04	0.3083	0.6917	0.2019	0.3185	0.7019	0.2120	0.3287	0.7121	0.2221
	$\alpha1$=0.05	0.2423	0.7577	0.1353	0.2525	0.7679	0.1455	0.2627	0.7781	0.1556
60	$\alpha1$=0.03	0.3180	0.6820	0.1653	0.3283	0.6922	0.1755	0.3386	0.7024	0.1857
	$\alpha1$=0.04	0.2398	0.7602	0.0907	0.2400	0.7704	0.1009	0.2502	0.7806	0.1911
	$\alpha1$=0.05	0.1894	0.8106	0.0498	0.1996	0.9007	0.0599	0.2008	0.9109	0.06
80	$\alpha1$=0.03	0.2806	0.7194	0.0907	0.2907	0.7295	0.0948	0.3008	0.7310	0.1909
	$\alpha1$=0.04	0.2147	0.7853	0.0408	0.2248	0.7954	0.0509	0.2349	0.8057	0.061
	$\alpha1$=0.05	0.1735	0.8265	0.0183	0.1836	0.8367	0.0285	0.1937	0.8469	0.0387
100	$\alpha1$=0.03	0.2637	0.7363	0.0498	0.2738	0.7466	0.0599	0.2839	0.7578	0.0611
	$\alpha1$=0.04	0.2054	0.7946	0.0183	0.3055	0.8047	0.0285	0.4056	0.8149	0.0387
	$\alpha1$=0.05	0.1687	0.8313	0.0067	0.1788	0.8415	0.0168	0.1889	0.8517	0.0269
120	$\alpha1$=0.03	0.2562	0.7438	0.0273	0.2562	0.7438	0.0273	0.2562	0.7438	0.0273
	$\alpha1$=0.04	0.2020	0.7980	0.0082	0.2020	0.7980	0.0082	0.2220	0.7980	0.0082
	$\alpha1$=0.05	0.1673	0.8327	0.0025	0.1673	0.8327	0.0025	0.1673	0.8327	0.0025

Table 2, shows the changes in the Reliability, Availability and Maintainability of the system for different values of failure rates and service rates for the Working State by keeping the other parameters constant. As the Failure rates increases keeping the service rate constant, Reliability and Availability of the system decreases but the Maintainability of the system increases.

Table 2: Sensitivity Analysis for the change of Working State in Service rate (0.7,0.8,0.9) and Failure (0.03,0.04,0.05) rate values

FAILURE RATE Vs SERVICE RATE										
TIME		μ1=0.7			μ1=0.8			μ1=0.9		
		R(t)	**M(t)**	**A(t)**	**R(t)**	**M(t)**	**A(t)**	**R(t)**	**M(t)**	**A(t)**
40	$\alpha1$=0.04	0.3083	0.6917	0.2019	0.3185	0.6815	0.2007	0.3287	0.6713	0.1995
	$\alpha1$=0.05	0.2423	0.7577	0.1353	0.2525	0.7475	0.1251	0.2627	0.7373	0.1149
	$\alpha1$=0.06	0.1950	0.8050	0.0907	0.2042	0.8028	0.0890	0.2144	0.7996	0.0800
60	$\alpha1$=0.04	0.2398	0.7602	0.0607	0.2400	0.7580	0.0595	0.2502	0.7499	0.05
	$\alpha1$=0.05	0.1894	0.8106	0.0498	0.1996	0.8004	0.0396	0.2098	0.7992	0.0294
	$\alpha1$=0.06	0.1557	0.8443	0.0273	0.1659	0.8341	0.0171	0.1761	0.8239	0.0069
80	$\alpha1$=0.04	0.2147	0.7853	0.0408	0.2248	0.7752	0.0307	0.2349	0.7651	0.0206
	$\alpha1$=0.05	0.1735	0.8265	0.0183	0.1836	0.8164	0.0172	0.1937	0.8063	0.0163
	$\alpha1$=0.06	0.1460	0.8540	0.0082	0.1561	0.8439	0.0069	0.1662	0.8338	0.0040
100	$\alpha1$=0.04	0.2054	0.7946	0.0183	0.2155	0.7845	0.0082	0.2256	0.7744	0.0020
	$\alpha1$=0.05	0.1687	0.8313	0.0067	0.1788	0.8212	0.0056	0.1889	0.8111	0.0035
	$\alpha1$=0.06	0.1436	0.8564	0.0025	0.1537	0.8463	0.0008	0.1638	0.8362	0.0002
120	$\alpha1$=0.04	0.2020	0.7980	0.0082	0.2020	0.7980	0.0082	0.2020	0.7980	0.0082
	$\alpha1$=0.05	0.1673	0.8327	0.0025	0.1673	0.8327	0.0025	0.1673	0.8327	0.0025
	$\alpha1$=0.06	0.1430	0.8570	0.0007	0.1430	0.8570	0.0007	0.1430	0.8570	0.0007

Table 3, depicts the changes in the Reliability and Maintainability of the system for different set of values of Arrival rate and Recovery rate by keeping the other parameters constant. As the recovery rate value increases by keeping the Arrival rate constant it is found that the Reliability of the system increases but the Maintainability of the system decreases.

Table 3: Sensitivity Analysis for the change of Arrival (0.1,0.2,0.3) and Recovery (0.01,0.02,0.03) rate values

ARRIVAL RATE Vs RECOVERY RATE							
TIME		λ=0.1		λ=0.2		λ=0.3	
		R(t)	M(t)	R(t)	M(t)	R(t)	M(t)
	β1=0.01	0.4014	0.5986	0.4117	0.5883	0.4222	0.5779
40	β1=0.02	0.4812	0.5188	0.4917	0.5084	0.5021	0.4890
	β1=0.03	0.5454	0.4546	0.5554	0.4443	0.5655	0.4340
60	β1=0.01	0.3180	0.6820	0.3288	0.6719	0.3395	0.6616
	β1=0.02	0.4299	0.5701	0.4301	0.5698	0.4411	0.5597
	β1=0.03	0.5137	0.4863	0.5239	0.4762	0.5341	0.4661
80	β1=0.01	0.2806	0.7194	0.2909	0.7092	0.3013	0.7009
	β1=0.02	0.4110	0.5890	0.4211	0.5787	0.4314	0.5685
	β1=0.03	0.5041	0.4959	0.5144	0.4858	0.5249	0.4757
100	β1=0.01	0.2637	0.7363	0.2741	0.7262	0.2845	0.7161
	β1=0.02	0.4040	0.5960	0.4142	0.5859	0.4244	0.5757
	β1=0.03	0.5012	0.4988	0.5113	0.4886	0.5214	0.4784
120	β1=0.01	0.2562	0.7438	0.2562	0.7438	0.2562	0.7438
	β1=0.02	0.4015	0.5985	0.4015	0.5985	0.4015	0.5985
	β1=0.03	0.5004	0.4996	0.5004	0.4996	0.5004	0.4996

Table 4, illustrates the changes in the Reliability and Maintainability of the system for different set of values of service rates and recovery rates keeping the other parameters constant. The table shows that as the Recovery rate values increases by keeping Service rate constant it is found that Reliability of the system increases whereas the Maintainability of the system decreases.

Table 4: Sensitivity Analysis for the change of Recovery rate (0.7,0.8,0.9) and Service rate (0.01,0.02,0.03) values

RECOVERY RATE Vs SERVICE RATE							
TIME		μ1=0.7		μ1=0.8		μ1=0.9	
		R(t)	M(t)	R(t)	M(t)	R(t)	M(t)
	β1=0.01	0.4014	0.5986	0.4115	0.5884	0.4216	0.5782
40	β1=0.02	0.4812	0.5188	0.4913	0.5086	0.5014	0.5004
	β1=0.03	0.5454	0.4546	0.5557	0.4444	0.5659	0.4342
60	β1=0.01	0.3180	0.6820	0.3283	0.6718	0.3285	0.6616
	β1=0.02	0.4299	0.5701	0.4300	0.5699	0.4311	0.5597
	β1=0.03	0.5137	0.4863	0.5239	0.4761	0.5341	0.4659
80	β1=0.01	0.2806	0.7194	0.2709	0.7092	0.2612	0.7009
	β1=0.02	0.4110	0.5890	0.4213	0.5789	0.4315	0.5688
	β1=0.03	0.5041	0.4959	0.5144	0.4858	0.5247	0.4757
100	β1=0.01	0.2637	0.7363	0.2739	0.7262	0.2841	0.7161
	β1=0.02	0.4040	0.5960	0.4142	0.5859	0.4244	0.5758
	β1=0.03	0.5012	0.4988	0.5113	0.4887	0.5214	0.4786
120	β1=0.01	0.2562	0.7438	0.2562	0.7438	0.2562	0.7438
	β1=0.02	0.4015	0.5985	0.4015	0.5985	0.4015	0.5985
	β1=0.03	0.5004	0.4996	0.5004	0.4996	0.5004	0.4996

Conclusion

RAM analysis of Er/M/1/N Queueing model with two different environmental states are studied in this paper. The state-transition diagram for the transient state of the r phase Erlangian queueing model is formed from which the differential-difference equations are obtained. A special case of N=5 is solved using Fourth-Order Runge-Kutta numerical method. It is observed that as time increases Reliability and Availability decreases, whereas Maintainability increases. In order to find the failure rate and the recovery rate, MTBF and MTTR of the Erlangian Queueing model was calculated. Sensitivity values becomes constant after 100 hours for different parametric values.

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Received: August 18, 2020
Accepted: Oktober 12, 2020

Statistical Analysis and Optimum Step Stress Accelerated Life Test Design for Nadarajah- Haghighi Distribution

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Abstract

In this study we have considered step stress accelerated life testing plan for complete data. The lifetimes of the failure items are assumed to follow Nadarajah-Haghighi distribution which is an extension of exponential distribution and has all the properties like Weibull, gamma exponentiated exponential distribution. The maximum likelihood estimates of the parameters and accelerated factors have been estimated and confidence intervals of these parameters are also obtained. Newton-Raphson iterative procedure is used to solve the non-linear equations which are not in closed form. Later, a simulation study has been performed to check the performance of the parameters and hence the theory of the paper.

Keywords: Nadarajah-Haghighi distribution, step-stress accelerated life testing plan, maximum likelihood estimation, simulation, R.

I. Introduction

In modern advanced technologies era, there is a very high competition among companies to maintain the value and honor in the market for their products. Every manufacturer and producer are trying their best to produce an item of high reliability that could stay longer and perform better which makes the lifetime of the products very high. Therefore, it is not only a very tedious but also a very time consuming hence the costly job for the researcher to predict the exact lifetime of the items in terms of hour, days, months or years. The analysis of the life and quality of the product must be done before the launch; therefore, they do not have sufficient time to obtain the failure lifetime of the selected specimens and analyze on the basis of them. So, to obtain the lifetime in quick span of time, the experimenter accelerates the process and obtains the failure time.

Step stress accelerated life test is one of the very important methods to accelerate the process to obtain failure times quickly. In this test, first we put the testing units at some stress (higher than use or normal stress). At a specific time point we observe the failed units and increase the stress level to higher than the previous one, and again we count the failed units and so on.Studies in [7] and [5] suggested that the cumulative effect of the applied stresses should be reflected by the life-stress model when dealing with data from accelerated tests with time-varying stresses. Based on this idea,[7, 8] proposed a cumulative damage (exposure) model which had gained acceptance in the reliability engineering field. Later [1] extended the results of [5] to the case where a prescribed censoring time is involved. Since then many researchers such as [9, 2, 3, 4, 10] studied SSALT with different censoring schemes and distributions.

In this paper we have considered that the lifetimes of the items follow Nadarjah-Haghighi (NH) lifetime distribution. In second section model and testing methods have been discussed. Maximum likelihood estimation (MLE) technique is used to estimate the parameters and acceleration factor and discussed in section 3. In section 4, approximate confidence intervals for the parameters are obtained. Section 5 talks about optimality criterion for the stress change time or the optimum time at what the stress have been changed or increased. Simulation study has been performed to validate the assumptions made in this study and is in section 6.

II. Model and Methods

Nadarajah and Haghighi (2011) proposed that a random variable X is said to follow the NH distribution with the probability density function (PDF) is given by

$$f(x;\alpha,\beta)=\alpha\beta(1+\beta x)^{\alpha-1}exp[1-(1+\beta x)^{\alpha}] \tag{1}$$

Where β is scale parameter and α is the shape parameter. The corresponding, cumulative distribution function (CDF), survival function (SF) and hazard rate function (HRF) are given by

$$F(x;\alpha,\beta)=1-exp[1-(1+\beta x)^{\alpha}] \tag{2}$$

$$S(x)=exp[1-(1+\beta x)^{\alpha}]$$

$$h(x)=\alpha\beta(1+\beta x)^{\alpha-1}$$

For $\alpha=1$, NH distribution is reduced to the exponential distribution. This distribution is an alternative to the Weibull, gamma and exponentiated exponential distributions with an attractive feature of always having the zero mode. NH distribution has closed form of survival and hazard rate functions like Weibull distribution, so it is a good choice for the lifetime data analyst.

Basic assumptions

1. In this test only two stress levels S_1 and $S_2 (S_1 < S_2)$ are used.
2. A random sample of n identical products is placed on the test initially under at stress level S_1 and run until time τ, then the stress is changed to S_2 and the test is continued until all products fail.
3. The lifetimes of the products are i.i.d. according to NH distribution at each level of stress.
4. The scale parameter β is a log-linear function of stress given by $log(\beta_i)=a+bS_i$, $i=1,2$. where $a$and $b$are unknown parameters depending on the nature of the product and the test method.
5. The cumulative exposure model given by equation (3) holds to reflect the effect of the applied stresses. For more detail reader may refer to Nelson (1990).

$$F(x)=\begin{cases}F_1(x), & 0<x<\tau\\ F_2\left(\frac{\beta_2}{\beta_1}\tau+x-\tau\right), & \tau\le x<\infty\end{cases} \tag{3}$$

The PDF of (3) is obtained as

$$f(x) = \begin{cases} f_1(x), & 0 < x < \tau \\ f_2\left(\frac{\beta_2}{\beta_1}\tau + x - \tau\right), & \tau \le x < \infty \end{cases} \tag{4}$$

Now using equations (1), (2), (3) and (4) he CDF and PDF for the test are given by

$$F(x) = \begin{cases} 1 - exp[1 - (1 + \beta_1 x)^\alpha], & 0 < x < \tau \\ 1 - exp\left[1 - \left\{1 + \beta_2\left(x - \tau\langle 1 - \frac{\beta_2}{\beta_1}\rangle\right)\right\}^\alpha\right], & \tau \le x < \infty \end{cases} \tag{5}$$

$$f(x) = \begin{cases} \alpha\beta_1(1 + \beta_1 x)^{\alpha-1} exp[1 - (1 + \beta_1 x)^\alpha], & 0 < x < \tau \\ \alpha\beta_2\left\{1 + \beta_2\left(x - \tau\langle 1 - \frac{\beta_2}{\beta_1}\rangle\right)\right\}^{\alpha-1} exp\left[1 - \left\{1 + \beta_2\left(x - \tau\langle 1 - \frac{\beta_2}{\beta_1}\rangle\right)\right\}^\alpha\right], & \tau \le x < \infty \end{cases} \tag{6}$$

III. Point Estimates of the Parameters using Maximum Likelihood Method

The ML method is used to determine the parameters that maximize the probability of the sample data. This method is considered to be more robust (with some exceptions) and yields estimates with good statistical properties. Also, it is an efficient method for quantifying uncertainty through confidence bounds. The MLE methods are versatile and are applicable to most of the models and to different types of data. However, the methodology for maximum likelihood estimation is simple, the implementation is mathematically intense. Since these estimators do not exist in closed form, numerical techniques are used to compute them.

For obtaining the MLE of the model parameters, let $x_{ij}, j = 1, 2, 3, \ldots n_i, i = 1,2$ be the observed failure times of a test unit $j$under stress level i, where n_1 denotes the number of units failed at the low stress S_1 and $n_2$denotes the number of units failed at higher stress S_2.

In this paper, the lifetime of the test item is assumed to follow the NH distribution with scale parameter β and shape parameter α. Therefore, the likelihood function can be written in the form

$$L(\beta_1, \beta_2, \alpha) = \prod_{j=1}^{n_1} \alpha\beta_1\left(1 + \beta_1 x_{1j}\right)^{\alpha-1} exp\left[1 - \left(1 + \beta_1 x_{1j}\right)^\alpha\right] \prod_{j=1}^{n_2} \alpha\beta_2\left\{1 + \beta_2\left(x_{2j} - \tau\langle 1 - \frac{\beta_2}{\beta_1}\rangle\right)\right\}^{\alpha-1} exp\left[1 - \left\{1 + \beta_2\left(x_{2j} - \tau\langle 1 - \frac{\beta_2}{\beta_1}\rangle\right)\right\}^\alpha\right] \tag{7}$$

The log-likelihood function corresponding to the above equation can be rewritten as

$$log\, L = nlog\alpha + n_1 log\beta_1 + n_2 log\beta_2 + (\alpha - 1)\sum_{j=1}^{n_1} log\left(1 + \beta_1 x_{1j}\right) + \sum_{j=1}^{n_1}\left[1 - \left(1 + \beta_1 x_{1j}\right)^\alpha\right] + (\alpha - 1)\sum_{j=1}^{n_2} log\left\{1 + \beta_2\left(x_{2j} - \tau\langle 1 - \frac{\beta_2}{\beta_1}\rangle\right)\right\} + \sum_{j=1}^{n_2}\left[1 - \left\{1 + \beta_2\left(x_{2j} - \tau\langle 1 - \frac{\beta_2}{\beta_1}\rangle\right)\right\}^\alpha\right] \tag{8}$$

Where, $n_1 + n_2 = n$

Now by using the life stress relationship $log(\beta_i) = a + bS_i$, $i = 1, 2$ in equation (8), the log-likelihood function ir deduced to the following equation:

$$\log L = l = n\log\alpha + na + (n_1 S_1 + n_2 S_2)b + (\alpha - 1)\sum_{j=1}^{n_1}\log\left[1 + x_{1j}e^{a+bS_1}\right] + \sum_{j=1}^{n_1}\left[1 - \left(1 + x_{1j}e^{a+bS_1}\right)^\alpha\right] + (\alpha - 1)\sum_{j=1}^{n_2}\log\left[1 + \left\{x_{2j} - \tau\left(1 - e^{b(S_2 - S_1)}\right)\right\}e^{a+bS_2}\right] + \sum_{j=1}^{n_2}\left[1 - \left\{1 + \left(x_{2j} - \tau\langle 1 - e^{b(S_2 - S_1)}\rangle\right)e^{a+bS_2}\right\}^\alpha\right] \tag{9}$$

Differentiating (9) partially w.r.t.a, b and α, we get

$$\frac{\partial l}{\partial a} = n + (\alpha - 1)\sum_{j=1}^{n_1} \frac{x_{1j}e^{a+bS_1}}{[1+x_{1j}e^{a+bS_1}]} + \alpha \sum_{j=1}^{n_1} x_{1j}e^{a+bS_1}\left(1 + x_{1j}e^{a+bS_1}\right)^{\alpha-1} + (\alpha - 1)\sum_{j=1}^{n_2} \frac{\{x_{2j}-\tau(1-e^{b(S_2-S_1)})\}e^{a+bS_2}}{[1+\{x_{2j}-\tau(1-e^{b(S_2-S_1)})\}e^{a+bS_2}]} + \alpha \sum_{j=1}^{n_2} \left(x_{2j} - \tau\langle 1 - e^{b(S_2-S_1)}\rangle\right)e^{a+bS_2}\left\{1 + \left(x_{2j} - \tau\langle 1 - e^{b(S_2-S_1)}\rangle\right)e^{a+bS_2}\right\}^{\alpha-1} \tag{10}$$

$$\frac{\partial l}{\partial b} = n_1S_1 + n_2S_2 + (\alpha - 1)\sum_{j=1}^{n_1} \frac{x_{1j}S_1e^{a+bS_1}}{[1+x_{1j}e^{a+bS_1}]} + \alpha \sum_{j=1}^{n_1} x_{1j}S_1\, e^{a+bS_1}\left(1 + x_{1j}e^{a+bS_1}\right)^{\alpha-1} + (\alpha - 1)\sum_{j=1}^{n_2} \frac{S_2(x_{2j}-\tau)e^{a+bS_2} - \tau\,(2S_2-S_1)e^{[a-b(2S_2-S_1)]}}{[1+\{x_{2j}-\tau(1-e^{b(S_2-S_1)})\}e^{a+bS_2}]} + \alpha \sum_{j=1}^{n_2} S_2\left(x_{2j} - \tau\right)e^{a+bS_2} - \tau\, e^{[a+b(2S_2-S_1)]}\left\{1 + \left(x_{2j} - \tau\langle 1 - e^{b(S_2-S_1)}\rangle\right)e^{a+bS_2}\right\}^{\alpha-1} \tag{11}$$

$$\frac{\partial l}{\partial \alpha} = \frac{n}{\alpha} + \sum_{j=1}^{n_1} \log\left[1 + x_{1j}\mathrm{e}^{a+bS_1}\right] - \sum_{j=1}^{n_1} \left(1 + x_{1j}\mathrm{e}^{a+bS_1}\right)^{\alpha} \log\left(1 + x_{1j}\mathrm{e}^{a+bS_1}\right) + \sum_{j=1}^{n_2} \log\left[1 + \left\{x_{2j} - \tau\left(1 - \mathrm{e}^{b(S_2-S_1)}\right)\right\}\mathrm{e}^{a+bS_2}\right] + \sum_{j=1}^{n_2} \left\{1 + \left(x_{2j} - \tau\langle 1 - \mathrm{e}^{b(S_2-S_1)}\rangle\right)\mathrm{e}^{a+bS_2}\right\}^{\alpha} \log\left\{1 + \left(x_{2j} - \tau\langle 1 - \mathrm{e}^{b(S_2-S_1)}\rangle\right)\mathrm{e}^{a+bS_2}\right\} \tag{12}$$

From (12) the MLE of α is given by the following equation:

$$\frac{n}{\alpha} + n_1[\log(\psi_1) - \psi_1^{\alpha}\log(\psi_1)] + n_2[\log(\psi_2) + \psi_2^{\alpha}\log(\psi_2)] = 0$$

$$\hat{\alpha} = \frac{n}{-n_1[\log(\psi_1) - \psi_1^{\alpha}\log(\psi_1)] - n_2[\log(\psi_2) + \psi_2^{\alpha}\log(\psi_2)]}$$

where,

$$\psi_1 = \left[1 + x_{1j}\mathrm{e}^{a+bS_1}\right]$$

and $\psi_2 = \left[1 + \left\{x_{2j} - \tau\left(1 - \mathrm{e}^{b(S_2-S_1)}\right)\right\}\mathrm{e}^{a+bS_2}\right]$

IV. The approximate confidence intervals for the parameters

The observed Fisher-information matrix can be written as follows:

$$F = -\begin{bmatrix} \frac{\partial^2 l}{\partial a^2} & \frac{\partial^2 l}{\partial a \partial b} & \frac{\partial^2 l}{\partial a \partial \alpha} \\ \frac{\partial^2 l}{\partial b \partial a} & \frac{\partial^2 l}{\partial b^2} & \frac{\partial^2 l}{\partial b \partial \alpha} \\ \frac{\partial^2 l}{\partial \alpha \partial a} & \frac{\partial^2 l}{\partial \alpha \partial b} & \frac{\partial^2 l}{\partial \alpha^2} \end{bmatrix}$$

for large samples, the point estimates of the parameters obtained by maximum likelihood method follow approximately normal distribution with mean (a, b, α) and varianceF^{-1}, therefore, $(\hat{a}, \hat{b}, \hat{\alpha}) \sim N(a, b, \alpha), F^{-1})$. Then the two sided $100(1 - \gamma)\%$ approximate confidence interval for the parameter of (a, b, α) can be written as

$\hat{a} \pm Z_{\gamma/2}\sqrt{var(\hat{a})}$; $\hat{b} \pm Z_{\gamma/2}\sqrt{var(\hat{b})}$; $\hat{\alpha} \pm Z_{\gamma/2}\sqrt{var(\hat{\alpha})}$

Where $Z_{\gamma/2}$ is the $(1-\gamma/2)th$ quantile of a standard normal distribution and $\sqrt{\text{var}(\hat{\alpha})}$, $\sqrt{var(\hat{a})}$ and $\sqrt{var(\hat{b})}$ is obtained by taking the square root of the diagonal elements of F^{-1}.

The elements of the information matrix Fcan be expressed by the following equations:

$$\frac{\partial^2 l}{\partial a^2} = (\alpha-1)\sum_{j=1}^{n_1}\frac{x_{1j}e^{a+bS_1}}{\left[1+x_{1j}e^{a+bS_1}\right]^2} + \alpha\sum_{j=1}^{n_1}\left[x_{1j}e^{a+bS_1}\left(1+x_{1j}e^{a+bS_1}\right)^{\alpha-1} + (\alpha-1)\left(x_{1j}e^{a+bS_1}\right)^2\left(1+x_{1j}e^{a+bS_1}\right)^{\alpha-2}\right] + (\alpha-1)\sum_{j=1}^{n_2}\frac{\{x_{2j}-\tau(1-e^{b(S_2-S_1)})\}e^{a+bS_2}}{\left[1+\{x_{2j}-\tau(1-e^{b(S_2-S_1)})\}e^{a+bS_2}\right]^2} + \alpha\sum_{j=1}^{n_2}\left[\left\{\left(x_{2j}-\tau\langle 1-e^{b(S_2-S_1)}\rangle\right)e^{a+bS_2}\right\}\left\{1+\left(x_{2j}-\tau\langle 1-e^{b(S_2-S_1)}\rangle\right)e^{a+bS_2}\right\}^{\alpha-1} + (\alpha-1)\left\{\left(x_{2j}-\tau\langle 1-e^{b(S_2-S_1)}\rangle\right)e^{a+bS_2}\right\}^2\left\{1+\left(x_{2j}-\tau\langle 1-e^{b(S_2-S_1)}\rangle\right)e^{a+bS_2}\right\}^{\alpha-2}\right] \tag{13}$$

$$\frac{\partial^2 l}{\partial b^2} = (\alpha-1)\sum_{j=1}^{n_1}\frac{S_1^2x_{1j}e^{a+bS_1}}{\left[1+x_{1j}e^{a+bS_1}\right]^2} + \alpha\sum_{j=1}^{n_1}\left[\begin{array}{c}\left(S_1^2x_{1j}e^{a+bS_1}\right)\left(1+x_{1j}e^{a+bS_1}\right)^{\alpha-1}\\ +(\alpha-1)S_1^2\left(x_{1j}e^{a+bS_1}\right)^2\left(1+x_{1j}e^{a+bS_1}\right)^{\alpha-2}\end{array}\right] + (\alpha-1)\sum_{j=1}^{n_2}\frac{S_2^2(x_{2j}-\tau)e^{a+bS_2}+\tau(2S_2-S_1)^2e^{a+b(2S_2-S_1)}+\tau(x_{2j}-\tau)(9S_2^2+S_1^2-6S_1S_2)e^{a+b(3S_2-S_1)}}{\left[1+\{x_{2j}-\tau(1-e^{b(S_2-S_1)})\}e^{a+bS_2}\right]^2} + \alpha\sum_{j=1}^{n_2}\left\{\left(\left[S_2^2\left(x_{2j}-\tau\right)e^{a+bS_2}+\tau(2S_2-S_1)^2e^{a+b(2S_2-S_1)}\right]\left[1+\left\{x_{2j}-\tau\left(1-e^{b(S_2-S_1)}\right)\right\}e^{a+bS_2}\right]^{\alpha-1}\right) + (\alpha-1)\left[\left\{S_2\left(x_{2j}-\tau\right)e^{a+bS_2}\right\}+\tau(2S_2-S_1)e^{a+b(2S_2-S_1)}\right]^2\left[1+\left(x_{2j}-\tau\langle 1-e^{b(S_2-S_1)}\rangle\right)e^{a+bS_2}\right]^{\alpha-2}\right\} \tag{14}$$

$$\frac{\partial l}{\partial \alpha^2} = -\frac{n}{\alpha^2} - \sum_{j=1}^{n_1}\left(\left[1+x_{1j}e^{a+bS_1}\right]\left[\log\left(1+x_{1j}e^{a+bS_1}\right)\right]^2\right) - \sum_{j=1}^{n_2}\left(\left[1+\left\{x_{2j}-\tau\left(1-e^{b(S_2-S_1)}\right)\right\}e^{a+bS_2}\right]^{\alpha}\left[\log\left\{1+\left(x_{2j}-\tau\langle 1-e^{b(S_2-S_1)}\rangle\right)e^{a+bS_2}\right\}\right]^2\right) \tag{15}$$

$$\frac{\partial^2 l}{\partial a\partial b} = \frac{\partial^2 l}{\partial b\partial a} = (\alpha-1)\sum_{j=1}^{n_1}\frac{S_1x_{1j}e^{a+bS_1}}{\left[1+x_{1j}e^{a+bS_1}\right]^2} + \alpha\sum_{j=1}^{n_1}\left[\begin{array}{c}\left(S_1x_{1j}e^{a+bS_1}\right)\left(1+x_{1j}e^{a+bS_1}\right)^{\alpha-1}\\ +(\alpha-1)S_1\left(x_{1j}e^{a+bS_1}\right)^2\left(1+x_{1j}e^{a+bS_1}\right)^{\alpha-2}\end{array}\right] + (\alpha-1)\sum_{j=1}^{n_2}\frac{\left[S_2(x_{2j}-\tau)e^{a+bS_2}+\tau(2S_2-S_1)e^{a+b(2S_2-S_1)}\right]}{\left[1+\{x_{2j}-\tau(1-e^{b(S_2-S_1)})\}e^{a+bS_2}\right]^2} + \alpha\sum_{j=1}^{n_2}\left\{\left(\left[S_2\left(x_{2j}-\tau\right)e^{a+bS_2}+\tau(2S_2-S_1)e^{a+b(2S_2-S_1)}\right]\left[1+\left\{x_{2j}-\tau\left(1-e^{b(S_2-S_1)}\right)\right\}e^{a+bS_2}\right]^{\alpha-1}\right) + (\alpha-1)\left[\left\{x_{2j}-\tau\left(1-e^{b(S_2-S_1)}\right)\right\}e^{a+bS_2}\right]\left[S_2\left(x_{2j}-\tau\right)e^{a+bS_2}+\tau\,(2S_2-S_1)e^{a+b(2S_2-S_1)}\right]\left[1+\left(x_{2j}-\tau\langle 1-e^{b(S_2-S_1)}\rangle\right)e^{a+bS_2}\right]^{\alpha-2}\right\} \tag{16}$$

$$\frac{\partial^2 l}{\partial a\partial \alpha} = \frac{\partial^2 l}{\partial \alpha\partial a} = \sum_{j=1}^{n_1}\frac{x_{1j}e^{a+bS_1}}{\left[1+x_{1j}e^{a+bS_1}\right]^2} + \sum_{j=1}^{n_1}\begin{array}{c}\left(x_{1j}e^{a+bS_1}\right)\left(1+x_{1j}e^{a+bS_1}\right)^{\alpha-1}\\ +\left[1+\alpha\, log\left(1+x_{1j}e^{a+bS_1}\right)\right]\end{array} + \sum_{j=1}^{n_2}\frac{\left[\left(x_{2j}-\tau\langle 1-e^{b(S_2-S_1)}\rangle\right)e^{a+bS_2}\right]}{\left[1+\{x_{2j}-\tau(1-e^{b(S_2-S_1)})\}e^{a+bS_2}\right]^2} + \sum_{j=1}^{n_2}\left\{\left(\left[\left(x_{2j}-\tau\langle 1-e^{b(S_2-S_1)}\rangle\right)e^{a+bS_2}\right]\left[1+\left\{x_{2j}-\tau\left(1-e^{b(S_2-S_1)}\right)\right\}e^{a+bS_2}\right]\right) + \left[1+\alpha\, log\left(\left(x_{2j}-\tau\langle 1-e^{b(S_2-S_1)}\rangle\right)e^{a+bS_2}\right)\right]\right\} \tag{17}$$

$$\frac{\partial^2 l}{\partial b \partial \alpha} = \frac{\partial^2 l}{\partial \alpha \partial b} = \sum_{j=1}^{n_1} \frac{S_1 x_{1j} e^{a+bS_1}}{[1+x_{1j}e^{a+bS_1}]} + \sum_{j=1}^{n_1} \left[\left(S_1 x_{1j} e^{a+bS_1}\right)\left(1 + x_{1j}e^{a+bS_1}\right)^{\alpha-1}\left\{1 + \alpha\, log\left(1 + x_{1j}e^{a+bS_1}\right)\right\}\right] + \sum_{j=1}^{n_2} \frac{[S_2(x_{2j}-\tau)e^{a+bS_2} + \tau(2S_2 - S_1)e^{a+b(2S_2-S_1)}]}{[1+\{x_{2j}-\tau(1-e^{b(S_2-S_1)})\}e^{a+bS_2}]} + \sum_{j=1}^{n_2} \left\{ \left[S_2(x_{2j}-\tau)e^{a+bS_2} + \tau(2S_2 - S_1)e^{a+b(2S_2-S_1)}\right]\left[1 + \left\{x_{2j} - \tau\left(1 - e^{b(S_2-S_1)}\right)\right\}e^{a+bS_2}\right]^{\alpha-1} + \left[1 + \alpha\, log\left(1 + \left(x_{2j} - \tau\langle 1 - e^{b(S_2-S_1)}\rangle\right)e^{a+bS_2}\right)\right]\right\} \quad (18)$$

V. Estimation of Optimal Stress Change Time

I. Asymptotic variance of MLEs of the model parameters

The asymptotic variance of $\hat{a}, \hat{b}$ and $\hat{\alpha}$ is given by the diagonal elements of the inverse of Fisher information matrix.

II. Generalized asymptotic variance of MLEs of the model parameters

The generalized asymptotic variance of$\hat{a}, \hat{b}$ and $\hat{\alpha}$ is obtained by the reciprocal of the determinant of Fisher information matrix.

i.e. $G_e A_s Var(\hat{a}, \hat{b}, \hat{\alpha}) = \frac{1}{det(F)} or \frac{1}{|F|}$

First, we obtain the optimum value of the stress change time τ either by minimizing the asymptotic or the generalized asymptotic variance. After that we would estimate the values of a, b and α by using the optimum value of τ and by maximizing the log likelihood function of the distribution. We obtain the optimum value of τ using the *optim()* function in R software. This function has several methods to minimise and gives the global minima of the objective function. The available methods in *optim()* are Nelder-Mead, BFGS, L-BFGS-B, CG, SANN and Brent.

VI. Simulation Study

Simulation study has been used to examine and validate the assumptions made in the study. The study has been performed using R-software/language. Here, in this study, point and confidence interval have been estimated along with their root mean square(s) and mean absolute error(s). Monte-Carlo simulation technique is used to perform simulation study as per the detailed steps presented below:

1. The random samples of sizes 30, 50, 75, 100, 125, 150 and 200 from are generated from NH distribution. To generate the random number from NH distribution, CDF inverse transformation method is used.
2. Two stress levels are fixed, $S_1$and S_2, and their respective values are 2 and 3.
3. First put all the testing units to stress $S_1$and run until the optimum stress change time τ=1.2 is attained. Then changed the level of stress to next level that is S_2at prefixed stress change time τ=1.2 and run the experiment.
4. For each sample, the acceleration factor and the parameters of the model are estimated in SSALT.
5. The above procedure from step 1-4 is repeated 10,000 times to avoid the randomness.
6. The Newton–Raphson method was used for solving the nonlinear equations given in ....

7. The RMSEs and MAEs of the estimators for acceleration factor and other parameters for all sample sizes are tabulated.
8. The confidence limit with confidence level γ=0.95 and γ =0.99 of the acceleration factor and other two parameters were constructed.
9. The results are summarized in Tables 1, 2 and 3. Table 1 presents the Estimates, RMSEs and MAEs of the estimators. The approximated confidence limits at 95% and 99% for the parameters and acceleration factor are presented in Table 2. Optimum value of stress change time is tabulated in Table 3.

Table 1:*The maximum likelihood estimates of parameters and their RMSEs and MAEs*

N	Parameters	Estimate	RMSE	MAE
30	$\hat{\alpha}$	2.5869	0.4424	0.0347
	$\hat{a}$	2.2382	0.6177	0.1391
	$\hat{b}$	-1.1074	0.1915	0.0189
50	$\hat{\alpha}$	2.5064	0.3259	0.0359
	$\hat{a}$	2.1863	0.2361	0.1254
	$\hat{b}$	-1.1509	0.1266	0.0463
75	$\hat{\alpha}$	2.5730	0.3974	0.0292
	$\hat{a}$	2.2180	0.4185	0.1469
	$\hat{b}$	-1.1209	0.1260	0.0190
100	$\hat{\alpha}$	2.5124	0.5276	0.0049
	$\hat{a}$	2.2944	0.3612	0.1175
	$\hat{b}$	-1.1360	0.1033	0.0327
125	$\hat{\alpha}$	2.5038	0.3201	0.0246
	$\hat{a}$	2.3044	0.3101	0.1057
	$\hat{b}$	-1.1459	0.1087	0.0373
150	$\hat{\alpha}$	2.5417	0.2343	0.0167
	$\hat{a}$	2.2318	0.2816	0.1415
	$\hat{b}$	-1.1218	0.0907	0.0198
200	$\hat{\alpha}$	2.5473	0.2141	0.0147
	$\hat{a}$	2.3031	0.1052	0.1879
	$\hat{b}$	-1.1686	0.0572	0.0624

Table 2:*Confidence interval of the estimators*

N	Confidence level	$\hat{\alpha}$		$\hat{a}$		$\hat{b}$	
		LCL	UCL	LCL	UCL	LCL	UCL
30	95%	1.8592	3.3145	1.2221	3.2542	-1.4223	-0.7924
	99%	1.5577	3.6160	0.8012	3.6751	-1.5528	-0.6619
50	95%	1.9703	3.0424	1.7979	2.5746	-1.3591	-0.9426
	99%	1.7482	3.2645	1.6370	2.7355	-1.4454	-0.8563
75	95%	1.9193	3.2266	1.5296	2.9063	-1.3281	-0.9136
	99%	1.6485	3.4974	1.2444	3.1915	-1.4140	-0.8277
100	95%	1.9735	3.0512	1.7002	2.8885	-1.3059	-0.9660
	99%	1.7502	3.2745	1.4541	3.1346	-1.3763	-0.8956
125	95%	1.9772	3.0303	1.7943	2.8144	-1.3246	-0.9671
	99%	1.7591	3.2484	1.5830	3.0258	-1.3987	-0.8930
150	95%	2.1563	2.9270	1.7686	2.6949	-1.2709	-0.9726
	99%	1.9966	3.0867	1.5767	2.8869	-1.3328	-0.9108
200	95%	2.1951	2.8994	2.1300	2.4761	-1.2626	-1.0745
	99%	2.0492	3.0453	2.0583	2.5478	-1.3016	-1.0355

Table 3:*Result of optimal design of step-stress ALT for different sample sizes*

n	nG.A.V.	$\hat{\tau}$	$\hat{\tau}'$
30	0.006035	1.2	1.25
50	0.000644	1.2	1.23
75	0.000401	1.2	1.24
100	0.000146	1.2	1.21
125	0.000128	1.2	1.20
150	0.000036	1.2	1.19
200	0.000089	1.2	1.21

VII. Conclusion

In this paper we have studied NH distribution under step stress model with complete data. First the testing units have been placed on test to obtain the failure times of these items and then using these data we have analysed the lifetimes of the items on normal stress condition or general use conditions. We have calculated MLEs of parameters, their respective RMSEs and MAEs and then approximate confidence intervals of these parameters were also derived using the MLEs of these parameters.
The simulation study shows that all our assumptions are true. We see that as the sample sizes increases the RMSE and MAE are getting smaller and confidence intervals are also getting narrower. Here optimality criteria for changing the stress time are also checked and at that time the estimation technique has been used to obtain the numerical value of the parameters.
Bayesian aspects of this study may be considered as future work or one also may use different censoring schemes with classical or Bayesian approach.

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Received: August 20, 2020
Accepted: Oktober 12, 2020

On Age Replacement Policy of a System Involving Minimal Repair

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Abstract

This paper made a survey on age replacement model involving minimal repair, and this was done by considering a parallel-series system with two subsystems, which are subsystems A and B, and each of the system is formed by three parallel units, therefore, the whole systems consist of six units. We constructed age replacement model involving minimal repair that will determine the optimal replacement time of the parallel-series system based on two different policies (Policy 1 and Policy 2). A numerical example was given to illustrate the characteristics of the age replacement models involving minimal repair constructed. From the results obtained, it was observed that policy 2 extends the optimal replacement time of a multi-component system, when compared to Policy 1.

Keywords: Optimal, Repair, Replacement, Rate, System, Time.

I. Introduction

The activities of maintaining military equipment, transportation, and civil structures requires high costs, for these reasons, this leaded the development of various maintenance policies that seek the optimal decision models for reducing the risk of a catastrophic breakdown of systems. Thus, maintenance has effect on system reliability, because it prolonged the life span of the systems. For most industrial equipment, maintenance policies are provided to reduce the incidence of system frowning to failure.

There is an extensive literature on the age replacement policy, for example, see Barlow and Proschan (1965), Elsayed (1996), Nakagawa (2005) and Pham (2003). Sandev and Aven (1999) studied the optimal replacement problem of a monotone system comprising n components, where the components are ‘‘minimally’’ repaired at failures. Jain *et al.* (2002) evaluated the expressions for expected cost for a system with replacement and minimal, and furthermore discussed the maintenance costs of various policies. Ouali and Yacout (2003) developed an optional replacement policy for the maintenance of two non-identical components connected in series configuration, where by each component is replaced correctively whenever it fails and preventively only if its age reaches or exceeds a preventive replacement age T when the other component fails. Chien and Sheu (2006) proposed age replacement policy for an operating system which is subjected to shocks that arrive according to a non-homogeneous Poisson process, and as shocks occur the system has two types of failure: type I failure (minor) or type II failure (catastrophic). Chen (2007) constructed a cache document replacement policy which content can be tailored to the specific requirements of a caching system. Wang *et al.* (2008) presented a condition-based order-replacement policy for a single-unit system, aiming to optimize the condition-based maintenance and the spare order management jointly. Aven and Castro (2008) presented a minimal repair replacement model of a one unit system subjected to two types of failures. Yaun and Xu (2011) studies a cold standby repairable system with two different components and one repairman who can take multiple vacations. Yusuf and Ali (2012) considered two parallel units in which both units operate simultaneously, and the system is subjected to two types of failures. Type I failure is minor and occur with the failure of a single component and is checked by minimal repairs, while type II failure is catastrophic in which both components failed and the system is replaced. Xu *et al.* (2012) investigated on replacement scheduling for non-repairable safety-related systems (SRS) with multiple components and states, and their aim is to determine the cost-minimizing time for replacing SRS while meeting the required safety. Wang *et al.* (2014) introduced a two-level inspection policy model for a single component plant system based on a three-stage failure process, such that the failure process divide the system's life into three stages: good, minor defective and severe defective stages. Zhao *et al.* (2014) answered the problem which replacement is better for continuous and discrete scheduled times. Chang (2014) considered a system which suffers one of two types of failure based on a specific random mechanism: type-I (repairable) failure is rectified by a minimal repair, and type-II (non-repairable) failure is removed by a corrective replacement. Firstly, he considered a modified random and age replacement policy in which the system is replaced at a planned time T, at a random working time, or at the first type-II failure, whichever occurs first. He further considered a system which work continuously for N jobs with random working times. Malki *et al.* (2015) investigated on age replacement policies for two-component parallel system with stochastic dependence. The stochastic dependence considered, is model by a one-sided domino effect. Coria *et al.* (2015) proposed an analytical optimization method for preventive maintenance (PM) policy with minimal repair at failure, periodic maintenance, and replacement for systems with historical failure time data influenced by a current PM policy. Yusuf *et al.* (2015) modified the work of Aven and Castro (2008) by introducing random working time Y. They constructed a modified random and age replacement model, for which the system is replaced at a planned time T, at a random working time Y, or at the first non-repairable type 2 failure whichever occurs first. Where they assumed that, if there is a component which fails and the repairman is on vacation, the failed component will wait for repair until the repairman is available.

The main contributions of this study are to develop age replacement models involving minimal repair for parallel-series system, which is subjected to two types of failures, so as to addressed (1) the problem of sudden failure of a multi-component system (2) avoid rising maintenance cost of a multi-component system, and (3) to provide some characteristics of the age replacement model involving minimal repair. The remainder of this paper is organized as follows: Section 2 discussed the methodology of the study. Section 3 discussed the proposed models. Section 4 presents the numerical results. Finally, section 5 discussed the conclusion and recommendations.

II. Methods

Reliability measures namely reliability function and failure rates are used to obtain the expressions of age replacement models based on some model assumptions. A numerical example was given for the purpose of investigating the characteristics of the models constructed.

Notations used

- $r_{ia}(t)$: Type I failure rate of unit A_i of subsystem A, for $i = 1, 2, 3$.
- $r_{ib}(t)$: Type II failure rate of unit B_i of subsystem B, for $i = 1, 2, 3$.
- $R_{ia}(t)$: reliability function of unit A_i of subsystem A, for $i = 1, 2, 3$.
- C_{ib} : cost of minimal repair of unit B_i of subsystem B due to Type II failure, for $i = 1, 2, 3$.
- C_p : cost of planned replacement of the system at time T.
- C_r : cost of un-planned replacement of the system due to Type I failure.
- T^*: Optimal replacement time of the system based on Policy 1.
- (T^*, τ^*) : Optimal pair replacement time of the system based on Policy 2.

III. Description of the system

A system comprising of two subsystems A and B in series is considered. Subsystem A consist of three active parallel units, which are A_1, A_2 and A_3. While, subsystem B consist of three active parallel units, which are B_1, B_2 and B_3. See figure 1 below. The three units A_1, A_2 and A_3 are subjected to Type I failure. While the three units B_1, B_2 and B_3 are subjected to Type II failure. The system will stop working completely, if it least one of the two subsystems (A and B) failed.

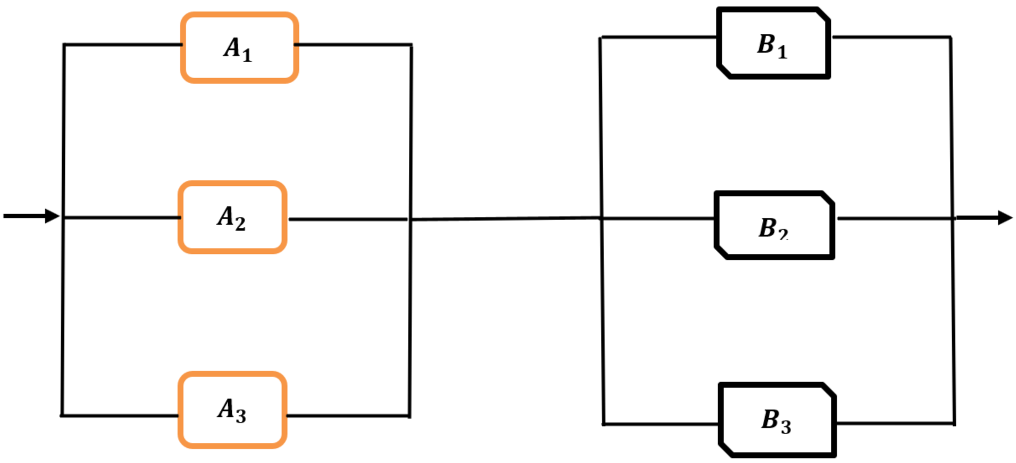

Figure 1. *Reliability block diagram of the system*

IV. Age Replacement Models

This section considers some of the fundamental replacement policies involving minimal repair.

Policy 1

Assumptions for this Policy 1:

1. Type I failure is un-repairable, while Type II failure is repairable.
2. Both the two failures are detected instantaneously.
3. All required resources are available when needed, which means that replacement/minima repair.
4. The system fails due to Type I failure, if all the three units of subsystem A fails due to Type I failure.
5. The system fails due to Type II failure, if all the three units of subsystem B fails due to Type II failure.
6. If the system failed due to Type I failure, the whole system will be replaced completely with new one.

7. If the system failed due to Type II failure, then the system is minimally repair, and allow the system to continue operating from where it stopped.
8. The system is replaced at a planned replacement time $T(T>0)$ after its installation or Type I failure of the system, whichever occurs first.

