Modelling Health Care Queue Management System Facing Patients' Impatience using Queuing Theory

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Abstract

In this paper, we study a finite capacity single server queuing model with balking and correlated reneging and discuss its application in queues at health care facilities. The reneging considered so far in the queuing literature is a function of system-state. In many practical situations, reneging may depend on factors other than the system-state. For instance, in health care systems the reneging of patients at two consecutive time marks may be correlated in a sense that if a patient reneges at a time mark then there is a probability that a patient may renege at the next time mark due to the factors like improper check-up, sluggish management, unnecessary costly prescription by doctors, etc. The steady-state solution of the model is derived by using matrix-decomposition method. Finally, the transient performance analysis of the model is performed using Runge-Kutta method.

Keywords: Queuing model, Correlated reneging, Balking, Health care queue management, Transient performance analysis

1 Introduction

Queues are the common sight of any service providing system. Consistent long queues, delay in services, and low service quality drastically affect the business and perception of any institution providing any sort of service. Customers in queues may tend to leave if they are subjected to wait longer than their patience level. This phenomenon is known as reneging in queuing terminology. This impacts the business entity to lose potential customers and profits. Thus, foreseeing long term benefits it becomes a matter of high importance for a business institution to avoid any such aversive situation even if it requires capital investment.

In critical systems like health care queue management systems, in order to improve the standard of service it is very important to assign sufficient resources to handle the patient behavior during long waiting times in queues [42]. Fomundam and Herrmann [10] work on the design of health care systems using queuing theory. Lakshmi and Shivakumar [35] review the practical use of queuing theory in health care management. Jeffery and James[16] propose a queuing theory-based method to estimate the ratio of patients leaving a hospital Emergency Department (ED) without treatment by considering them as balked or reneged patients. Obulor and Eke [40] use queuing model to analyse the appointment scheduling process to have reduced waiting times for patients and reduced idle times for health staff. The authors in [2] use queuing theory to discuss latest mathematical modelling techniques in the planning of health care systems.

Queuing systems with customers' impatience are used to model and analyse a number of real life systems like hospitals handing critical patients, modelling computer-communication systems with packet loss, impatient telephone switch board customers, and perishable inventory systems. The study of queuing models with impatience starts in late 1950s. Haight [11] first uses the concept of balking in a single server queuing model. Haight [12] then studies a single server queuing system with reneging. The steady-state analysis of the model is performed. Ancker and Gafarian [3] investigate an M/M/1/N queuing model with balking and reneging where reneging of customers was in accordance with exponential distribution and the balking probability for an arriving customers is n/N, where n is number of customers in the system and N is the system capacity. Anker and Gafarian [4] examine an infinite capacity single server Markovian queuing model with balking and reneging. For the arriving customers probability of balking is $1-(\beta/n)$; n=1,2,3,... where n is the number of customers present in the system and β is a measure of customer's desire to join the queue. Subba Rao [43] analyse an M/G/1/N queuing system with balking, reneging and interruptions. She uses supplementary variable technique and discrete transforms to obtain the solution of the model. Since then, a lot of research are written on queuing models with reneging and balking. Kumar et al. [18] consider a single server infinite capacity Markovian queuing system with balking and analysed its transient solution. Kumar and Sharma [20] bring the new concept of retention of reneging customers in queuing theory. They study an M/M/1/N queuing system with reneging and retention of reneging customers and obtained the steady-state solution of the model. Kumar and Sharma [29] obtain the transient solution of a single server queuing model with reneging and retention of reneging customers. Kumar and Sharma [30] study an M/M/c queuing model with balking, reneging and retention of reneging customers and derive its transient solution. Yang and Wu [45] include the concept of retention of reneging customers in a finite capacity queuing system with working breakdowns. They use the matrix-decomposition method for the steady-state solutions. Kumar and Soodan [32] consider a single server queuing model with correlated arrivals and reneging and analysed its transient behavior. For more insights on queuing models with reneging and retention of reneging customers one can refer [19, 21, 22, 23, 24, 25, 26, 27, 28, 31, 44]

The majority of work done so far in the literature considers reneging as a function of system-state (such as queue length or time in the queue). But, in many real life scenarios the reneging of customers may depend on other factors also. Kumar and Soodan [33, 34] introduce the concept of correlated reneging in queuing theory. In the scenarios like health care systems, launching of quality branded product, movie of a famous actor, etc. the length of the queue does not demoralize the customers in the queue. In such high-end products and services, customers wish to stay in the queue. But, for these services and products to survive in market they have to maintain the perception in the masses. The perception impacts the decision-making of customers where the similar customers appear in conjunction (physically or virtually). If the perception about the product or service goes negative the word-of mouth publicity critically influences the customers to renege.