Based on the assumptions of Policy 1, we have the probability that the system will be replaced at planned time T before Type I failure occurs, as

$$R(T)=1-\big(1-R_1(T)\big)\big(1-R_2(T)\big)\big(1-R_1(T)\big), \tag{1}$$

where

$$R_i(T)=e^{-\int_0^T r_{ia}(t)dt},\ for\ i=1,\ 2,\ 3. \tag{2}$$

The cost of unplanned replacement of the system in one replacement cycle is

$$C_r\big(1-R(T)\big). \tag{3}$$

The cost of planned replacement of the system in one replacement cycle is

$$C_pR(T). \tag{4}$$

The cost of minimal repair of unit B_1 of subsystem B in one replacement cycle is

$$\int_0^T C_{1b}r_{1b}(t)dt. \tag{5}$$

The cost of minimal repair of unit B_2 of subsystem B in one replacement cycle is

$$\int_0^T C_{2b}r_{2b}(t)dt. \tag{6}$$

The cost of minimal repair of unit B_3 of subsystem B in one replacement cycle is

$$\int_0^T C_{3b}r_{3b}(t)dt. \tag{7}$$

Based on this policy 1, we have the total replacement cost rate of the system in one replacement cycle as

$$C(T)=\frac{C_r(1-R(T))+C_pR(T)+\int_0^T C_{1b}r_{1b}(t)dt+\int_0^T C_{2b}r_{2b}(t)dt+\int_0^T C_{3b}r_{3b}(t)dt}{\int_0^T R(t)dt}. \tag{8}$$

Noting that, $C(T)$ is adopted as the objective function of an optimization problem, and the aim is to determine an optimal replacement time T^* that minimizes$C(T)$.

Policy 2

Assumptions for this Policy 2:

1. Both Type I failure and Type II failure are repairable, where the failure each of the six units is rectify by minimal repair.
2. Both the two failures are detected instantaneously.
3. All required resources are available when needed, which means that there is no waiting time.
4. The system fails due to Type I failure, if all the three units of subsystem A fails due to Type I failure.
5. The system fails due to Type II failure, if all the three units of subsystem B fails due to Type II failure.
6. If the system fails due to Type II failure, we minimally repair the system, and allow the system to continue operating from where it stopped.
7. On the first Type I failure after a given system age τ, an un-planned replacement of the system is carried out. However, if, for given T, such that, $\tau<T$, there is no replacement in $[\tau,T]$, then at time T, a planned replacement of the system is carried out.
8. If the system fails due to Type I failure before a given time τ, we minimally repair the system, and allow the system to continue operating from where it stopped.

Based on the assumptions of Policy 2, we have the probability that the system will be replaced at planned time T before the first Type I failure of the system after a given time τ occurs, as

$$R(T-\tau)=1-\big(1-R_1(T-\tau)\big)\big(1-R_2(T-\tau)\big)\big(1-R_1(T-\tau)\big), \tag{9}$$

$$R_i(T-\tau)=e^{-\int_0^{T-\tau} r_{ia}(t)dt},\ for\ i=1,\ 2,\ 3. \tag{10}$$

The cost of unplanned replacement of the system in one replacement cycle is

$$C_r\big(1 - R(T-\tau)\big). \tag{11}$$

The cost of planned replacement of the system in one replacement cycle is

$$C_p R(T-\tau). \tag{12}$$

The cost of minimal repair of unit A_1 of subsystem A before given time τ in one replacement cycle is

$$\int_0^\tau C_{1a} r_{1a}(t)dt. \tag{13}$$

The cost of minimal repair of unit A_2 of subsystem A before given time τ in one replacement cycle is

$$\int_0^\tau C_{2a} r_{2a}(t)dt. \tag{14}$$

The cost of minimal repair of unit A_3 of subsystem A before given time τ in one replacement cycle is

$$\int_0^\tau C_{3a} r_{3a}(t)dt. \tag{15}$$

The cost of minimal repair of unit B_1 of subsystem B before planned time T in one replacement cycle is

$$\int_0^T C_{1b} r_{1b}(t)dt. \tag{16}$$

The cost of minimal repair of unit B_2 of subsystem B before planned time T in one replacement cycle is

$$\int_0^\tau C_{2b} r_{2b}(t)dt. \tag{17}$$

The cost of minimal repair of unit B_3 of subsystem B before planned time T in one replacement cycle is

$$\int_0^\tau C_{3b} r_{3b}(t)dt. \tag{18}$$

Based on this policy 2, we have the total replacement cost rate of the system in one replacement cycle as

$$C(T,\tau) = \frac{C_r\big(1 - R(T-\tau)\big) + C_p R(T-\tau) + \int_0^\tau C_{1a} r_{1a}(t)dt + \int_0^\tau C_{2a} r_{2a}(t)dt + \int_0^\tau C_{3a} r_{3a}(t)dt}{\tau + \int_0^{T-\tau} R(t)dt}$$

$$\frac{+\int_0^T C_{1b} r_{3b}(t)dt + \int_0^T C_{2b} r_{2b}(t)dt + \int_0^T C_{3b} r_{3b}(t)dt}{\tau + \int_0^{T-\tau} R(t)dt}. \tag{19}$$

Noting that, $C(T,\tau)$ is adopted as the objective function of an optimization problem, and the aim is to determine the optimal pair replacement time (T^*, τ^*) that minimizes $C(T,\tau)$.

V. Numerical example

Let the rate of Type I failure of the three units of subsystem A follows Weibull distribution:

$$r_{ia}(t) = \lambda_{ia} \propto_{ia} t^{\propto_{ia}-1},\ t \geq 0\ ,\ i = 1,2,3. \tag{20}$$

where $\propto_{ia} > 1$.

Again, let the rate of Type II failure of the three units of subsystem B follows Weibull distribution:

$$r_{ib}(t) = \lambda_{ib} \propto_{ib} t^{\propto_{ib}-1},\ t \geq 0\ ,\ i = 1,2,3. \tag{21}$$

where $\propto_{ib} > 1$.

Let the set of parameters and cost of repair/replacement be used throughout this particular example:

1. $\propto_{ia} = 3$, for i = 1, 2, 3.
2. $\lambda_{ia} = 0.008$, for i = 1, 2, 3.
3. $\propto_{ib} = 3.5$, for i = 1, 2, 3.

4. $\lambda_{ib} = 0.00025$, for i = 1, 2, 3.
5. $C_{ia} = 7$, for i = 1, 2, 3.
6. $C_{ib} = 5$, for i = 1, 2, 3.
7. $C_r = 75$ and $C_p = 50$.

By substituting the parameters in equations 20 and 21, we obtained the failure rates as follows

$$r_{ia}(t) = 0.024t^2, \text{ for } i = 1, 2, 3, \tag{22}$$

and

$$r_{ib}(t) = 0.000875t^{2.5}, \text{ for } i = 1, 2, 3. \tag{23}$$

The tables and the graphs below, are the results obtained by substituting the cost of repair/replacement and equations (22) to (23) in the cost rates C(T) and C(T, τ).

Table 1: *Values of $C(T)$ and C(T, τ) versus planned replacement T*

T	1	2	3	4	5	6	7	8	9	10
C(T)	250.01	125.07	84.05	66.86	64.77	74.12	85.53	89.01	90.39	95.90

Table 2 : *Optimal replacement time of the system from C(T) as C_p decreases.*

C_p	50	40	30	20	10
T^*	5	4	4	4	3

Table 3 : *Optimal replacement time of the system from C(T) as C_r increases.*

C_r	75	85	95	105	115
T^*	5	5	4	4	4

Table 4: *The values of C(T, τ) versus planned replacement T.*

T	**T**	**C(T, τ)**
1	0.5	83.36
2	1	41.79
3	1.5	28.08
4	2	21.45
5	2.5	17.85
6	3	15.94
7	3.5	15.00
8	4	14.56
9	4.5	14.40
10	5	14.44
11	5.5	14.64
12	6	14.93
13	6.5	15.30
14	7	15.73
15	7.5	16.22

Table 5 : *Optimal replacement time of the system from C(T, τ) as C_p decreases.*

C_p	50	40	30	20	10
(T^*, τ^*)	(10, 5)	(7, 3.5)	(6, 3)	(5, 2.5)	(4, 2)

Table 6 : *Optimal replacement time of the system from C(T, τ) as C_r increases.*

C_r	75	85	95	105	115
(T^*, τ^*)	(10, 5)	(9, 4.5)	(7, 3.5)	(7, 3.5)	(7, 3.5)

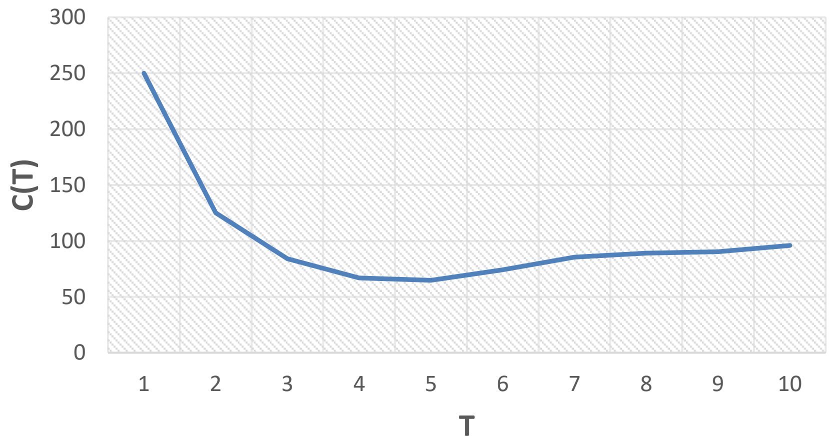

Figure 2. *Plot of C(T) against T*.

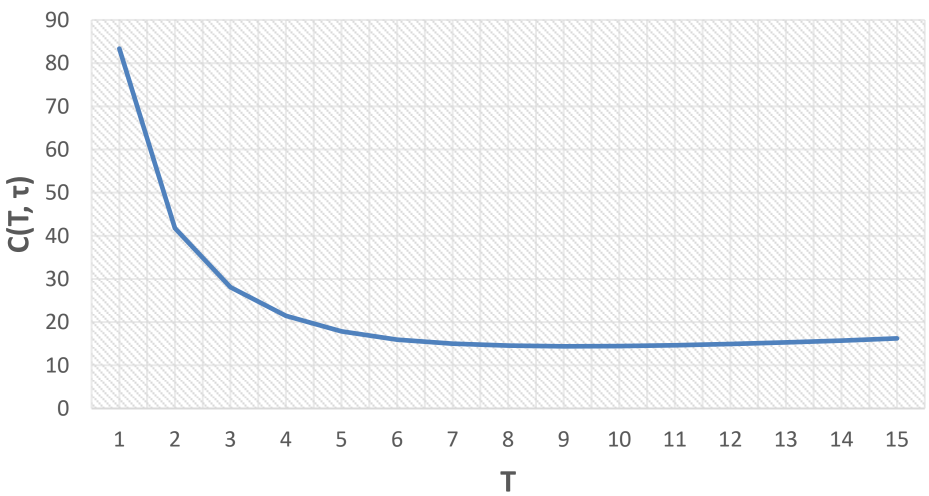

Figure 3. Plot of $C(T,\tau)$ against T.

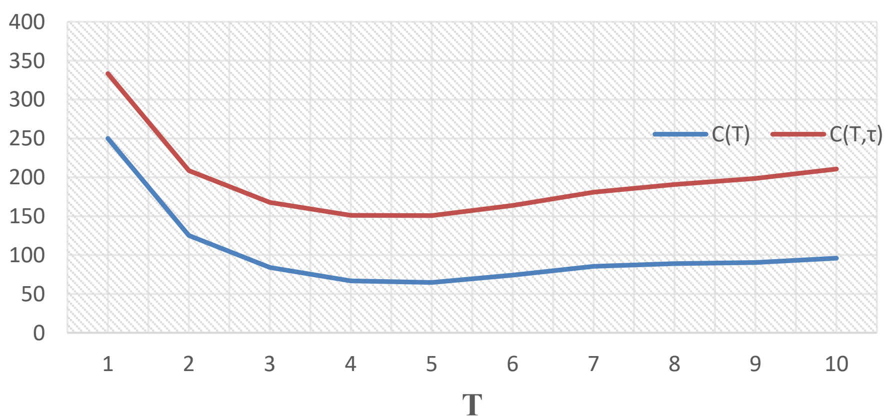

Figure 4: Plot of C(T) and $C(T,\tau)$ agianst T.

Some observations from the results obtained are as follows

1. Observe from table 1, we have the optimal replacement time for the system based on Policy 1 as 5, that is, $T^* = 5$, with minimal cost rate $C(T^* = 5) = 64.77$. See figure 2 below for the plot of $C(T)$ versus T.
2. Observe from table 2, that the optimal replacement time of the system based on Policy 1, sometimes decreases slightly as the cost of planned replacement (C_p) decreases.
3. Observe from table 3, that the optimal replacement time of the system based on Policy 1, sometimes decreases slightly as the cost of un-planned replacement (C_r) increases.
4. Observe from table 4, we have the optimal replacement time for the system based on Policy 2 as (9, 4.5), with minimal cost rate $C(T^* = 9, \tau^* = 4.5) = 14.40$. See figure 3 below for the plot of $C(T, \tau)$ versus T.
5. Observe from table 5, that the optimal replacement time of the system based on Policy 2, sometimes decreases slightly as the cost of planned replacement (C_p) decreases.
6. Observe from table 6, that the optimal replacement time of the system based on Policy 2, sometimes decreases slightly as the cost of un-planned replacement (C_r) increases.
7. Observe from figure 4, we have, $C(T) < C(T, \tau)$.
8. Observe from tables 2, 3, 5 and 6, that the optimal replacement time obtained from Policy 2 is higher than that of Policy 1.

VI. Conclusion and recommendations

This paper gives a survey on some important maintenance policies involving minimal repairs and replacements of multi-component systems. In this paper, we constructed age replacement models with minimal repair for a parallel-series system based on two different policies (Policy 1 and Policy 2), such that the system contained two subsystems, which are subsystem A and subsystem B. We assumed that two subsystems are formed by three units. It was also assumed that subsystem A is subjected to Type I failure and subsystem B is subjected to Type II failure. Finally, a numerical example was given, to investigate the characteristics of the age replacement models with minimal repair constructed for a multi-component system, where from the results, it was also observed that, the optimal replacement time obtained from Policy 2 is higher than the optimal replacement time obtained from Policy 1. The results obtained would be useful for the practical maintenance of multi-component systems.

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Received: August 26, 2020
Accepted: November 15, 2020

Key Management and Distribution for Mobile Ad-Hoc Network

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Abstract

Security has become the principal concern in mobile ad-hoc network. Secure communication depends on using cryptographic mechanisms. Cryptographic mechanism involves symmetric key and asymmetric key approaches. The symmetric key approach is more reliable except the key distribution phase. Asymmetric key approach gives robust security, but it results in high computational, high communications and high storage overhead. The propose research uses both the concepts. The symmetric key approach for reliable data exchange and asymmetric key approach for key management and distribution to achieve robust security in constraint based mobile ad-hoc network.

Keywords: Key management, Key distribution, Mobile ad-hoc network, Security.

I. Introduction

In a region where communication area and existing infrastructure is limited and inconvenience, mobile ad-hoc network is one of the solutions [1]. A mobile ad-hoc network also known as MANET is an assortment of several devices across the temporary network without any assist with the centralized administration. In such kind of network all mobile devices works as a host as well as a router. In such environment's routing protocol is required because two hosts that wish to communicate may not be capable to transmit packet directly [2]. Due to the resource constrains the nature of mobile Ad-hoc network key management is very crucial and challenging. Key management combines the security concepts like confidentiality, authentication, and key confirmation, [3]-[6] generally defines as security goals. To attain robust security, it is important to encrypt messages with strongly secure key [7]. The secure exchange of secret key is the major issue related to the symmetric cryptography implementation [8]. The asymmetric one is better than the symmetric in term of providing robust security [8]. The traditional key management schemes are insufficient for mobile Ad-hoc network. To implement ideal key management one can, have to know the basic characteristics of mobile Ad-hoc network. As well as also study the requirements of key managements.

II. Characteristics of Ad-hoc Network

Following are the core characteristics of mobile Ad-hoc network.

- ***Dynamic Topology***: Topology means physical arrangements of the node across the network. In Ad-hoc network nodes are mobile hence the topology may change frequently.

- ***Bandwidth Limitation***: Bandwidth is a transmission capacity of the network. The mobile-Ad-hoc network offer less bandwidth than traditional network. Due to this reason number of messages and packet size is limited.
- ***Energy Constrain***: All nodes across the mobile Ad-hoc networks are battery operated hence they have limited power source. Due to limited power complex algorithms may not be possible to implement in an Ad - hoc network.
- ***Physical Security***: The mobile Ad-hoc network does not have fixed infrastructure; hence nodes are being physically compromised by theft. Security is a big issue in mobile Ad-hoc network. The cryptographic key is one of the solutions of security issues. It is necessary to understand the characteristic of mobile Ad-hoc network before implementing the cryptographic key. The proposed work shows the key management in mobile Ad-hoc network.

III. Literature Review

Literature review shows that the author has studied various aspects of the key management and distribution for mobile ad-hoc networking.

The base concepts of key managements specifically covered secure communication for key materials exchange explained by Menezes et al., 1996 [9]. Based on the knowledge of neighbor discovery random key per distribution scheme for secure communication is proposed by Eschenauer and Gligor [2002] [10], the scheme is based on the exact location of the node.

Pairwise communication suggests by Pietro et al. [2003] [11]. It is based on random key assignments. The concept is later on extended to pseudo random key generation for energy efficient key management. Based on four sets of keys Zhu et al. [2003] [12] introduced a LEAP security mechanism for neighbor compromised node. Link layer key management encryption scheme TinySec proposed by Karlof et al., 2004 [13]. Hu et al., 2004 [14] presents the trusted Certificate Authority (CA) for public key cryptography. CA is responsible to revoke key as a key will compromised. Authentication is the base for secure communication [13], without robust authentication mechanism confidentiality, data integrity, and non-repudiation are hard to define. There is diversity of symmetric and asymmetric algorithms available, including DES, AES, IDEA, RSA, and EIGamal [9]. These cryptographic algorithms are the security primitives that are widely used in wired and wireless networks. They can also be used in MANETs and help to achieve the security in its unique network settings [15][16]. Asymmetric Key Cryptography is complex, slow and power hungry, and as such not at all suitable for use in ultra-low power environments [17].

The followings are the basic key management approaches used popularly in mobile Ad-hoc network. Basic Key Management, by Eschenauer and Gligor [2002] [10], Random Key Pre-distribution, by Chan et al. [2003], [18] Random Key Assignment, by Pietro et al. [2003] [11], Establishing Pairwise Keys, by Liu et al. [2003] [19], Pairwise Key Pre-distribution, by Du et al. [2003] [19] [20], Deployment Knowledge, by Du et al. [2004] [20], Group Key Management, by Eltoweissy et al. [2005], [21] Location-Based Keys, by Zhang et al. [2005] [22].

IV. Motivation

Although substantial expansions have been made towards the key management and distribution in the mobile ad-hoc network, robust security measures remain insufficient. Most of the explanation offered in literature addresses the key management and distribution in traditional network, but these may not be exactly fitted into the mobile ad-hoc network.

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Thus, there is a need for a better scheme for key management and distribution in mobile ad-hoc network which can provide less computational, less communication and less storage overhead.

V. Security attacks

Compare with traditional network mobile ad-hoc network is more vulnerable to security attacks. There are two types of security attacks. One is passive and another is active [2][23][24].

- ***Passive Attack***: In passive attack intruder may not have enough knowledge to alter the captured data. The attacker only listen the communication without any kind of modification. These kinds of captures are the example of eavesdropping. These attacks break the confidentiality and are difficult to detect. Utilization of powerful encryption methods is one of the solutions of these attacks.
- ***Active Attack***: In active attack intruder may have enough knowledge to alter the captured data. The attacker listens as well as may alter the communication. These attacks break the authentication. Utilization of powerful MAC algorithms or any message digest algorithms are one of the solutions of these attacks.

VI. Key Management Requirements

Following are the base requirements for key management.

- ***Confidentiality***: Confidentiality means key information remains secrete between a source node and destination node. No one can know the key information exchange between the source and destination node. The various cryptographic algorithms are used to maintain confidentiality.
- ***Confidentiality***: Confidentiality means key information remains secrete between a source node and destination node. No one can know the key information exchange between the source and destination node. The various cryptographic algorithms are used to maintain confidentiality.
- ***Authentication***: Only the authorized nodes can gain the cryptographic key materials, no one else. The various MAC algorithms as well as message digest mechanisms are used to maintain authentication.
- ***Key-confirmation***: Key establishment protocols are responsible to ensure key confirmation. Key confirmation ensures that the key materials being exchanged are between the authorized nodes. Key confirmation uses the concepts of nonce [25].
- ***Key freshness***: It ensures new and unique independent keys are used for different sessions. The concepts of new and independent key ensure the forward and backward secrecy.
- ***Forward secrecy***: It restricts opponent from discovering subsequent keys from a compromised contiguous subset of old keys.
- ***Backward secrecy***: It restricts opponent from discovering preceding keys from a compromised contiguous subset of old keys.
- ***Key independence***: It subsumes the forward and backward secrecy. Key independence ensures that an opponent who knows a proper subset of keys cannot discover any other keys.
- ***Availability***: It ensures that whenever the network expect keying materials it is ready to use.
- ***Survivability***: Survivability is the ability of the key management to remain available even in the presence of threats and failures.
- ***Scalability***: It ensures network to allow numbers of nodes according to the requirements. The network can have the ability to add or remove nodes.
- ***Resistance***: The ability to protect against tolerates attacks.
- ***Recovery***: Ability to recover the information unavailable due to damage. The self-healing

and mutual healing mechanisms are used to implement recovery.

- ***Efficiency***: Key management schemes should be efficient in communication, computation and storage overhead

VII. Key Managements and Distribution

The basic difficulty in mobile ad-hoc network is to maintain secure communication by surroundings up secret keys between communicating nodes [1]-[6], [9]-[12]. In general this phenomenon is called key distribution [1]-[6], [9]-[12]. One of the popularly used techniques is trusted server scheme. The trusted server scheme is depends on a trusted server like karberos [Neuman and Tso, 1994] [26]. Since there is no fix trusted infrastructure in mobile ad-hoc network trusted server scheme is inappropriate [1]-[6], [9]-[13]. The second approach is based on Asymmetric and Symmetric key cryptography [27]. Asymmetric key cryptography is also known as public key cryptography. However, due to the resource constraint nature of the mobile ad-hoc deices this scheme is not much more fruitful for entire data communication [1]-[6], [9]-[16]. Asymmetric key algorithms like Diffe-Hellman [Diffie and Hellman, 1976] and RSA [Rivest et al., 1978] require high computation resources which is not feasible to transmit large amount of data in mobile ad-hoc network.

Nowadays, security is an important issue in almost every network [28]. Cryptography is a significant and dominant tool for secure communication. It transmits the cipher text across the network. The source node converts plain text into cipher text, the mechanism is known as encryption. The destination node converts received cipher text into plain text the mechanism is known as decryption. The specific key value is used for encryption and decryption. Symmetric key and Asymmetric key algorithms are used to implement the concepts of the encryption and decryption. The symmetric key algorithms use the same key for encryption and decryption. Asymmetric key algorithms use different key for encryption and decryption. So, there is no need to exchange the key value across the network. Hence it maintains the confidentiality.

The proposed scheme suggests that to use symmetric key algorithms for encryption and decryption for the data value. The problem of securely transmit key value between source and destination node will be resolved by utilizing the asymmetric key cryptography. Hence it maintains confidentiality for key exchange. Means key value is only known to intended source and destination node only.

VIII. Conclusion

After evaluating large literature, the author suggests to use Symmetric key cryptography for large amount of data. While the Asymmetric key cryptography for key exchange. This key is used for symmetric cryptography. This kind of implementation gives robust security in constraint based ad-hoc network.

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Received: August 27, 2020
Accepted: November 15, 2020

Modified Group Lottery Scheduling Algorithm for Ready Queue Mean Time Estimation in Multiprocessor Environment

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Abstract

The problem of ready queue mean time estimation in the multiprocessor environment was discussed by Shukla et. al. [5] and several others. In recent years, most of the existing and relating contributions assume that all processes in the ready queue might have been completed before a particular instant of time occur like a sudden failure or interrupt. Due to this, data of time consumed by processes remain available. The idea of improvement in this paper is to assume that at the instant of occurrence of breakdown, some processes are partially completed and remaining is completely processed. Under this situation, the time computation and allocation strategies need to be re-designed. Therefore this has been taken into account in this paper with a proposal of a modified scheme. It contains arbitrary, Type-A, and Type- B allocations of sample units to the processors. Confidence intervals for the sample mean values are calculated and simulated over many samples using cumulative probabilities. It was found that Type-A allocation has the lowest variance.

Keywords: CPU, Scheduling, Lottery Scheduling, Estimation, Sampling, Probability, Allocation, Simulation.

I. Introduction

The challenging task of an operating system is CPU scheduling algorithms where various non-probabilities based traditional schemes are operational. These can simply be handled easily by processors while probabilistic scheduling schemes have to face the difficulty of resource management, system performance, and low system overhead. Lottery scheduling is one such probability-based scheme first introduced by Carl A. Waldspurger [12]. Shukla, Jain, and Choudhary [4] have initiated the problem of estimation of ready queue processing time by suggesting SL scheduling algorithm in a multiprocessor environment. The contribution contains a sample-based estimation of ready queue mean time which likely to be spent while completes exhaust of ready queue occurs. It reveals the approach of systematic sampling which has some limitations in terms of efficiency of the predicted value. Shukla et. al. [6] extended similar problem under the approach of lottery scheduling. Content of contribution stands for randomly selected processes from the ready queue for forecasting the sample-based mean time. The limitation of lottery scheduling appears due to the reason that processes happen to be of any size may appear in any order before multiprocessors. Shukla and Jain [7] extended the ready queue processing time estimation approach to the care of probability proportional to size-dependent lottery scheduling which provides better prediction than earlier. Following the similar approach, Shukla and Jain [8] used factor type estimation method for estimating mean ready queue processing time in setup of lottery scheduling under a multiprocessor environment. Shukla and Jain [9] extended approach using ratio type estimation method and advocated for better efficiency under constraints. A similar approach adopted in Jain and Shukla [10] and Shukla and Jain [11] with additive features. An exhaustive review of the problem of ready queue mean time estimation is due to Shukla and More [1] and some suggestive contributions are due to Shukla and More [2] [3]. Sampling technique concepts and applications are in Cochran [13].

Shukla D., Jain, and Choudhary [5] discussed GL scheduling which assumes the processes present in all processors in the time session (0-T) have been completely processed at instant T and their compound predictive estimate of average processing time could be obtained. Such an estimate is useful for forecasting the expected time required to vacate the entire ready queue. This helps in backup management while sudden failure (or disaster) occurs. But it doesn't cover the case when a sudden failure occurs during the processing of these jobs (processes). How estimation will be in a situation when the last process is partially processed and kept on hold. This paper takes into account this problem and provides a solution

II. GL Scheduling Scheme (due to Shukla, Jain, and Choudhary [5]):

Step 1: Assume multiple processors Q_1, Q_2, Q_3……Q_r, each draws random samples of jobs from corresponding ready queues. Processes in the i^{th} ready queue are homogeneous concerning certain characteristics whereas in the usual waiting queue they are present in any order of size measure.

Step 2: The CPU restricts a session of time duration T. All N ready queue processes are divided into r groups each of size containing N_i processes ($\sum N_i = N$). This division is based on size measure.

Step 3: All N processes are allotted token of numbers and each processor draws a random number. If the random number of i^{th} processor matches the allotted random number to the j^{th} process of the i^{th} group then it is selected for processing (i=1, 2, 3….r, j=1, 2, 3…..N_i).

Step 4: Let k_1 processes received from the first group, k_2 processes from the second group, and so on, the k_r^{th} received processes from r^{th} group in a random manner using lottery procedure [$\sum k_i = k$] in a session of fixed time T where k is the total sample size.

Step 5: At the end of a session, the CPU provides processed time data for k_1, k_2, k_3….k_r jobs as (t_{11}, t_{12}, t_{13}…... t_{21}, t_{22}, t_{23}…., .t_{i1}, t_{i2}, t_{i3}…) where t_{ij} are the time consumed by j^{th} job.

III. Modified Group Lottery Scheduling (MGLS) Scheme

The proposed contribution is an extension of the previous algorithm suggested by Shukla et. al. [5], with the idea of improvement to include the processing time of those processes that remained partially processed due to sudden system breakdown or occurrence of an interrupt. Following are steps of the proposed scheme:

Step 1: Assume r processors Q_1, Q_2, Q_3, Q_4……….Q_r, in a system each, receives random samples from corresponding linked ready queues. Processes in corresponding ready queues are of homogeneous concerning a specific characteristic. If any event wait appears, that process moves to a waiting/blocked/suspended queue.

Step 2: Total N processes assumed present in the system are divided into r groups of ready queues with the assumption that i^{th} group (or ready queue) has N_i processes ($\sum N_i = N$).

Step 3: All N processes in the system are assigned token of numbers. Processors generate random numbers whose matching occurs with token assigned to processes. If i^{th} processor random number matches to the token number of j^{th} process then j^{th} assigns to i^{th} processor.

Step 4: Using (3), suppose total k_r processes selected from r^{th} group of the ready queue in a random manner and assigned to Q_rth processor. The total sample size is $k = \sum k_i$ where $i = 1,2,3,\ldots\ldots.r$, $j = 1,2,3,\ldots\ldots\ldots N_i$

Step 5: Let t_{ij} denote time consumed by the jth process assigned to ith processor.

Step 6: At instant time T, out of total k_i processes present in ith processor, assume k_{i-1} have completely processed but the last one is partially processed with time t_i^* in all Q_1, Q_2, Q_3 …Q_r. The set of time (t_1^*, t_2^*, t_3^*.......t_r^*) is the time consumed by partially processed jobs.

Step 7: Processes within the processor are divided into two parts. The Part A being sub-group of completely processed and part B for unprocessed (t_i^*)

Step 8: Overall mean time, $\overline{mt} = \frac{1}{N}\sum\sum t_{ij}$, $\overline{mt}_i = \frac{1}{N_i}\sum_{j=1}^{N_i}(t_{ij})$ (for ith ready queue), $S_i^2 = \frac{1}{N_{i-1}}\sum_{j=1}^{N_i}(t_{ij} - \overline{mt}_i)^2$ (for ith ready queue) and $S^2 = \frac{1}{N-1}\sum_{i=1}^{r}\sum_{j=1}^{N_i}(t_{ij} - \overline{mt})^2$ under assumption while all N completely processed before occurring T but under step (6) it does not happen.

Note: The steps 5, 6, and 7 are the idea of improvement in this paper over the Shukla et. al. [5].

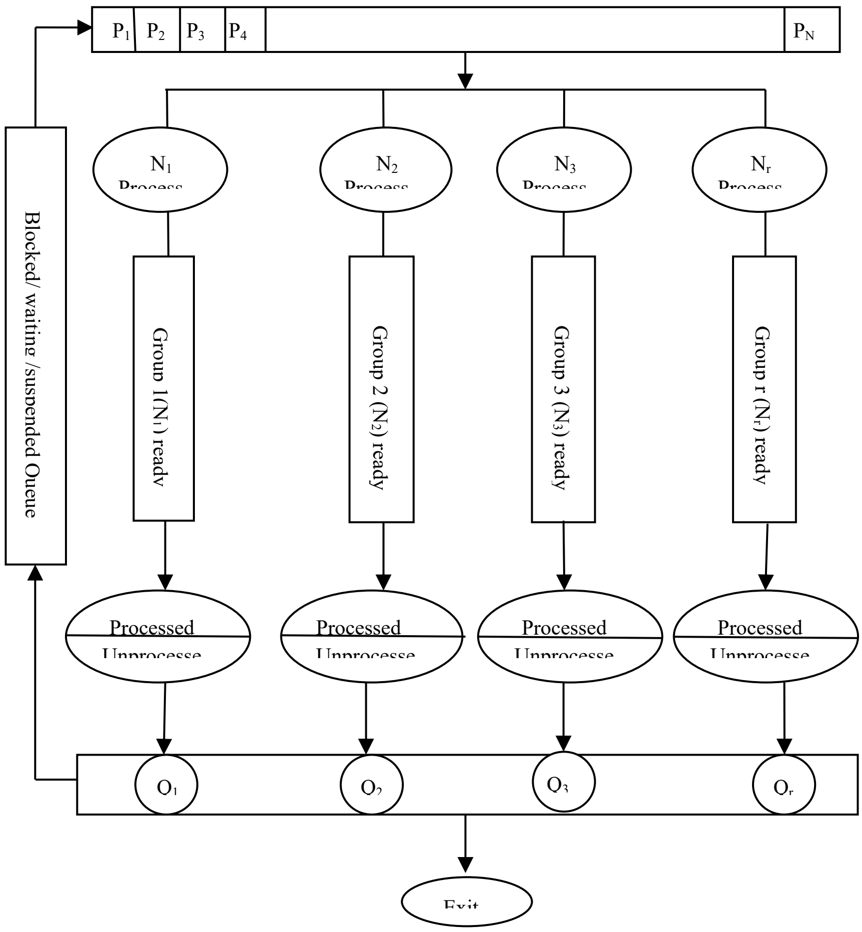


Figure 1: *Setup of ready queue and multiprocessor environment*

IV. Estimation Procedure under Arbitrary Allocation

The Modified Group Lottery Scheduling algorithm (MGLS) provides the estimation of mean time likely to consume by the N processes in the ready queue while occurrences of time T. For ith ready queue (group), the mean time is spited into:

(a) $\bar{t}_i' = \left(\frac{1}{(k_i-1)}\right)\sum_{j=1}^{k_i-1}(t_{ij})$ (for processed part A of sample not including unprocessed)

(b) $\bar{t}^* = \frac{1}{r}\sum_{j=1}^{r}(t_i^*)$ (for unprocessed part B jobs in all r samples)

(c) The mean time estimator is $\bar{u} = [\sum_{i=1}^{r} w_i \bar{t}_i{}' + \bar{t}^*]/2$ where $w_i = \frac{N_i}{N}$

(d) The mean square of time $\bar{t}_i$ for i^{th} group is $S_i^2 = \frac{1}{(N_i-1)} \sum_{j=1}^{N_i} (t_{ij} - \bar{t}_i)^2 = \left(\frac{1}{(N_i-1)}\right) \sum_{j=1}^{N_i} (t_{ij} - \overline{mt}_i)^2$ Where $\bar{t}_i = \frac{1}{N_i} \sum_{j=1}^{N_i} t_{ij}$

(e) $S^2 = \frac{1}{(N-1)} \sum_{i=1}^{r} \sum_{j=1}^{N_i} (t_{ij} - \bar{t})$ where $\bar{t} = \frac{1}{N} \sum_{i=1}^{r} \sum_{j=1}^{N_i} (t_{ij}) = \overline{mt}$

(f) Variance of estimator $\bar{u}$ is $V(\bar{u})_{arbit} = V[\sum_{i=1}^{r} w_i \bar{t}_i{}' + \bar{t}^*] = \sum_{i=1}^{r} w_i^2 V(\bar{t}_i{}') + V(\bar{t}^*)$

$$= \sum_{i=1}^{r} \left(\frac{1}{(k_i-1)} - \frac{1}{N_i}\right) w_i^2 S_i^2 + \left[\left(\frac{1}{r} - \frac{1}{N}\right) S^2\right] \quad (4.1)$$

This estimator $\bar{u}$ and variance V ($\bar{u}$) $_{arbit}$ is based on arbitrary allocation of processes to the processors.

V. Types of Allocations:

Type-A Allocation: Based on prior information of processor speed

The choice of k_i depends on the speed of processors. A fast processor can randomly pick a larger number of jobs from the group of ready queue samples. Let priority known processor speed are S_1^*, S_2^*, S_3^*.......S_r^* for Q_1, Q_2, Q_3 ….Q_r respectively, and $\sum_{i=1}^{r} S_i^* = S^*$ holds.

Let $k_i \ \alpha \ S_i^*$, $k_i = MS_i^*$, $\sum k_i = \sum MS_i^*$, $k = M S^*$, $M = (k/s^*)$, $k_i = (\frac{k}{S^*}) S_i^*$ (M is any constant) (5.1) Substituting (5.1) in (4.1) one can get

$$V(\bar{u})_I = \sum_{i=1}^{r} \left[\left(\frac{1}{\left(\left(\frac{k}{S^*}\right)S_i^* - 1\right)} - \frac{1}{N_i}\right) w_i^2 S_i^2\right] + \left[\left(\frac{1}{r} - \frac{1}{N}\right) S^2\right] = \sum_{i=1}^{r} \left[\left(\frac{S^*}{(k S_i^* - S^*)} - \frac{1}{N_i}\right) w_i^2 S_i^2\right] + \left[\left(\frac{1}{r_*} - \frac{1}{N}\right) S^2\right]$$

$$V(\bar{u})_I = \sum_{i=1}^{r} \left[\left(\frac{S^* (w_i^2 S_i^2)}{(k S_i^* - S^*)}\right)\right] - \frac{1}{N} \sum_{i=1}^{r} w_i S_i^2 + \left[\left(\frac{1}{r} - \frac{1}{N}\right) S^2\right] \quad (5.2)$$

Type-B Allocation: Based on prior information of variation (S_i^2) in ready queue:

The S_i^2 for i^{th} group is defined in section 4.0 as under

$$S_i^2 = \sum_{j=1}^{N_i} \frac{1}{(N_i-1)} (t_{ij} - \bar{t}_i) = \left(\frac{1}{(N_i-1)}\right) \sum_{j=1}^{N_i} (t_{ij} - \overline{mt}_i)^2$$

Consider $k_i \ \alpha \ S_i^*$ and $k_i \ \alpha \ S_i$ together where S_i refers to variability among processes in i^{th} queue related to a characteristic (e.g. expected time of process) and assumed known.

Then, $k_i \ \alpha \ S_i^* S_i$, $k_i = M^* \ S_i^* S_i$ where M is constant $\sum k_i = M^* \sum S_i^* S_i$,

$$M^* = \frac{k}{\sum S_i^* S_i} \text{ and } k_i = \left[\frac{k}{\sum S_i^* S_i}\right] S_i^* S_i \quad (5.3)$$

The variance under Type-B allocation could be obtained by substituting (5.3) in expression (4.1)

$$V(\bar{u})_{II} = \sum_{i=1}^{r} \left[\left(\frac{k S_i^* S_i - \sum S_i^* S_i}{\sum S_i^* S_i}\right) w_i^2 S_i^2\right] - \left[\frac{1}{N} \sum_{i=1}^{r} w_i S_i^2\right] + \left[\left(\frac{1}{r} - \frac{1}{N}\right) S^2\right] \quad (5.4)$$

VI. Numerical Illustration:

Consider a small data setup with 30 processes in the ready queue whose expected processing time (t_{ij}) are given in table 1. This numerical table 1 is to justify the computations, expressions, results.

Table 1: *Total Processes Data*

Total Processes Data											
Process	CPU Time	Process	CPU Time	Process	CPU Time	Process	CPU Time	Process	CPU Time	Process	CPU Time
Proc$_{1}$	30	Proc$_{6}$	60	Proc$_{11}$	138	Proc$_{16}$	89	Proc$_{21}$	143	Proc$_{26}$	79
Proc$_{2}$	20	Proc$_{7}$	33	Proc$_{12}$	43	Proc$_{17}$	123	Proc$_{22}$	29	Proc$_{27}$	46
Proc$_{3}$	142	Proc$_{8}$	43	Proc$_{13}$	109	Proc$_{18}$	67	Proc$_{23}$	147	Proc$_{28}$	59
Proc$_{4}$	40	Proc$_{9}$	101	Proc$_{14}$	26	Proc$_{19}$	58	Proc$_{24}$	94	Proc$_{29}$	72
Proc$_{5}$	59	Proc$_{10}$	69	Proc$_{11}$	138	Proc$_{16}$	89	Proc$_{21}$	143	Proc$_{26}$	79

Assume there are three processors Q_1, Q_2, Q_3 (r=3) having known processing speed S_1^*, S_2^*, S_3^* respectively. Ready queues are divided into three groups as under as in Table 2, Table 3 and 4.

Table 2: First Group Data (below 50 CPU time)

Ready Queue Group 1										
Process	Proc1	Proc2	Proc4	Proc7	Proc8	Proc12	Proc14	Proc22	Proc27	Proc30
CPUTime	30	20	40	33	43	43	26	29	46	22

Table 3: Second Group Data (above 50 but below 100 CPU time)

Ready Queue Group 2												
Process	Proc5	Proc6	Proc10	Proc15	Proc16	Proc18	Proc19	Proc20	Proc24	Proc26	Proc28	Proc29
CPUTime	59	60	69	74	89	67	58	84	94	79	59	72

Table 4: Third Group Data (above 100 CPU time)

Ready Queue Group 3								
Process	Proc3	Proc9	Proc11	Proc13	Proc17	Proc21	Proc23	Proc25
CPUTime	112	101	138	109	123	143	147	131

Table 5: Available Speed of the Processor

Processor's Speeds				
Processors	Q_1	Q_2	Q_3	Total available speed
Speed	$S_1^*=2.5$	$S_2^*=3.0$	$S_3^*=5.5$	11.0

Table 6: Parameters of all N Processes in System

Parameters of all N Processes in System			
Complete N	**Group 1 (Table 6.2)**	**Group 2 (Table 6.3)**	**Group 3 (Table 6.4)**
Mean time $\bar{t} = \frac{1}{k_i}\sum_{i=1} t_{ij}$ = 73.33	$w_1 = \frac{N_1}{N} = 0.33$	$w_2 = \frac{N_2}{N} = 0.4$	$w_3 = \frac{N_3}{N} = 0.26$
Mean square $S^2 = 1461.8484$	Mean time ($\overline{mt_1}$) = $\bar{t}_1$ = 33.20	Mean time ($\overline{mt_2}$) = $\bar{t}_2$ = 72.0	Mean time ($\overline{mt_3}$) = $\bar{t}_3$ = 125.50
	Square of mean time $(\overline{mt_1})^2 = 1102.24$	Square of mean time $(\overline{mt_2})^2 = 5184$	Square of mean time $(\overline{mt_3})^2 = 15750.25$
	Total sum of square $\sum_{j=1}^{N_i} t_{1j}{}^2 = 11804$	Total sum of square $\sum_{j=1}^{N_i} t_{2j}{}^2 = 63890$	The total sum of square $\sum_{j=1}^{N_i} t_{3j}{}^2 = 128018$
	Mean square $S_1^2 = 86.8444$ and $S_1 = 9.32$	Mean square $S_2^2 = 152.9090$ and $S_2 = 12.37$	Mean square $S_3^2 = 288$ and $S_3 = 16.97$

VII. Calculation for Arbitrary Allocation

Table 6 reveals parametric values of all three queues assuming if all N have been processed before occurrences of instant breakdown T. Parameters S_i^2, S^2, $\overline{t_1}$, $\overline{t_2}$, $\overline{t_3}$, and $\bar{t}$ have been calculated at the entire level. Moving on at the sample level, the arbitrary allocation k_1, k_2, k_3 is adopted for sample size $k = \sum k_i = 12$. In table 7, sample values $k_1 = 4$, $k_2 = 4$, $k_3 = 4$ considered for total random sample size k=12 drawn from N=30.

Variance of estimator $\bar{u}$ is $V(\bar{u})_{arbit} = V\left[\sum_{i=1}^{r} w_i \bar{t}_i' + \bar{t}^*\right] = \sum_{i=1}^{r} w_i^2 V(\bar{t}_i') + V(\bar{t}^*)$

$$= \sum_{i=1}^{r}\left(\frac{1}{(k_i-1)} - \frac{1}{N_i}\right) w_i^2 S_i^2 + \left[\left(\frac{1}{r} - \frac{1}{N}\right) S^2\right]$$

Table 7: Variances Calculation under Arbitrary Allocations (S_i^2 and S^2 known)

Variance under Arbitrary Allocation
$k_1 = 4$, $k_2 = 4$, $k_3 = 4$
$V(\bar{u})_{arbit} = 446.442$

Calculation for Type-A and Type-B allocations:

Consider following available data for variability and processor speed, both are assumed priory known. Table 8 has similar content relating to S_i^*

Table 8: Prior knowledge of Speed and Variability

Prior knowledge of Speed and Variability			
Processors	**Speed (S_i^*)**	**Variability (S_i)**	**$S_i^* S_i$**
Processor 1	$S_1^* = 2.5$	$S_1 = 9.3$	23.25
Processor 2	$S_2^* = 3.0$	$S_2 = 12.3$	36.9
Processor 3	$S_3^* = 5.5$	$S_3 = 16.9$	92.95
Total	$(S^*) = 11.0$		$\sum S_i^* S_i = 153.1$

Case 1: For Type-A allocation using (5.1), $k_i = (k/S^*) S_i^*$, $S^* = \sum S_i^*$, $k = \sum k_i$, For pre-fixed k = 12, its division in three parts is in table 9 .

Table 9: Allocation under Type -A

Allocation under Type -A			
k_1	$= (k/S^*)S_1^*$	= 2.72	= 3 (from first ready queue)
k_2	$= (k/S^*)S_2^*$	= 3.27	= 3 (from second ready queue)
k_3	$= (k/S^*)S_1^3$	= 6.0	= 6 (from third ready queue)
Total $k = (k_1+k_2+k_3)$			k = 12

Case 2: For Type-B allocation using (5.3), $k_i = \left[\frac{k}{\sum S_i^* S_i}\right](S_i^* S_i)$, and k = 12 is divided in three parts as shown in table 10.

Table 10: Allocation under Type- B

Allocation under Type-B			
k_1	=	$[k/(\sum S_i^* S_i)]$= 2.20	= 2 (from first ready queue)
k_2	=	$[k/(\sum S_i^* S_i)]$= 1.98	= 2 (from second ready queue)
k_3	=	$[k/(\sum S_i^* S_i)]$= 7.87	= 8 (from third ready queue)
Total k = $(k_1+k_2+k_3)$			k = 12

Calculation of Variance under Type-A allocation:

$$V(\bar{u})_I = \sum_{i=1}^{r} [S^* (w_i^2 S_i^2) / (kS_i^* - S^*)] - \frac{1}{N}\sum w_i S_i^2 + \left(\frac{1}{r} - \frac{1}{N}\right) S^2$$
$$= S^*\{[w_1^2S_1^2/(kS_1^* - S^*)] + [w_2^2S_2^2/(kS_2^* - S^*)] + [w_3^2S_3^2/(kS_3^* - S^*)]\} - \frac{1}{N}[w_1S_1^2 + w_2S_2^2 + w_3S_3^2]$$
$$+ \left(\frac{1}{r} - \frac{1}{N}\right) \frac{1}{N-1} [\sum_{i=1}^{r}\sum_{j=1}^{N_i}(t_{ij} - \bar{t})] \text{ when } r = 3 \quad 7.1)$$

Calculation of Variance under Type-B allocation:

$$V(\bar{u})_{II} = \sum_{i=1}^{r}[(kS_i^* S_i - \sum S_i^* S_i)/\sum S_i^* S_i] w_i^2 S_i^2 - \frac{1}{N}\sum_{i=1}^{r} w_i S_i^2 + \left[\left(\frac{1}{r} - \frac{1}{N}\right) S^2\right]$$
$$= [(kS_1^*S_1 - \sum S_1^*S_1)/\sum S_1^*S_1]\, w_1^2S_1^2 + [(kS_2^*S_2 - \sum S_2^*S_2)/\sum S_2^*S_2]\, w_2^2S_2^2 + [(kS_3^*S_3 - \sum S_3^*S_3)/\sum S_3^*S_3]$$
$$w_3^2S_3^2 - \frac{1}{N}[w_1S_1^2 + w_2S_2^2 + w_3S_3^2] + \left(\frac{1}{r} - \frac{1}{N}\right)\frac{1}{N-1} [\sum_{i=1}^{r}\sum_{j=1}^{N_i}(t_{ij} - \bar{t})] \text{ when } r = 3 \quad (7.2)$$

Table 11: Comparison of Variances under different Allocations

Comparison of Variances under different Allocations		
Variance under Type-A Allocation	**Variance under Type-B Allocation**	**Variance under Arbitrary Allocation**
k_1 =3, k_2= 3, k_3= 6	k_1 =2, k_2= 2, k_3= 8	k_1 =4, k_2= 4, k_3= 4
$V(\bar{u})_I$ = 442.08	$V(\bar{u})_{II}$ = 611.452	$V(\bar{u})_{arbit}$= 446.442

Table 8 contains the assumption that three S_i^2 (i = 1, 2, 3) are priory known (or guessed) and so the variance $V(\bar{u})_I$ is lowest under the type-A allocation (while S_i^2 and S^2 known) in comparison to Type-B and Arbitrary allocation.

Estimate of Variance :

The value $S_i^2 = \left(\frac{1}{(N_i-1)}\right)\sum_{j=1}^{N_i}(t_{ij} - \bar{t}_i)^2$ suppose not known then they are to be replaced by sample value estimates. The sample based estimate of S^2 and S_i^2 are defined like $(es)^2$ and $(es_i)^2$ with expressions are as under:

$$(es_i)^2 = \left(\frac{1}{(k_i-1)}\right)\sum_{j=1}^{k_i-1}(t_{ij} - \bar{t}_i) \quad \text{and } (es)^2 = \left(\frac{1}{[(k-r)-1]}\right)\sum_{i=1}^{r}\sum_{j=1}^{[k-r-1]}(t_{ij} - \bar{t}_i)^2 \quad (7.3.1)$$

$$Est[V(\bar{u})_{arbit}] = \sum_{i=1}^{r}\left(\frac{1}{(k_i-1)} - \frac{1}{N_i}\right) w_i^2 (es_i)^2 + [(\frac{1}{r} - \frac{1}{N})(es)^2] \quad (7.3.2)$$

$$Est[V(\bar{u})_I] = \sum_{i=1}^{r}[S^* (w_i^2 (es)^2) / (kS_i^* - S^*)] - \frac{1}{N}\sum w_i (es_i)^2 + \left(\frac{1}{r} - \frac{1}{N}\right)(es)^2 \quad (7.3.3)$$

$$Est[V(\bar{u})_{II}] = [(\sum_{i=1}^{r}[k S_i^* (es_i) - \sum S_i^* (es_i))/\sum S_i^* (es_i)] w_i^2 (es_i)^2 - \frac{1}{N}\sum_{i=1}^{r} w_i (es_i)^2 +$$
$$\left[\left(\frac{1}{r} - \frac{1}{N}\right)(es)^2\right] \quad (7.3.4)$$

Calculations of estimated values are in table 7.6 and 7.7 on the 10 samples.

Table 12: Calculations of Sample Mean and Estimate of Variance under Arbitrary Allocation (Section 4.0) in 10 samples (when S_i^2 and S^2 unknown)

(*Partially processed job containing a part of the processing time and unprocessed due time)

Calculations of Sample Mean and Estimate of Variance under Arbitrary Allocation							
Random Sample No.	Sampled Selected with Processing Time (k=9) Group1 $K_1=4$	Group2 $K_2=4$	Group3 $K_3=4$	Processed $\sum w_i \bar{t}_{i'}$	Unprocessed $(t_1^*+t_2^*+t_3^*)/3$ $es^2=\frac{1}{(r-1)}\sum_{i=1}^{r}(t_i^* - \bar{t}^*)^2$	Sample Mean $(\bar{u})$	$V(\bar{u})_{arbit}$
1.	30,43,33,30* Mean=35.33 $t_1^*=25$ $(es_1)^2=46.33$	60,84,67,59* Mean=70.33 $t_2^*=39$ $(es_2)^2=152.33$	138,112,109,101* Mean=119.6 $t_3^*=61$ $(es_3)^2=254.33$	70.88	41.6 $(es)^2=37.66$	56.24	112.478
2.	33,46,40,20* Mean=39.6 $t_1^*=15$ $(es_1)^2=50.26$	69,58,59,60* Mean=62 $t_2^*=35$ $(es_2)^2=37$	109,101,112,143* Mean=107.33 $t_3^*=88S$ $(es_3)^2=32.33$	65.77	46 $(es)^2=1423$	55.88	430.07
3.	20,46,30,40* Mean=32 $t_1^*=25$ $(es_1)^2=172$	59,72,79,69* Mean=70 $t_2^*=39$ $(es_2)^2=103$	147,138,101,123* Mean=128.6 $t_3^*=56$ $(es_3)^2=594.33$	71.99	40 $(es)^2=241$	55.99	86.66
4.	40,22,26,33* Mean=29.33 $t_1^*=23$ $(es_1)^2=89.33$	74,84,60,58* Mean=72.66 $t_2^*=29$ $(es_2)^2=146.79$	131,109,123,112* Mean=121 $t_3^*=67$ $(es_3)^2=124$	70.20	39.77 $(es)^2=557$	54.98	176.44
5.	43,29,30,20* Mean=34 $t_1^*=15$ $(es_1)^2=61$	79,67,58,60* Mean=68 $t_2^*=35$ $(es_2)^2=111$	123,143,112,101* Mean= 126 $t_3^*=65$ $(es_3)^2=247$	71.18	38.33 $(es)^2=634$	54.75	198.63
6.	20,22,29,43* Mean=23.66 $t_1^*=28$ $(es_1)^2=22.80$	59,72,84,67* Mean=71.66 $t_2^*=47$ $(es_2)^2=156.33$	101,109,123,131* Mean= 111 $t_3^*=81$ $(es_3)^2=124$	65.33	52 $(es)^2=721$	58.66	224.36
7.	30,29,20,26* Mean=26.33 $t_1^*=19$ $(es_1)^2=30.33$	59,69,72,58* Mean=66.66 $t_2^*=38$ $(es_2)^2=46.33$	101,147,109,112* Mean=119 $t_3^*=66$ $(es_3)^2=604$	66.29	41 $(es)^2=559$	53.64	176.34
8.	30,26,33,29* Mean=29.66 $t_1^*=24$ $(es_1)^2=12.33$	72,58,74,60* Mean=68 $t_2^*=44$ $(es_2)^2=76$	112,131,101,123* Mean=114.66 $t_3^*=68$ $(es_3)^2=230.33$	66.79	45.33 $(es)^2=486$	56.06	151.44
9.	40,29,30,46* Mean=33 $t_1^*=26$ $(es_1)^2=37$	60,58,67,79* Mean=61.66 $t_2^*=49$ $(es_2)^2=23.57$	109,112,131,101* Mean= 117.33 $t_3^*=79$ $(es_3)^2=142.33$	66.05	51.33 $(es)^2=707$	58.69	215.38
10.	20,43,40,22* Mean=34.33 $t_1^*=16$ $(es_1)^2=156.5$	79,58,60,59* Mean=65.66 $t_2^*=34$ $(es_2)^2=134.33$	123,101,112,143* Mean= 112 $t_3^*=73$ $(es_3)^2=121$	66.71	41 $(es)^2=849$	53.85	265.19

Table 13: Estimated values of Variances over 10 samples as per table 6.7 (when S_i^2 and S^2 are unknown)

Sample Number	1	2	3	4	5	6	7	8	9	10
Sample Mean ($\bar{u}$)	56.24	55.88	55.99	54.98	54.75	58.66	53.64	56.06	58.69	53.85
Est[$V(\bar{u})_{arbit}$]	112.478	430.07	86.66	176.44	198.63	224.36	176.34	151.44	215.38	265.19
Est[$V(\bar{u})_I$]	113.65	431.86	90.26	180.95	201.02	227.11	175.22	151.93	216.11	271.09
Est[$V(\bar{u})_{II}$]	242.29	453.07	333.11	261.55	317.58	308.78	405.65	253.46	273.94	349.22

Calculation of Confidence Interval (CI):

A. The 95% Confidence Interval of the sample mean $\bar{\mathbf{u}}$ is defined as:
Probability [$(\bar{\mathbf{u}}) \pm \mathbf{1.96}\ \sqrt{\mathbf{v}(\bar{\mathbf{u}})}$] = 0.95. The interpretation of C.I. is that it is an interval where the chance of laying the unknown true value of mean time is 95%.

B. In another way, the 95% chance is that unknown mean processing time of all N processes will lie in the confidence interval.

C. Table 8, 9, and 10 present the computation of confidence intervals for different types of allocations. When S_i^2, S^2 treated unknown.

Table 14: Confidence Interval Calculation under Arbitrary Allocation [using Table 6 and 7]

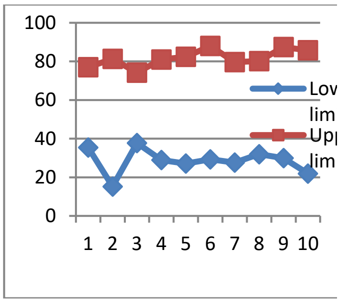
Sample Number	1	2	3	4	5	6	7	8	9	10
Sample Mean ($\bar{u}$)	56.24	55.88	55.99	54.98	54.75	58.66	53.64	56.06	58.69	53.85
Est.[$V(\bar{u})_{arbit}$]	112.478	430.07	86.66	176.44	198.63	224.36	176.34	151.44	215.38	265.19
Estimate of Confidence Interval for Est[$V(\bar{u})_{arbit}$]	(35.45, 77.02)	(15.23, 81.28)	(37.74, 74.23)	(28.94, 81.01)	(27.12, 82.37)	(29.30, 88.01)	(27.61, 79.66)	(31.94, 80.17)	(29.92, 87.45)	(21.93, 85.76)

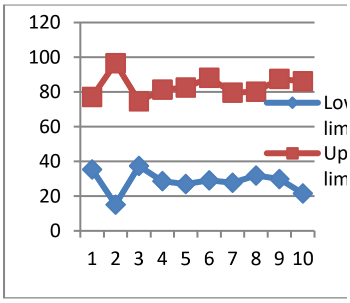
Table 15: Confidence Interval Calculation for Type-A Allocation [using Table 9 and 10]

Sample Number	1	2	3	4	5	6	7	8	9	10
Sample Mean ($\bar{u}$)	56.24	55.88	55.99	54.98	54.75	58.66	53.64	56.06	58.69	53.85
Est.$V(\bar{u})_I$	113.65	431.86	90.26	180.95	201.02	227.11	175.22	151.93	216.11	271.09
Estimate of Confidence Interval for Est[$V(\bar{u})_I$]	(35.34, 77.13)	(15.14, 96.61)	(37.36, 74.61)	(28.61, 81.34)	(26.96, 82.53)	(29.12, 88.19)	(27.69, 79.58)	(31.90, 80.21)	(29.87, 87.5)	(21.57, 86.12)

Table 16: Confidence Interval Calculation for Type-B Allocation [using Table 11 and 12]

Sample Number	1	2	3	4	5	6	7	8	9	10
Sample Mean ($\bar{u}$)	56.24	55.88	55.99	54.98	54.75	58.66	53.64	56.06	58.69	53.85
Est.[$V(\bar{u})_{II}$]	242.29	453.07	333.11	261.55	317.58	308.78	405.65	253.46	273.94	349.22
Estimate of Confidence Interval for Est[$V(\bar{u})_{II}$]	(25.73, 86.74)	(14.16, 97.59)	(20.21, 91.76)	(23.28, 86.67)	(19.82, 89.67)	(24.21, 93.1)	(14.16, 93.11)	(24.85, 87.26)	(26.24, 91.13)	(17.22, 90.47)





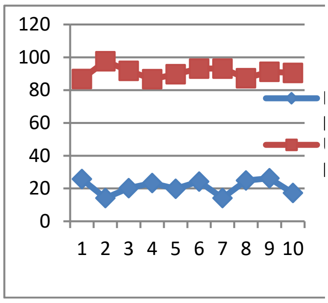


Fig. 2: Fig. 3: & Fig 4: Graphical Representation of Estimated CI under Arbitrary, Type-A and Type-B Allocation over 10 samples

The graphical representation in Fig. 2, 3, 4 shows wide gap between the upper and lower limit. The Fig 2 shows the smallest length interval.

8.1 Simulation of Confidence Interval under Arbitrary Allocation:

8.1.1 Simulation Algorithm:

Step I: Draw a random sample of size k.

Step II: Compute the lower limit and upper limit of confidence interval (CI) under three allocations.

Step III: Repeat step I and II for d times (here d =200 considered)

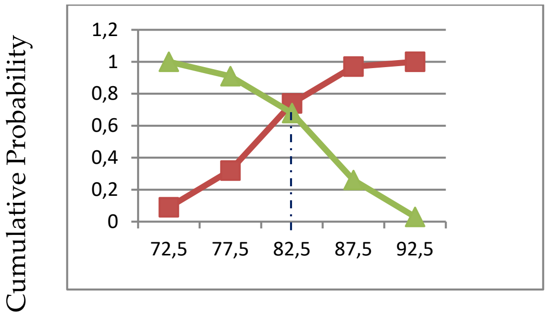
Step IV: Let f_i be the frequency of i^{th} class interval for lower limit (LL) of CI over d=200 samples. Calculate probabilities pi = (f_i/d) = (frequency of class interval /Total frequency d). Similar is for upper limit (UL) CI.

Step V: Compute the Less than Type (LTT) and more than Type (MTT) cumulative probabilities overall d samples for lower limit (LL) and upper limit (UL) of confidence intervals.

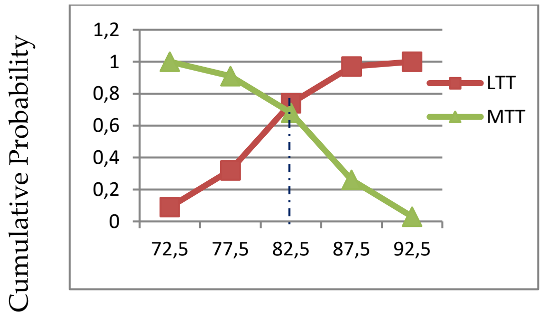
Step VI: Plot data of step IV on the graph. The perpendicular from point of intersection on the x-axis is the simulated value of lower limit and upper limit of a confidence interval for unknown parameters required to be estimated.

Table 17: Cumulative Probability-based Simulation for Arbitrary Allocation (over d=200)

The lower limit of Confidence Interval					**The upper limit of Confidence Interval**				
Class Interval (LL)	**Mid-value of class interval**	**Probability P_i**	**Cumulative probabilities LTT**	**MTT**	**Class Interval (UL)**	**Mid-value of class interval**	**Probability P_i**	**Cumulative probabilities LTT**	**MTT**
10-15	12.5	0.01	0.01	1	70-75	72.5	0.09	0.09	1
15-20	17.5	0.12	0.13	0.99	75-80	77.5	0.23	0.32	0.91
20-25	22.5	0.15	0.28	0.87	80-85	82.5	0.42	0.74	0.68
25-30	27.5	0.43	0.71	0.72	85-90	87.5	0.23	0.97	0.26
30-35	32.5	0.18	0.89	0.29	90-95	92.5	0.03	1.00	0.03
35-40	37.5	0.10	0.99	0.01	Total		1.00		
40-45	42.5	0.01	1.00	0					
Total		1.00							



Class Interval (using Table 8.4 Lower Limit)



Class Interval (using Table 8.4 Upper Limit)

Fig 5: & Fig 6: Graphical representation for LTT & MTT for Arbitrary Allocation

Table 18: Simulated values of C I under Arbitrary Allocation (using Table 12, Fig 5 & Fig. 6)

Simulated values of Lower Limit of C I	Simulated values of Upper Limit of C I
24.5	79.5

Fig. 5.and Fig. 6 is revealing point of intersection of two curves. The final value is determined by perpendicular drawn on the X-axis. The table 18 contains the estimated value, based on perpendicular, which is (24.5, 79.5).

Simulation of Confidence Interval under Type-A Allocation:

Table 19 Sample mean and variance calculation for Type-A allocation (over 10 samples)

Sample Number	Sampled Selected with Processing Time (k=9) Group1 $K_1=(3)$	Group2 $K_2=(3)$	Group3 $K_3=(6)$	Processed $\sum w_i \bar{t}_{i'}$	Unprocessed $(t_1^*+t_2^*+t_3^*)/3$ $es^2=\frac{1}{(r-1)}\sum_{i=1}^{r}(t_i^*-\bar{t}^*)^2$	Sample Mean $(\bar{u})$	$V(\bar{u})_I$
1.	30,43,33* Mean=36.5 t_1^*=25 $(es_1)^2$= 42.25	60,84,67* Mean=72 t_2^*=37 $(es_2)^2$= 144	138,112,109, 101,143,123* Mean=120.6 t_3^*=83 $(es_3)^2$= 279.44	72.19	48.33 $(es)^2$=937.8	60.26	293.31
2.	33,46,40* Mean=39.5 t_1^*=20 $(es_1)^2$= 42.25	69,58,59* Mean=63.5 t_2^*=34 $(es_2)^2$= 30.25	109,101,112, 143,147,131* Mean=122.4 t_3^*=81 $(es_3)^2$= 355.04	70.25	45 $(es)^2$= 1021	57.62	312.19
3.	20,46,30* Mean=33 t_1^*=20 $(es_1)^2$= 169	59,72,79* Mean=65.5 t_2^*=49 $(es_2)^2$= 42.25	147,138,101, 123,112,109* Mean=124.2 t_3^*=59 $(es_3)^2$=279.76	68.91	42.66 $(es)^2$= 274.12	55.78	94.93
4.	40,22,26* Mean=31 t_1^* =20 $(es_1)^2$=81	74,84,60* Mean=79 t_2^*=31 $(es_2)^2$= 25	131,109,123, 112,101,143* Mean=115.2 t_3^*=100 $(es_3)^2$=112.19	71.78	50.33 $(es)^2$=1880.83	61.05	570.43
5.	43,29,30* Mean=39 t_1^*= 15 $(es_1)^2$ = 176	79,67,58* Mean=73 t_2^*=35 $(es_2)^2$= 36	123,143,112, 101,109,147* Mean=117.6 t_3^*=75 $(es_3)^2$=211.04	72.64	41.66 $(es)^2$=934.16	57.15	292.54
6.	20,22,29* Mean=21	59,72,84* Mean=65.5	101,109,123, 131,143,112*	64.69	52 $(es)^2$=964	58.34	356.33

	$t_1^*=20$ $(es_1)^2= 1$	$t_2^*=54$ $(es_2)^2=$ 42.25	Mean121.4 $t_3^*=82$ $(es_3)^2=226.24$				
7.	30,29,20* Mean=29.5 $t_1^*=25$ $(es_1)^2= 0.25$	59,69,72* Mean=64 $t_2^*=42$ $(es_2)^2= 25$	101,147,109, 112,138,123* Mean=121.4 $t_3^*=73$ $(es_3)^2=317.84$	66.89	46.66 $(es)^2=593.26$	56.77	192.63
8.	30,26,33* Mean=28 $t_1^* =22$ $(es_1)^2= 4$	72,58,74* Mean=65 $t_2^*=50$ $(es_2)^2= 49$	112,131,101, 123,109,131* Mean=115.2 $t_3^*=90$ $(es_3)^2=112.16$	65.19	54 $(es)^2=1168$	59.59	353.95
9.	40,29,30* Mean=34.5 $t_1^*=21$ $(es_1)^2=$ 30.25	60,58,67* Mean=59 $t_2^*= 47$ $(es_2)^2= 1$	109,112,131, 123,143,101* Mean=123.6 $t_3^*=79$ $(es_3)^2=155.84$	67.11	49 $(es)^2= 844$	58.05	255.55
10.	20,43,40* Mean=31.5 $t_1^*=30$ $(es_1)^2=$ 132.25	79,58,60* Mean=68.5 $t_2^*=35$ $(es_2)^2 =$ 110.25	123,101,112, 143,147,138* Mean=125.2 $t_3^*=78$ $(es_3)^2 = 311.36$	66.12	47.66 $(es)^2=697.28$	56.89	223.97

Table 20: Confidence Interval for Type-A Allocation (using Table 19)

Confidence Interval for Type-A Allocation										
Sample Number	1	2	3	4	5	6	7	8	9	10
Sample Mean ($\bar{u}$)	60.26	57.62	55.78	61.05	57.15	58.34	56.77	59.59	58.05	56.89
Est.[V($\bar{u}$)$_I$]	293.31	312.19	94.93	570.43	292.54	356.33	192.63	353.95	255.55	223.97
Estimate of confidence interval for Est[V($\bar{u}$)$_I$]	(26.69, 93.82)	(22.98, 92.25)	(36.68, 74.87)	(14.23, 106.61)	(23.62, 90.67)	(21.34, 95.33)	(29.56, 83.97)	(22.71, 96.46)	(26.71, 89.38)	(27.55, 86.22)

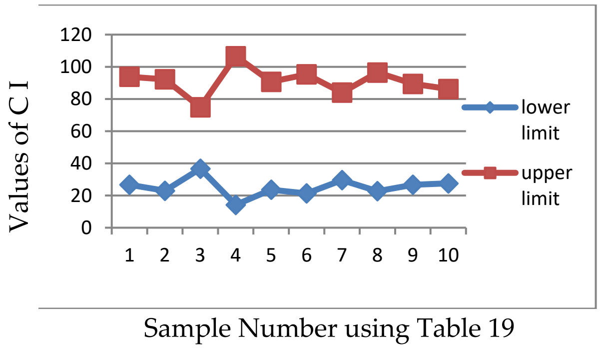
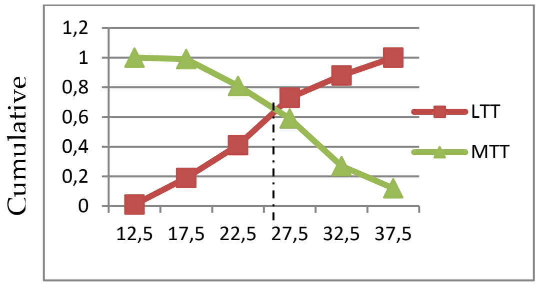


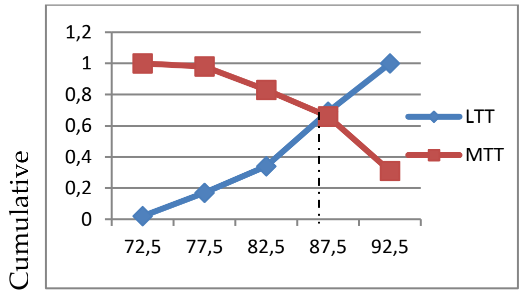
Fig 7: Graphical Representation of Confidence Interval for Type-A Allocation

Table 21: Cumulative Probabilities Simulation for Type-A Allocation (over d=200)

The lower limit of Confidence Interval					The upper limit of Confidence Interval				
Class Interval (LL)	Mid-value of class interval	Probability P_i	Cumulative probabilities LTT	MTT	Class Interval (UL)	Mid-value of class interval	Probability P_i	Cumulative Probabilities LTT	MTT
10-15	12.5	0.01	0.01	1	70-75	72.5	0.02	0.02	1
15-20	17.5	0.18	0.19	0.99	75-80	77.5	0.15	0.17	0.98
20-25	22.5	0.22	0.41	0.81	80-85	82.5	0.17	0.34	0.83
25-30	27.5	0.32	0.73	0.59	85-90	87.5	0.35	0.69	0.66
30-35	32.5	0.15	0.88	0.27	90-95	92.5	0.31	1.00	0.31
35-40	37.5	0.12	1.00	0.12	Total		1.00		
Total		1.0							



Class Interval (using Table 21 Lower Limit)



Class Interval (using Table 21 Upper Limit)

Fig 8: & Fig 9: Graphical representation for Lower limit & Upper limit for Type-A allocation

Table 22: Simulated values of CI under Type-A Allocation (using Table 9, Fig 8 & Fig. 9)

Simulated values of Lower Limit of C I	Simulated values of Upper Limit of C I
23.5	83.5

Fig. 8 and Fig. 9 are revealing point of intersection of two curves. The final value is determined by perpendicular drawn on the X-axis. Table 22 contains the estimated value, based on the perpendicular, which is (23.5, 83.5).

Simulation of Confidence Interval for Type-B Allocation:

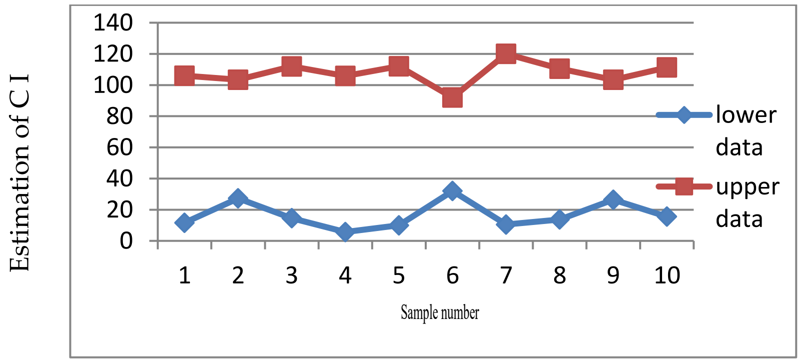
Table 23: Sample Mean and Variance Calculation for Type-B Allocation (over 10 samples)

Random sample	Sampled Selected with Processing Time (k=9) Group1 $K_1=(2)$	Group2 $K_2=(2)$	Group3 $K_3=(8)$	Processed $\sum w_i \bar{t}_{i'}$	Unprocessed $(t_1^*+t_2^*+t_3^*)/3$ $es^2=\frac{1}{(r-1)}\sum_{i=1}^{r}(t_i^*-\bar{t}^*)^2$	Sample Mean $(\bar{u})$	$V(\bar{u})_{II}$
1.	30,20* Mean=30 $t_1^*=20$ $(es_1)^2=30$	59,60* Mean=59 $t_2^*=60$ $(es_2)^2=59$	123,101,112,143, 147,138,109,131* Mean=124.71 $t_3^*=131$ $(es_3)^2=331.48$	65.92	51.66 $(es)^2=1909.36$	58.79	579.42
2.	40,33* Mean=40 $t_1^*=33$ $(es_1)^2=40$	69,74* Mean=69 $t_2^*=74$ $(es_2)^2=69$	123,101,112,143, 147,138,131,109* Mean=127.85 $t_3^*=109$ $(es_3)^2=286.27$	74.04	56.66 $(es)^2=1234.46$	65.35	377.46
3.	43,20* Mean=43 $t_1^*=20$ $(es_1)^2=43$	67,58* Mean=67 $t_2^*=58$ $(es_2)^2=67$	123,101,112,143, 147,109,131,138* Mean=123.71 $t_3^*=138$ $(es_3)^2=306.23$	73.15	53.33 $(es)^2=2033.86$	63.24	617.62
4.	40,29* Mean=40 $t_1^*=29$ $(es_1)^2=40$	33,58* Mean=33 $t_2^*=58$ $(es_2)^2=33$	123,101,112,143, 138,109,131,147* Mean=122.42 $t_3^*=147$ $(es_3)^2=247.95$	58.22	53.33 $(es)^2=2158.86$	55.77	652.91
5.	46,22* Mean=46 $t_1^*=22$ $(es_1)^2=46$	58,59* Mean=58 $t_2^*=59$ $(es_2)^2=58$	123,101,112,147, 138,109,131,143* Mean= 123 $t_3^*=143$ $(es_3)^2=277.66$	70.36	51.66 $(es)^2=2234.36$	61.01	677.44
6.	30,40* Mean=30 $t_1^*=40$ $(es_1)^2=30$	59,72* Mean=59 $t_2^*=72$ $(es_2)^2=59$	101,143,147,138, 109,131,143,112* Mean=130.28 $t_3^*=112$ $(es_3)^2=328.90$	67.37	56.66 $(es)^2=759.46$	62.01	234.36

7.	$43,26^*$ Mean=43 $t_1^*=26$ $(es_1)^2=43$	$59,69^*$ Mean=59 $t_2^*=69$ $(es_2)^2=59$	112,143,147,138, $109,131,101,123^*$ Mean= 125.85 $t_3^*=123$ $(es_3)^2= 336.90$	70.51	60 $(es)^2=2575$	65.25	779.92
8.	$26,30^*$ Mean=26 $t_1^*=30$ $(es_1)^2=26$	$69,58^*$ Mean=69 $t_2^*=58$ $(es_2)^2=69$	123,101,112,143, $147,138,109,131^*$ Mean=124.71 $t_3^*=131$ $(es_3)^2= 331.48$	68.60	55 $(es)^2=1975$	61.8	601.96
9.	$22,29^*$ Mean=22 $t_1^*=29$ $(es_1)^2=22$	$94,59^*$ Mean=94 $t_2^*=59$ $(es_2)^2=94$	123,101,112,143, $147,138, 131,109^*$ Mean= 127.85 $t_3^*=109$ $(es_3)^2=286.27$	78.10	51.66 $(es)^2=1259.36$	64.88	385.75
10.	$20,33^*$ Mean=20 $t_1^*=33$ $(es_1)^2=20$	$59,79^*$ Mean=59 $t_2^*=79$ $(es_2)^2=59$	123,101,112,143, $147,109,131,138^*$ Mean=123.71 $t_3^*=138$ $(es_3)^2= 307.47$	62.36	64 $(es)^2=1948$	63.18	590.45

Table 24: Confidence Interval for Type-B Allocation (using Table 10.1)

Confidence Interval for Type-B Allocation										
Random sample	1	2	3	4	5	6	7	8	9	10
Sample Mean ($\bar{u}$)	58.79	65.35	63.24	55.77	61.01	62.01	65.25	61.8	64.88	63.18
Est.[$V(\bar{u})_{II}$]	579.42	377.46	617.62	652.91	677.44	234.36	779.92	601.96	385.75	590.45
Estimate of confidence interval for Est.[$V(\bar{u})_{II}$]	(11.61, 105.96)	(27.27, 103.42)	(14.53, 111.94)	(5.68,105.85)	(9.99, 112.02)	(32.00, 92.01)	(10.51, 119.98)	(13.71, 110.5)	(26.38, 103.37)	(15.55, 111.26)

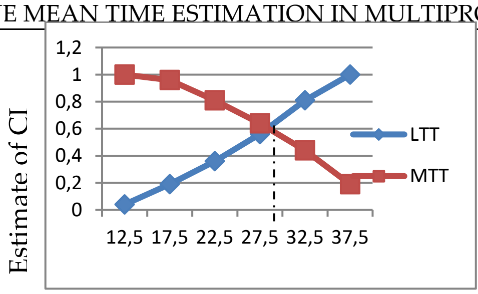


Sample Number using Table 10.1

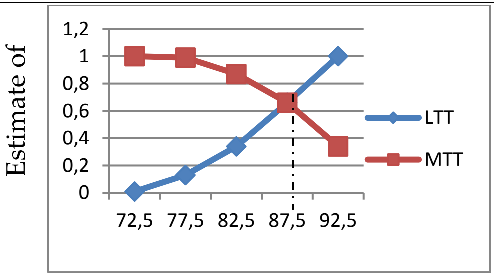
Fig 10: Graphical Representation for Type-B Allocation

Table 25: Cumulative Probabilities Simulation for Type-B Allocation (over d=200)

The lower limit of the confidence interval					The upper limit of the confidence interval				
Class Interval (LL)	Mid-value of class interval	Probability P_i	Cumulative probabilities LTT	MTT	Class Interval (UL)	Mid-value of class interval	Probability P_i	Cumulative probabilities LTT	MTT
10-15	12.5	0.04	0.04	1	70-75	72.5	0.01	0.01	1
10-15	17.5	0.15	0.19	0.96	75-80	77.5	0.12	0.13	0.99
15-20	22.5	0.17	0.36	0.81	80-85	82.5	0.21	0.34	0.87
20-25	27.5	0.20	0.56	0.64	85-90	87.5	0.32	0.66	0.66
25-30	32.5	0.25	0.81	0.44	90-95	92.5	0.34	1.00	0.34
30-35	37.5	0.19	1.00	0.19	Total		1.00		
Total		1							



Class Interval (using Table 10.3 Lower Limit)



Class Interval (using Table 10.3 Upper Limit)

Fig 11: & Fig 12: Graphical representation for Lower limit & Upper limit Type-B allocation Confidence Interval

Table 26: Simulated values of CI under Type-B Allocation

Simulated values of Lower Limit of C I	Simulated values of Upper Limit of C I
25.5	84.5

Fig. 11 and Fig. 12 are revealing point of intersection of two curves. Final value is determined by perpendicular drawn on the X-axis. Table 26 contains the estimated value, based on the perpendicular, which is (25.5, 84.5).

11. Results, Discussion and Conclusion:

The comparative analysis is stated in table 27

Table 27: Comparative Analysis of Variance and Confidence Interval Range

Strategy	True Value of Mean	Variance of Mean	95% Confidence Interval CI
Arbitrary allocation	73.33	450.92	[24.5, 79.5]
Type-A allocation	73.33	442.08	[23.5, 83.5]
Type-B allocation	73.33	611.452	[25.5, 84.5]

Algorithm MGLS considers a possibility that some processes remain unprocessed while time instant T occurs which was not considered in GL scheduling [5]. As a consequence, the processes in a sample drawn are divided into two parts A and B. The part A incorporates those who processed and part B has partially processed at the breakdown instant T.

Specific assumption herein is that the last process remains unfinished while T appears in every processor. Estimation procedure proposed herein is such as from whole population of jobs in system, some processes are randomly selected and using the sample estimates mean time and variance of the mean time of processed jobs, as well as the variance of partially processed jobs. The estimation procedure is categorized for arbitrary allocation of sample units to processors.

Further, content has two special cases Type-A allocation and Type-B allocation. The Type-A allocation is based on available prior information of processor speed and Type-B allocation is based on available prior information of variability along with processor speed. In all types of allocations, attempt has been made to find out which allocation will provide the lowest variance (efficient).

For the sake of convenience and simplicity, 30 processes present in system have been considered where groups of ready queues are formed. In particular, three groups Group 1, Group 2, and

Group 3 are formed having some processes according to pre-determined CPU time. Table 5 shows the pre-defined speed of processors. For the arbitrary allocation of sampled processes, the sample mean and variance are calculated with the setup shown in table 12 and subsequently in table 19 and table 23. For the special cases, the processor speed and variability of processors is considered. The variance of the Type-A and Type-B allocation is calculated and compared. This can be seen in Table 4. Table 5 which reveal the comparison between them relating to variance of allocations.

The simulation procedure is proposed and the confidence intervals Prob.[$(\bar{u}) \pm 1.96\sqrt{V(\bar{u})}$] are calculated and represented in graphical form. Over a large number of samples, the confidence interval of Type-A and Type-B allocation are calculated and displayed in graphical representation. For obtaining a single-valued result, it has been introduced the calculation of cumulative probabilities and the LTT and MTT probabilities of lower and upper limits of the confidence interval are measured. Observing all the calculated data and the final table, one can conclude that the Type-A allocation is an efficient scheme to find out the predictive estimate and it is the best one among all who tested.

It was found that estimation of mean times lies within the length of the confidence interval. The improvement suggests over [5] is fruitful and provides better results. The sample-based procedure of estimation of the mean time is more efficient under the Type-A allocation scheme. Such estimates are useful when the system fails suddenly and the system manager needs time estimation for processing the remaining jobs in the queue. This approach helps in the immediate arrangement of resources while disaster management required.

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Received: August 27, 2020
Accepted: November 15, 2020

EM Algorithm for Estimating the Burr XII Parameters in Partially Accelerated Life Tests

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Abstract

In this paper, I present maximum likelihood estimation via the expectation-maximization algorithm to estimate the Burr XII parameters and acceleration factor in step-stress partially accelerated life tests under multiple censored data. In addition, the asymptotic variance and covariance matrix of the estimators are derived by using the complete and missing information matrices, and confidence intervals of the parameters are obtained. The simulation results show that the maximum likelihood estimation via the expectation-maximization algorithm performs well in most cases in terms of the absolute relative bias, the root mean square error, and the coverage rate. Furthermore, a numerical example is also given to demonstrate the performance of the proposed method.

Keywords: partially accelerated life test, acceleration factor, Burr XII distribution, maximum likelihood estimation, EM algorithm

I. Introduction

Generally, life testing of products under normal conditions usually requires a long period of time. Long-term testing will increase the test cost and will take a lot of time. Accelerated life test (ALT) is one of the solutions that can avoid above problems. ALT has been successfully applied to obtain information about product life quickly and economically under more severe operating conditions. Stress conditions, such as, cycling rate, load, voltage, pressure, vibration, and temperature are the most common methods in practice. The acceleration factor in ALT is usually assumed to be a known value. On the contrary, the acceleration factor in partial accelerated life testing (PALT) is usually assumed as an unknown value. Constant stress, step stress and progressive stress are three major stress types of PALT. Progressive stress is a more complicated PALT approach among these major stress types. In a constant-stress test, test units are run at some unchanged constant level of stress. In a step-stress test, the level of stress can be changed at a specified time, and this kind of test method is called step-stress partially accelerated life test (SS-PALT).

The Burr XII distribution is widely applied in reliability engineering because of its many advantages. Rodriguez (1977) showed that the area in the ($\sqrt{\beta_1}, \beta_2$) plane corresponding to the Burr XII distribution is wide and it covers various well-known distributions. Zimmer et al. (1998) presented the statistical and probabilistic properties of the Burr XII distribution, and described its connection with other distributions used in reliability analysis. The Burr XII distribution has been applied in reliability analysis widely. Wingo (1993) formatted the MLE to fit the Burr XII distribution through the use of multiple censored data. Ali Mousa (1995) estimated the parameters of the Burr XII distribution with Type II censored data for an ALT model by using the Bayes method. Wang et al. (1996) presented the MLE for obtaining point and interval estimates of the Burr XII parameters. Watkins (1999) developed an algorithm for calculating the MLE of the three-parameter Burr XII distribution. As to the parameter estimation of the Burr XII distribution in SS-PALT, Abd-Elfattah et al. (2008) investigated the maximum likelihood method for the parameters of the Burr XII distribution in SS-PALT under type I censored data. Abdel-Ghaly et al. (2008) considered the estimation problem of the Burr-XII distribution in SS-PALT using censored data. Abdel-Hamid (2009) estimated the parameters of the Burr XII distribution with progressive Type II censoring for a CS-PALT model by using the MLE method. Cheng and Wang (2012) compared the performance of the maximum likelihood estimates of the Burr XII parameters for CS-PALT. So, it has been shown that the Burr XII distribution is a flexible model and is recommended for modeling in the reliability analysis and ALTs.

The MLE via the Newton-Raphson algorithm is very sensitive to its initial parameter estimation value. Other options can be adopted to avoid the above problem, for example, the expectation-maximization (EM) algorithm. EM algorithm is an iterative algorithm approach applied in a variety of incomplete data problems (Dempster et al., 1977). EM algorithm can be used in data sets with missing values, censored and grouped observations, or models with truncated distributions. EM algorithm involves two steps, the E-step and the M-step. In the E-step, the expected values of the complete data sufficient statistics are computed. In the M-step, parameter estimates that maximize the complete data likelihood are solved by using the conditional expected value that computed in the E-step. Both steps of the iterations are repeated until the parameter estimates converge. The development and application of EM algorithms are getting more and more mature. Louis (1982) derived a procedure for extracting the observed information matrix when EM algorithm is used to find maximum likelihood estimates in incomplete data problems. In reliability analysis, EM algorithm has been commonly used. Ng et al. (2002) presented the MLE via EM algorithm to estimate the lognormal and the Weibull parameters with progressively type II censored data. Acusta et al. (2002) proposed an estimator of the probability density function when the data is randomly censored, obtained through an EM algorithm, for solving a maximum likelihood problem. Balakrishnan and Kim (2004) used EM algorithm to find the maximum likelihood estimates under type II right censored samples from a bivariate normal distribution. Park (2005) presented the MLE via EM algorithm to estimate the exponential and lognormal parameters with complex data including: fully-observed, censored, and partially-masked. Cheng and Wang (2012) presented the performance of the maximum likelihood estimates of the Burr XII parameters for CS-PALT by using EM algorithm.

In this paper, I present the performance of the maximum likelihood estimates via EM algorithm for the Burr XII parameters in SS-PALT under multiple censored data in terms of the absolute relative bias, the root mean square error, and the coverage rate. The asymptotic variance and covariance matrix of the estimators are also derived. Then, the confidence intervals of the parameters can be obtained. In addition, an illustrative example is used to demonstrate the proposed method.

II. Model in step-stress PALT under multiple censored data

The probability density function and cumulative distribution function of the two-parameter Burr XII distribution are given by

$$f\left(t;c,k\right)=\frac{kct^{c-1}}{\left(1+t^{c}\right)^{k+1}} \qquad , \quad t>0,\ \ c>0,\ \ k>0 \tag{1}$$

$$F\left(t;c,k\right)=1-\frac{1}{\left(1+t^{c}\right)^{k}} \quad , \ t>0, \ c>0, \ k>0 \tag{2}$$

where the parameters *c* and *k* are the shape parameters of the distribution.

In SS-PALT, the test unit is first run at normal condition and if the unit does not fail or be censored before the specified time, τ , the test is switched to a stress condition for testing until the unit fails or be censored. Then, the total lifetime *X* of the unit in SS-PALT is given by

$$X=\begin{cases} T, & T\le\tau \\ \tau+\beta^{-1}\left(T-\tau\right), & T>\tau \end{cases} \tag{3}$$

where *T* is the lifetime of an unit at normal condition, τ is the stress change time and β is the acceleration factor ($\beta>1$). I assume that the lifetime of the test unit follows a two-parameter Burr XII distribution. Therefore, the CDF and PDF of total lifetime *X* of an item are given by

$$F(x;c,k,\beta)=\begin{cases} 0, & x<0 \\ 1-\dfrac{1}{(1+x^{c})^{k}}, & 0<x\le\tau \\ 1-\dfrac{1}{\left\{1+\left[\tau+\beta\left(x-\tau\right)\right]^{c}\right\}^{k}}, & x>\tau \end{cases} \tag{4}$$

where $c>0$, $k>0$, $\beta>1$, and

$$f(x;c,k,\beta)=\begin{cases} 0, & x<0 \\ \dfrac{kcx^{c-1}}{(1+x^{c})^{k+1}}, & 0<x\le\tau \\ \dfrac{\beta kc\left[\tau+\beta\left(x-\tau\right)\right]^{c-1}}{\left\{1+\left[\tau+\beta\left(x-\tau\right)\right]^{c}\right\}^{k+1}}, & x>\tau \end{cases} \tag{5}$$

Suppose that there are n_{1f} failures and n_{1c} units with censoring at normal condition. Also, I assume that there are n_{2f} failures and n_{2c} units with censoring at stress condition. Let $\delta_{i,(1,f)}$, $\delta_{i,(1,c)}$, $\delta_{i,(2,f)}$, $\delta_{i,(2,c)}$ be indicator functions, which (1, *f*) of the indicator function denotes that the sample unit fails before the stress change time, τ , and (1, *c*) denotes that the unit is censored before the time, τ . Also, (2, *f*) denotes that the unit fails after the time, τ . (2, *c*) denotes that the unit is censored after the time, τ . Furthermore, the equations are obtained as follows.

$\sum_{i=1}^{n}\delta_{i,(1,f)}=n_{1f}$, $\sum_{i=1}^{n}\delta_{i,(1,c)}=n_{1c}$, $\sum_{i=1}^{n}\delta_{i,(2,f)}=n_{2f}$, $\sum_{i=1}^{n}\delta_{i,(2,c)}=n_{2c}$, $n_1=n_{1f}+n_{1c}$, and $n_2=n_{2f}+n_{2c}$.

III. Complete-data likelihood function via EM algorithm

Let $\mathbf{y}=\left(\mathbf{y}_1^T,\ldots,\mathbf{y}_n^T\right)^T$ denote the observed data where $\mathbf{y}_i=\left(d_i,\delta_i\right)^T$ and δ_i = 0 (censored) or 1 (failure). As seen in the observations, x_i is censored or uncensored at d_i (i = 1,…,n). Then, the probability density function of the Burr XII distribution, given $x_i>d_i$ is calculated as follows:

Let $a_i=\tau+\beta\left(x_i-\tau\right)$ $A_i=\tau+\beta\left(X_i-\tau\right)$ $D_i=\tau+\beta\left(d_i-\tau\right)$

$$f(x_i\left|x_i>d_i\right.)=\frac{f(x_i)}{1-F(d_i)}=\begin{cases} kc\left(1+d_i^c\right)^k\dfrac{x_i^{c-1}}{\left(1+x_i^c\right)^{k+1}}, & x_i>d_i,\quad d_i<\tau \\ \left(1+D_i^c\right)^k\beta kc\dfrac{a_i^{c-1}}{\left(1+a_i^c\right)^{k+1}}, & x_i>d_i,\quad d_i>\tau \end{cases} \tag{6}$$

the complete data likelihood function of the Burr XII distribution can be expressed as

$$L_c\left(c,k,\beta\right)=\prod_{i=1}^{n}f_c\left(x_i;c,k,\beta\right)=\prod_{i=1}^{n}f(x_i)^{\delta_{i,(1,f)}}f(x_i)^{\delta_{i,(1,c)}}f(x_i)^{\delta_{i,(2,f)}}f(x_i)^{\delta_{i,(2,c)}} \tag{7}$$

the complete data log-likelihood function of the Burr XII distribution is then expressed as

$$\begin{aligned}
\log\left[L_c\left(c,k,\beta\right)\right]&=\sum_{i=1}^{n}\log\left[f_c\left(x_i;c,k,\beta\right)\right]\\
&=n\log(k)+n\log(c)+n_2\log(\beta)\\
&\quad+(c-1)\sum_{i=1}^{n}\delta_{i,(1,f)}\log(x_i)-(k+1)\sum_{i=1}^{n}\delta_{i,(1,f)}\log\left(1+x_i^c\right)\\
&\quad+(c-1)\sum_{i=1}^{n}\delta_{i,(1,c)}\log(x_i)-(k+1)\sum_{i=1}^{n}\delta_{i,(1,c)}\log\left(1+x_i^c\right)\\
&\quad+(c-1)\sum_{i=1}^{n}\delta_{i,(2,f)}\log(a_i)-(k+1)\sum_{i=1}^{n}\delta_{i,(2,f)}\log\left(1+a_i^c\right)\\
&\quad+(c-1)\sum_{i=1}^{n}\delta_{i,(2,c)}\log(a_i)-(k+1)\sum_{i=1}^{n}\delta_{i,(2,c)}\log\left(1+a_i^c\right)
\end{aligned} \tag{8}$$

then, the Q-function of the Burr XII distribution is obtained as

$$\begin{aligned}
&E\left\lfloor \log L_c\left(c,k,\beta\right) \mid \mathbf{y}\right\rfloor = n\log\left(k\right) + n\log\left(c\right) + n_2\log\left(\beta\right) \\
&+\left(c-1\right)\sum_{i=1}^{n}\delta_{i,(1,f)}\log\left(d_i\right) - \left(k+1\right)\sum_{i=1}^{n}\delta_{i,(1,f)}\log\left(1+d_i^c\right) \\
&+\left(c-1\right)\sum_{i=1}^{n}\delta_{i,(1,c)}E\left[\log\left(X_i\right)\middle|X_i > d_i\right] - \left(k+1\right)\sum_{i=1}^{n}\delta_{i,(1,c)}E\left[\log\left(1+X_i^c\right)\middle|X_i > d_i\right] \\
&+\left(c-1\right)\sum_{i=1}^{n}\delta_{i,(2,f)}\log\left(D_i\right) - \left(k+1\right)\sum_{i=1}^{n}\delta_{i,(2,f)}\log\left(1+D_i^c\right) \\
&+\left(c-1\right)\sum_{i=1}^{n}\delta_{i,(2,c)}E\left[\log\left(A_i\right)\middle|X_i > d_i\right] - \left(k+1\right)\sum_{i=1}^{n}\delta_{i,(2,c)}E\left[\log\left(1+A_i^c\right)\middle|X_i > d_i\right]
\end{aligned} \tag{9}$$

For the E-step, $Q\left(\boldsymbol{\psi};\boldsymbol{\psi}_{(m)}\right)$ can be calculated, where $\boldsymbol{\psi}$ denotes the set of parameters, *c*, *k* and *β* and $\boldsymbol{\psi}_{(m)}$ denotes the set of estimates, $c_{(m)}$, $k_{(m)}$ and $\beta_{(m)}$, in *m*-th iteration.

$$\begin{aligned}
&Q\left(\boldsymbol{\psi};\boldsymbol{\psi}_{(m)}\right) = E_{\boldsymbol{\psi}_{(m)}}\left[\log L_c\left(c,k,\beta\right) \mid \mathbf{y}\right] \\
&= n\log\left(k\right) + n\log\left(c\right) + n_2\log\left(\beta\right) \\
&+\left(c-1\right)\sum_{i=1}^{n}\delta_{i,(1,f)}\log\left(d_i\right) - \left(k+1\right)\sum_{i=1}^{n}\delta_{i,(1,f)}\log\left(1+d_i^c\right) \\
&+\left(c-1\right)\sum_{i=1}^{n}\delta_{i,(1,c)}E_{\boldsymbol{\psi}_{(m)}}\left[\log\left(X_i\right)\middle|X_i > d_i\right] - \left(k+1\right)\sum_{i=1}^{n}\delta_{i,(1,c)}E_{\boldsymbol{\psi}_{(m)}}\left[\log\left(1+X_i^c\right)\middle|X_i > d_i\right] \\
&+\left(c-1\right)\sum_{i=1}^{n}\delta_{i,(2,f)}\log\left(D_i\right) - \left(k+1\right)\sum_{i=1}^{n}\delta_{i,(2,f)}\log\left(1+D_i^c\right) \\
&+\left(c-1\right)\sum_{i=1}^{n}\delta_{i,(2,c)}E_{\boldsymbol{\psi}_{(m)}}\left[\log\left(A_i\right)\middle|X_i > d_i\right] - \left(k+1\right)\sum_{i=1}^{n}\delta_{i,(2,c)}E_{\boldsymbol{\psi}_{(m)}}\left[\log\left(1+A_i^c\right)\middle|X_i > d_i\right]
\end{aligned} \tag{10}$$

For the M-step, $\boldsymbol{\psi}_{(m+1)}$ is the specific value of $\boldsymbol{\psi}\in\Omega$ that maximizes $Q\left(\boldsymbol{\psi};\boldsymbol{\psi}_{(m)}\right)$; that is, $Q\left(\boldsymbol{\psi}_{(m+1)};\boldsymbol{\psi}_{(m)}\right) \geq Q\left(\boldsymbol{\psi};\boldsymbol{\psi}_{(m)}\right)$. The E and M steps repeatedly iterative compute until the estimates of parameters converge to the default value. The above term in equation (10), $E_{\boldsymbol{\psi}(m)}[log(X_i)\,|X_i > d_i]$, can be directly solved by using Monte Carlo method. However the other terms, $E_{\boldsymbol{\psi}_{(m)}}\left\lfloor \log\left(1+X_i^c\right) \mid X_i > d_i\right\rfloor$, $E_{\boldsymbol{\psi}(m)}[log(A_i)\,|X_i > d_i]$ and $E_{\boldsymbol{\psi}(m)}[log(1 + A_i^c)\,|X_i > d_i]$ can not be directly solved using Monte Carlo method because the unknown parameter, *c* and *β*, exists within the terms, $log(1 + X_i^c)$, $log(A_i)$ and $log(1 + A_i^c)$, where $A_i = \tau + \beta(X_i - \tau)$. To decompose these terms, Taylor series expansion can be applied to decompose these terms, $log(1 + X_i^c)$, $log(A_i)$ and $log(1 + A_i^c)$, and then Monte Carlo method can be applied to compute the integral.

For the Burr XII distribution, the variance-covariance matrix of parameters *c*, *k* and *β* is obtained as

$$\begin{bmatrix} Var(\hat{c}) & Cov(\hat{c},\hat{k}) & Cov(\hat{c},\hat{\beta}) \\ Cov(\hat{c},\hat{k}) & Var(\hat{k}) & Cov(\hat{k},\hat{\beta}) \\ Cov(\hat{c},\hat{\beta}) & Cov(\hat{k},\hat{\beta}) & Var(\hat{\beta}) \end{bmatrix} = (-1)\times \begin{bmatrix} E\left(\frac{\partial^2 \log L}{\partial c^2}\right) & E\left(\frac{\partial^2 \log L}{\partial c \partial k}\right) & E\left(\frac{\partial^2 \log L}{\partial c \partial \beta}\right) \\ E\left(\frac{\partial^2 \log L}{\partial c \partial k}\right) & E\left(\frac{\partial^2 \log L}{\partial k^2}\right) & E\left(\frac{\partial^2 \log L}{\partial k \partial \beta}\right) \\ E\left(\frac{\partial^2 \log L}{\partial c \partial \beta}\right) & E\left(\frac{\partial^2 \log L}{\partial k \partial \beta}\right) & E\left(\frac{\partial^2 \log L}{\partial \beta^2}\right) \end{bmatrix}^{-1} \tag{11}$$

where E symbolizes expectation and L denotes log-likelihood function. The observed information (I_{obs}) can be used to construct the variance-covariance matrix and confidence intervals for c, k and β. Complete (I_{comp}) and missing (I_{miss}) information can be used to calculate the rate of convergence of EM algorithm. Louis (1982) showed that the observed information presents the difference between complete information and missing information within the framework of EM algorithm. The equation is expressed as I_{obs} $I_{obs} = I_{comp}$ -I_{miss}. I_{comp} and I_{miss} are obtained in Appendix. Therefore, the variance-covariance matrix of parameters c, k and β can be obtained by inverting the observed information matrix and is given by

$$\begin{bmatrix} Var(\hat{c}) & Cov(\hat{c},\hat{k}) & Cov(\hat{c},\hat{\beta}) \\ Cov(\hat{c},\hat{k}) & Var(\hat{k}) & Cov(\hat{k},\hat{\beta}) \\ Cov(\hat{c},\hat{\beta}) & Cov(\hat{k},\hat{\beta}) & Var(\hat{\beta}) \end{bmatrix} = [I_{comp}(c,k,\beta;\mathbf{y}) - I_{miss}(c,k,\beta;\mathbf{y})]^{-1} \tag{12}$$

Thus, an approximate (1-α)100% confidence intervals for c, k and β are obtained as

$$\hat{c} \pm z_{\frac{\alpha}{2}}\sqrt{var(\hat{c})}\ ,\hat{k} \pm z_{\frac{\alpha}{2}}\sqrt{var(\hat{k})} \text{ and } \hat{\beta} \pm z_{\frac{\alpha}{2}}\sqrt{var(\hat{\beta})} \tag{13}$$

where $z_{\frac{\alpha}{2}}$ is a standard normal variate.

IV. Observed-data likelihood function via BFGS algorithm

The MLE based on observed-data likelihood function of the Burr XII distribution with multiple censored data in a SS-PALT is given by

$$L = \prod_{i=1}^{n} f(x_{i;1,f})\left[1-F(x_{i;1,c})\right] f(x_{i;2,f})\left[1-F(x_{i;2,c})\right]. \tag{14}$$

The log-likelihood function is obtained as

$$\begin{aligned} \log L =& n_{1f}\log(c) + n_{1f}\log(k) + (c-1)\sum_{i=1}^{n}\log(x_{i;1,f}) - (k+1)\sum_{i=1}^{n}\log(1+x_{i;1,f}^{c}) - k\sum_{i=1}^{n}\log(1+x_{i;1,c}^{c}) \\ &+ n_{2f}\log(\beta) + n_{2f}\log(c) + n_{2f}\log(k) + (c-1)\sum_{i=1}^{n}\log(a_{i;2,f}) - (k+1)\sum_{i=1}^{n}\log(1+a_{i;2,c}^{c}) - k\sum_{i=1}^{n}\log(1+a_{i;2,c}^{c}), \end{aligned} \tag{15}$$

where $a_{i;2,f} = \tau + \beta(x_{i;2,f} - \tau)$ and $a_{i;2,c} = \tau + \beta(x_{i;2,c} - \tau)$.

The estimates of c, k, and β are obtained by setting the first partial derivatives of the log-likelihood to zero with respect to *c*, *k* , and *β*, respectively. The simultaneous equations are given as follows:

$$\begin{aligned}\partial \log L / \partial c &= n_{1f} c^{-1} + \sum_{i=1}^{n} \log\left(x_{i;1,f}\right) - (k+1)\sum_{i=1}^{n} \log\left(x_{i;1,f}\right) x_{i;1,f}^{c}\left(1+x_{i;1,f}^{c}\right)^{-1} \\ &\quad - k\sum_{i=1}^{n} \log\left(x_{i;1,c}\right) x_{i;1,c}^{c}\left(1+x_{i;1,c}^{c}\right)^{-1} + n_{2f}c^{-1} + \sum_{i=1}^{n}\log\left(a_{i;2,f}\right) \\ &\quad -(k+1)\sum_{i=1}^{n}\log\left(a_{i;2,f}\right)a_{i;2,f}^{c}\left(1+a_{i;2,f}^{c}\right)^{-1} - k\sum_{i=1}^{n}\log\left(a_{i;2,c}\right)a_{i;2,c}^{c}\left(1+a_{i;2,c}^{c}\right)^{-1} = 0,\end{aligned} \tag{16}$$

$$\begin{aligned}\partial \log L / \partial k &= n_{1f}k^{-1} - \sum_{i=1}^{n}\log\left(1+x_{i;1,f}^{c}\right) - \sum_{i=1}^{n}\log\left(1+x_{i;1,c}^{c}\right) \\ &\quad + n_{2f}k^{-1} - \sum_{i=1}^{n}\log\left(1+a_{i;2,f}^{c}\right) - \sum_{i=1}^{n}\log\left(1+a_{i;2,c}^{c}\right) = 0,\end{aligned} \tag{17}$$

$$\begin{aligned}\partial \log L / \partial \beta &= n_{2f}\beta^{-1} + (c-1)\sum_{i=1}^{n}\left(x_{i;2,f}-\tau\right)a_{i;2,f}^{-1} - (k+1)\sum_{i=1}^{n} c a_{i}^{c-1}\left(x_{i;2,f}-\tau\right)\left(1+a_{i;2,f}^{c}\right)^{-1} \\ &\quad - k\sum_{i=1}^{n} c a_{i;2,c}^{c-1}\left(x_{i;2,c}-\tau\right)\left(1+a_{i;2,c}^{c}\right)^{-1} = 0.\end{aligned} \tag{18}$$

BFGS algorithm is then applied for solving these simultaneous equations to obtain the estimated values of c , k , and *β*. The initial estimates of the parameters are chosen using pseudo complete estimates which the samples are completely treated as failures. The asymptotic variance-covariance matrix of *c*, *k* , and *β* is established as

$$Var\left(\hat{\boldsymbol{\psi}}\right) = I_{obs}^{-1}\left(\boldsymbol{\psi};\mathbf{x}\right) = \left[-\partial^{2}\log L\left(\boldsymbol{\psi}\right)/\partial\boldsymbol{\psi}\partial\boldsymbol{\psi}^{T}\right]^{-1}, \tag{19}$$

where $\boldsymbol{\psi}$ denotes the set of *c*, *k*, and *β*. Thus, the approximate (1-α)100% confidence intervals for *c*, *k*, and *β* are obtained as

$$\hat{c} \pm z_{\alpha/2}\sqrt{var(\hat{c})}\ ,\ \hat{k} \pm z_{\alpha/2}\sqrt{var\left(\hat{k}\right)}\ \text{ and }\ \hat{\beta} \pm z_{\alpha/2}\sqrt{var\left(\hat{\beta}\right)}\ , \tag{20}$$

where $z_{\alpha/2}$ is the $100(1-\alpha/2)$ percentile of the standard normal distribution.

V. Simulation study

The method in Wang, Cheng and Lu (2012) was used for generating multiple censored samples. Censored samples were randomly generated from the Burr XII distribution with specified values of *c*, *k* and *β*. The simulation included the following conditions: sample sizes *n* = 100, 200; the stress change time, *τ* = 0.5, 1.5; censoring level *CL* = 0.2. Here we considered (*c*, *k*, *β*) = (1, 0.5, 1.25), (1, 0.5, 2), (1, 1, 1.25), (1, 1, 2), (2, 0.5, 1.25), (2, 0.5, 2),(2, 1, 1.25), (2, 1, 2), (2, 2, 1.25) and (2, 2, 2) as true parameter values. For each data set, 1000 replications are simulated. To assess the performance of the MLE via EM algorithm, I consider three major measures including the absolute relative bias (ARB), the root mean squared error (RMSE) , and the coverage rate (CR).

They are defined as follows:

1) $\mathrm{ARB}(\hat{c}) = N^{-1}\sum_{i=1}^{N}\left|(\hat{c}_i - c)/c\right|$, $\mathrm{ARB}(\hat{k}) = N^{-1}\sum_{i=1}^{N}\left|(\hat{k}_i - k)/k\right|$ and $\mathrm{ARB}(\hat{\beta}) = N^{-1}\sum_{i=1}^{N}\left|(\hat{\beta}_i - \beta)/\beta\right|$,
2) $RMSE(\hat{c}) = N^{-1}\sum_{i=1}^{N}(\hat{c}_i - c)^2$, $RMSE(\hat{k}) = N^{-1}\sum_{i=1}^{N}(\hat{k}_i - k)^2$ and $RMSE(\hat{\beta}) = N^{-1}\sum_{i=1}^{N}(\hat{\beta}_i - \beta)^2$,
3) The coverage rate at the 95% confidence intervals for c, k and β is based on N simulations, where $\overline{c} = N^{-1}\sum_{i=1}^{N}\hat{c}_i$, $\overline{k} = N^{-1}\sum_{i=1}^{N}\hat{k}_i$, $\overline{\beta} = N^{-1}\sum_{i=1}^{N}\hat{\beta}_i$, and N = 1,000.

The simulation results for the multiple censored with CL=0.2 for sample sizes 100 and 200 are presented in Tables 1-2. The following conclusions were observed.

1) For the sample size of 100 in Table 1, EM algorithm provides lower levels of ARB and RMSE for parameters c, k, and β than BFGS algorithm does in most scenarios. EM algorithm estimates perform better than BFGS algorithm does, the proportion accounting for 68.3% (41 cases/60 cases) for ARB and 71.7% (43 cases/60 cases) for RMSE. This indicates that EM algorithm performs better than BFGS algorithm does in this simulation study.
2) For the sample size of 100 in Table 1, the 95% C.I. is calculated for parameters c, k, and β. In most scenarios, EM algorithm provides higher levels of CR for parameters c, k, and β than BFGS algorithm does. EM algorithm estimates perform better than BFGS algorithm does, the proportion accounting for 100% (60 cases/60 cases). The average values of CR are 95.6% for EM algorithm and 72.0% for BFGS algorithm. This indicates that EM algorithm performs better than BFGS algorithm does in this simulation study.
3) For the sample size of 200 in Table 2, the results are similar with those for the sample size of 100. EM algorithm estimates perform better than BFGS algorithm does, the proportion accounting for 58.3% (35 cases/60 cases) for ARB and 65.0% (39 cases/60 cases) for RMSE. EM algorithm estimates perform better than BFGS algorithm does, the proportion accounting for 73.3% (44 cases/60 cases) for CR. The average values of CR are 93.9% for EM algorithm and 88.6% for BFGS algorithm.
4) With the sample size of complete data increasing from 100 to 200, EM algorithm and BFGS algorithm estimates for parameters c, k, and β are more accurate and have fewer errors, and lower ARB and RMSE.

Table 1: *ARB, RMSE and CR of the estimates with n = 100.*

k	c	β	τ	*Parameters*	BFGS algorithm			EM algorithm		
					ARB	RMSE	CR (%)	ARB	RMSE	CR (%)
				k	0.1480	0.1728	62.4	0.1461	0.1677	90.7
1	0.5	1.25	0.5	c	0.1084	0.0692	89.2	0.1010	0.0652	99.7
				β	0.3148	0.4633	44.9	0.2644	0.4025	87.7
				k	0.1479	0.1717	63.0	0.1347	0.1601	91.4
1	0.5	2	0.5	c	0.1036	0.0661	90.6	0.1015	0.0652	99.2
				β	0.2947	0.6911	49.2	0.2544	0.6264	89.2
				k	0.1288	0.1505	74.4	0.1265	0.1487	97.4
1	1	1.25	0.5	c	0.1046	0.1384	90.2	0.1021	0.1349	99.5
				β	0.2272	0.3456	64.7	0.2045	0.3117	95.8
				k	0.1296	0.1543	72.1	0.1150	0.1407	98.3
1	1	2	0.5	c	0.0981	0.1280	90.9	0.1028	0.1315	99.7
				β	0.2235	0.5401	67.1	0.2299	0.5481	94.2
				k	0.1431	0.3294	64.2	0.1505	0.3404	92.0
2	0.5	1.25	0.5	c	0.0871	0.0558	93.4	0.0831	0.0523	99.6
				β	0.2383	0.3631	59.0	0.2080	0.3209	96.3
				k	0.1369	0.3176	67.4	0.1217	0.2849	95.3

2	0.5	2	0.5	*c*	0.0856	0.0549	93.7	0.0903	0.0578	99.3
				β	0.2435	0.5978	57.7	0.2433	0.5905	94.5
				k	0.1423	0.3341	65.8	0.1340	0.3145	96.9
2	1	1.25	0.5	*c*	0.0921	0.1204	90.6	0.0879	0.1130	99.8
				β	0.2343	0.3538	64.9	0.2020	0.3101	98.3
				k	0.1425	0.3354	64.6	0.1128	0.2747	98.1
2	1	2	0.5	*c*	0.0963	0.1259	88.0	0.1069	0.1385	99.2
				β	0.2459	0.5962	64.0	0.2257	0.5507	94.6
				k	0.1827	0.4300	55.7	0.1325	0.3338	98.2
2	2	1.25	0.5	*c*	0.0849	0.2105	92.2	0.1016	0.2526	99.8
				β	0.2377	0.4180	79.7	0.1904	0.3279	99.4
				k	0.1807	0.4256	54.8	0.1610	0.4157	99.6
2	2	2	0.5	*c*	0.0805	0.2029	94.2	0.1295	0.3319	99.6
				β	0.2329	0.6224	77.9	0.2243	0.5268	95.3
				k	0.1605	0.0935	60.0	0.1508	0.0891	91.4
0.5	1	1.25	1.5	*c*	0.1313	0.1734	82.5	0.1339	0.1712	98.1
				β	0.2792	0.4255	55.4	0.2310	0.3533	91.5
				k	0.1573	0.0927	62.9	0.1461	0.0874	91.9
0.5	1	2	1.5	*c*	0.1227	0.1635	85.9	0.1234	0.1628	99.0
				β	0.2592	0.6214	59.5	0.2373	0.5795	91.2
				k	0.1226	0.0731	77.3	0.1672	0.0981	90.4
0.5	2	1.25	1.5	*c*	0.0987	0.2407	89.2	0.1331	0.3307	99.5
				β	0.2117	0.3364	74.0	0.2007	0.3091	97.9
				k	0.1217	0.0723	78.3	0.1489	0.0896	92.6
0.5	2	2	1.5	*c*	0.0948	0.2315	90.1	0.1333	0.3554	99.5
				β	0.2045	0.5196	75.5	0.2186	0.5354	95.1
				k	0.1592	0.1822	56.1	0.1542	0.1767	87.5
1	0.5	1.25	1.5	*c*	0.1066	0.0682	89.9	0.1047	0.0674	98.9
				β	0.3348	0.5185	42.8	0.2703	0.4126	88.3
				k	0.1481	0.1713	62.5	0.1411	0.1641	90.3
1	0.5	2	1.5	*c*	0.1019	0.0650	91.4	0.1069	0.0675	99.1
				β	0.3152	0.7462	43.4	0.2740	0.6668	88.9
				k	0.1511	0.1756	58.7	0.1517	0.1774	89.0
1	1	1.25	1.5	*c*	0.0927	0.1221	92.7	0.0927	0.1242	99.0
				β	0.2241	0.3389	65.0	0.2157	0.3251	96.6
				k	0.1499	0.1738	61.3	0.1370	0.1601	92.4
1	1	2	1.5	*c*	0.0971	0.1256	91.6	0.1046	0.1330	99.5
				β	0.2300	0.5519	61.8	0.2238	0.5417	94.7
				k	0.1721	0.1883	52.9	0.1706	0.1876	89.0
1	2	1.25	1.5	*c*	0.0879	0.2233	91.4	0.0852	0.2125	99.1
				β	0.2461	0.4011	69.6	0.2206	0.3670	99.1
				k	0.1700	0.1872	52.4	0.1534	0.1738	89.8
1	2	2	1.5	*c*	0.0882	0.2284	92.9	0.0894	0.2292	99.1
				β	0.2402	0.6245	66.4	0.2250	0.6059	98.5

Table 2: *ARB, RMSE and CR of the estimates with n = 200.*

k	*c*	*β*	*τ*	*Parameters*	BFGS algorithm			EM algorithm		
					ARB	RMSE	CR (%)	ARB	RMSE	CR (%)
				k	0.1508	0.1660	78.2	0.1473	0.1616	81.2
1	0.5	1.25	0.5	*c*	0.0711	0.0455	99.2	0.0743	0.0478	98.8
				β	0.2466	0.3757	85.2	0.1990	0.3043	93.0
				k	0.1484	0.1650	76.4	0.1334	0.1496	85.9
1	0.5	2	0.5	*c*	0.0693	0.0444	99.0	0.0760	0.0479	99.2
				β	0.2342	0.5641	85.8	0.2042	0.5030	92.1
				k	0.1408	0.1581	100.0	0.1352	0.1515	89.0
1	1	1.25	0.5	*c*	0.0753	0.0971	98.6	0.0759	0.0974	99.1
				β	0.1647	0.2510	97.9	0.1561	0.2368	98.2
				k	0.1369	0.1563	99.2	0.1206	0.1410	91.7
1	1	2	0.5	*c*	0.0716	0.0922	99.0	0.0799	0.1033	99.1
				β	0.1650	0.4029	98.6	0.1721	0.4149	95.9
				k	0.1426	0.3155	66.7	0.1351	0.2970	81.8

2	0.5	1.25	0.5	c	0.0630	0.0396	97.9	0.0612	0.0379	99.0
				β	0.1960	0.2975	74.1	0.1781	0.2736	97.2
				k	0.1441	0.3176	63.5	0.1220	0.2738	88.4
2	0.5	2	0.5	c	0.0612	0.0394	97.9	0.0707	0.0445	98.8
				β	0.1968	0.4826	74.4	0.2002	0.4909	94.5
				k	0.1448	0.3255	70.4	0.1244	0.2817	91.8
2	1	1.25	0.5	c	0.0640	0.0830	97.2	0.0641	0.0817	99.4
				β	0.1880	0.2884	89.4	0.1548	0.2302	99.3
				k	0.1445	0.3240	72.2	0.1029	0.2425	97.4
2	1	2	0.5	c	0.0650	0.0827	97.8	0.0761	0.0958	98.3
				β	0.1819	0.4532	91.1	0.1733	0.4092	96.5
				k	0.1556	0.3650	97.0	0.1130	0.2798	97.6
2	2	1.25	0.5	c	0.0637	0.1588	99.9	0.0790	0.1994	99.2
				β	0.1594	0.2723	99.4	0.1446	0.2255	99.3
				k	0.1534	0.3619	96.9	0.1172	0.2882	99.3
2	2	2	0.5	c	0.0651	0.1621	99.7	0.1089	0.2649	98.6
				β	0.1632	0.4386	99.6	0.1880	0.4362	96.8
				k	0.1656	0.0921	84.4	0.1525	0.0857	82.7
0.5	1	1.25	1.5	c	0.0931	0.1204	98.9	0.0908	0.1147	98.5
				β	0.2053	0.3156	94.9	0.1715	0.2636	94.0
				k	0.1599	0.0896	82.9	0.1475	0.0842	85.5
0.5	1	2	1.5	c	0.0845	0.1135	98.6	0.0966	0.1262	99.3
				β	0.2012	0.4938	94.0	0.1805	0.4461	92.5
				k	0.1351	0.0761	90.2	0.1570	0.0884	83.5
0.5	2	1.25	1.5	c	0.0704	0.1716	99.2	0.0823	0.2116	99.6
				β	0.1619	0.2558	95.6	0.1430	0.2146	98.2
				k	0.1337	0.0752	92.4	0.1446	0.0827	87.1
0.5	2	2	1.5	c	0.0698	0.1717	99.2	0.0892	0.2341	99.2
				β	0.1427	0.3658	94.2	0.1450	0.3584	96.6
				k	0.1569	0.1722	66.2	0.1466	0.1617	75.5
1	0.5	1.25	1.5	c	0.0695	0.0449	98.5	0.0719	0.0448	99.5
				β	0.2454	0.3697	71.9	0.1939	0.3028	95.6
				k	0.1616	0.1754	61.3	0.1431	0.1575	83.0
1	0.5	2	1.5	c	0.0686	0.0438	99.0	0.0752	0.0475	99.0
				β	0.2301	0.5617	74.0	0.1996	0.4927	92.8
				k	0.1542	0.1683	67.5	0.1509	0.1663	75.6
1	1	1.25	1.5	c	0.0677	0.0871	98.6	0.0688	0.0875	99.1
				β	0.1779	0.2689	83.7	0.1609	0.2451	98.2
				k	0.1545	0.1697	66.5	0.1367	0.1544	80.2
1	1	2	1.5	c	0.0666	0.0842	99.1	0.0752	0.0935	99.6
				β	0.1714	0.4153	85.4	0.1796	0.4242	97.3
				k	0.1436	0.1596	73.3	0.1347	0.1509	80.4
1	2	1.25	1.5	c	0.0603	0.1521	99.6	0.0616	0.1566	99.8
				β	0.1864	0.2935	85.0	0.1612	0.2483	98.4
				k	0.1395	0.1547	75.4	0.1225	0.1384	88.5
1	2	2	1.5	c	0.0641	0.1601	98.6	0.0706	0.1787	99.5
				β	0.1742	0.4363	86.2	0.1722	0.4117	97.5

VI. Illustrative example

To illustrate the proposed MLEs via EM algorithm for the Burr XII distribution in SS-PALT, one data set from a light-emitting diode (LED) life test was used. The life test data with 1,000 hours of unit are as follows:

0.02*, 0.03*, 0.08*, 0.11*, 0.13*, 0.14*, 0.15*, 0.19*, 0.21*, 0.25*, 0.25*, 0.27, 0.28*, 0.31, 0.33, 0.35, 0.37*, 0.42, 0.43*, 0.44*, 0.46, 0.46, 0.49, 0.51, 0.51, 0.55*, 0.56, 0.58, 0.58*, 0.59, 0.59*, 0.6, 0.71, 0.71*, 0.73, 0.73, 0.73, 0.78, 0.79*, 0.81, 0.84, 0.87, 0.89, 0.9, 0.92, 0.92, 0.95, 1.01, 1.02, 1.06, 1.07, 1.08, 1.24, 1.24*, 1.25, 1.26, 1.31, 1.5*, 1.51*, 1.52*, 1.53*, 1.54, 1.55*, 1.56, 1.57*, 1.64, 1.64*, 1.65*, 1.67, 1.69, 1.7*, 1.83, 1.91, 2.03, 2.1*, 2.36, 2.78, 4.67

There are 78 samples with stress change time, τ = 1.5 and censoring level CL = 0.4. The samples of failure and censoring in the two phases of SS-PALT, respectively, are 36 failures in phase 1, 21 censoring in phase 1, 11 failures in phase 2 and 10 censoring in phase 2. The symbol "*" denotes multiple censored values. The histogram of the samples is illustrated in Figure 1 and the plot of the probability density function is illustrated in Figure 2. The initial estimates for the parameters were chosen by using pseudo complete estimates. Here, the pseudo complete estimates are computed from the samples which are completely treated as failures. Using the MLE with EM algorithm, the estimates are converged to 2.538 for c, 0.776 for k and 1.795 for β. The information matrices based on EM algorithm are obtained as

$$I_{comp} = \begin{bmatrix} 17.1297 & 25.1871 & 3.9065 \\ 25.1871 & 129.4696 & 10.3786 \\ 3.9065 & 10.3786 & 3.5688 \end{bmatrix}$$

$$I_{miss} = \begin{bmatrix} 6.0889 & 14.6162 & 1.6656 \\ 14.6162 & 51.4559 & 4.4966 \\ 1.6656 & 4.4966 & 1.4097 \end{bmatrix}$$

$$I_{obs} = I_{comp} - I_{miss} = \begin{bmatrix} 11.0408 & 10.5709 & 2.2409 \\ 10.5709 & 78.0137 & 5.8820 \\ 2.2409 & 5.8820 & 2.1591 \end{bmatrix}$$

Then, the asymptotic variance-covariance matrix based on EM algorithm can be obtained as

$$I_{obs}^{-1} = \begin{bmatrix} 0.1191 & -0.0086 & -0.1003 \\ -0.0086 & 0.0168 & -0.0367 \\ -0.1003 & -0.0367 & 0.6673 \end{bmatrix}$$

Then, the 95% confidence intervals, (1.862, 3.214) for c, (0.521, 1.031) for k and (0.194, 3.396) for β are obtained. The rates of convergence of c, k and β computed by $J\left(\hat{\boldsymbol{\psi}}\right) = I_{miss}\left(\hat{\boldsymbol{\psi}}\right) / I_{comp}\left(\hat{\boldsymbol{\psi}}\right)$ are 0.355 for c, 0.397 for k and 0.395 for β, respectively.

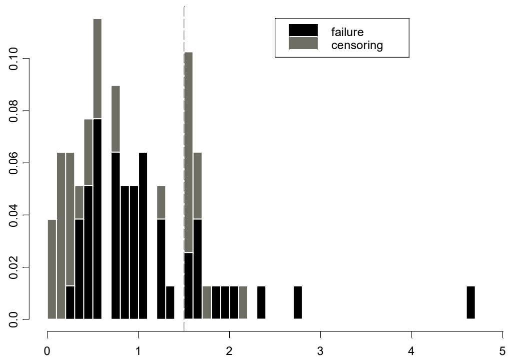

Figure 1: *Histogram of the samples*

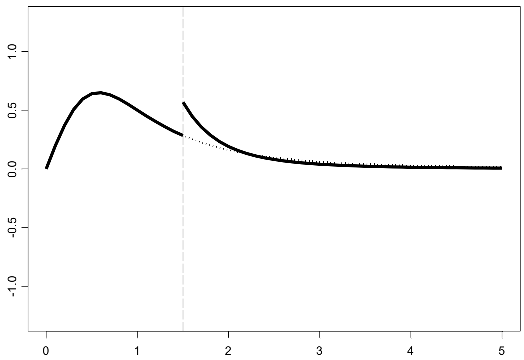

Figure 2: *Probability density plot*

VII. Conclusion

The lifetime of products under normal conditions usually requires a long period of time, which makes the test costly. Accelerated life test is used to obtain information about the lifetime of products quickly and economically under more severe operation conditions. In this paper, I present maximum likelihood estimation via EM algorithm to estimate the Burr XII parameters and acceleration factor in SS-PALT under multiple censored data. Simulation results show that the MLE via EM algorithm perform well in most cases in terms of the absolute relative bias, the root mean square, and the coverage rate. The simulation results and a real data analysis show the MLE via EM algorithm is a better alternative for estimating the Burr XII parameter in SS-PALT with multiple censored data.

Appendix:

The second partials of the complete data log-likelihood function for calculating elements of the complete information matrix are calculated. Then, the expected values of the second partials of the complete data log-likelihood function are obtained as

$$E_{\Psi}\left[\frac{\partial^2 \log L_c}{\partial c^2}|\mathbf{y}\right]$$

$$=\frac{-n}{c^2}-(k+1)\sum_{i=1}^{n}\delta_{i,(1,f)}\frac{d_i^c \log(d_i)^2}{\left(1+d_i^c\right)^2}-(k+1)\sum_{i=1}^{n}\delta_{i,(1,c)}E_{\Psi}\left[\frac{X_i^c \log(X_i)^2}{\left(1+X_i^c\right)^2}\mid X_i>d_i\right]$$

$$-(k+1)\sum_{i=1}^{n}\delta_{i,(2,f)}\frac{D_i^c \log(D_i)^2}{\left(1+D_i^c\right)^2}-(k+1)\sum_{i=1}^{n}\delta_{i,(2,c)}E_{\Psi}\left[\frac{A_i^c \log(A_i)^2}{\left(1+A_i^c\right)^2}\mid X_i>d_i\right]$$

$$E_{\Psi}\left[\frac{\partial^2 \log L_c}{\partial k^2}|\mathbf{y}\right]=-\frac{n}{k^2}$$

$$E_{\Psi}\left[\frac{\partial^2 \log L_c}{\partial \beta^2}|\mathbf{y}\right]=\frac{-n_2}{\beta^2}-(c-1)\sum_{i=1}^{n}\delta_{i,(2,f)}\frac{(d_i-\tau)^2}{D_i^2}$$

$$-(k+1)c(c-1)\sum_{i=1}^{n}\delta_{i,(2,f)}\frac{(d_i-\tau)^2 D_i^{c-2}}{1+D_i^c}+(k+1)c^2\sum_{i=1}^{n}\delta_{i,(2,f)}\frac{(d_i-\tau)^2 D_i^{2c-2}}{\left(1+D_i^c\right)^2}$$

$$-(c-1)\sum_{i=1}^{n}\delta_{i,(2,c)}E_{\Psi}\left[\frac{(X_i-\tau)^2}{A_i^2}|X_i>d_i\right]$$

$$-(k+1)c(c-1)\sum_{i=1}^{n}\delta_{i,(2,c)}E_{\Psi}\left[\frac{(X_i-\tau)^2 A_i^{c-2}}{1+A_i^c}|X_i>d_i\right]$$

$$+(k+1)c^2\sum_{i=1}^{n}\delta_{i,(2,c)}E_{\Psi}\left[\frac{(X_i-\tau)^2 A_i^{2c-2}}{\left(1+A_i^c\right)^2}|X_i>d_i\right]$$

$$E_{\boldsymbol{\psi}}\left[\frac{\partial^2 \log L_c}{\partial c \partial k}|\mathbf{y}\right]=-\sum_{i=1}^{n}\delta_{i,(1,f)}\frac{d_i^c\log\left(d_i\right)}{1+d_i^c}-\sum_{i=1}^{n}\delta_{i,(1,c)}E_{\boldsymbol{\psi}}\left[\frac{X_i^c\log\left(X_i\right)}{1+X_i^c}|X_i>d_i\right]$$

$$-\sum_{i=1}^{n}\delta_{i,(2,f)}\frac{D_i^c\log\left(D_i\right)}{1+D_i^c}-\sum_{i=1}^{n}\delta_{i,(2,c)}E_{\boldsymbol{\psi}}\left[\frac{A_i^c\log\left(A_i\right)}{1+A_i^c}|X_i>d_i\right]$$

$$E_{\boldsymbol{\psi}}\left[\frac{\partial^2 \log L_c}{\partial c \partial \beta}|\mathbf{y}\right]=\sum_{i=1}^{n}\delta_{i,(2,f)}\frac{d_i-\tau}{D_i}-\left(k+1\right)c\sum_{i=1}^{n}\delta_{i,(2,f)}\frac{\left(d_i-\tau\right)D_i^{c-1}\log\left(D_i\right)}{1+D_i^c}$$

$$-\left(k+1\right)\sum_{i=1}^{n}\delta_{i,(2,f)}\frac{\left(d_i-\tau\right)D_i^{c-1}}{1+D_i^c}+\left(k+1\right)c\sum_{i=1}^{n}\delta_{i,(2,f)}\frac{\left(d_i-\tau\right)D_i^{2c-1}\log\left(D_i\right)}{\left(1+D_i^c\right)^2}$$

$$+\sum_{i=1}^{n}\delta_{i,(2,c)}E_{\boldsymbol{\psi}}\left[\frac{X_i-\tau}{A_i}|X_i>d_i\right]-\left(k+1\right)c\sum_{i=1}^{n}\delta_{i,(2,c)}E_{\boldsymbol{\psi}}\left[\frac{\left(X_i-\tau\right)A_i^{c-1}\log\left(A_i\right)}{1+A_i^c}|X_i>d_i\right]$$

$$-\left(k+1\right)\sum_{i=1}^{n}\delta_{i,(2,c)}E_{\boldsymbol{\psi}}\left[\frac{\left(X_i-\tau\right)A_i^{c-1}}{1+A_i^c}|X_i>d_i\right]$$

$$+\left(k+1\right)c\sum_{i=1}^{n}\delta_{i,(2,c)}E_{\boldsymbol{\psi}}\left[\frac{\left(X_i-\tau\right)A_i^{2c-1}\log\left(A_i\right)}{\left(1+A_i^c\right)^2}|X_i>d_i\right]$$

$$E_{\boldsymbol{\psi}}\left[\frac{\partial^2 \log L_c}{\partial k \partial \beta}|\mathbf{y}\right]=-c\sum_{i=1}^{n}\delta_{i,(2,f)}\frac{\left(d_i-\tau\right)D_i^{c-1}}{1+D_i^c}-c\sum_{i=1}^{n}\delta_{i,(2,c)}E_{\boldsymbol{\psi}}\left[\frac{\left(X_i-\tau\right)A_i^{c-1}}{1+A_i^c}|X_i>d_i\right]$$

The expected values of the second partials of the complete data log-likelihood function can also be computed by using Monte Carlo integral. Then, the complete information becomes

$$I_{comp}\left(\boldsymbol{\psi};\mathbf{y}\right)=E_{\boldsymbol{\psi}}\left\{I_{comp}\left(\boldsymbol{\psi};\mathbf{x}\right)|\mathbf{y}\right\}$$

$$=\left(-1\right)\times\begin{bmatrix} E_{\boldsymbol{\psi}}\left\{\frac{\partial^2 \log L_c}{\partial c^2}|\mathbf{y}\right\} & E_{\boldsymbol{\psi}}\left\{\frac{\partial^2 \log L_c}{\partial ck}|\mathbf{y}\right\} & E_{\boldsymbol{\psi}}\left\{\frac{\partial^2 \log L_c}{\partial c\beta}|\mathbf{y}\right\} \\ E_{\boldsymbol{\psi}}\left\{\frac{\partial^2 \log L_c}{\partial ck}|\mathbf{y}\right\} & E_{\boldsymbol{\psi}}\left\{\frac{\partial^2 \log L_c}{\partial k^2}|\mathbf{y}\right\} & E_{\boldsymbol{\psi}}\left\{\frac{\partial^2 \log L_c}{\partial k\beta}|\mathbf{y}\right\} \\ E_{\boldsymbol{\psi}}\left\{\frac{\partial^2 \log L_c}{\partial c\beta}|\mathbf{y}\right\} & E_{\boldsymbol{\psi}}\left\{\frac{\partial^2 \log L_c}{\partial k\beta}|\mathbf{y}\right\} & E_{\boldsymbol{\psi}}\left\{\frac{\partial^2 \log L_c}{\partial \beta^2}|\mathbf{y}\right\} \end{bmatrix}$$

Now, the missing information matrix by using the likelihood function of X given Y can be derived and is given as follows

$$k\left(\mathbf{x}\left|\mathbf{y};\boldsymbol{\psi}\right.\right)=\prod_{i=1}^{n}f(x_i\left|x_i>d_i\right.)^{\delta_{i,(1,c)}}f(x_i\left|x_i>d_i\right.)^{\delta_{i,(2,c)}}$$

Then, the log-likelihood function of X given Y is expressed as

$$\log k\left(\mathbf{x}|\mathbf{y};\boldsymbol{\psi}\right)$$
$$=\sum_{i=1}^{n}\delta_{i,(1,c)}\left[\log(k)+\log(c)+k\log\left(1+d_i^c\right)+(c-1)\log(x_i)-(k+1)\log\left(1+x_i^c\right)\right]$$
$$+\sum_{i=1}^{n}\delta_{i,(2,c)}\left[\log(\beta)+\log(k)+\log(c)+k\log\left(1+D_i^c\right)+(c-1)\log(a_i)-(k+1)\log\left(1+a_i^c\right)\right]$$

The second partials of the log-likelihood functions for calculating elements of missing information matrix can be calculated. The expected values of the second partials of the log-likelihood function of X given Y are calculated as

$$E_{\boldsymbol{\psi}}\left[\frac{\partial^2\log k\left(\mathbf{x}|\mathbf{y};\boldsymbol{\psi}\right)}{\partial c^2}|\mathbf{y}\right]$$
$$=-\frac{n_{1c}}{c^2}+k\sum_{i=1}^{n}\delta_{i,(1,c)}\frac{d_i^c\log(d_i)^2}{\left(1+d_i^c\right)^2}-(k+1)\sum_{i=1}^{n}\delta_{i,(1,c)}E_{\boldsymbol{\psi}}\left[\frac{X_i^c\log(X_i)^2}{\left(1+X_i^c\right)^2}|X_i>d_i\right]$$
$$-\frac{n_{2c}}{c^2}+k\sum_{i=1}^{n}\delta_{i,(2,c)}\frac{D_i^c\log(D_i)^2}{\left(1+D_i^c\right)^2}-(k+1)\sum_{i=1}^{n}\delta_{i,(2,c)}E_{\boldsymbol{\psi}}\left[\frac{A_i^c\log(A_i)^2}{\left(1+A_i^c\right)^2}|X_i>d_i\right]$$
$$E_{\boldsymbol{\psi}}\left[\frac{\partial^2\log k\left(\mathbf{x}|\mathbf{y};\boldsymbol{\psi}\right)}{\partial k^2}|\mathbf{y}\right]=-\frac{n_{1c}+n_{2c}}{k^2}$$
$$E_{\boldsymbol{\psi}}\left[\frac{\partial^2\log k\left(\mathbf{x}|\mathbf{y};\boldsymbol{\psi}\right)}{\partial \beta^2}|\mathbf{y}\right]$$
$$=\frac{-n_{2c}}{\beta^2}+kc(c-1)\sum_{i=1}^{n}\delta_{i,(2,c)}\frac{(d_i-\tau)^2D_i^{c-2}}{\left(1+D_i^c\right)}-kc^2\sum_{i=1}^{n}\delta_{i,(2,c)}\frac{(d_i-\tau)^2D_i^{2c-2}}{\left(1+D_i^c\right)^2}$$
$$-(c-1)\sum_{i=1}^{n}\delta_{i,(2,c)}E_{\boldsymbol{\psi}}\left[\frac{(X_i-\tau)^2}{A_i^2}|X_i>d_i\right]$$
$$-(k+1)c(c-1)\sum_{i=1}^{n}\delta_{i,(2,c)}E_{\boldsymbol{\psi}}\left[\frac{(X_i-\tau)^2A_i^{c-2}}{1+A_i^c}|X_i>d_i\right]$$
$$+(k+1)c^2\sum_{i=1}^{n}\delta_{i,(2,c)}E_{\boldsymbol{\psi}}\left[\frac{(X_i-\tau)^2A_i^{2c-2}}{\left(1+A_i^c\right)^2}|X_i>d_i\right]$$
$$E_{\boldsymbol{\psi}}\left[\frac{\partial^2\log k\left(\mathbf{x}|\mathbf{y};\boldsymbol{\psi}\right)}{\partial c\partial k}|\mathbf{y}\right]=\sum_{i=1}^{n}\delta_{i,(1,c)}\frac{d_i^c\log(d_i)}{1+d_i^c}-\sum_{i=1}^{n}\delta_{i,(1,c)}E_{\boldsymbol{\psi}}\left[\frac{X_i^c\log(X_i)}{1+X_i^c}|X_i>d_i\right]$$
$$+\sum_{i=1}^{n}\delta_{i,(2,c)}\frac{D_i^c\log(D_i)}{1+D_i^c}-\sum_{i=1}^{n}\delta_{i,(2,c)}E_{\boldsymbol{\psi}}\left[\frac{A_i^c\log(A_i)}{1+A_i^c}|X_i>d_i\right]$$

$$E_{\boldsymbol{\psi}}\left[\frac{\partial^2 \log k\left(\mathbf{x}|\mathbf{y};\boldsymbol{\psi}\right)}{\partial c\partial \beta}|\mathbf{y}\right]=kc\sum_{i=1}^{n}\delta_{i,(2,c)}\frac{D_i^{c-1}\left(d_i-\tau\right)\log\left(D_i\right)}{1+D_i^c}+k\sum_{i=1}^{n}\delta_{i,(2,c)}\frac{\left(d_i-\tau\right)D_i^{c-1}}{1+D_i^c}$$

$$-kc\sum_{i=1}^{n}\delta_{i,(2,c)}\frac{\left(d_i-\tau\right)D_i^{2c-1}\log\left(D_i\right)}{\left(1+D_i^c\right)^2}+\sum_{i=1}^{n}\delta_{i,(2,c)}E_{\boldsymbol{\psi}}\left[\frac{X_i-\tau}{A_i}|X_i>d_i\right]$$

$$-\left(k+1\right)c\sum_{i=1}^{n}\delta_{i,(2,c)}E_{\boldsymbol{\psi}}\left[\frac{A_i^{c-1}\left(X_i-\tau\right)\log\left(A_i\right)}{1+A_i^c}|X_i>d_i\right]$$

$$-\left(k+1\right)\sum_{i=1}^{n}\delta_{i,(2,c)}E_{\boldsymbol{\psi}}\left[\frac{\left(X_i-\tau\right)A_i^{c-1}}{1+A_i^c}|X_i>d_i\right]$$

$$+\left(k+1\right)c\sum_{i=1}^{n}\delta_{i,(2,c)}E_{\boldsymbol{\psi}}\left[\frac{\left(X_i-\tau\right)A_i^{2c-1}\log\left(A_i\right)}{\left(1+A_i^c\right)^2}|X_i>d_i\right]$$

$$E_{\boldsymbol{\psi}}\left[\frac{\partial^2 \log k\left(\mathbf{x}|\mathbf{y};\boldsymbol{\psi}\right)}{\partial k\partial \beta}|\mathbf{y}\right]$$

$$=c\sum_{i=1}^{n}\delta_{i,(2,c)}\left[\frac{\left(d_i-\tau\right)D_i^{c-1}}{1+D_i^c}\right]-c\sum_{i=1}^{n}\delta_{i,(2,c)}E_{\boldsymbol{\psi}}\left[\frac{\left(X_i-\tau\right)A_i^{c-1}}{1+A_i^c}|X_i>d_i\right]$$

The expected values of the second partials of the log-likelihood functions can also be computed by using Monte Carlo integral. Thus, the missing information matrix can be computed from equations (22-26) and is expressed as following:

$$I_{miss}\left(\boldsymbol{\psi};\mathbf{y}\right)=E_{\boldsymbol{\psi}}\left\{I_{miss}\left(\boldsymbol{\psi};\mathbf{x}\right)|\mathbf{y}\right\}$$

$$=\left(-1\right)\times\begin{bmatrix} E_{\boldsymbol{\psi}}\left\{\frac{\partial^2 \log k\left(\mathbf{x}|\mathbf{y};\boldsymbol{\psi}\right)}{\partial c^2}|\mathbf{y}\right\} & E_{\boldsymbol{\psi}}\left\{\frac{\partial^2 \log k\left(\mathbf{x}|\mathbf{y};\boldsymbol{\psi}\right)}{\partial ck}|\mathbf{y}\right\} & E_{\boldsymbol{\psi}}\left\{\frac{\partial^2 \log k\left(\mathbf{x}|\mathbf{y};\boldsymbol{\psi}\right)}{\partial c\beta}|\mathbf{y}\right\} \\ E_{\boldsymbol{\psi}}\left\{\frac{\partial^2 \log k\left(\mathbf{x}|\mathbf{y};\boldsymbol{\psi}\right)}{\partial ck}|\mathbf{y}\right\} & E_{\boldsymbol{\psi}}\left\{\frac{\partial^2 \log k\left(\mathbf{x}|\mathbf{y};\boldsymbol{\psi}\right)}{\partial k^2}|\mathbf{y}\right\} & E_{\boldsymbol{\psi}}\left\{\frac{\partial^2 \log k\left(\mathbf{x}|\mathbf{y};\boldsymbol{\psi}\right)}{\partial k\beta}|\mathbf{y}\right\} \\ E_{\boldsymbol{\psi}}\left\{\frac{\partial^2 \log k\left(\mathbf{x}|\mathbf{y};\boldsymbol{\psi}\right)}{\partial c\beta}|\mathbf{y}\right\} & E_{\boldsymbol{\psi}}\left\{\frac{\partial^2 \log k\left(\mathbf{x}|\mathbf{y};\boldsymbol{\psi}\right)}{\partial k\beta}|\mathbf{y}\right\} & E_{\boldsymbol{\psi}}\left\{\frac{\partial^2 \log k\left(\mathbf{x}|\mathbf{y};\boldsymbol{\psi}\right)}{\partial \beta^2}|\mathbf{y}\right\} \end{bmatrix}$$

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Received: September 11, 2020
Accepted: November 25, 2020

Correlated Reneging in an Optional Service Markovian Queue With Working Vacations

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Abstract

This paper studies an $M/M/1$ queueing system with second optional service, correlated reneging and working vacations. All arriving customers require the first essential service whereas only a portion of them require a second optional service. The matrix geometric method is used to compute the stationary probability distribution of the system size. Further, various system performance measures are obtained and a cost optimization problem is considered using bat algorithm (BA). A variety of numerical illustrations are summarized in tables and graphs to provide an insight into the performance characteristics of the studied model.

Keywords: first essential service, second optional service, single working vacation, multiple working vacations, correlated reneging

I. Introduction

Vacation queues have been one of the intensive research topics for long time. There has been a considerable attention paid to the queueing models with server vacations, see Doshi [4]. During the vacation period, the server can be utilized for ancillary work, for example, in web services, file transfer services, manufacturing systems, etc. Such queueing model was first introduced by Servi and Finn [17] in an $M/M/1$ queueing system with working vacations and applied those results to analyze the performance of gate way router in communication networks. Later, Selvaraju and Goswami [16] have considered a single server impatient customers Markovian queueing system with single working vacation (SWV) and multiple working vacations (MWV). Rajadurai et al. [14] gave an analysis of a single server feedback retrial queueing system with subject to server breakdown and repair under MWV policy using probability generating function technique.

As for optional service, Madan [10] first investigated an $M/G/1$ queueing system with second optional service (SOS), in which some of the customers may require a SOS immediately after completion of the first essential service (FES). Using matrix geometric method, Jain and Chauhan [5] was able to approximate working vacation (WV) queueing system with SOS and unreliable server. Batch arrival bulk service queue with unreliable server, SOS and two different types of vacations has been investigated by Ayyappan and Supraja [2]. Manoharan and Sasi [11] discussed an $M/G/1$ queueing system with SOS and second optional vacation. A retrial queueing system with SOS under Erlang services has been investigated by Sekar et al. [15] using matrix geometric method.

Naturally, server vacations increase the waiting time of the customers. Due to longer wait in the queue, some customers may get discouraged and may decide to leave the queue without receiving the service (reneging). Such type of situations occur in real life like customers waiting in call centers, hospital emergency rooms, web services, programs waiting to be processed on a computer, etc. Queueing models with reneging have been investigated by many authors like Ancker and Gafarian [1], Baruah et al. [3], etc. The transient and steady-state behavior of the $M/M/1$ queue having customers' impatience with threshold has been discussed in Sharma et al. [18]. Mohan [12] introduced the concept of correlation in gambler's ruin problem. In Kim and Kim [7] , waiting time distribution of an $M/M/1$ queue was investigated where the inter arrival time between the n^{th} and $(n+1)^{th}$ customers and the service time of the n^{th} customer are correlated random variables with Downton's bivariate exponential distribution. A catastrophic queueing model with correlated input for the cell traffic generated by new broadband services has been studied by Jain and Kumar [6]. Kumar [8] studied a catastrophic-cum-restorative queueing problem with correlated input and impatient customer. There is another concept of correlated reneging wherein a customer's reneging at any time instant depends solely on the previous time instants' reneging or non-reneging. Transient numerical analysis of a single server queueing model with correlated reneging, balking and feedback has been carried out by Kumar and Soodan [9].

Existing literature shows frequent research topics related with WVs and SOS. However, a research gap observes no previous work on SOS, WVs in a queue with correlated reneging. As these topics are important in the real life situations, we consider an infinite capacity single server queueing system with SOS, WVs and correlated reneging. We have used matrix geometric method to obtain the steady state system length distributions. Some performance measures have been discussed. We employ the recently developed bat algorithm which was introduced by Yang [19] to achieve the optimal values of decision parameters and the expected cost. Particular cases of the model have been given. Later, a variety of numerical illustrations have been presented through tables and graphs.

The remainder of the model is structured as follows: Model description and practical justification of the model are presented in Section 2. In Section 3, the mathematical formulation of the model is given. Matrix geometric solution is given in Section 4. Section 5 is devoted to some performance measures, cost model and special cases of the model. Numerical investigations are given in the form of tables and graphs in Section 6. Finally, Section 7 concludes our paper.

II. Model Description

Consider a single server queueing system with SOS, WVs and correlated reneging. The model under consideration is schematically represented in Figure 1.

The queueing model is based on the following assumptions.

1. Customers arrive according to a Poisson process with rate λ.
2. The FES is provided to all customers. Immediately after completion of FES, a customer may demand SOS with probability r or he may leave the system with the complementary probability $(1-r)$. The service times of both FES and SOS are exponentially distributed with parameters μ_1 and μ_2, respectively.
3. At the end of a service, if there is no customer in the system, the server begins a WV of random length which is exponentially distributed with parameter θ. During WV, service is provided according to a Poisson distribution with parameter η. In SWV, when the server returns from WV period and finds no customer in the system, it does not take another WV but remains idle until the next arrival. But MWV policy requires the server to keep taking vacations until it finds at least one customer waiting in the system at a vacation completion instant. When the server returns from its vacation and finds at least one customer in the

system, it switches its service rate from η to μ_1 and a busy period starts; otherwise, it immediately leaves for another WV.

4. During WV, customers become impatient and they may renege from the queue. The reneging of customers can take place only at the transition marks t_0, t_1, t_2,... where $K_m = t_m - t_{m-1}$, $m = 1,2,3,..$, are random variables with $P[K_m \leq x] = 1 - exp(-\alpha x); \alpha > 0, m = 1,2,3, \ldots$ $i.e.$, the distribution of inter-transition marks is negative exponential with parameter α. The average reneging rate of a customer is given by $\alpha_n = n\alpha, n \geq 1$.
5. The reneging at two consecutive transition marks is governed by the following transition probability matrix:

$$\begin{array}{cc} & \text{To} \quad t_r \\ & \begin{array}{cc} 0 & 1 \end{array} \\ \text{From } t_{r-1} \begin{array}{c} 0 \\ 1 \end{array} & \left\| \begin{array}{cc} q_{00} & q_{01} \\ q_{10} & q_{11} \end{array} \right\| \end{array}$$

where $q_{00}+q_{01} = 1$ and $q_{10} + q_{11} = 1$.

Here, 0 refers to no reneging and 1 refers to the occurrence of reneging.

Thus, the reneging at two consecutive transition marks is correlated.

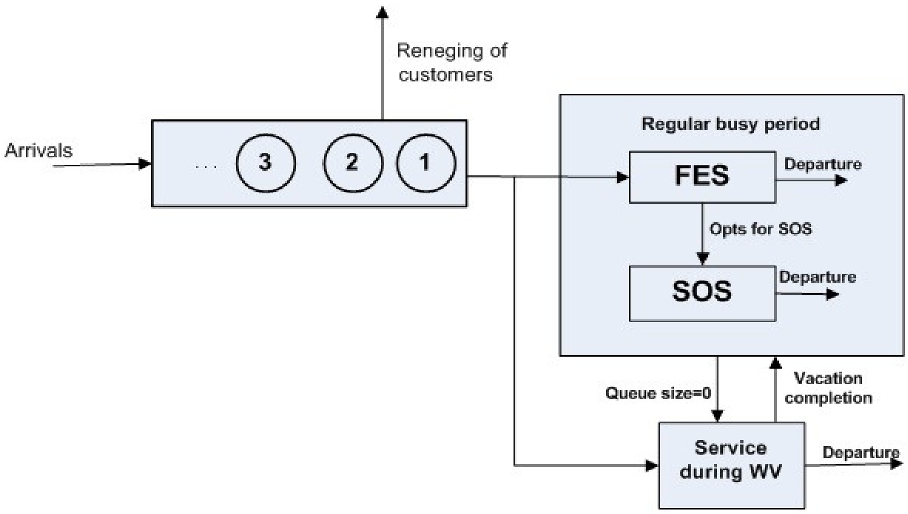

Figure 1: *General structure of the model.*

2.1 Practical Justification of the Model

The above discussed model has real time applications in electronic commerce (also known as E-commerce) which is a process of buying and selling of products, making money transfers and transferring data over an electronic medium. The whole E-commerce process can be divided into three main components, viz. receiving orders, processing order information and shipping. Cross-selling is a sales technique to increase sales by suggesting additional items to customers. When the sales are at their lowest, E-commerce merchant carries out the tasks like contacting suppliers and important clients, managing accounts up to date, etc. The speed of a service process will take a hit during this time and the customers may cancel the orders as they anticipate longer wait. If an order is canceled (not canceled) at any time instant, then there is a chance that an order may or may not be canceled at next time instant. Here, orders, selling of products, cross-selling, maintaining accounts, canceling of orders at time instants can be represented by the arrivals, FES, SOS, WV, correlated reneging, respectively in basic queueing situations.

III. Mathematical Formulation of the Model

At time t, let $N(t)$ be the number of customers in the queue, $J(t)$ be the state of the server, which is defined as

$$J(t)=\begin{cases} 0, \textit{the server is on } WV, \\ 1, \textit{the server is idle or busy } (SWV)\& \\ \quad \textit{the server is busy}(MWV) \end{cases}$$

and $S(t)$ be the state of the customer which is given as

$$S(t)=\begin{cases} 0, \ \textit{no reneging}, \\ 1, \ \textit{occurrence of reneging}. \end{cases}$$

The process $\{L(t), J(t), S(t), t \geq 0\}$ defines a continuous-time Markov process with state space χ $=\{(n, j, s): n \geq 0, j = 0,1, s = 0,1\}$. For mathematical formulation purpose, we define the following steady-state probabilities:

$E_{0,0,0}(E_{0,0,1})$ = Probability that the queue is empty, the server is idle, server is on WV, and a customer has not reneged (reneged) at the previous transition mark.

$E_{0,1,0}(E_{0,1,1})$ = Probability that the queue is empty, the server is idle, server is in busy state, and a customer has not reneged (reneged) at the previous transition mark.

$P_{n,0,0}(P_{n,0,1})$ = Probability of n customers in the queue, the server is not idle, server is on WV, and a customer has not reneged (reneged) at the previous transition mark.

$P_{n,1,0}(P_{n,1,1})$ = Probability of n customers in the queue, the server is not idle, server is rendering FES, and a customer has not reneged (reneged) at the previous transition mark.

$Q_{n,1,0}(Q_{n,1,1})$ = Probability of n customers in the queue, the server is not idle, server is rendering SOS, and a customer has not reneged (reneged) at the previous transition mark.

Steady-state equations:

$$(\lambda + \omega\theta)E_{0,0,0} = \eta P_{0,0,0} + (1-r)\mu_1 P_{0,1,0} + \mu_2 Q_{0,1,0}, \tag{1}$$
$$(\lambda + \omega\theta)E_{0,0,1} = \eta P_{0,0,1} + (1-r)\mu_1 P_{0,1,1} + \mu_2 Q_{0,1,1}, \tag{2}$$
$$(\lambda + \theta + \eta)P_{0,0,0} = \eta P_{1,0,0} + \lambda E_{0,0,0}, \tag{3}$$
$$(\lambda + \theta + \eta)P_{0,0,1} = \eta P_{1,0,1} + \lambda E_{0,0,1} + \alpha[q_{11}P_{1,0,1} + q_{01}P_{1,0,0}], \tag{4}$$
$$(\lambda + \theta + \eta + n\alpha)P_{n,0,0} = \lambda P_{n-1,0,0} + \eta P_{n+1,0,0} + n\alpha[q_{00}P_{n,0,0} + q_{10}P_{n,0,1}], n \geq 1, \tag{5}$$
$$(\lambda + \theta + \eta + n\alpha)P_{n,0,1} = \lambda P_{n-1,0,1} + \eta P_{n+1,0,1} + (n+1)\alpha[q_{01}P_{n+1,0,0} + q_{11}P_{n+1,0,1}], n \geq 1, \tag{6}$$
$$\lambda E_{0,1,0} = \omega\theta E_{0,0,0}, \tag{7}$$
$$\lambda E_{0,1,1} = \omega\theta E_{0,0,1}, \tag{8}$$
$$(\lambda + r\mu_1 + (1-r)\mu_1)P_{0,1,0} = \theta P_{0,0,0} + (1-r)\mu_1 P_{1,1,0} + \mu_2 Q_{1,1,0} + \omega\lambda E_{0,1,0}, \tag{9}$$
$$(\lambda + r\mu_1 + (1-r)\mu_1)P_{0,1,1} = \theta P_{0,0,1} + (1-r)\mu_1 P_{1,1,1} + \mu_2 Q_{1,1,1} + \omega\lambda E_{0,1,1}, \tag{10}$$
$$(\lambda + r\mu_1 + (1-r)\mu_1)P_{n,1,0} = \lambda P_{n-1,1,0} + (1-r)\mu_1 P_{n+1,1,0} + \mu_2 Q_{n+1,1,0} + \theta P_{n,0,0}, n \geq 1, \tag{11}$$
$$(\lambda + r\mu_1 + (1-r)\mu_1)P_{n,1,1} = \lambda P_{n-1,1,1} + (1-r)\mu_1 P_{n+1,1,1} + \mu_2 Q_{n+1,1,1} + \theta P_{n,0,1}, n \geq 1, \tag{12}$$
$$(\lambda + \mu_2)Q_{0,1,0} = r\mu_1 P_{0,1,0}, \tag{13}$$
$$(\lambda + \mu_2)Q_{n,1,0} = r\mu_1 P_{n,1,0} + \lambda Q_{n-1,1,0}, n \geq 1, \tag{14}$$
$$(\lambda + \mu_2)Q_{0,1,1} = r\mu_1 P_{0,1,1} \tag{15}$$
$$(\lambda + \mu_2)Q_{n,1,1} = r\mu_1 P_{n,1,1} + \lambda Q_{n-1,1,1}, n \geq 1. \tag{16}$$

Here, $\omega = 1$ or 0 correspond to the steady-state equations for SWV or MWV.

IV. Matrix Geometric Solution

Matrix geometric method is used for the analysis of quasi-birth-death (QBD) process with continuous time Markov chains whose transition rate matrices have a repetitive block structure. The method was developed by Neuts [13]. The transition rate matrix Q of the Markov chain corresponding to the coefficients of equations (1) to (16) has the block tridiagonal form given by:

$$\mathbf{Q} = \begin{pmatrix} \mathbf{A}_0 & \mathbf{C}_0 & & & & & & & \\ \mathbf{B}_1 & \mathbf{A}_1 & \mathbf{C}_1 & & & & & & \\ & \mathbf{B}_2 & \mathbf{A}_2 & \mathbf{C}_1 & & & & & \\ & & \mathbf{B}_3 & \mathbf{A}_3 & \mathbf{C}_1 & & & & \\ & & & \vdots & \vdots & \vdots & & & \\ & & & & \vdots & \vdots & \vdots & & \\ & & & & & \mathbf{B}_{N-1} & \mathbf{A}_{N-1} & \mathbf{C}_1 & \\ & & & & & & \mathbf{B}_N & \mathbf{A}_N & \mathbf{C}_1 \\ & & & & & & \vdots & \vdots & \vdots \end{pmatrix}$$

The transition rate matrix Q of the QBD process has the sub-matrices given as:

$$\mathbf{A}_0 = \begin{cases} \begin{pmatrix} -(\lambda+\theta) & 0 & \theta & 0 \\ 0 & -(\lambda+\theta) & 0 & \theta \\ 0 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & -\lambda \end{pmatrix} (for\ SWV), \\ \begin{pmatrix} -\lambda & 0 \\ 0 & -\lambda \end{pmatrix} (for\ MWV), \end{cases} ,$$

$$\mathbf{C}_0 = \begin{cases} \begin{pmatrix} \lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 & 0 \end{pmatrix} (for\ SWV), \\ \begin{pmatrix} \lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 & 0 \end{pmatrix} (for\ MWV), \end{cases}$$

$$\mathbf{C}_1 = \begin{pmatrix} \lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda \end{pmatrix}, A_1 = \begin{pmatrix} \delta_1 & 0 & \theta & 0 & 0 & 0 \\ 0 & \delta_1 & 0 & \theta & 0 & 0 \\ 0 & 0 & \delta_3 & 0 & r\mu_1 & 0 \\ 0 & 0 & 0 & \delta_3 & 0 & r\mu_1 \\ 0 & 0 & 0 & 0 & \delta_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & \delta_4 \end{pmatrix},$$

$$\mathbf{B}_1 = \begin{cases} \begin{pmatrix} \eta & 0 & 0 & 0 \\ 0 & \eta & 0 & 0 \\ (1-r)\mu_1 & 0 & 0 & 0 \\ 0 & (1-r)\mu_1 & 0 & 0 \\ \mu_2 & 0 & 0 & 0 \\ 0 & \mu_2 & 0 & 0 \end{pmatrix} (for\ SWV), \\ \begin{pmatrix} \eta & 0 \\ 0 & \eta \\ (1-r)\mu_1 & 0 \\ 0 & (1-r)\mu_1 \\ \mu_2 & 0 \\ 0 & \mu_2 \end{pmatrix} (for\ MWV), \end{cases}$$

$$\mathbf{B}_i=\begin{pmatrix}\eta & (i-1)\alpha q_{01} & 0 & 0 & 0 & 0\\ 0 & \eta+(i-1)\alpha q_{11} & 0 & 0 & 0 & 0\\ 0 & 0 & (1-r)\mu_1 & 0 & 0 & 0\\ 0 & 0 & 0 & (1-r)\mu_1 & 0 & 0\\ 0 & 0 & \mu_2 & 0 & 0 & 0\\ 0 & 0 & 0 & \mu_2 & 0 & 0\end{pmatrix}, 2\le i\le N-1,$$

$$\mathbf{B}_i=\begin{pmatrix}\eta & (N-1)\alpha q_{01} & 0 & 0 & 0 & 0\\ 0 & \eta+(N-1)\alpha q_{11} & 0 & 0 & 0 & 0\\ 0 & 0 & (1-r)\mu_1 & 0 & 0 & 0\\ 0 & 0 & 0 & (1-r)\mu_1 & 0 & 0\\ 0 & 0 & \mu_2 & 0 & 0 & 0\\ 0 & 0 & 0 & \mu_2 & 0 & 0\end{pmatrix}, i\ge N,$$

$$\mathbf{A}_i=\begin{pmatrix}\delta_2+(i-1)\alpha q_{00} & 0 & \theta & 0 & 0 & 0\\ (i-1)\alpha q_{10} & \delta_2 & 0 & \theta & 0 & 0\\ 0 & 0 & \delta_3 & 0 & r\mu_1 & 0\\ 0 & 0 & 0 & \delta_3 & 0 & r\mu_1\\ 0 & 0 & 0 & 0 & \delta_4 & 0\\ 0 & 0 & 0 & 0 & 0 & \delta_4\end{pmatrix}, 2\le i\le N-1,$$

$$\mathbf{A}_i=\begin{pmatrix}\delta_5+(N-1)\alpha q_{00} & 0 & \theta & 0 & 0 & 0\\ (N-1)\alpha q_{10} & \delta_2 & 0 & \theta & 0 & 0\\ 0 & 0 & \delta_3 & 0 & r\mu_1 & 0\\ 0 & 0 & 0 & \delta_3 & 0 & r\mu_1\\ 0 & 0 & 0 & 0 & \delta_4 & 0\\ 0 & 0 & 0 & 0 & 0 & \delta_4\end{pmatrix}, i\ge N.$$

where $\delta_1=-(\lambda+\theta+\eta)$,$\delta_2=-(\lambda+\theta+\eta+(i-1)\alpha)$, $\delta_3=-(\lambda+\mu_1)$, $\delta_4=-(\lambda+\mu_2)$ and $\delta_5=-(\lambda+\theta+\eta+(N-1)\alpha)$.

Let $\mathbf{P}$ be the corresponding steady state probability vector of $\mathbf{Q}$. By partitioning the vector $\mathbf{P}$ as $\mathbf{P}=\{\mathbf{P}_0,\mathbf{P}_1,\mathbf{P}_2,\dots\}$ where
$\mathbf{P}_0$=$[E_{0,0,0},E_{0,0,1},E_{0,1,0},E_{0,1,1}]$, (for SWV) and $\mathbf{P}_0$=$[E_{0,0,0},E_{0,0,1}]$, (for MWV),
$\mathbf{P}_{i+1}$=$[P_{i,0,0},P_{i,0,1},P_{i,1,0},P_{i,1,1},Q_{i,1,0},Q_{i,1,1}]$, $i\ge 0$.

According to Neuts [13] , the system is stable and the steady state probability vector exists if and only if $\mathbf{YC}_1\mathbf{e}_6<\mathbf{YB}_N\mathbf{e}_6$ where $\mathbf{Y}$ is an invariant probability of the matrix $\mathbf{M}$=$\mathbf{A}_N+\mathbf{B}_N+\mathbf{C}_1$. $\mathbf{e}_n$ denotes a column vector with size n, and all elements equal to 1. $\mathbf{Y}$ satisfies the equations $\mathbf{YM}=0$ and $\mathbf{Ye}_6=1$.

Apparently, when the stability condition is satisfied, the sub-vectors of $\mathbf{P}$, corresponding to different levels satisfy

$$\mathbf{P}_n=\mathbf{P}_N\mathbf{R}^{n-N}, n\ge N, \tag{17}$$

where the matrix R is the minimal non-negative solution of the matrix quadratic equation

$$\mathbf{C}_1+\mathbf{RA}_N+\mathbf{R}^2\mathbf{B}_N=\mathbf{0}, \tag{18}$$

which can be obtained by using the following iterative procedure.

Computational algorithm for R:

Step 1: Set $k=1$.
Step 2: Set $\mathbf{U}=\mathbf{A}_N$ and calculate $\mathbf{G}=(\mathbf{I}-\mathbf{U})^{-1}\mathbf{B}_N$.
Step 3: Increment k by 1.
Step 4: Replace $\mathbf{U}=\mathbf{A}_N+\mathbf{C}_1\mathbf{G}$ and $\mathbf{G}=(\mathbf{I}-\mathbf{U})^{-1}\mathbf{B}_N$.
Step 5: Repeat Steps 3 and 4 until $\parallel \mathbf{e}_n-\mathbf{Ge}_n\parallel_\infty<\epsilon$, where ϵ is a stopping tolerance.
Step 6: Calculate $\mathbf{R}=\mathbf{C}_1(\mathbf{I}-\mathbf{U})^{-1}$.

From the equation $\mathbf{PQ}=\mathbf{0}$, the governing system of difference equations can be given as

$$\mathbf{P}_0\mathbf{A}_0+\mathbf{P}_1\mathbf{B}_1=\mathbf{0}, \tag{19}$$

$$\mathbf{P}_0\mathbf{C}_0 + \mathbf{P}_1\mathbf{A}_1 + \mathbf{P}_2\mathbf{B}_2 = \mathbf{0}, \tag{20}$$
$$\mathbf{P}_{n-1}\mathbf{C}_1 + \mathbf{P}_\mathbf{n}\mathbf{A}_n + \mathbf{P}_{n+1}\mathbf{B}_{n+1} = \mathbf{0}, 2 \le n \le N-1, \tag{21}$$
$$\mathbf{P}_{n-1}\mathbf{C}_1 + \mathbf{P}_n\mathbf{A}_N + \mathbf{P}_{n+1}\mathbf{B}_N = \mathbf{0}, n \ge N, \tag{22}$$
and the normalizing condition
$$\sum_{n=0}^{\infty} \mathbf{P}_n\mathbf{e}_n = 1. \tag{23}$$
From equations (19) to (22), after some mathematical manipulations, we get
$$\mathbf{P}_{n-1} = \mathbf{P}_n\boldsymbol{\phi}_n, 1 \le n \le N, \tag{24}$$
$$\mathbf{P}_N[\boldsymbol{\phi}_N\mathbf{C}_1 + \mathbf{A}_N + \mathbf{R}\mathbf{B}_N] = \mathbf{0}. \tag{25}$$
where
$$\boldsymbol{\phi}_1 = -\mathbf{B}_1(\mathbf{A}_0^{-1}), \boldsymbol{\phi}_2 = -\mathbf{B}_2[\boldsymbol{\phi}_1\mathbf{C}_0 + \mathbf{A}_1]^{-1}, \boldsymbol{\phi}_n = -\mathbf{B}_\mathbf{n}(\mathbf{A}_{\mathbf{n-1}} + \boldsymbol{\phi}_{\mathbf{n-1}}\mathbf{C}_\mathbf{1})^{-\mathbf{1}}, 3 \le n \le N.$$
solving equations (23) and (24), we get
$$\mathbf{P}_N[\sum_{j=1}^{N} \prod_{i=N}^{m} \boldsymbol{\phi}_i + (\mathbf{I}-\mathbf{R})^{-1}]\mathbf{e}_n = 1. \tag{26}$$
Solving equations (25) and (26), we obtain $\mathbf{P}_N$. We use equations (17) and (24) to get $\mathbf{P}_n$ for $n \ge 0$.

V. Performance Measures

- Expected number of customers in the queue, when the server is busy in FES and SOS, respectively are
$$E[QF] = \sum_{n=1}^{\infty} nP_{n,1,0} + \sum_{n=1}^{\infty} nP_{n,1,1}; E[QS] = \sum_{n=1}^{\infty} nQ_{n,1,0} + \sum_{n=1}^{\infty} nQ_{n,1,1}.$$
- Expected number in the queue, when the server is in WV is given as
$$E[Q_{WV}] = \sum_{n=1}^{\infty} nP_{n,0,0} + \sum_{n=1}^{\infty} nP_{n,0,1}.$$
- Expected number of customers in the system is
$$E[L] = \sum_{n=0}^{\infty} (n+1)[P_{n,1,0} + P_{n,1,1}] + \sum_{n=0}^{\infty} (n+1)[P_{n,0,0} + P_{n,0,1}] + \sum_{n=0}^{\infty} (n+1)[Q_{n,1,0} + Q_{n,1,1}].$$
- Expected reneging rate of the customer is
$$E[RC] = \sum_{n=1}^{\infty} n\alpha P_{n,0,0} + \sum_{n=1}^{\infty} n\alpha P_{n,0,1}.$$
- Expected number of customers served is
$$ECS = \sum_{n=0}^{\infty} \eta(P_{n,0,0} + P_{n,0,1}) + \sum_{n=0}^{\infty} \mu_1(P_{n,1,0} + P_{n,1,1}) + \sum_{n=0}^{\infty} r\mu_2(Q_{n,1,0} + Q_{n,1,1}).$$
- Probability that the server is on WV is
$$P_{WV} = \sum_{n=0}^{\infty} P_{n,0,0} + \sum_{n=0}^{\infty} P_{n,0,1}.$$
- Probability that the server is idle is
$$P_0 = E_{0,0,0} + E_{0,0,1} + E_{0,1,0} + E_{0,1,1} (for\ SWV); P_0 = E_{0,0,0} + E_{0,0,1} (for\ MWV)$$
- Probability that the server is busy with FES and SOS is
$$PBF = \sum_{n=0}^{\infty} P_{n,1,0} + \sum_{n=0}^{\infty} P_{n,1,1}; PBS = \sum_{n=0}^{\infty} Q_{n,1,0} + \sum_{n=0}^{\infty} Q_{n,1,1}.$$

5.1 Special Cases of the Model

Case 1: Taking particular values of the parameters as $\alpha = 0$, $r = 0$, $\mu_2 = 0$ and $\omega = 0$, our model reduces to $M/M/1$ queueing model with MWV and results match with Servi and Finn [17].
Case 2: The present model reduces to an $M/M/1$ queueing model with SWV and MWV by taking values of the parameters as $\alpha = 0$, $r = 0$ and $\mu_2 = 0$. Results match with Selvaraju and Goswami [16] (by taking $\alpha = 0$ in their paper).

5.2 Cost Model

This section develops a cost model in order to carry out an economic analysis of the queueing system under consideration. We formulate an expected cost function per unit time, where the service rate in FES (μ_1) and that in SOS (μ_2) are decision variables.

Let us define

$c_1 \equiv$ Cost per unit time when the customer waits for the service,

$c_2 \equiv$ Cost per unit time when the server is on WV,

$c_3 \equiv$ Cost per unit time when the customer reneges,

$c_4 \equiv$ Cost per unit time when the server is busy with SOS.

Using the above cost parameters, the following cost optimization problem is designed as

$$minimize\ \tau_c[\mu_1, \mu_2] = c_1\mu_1 E[L] + c_2\eta P_{WV} + c_3 E[RC] + c_4\mu_2.$$

Let Rev be the revenue earned by providing service to a customer, τ_r be the total expected revenue per unit time of the system and τ_p be the total expected profit per unit time of the system. Thus,

$$\tau_r = Rev \times ECS, \tau_p = \tau_r - \tau_c.$$

5.3 Bat Algorithm

Bat algorithm is an innovative technique proving to give better solution than many popular traditional and heuristic algorithms for solving complex engineering problems. The bat algorithm is a meta-heuristic algorithm for global optimization. It was inspired by the echolocation behavior of micro bats, with varying pulse rates of emission and loudness. The bat algorithm was developed by Yang in 2010.

The bat algorithm works with the following three idealized rules

1. All bats use the echolocation to detect the distance from a food source and also have the knowledge to distinguish between foods/victims and background barriers.
2. Bats fly randomly in the surroundings with velocity $\mathbf{V_i}$ at position $\mathbf{x_i}$ with a frequency f_{min}, varying wavelength W and loudness L_0 in search for prey. They can automatically regulate the frequency(or wavelength) of their emitted pulses and change the rate of pulse emission (p) correspondingly in the range between 0 and 1, depending on the proximity of their target.
3. Though the loudness can vary in a variety of ways, we consider that the loudness varies from a large (positive) L_0 to a minimum constant value L_{min}.

In addition to these assumptions, for simplicity, the frequency f is taken in a range $[f_{min}, f_{max}]$ corresponds to a range of wavelengths $[W_{min}, W_{max}]$. We can either use wave lengths or frequencies for implementation, we use $f_{min} = 0$ and f_{max} depending on the domain size of the problem of interest. Therefore, with the help of the mentioned assumptions, the updated equations for frequency f_i, position $\mathbf{x_i}$ and velocity $\mathbf{V_i}$ are as follows

$$f_i = f_{min} + (f_{max} - f_{min})\boldsymbol{\vartheta},$$
$$\mathbf{V_i^{t+1}} = \mathbf{V_i^t} + (\mathbf{x}_i^t - \mathbf{x}^*)f_i,$$
$$\mathbf{x}_i^{t+1} = \mathbf{x}_i^t + \mathbf{V_i^{t+1}}.$$

where

- $\boldsymbol{\vartheta} \in [0,1]$ is a uniformly distributed random vector.
- f_i is the frequency that i^{th} bat emits and f_{min}, f_{max} are the lower and upper bounds of frequencies, respectively.
- V_i^t is the velocity of i^{th} bat after t generations.
- x_i^t is the position of i^{th} bat after t generations.
- x^* is the current best position (solution) of the fitness function among all the bats.

After selecting a solution among the current best solutions, for the local search we use the random walk for each bat. Hence, the new position updating formula is generated locally and is expressed as

$$\mathbf{x}_{new} = \mathbf{x}_{old} + \epsilon_1 L^{(t)}$$

where $\epsilon_1 \in [-1,1]$ is a random number and $L^{(t)} =< L_i^t >$ is the average loudness of all the bats at time instant t. Now to control the step size, the new position updating formula is rewritten as

$$\mathbf{x}_{new} = \mathbf{x}_{old} + \varsigma \epsilon_t L^{(t)}$$

where, the value of ϵ_t is taken from the Gaussian normal distribution N(0,1), and ς is a scaling factor having standard value 0.001.

VI. Numerical Investigations

This section is devoted to study numerically the performance measures and cost profit aspects associated with the model using Mathematica software. The parameters of the model are assumed to be $\lambda = 0.8$, $\mu_1 = 3.5$, $\mu_2 = 3.0$, $\eta = 2.5$, $\theta = 0.5$, $\alpha = 0.7$, $r = 0.6$, $q_{00} = 0.6$, $q_{11} = 0.5$. For the economic analysis of the system, we fix the different costs as $c_1 = 5$, $c_2 = 4$, $c_3 = 3$, and $c_4 = 2$, for bat algorithm, we assume $f_{min} = 0$, $f_{max} = 2$, $L = 0.5$, $p = 0.5$, lower and upper bounds of μ_1 and μ_2 are taken as [1.5, 4.5] and [1.0, 4.0], respectively.

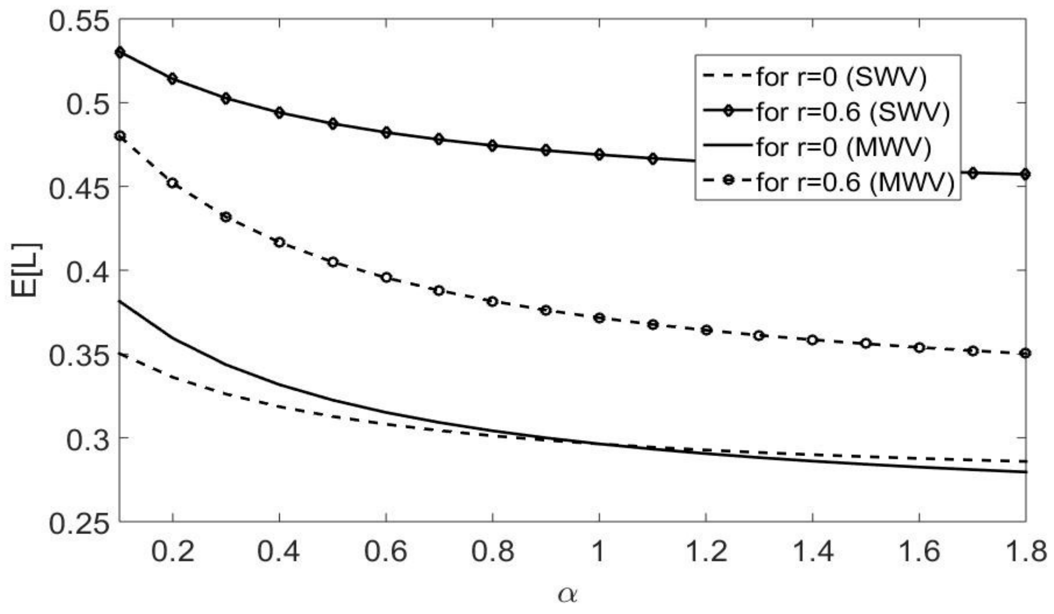

Figure 2: *Effect of α on $E[L]$ for different r.*

Table 1: *Effect of r on performance measures.*

		SWV		MWV	
	Cases	$E[L]$	P_0	$E[L]$	P_0
$r = 0$	Correlated reneging	0.30435	0.75028	0.30917	0.73764
	No reneging	0.37128	0.72819	0.41401	0.70354
$r = 0.3$	Correlated reneging	0.38191	0.70939	0.34312	0.72109
	No reneging	0.45345	0.68633	0.46087	0.68318
$r = 0.6$	Correlated reneging	0.47796	0.66375	0.38789	0.70126
	No reneging	0.55452	0.63993	0.52144	0.65913

Table 2: *Effect of r and μ_1 on performance measures.*

		SWV		MWV	
	Cases	$E[L]$	P_0	$E[L]$	P_0
$r = 0$	$\mu_1 = 3.6$	0.29911	0.75334	0.30689	0.73883
	$\mu_1 = 3.8$	0.29826	0.75892	0.30288	0.74099
	$\mu_1 = 4.0$	0.29757	0.76387	0.29944	0.74289
$r = 0.3$	$\mu_1 = 3.6$	0.35797	0.71279	0.34001	0.72252
	$\mu_1 = 3.8$	0.35550	0.71900	0.33453	0.72508
	$\mu_1 = 4.0$	0.35341	0.72449	0.32985	0.72734
$r = 0.6$	$\mu_1 = 3.6$	0.43122	0.66757	0.38355	0.70298
	$\mu_1 = 3.8$	0.42638	0.67451	0.37591	0.70607
	$\mu_1 = 4.0$	0.42227	0.68065	0.36941	0.70878

Table 3: *Effect of q_{11} and q_{00} on τ_c, τ_r and τ_p.*

	SWV			MWV		
	τ_c	τ_r	τ_p	τ_c	τ_r	τ_p
$q_{11} = 0.2$	15.7559	41.3303	25.5743	15.1813	37.6903	22.5090
$q_{11} = 0.4$	15.6551	41.1014	25.4463	15.0058	37.2814	22.2756
$q_{11} = 0.6$	15.5365	40.8314	25.2950	14.7985	36.7970	21.9986
$q_{00} = 0.3$	15.4533	40.6499	25.1966	14.6526	36.4701	21.8175
$q_{00} = 0.5$	15.5335	40.8292	25.2957	14.7933	36.7929	21.9997
$q_{00} = 0.7$	15.6963	41.1860	25.4897	15.0775	37.4326	22.3551

Table 4: *Effect of λ, q_{11} and r on optimum cost.*

	μ_1^*	μ_2^*	τ_c^*
$\lambda = 0.6$	2.2495	1.6517	9.8199
$\lambda = 0.8$	2.5264	2.1841	13.2489
$\lambda = 1.0$	2.9319	2.6845	17.2492
$q_{11} = 0.4$	2.5313	2.2132	13.6841
$q_{11} = 0.6$	2.5176	2.2015	13.5692
$q_{11} = 0.8$	2.4924	2.1524	13.4325
$r = 0.2$	1.8532	1.3842	10.7163
$r = 0.4$	2.4609	1.9185	11.9251
$r = 0.6$	2.7886	2.4301	12.1792

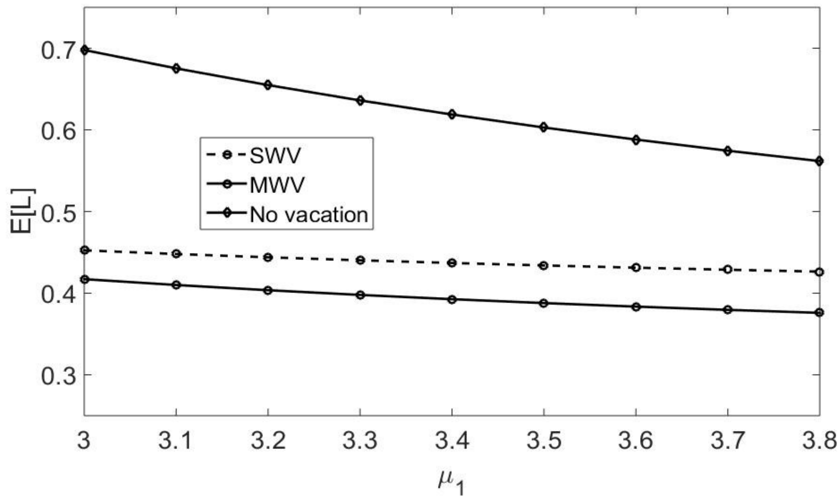

Figure 3: *Effect of* μ_1 *on* $E[L]$.

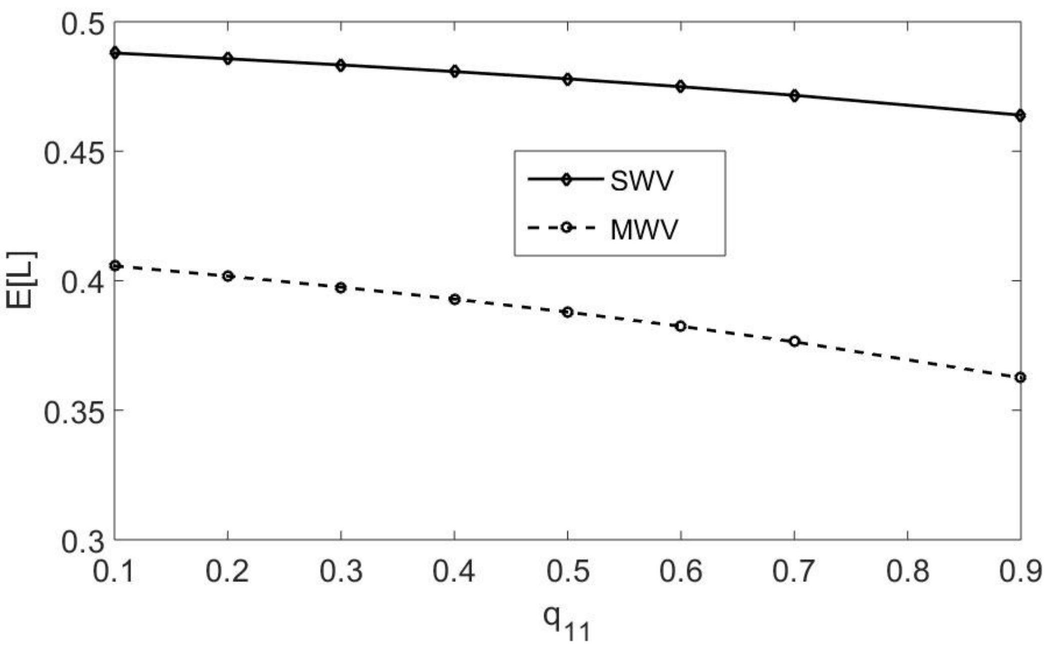

Figure 4: *Effect of* q_{11} *on* $E[L]$.

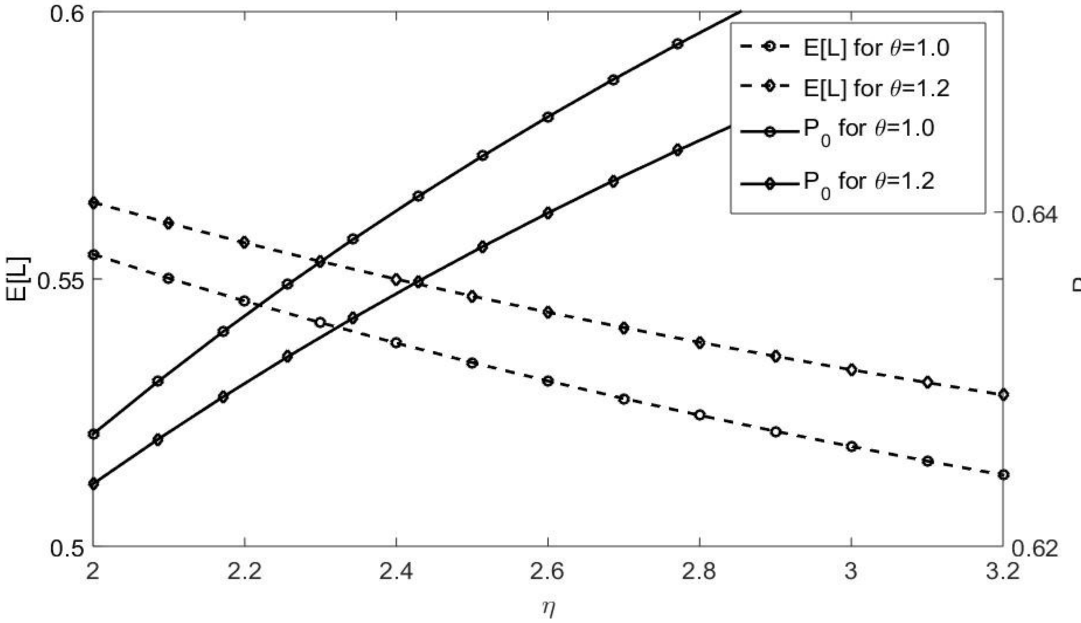

Figure 5: *Effect of η on $E[L]$ and P_0 for different values of θ.*

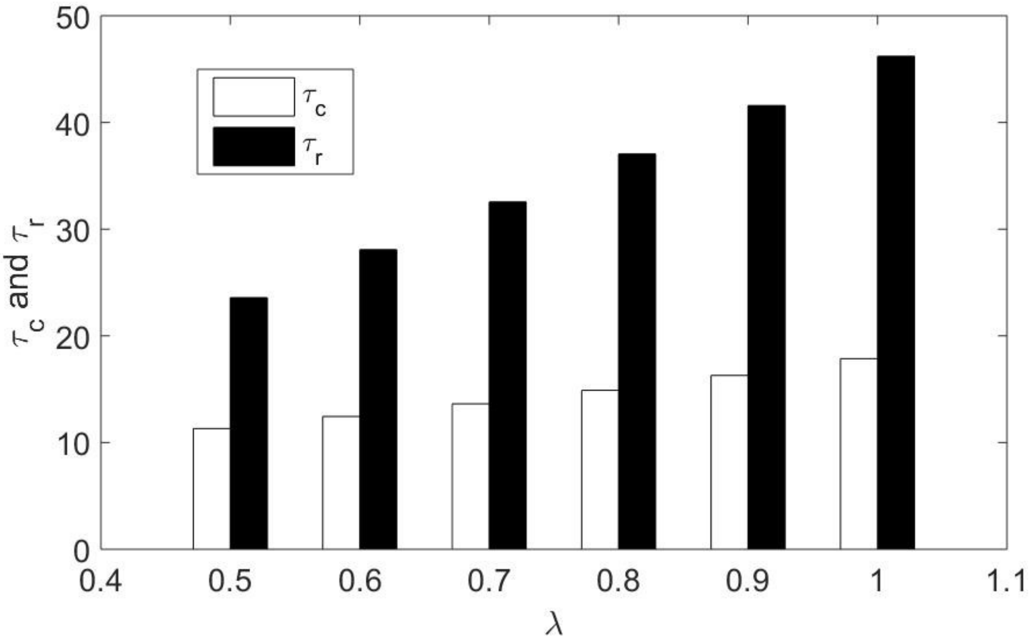

Figure 6: *Effect of λ on τ_c and τ_r.*

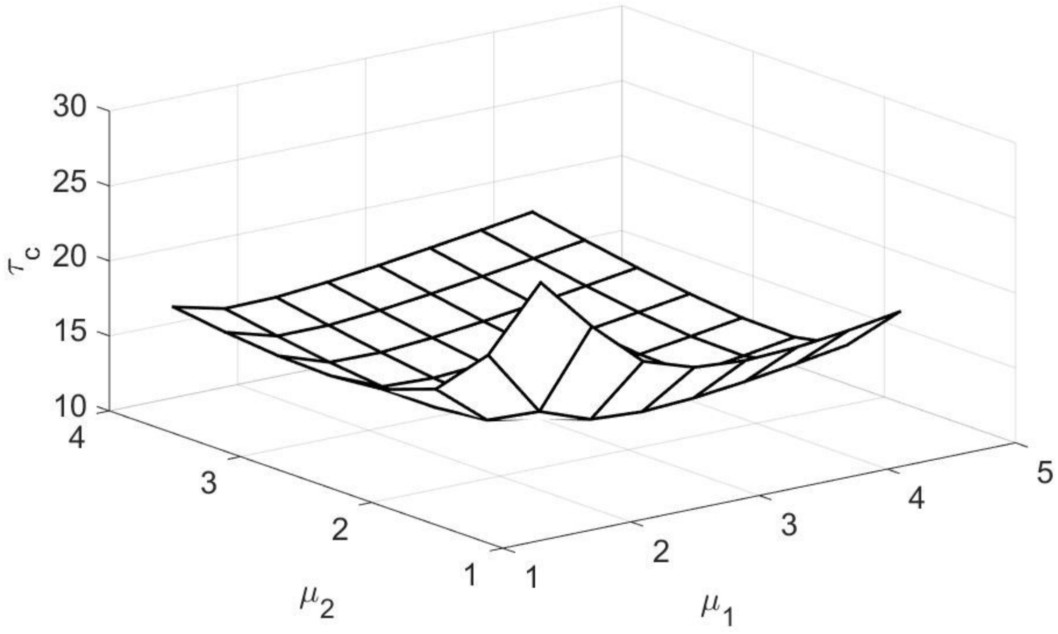

Figure 7: *Effect of μ_1 and μ_2 on τ_c.*

The effect of reneging rate (α) on average system length ($E[L]$) is shown in Figure 2. It is clear that as α increases, $E[L]$ decreases for both the models. In absence of SOS ($r = 0$) , we see an interesting behavior; for $\alpha < 1$ system length in SWV is smaller, for $\alpha > 1$ system length in MWV is smaller and at $\alpha = 1$ they coincide. In presence of an optional service, obviously MWV gives smaller system size because of the predominant effect of reneging.

In Tables 1 and 2, for fixed μ_1, as SOS probability (r) increases, the average system length ($E[L]$) increases and idle probability P_0 decreases. Further, for fixed r, as μ_1 increases a completely opposite trend is observed.

Table 3 illustrates the impact of reneging (non-reneging) probabilities of customers at both transition marks q_{11} (q_{00}) on total expected cost (τ_c), total expected revenue (τ_r) and total expected profit (τ_p) for both the models. As expected, an increase in q_{11}, decrease τ_c, τ_r and τ_p. This is because of the significant number of lost customers. On the other hand, opposite trend is observed for q_{00}. Therefore, it reveals the fact that q_{00} has positive effect on the economy of the system as it enforces the customers to be held in the system.

In table 4, using bat algorithm, the effect of λ, q_{11} and r on optimal service rates (μ_1^* and μ_2^*) and minimum expected cost (τ_c^*) is shown for MWV model. We observe that

- when arrival rate (λ) increases,μ_1^*, μ_2^* and τ_c^* increase as expected in the view of stability of the system.
- most importantly, increase in reneging probability (q_{11}), substantially reduce the optimal service rates and minimum cost due to lost customers.
- as r increases,μ_1^*, μ_2^* and τ_c^* increase. This agrees with our intuition.

In Figure 3, we show the effect of μ_1 on the system lengths for the model with SWV, MWV and no vacation. The graphs show the larger system lengths in the absence of vacation. This is explained by the fact that reneging occurs only during WV. When there is no WV, customers are remain in the system till they get served.

Figure 4 depicts that an increase in q_{11}, decreases the system length ($E[L]$), which is obviously true. Through Figure 5 we demonstrate the effect of the service rate in WV period (η) on $E[L]$ and P_0 for different values of vacation rate (θ) in SWV model. It is quite obvious that for a fixed θ, increase in η, decreases $E[L]$ and increases P_0. Moreover, upon increasing of θ, reverse trend is observed.

The impact of arrival rate λ on τ_c and τ_r for MWV policy is shown in Figure 6. We observe that, τ_c and τ_r increase with the increasing of λ. This is quite reasonable, the bigger the arrival rate, the greater the total expected cost and the total expected revenue. In Figure 7, we portray the three-dimensional surface plot generated through the joint variation of decision parameters μ_1 and μ_2 for MWV model. It prompts the convex nature of τ_c with respect to μ_1 and μ_2. As per the restriction of the system resources, the analyst can design parameters for the optimal service cost.

VII. Conclusion

In this paper, we have carried out an analysis of infinite buffer single server queueing system with SOS and correlated reneging under single and multiple working vacation policies. Using matrix geometric method, we derived the steady-state probabilities of the system. Some performance measures are developed. A cost model was established, and bat algorithm is applied to determine the optimal values of service rates in FES and SOS with the aim of minimizing the expected cost per unit time. The effects of various parameters on the system performance measures were explored by numerical experiments. Our study shows that

- increasing the service rates reduces the average system length.
- increase of the non-reneging probability of the customer at both transition marks, increases the expected system length and it shows the positive effect on the economy of the system as it increases the revenue.
- MWV model has lower system lengths for higher reneging rates due to the departures of customers by the way of reneging.

According to the analysis of expected system length by numerical examples, we find that our model represents some practical problems reasonably. The obtained results have potential applications in modeling computer and telecommunication systems, computer networks, manufacturing, and so on. So, the service companies may design the reasonable WV rate, service rates and correlated reneging rates to enable the companies to operate more flexibly and efficiently. To make the system modeling more closer to real world problems, we extend our model to consider general service times and server breakdowns.

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Received: September 11, 2020
Accepted: November 25, 2020

A Three Unit Warm and Cold Standby System Model of Discrete Parametricmarkov Chain

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Abstract

The paper deals with cost benefit analysis of a threeidentical unit cold and warm standby system model. Each unit has two modes- normal (N) and total failure (F). The warm standby unit becomes operative instantaneously upon the failure of an operative unit, whereas cold standby unit needs activation to become operative or to be warm standby. A single repairman is always available to repair a failed unit. The activation of a cold standby unit is made by the operator itself. The distributions of time to failure, time to repair and activation time are taken as independent random variables of discrete nature having geometric distributions with different parameters.

Keywords: Regenerative point, reliability, MTSF, availability, geometric distribution, Markov-Chain.

I. Introduction

Two-unit cold standby redundant system models have been analyzed widely in the literature of reliability by various authors including [1,3,5,8]. In these system models the authors have assumed that the standby unit starts operation instantaneously with the help of a switching device when the operative unit fails. In real life, the situations arise in many times when the standby unit does not work instantaneously and take a significant time to be operative and this time may be called activation time of cold standby unit. Gupta et al. [6] analyzed a two-unit standby system model assuming that the cold standby unit goes for activation before starting its operation on line. During the activation time of cold standby unit the system remains down and no output is obtained by the system till the activation is completed and standby unit starts working. To avoid the situation of down time of the system due to activation, the warm standby redundancies have been considered in the literature of reliability as a warm standby unit becomes operative instantaneously without any activation. But the drawback of the warm standby unit is that it can fail during its standby state. So, one is to prefer a warm standby over the cold standby when during the down period of the system due to activation of cold standby there is a great unbearable loss.

Some authors including [4,7,9,10] analyzed the two-unit warm standby redundant system models using different concepts. All the above system models have been analyzed by considering continuous distributions of all the random variables involved.

The purpose of the present paper is to consider both types of standbys warm and cold simultaneously in a single system i.e. a three unit redundant system. As soon as the operating unit fails, the warm standby unit becomes operative instantaneously whereas the cold standby unit needs activation before coming into operation or warm standby unit. The warm standby unit may also fail while it is in standby position whereas the cold standby unit can't fail during its standby state. This system model is based on discrete parametric Markov-Chain. Gupta and Varshney [2] introduced the concept of discrete parametric Markov-Chain in analyzing the system models in the field of reliability modeling. The following economic related measures of system effectiveness are obtained by using regenerative point technique-

i) Transition probabilities and mean sojourn times in various states.
ii) Reliability and mean time to system failure.
iii) Point-wise and steady-state availability of the system during time (0, t-1).
iv) Expected busy period of repairman during time (0, t-1).
v) Net expected profit incurred by the system during a finite and steady-state are obtained.

II. Model Description and Assumptions

1. The system consists of three identical units.Initially one unit is operative and rests two are kept in spare as cold and warm standbys.
2. Each unit has two modes- Normal (N) and Total failure (F).
3. Upon failure of an operating unit, the warm standby unit becomes operative instantaneously whereas the cold standby unit requires activation time before coming into operation/warm standby.
4. A switching device is used to start the activation of cold standby unit and to put a warm standby into operation which is always perfect and instantaneous.
5. A single repairman is always available with the system to repair a failed unit and a repaired unit becomes either operative, warm standby or cold standby as per the situations.
6. The activation action of a cold standby unit is carried out by the operator itself and there is no need for a separate human being at the system for this purpose. After completion of activation unit becomes operative or warm standby as the requirement.
7. The time to failure, time to activation and repair time follow geometric distributions with different parameters.

III. Notations and States of the System

a) Notations :

pq^x : p.m.f. of failure time of a unit; $p+q=1$.

rs^x : p.m.f. of repair time byrepairman of failed unit and $r+s=1$.

cd^x : p.m.f. of time to activate of a cold standby unitrespectively; $c+d=1$.

$q_{ij}(\bullet), Q_{ij}(\bullet)$: p.m.f. and C.d.f. of one step or direct transition time from state S_i to S_j.

p_{ij} : steady state transition probability from state S_i to S_j.

$$p_{ij} = Q_{ij}(\infty)$$

$Z_i(t)$: probability that the system sojourn in state S_i up to epoch (t-1).

ψ_i : mean sojourn time in state S_i.

$*, h$: symbol and dummy variable used in geometric transform e. g.

$$GT\left[q_{ij}(t)\right] = q_{ij}^*(h) = \sum_{t=0}^{\infty} h^t q_{ij}(t)$$

b) Symbols for the States of the System:

$N_0 / N_{ws} / N_{cs}$:unit in normal mode and operative/warm standby/cold standby.

N_{ca} : the cold standby unit in normal mode and under activation.

F_r / F_{wr} : unit in total failure (F) mode and under repair/waits for repair.

The transition diagram of the system model is shown in fig. 1.

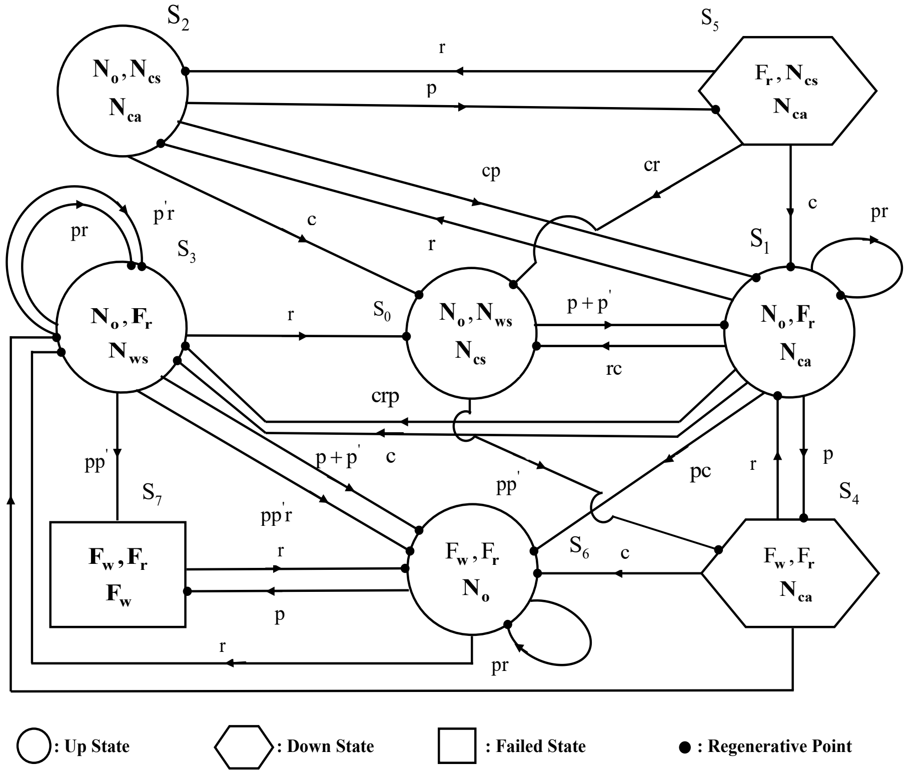

Figure 1

With the help of above symbols the possible states of the system are:

$$S_0 \equiv \begin{pmatrix} N_o, N_{ws} \\ N_{cs} \end{pmatrix}, \quad S_1 \equiv \begin{pmatrix} N_o, F_r \\ N_{ca} \end{pmatrix}, \quad S_2 \equiv \begin{pmatrix} N_o, N_{cs} \\ N_{ca} \end{pmatrix}, \quad S_3 \equiv \begin{pmatrix} N_o, F_r \\ N_{ws} \end{pmatrix}$$

$$S_4 \equiv \begin{pmatrix} F_w, F_r \\ N_{ca} \end{pmatrix}, \quad S_5 \equiv \begin{pmatrix} F_r, N_{cs} \\ N_{ca} \end{pmatrix}, \quad S_6 \equiv \begin{pmatrix} F_w, F_r \\ N_o \end{pmatrix}, \quad S_7 \equiv \begin{pmatrix} F_w, F_r \\ F_w \end{pmatrix}$$

The states S_0, S_1, S_2, S_3, S_6 are up states; S_4, S_5 are down states and S_7 is failed state.

IV. Transition Probabilities

Let $Q_{ij}(t)$ be the probability that the system transits from state S_i to S_j during time interval (0, t) i.e., if T_{ij} is the transition time from state S_i to S_j then

$$Q_{ij}(t) = P\lfloor T_{ij} \le t \rfloor$$

By using simple probabilistic arguments we have,

$$Q_{01}(t) = \frac{(pq' + p'q)}{1-(qq')}\left[1-(qq')^{(t+1)}\right], \qquad Q_{04}(t) = \frac{pp'}{1-(qq')}\left[1-(qq')^{(t+1)}\right]$$

$$Q_{10}(t) = \frac{rcq}{1-sdq}\left[1-(sdq)^{t+1}\right], \qquad Q_{11}(t) = \frac{prd}{1-qds}\left[1-(qds)^{t+1}\right]$$

$$Q_{12}(t) = \frac{rdq}{1-sdq}\left[1-(sdq)^{t+1}\right] \qquad Q_{13}(t) = \frac{c(rp+qs)}{1-dqs}\left[1-(dqs)^{t+1}\right]$$

$$Q_{14}(t) = \frac{pds}{1-qds}\left[1-(qds)^{t+1}\right], \qquad Q_{16}(t) = \frac{pcs}{1-qds}\left[1-(qds)^{t+1}\right]$$

$$Q_{20}(t) = \frac{cq}{1-qd}\left[1-(qd)^{t+1}\right], \qquad Q_{21}(t) = \frac{cp}{1-dq}\left[1-(dq)^{t+1}\right]$$

$$Q_{25}(t)=\frac{pd}{1-qd}\left[1-(qd)^{t+1}\right], \qquad Q_{30}(t)=\frac{rqq'}{1-sqq'}\left[1-(sqq')^{t+1}\right]$$

$$Q_{33}(t)=\frac{r(pq'+p'q)}{1-sqq'}\left[1-(sqq')^{t+1}\right], \qquad Q_{36}(t)=\frac{s(pq'+p'q)}{1-sqq'}\left[1-(sqq')^{t+1}\right]$$

$$Q_{37}(t)=\frac{spp'}{1-sqq'}\left[1-(sqq')^{t+1}\right], \qquad Q_{41}(t)=\frac{rd}{1-ds}\left[1-(ds)^{t+1}\right]$$

$$Q_{43}(t)=\frac{cr}{1-ds}\left[1-(ds)^{t+1}\right], \qquad Q_{46}(t)=\frac{cs}{1-ds}\left[1-(ds)^{t+1}\right]$$

$$Q_{50}(t)=\frac{cr}{1-ds}\left[1-(ds)^{t+1}\right], \qquad Q_{51}(t)=\frac{cs}{1-ds}\left[1-(ds)^{t+1}\right]$$

$$Q_{52}(t)=\frac{rd}{1-ds}\left[1-(ds)^{t+1}\right], \qquad Q_{63}(t)=\frac{qr}{1-qs}\left[1-(qs)^{t+1}\right]$$

$$Q_{66}(t)=\frac{pr}{1-qs}\left[1-(qs)^{t+1}\right], \qquad Q_{67}(t)=\frac{ps}{1-qs}\left[1-(qs)^{t+1}\right]$$

$$Q_{76}(t)=1-s^{t+1} \tag{1-26}$$

The steady state transition probabilities from state S_i to S_j can be obtained from (1-26) by taking $t\to\infty$, as follows:

$$p_{01}=\frac{(pq'+p'q)}{1-(qq')}, \quad p_{04}=\frac{pp'}{1-(qq')}, \quad p_{10}=\frac{rcq}{1-sdq}, \quad p_{11}=\frac{rdp}{1-sdq}$$

$$p_{12}=\frac{rdq}{1-sdq}, \quad p_{13}=\frac{c(rp+sq)}{1-sdq}, \quad p_{14}=\frac{sdp}{1-sdq}, \quad p_{16}=\frac{scp}{1-sdq}$$

$$p_{20}=\frac{cq}{1-dq}, \quad p_{21}=\frac{cp}{1-dq}, \quad p_{25}=\frac{pd}{1-qd}, \quad p_{30}=\frac{rqq'}{1-sqq'}$$

$$p_{33}=\frac{r(pq'+p'q)}{1-sqq'}, \quad p_{36}=\frac{s(pq'+p'q)}{1-sqq'}, \quad p_{37}=\frac{spp'}{1-sqq'}, \quad p_{41}=\frac{rd}{1-sd}$$

$$p_{43}=\frac{rc}{1-sd}, \quad p_{46}=\frac{sc}{1-sd}, \quad p_{50}=\frac{rc}{1-sd}, \quad p_{51}=\frac{sc}{1-sd}$$

$$p_{52}=\frac{rd}{1-sd}, \quad p_{63}=\frac{rq}{1-sq}, \quad p_{66}=\frac{pr}{1-sq}, \quad p_{67}=\frac{sp}{1-sq}$$

$$p_{76}=1-s^{t+1}$$

We observe that the following relations hold-

$$p_{76}=1, \qquad p_{01}+p_{04}=1, \qquad p_{10}+p_{11}+p_{12}+p_{13}+p_{14}+p_{16}=1$$

$$p_{20}+p_{21}+p_{25}=1, \qquad p_{30}+p_{33}+p_{36}+p_{37}=1, \qquad p_{41}+p_{43}+p_{46}=1$$

$$p_{50}+p_{51}+p_{52}=1 \qquad p_{63}+p_{66}+p_{67}=1 \tag{26-34}$$

V. Mean SojournTimes

Let T_i be the sojourn time in state S_i (i=0-7) then ψ_i mean sojourn time in state S_i is given by

$$\psi_i=\sum_{t=1}^{\infty}P\left[T\ge t\right]$$

In particular,

$$\psi_0=\frac{qq'}{1-qq'}, \qquad \psi_1=\frac{sdq}{1-sdq}, \qquad \psi_2=\frac{cq}{1-dq}, \qquad \psi_3=\frac{sqq'}{1-sqq'}$$

$$\psi_4 = \frac{ds}{1-ds}, \qquad \psi_5 = \frac{ds}{1-ds}, \qquad \psi_6 = \frac{qs}{1-qs}, \qquad \psi_7 = \frac{s}{r} \qquad (35\text{-}42)$$

VI. Methodology for Developing Equations

In order to obtain various interesting measures of system effectiveness we developed the recurrence relations for reliability, availability and busy period of repairman as follows-

a) Reliability of the system-

Here we define $R_i(t)$ as the probability that the system does not fail up to epochs 0, 1, 2,.., (t-1) when it is initially started from up state S_i. To determine it, we regard the failed states S_7 as absorbing state. Now, the expression for $R_i(t)$; i=0, 1, 2, 3, 4,5,6; we have the following set of convolution equations.

$$\begin{aligned} R_0(t) &= (qq')^t + \sum_{u=0}^{t-1} q_{01}(u) R_1(t-1-u) + \sum_{u=0}^{t-1} q_{04}(u) R_4(t-1-u) \\ &= Z_0(t) + q_{01}(t-1) © R_1(t-1) + q_{04}(t-1) © R_4(t-1) \end{aligned}$$

Similarly,

$$\begin{aligned} R_1(t) &= Z_1(t) + q_{10}(t-1)©R_0(t-1) + q_{11}(t-1)©R_1(t-1) + q_{12}(t-1)©R_2(t-1) \\ &\quad + q_{13}(t-1)©R_3(t-1) + q_{14}(t-1)©R_4(t-1) + q_{16}(t-1)©R_6(t-1) \\ R_2(t) &= Z_2(t) + q_{20}(t-1)©R_0(t-1) + q_{21}(t-1)©R_1(t-1) + q_{25}(t-1)©R_5(t-1) \\ R_3(t) &= Z_3(t) + q_{30}(t-1)©R_0(t-1) + q_{33}(t-1)©R_3(t-1) + q_{36}(t-1)©R_6(t-1) \\ R_4(t) &= Z_4(t) + q_{41}(t-1)©R_1(t-1) + q_{43}(t-1)©R_3(t-1) + q_{46}(t-1)©R_6(t-1) \\ R_5(t) &= Z_5(t) + q_{50}(t-1)©R_0(t-1) + q_{51}(t-1)©R_1(t-1) + q_{52}(t-1)©R_2(t-1) \\ R_6(t) &= Z_6(t) + q_{63}(t-1)©R_3(t-1) + q_{66}(t-1)©R_6(t-1) \end{aligned} \qquad (43\text{-}49)$$

Where,

$$Z_1(t) = (sdq)^t, \qquad Z_2(t) = (qd)^t, \qquad Z_3(t) = (sqq')^t, \qquad Z_4(t) = (sd)^t$$

$$Z_5(t) = (sd)^t, \qquad Z_6(t) = (sq)^t$$

b) Availability of the System-

Let $A_i(t)$ be the probability that the system is up at epoch (t-1), when it initially started from state S_i. Then, by using simple probabilistic arguments, as in case of reliability the following recurrence relations can be easily developed for $A_i(t)$; i=0 to 7.

$$\begin{aligned} A_0(t) &= Z_0(t) + q_{01}(t-1)©A_1(t-1) + q_{04}(t-1)©A_4(t-1) \\ A_1(t) &= Z_1(t) + q_{10}(t-1)©A_0(t-1) + q_{11}(t-1)©A_1(t-1) + q_{12}(t-1)©A_2(t-1) \\ &\quad + q_{13}(t-1)©A_3(t-1) + q_{14}(t-1)©A_4(t-1) + q_{16}(t-1)©A_6(t-1) + q_{17}(t-1)©A_7(t-1) \\ A_2(t) &= Z_2(t) + q_{20}(t-1)©A_0(t-1) + q_{21}(t-1)©A_1(t-1) + q_{25}(t-1)©A_5(t-1) \\ A_3(t) &= Z_3(t) + q_{30}(t-1)©A_0(t-1) + q_{33}(t-1)©A_3(t-1) + q_{36}(t-1)©A_6(t-1) \\ &\quad + q_{37}(t-1)©A_7(t-1) \end{aligned}$$

$$\begin{aligned}
&A_4(t) = q_{41}(t-1)©A_1(t-1) + q_{43}(t-1)©A_3(t-1) + q_{46}(t-1)©A_6(t-1) \\
&A_5(t) = q_{50}(t-1)©A_0(t-1) + q_{51}(t-1)©A_1(t-1) + q_{52}(t-1)©A_2(t-1) \\
&A_6(t) = Z_6(t) + q_{63}(t-1)©A_3(t-1) + q_{66}(t-1)©A_6(t-1) + q_{67}(t-1)©A_7(t-1) \\
&A_7(t) = q_{76}(t-1)©A_6(t-1)
\end{aligned} \tag{50-57}$$

Where, The values of $Z_i(t)$; i=0 to 3 are same as given in section 6(a). $Z_6(t) = q^t s^t$

c) Busy Period of Repairman

Let $B_i(t)$ be the probability that the repairman is busy in the repairof a failed unit at epoch t-1, when it initially started from state S_i. Then, by using simple probabilistic arguments, as in case of reliability the following recurrence relations can be easily developed for $B_i(t)$; i=0 to 7.

$$\begin{aligned}
&B_0(t) = q_{01}(t-1)©B_1(t-1) + q_{04}(t-1)©B_4(t-1) \\
&B_1(t) = Z_1(t) + q_{10}(t-1)©B_0(t-1) + q_{11}(t-1)©B_1(t-1) + q_{12}(t-1)©B_2(t-1) + q_{13}(t-1)©B_3(t-1) \\
&+q_{14}(t-1)©B_4(t-1) + q_{16}(t-1)©B_6(t-1) + q_{17}(t-1)©B_7(t-1) \\
&B_2(t) = q_{20}(t-1)©B_0(t-1) + q_{21}(t-1)©B_1(t-1) + q_{25}(t-1)©B_5(t-1) \\
&B_3(t) = Z_3(t) + q_{30}(t-1)©B_0(t-1) + q_{33}(t-1)©B_3(t-1) + q_{36}(t-1)©B_6(t-1) \\
&\qquad +q_{37}(t-1)©B_7^{I}(t-1) \\
&B_4(t) = Z_4(t) + q_{41}(t-1)©B_1(t-1) + q_{43}(t-1)©B_3(t-1) + q_{46}(t-1)©B_6(t-1) \\
&B_5(t) = Z_5(t) + q_{50}(t-1)©B_0(t-1) + q_{51}(t-1)©B_1(t-1) + q_{52}(t-1)©B_2(t-1) \\
&B_6(t) = Z_6(t) + q_{63}(t-1)©B_3(t-1) + q_{66}(t-1)©B_6(t-1) + q_{67}(t-1)©B_7(t-1) \\
&B_7(t) = Z_7(t) + q_{76}(t-1)©B_6(t-1)
\end{aligned} \tag{58-65}$$

Where, $Z_7(t) = s^t$.

VII. Analysis of Reliability and MTSF

Taking geometric transform of (43-46) and simplifying the resulting set of algebraic equations for $R_0^*(h)$ we get

$$R_0^*(h) = \frac{N_1(h)}{D_1(h)} \tag{66}$$

Where,

$$\begin{aligned}
N_1(h) = {} & [(1-hq_{11}^*)(1-h^2q_{25}^*q_{52}^*)(1-hq_{33}^*-h^2q_{36}^*q_{63}^*) - h^2q_{12}^*q_{21}^*(1-hq_{33}^*) - h^3q_{12}^*q_{25}^*q_{51}^*(1-hq_{33}^*-h^2q_{36}^*q_{63}^*) \\
& +hq_{14}^*(1-hq_{33}^*)(hq_{41}^*+hq_{46}^*) + h^2q_{14}^*q_{36}^*(1-hq_{46}^*) + h^2q_{13}^*q_{36}^* + h^3q_{14}^*q_{25}^*q_{52}^*(h^3q_{36}^*q_{41}^*q_{63}^* \\
& -hq_{46}^*(1-hq_{33}^*)) - h^3q_{16}^*q_{25}^*q_{52}^*(1-q_{33}^*)]Z_0^* + [hq_{01}^*(1-h^2q_{25}^*q_{52}^*)(1-hq_{33}^*-h^2q_{36}^*q_{63}^*) - hq_{04}^* \\
& (1-hq_{33}^*)(1-h^2q_{25}^*q_{52}^*)]Z_1^* + [h^2q_{01}^*q_{12}^*(1-hq_{33}^*-h^2q_{36}^*q_{63}^*) - h^2q_{04}^*q_{12}^*\{h^2q_{51}^*q_{63}^* + hq_{51}^*(1-hq_{33}^*)\}]Z_2^* \\
& +[hq_{01}^*\{hq_{13}^* - hq_{14}^*(hq_{43}^*-h^2q_{46}^*q_{63}^*) - h^3q_{14}^*q_{25}^*q_{52}^*(hq_{43}^*-h^2q_{46}^*q_{63}^*)\} + hq_{16}^*(1-h^2q_{25}^*q_{52}^*) \\
& -hq_{04}^*(hq_{63}^*(1-hq_{11}^*)(1-h^2q_{25}^*q_{52}^*) - hq_{12}^*(h^2q_{21}^*q_{63}^* + (1-hq_{33}^*) + hq_{41}^*q_{63}^*(hq_{14}^*+h^2q_{25}^*q_{52}^*)]Z_3^* \\
& +[h^2q_{01}^*q_{14}^*(1-h^2q_{25}^*q_{52}^*)\{h^2q_{36}^*q_{63}^*(1-h^2q_{25}^*q_{52}^*) + (1-q_{33}^*)\} + hq_{14}^*(1-hq_{33}^*)(1-h^2q_{25}^*q_{52}^*)]Z_4^* \\
& +[h^3q_{01}^*q_{12}^*q_{25}^*(1-hq_{33}^*-h^2q_{36}^*q_{63}^*)]Z_5^* + [h^2q_{01}^*q_{36}^*(1-h^2q_{25}^*q_{52}^*)(hq_{13}^*+h^2q_{14}^*q_{43}^*) - h^2q_{36}^*q_{63}^*] \\
& -h^3q_{01}^*q_{12}^*q_{25}^*(1-hq_{33}^*)(h^2q_{14}^*q_{46}^* + hq_{16}^*) + hq_{01}^*(1-hq_{33}^*)(hq_{16}^*+h^2q_{14}^*q_{46}^*)
\end{aligned}$$

$$\begin{aligned}
D_1(h) = {} & [(1-hq_{11}^*)(1-h^2q_{25}^*q_{52}^*)(1-hq_{33}^*-h^2q_{36}^*q_{63}^*) - h^2q_{12}^*q_{21}^*(1-hq_{33}^*) - h^3q_{12}^*q_{25}^*q_{51}^*(1-hq_{33}^*-h^2q_{36}^*q_{63}^*) \\
& +hq_{14}^*(1-hq_{33}^*)(hq_{41}^*+hq_{46}^*) + h^2q_{14}^*q_{36}^*(1-hq_{46}^*) + h^2q_{13}^*q_{36}^* + h^3q_{14}^*q_{25}^*q_{52}^*(h^3q_{36}^*q_{41}^*q_{63}^*
\end{aligned}$$

$$
\begin{aligned}
&-hq_{46}^*(1-hq_{33}^*))-h^3q_{16}^*q_{25}^*q_{52}^*(1-q_{33}^*)]-hq_{10}^*[hq_{01}^*(1-h^2q_{25}^*q_{52}^*)(1-hq_{33}^*-q_{36}^*q_{63}^*)-h^2q_{04}^* \\
&(1-hq_{33}^*)(1-h^2q_{25}^*q_{52}^*)]-q_{20}^*[h^2q_{01}^*q_{12}^*(1-hq_{33}^*-h^2q_{36}^*q_{63}^*)-h^2q_{04}^*q_{12}^*\{h^2q_{51}^*q_{63}^*+hq_{51}^*(1-hq_{33}^*)\}] \\
&-q_{30}^*[hq_{01}^*\{hq_{13}^*-hq_{14}^*(hq_{43}^*-h^2q_{46}^*q_{63}^*)-h^3q_{14}^*q_{25}^*q_{52}^*(hq_{43}^*-h^2q_{46}^*q_{63}^*)\}+hq_{16}^*(1-h^2q_{25}^*q_{52}^*) \\
&-hq_{04}^*(hq_{63}^*(1-hq_{11}^*)(1-h^2q_{25}^*q_{52}^*)-hq_{12}^*(h^2q_{21}^*q_{63}^*+(1-hq_{33}^*)+h^2q_{41}^*q_{63}^*(hq_{14}^*+h^2q_{25}^*q_{52}^*)] \\
&-q_{50}^*[h^3q_{01}^*q_{12}^*q_{25}^*(1-hq_{33}^*-h^2q_{36}^*q_{63}^*)]
\end{aligned}
$$

Collecting the coefficient of h^t from expression (66), we can get the reliability of the system $R_0(t)$. The MTSF is given by-

$$
E(T)=\lim_{h\to 1}\sum_{t=1}^{\infty}h^tR(t)=\frac{N_1(1)}{D_1(1)}-1 \tag{67}
$$

$$
\begin{aligned}
N_1(1)=&[(1-p_{11})(1-p_{25}p_{52})(1-p_{33}-p_{36}p_{63})-p_{12}p_{21}(1-p_{33})-p_{12}p_{25}p_{51}(1-p_{33}-p_{36}p_{63}) \\
&+p_{14}(1-p_{33})(p_{41}+p_{46})+p_{14}p_{36}(1-p_{46})+p_{13}p_{36}+p_{14}p_{25}p_{52}\{p_{36}p_{41}p_{63} \\
&-p_{46}p_{61}(1-p_{33})\}-p_{16}p_{25}p_{52}(1-p_{33})]\psi_0+[p_{01}(1-p_{25}p_{52})(1-p_{33}-p_{36}p_{63})-p_{04} \\
&(1-p_{33})(1-p_{25}p_{52})]\psi_1+[p_{01}p_{12}(1-p_{33}-p_{36}p_{63})-p_{04}p_{12}\{p_{51}p_{63}+p_{51}(1-p_{33})\}]\psi_2 \\
&+[p_{01}\{p_{13}-p_{14}(p_{43}-p_{46}p_{63})-p_{14}p_{25}p_{52}(p_{43}-p_{46}p_{63})\}+p_{16}(1-p_{25}p_{52}) \\
&-p_{04}\{p_{63}(1-p_{11})(1-p_{25}p_{52})-p_{12}p_{21}p_{63}+(1-p_{33})+p_{12}p_{41}p_{63}(p_{14}+p_{25}p_{52})\}]\psi_3 \\
&+[p_{01}p_{14}((1-p_{25}p_{52})(p_{36}p_{63}(1-p_{25}p_{52})+(1-p_{33}))+p_{14}(1-p_{33})(1-p_{25}p_{52})]\psi_4 \\
&+[p_{01}p_{12}p_{25}(1-p_{33}-p_{36}p_{63})]\psi_5+[p_{01}p_{36}(1-p_{25}p_{52})(p_{13}+p_{14}p_{43})-p_{36}p_{63}] \\
&-p_{12}p_{21}-p_{14}p_{41}(1+p_{25}p_{51})\}]\psi_6
\end{aligned}
$$

$$
\begin{aligned}
D_1(1)=&[(1-p_{11})(1-p_{25}p_{52})(1-p_{33}-p_{36}p_{63})-p_{12}p_{21}(1-p_{33})-p_{12}p_{25}p_{51}(1-p_{33}-p_{36}p_{63}) \\
&+p_{14}(1-p_{33})(p_{41}+p_{46})+p_{14}p_{36}(1-p_{46})+p_{13}p_{36}+p_{14}p_{25}p_{52}\{p_{36}p_{41}p_{63} \\
&-p_{46}(1-p_{33})\}-p_{16}p_{25}p_{52}(1-p_{33})]-p_{10}[p_{01}(1-p_{25}p_{52})(1-p_{33}-p_{36}p_{63})-p_{04} \\
&(1-p_{33})(1-p_{25}p_{52})]-p_{20}[p_{01}p_{12}(1-p_{33}-p_{36}p_{63})-p_{04}p_{12}\{p_{51}p_{63}+p_{51}(1-p_{33})\}] \\
&-p_{30}[p_{01}\{p_{13}-p_{14}(p_{43}-p_{46}p_{63})-p_{14}p_{25}p_{52}(p_{43}-p_{46}p_{63})\}+p_{16}(1-p_{25}p_{52}) \\
&-p_{04}(p_{63}(1-p_{11})(1-p_{25}p_{52})-p_{12}(p_{21}p_{63}+(1-p_{33})+p_{41}p_{63}(p_{14}+p_{25}p_{52})] \\
&-p_{50}[p_{01}p_{12}p_{25}(1-p_{33}-p_{36}p_{63})]
\end{aligned}
$$

VIII. Availability Analysis

On taking geometric transform of (50-57) and simplifying the resulting equations for we get,

$$
A_0^*(h)=\frac{N_2(h)}{D_2(h)} \tag{68}
$$

Where,

$$
N_2(h)=\begin{vmatrix}
Z_0 & -hq_{01}^* & 0 & 0 & -hq_{04}^* & 0 & 0 & 0 \\
Z_1 & 1-hq_{11}^* & -hq_{12}^* & -hq_{13}^* & -hq_{14}^* & 0 & -hq_{16}^* & 0 \\
Z_2 & -hq_{21}^* & 1 & 0 & 0 & -hq_{25}^* & 0 & 0 \\
Z_3 & 0 & 0 & -hq_{33}^* & 0 & 0 & -hq_{36}^* & -hq_{37}^* \\
0 & -hq_{41}^* & 0 & -hq_{43}^* & 1 & 0 & -hq_{46}^* & 0 \\
0 & -hq_{51}^* & -hq_{52}^* & 0 & 0 & 1 & 0 & 0 \\
Z_6 & 0 & 0 & -hq_{63}^* & 0 & 0 & 1-hq_{66}^* & -hq_{67}^* \\
0 & 0 & 0 & 0 & 0 & 0 & -hq_{76}^* & 1
\end{vmatrix}
$$

and

$$D_2(h)=\begin{vmatrix} 1 & -hq_{01}^* & 0 & 0 & -hq_{04}^* & 0 & -hq_{06}^* & 0 \\ -hq_{10}^* & 1-hq_{11}^* & -hq_{12}^* & -hq_{13}^* & -hq_{14}^* & 0 & -hq_{16}^* & 0 \\ -hq_{20}^* & -hq_{21}^* & 1 & 0 & 0 & -hq_{25}^* & 0 & 0 \\ -hq_{30}^* & 0 & 0 & 1-hq_{33}^* & 0 & 0 & -hq_{36}^* & -hq_{37}^* \\ 0 & -hq_{41}^* & 0 & -hq_{43}^* & 1 & 0 & -hq_{46}^* & 0 \\ -hq_{50}^* & -hq_{51}^* & -hq_{52}^* & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -hq_{63}^* & 0 & 0 & 1-hq_{66}^* & -hq_{67}^* \\ 0 & 0 & 0 & 0 & 0 & 0 & -hq_{76}^* & 1 \end{vmatrix}$$

The steady state availabilities of the system due to operation of unit -

$$A_0=\lim_{t\to\infty}A_0(t)=\lim_{h\to 1}(1-h)\frac{N_2(h)}{D_2(h)}$$

But $D_2(h)$ at h=1 is zero, therefore by applying L. Hospital rule, we get

$$A_0=-\frac{N_2(1)}{D_2'(1)} \tag{69}$$

Where,

$$N_2(1)=u_0\psi_0+u_1\psi_1+u_2\psi_2+u_3\psi_3+u_6\psi_6$$

and

$$D_2'(1)=-\lfloor u_0\psi_0+u_1(\psi_1+p_{12}\psi_2+p_{14}\psi_4)+u_3\psi_3+u_5\psi_5+u_6(\psi_6+p_{76}\psi_7)\rfloor$$

Where,

$u_i=U_i^*(0)$ and $U_i^*(h)$; i=0, 1,…, 7 are the minors of the elements of first column of $D_2(h)$.

Now the expected uptime of the system due to operative unit upto epoch (t-1) are given by

$$\mu_{up}(t)=\sum_{x=0}^{t-1}A_0(x)$$

So that

$$\mu_{up}^*(h)=\frac{A_0^*(h)}{(1-h)} \tag{70}$$

IX. Busy Period Analysisof Repairman

On taking geometric transforms of (58-65) and simplifying the resulting equations,we get

$$B_0^*(h)=\frac{N_4(h)}{D_2(h)} \tag{71}$$

Where,

$$N_3(h)=h\lfloor U_1^*(Z_1^*+q_{12}^*Z_2^*+q_{14}^*Z_4^*)+U_3^*Z_3^*+U_5^*Z_5^*+U_6^*(Z_6^*+q_{76}^*Z_7)\rfloor$$

and $D_2(h)$ is same as in availability analysis.

In the long run the respective probabilities that the repairman is busy in the repair of a failed unit are given by-

$$B_0=\lim_{t\to\infty}B_o(t)=\lim_{h\to 1}(1-h)\frac{N_3(h)}{D_2(h)}$$

But $D_2(h)$ at h=1 is zero, therefore by applying L. Hospital rule, we get

$$B_0=-\frac{N_3(1)}{D_2'(1)} \tag{72}$$

Where,

$$N_3(1) = \lfloor u_1 \left(\psi_1 + p_{12}\psi_2 + p_{14}\psi_4\right) + u_3\psi_3 + u_5\psi_5 + u_6\left(\psi_6 + p_{76}\psi_7\right) \rfloor$$

and $D_2'(1)$ is same as in availability analysis.

Now the expected busy period of the repairman in repair of a failed unit up to epoch (t-1) are respectively given by-

$$\mu_b(t) = \sum_{x=0}^{t-1} B_0(x), \tag{73}$$

X. Profit Function Analysis

We are now in the position to obtain the net expected profit incurred up to epoch (t-1) by considering the characteristics obtained in earlier section. Let us consider,

$K_0 =$ revenue per-unit time by the system due to operative unit.

$K_1 =$ cost per-unit time when repairman is busy in repair of failed unit.

Then, the net expected profit incurred up to epoch (t-1) is given by

$$P(t) = K_0\mu_{up}(t) - K_1\mu_b(t) \tag{74}$$

The expected profit per unit time in steady state is given by-

$$\begin{aligned} P &= \lim_{t\to\infty}\frac{P(t)}{t} = \lim_{h\to 1}(1-h)^2 P^*(h) \\ &= K_0 \lim_{h\to 1}(1-h)^2 \frac{A_0^*(h)}{(1-h)} - K_1 \lim_{h\to 1}(1-h)^2 \frac{B_0^*(h)}{(1-h)} \\ &= K_0A_0 - K_1B_0 \end{aligned} \tag{75}$$

XI. Graphical Representation

The curves for MTSF and profit function have been drown for different values of failure parameters. Fig. 2 depicts the variation in MTSF with respect to failure rate (p) for different values of repair rate (r) of a unit and activation rate (c) when values of other parameters are kept fixed as $p' = 0.99$ and $q' = 0.01$. From the curves we conclude that expected life of the system decrease with increase in p. Further, increases as the values of r and c increases.

Similarly, Fig. 3 reveals the variations in profit (P) with respect to p for varying values of r and c, when other parameters are kept fixed as $p' = 0.99$, $q' = 0.01$, $K_0 = 50$, and $K_1 = 450$. From the figure it is clearly observed from the smooth curves, that the system is profitable if the value of parameter p is smaller than 0.062, 0.102 and 0.112 respectively for $r = 0.1$, 0.15 and 0.2 for fixed value of $c = 0.95$. From dotted curves, we conclude that system is profitable only if value of parameter p is smaller than 0.052, 0.078 and 0.118 respectively for $r = 0.1$, 0.15 and 0.2 for fixed value of $c = 0.94$.

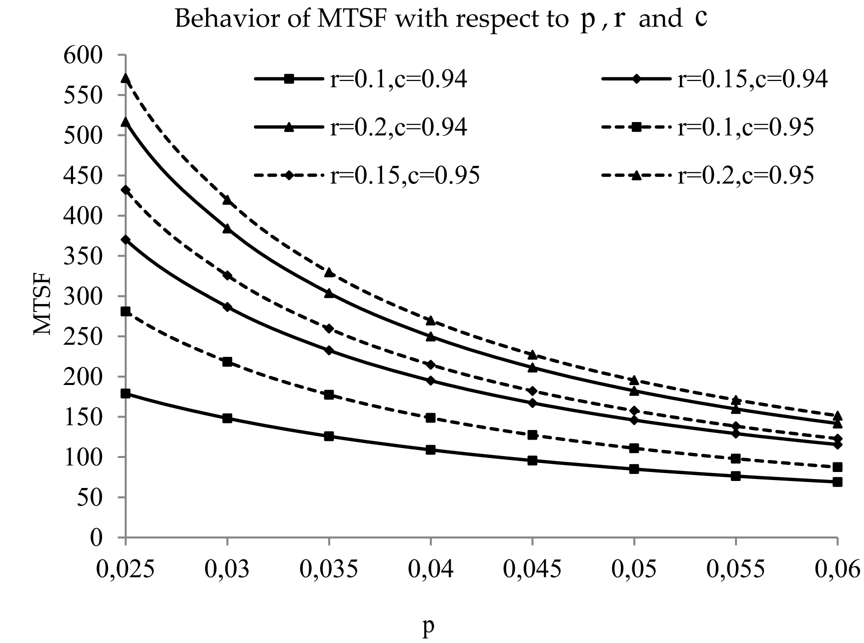

Figure 2

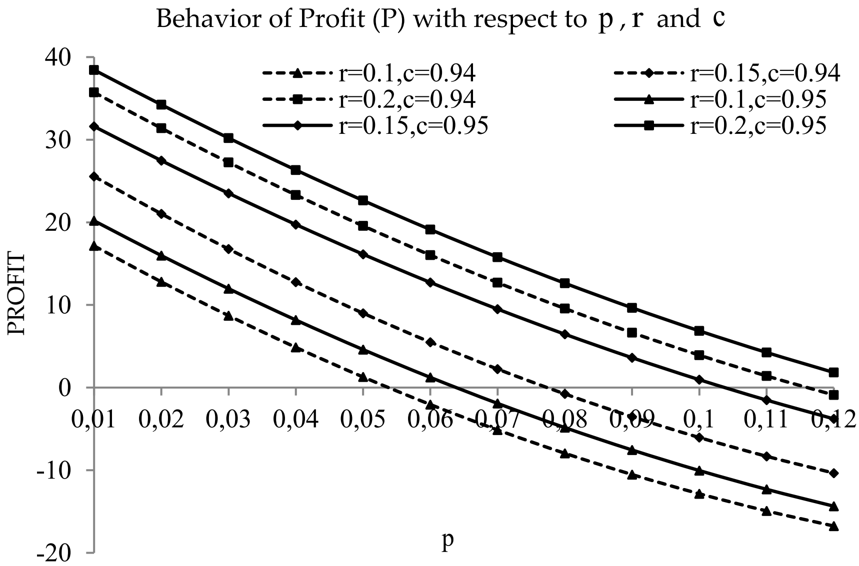

Figure 3

XII. Conclusions

1. It is indicated in fig.2 that we can easily obtain the upper limit of "p" to achieve at least a particular value of MTSF. As an illustration to get at least MTSF 150 unit, the failure rate "p" must be less than 0.025, 0.031 and 0.037 respectively for repair rate r = 0.1, 0.15 and 0.2 when activation rate is kept fixed as c = 0.94. Similarly, when c = 0.95 is kept fixed as "p" must be less than 0.253, 0.034 and 0.040 corresponding to r = 0.1, 0.15 and 0.20.
2. In fig. 3 it is reveled from the dotted curves that the system is profitable only if failure rate "p" is greater than 0.052, 0.078 and 0.118 respectively for r = 0.1, 0.15 and 0.2 for fixed value of c =

0.94. From smooth curves, we conclude that system is profitable only if value of parameter "p" is greater than 0.062, 0.102 and 0.112 respectively for r = 0.1, 0.15 and 0.2 for fixed value of c = 0.95.

XIII. Acknowledgement

The first author Mr. Shubham Gupta is thankful to University Grants Commission, New Delhi for providing the financial assistance in the form of J.R.F. awarded vide letter Ref. No. 190510175219 dated 29-11-2019.

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Received: August 21, 2020
Accepted: November 25, 2020

A Matrix Method for Transient Solution of an M/M/2/N Queuing System with Heterogeneous Servers and Retention of Reneging Customers

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Abstract

In this paper, a finite capacity two heterogeneous servers' queuing system with retention of reneging customers is studied. The explicit transient probabilities of system size are obtained using matrix method. Further, the time-dependent mean and variance are presented. Finally, a numerical example is provided to show the behavior of the system.

Keywords: Retention of reneging customers, Heterogeneous servers, matrix method, Transient solution

1 Introduction

Queuing theory emerges as proficient instrument in solving the difficulties of clogging in telecommunication systems, computer-communication systems, service systems and traffic systems. Most of the work done in multi-server queuing, researchers have assumed equal service rate for all the servers. This assumption is validated only in mechanically or electronically controlled systems. But, when the servers are human they will perform with different rates as per their abilities. For example, counters of a library where different library assistants work on different rates can be accurately demonstrated using heterogeneous multi-server queuing systems. Kumar et al. [14] state that it is quite difficult to obtain the analytical results for queuing systems multi-heterogeneous servers. Morse [20] was the first to incorporate the idea of heterogeneity in service. Gumbel [11] derived the expressions for steady-state system size probabilities and the expected queue length for non-homogeneous multiple server queuing model. Saaty [21] obtained time-independent probabilities for heterogeneous server queuing system. He extended the Morse's model with two different service rates. The same model with two types of queue disciplines was studied by Krishnamoorthy [13]. Godini [10] considered a heterogeneous server $M/M/S$ queuing system. Singh [29] analyzed three heterogeneous servers' $M/M/3$ queue where the first server is faster than other two and second server is faster than third server. He also obtained steady-state results and compared them with the existing three server homogeneous system. Cooper [5] studied $M/M/S$ queuing system with different service rates. He also obtained the steady-state to analyze the performance measures like number of customers in the system and server utilization. Queuing system with two types of processors was studied by Trivedi [32] to get the steady-state results. Sharma and Dass [23] did the busy period analysis of $M/M/2/N$ queuing system with heterogeneous servers and also acquired the expression for customers' number in the system and density function. For the same model Sharma and Dass [24] achieved the steady-state results for number of customers using Laplace transform and matrix method. Dharmaraja [6] obtained the transient solutions for the model already studied by Trivedi's [32].

In our daily life, a customer may not be allowed to be served at time instance that he joins the queue, and he has to wait some time duration until his service process is started. During waiting time period, he may become impatient when the waiting time is higher than his expected service time duration and may leave the system before getting service. Telephone switchboard customers, perishable goods, inventory systems, and hospital emergency rooms' handling of critical patients are the most prominent examples for above mentioned situations. Taking an example of a call centre where a customer who is told to hold on for some time to contact customer care officer may renege if he becomes impatient before his connection with customer care officer is established if his waiting time more than his patience level. This behavior can be observed in train ticket booking also. A customer in queue may renege after waiting for some time. Both balking and reneging influence the performance of the queuing system. Thus, many researchers have shown their keen interest in these two concepts. Singh [28] has analyzed an $M/M/2$ queuing system with heterogeneity and balking, and furthermore, the results with corresponding two-server homogeneous system are compared. About-El-Ata [1] also studied an $M/M/2$ queuing system with balking and heterogeneity. Al-seedy [4] attempted to obtain the transient solutions for system size probabilities of an $M/M/2$ queue with balking, heterogeneous servers, and an additional server is set up for longer queues. El-Paoumy [7] has analyzed a finite capacity queuing system what has batch arrival, balking, reneging and two heterogeneous servers. Yue and Yue [34] have studied a two heterogeneous servers queuing system with balking, single vacation, and under Bernoulli schedules. A two heterogeneous servers queuing system was discussed by Yue et al. [33] by adding the feature of balking. They implemented the condition that first server is reliable and second server is subject to breakdown by extending the model of Singh [28]. Matrix-geometric method was used to derive the steady-state results for the system size probabilities. Furthermore, they have presented the performance measures such as the mean system size, and the average balking rate. El-Sherbiny [9] has studied a finite capacity two heterogeneous server queuing system with two general different balk functions to derive the steady-state results, and he probability generating function technique along with hypergeometric function. Kumar and Sharma [16] obtained the steady-state probabilities of number of customers in the system and some performance measures for an $M/M/2/N$ queue with discouraged arrivals, two-heterogeneous servers, reneging and retention of reneged customers. This was an extending of Kumar and Sharma [17] model. Impatient behavior on a two heterogeneous servers queuing system was studied by Ammar [2] to obtain the transient and steady-state results along with some performance measures. a two heterogeneous servers $M/M/2/N$ queue subject to reverse balking and reneging has been studied by Som and Kumar [30], and they presented the steady-state expressions and some performance measures for that model. Kumar and Sharma [18] have recently obtained the transient and steady-state system size probabilities for a heterogeneous servers' $M/M/2$ queuing system with retention of reneged customers. Furthermore, they have presented mean and variance as the performance measures and numerical illustrations also are provided.

Even though researchers usually consider infinite capacity queuing systems, it may not appear in real life situations. For example, an e-mail server system has to limit its waiting line for mails considering available limit of memory. It never becomes infinite and should be limited to some finite capacity. The transient solution of a finite capacity $M/M/1$ queue was obtained by Takacs [?] by making use of a technique involving eigenvectors and eigenvalues. Sharma and Gupta [25] have used Chebyshev polynomials to analyze an $M/M/1/N$ queuing system in transient state. Sharma and Maheswar [27] have applied the matrix geometric method to derive the time dependent results for $M/M/1/N$ queue. A finite capacity correlated two server Markovian queuing system was analyzed by Sharma and Maheswar [22] by using matrix method. Ammar et al. [3] obtained the transient solutions for an $M/M/1/N$ queuing system with discouraged arrivals and reneging by using computable matrix technique. Yue and yue [35] derived the steady state expressions for a finite capacity multi-server queuing system with simultaneous balking, reneging, and synchronous vacations of servers. An $M/M/c/N$ queuing system with balking and retention of reneged customers was analyzed by Kumar [15] and he used probability generating function technique to derive steady-state solutions with some performance measures for that model. Sharma [22] obtained the transient solution for two-heterogeneous servers' queuing system in general form by making use of computable matrix technique. A finite capacity two heterogeneous servers queuing system with general balk function, reneging was studied by El-Paoumy and Nabwey [8] to obtain the steady-state expressions.Recently, Isguder and Kocer [12] have studied finite capacity queueing system with recurrent input and two heterogeneous servers and they derived the steady-state expressions for system size.

The applicability of this model can be seen in communication and computer systems. Messages arrive to communication device as data packets are transferred through one of the communication channels which are working with different rates. The model we are considering has two heterogeneous processors and one of them is faster. If the data packets takes too much time to transmit, then the sender may recall the message sent. This is known as reneging in queuing theory. Customer retention strategies can be applied to reduce the dropping of the packets. Capacity of memory of both processors are limited to finite value. Therefore, this system has to limit their waiting room capacity for some finite number of messages.

The application as discussed above motivates us to analyze the behavior of a finite capacity two-heterogeneous servers' queuing system with retention of reneging customers. From the literature servey, it has been noticed that transient solution of the queuing model considered in this paper has not been obtained by using matrix method. Hence, we study and $M/M/2/N$ queuing system with two-heterogeneous servers and retention of reneging customers, and obtained its transient solution by employing matrix method.

Rest of the paper has been arranged as follows; In section 3, the model is described. Section 4 provides the transient solution. In section 5, time-dependent mean and variance are presented. Section 6 deals numerical illustrations. Finally, the paper is concluded in section 7.

2 Model Description

A finite capacity two-heterogeneous queueing system with impatient customers is considered. Arrivals occur to the system in accordance with Poisson process with rate λ. The system has two severs and they have different exponentially distributed service rates μ_1 (server-1) and μ_2 (server-2) such that $\mu_1 < \mu_2$. In this model, we consider modified queue discipline i.e. an arriving customer goes to the server-1 if there is no customers in the system. Otherwise, it joins the server who is free. After joining the queue, the arrivals activates an individual timer, exponentially distributed with parameter ξ. If the customer's service has not been started before the customer's timer expires, he abandons the system with probability p or may remain in the queue for his service with probability $q(= 1 - p)$. The reneging rate when there are n customers in the queue is given by $(n - 2)\xi p$. The number of customers in the system is limited to N. It is assumed that inter-arrival times, service times are mutually independent and the service discipline is First-In, First-Out(FIFO).

Let $\{X(t), t \geq 0\}$ denotes the number of customers in the system at time t, and $P_n(t)$, $n = 0,1,2,3,4,\ldots.,N$ be the time-dependent probabilities for the number of customers at time t. Initially, it is assumed that there are i customers in the queuing system.

Then, the set of forward Kolmogorov differential difference equations governing the process are given by

$$P_0'(t) = -\lambda P_0(t) + \mu_1 P_1(t) \tag{1}$$

$$P_1'(t) = \lambda P_0(t) - (\lambda + \mu_1)P_1(t) + (\mu_1 + \mu_2)P_2(t) \tag{2}$$

$$P_2'(t) = \lambda P_1(t) - (\lambda + \mu_1 + \mu_2)P_2(t) + (\mu_1 + \mu_2 + \xi p)P_3(t) \tag{3}$$

$$P_n'(t) = \lambda P_{n-1}(t) - (\lambda + \mu_1 + \mu_2 + (n-2)\xi p)P_n(t)$$
$$+(\mu_1 + \mu_2 + (n-1)\xi p)P_{n+1}(t); n = 3,4,5.\dots, N-1 \tag{4}$$

$$P_N'(t) = \lambda P_{N-1}(t) - (\mu_1 + \mu_2 + (N-2)\xi p)P_N(t) \tag{5}$$

3 Transient solutions

In this section, the transient solution of the above described model is derived by employing matrix method.

4.1 Evaluation of $P_n(t)$

Taking Laplace transform of the equations (1)-(5), we have

$$AP(s) = P(0) \tag{6}$$

Where A is a tridiagonal matrix of order $(N+1) \times (N+1)$, and $P(s)$ and $P(0)$ are column vectors of order $N+1$. Matrix A is given by

$$\begin{array}{c} \\ \\ \\ 0.75 \\ \\ \\ \\ \end{array}\begin{pmatrix} s+\lambda & -\mu_1 & .. & .. & .. & .. & 0 \\ -\lambda & s+\lambda+\mu_1 & -(\mu_1+\mu_2) & .. & .. & .. & 0 \\ 0 & -\lambda & s+\lambda+\mu_1+\mu_2 & -(\mu_1+\mu_2+\xi p) & .. & .. & 0 \\ .. & .. & .. & .. & .. & .. & .. \\ .. & .. & .. & .. & .. & .. & .. \\ 0 & .. & .. & .. & .. & s+\lambda+\mu_1+\mu_2+(N-3)\xi p & -[\mu_1+\mu_2+(N-2)\xi p] \\ 0 & .. & .. & .. & .. & -\lambda & -[\mu_1+\mu_2+(N-2)\xi p] \end{pmatrix}$$

and

$$P(s) = \left[\hat{P}_0(s), \hat{P}_1(s), \dots\dots\dots, \hat{P}_N(s)\right]^T,$$
$$P(0) = [P_0(0), P_1(0), \dots\dots\dots, P_N(0)]^T,$$

where $\hat{P}_n(s)$ is the Laplace transform of $P_n(t)$.

The matrix A can be transformed into the symmetric tridiagonal form by the diagonal matrix

$$M = dg[d_0, d_1, d_2, \dots\dots\dots, d_N]$$

with

$$d_0 = 1$$

$$d_n = \prod_{k=1}^{n} \sqrt{\frac{\mu_1 + (1-\delta_{1k})\mu_2 + (n-2+\delta_{1k})\xi p}{\lambda}}, 1 \le n \le N$$

Using the diagonal matrix M, a symmetric tridiagonal matrix, $sI + B = MAM^{-1}$, is obtained. Diagonal entries of this matrix are same as in matrix A and off diagonal entries in the nth row are represented by

$-\sqrt{\lambda(\mu_1 + (1-\delta_{1n})\mu_2 + (n-2+\delta_{1n})\xi p)}$ and $-\sqrt{\lambda(\mu_1 + \mu_2 + (n-1)\xi p)}$ respectively. This matrix and matrix **A** have same eigenvalues.

where matrix B is given by

$$\begin{pmatrix} \lambda & -\sqrt{\lambda\ \mu_1} & 0 & \cdot\cdot & \cdot\cdot & \cdot\cdot & 0 \\ -\sqrt{\lambda\ \mu_1} & \lambda+\mu_1 & -\sqrt{\lambda\ (\mu_1+\mu_2)} & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ 0 & -\sqrt{\lambda\ (\mu_1+\mu_2)} & \lambda+\mu_1+\mu_2 & -\sqrt{\lambda\ (\mu_1+\mu_2+\xi\ p)} & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ 0.68\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & s+\lambda+\mu_1+\mu_2+(N-3)\xi\ p & -\sqrt{\lambda\ (\mu_1+\mu_2+(N-2)\xi\ p)} \\ 0 & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & -\sqrt{\lambda\ (\mu_1+\mu_2+(N-2)\xi\ p)} & -[\mu_1+\mu_2+(N-2)\xi\ p\ ] \end{pmatrix}$$

Two matrices $A_n(s)$ and $B_n(s)$ are defined with the determinants $T_n(s)$ and $U_n(s)$. They represent the bottom right and top left $(n \times n)$ square matrices of the matrix $sI + B$ respectively. The determinants $T_n(s)$ and $U_n(s)$ satisfy the following difference equations,

$$T_n(s) = [s + \lambda + \mu_1 + (1 - \delta_{Nn})\mu_2 + (N - n - 1 + \delta_{Nn})\xi p]T_{n-1}(s)$$
$$-[\lambda(\mu_1 + \mu_2 + (N - n)\xi p)]T_{n-2}(s)$$
$$U_n(s) = [s + \lambda + \mu_1 + (1 - \delta_{2n})\mu_2 + (n - 3 + \delta_{2n})\xi p]U_{n-1}(s)$$
$$-[\lambda(\mu_1 + (1 - \delta_{2n})\mu_2 + (n - 3 + \delta_{2n})\xi p)]U_{n-2}(s)$$

with the initial conditions

$$T_0(s) = 1 = U_0(s)$$

$$T_1(s) = s + \lambda + \mu_1 + \mu_2 + (N - 2)\xi p$$
$$U_1(s) = s + \lambda$$

Using the Lemma (1) and (2) of Lewis [19], we are able to derive the following results

$$(sI + B)^{-1} = \frac{C}{|sI+B|}, C = \left(C_{ij}(s)\right),$$

$$C_{ij}(s) = \sqrt{\prod_{r=j+1}^{i} \lambda[\mu_1 + (1 - \delta_{1r})\mu_2 + (r - 2 + \delta_{1r})\xi p]}U_j(s)T_{N-i}(s), i > j$$
$$= U_i(s)T_{N-j}(s), i = j$$
$$= \sqrt{\prod_{r=i+1}^{j} \lambda[\mu_1 + (1 - \delta_{1r})\mu_2 + (r - 2 + \delta_{1r})\xi p]}U_i(s)T_{N-j}(s), i < j$$

and

$$|A| = |sI + B| = T_N(s) = U_N(s)$$

The equation (6) is rearranged as follows,

$$P(s) = A^{-1}P(0)$$
$$= M^{-1}(sI + B)^{-1}MP(0)$$

$$P_n(s) = \frac{\sum_{j=0}^{N} d_n^{-1}C_{nj}(s)d_jP_j(0)}{|sI+B|}$$
$$= \frac{d_n^{-1}d_iC_{ni}(s)}{|sI+B|}$$

where

$$d_n^{-1}d_i = \prod_{k=n+1}^{i} \sqrt{\frac{\mu_1+(1-\delta_{1kr})\mu_2+(k-2+\delta_{1k})\xi p}{\lambda}}, i > n$$
$$= 1, i = n,$$
$$= \frac{1}{\prod_{k=i+1}^{n}\sqrt{\frac{\mu_1+(1-\delta_{1k})\mu_2+(k-2+\delta_{1k})\xi p}{\lambda}}}, n > i$$

Then, we can derive the following expression for $P_n(s)$,

$$\hat{P}_n(s) = \prod_{r=n+1}^{i} [\mu_1 + (1 - \delta_{1r})\mu_2 + (r - 2 + \delta_{1r})\xi p]\frac{U_n(s)T_{N-i}(s)}{|sI+B|}, n < i$$
$$= \frac{U_n(s)T_{N-n}(s)}{|sI+B|}, n = i$$
$$= \lambda^{n-i}\frac{U_n(s)T_{N-i}(s)}{|sI+B|}, n > i, 0 \le i, n \le N \qquad (7)$$

Since symmetric tridiagonal matrix B is a diagonally dominant matrix, eigenvalues of its are real, positive and distinct.

Let $\alpha_m (m = 0,1,2, \dots, N)$ be an eigenvalue of matrix B, and $\alpha_0 = 0$, Then obviously, we have

$$|sI + B| = s \prod_{m=1}^{N} (s + \alpha_m) \tag{8}$$

Substituting the equation (8) in (??), and making use of partial fraction decomposition, we derive

$$\begin{aligned}
\hat{P}_n(s) &= \prod_{r=n+1}^{i} [\mu_1 + (1-\delta_{1r})\mu_2 + (r-2+\delta_{1r})\xi p]\left(\frac{\pi_n}{s} + \sum_{k=1}^{N} \frac{A_{nk}}{s+\alpha_k}\right), \\
& 0 \le n < i \\
&= \frac{\pi_n}{s} + \sum_{k=1}^{N} \frac{B_{nk}}{s+\alpha_k}, n = i \\
&= \lambda^{n-i}\left(\frac{\pi_n}{s} + \sum_{k=1}^{N} \frac{B_{nk}}{s+\alpha_k}\right), i < n \le N
\end{aligned} \tag{9}$$

where

$$\begin{aligned}
\pi_n &= \frac{U_n(0)T_{N-i}(0)}{\prod_{j=1}^{N} \alpha_j}, 0 \le n \le i \\
&= \frac{U_i(0)T_{N-n}(0)}{\prod_{j=1}^{N} \alpha_j}, i \le n \le N \\
A_{nk} &= \frac{U_n(-\alpha_k)T_{N-i}(-\alpha_k)}{(-\alpha_k)\prod_{j=1, j\neq k}^{N}(\alpha_j - \alpha_k)} \\
B_{nk} &= \frac{U_i(-\alpha_k)T_{N-n}(-\alpha_k)}{(-\alpha_k)\prod_{j=1, j\neq k}^{N}(\alpha_j - \alpha_k)}
\end{aligned}$$

The inversion of the equation (??) yields

$$\begin{aligned}
P_n(t) &= \prod_{r=n+1}^{i} [\mu_1 + (1-\delta_{1r})\mu_2 + (r-2+\delta_{1r})\xi p](\pi_n + \sum_{k=1}^{N} A_{nk} e^{-\alpha_k t}), \\
& 0 \le n < i \\
&= \pi_i + \sum_{k=1}^{N} B_{ik} e^{-\alpha_k t}, n = i \\
&= \lambda^{n-i}(\pi_n + \sum_{k=1}^{N} B_{nk} e^{-\alpha_k t}), i < n \le N
\end{aligned}$$

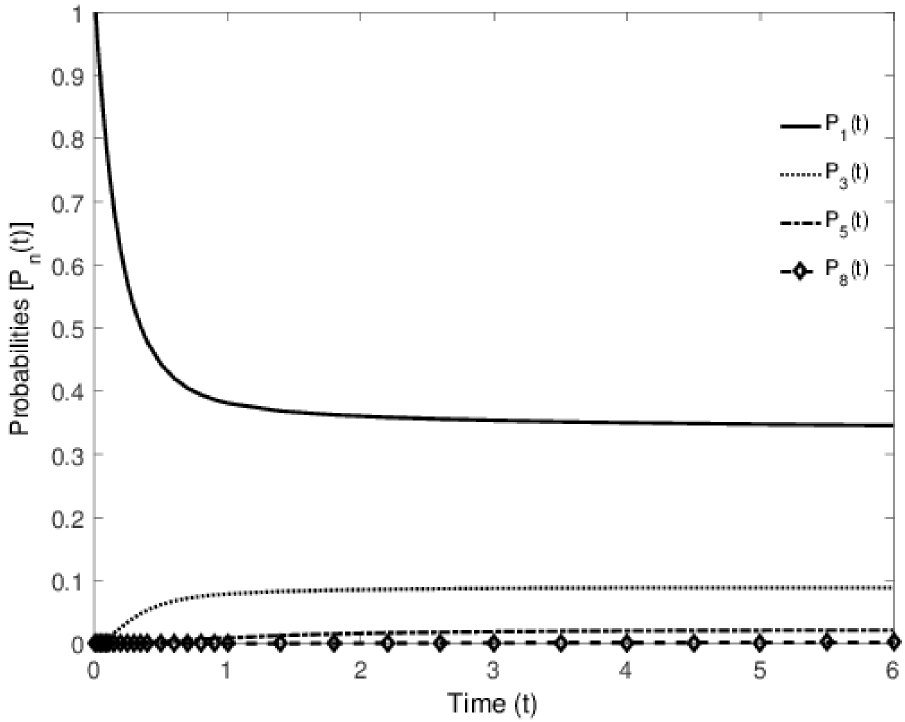

Figure 1: Variation in transient probabilities with time

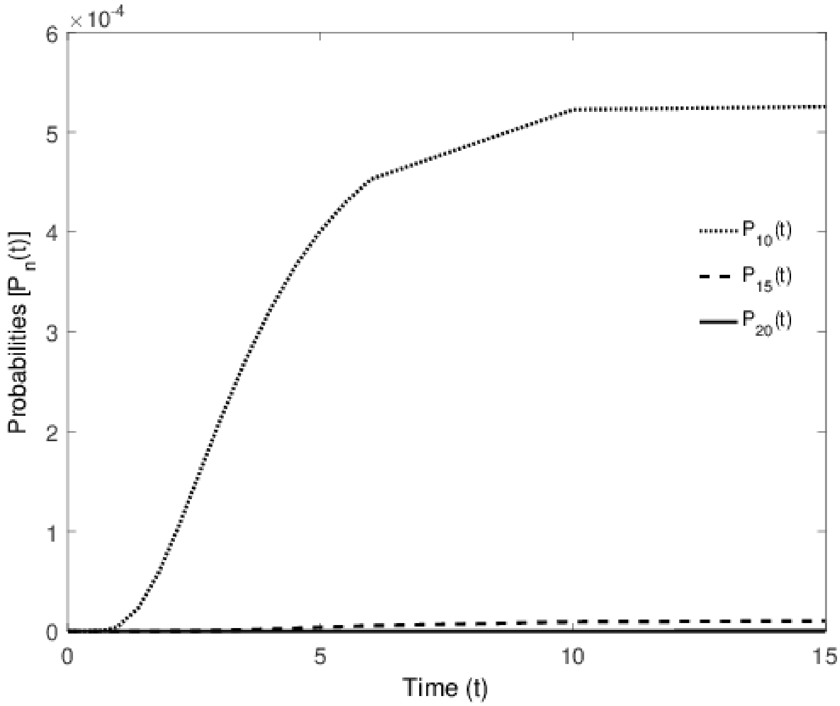

Figure 2: Variation in transient probabilities with time

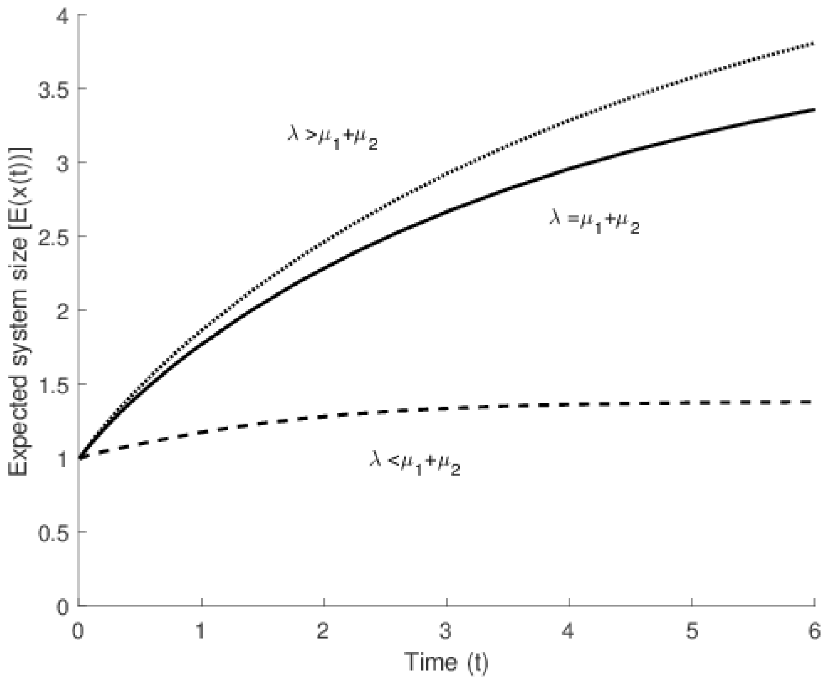

Figure 3: Comparison of expected system size $[E(X(t))]$ against time (t) for varying arrival rate (λ)

5 Time dependent mean and variance

In this section, time dependent expected value and variance of the system size distribution are derived.

5.1 Mean

Let $X(t)$ denotes the number of jobs in the system at time t. The average number of jobs in the system at time t is given by

$$E(X(t)) = \sum_{j=1}^{N} jP_n(t)$$
$$= \sum_{j=1}^{i} jP_j(t) + nP_n(t) + \sum_{j=n+1}^{N} jP_j(t)$$
$$= \sum_{j=1}^{i} j \prod_{r=j+1}^{i} [\mu_1 + (1-\delta_{1r})\mu_2 + (r-2+\delta_{1r})\xi p]$$
$$\times \left(\pi_j + \sum_{k=1}^{N} A_{jk}e^{-\alpha_k t}\right)$$
$$+n\pi_n + \sum_{k=1}^{N} B_{nk}e^{-\alpha_k t}$$
$$+\sum_{j=n+1}^{N} j\lambda^{j-i}\left(\pi_j + \sum_{k=1}^{N} B_{jk}e^{-\alpha_k t}\right)$$

5.2 Variance

Let $X(t)$ denotes the number of jobs in the system at time t. The variance of jobs in the system at time t is given by

$$Var(X(t)) = \sum_{j=1}^{N} j^2P_n(t) - [E(X(t))]^2$$
$$= \sum_{j=1}^{i} j^2P_j(t) + n^2P_n(t) + \sum_{j=n+1}^{N} j^2P_j(t) - [E(X(t))]^2$$
$$= \sum_{j=1}^{i} j^2 \prod_{r=j+1}^{i} [\mu_1 + (1-\delta_{1r})\mu_2 + (r-2+\delta_{1r})\xi p]$$
$$\times \left(\pi_j + \sum_{k=1}^{N} A_{jk}e^{-\alpha_k t}\right)$$
$$+n^2\pi_n + \sum_{k=1}^{N} B_{nk}e^{-\alpha_k t}$$
$$+\sum_{j=n+1}^{N} j^2\lambda^{j-i}\left(\pi_j + \sum_{k=1}^{N} B_{jk}e^{-\alpha_k t}\right) - [E(X(t))]^2$$

6 Numerical illustrations

The numerical examples which illustrate the functioning of concerned model in transient state are presented in this section.

Figures 1 and 2 presents the behaviour of the probabilities $P_n(t)$ against time t for varying values of n with parameters $\lambda = 1.8$, $\mu_1 = 1.5$, $\mu_2 = 2$, $\xi = 0.1$, $p = 0.4$ and initial value $i = 1$. It can be noticed that all the probabilities tend to settle at steady-state when time progresses.

Figure 3 is plotted to describe the comparison of the expected system sizes $E(X(t))$ with same parameter values and three types of arrivals. Here, if $\lambda < \mu_1 + \mu_2$, it can be seen that expected system size of the queue reaches its steady state with time t. But, for other two cases, it rapidly increases expected number of customers in the system when time progresses.

7 Conclusion

A finite capacity two-heterogeneous servers' queuing system with retention of reneging customers is studied. The matrix method is used to derive the transient solution. Additionally, mean and variance of the system size are presented as the performance measures. Finally, numerical analysis is added to express the behaviour of system size probabilities and expected system size against time

Acknowledgements

One of the authors, M.I.G. Suranga Sampath, would like to acknowledge the financial assistance provided by NSFs of China under No. 11271013 and the Fundamental Research Funds for the Central Universities under HUST:2016YXMS003. He along with other coauthors would also like to thank the School of Mathematics, Shri Mata Vaishno Devi University, Katra for providing necessary facilities for the preparation of this manuscript.

Declaration of Conflicting Interests

The Authors declare that there is no conflict of interest

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Received: November 16, 2020
Accepted: December 15, 2020

Sensitivity Analysis of Three Different Series – Parallel Dynamo Configurations

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Abstract

This paper deals with the sensitivity analysis of three configurations arranged in series-parallel. Configuration I consist of six units in which four are on operation while two are on standby. Configuration II consist of seven units with three of the units are on standby while the remaining four are on operation. Configuration III comprises of two subsystems C and D with three unit in each subsystem with a unit on standby. Units in each configuration provide 25MW. Both the failure and repair time are assumed exponentially distributed. System of first order linear differential difference equations is obtained using the transition diagram. Explicit expressions of the system availability, Mean Time To Failure (MTTF), busy period due to partial failure, busy period due to complete failure and profit were derived. Furthermore results of sensitivity of the system availability, MTTF and profit were determined. The obtained results were analyzed and compared, configuration I was found to be the optimal configuration.

Keywords: Sensitivity, Reliability, Dynamo, Availability, Configurations, Series - Parallel.

I. Introduction

Every manufacturer expects the performance of his/her engines with full efficiency within the designed limits. However, in real life users have the tendency to operate the system beyond even their control limits and such situations are termed as abnormal condition. In the system design, redundancy is found in almost all types of systems that plays an important role for improvement of reliability and availability of the system. Sometimes, it is difficult to keep a high cost identical unit in standby situation; therefore, a duplicate unit may be kept as spare for use in emergency and to provide services to the customers for a considerable period. Each unit can perform same kind of functions, but their degree of reliability and desirability may differ from unit to unit, Kumar *et al.* (2020). High system reliability and availability of electrical system plays a vital role towards industrial growth as the profit is directly dependent on production volume which depends upon system performance. Due to their prevalence in domestic, manufacturing, and industrial systems, many researchers have studied reliability and availability problem of different electrical systems.

A great number of models have been introduced to describe the behaviour and performance of electrical system that is subject to failure. For this reason, many researchers have studied reliability problem of different electrical systems.Redundancy technique is widely used to improve system reliability. However, in the real world situation, many systems are load-sharing, such as electric generators sharing an electrical load in a power plant, cables in a suspension bridge, and valves or pumps in a hydraulic system, Chunbo *et al.* (2015). To cite few, Chauhan and Malik (2017) focused on the evaluation of reliability and MTSF of a parallel system with Weibull failure laws. Abdul Kareem and Singh (2019) worked on cost assessment of complex repairable system consisting two subsystems in series configuration using Gumbel Hougaard family copula. Rajesh *et al.* (2018) have studied the reliability and availability for a three unit gas turbine power generating system with seasonal effect and FCFS repair pattern. Dalip *et al.* (2014) have also studied reliability and economic analysis of a power generating system comprising one gas and one gas steam turbine with random inspection. However, situations may be there, where the two units may be dissimilar but the nature of the work done by them is the same. Such a situation was discussed by Singh and Taneja (2013) and (2014) for a gas turbine power plant. However, they did not consider the parameter 'Temperature' which also affects the working/function and efficiency of a gas turbine system. One such situation was discussed by Rajesh *et al.* (2018) where effects of temperature on production of a system comprising one gas turbine and one steam turbine have been taken into account. Such a system necessarily goes to down mode on failure of gas turbine irrespective of operability of steam turbine, as steam turbine cannot work without working of gas turbine. However, this problem can be overcome to some extent if number of gas turbine is increased, i.e., redundancy is introduced._Yusuf (2016) presented an article on reliability evaluation of parallel system with two types of preventive maintenance. Ram M. and Kumar (2015) discussed on performability/performance analysis of a system under 1-out-of -2: G scheme with perfect reworking, Wang *et al.* (2003) have studied cost benefit analysis of series systems with warm standby components, Tseng *et al.* (2013) studied comparative analysis of three systems with imperfect coverage and standby switching failures and Wang and Chin (2006) also discussed on cost benefit analysis of series systems with cold standby components and a repairable service station. In their research paper Wang and Chin (2006) considered three configurations as follows:

The first configuration is a serial system of one primary 30 MW component with one cold standby 30MW component. The second configuration is a serial system of two primary 15MW components and one cold standby 15MW component. The last configuration is a serial system of three primary 10MW components with two interchangeable cold standby 10MW components. Each standby unit can replace either one of the failed components and the total of 30MW is expected in all the three configurations. Lastly, Wang and Kuo (2018) have studied cost benefit analysis of three systems with imperfect coverage and standby switching failures. In the paper, data center require a 30MW power electricity, and they assumed that the electricity generation capacity of generators is available in units of 30MW, 15MW, and 10MW. To provide reliable and stable power supply, there are standby generators, and all the active and standby generators are continuously monitored by a fault detecting device to identify if they fail. They also assumed that standby generators are allowed to fail while inactive before they are put into full operation. Goyal *et al*. (2017) published a research work on Sensitivity analysis of a three unit series system under k-out of-n redundancy. Considering reliability, as one of the performance measure, the authors have designed a complex system which consists of three subsystems, namely, A, B and C in series configuration. The subsystem A consists of n numbers of units which are arranged in parallel configuration, subsystem B consists of two sub-subsystems X and Y align parallel to one another, where X is a type of 1-out-of-n. Failure and repair rates are assumed to follow the general distribution.

In this research work, some relevant literature related to reliability analysis and performance evaluation of dynamo system configurations were reviewed which mostly focused on the cost benefit analysis of the system. Relevant literature that has to do with system modeling and how the model would be applied to solved practical system and improved efficiency as studied by many scholars were reviewed. This research paper further enhanced the work of the previous researchers. 100MW was considered as the total output and the three configurations have uniform of 25MW in all the units of the configurations. Furthermore, some practical applications are also addressed.

II.Notation, Assumption and Description of Three Configurations

Notations

λ : *Failure rate*
μ : *Repair rate*
$Av_{i;}$ i = 1,2,3 Availability of system
$MTTF_{i;}$ i = 1,2,3 Mean time to failure
Q(t) = Probability row vector

Assumptions

1. Systems have redundant standby units
2. Repair is immediate
3. Switching from standby level to operation stage is perfect

Description of the three configurations

Configuration I consists of six units each of the unit has 25MW arranged in series-parallel. Out of the six units four are on operational stage while two are on standby. The failure of A_1 or A_6 causes the complete failure of the system. Configuration II has seven sub-components/ units with 25MW each arranged in series–parallel, three of the units are on standby while the remaining four are on operation stages, the failure of the system is said to have occur if B_2 and A_1 or A_2 fails. Configuration III comprises of two subsystems C and D with three units in each subsystem and out three units there is one standby with 25MW in each unit. Out of the six units in total four are on operation while two are on standby. The system will collapse if C_1 and C_2 or C_5 and C_6 fail. The parameter λ represents the failure rate in all the three configurations. Whenever active unit fails, it will immediately be replaced by a standby and the failed unit is taken for repair which is represented by μ.

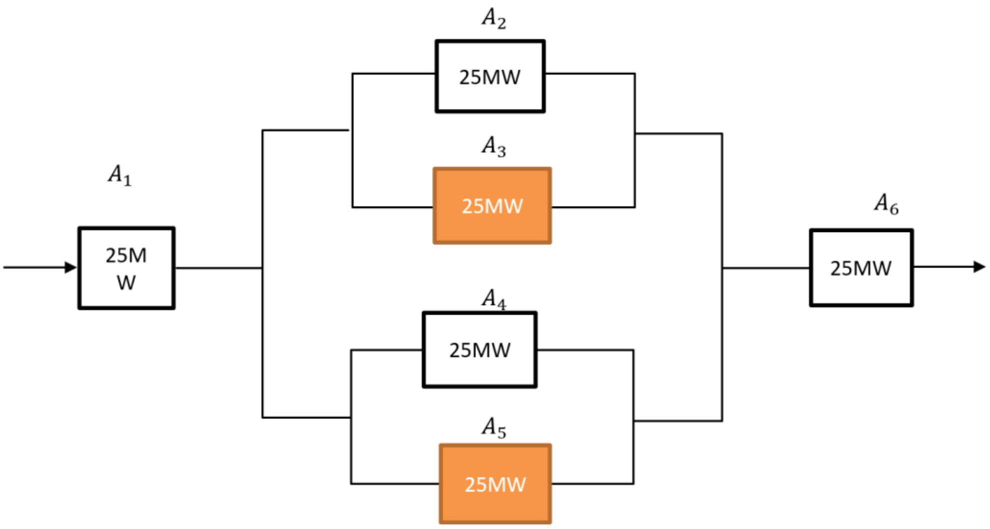

Figure 1: Block diagram of Configuration I

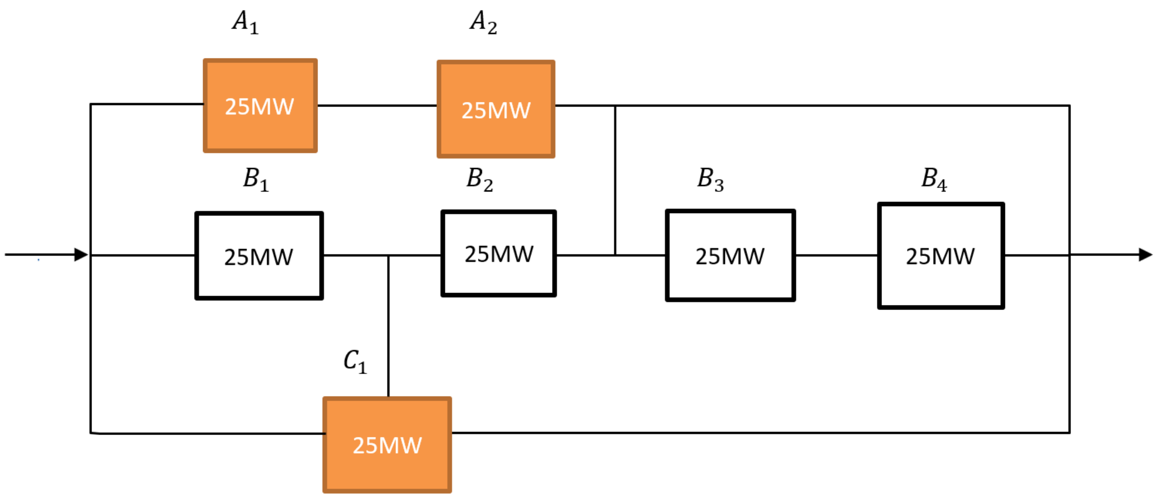

Figure 2: Block diagram of Configuration II

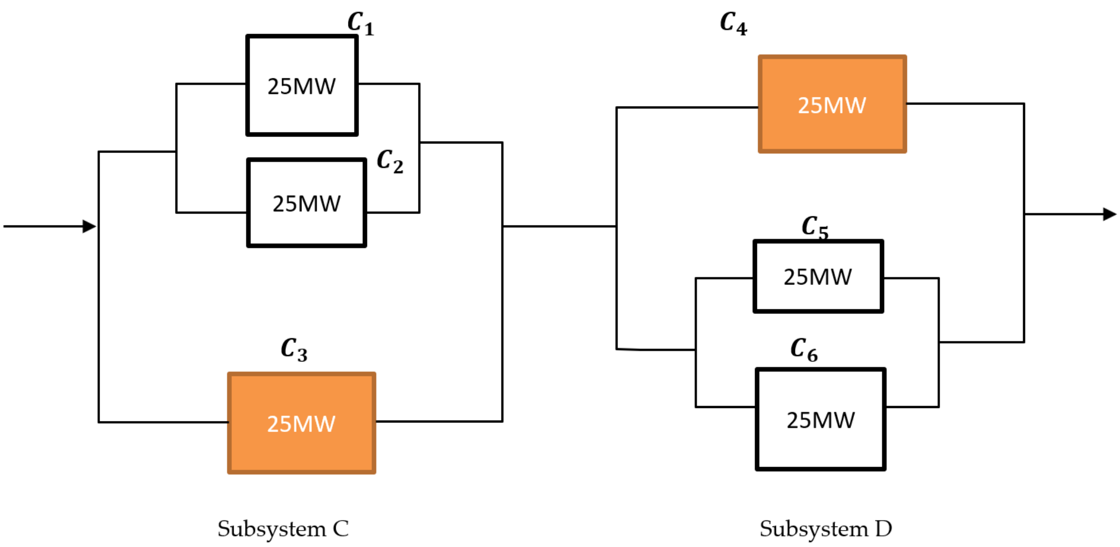

Figure 3: Block diagram of Configuration III

III. Models Formulation

Availability and Meantime to Failure of Configuration I

According to Wang *et al.* (2006), let *Q* (*t*) be the probability that at time t there are n components working in the system. Then the initial conditions for this problem are stated as follows:

$$Q(0) = [Q_0(0), Q_1(0), Q_2(0), \dots, Q_{16}(0)] = [1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]$$

The following differential equations are obtained:

$$\begin{aligned}
Q_0^I(t) &= -4\lambda Q_0(t) + \mu Q_1(t) + \mu Q_2(t) + \mu Q_7(t) + \mu Q_8(t)\\
Q_1^I(t) &= -(4\lambda + \mu)Q_1(t) + \lambda Q_0(t) + \mu Q_3(t) + \mu Q_4(t) + \mu Q_9(t) + \mu Q_{10}(t)\\
Q_2^I(t) &= -(4\lambda + \mu)Q_2(t) + \lambda Q_0(t) + \mu Q_5(t) + \mu Q_6(t) + \mu Q_{11}(t) + \mu Q_{12}(t)\\
Q_3^I(t) &= -(\lambda + \mu)Q_3(t) + \lambda Q_1(t) + \mu Q_{13}(t)\\
Q_4^I(t) &= -(\lambda + \mu)Q_4(t) + \lambda Q_1(t) + \mu Q_{14}(t)\\
Q_5^I(t) &= -(\lambda + \mu)Q_5(t) + \lambda Q_2(t) + \mu Q_{15}(t)\\
Q_6^I(t) &= -(\lambda + \mu)Q_6(t) + \lambda Q_2(t) + \mu Q_{16}(t)\\
Q_7^I(t) &= -\mu Q_7(t) + \lambda Q_0(t)\\
Q_8^I(t) &= -\mu Q_8(t) + \lambda Q_0(t)\\
Q_9^I(t) &= -\mu Q_9(t) + \lambda Q_1(t)\\
Q_{10}^I(t) &= -\mu Q_{10}(t) + \lambda Q_1(t)\\
Q_{11}^I(t) &= -\mu Q_{11}(t) + \lambda Q_2(t)\\
Q_{12}^I(t) &= -\mu Q_{12}(t) + \lambda Q_2(t)\\
Q_{13}^I(t) &= -\mu Q_{13}(t) + \lambda Q_3(t)\\
Q_{14}^I(t) &= -\mu Q_{14}(t) + \lambda Q_4(t)\\
Q_{15}^I(t) &= -\mu Q_{15}(t) + \lambda Q_5(t)\\
Q_{16}^I(t) &= -\mu Q_{16}(t) + \lambda Q_6(t)
\end{aligned} \tag{1}$$

The differential equations in (1) above can be written in the matrix form as

$$Q^I(t) = T_1 Q(t) \tag{2}$$

where

$$\mathrm{T}_1 = \begin{bmatrix}
-4\lambda & \mu & \mu & 0 & 0 & 0 & 0 & \mu & \mu & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\lambda & -y_1 & 0 & \mu & \mu & 0 & 0 & 0 & 0 & \mu & \mu & 0 & 0 & 0 & 0 & 0 & 0\\
\lambda & 0 & -y_1 & 0 & 0 & \mu & \mu & 0 & 0 & 0 & 0 & \mu & \mu & 0 & 0 & 0 & 0\\
0 & \lambda & 0 & -y_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu & 0 & 0 & 0\\
0 & \lambda & 0 & 0 & -y_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu & 0 & 0\\
0 & 0 & \lambda & 0 & 0 & -y_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu & 0\\
0 & 0 & \lambda & 0 & 0 & 0 & -y_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu\\
\lambda & 0 & 0 & 0 & 0 & 0 & 0 & -\mu & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu & 0 & 0 & 0 & 0 & 0\\
0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu & 0 & 0 & 0 & 0\\
0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu & 0 & 0 & 0\\
0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu & 0 & 0\\
0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu & 0\\
0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu
\end{bmatrix}$$

$y_1 = (4\lambda + \mu)$ and $y_2 = (\lambda + \mu)$

Equation (2) above can be written in the matrix form as:

$$\begin{pmatrix} Q_0^I(t) \\ Q_1^I(t) \\ Q_2^I(t) \\ Q_3^I(t) \\ Q_4^I(t) \\ Q_5^I(t) \\ Q_6^I(t) \\ Q_7^I(t) \\ Q_8^I(t) \\ Q_9^I(t) \\ Q_{10}^I(t) \\ Q_{11}^I(t) \\ Q_{12}^I(t) \\ Q_{13}^I(t) \\ Q_{14}^I(t) \\ Q_{15}^I(t) \\ Q_{16}^I(t) \end{pmatrix} = \begin{pmatrix} -4\lambda & \mu & \mu & 0 & 0 & 0 & 0 & \mu & \mu & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda & -y_1 & 0 & \mu & \mu & 0 & 0 & 0 & 0 & \mu & \mu & 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda & 0 & -y_1 & 0 & 0 & \mu & \mu & 0 & 0 & 0 & 0 & \mu & \mu & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & -y_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & -y_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 & -y_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & \lambda & 0 & 0 & 0 & -y_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu \\ \lambda & 0 & 0 & 0 & 0 & 0 & 0 & -\mu & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu \end{pmatrix}$$

In the steady state all the derivative equal to zero, thus from equation (2) above, $T_1Q(\circledcirc) = 0$ is obtained.

$$\begin{pmatrix} -4\lambda & \mu & \mu & 0 & 0 & 0 & 0 & \mu & \mu & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda & -y_1 & 0 & \mu & \mu & 0 & 0 & 0 & 0 & \mu & \mu & 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda & 0 & -y_1 & 0 & 0 & \mu & \mu & 0 & 0 & 0 & 0 & \mu & \mu & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & -y_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & -y_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 & -y_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & \lambda & 0 & 0 & 0 & -y_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu \\ \lambda & 0 & 0 & 0 & 0 & 0 & 0 & -\mu & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu \end{pmatrix} \begin{pmatrix} Q_0(\infty) \\ Q_1(\infty) \\ Q_2(\infty) \\ Q_3(\infty) \\ Q_4(\infty) \\ Q_5(\infty) \\ Q_6(\infty) \\ Q_7(\infty) \\ Q_8(\infty) \\ Q_9(\infty) \\ Q_{10}(\infty) \\ Q_{11}(\infty) \\ Q_{12}(\infty) \\ Q_{13}(\infty) \\ Q_{14}(\infty) \\ Q_{15}(\infty) \\ Q_{16}(\infty) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (3)$$

Using the normalizing condition

$$\sum_{i=0}^{16} Q_i(\infty) = 1 \quad (4)$$

Following Wang et al (2006) Equation (4) is substituted in the last row of (3) to obtain

$$\begin{pmatrix} -4\lambda & \mu & \mu & 0 & 0 & 0 & 0 & \mu & \mu & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda & -y_1 & 0 & \mu & \mu & 0 & 0 & 0 & 0 & \mu & \mu & 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda & 0 & -y_1 & 0 & 0 & \mu & \mu & 0 & 0 & 0 & 0 & \mu & \mu & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & -y_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & -y_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 & -y_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & \lambda & 0 & 0 & 0 & -y_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu \\ \lambda & 0 & 0 & 0 & 0 & 0 & 0 & -\mu & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} Q_0(\infty) \\ Q_1(\infty) \\ Q_2(\infty) \\ Q_3(\infty) \\ Q_4(\infty) \\ Q_5(\infty) \\ Q_6(\infty) \\ Q_7(\infty) \\ Q_8(\infty) \\ Q_9(\infty) \\ Q_{10}(\infty) \\ Q_{11}(\infty) \\ Q_{12}(\infty) \\ Q_{13}(\infty) \\ Q_{14}(\infty) \\ Q_{15}(\infty) \\ Q_{16}(\infty) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad (5)$$

System of linear differential equations given in equation (5) above was solved using MATLAB package to obtain the explicit solution of $Q_0(\infty)$, $Q_1(\infty)$,$Q_2(\infty)$,…, $Q_{16}(\infty)$

AV1$(\infty) = Q_0(\infty) + Q_1(\infty) + Q_2(\infty) + Q_3(\infty) + Q_4(\infty) + Q_5(\infty) + Q_6(\infty) = \frac{\mu^3 + 2\lambda\mu^2 + 4\lambda^2\mu}{4\lambda^3 + 8\lambda^2 + 4\lambda\mu^2 + \mu^3}$

Now to evaluate the MTTF1, the rows and column of the absorbing (failure) state were deleted and the new matrix M1 was transposed as given in the equation (6) below, Wang *et al.* (2006):

E[T$_{Q(0) \to Q(absorbing)}$] = Q(0) ($-M_1^{-1}$)[1,1,1,1,1,1,1]^T (6)

Where,

$$M_1 = \begin{pmatrix} -4\lambda & \lambda & \lambda & 0 & 0 & 0 & 0 \\ \mu & -y_1 & 0 & \lambda & \lambda & 0 & 0 \\ \mu & 0 & -y_1 & 0 & 0 & \lambda & \lambda \\ 0 & \mu & 0 & -y_2 & 0 & 0 & 0 \\ 0 & \mu & 0 & 0 & -y_2 & 0 & 0 \\ 0 & 0 & \mu & 0 & 0 & -y_2 & 0 \\ 0 & 0 & \mu & 0 & 0 & 0 & -y_2 \end{pmatrix}.$$

From equation (6) we have:

E[T$_{Q(0) \to Q(absorbing)}$] = MTTF1 = $\frac{(\lambda + \mu) + 2\lambda}{8\lambda^2 + 5\lambda\mu + \mu^2} + \frac{4\lambda^2 + 3\lambda\mu + \mu^2}{2\lambda(8\lambda^2 + 5\lambda\mu + \mu^2)}$

Availability and Meantime to Failure of Configuration II

To further investigate the availability of configuration II, $Q_i(t), i = 0,1,2,3,\dots,10$ were defined to be the probabilities that the system at time t ≥ 0 is in state Si . Let Q(t) be the probability row vector at time t ≥ 0. The initial condition for this problem is

$Q(0) = [Q_0(0), Q_1(0), Q_2(0), \dots, Q_{10}(0)] = [1,0,0,0,0,0,0,0,0,0,0]$.

Then the following differential equations are obtained:

$$\frac{dQ_0}{dt}(t) = -8\lambda Q_0(t) + \mu Q_1(t) + \mu Q_2(t)$$
$$\frac{dQ_1}{dt}(t) = -(8\lambda + \mu)Q_1(t) + 4\lambda Q_0(t) + \mu Q_3(t) + \mu Q_4(t)$$
$$\frac{dQ_2}{dt}(t) = -(8\lambda + \mu)Q_2(t) + 4\lambda Q_0(t) + \mu Q_5(t) + \mu Q_6(t)$$
$$\frac{dQ_3}{dt}(t) = -(8\lambda + \mu)Q_3(t) + 4\lambda Q_1(t) + \mu Q_7(t) + \mu Q_8(t)$$

$$\frac{dQ_4}{dt}(t) = -(8\lambda + \mu)Q_4(t) + 4\lambda Q_1(t) + \mu Q_9(t) + \mu Q_{10}(t)$$
$$\frac{dQ_5}{dt}(t) = -\mu Q_5(t) + 4\lambda Q_2(t)$$
$$\frac{dQ_6}{dt}(t) = -\mu Q_6(t) + 4\lambda P_2(t)$$
$$\frac{dQ_7}{dt}(t) = -\mu Q_7(t) + 4\lambda Q_3(t)$$
$$\frac{dQ_8}{dt}(t) = -\mu Q_8(t) + 4\lambda Q_3(t)$$
$$\frac{dQ_9}{dt}(t) = -\mu Q_9(t) + 4\lambda Q_4(t)$$
$$\frac{dQ_{10}}{dt}(t) = -\mu Q_{10}(t) + 4\lambda Q_4(t) \qquad (7)$$

With initial conditions $Q(0) = [Q_0(0), Q_1(0), Q_2(0), \dots, Q_{10}(0)] = [1,0,0,0,0,0,0,0,0,0,0]$. Equation (7) could be written in the form of matrix as given in equation (8) below:

$$Q^I(t) = T_2 Q(t) \qquad (8)$$

$$\begin{pmatrix} \frac{dQ_0}{dt}(t) \\ \frac{dQ_1}{dt}(t) \\ \frac{dQ_2}{dt}(t) \\ \frac{dQ_3}{dt}(t) \\ \frac{dQ_4}{dt}(t) \\ \frac{dQ_5}{dt}(t) \\ \frac{dQ_6}{dt}(t) \\ \frac{dQ_7}{dt}(t) \\ \frac{dQ_8}{dt}(t) \\ \frac{dQ_9}{dt}(t) \\ \frac{dQ_{10}}{dt}(t) \end{pmatrix} = \begin{pmatrix} -8\lambda & \mu & \mu & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4\lambda & -y_3 & 0 & \mu & \mu & 0 & 0 & 0 & 0 & 0 & 0 \\ 4\lambda & 0 & -y_3 & 0 & 0 & \mu & \mu & 0 & 0 & 0 & 0 \\ 0 & 4\lambda & 0 & -y_3 & 0 & 0 & 0 & \mu & \mu & 0 & 0 \\ 0 & 4\lambda & 0 & 0 & -y_3 & 0 & 0 & 0 & 0 & \mu & \mu \\ 0 & 0 & 4\lambda & 0 & 0 & -\mu & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4\lambda & 0 & 0 & 0 & -\mu & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4\lambda & 0 & 0 & 0 & -\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 4\lambda & 0 & 0 & 0 & 0 & -\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 4\lambda & 0 & 0 & 0 & 0 & -\mu & 0 \\ 0 & 0 & 0 & 0 & 4\lambda & 0 & 0 & 0 & 0 & 0 & -\mu \end{pmatrix} \begin{pmatrix} Q_0(t) \\ Q_1(t) \\ Q_2(t) \\ Q_3(t) \\ Q_4(t) \\ Q_5(t) \\ Q_6(t) \\ Q_7(t) \\ Q_8(t) \\ Q_9(t) \\ Q_{10}(t) \end{pmatrix}$$

Where, $y_3 = (8\lambda + \mu)$

To calculate the state probabilities, all derivatives of state are equal to zero. This will enable us to compute steady state availability by equating the left hand side of equation (8) to zero. Now we have

$$T_2 Q(\infty) = 0 \qquad (9)$$

Thus, equation (9) above could be written in matrix form as:

$$\begin{pmatrix} -8\lambda & \mu & \mu & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4\lambda & -y_3 & 0 & \mu & \mu & 0 & 0 & 0 & 0 & 0 & 0 \\ 4\lambda & 0 & -y_3 & 0 & 0 & \mu & \mu & 0 & 0 & 0 & 0 \\ 0 & 4\lambda & 0 & -y_3 & 0 & 0 & 0 & \mu & \mu & 0 & 0 \\ 0 & 4\lambda & 0 & 0 & -y_3 & 0 & 0 & 0 & 0 & \mu & \mu \\ 0 & 0 & 4\lambda & 0 & 0 & -\mu & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4\lambda & 0 & 0 & 0 & -\mu & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4\lambda & 0 & 0 & 0 & -\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 4\lambda & 0 & 0 & 0 & 0 & -\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 4\lambda & 0 & 0 & 0 & 0 & -\mu & 0 \\ 0 & 0 & 0 & 0 & 4\lambda & 0 & 0 & 0 & 0 & 0 & -\mu \end{pmatrix} \begin{pmatrix} Q_0(t) \\ Q_1(t) \\ Q_2(t) \\ Q_3(t) \\ Q_4(t) \\ Q_5(t) \\ Q_6(t) \\ Q_7(t) \\ Q_8(t) \\ Q_9(t) \\ Q_{10}(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Using the normalizing condition below, it follows that

$$\sum_{i=0}^{10} Q_i(\infty) = 1 \tag{10}$$

Solving equation (10) above to obtain the explicit solution of $Q_0(\infty)$, $Q_1(\infty)$, $Q_2(\infty)$, …, $Q_{10}(\infty)$, the explicit equation for steady state availability is therefore obtained as follows:

$$\mathrm{AV}_2(\infty) = Q_0(\infty) + Q_1(\infty) + Q_2(\infty) + Q_3(\infty) + Q_4(\infty) = \frac{\mu^3 + 8\lambda\mu^2 + 32\lambda^2\mu}{256\lambda^3 + 64\lambda^2 + 8\lambda\mu^2 + \mu^3} \tag{11}$$

Now to evaluate the MTTF for configuration II, following Wang and Kuo (2000) and Wang *et al.* (2006), the MTTF of the system could be obtained by deleting the rows and column of the absorbing (failure) state and transposing the new matrix M_2. The expected time to reach an absorbing state is given in equation (12) below:

$$E[T_{Q(0)\rightarrow Q(\text{absorbing})}] = Q(0)\,(-M_2^{-1})[1,1,1,1,1]^T \tag{12}$$

Where

$$M_2 = \begin{pmatrix} -8\lambda & 4\lambda & 4\lambda & 0 & 0 \\ \mu & -y_3 & 0 & 4\lambda & 4\lambda \\ \mu & 0 & -y_3 & 0 & 0 \\ 0 & \mu & 0 & -y_3 & 0 \\ 0 & \mu & 0 & 0 & -y_3 \end{pmatrix}$$

Now for the second system, the explicit expression/equation of MTTF_2 is given by equation (13) below:

$$E[T_{Q(0)\rightarrow Q(\text{absorbing})}] = \mathrm{MTTF}_2 = \frac{8\lambda + \mu}{128\lambda^2 + 16\lambda\mu + \mu^2} + \frac{128\lambda^2 + 24\lambda\mu + 2\mu^2}{8(128\lambda^3 + 16\lambda^2\mu + \lambda\mu^2)} + \frac{512\lambda^3 + 128\lambda^2\mu + 16\lambda\mu^2 + \mu^3}{32\lambda(128\lambda^3 + 16\lambda^2\mu + \lambda\mu^2)} \tag{13}$$

Availability and Meantime to Failure of Configuration III

For the analysis of availability case of configuration III, $Q_i(t), i = 0,1,2,3,\ldots,7$ are defined to be the probability that the system at time $t \geq 0$ is in the state S_i. Let Q(t) also be the probability row vector at time $t \geq 0$. The initial condition for this problem is:

$Q(0) = [Q_0(0), Q_1(0), Q_2(0), \ldots, Q_7(0)] = [1,0,0,0,0,0,0,0]$.

$\frac{dQ_0}{dt}(t) = -8\lambda Q_0(t) + \mu Q_1(t) + \mu Q_2(t)$

$\frac{dQ_1}{dt}(t) = -(8\lambda + \mu)Q_1(t) + 4\lambda Q_0(t) + \mu Q_3(t) + \mu Q_4(t)$

$\frac{dQ_2}{dt}(t) = -(8\lambda + \mu)Q_2(t) + 4\lambda Q_0(t) + \mu Q_3(t) + \mu Q_5(t)$

$\frac{dQ_3}{dt}(t) = -(8\lambda + 2\mu)Q_3(t) + 4\lambda Q_1(t) + 4\lambda Q_2(t) + \mu Q_6(t) + \mu Q_7(t)$

$$\frac{dQ_4}{dt}(t) = -\mu Q_4(t) + 4\lambda Q_1(t)$$
$$\frac{dQ_5}{dt}(t) = -\mu Q_5(t) + 4\lambda Q_2(t)$$
$$\frac{dQ_6}{dt}(t) = -\mu Q_6(t) + 4\lambda Q_3(t)$$
$$\frac{dQ_7}{dt}(t) = -\mu Q_7(t) + 4\lambda Q_3(t) \quad (14)$$

Equation (14) is rewritten in the matrix form as presented in equation (15) below:

$$Q^I(t) = T_3 Q(t) \quad (15)$$

Where,

$$T_3 = \begin{bmatrix} -8\lambda & \mu & \mu & 0 & 0 & 0 & 0 & 0 \\ 4\lambda & -y_3 & 0 & \mu & \mu & 0 & 0 & 0 \\ 4\lambda & 0 & -y_3 & \mu & 0 & \mu & 0 & 0 \\ 0 & 4\lambda & 4\lambda & -y_4 & 0 & 0 & \mu & \mu \\ 0 & 4\lambda & 0 & 0 & -\mu & 0 & 0 & 0 \\ 0 & 0 & 4\lambda & 0 & 0 & -\mu & 0 & 0 \\ 0 & 0 & 0 & 4\lambda & 0 & 0 & -\mu & 0 \\ 0 & 0 & 0 & 4\lambda & 0 & 0 & 0 & -\mu \end{bmatrix}$$

Where $y_4 = (8\lambda + 2\mu)$

Initial conditions are considered as given in the following equation: $Q(0) = [Q_0(0), Q_1(0), Q_2(0), \dots, Q_7(0)] = [1,0,0,0,0,0,0,0]$.

To obtain the steady state probabilities, right hand side of (15) is equated to zero such that

$$T_3 Q(\infty) = 0 \quad (16)$$

Thus, (16) can be written in matrix form as follows:

$$\begin{pmatrix} -8\lambda & \mu & \mu & 0 & 0 & 0 & 0 & 0 \\ 4\lambda & -y_3 & 0 & \mu & \mu & 0 & 0 & 0 \\ 4\lambda & 0 & -y_3 & \mu & 0 & \mu & 0 & 0 \\ 0 & 4\lambda & 4\lambda & -y_4 & 0 & 0 & \mu & \mu \\ 0 & 4\lambda & 0 & 0 & -\mu & 0 & 0 & 0 \\ 0 & 0 & 4\lambda & 0 & 0 & -\mu & 0 & 0 \\ 0 & 0 & 0 & 4\lambda & 0 & 0 & -\mu & 0 \\ 0 & 0 & 0 & 4\lambda & 0 & 0 & 0 & -\mu \end{pmatrix} \begin{pmatrix} Q_0(\infty) \\ Q_1(\infty) \\ Q_2(\infty) \\ Q_3(\infty) \\ Q_4(\infty) \\ Q_5(\infty) \\ Q_6(\infty) \\ Q_7(\infty) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Solving (16) using normalizing condition

$$\sum_{i=0}^{7} Q_i(\infty) = 1 \quad (17)$$

$Q_0(\infty)$, $Q_1(\infty)$,$Q_2(\infty)$, $Q_3(\infty)$, $Q_4(\infty)$,$Q_5(\infty)$, $Q_6(\infty)$, $Q_7(\infty)$ are obtained. Therefore, the explicit expression/equation of AV_3(◎) is given by

$$AV_3(\infty) = Q_0(\infty) + Q_1(\infty) + Q_2(\infty) + Q_3(\infty) = \frac{\mu^3 + 8\lambda\mu^2 + 16\lambda^2\mu}{128\lambda^3 + 48\lambda^2 + 8\lambda\mu^2 + \mu^3} \quad (18)$$

To compute MTTF for configuration III, follow similar argument used in configurations I and II. The rows and column of the absorbing states of the matrix T_3 are therefore deleted and take the transpose to obtain a new matrix M_3

$$E[T_{Q(0)\rightarrow Q(\text{absorbing})}] = Q(0)\,(-M_3^{-1})[1,1,1,1]^T \tag{19}$$

Where, $M_3 = \begin{pmatrix} -8\lambda & 4\lambda & 4\lambda & 0 \\ \mu & -y_3 & 0 & 4\lambda \\ \mu & 0 & -y_3 & 4\lambda \\ 0 & \mu & \mu & -y_4 \end{pmatrix}$

The explicit expression/equation of $MTTF_3$ is therefore obtained as follows:

$$E[T_{Q(0)\rightarrow Q(\text{absorbing})}] = MTTF_3 = \frac{1}{2(8\lambda+\mu)} + \frac{4\lambda+\mu}{4(8\lambda^2++\lambda\mu)} + \frac{32\lambda^2+8\lambda\mu+\mu^2}{32\lambda(8\lambda^2+\lambda\,\mu)} \tag{20}$$

IV. Discussion

Sensitivity Analysis of Three Configurations

In this section, numerical comparisons for the result of sensitivity analysis for all the developed models were presented. Computer software, MATLAB is used to compute the three configurations in terms of their sensitivity analysis. From the results of system one it has been observed that configuration I is far better than all the remaining configurations as we can observed in table 1 through 2 below. It can be see that availability of configuration I is compared with that of configuration II and configuration III in terms of failure rate λ and repair rate μ. Furthermore, virtually all the configurations were compared with configuration I in terms of their **MTTF** with effect of failure rate λ and repair rate μ that is table 5 to table 6 below. It is also observed that configuration I retain its optimality.

Similarly, configuration I was compared with all the remaining configurations and turn to be the best in terms of Profit and table 3 to table 4 below clearly justify that the configuration I is the optimal. However, in the sensitivity results obtained from table 1 through table 6 with the help of Bar chat (i.e. Figure 1 – 18) below, availability versus failure and repair rate, Profit versus failure and repair rates and **MTTF** versus failure and repair, it can be justified that configuration I was the best because as one can observed from all tables below. Despite the fact that failure increases in table 1 for instance configuration I has the maximum availability and similarly with repair it shows more increasing trends which is far better than the remaining configurations and this bridge the practical gap that remains untouched.

Table 1: Variation of Availability with respect to μ for the three Configurations for different values of λ

μ	λ = 0.1			λ = 0.5			λ = 0.9		
	Configuration			Configuration			Configuration		
	I	II	III	I	II	III	I	II	III
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.1111	0.4316	0.1383	0.1507	0.1668	0.0278	0.0285	0.1042	0.0154	0.0157
0.2222	0.5664	0.2710	0.3017	0.2681	0.0555	0.0581	0.1804	0.0309	0.0317
0.3333	0.6473	0.3907	0.4331	0.3377	0.0833	0.0886	0.2389	0.0463	0.0481
0.4444	0.7030	0.4935	0.5398	0.3899	0.1109	0.1196	0.2856	0.0617	0.0649
0.5556	0.7439	0.5788	0.6239	0.4316	0.1383	0.1507	0.3240	0.0771	0.0818
0.6667	0.7750	0.6483	0.6897	0.4663	0.1656	0.1818	0.3566	0.0925	0.0989
0.7778	0.7995	0.7044	0.7411	0.4961	0.1925	0.2126	0.3847	0.1078	0.1161
0.8889	0.8193	0.7497	0.7817	0.5222	0.2191	0.2430	0.4094	0.1231	0.1334
1.0000	0.8356	0.7864	0.8140	0.5455	0.2453	0.2727	0.4316	0.1383	0.1507

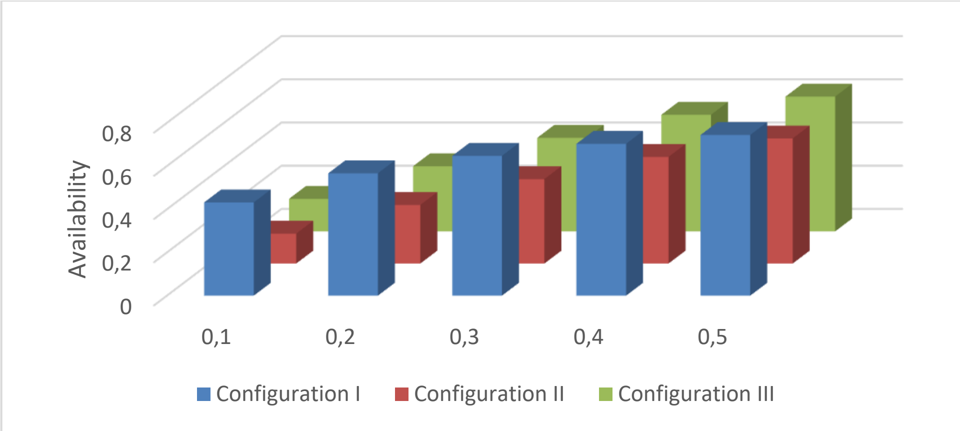

Figure 1: Availability against μ for λ= 0.1

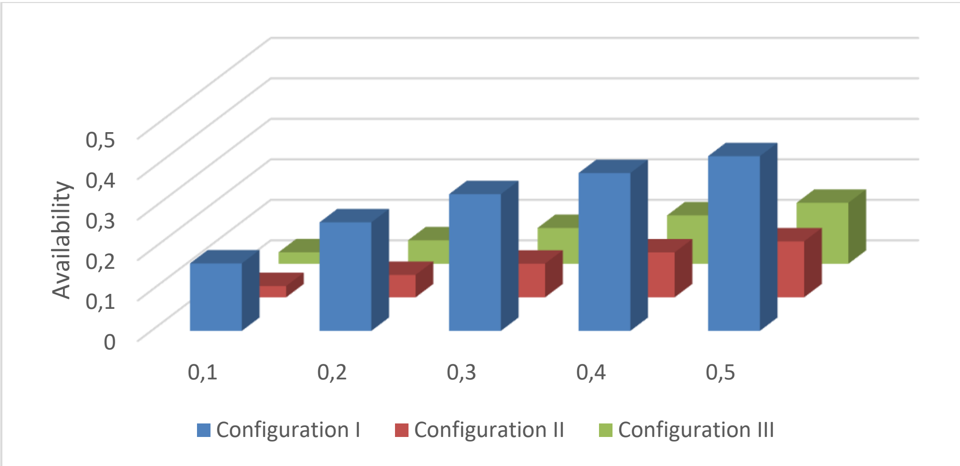

Figure 2: Availability against μ for λ = 0.5

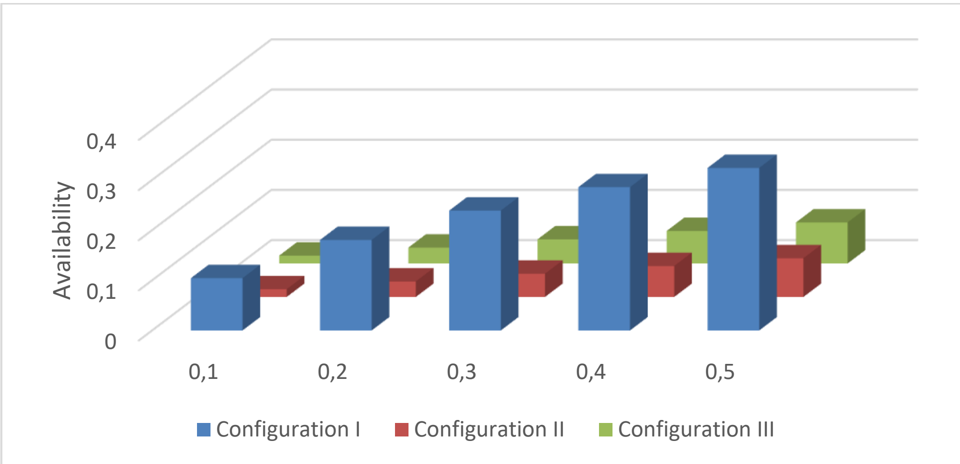

Figure 3: Availability against μ for λ = 0.9

Table 2: Variation of Availability with respect to λ for the three Configurations for different values of μ

λ	μ = 0.3			μ = 0.6			μ = 0.9		
	Configuration			Configuration			Configuration III		
	I	II	III	I	II	III	I	II	III
0.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.1111	0.6053	0.3244	0.3611	0.7388	0.5679	0.6133	0.8057	0.7186	0.7539
0.2222	0.4686	0.1676	0.1841	0.6053	0.3244	0.3611	0.6853	0.4591	0.5047
0.3333	0.3922	0.1122	0.1211	0.5246	0.2217	0.2460	0.6053	0.3244	0.3611
0.4444	0.3399	0.0843	0.0898	0.4686	0.1676	0.1841	0.5479	0.2482	0.2760
0.5556	0.3008	0.0675	0.0712	0.4262	0.1345	0.1464	0.5040	0.2003	0.2215
0.6667	0.2701	0.0562	0.0589	0.3922	0.1122	0.1211	0.4686	0.1676	0.1841
0.7778	0.2452	0.0482	0.0502	0.3639	0.0963	0.1032	0.4392	0.1440	0.1572
0.8889	0.2247	0.0422	0.0437	0.3399	0.0843	0.0898	0.4141	0.1262	0.1369
1.0000	0.2073	0.0375	0.0388	0.3191	0.0749	0.0794	0.3922	0.1122	0.1211

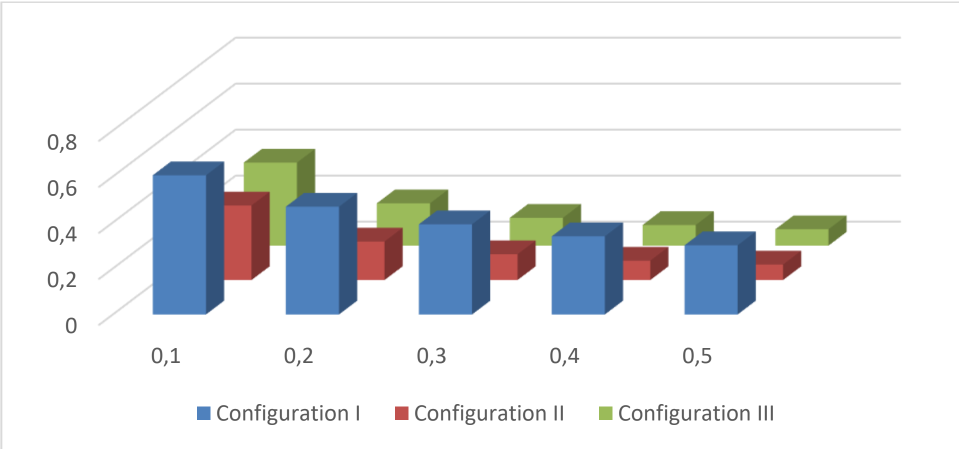

Figure 4: Availability against λ for μ = 0.3

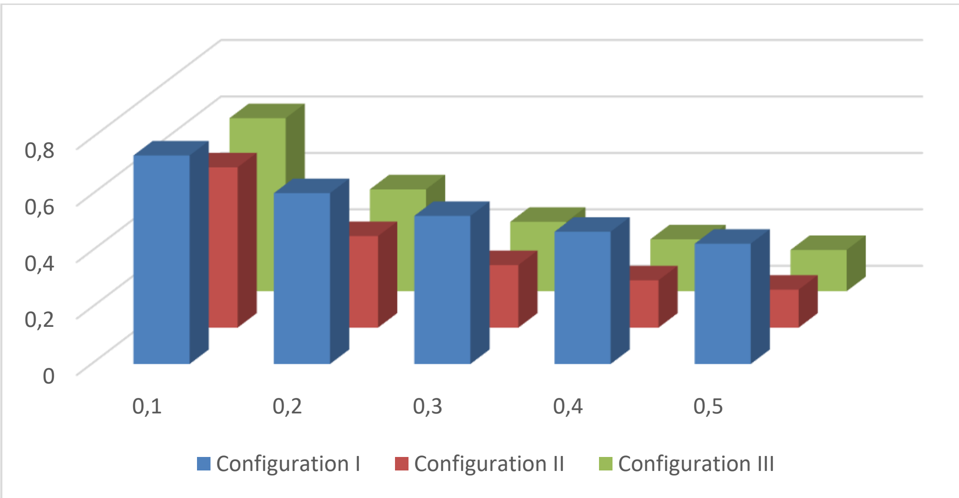

Figure 5: Availability against λ for μ = 0.6

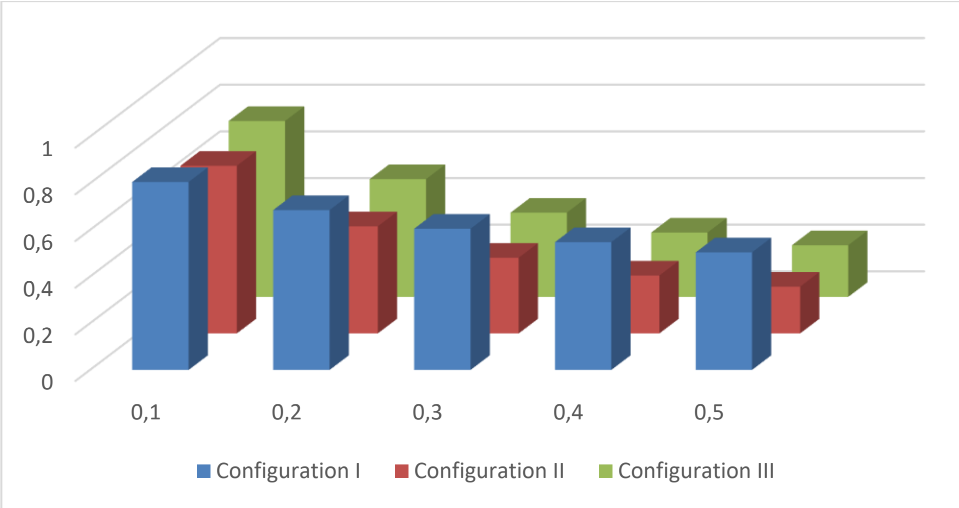

Figure 6: Availability against λ for $\mu = 0.9$

Table 3: Variation of Profit* 10^6 with respect to μ for the three Configurations for different values of λ

μ	$\lambda = 0.1$			$\lambda = 0.5$			$\lambda = 0.9$		
	Configuration			Configuration			Configuration		
	I	II	III	I	II	III	I	II	III
0.1111	2.1574	0.6912	0.7532	0.8337	0.1384	0.1420	0.7785	0.7671	0.5205
0.2222	2.8318	1.3545	1.5082	1.3399	0.2773	0.2903	1.582	1.5387	0.9016
0.3333	3.2362	1.9534	2.1650	1.6879	0.4158	0.4426	2.403	2.3100	1.1941
0.4444	3.5149	2.4673	2.6986	1.9491	0.5539	0.5974	3.2382	3.0808	1.4275
0.5556	3.7191	2.8939	3.1193	2.1574	0.6912	0.7532	4.0849	3.8506	1.6198
0.6667	3.8749	3.2413	3.4480	2.3310	0.8274	0.9087	4.9401	4.6192	1.7825
0.7778	3.9976	3.5220	3.7053	2.4799	0.9621	1.0628	5.8013	5.3860	1.9231
0.8889	4.0964	3.7484	3.9082	2.6106	1.0951	1.2146	6.666	6.1505	2.0469
1.0000	4.1778	3.9316	4.0696	2.7270	1.2260	1.3632	7.5319	6.9121	2.1574

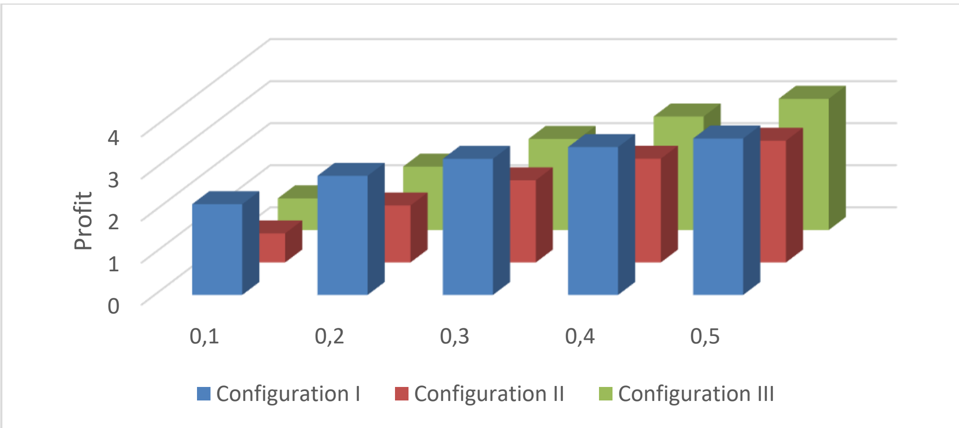

Figure 7: Profit against μ for $\lambda = 0.1$

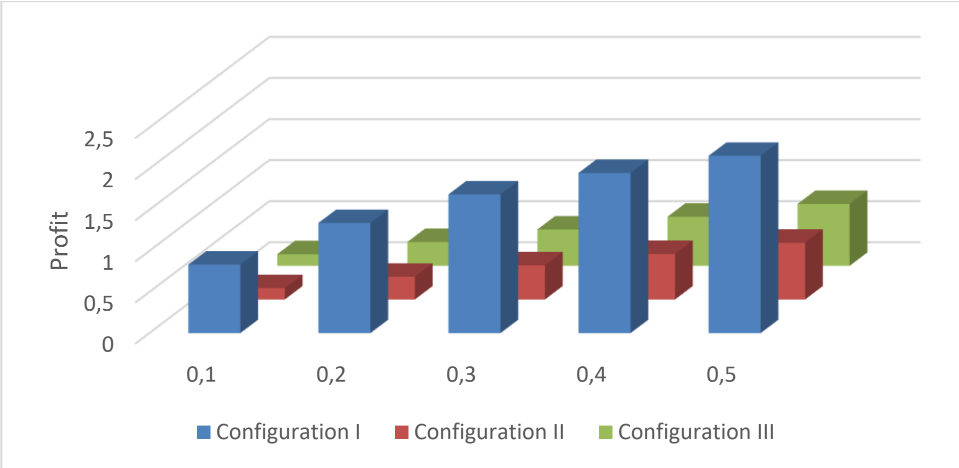

Figure 8: Profit against μ for $\lambda = 0.5$

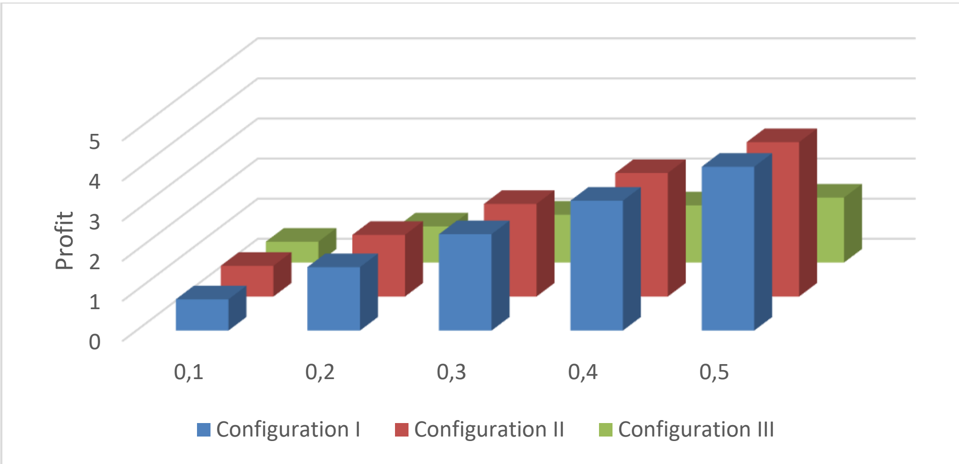

Figure 9: Profit against μ for $\lambda = 0.9$

Table 4: Variation of Profit*10^6 with respect to λ for the three Configurations for different values of μ

λ	μ = 0.3			μ = 0.6			μ = 0.9		
	Configuration			Configuration			Configuration		
	I	II	III	I	II	III	I	II	III
0.1111	3.0264	1.6216	1.8054	3.6939	2.8392	3.0664	4.0284	3.5927	3.7691
0.2222	2.3429	0.8376	0.9203	3.0264	1.6216	1.8054	3.4264	2.2950	2.5230
0.3333	1.9605	0.5608	0.6052	2.6228	1.1083	1.2296	3.0264	1.6216	1.8054
0.4444	1.6990	0.4210	0.4484	2.3429	0.8376	0.9203	2.7393	1.2406	1.3798
0.5556	1.5035	0.3369	0.3554	2.1306	0.6720	0.7314	2.5195	1.0009	1.1071
0.6667	1.3501	0.2807	0.2940	1.9605	0.5608	0.6052	2.3429	0.8376	0.9203
0.7778	1.2259	0.2406	0.2506	1.8193	0.4810	0.5154	2.1958	0.7195	0.7855
0.8889	1.1229	0.2105	0.2183	1.6990	0.4210	0.4484	2.0701	0.6304	0.6840
1.0000	1.0361	0.1870	0.1933	1.5949	0.3743	0.3966	1.9605	0.5608	0.6052

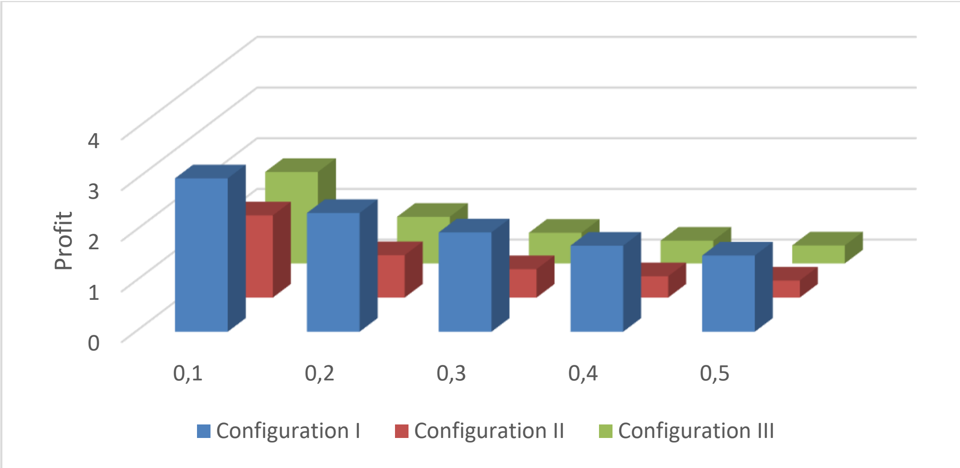

Figure 10: Profit against λ for $\mu = 0.3$

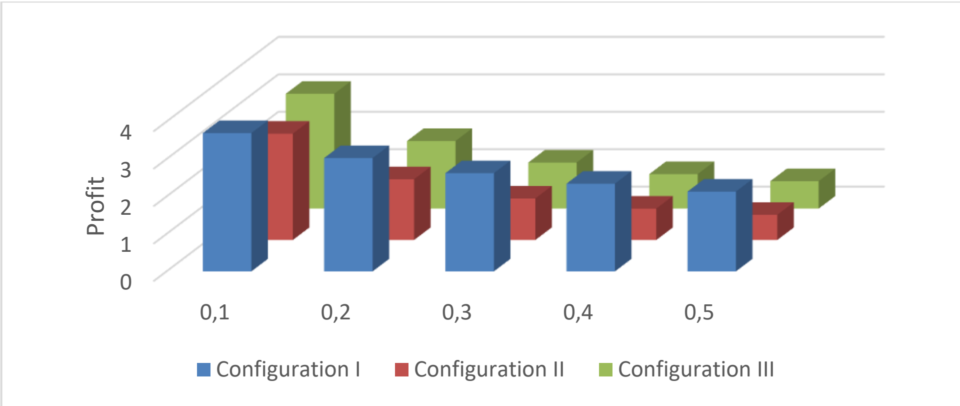

Figure 11: Profit against λ for $\mu = 0.6$

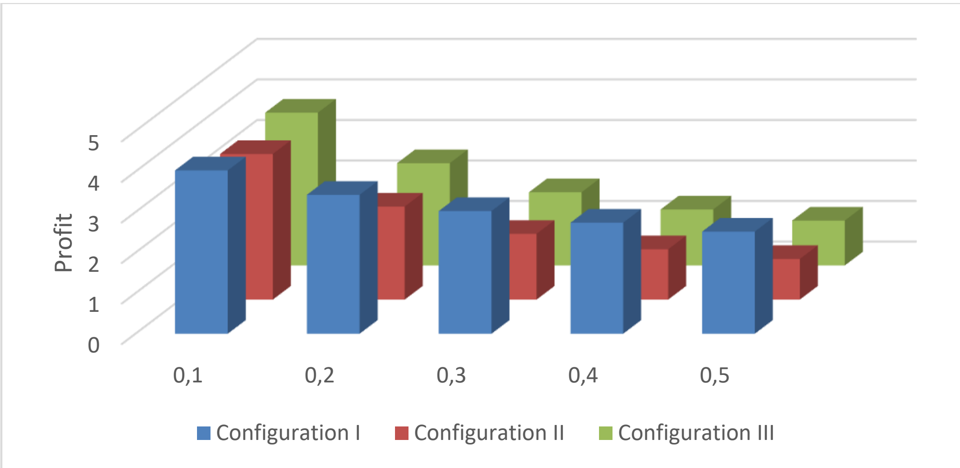

Figure 12: Profit against λ for $\mu = 0.9$

Table 5: Variation of MTTF with respect to μ for the three Configurations for different values of λ

μ	$\lambda = 0.1$			$\lambda = 0.5$			$\lambda = 0.9$		
	Configuration			Configuration			Configuration		
	I	II	III	I	II	III	I	II	III
0.1111	5.6761	3.3914	3.3960	1.2183	0.6355	0.6355	0.6843	0.3504	0.3505
0.2222	5.4158	3.6692	3.6836	1.1919	0.6460	0.6462	0.6751	0.3537	0.3537
0.3333	5.2795	3.9574	3.9828	1.1698	0.6567	0.6571	0.6668	0.3569	0.3570
0.4444	5.2001	4.2544	4.2907	1.1511	0.6674	0.6681	0.6593	0.3602	0.3603
0.5556	5.1501	4.5590	4.6050	1.1352	0.6783	0.6792	0.6524	0.3635	0.3637
0.6667	5.1166	4.8700	4.9242	1.1216	0.6892	0.6905	0.6462	0.3668	0.3671
0.7778	5.0931	5.1862	5.2475	1.1099	0.7002	0.7019	0.6406	0.3701	0.3705
0.8889	5.0761	5.5069	5.5738	1.0998	0.7114	0.7134	0.6354	0.3735	0.3739
1.0000	5.0633	5.8312	5.9028	1.0909	0.7226	0.7250	0.6307	0.3768	0.3773

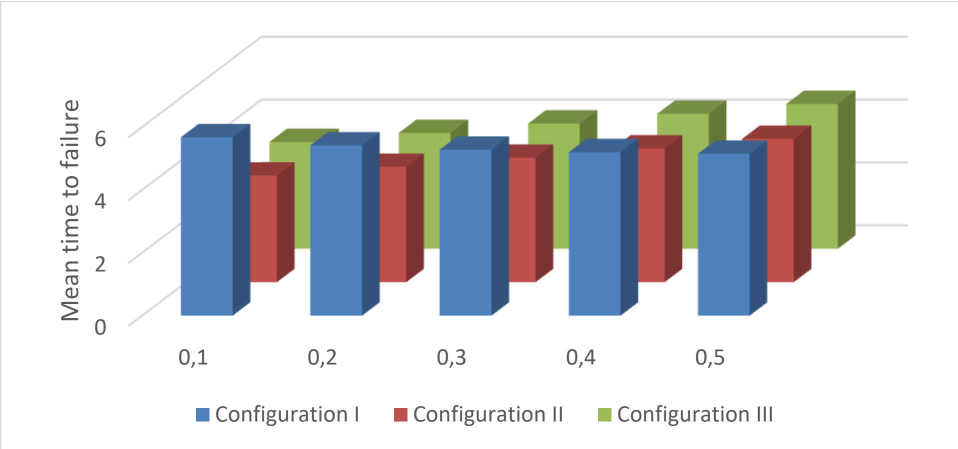

Figure 13: Profit against μ for $\lambda = 0.1$

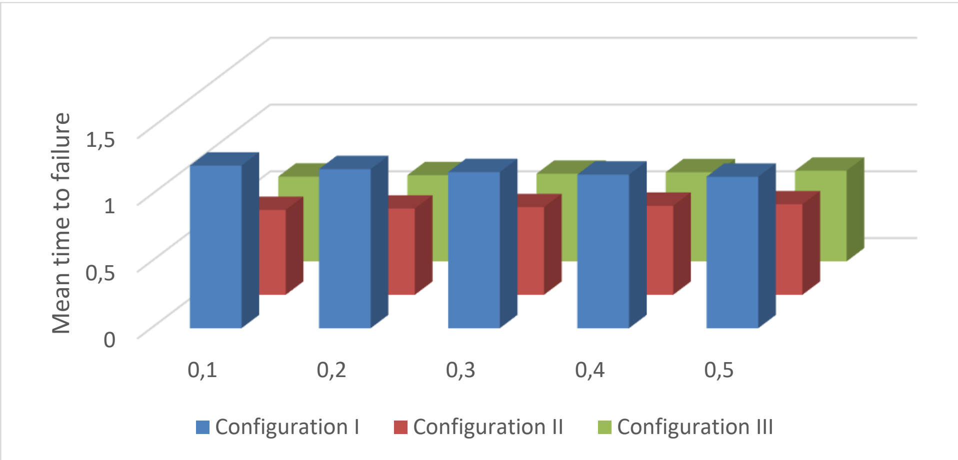

Figure 14: Profit against μ for $\lambda = 0.5$

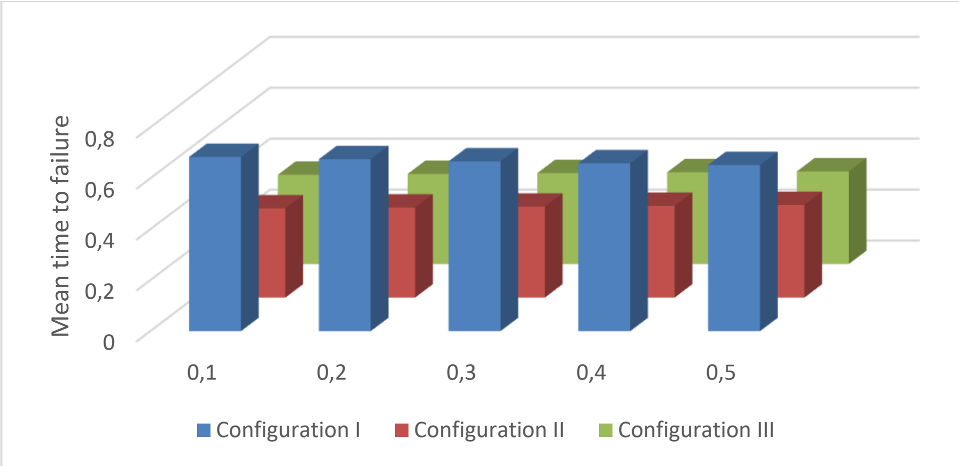

Figure 15: Profit against μ for λ = 0.9

Table 6: Variation of MTTF with respect to λ for the three Configurations for different values of μ

λ	μ= 0.3			μ= 0.6			μ= 0.9		
	Configuration			Configuration			Configuration		
	I	II	III	I	II	III	I	II	III
0.1111	4.8126	3.4128	3.4299	4.6403	4.0644	4.1046	4.5789	4.7509	4.8076
0.2222	2.5215	1.5526	1.5555	2.4063	1.7064	1.7150	2.3508	1.8666	1.8813
0.3333	1.7254	1.0020	1.0029	1.6482	1.0687	1.0718	1.6042	1.1376	1.1433
0.4444	1.3152	0.7392	0.7396	1.2608	0.7763	0.7777	1.2262	0.8143	0.8171
0.5556	1.0638	0.5855	0.5858	1.0236	0.6091	0.6099	0.9961	0.6331	0.6347
0.6667	0.8935	0.4847	0.4849	0.8627	0.5010	0.5015	0.8405	0.5175	0.5185
0.7778	0.7704	0.4135	0.4136	0.7461	0.4254	0.4257	0.7279	0.4375	0.4381
0.8889	0.6773	0.3605	0.3606	0.6576	0.3696	0.3698	0.6424	0.3788	0.3793
1.0000	0.6043	0.3196	0.3196	0.5880	0.3267	0.3269	0.5751	0.3340	0.3343

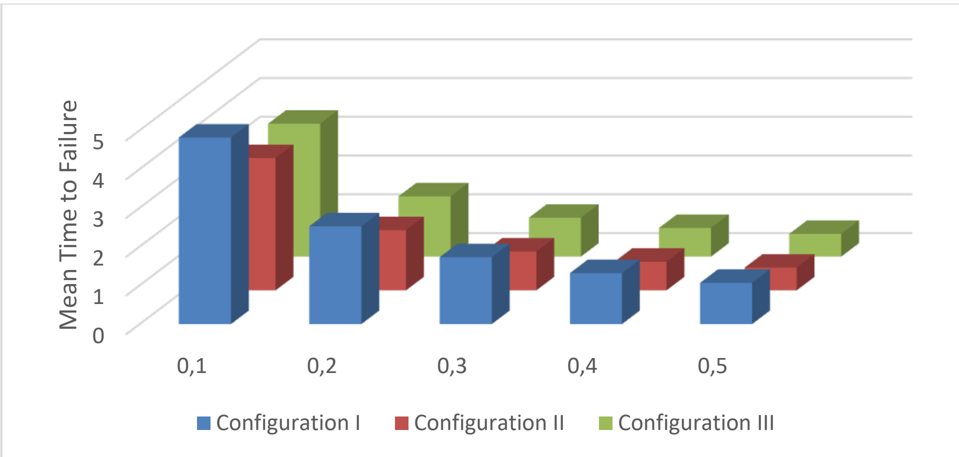

Figure 16: Profit against λ for μ= 0.3

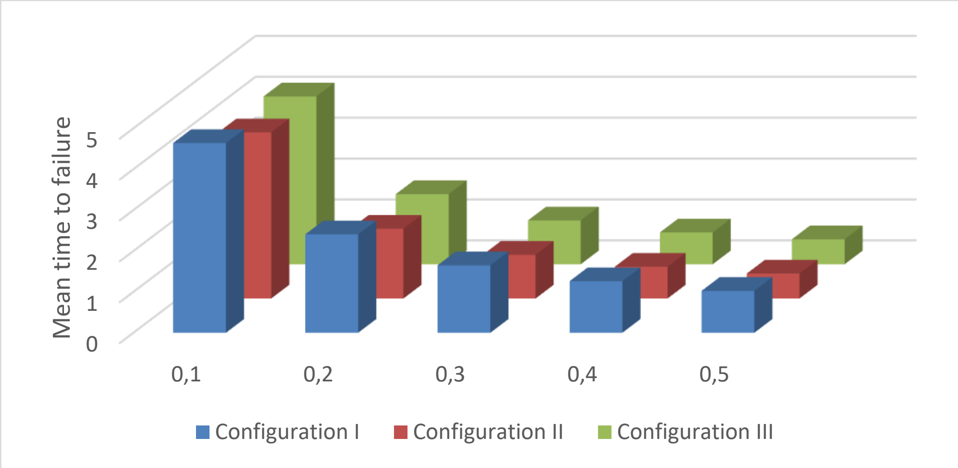

Figure 17: Profit against λ for μ= 0.6

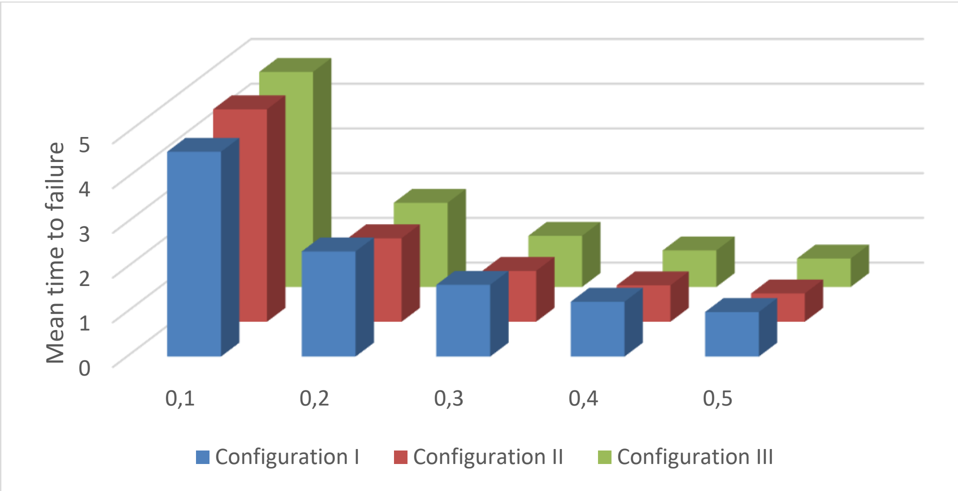

Figure 18: Profit against λ for μ= 0.

Conclusion

In this paper, three different series - parallel dynamo configurations were constructed with standby in each of the configuration and a repairable service station to study the sensitivity analysis of the three configurations under probability. The explicit expressions/equations for **MTTF** and Availability for the three configurations were developed and performed a sensitivity analysis base on the numerical values fixed. It was found out that the optimal configuration using the sensitivity analysis by fixing both λ and μ is configuration I.

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Received: July 17, 2020
Accepted: October 15, 2020

ISSN 1932-2321