For example, consider a health care system that analogs to a queuing system in which the arrival of patients in the medical facility is similar to the arriving customers, the diagnosing of patients by a doctor is similar to the servicing customers, and the abandoning of the patient from the health care system before the consultation of doctor as reneging customers. At times the reneging of patients could be bursty due to many reasons like improper check-up, sluggish management, unnecessary costly prescription by doctors, etc. which may prevail a bad perception among masses. That is, for a patient reneging at any time instant, there would be a probability of a patient to renege at the next time instant influenced by the decision of earlier patient. Thus, the probability of reneging is dependent on a recently reneged customer where the similar customers appear in conjunction to share their views and experiences. So, influenced by the decision of earlier patient, other patients may also decide to renege.

We referred this form of reneging as correlated reneging. Sometimes, it happens that the arriving patient leaves the system before joining it. This situation is known as balking in queuing theory.

Mohan [37] first introduces the concept of correlation in gambler's ruin problem. Murari [38] studies a queuing system with correlated arrivals and general service time distribution. Mohan and Murari [39] obtain the transient solution of a queuing model with correlated arrivals and variable service capacity. Conolly [7] considers a queuing system having services depending on inter-arrival times. Conolly and Hadidi [8] consider a model having arrival pattern impacting the service pattern. They examine the initial busy period, state and output processes. Cidon et al. [5] consider a queue in which service time is correlated to inter-arrival time. They study this correlation in case of communication systems and showed the impact through numerical results by comparing with less reliable models. Patuwo et al. [41] work on serial correlation in the arrivals. They study the consequences of correlation on mean queuing performances. They found that positive serial correlations may have vital influence on mean queue length. Drezner [9] performs the performance analysis of $M^c/G/1$ queues. Adan and Kulkarni [1] study a single server queue in which both the inter-arrival times and service times rely on same discrete-time Markov chain, a generalisation of $MAP/G/1$ queuing model. They also obtain the waiting time, steady-state and queue length distribution along with moments for this model. Iravani and Luangkesorn [15] study a model of parallel queues with correlated arrivals and bulk services. To obtain the performance measures they use the matrix geometric method. Hwang and Sohraby [14] consider a correlated queue of packets moving in transmission line with finite capacity. Numerical examples are illustrated to exhibit the importance of correlation on system performances. Hunter [13] studies the consequences of correlated arrivals on the steady-state queue length process for single server queuing model. He extends the concept to four different models and compared their condition for stability and queuing behavior. Kamoun and ali [17] consider a single server queuing model with finite capacity and correlated arrival in which the packets are submitted to random interruptions. The approximations for the queue length distributions of products in machine repair problem are obtained. Claeys et al. [6] study a discrete-time $D-BMAP/G^{l,c}/1$ queue and obtain various performance measures associated with buffer content. They illustrate that the correlated in arrivals cannot be neglected for the evaluation of performance measures and buffer management. Lambert et al. [36] study a discrete time $D-MAP/PH/1$ queuing model and develop an algorithm to deliberate the queue length and delay distribution of customers. They also give some advice to design optical buffers.

In this paper, we study a finite capacity single server queuing model with balking and correlated reneging and discuss their application in queues at health care facilities. Rest of the paper is as follows: In section 2, the stochastic queuing model is described. In section 3, the mathematical model is presented. In section 4, steady-state analysis of the model is done. Section 5 deals with the transient performance analysis of the model. Finally, the paper is concluded in section 6.

2 Stochastic queuing model

We consider a finite capacity single server Markovian queuing model with balking and correlated reneging. The state-transition diagram of the queuing model is shown in figure 1. Patients arrive at a health care facility one by one in accordance with Poisson process with parameter λ. There is a single queue and a single server. The service-times are independently, identically and exponentially distributed with parameter μ. On arrival, the incoming patient may decide to not join the queue (i.e. balk) with certain probability (say, $1-\beta$). This means the arriving patient may join the queue with probability β. The capacity of the system is finite (say, K), and K=N+1, where N is the capacity of the queue. After joining the queue and waiting for some time, a patient may leave the queue (renege) without the check-up. The reneging of the patients can take place only at the transition marks $t_0, t_1, t_2, \ldots$ where $\theta_r = t_r - t_{r-1}, r = 1,2,3\ldots$, are random variables with $P[\theta_r \le x] = 1 - exp(-\xi x); \xi \ge 0, r = 1,2,3, \ldots$ That is, the distribution of inter-transition marks is negative exponential with parameter ξ. The reneging at two consecutive transition marks is governed by the following transition probability matrix:

$$\begin{array}{ccc} & & \text{to } t_r \\ & & \begin{array}{cc} 0 & \quad 1 \end{array} \\ \text{from } t_{r-1} & \begin{array}{c} 0 \\ 1 \end{array} & \left\| \begin{array}{cc} p_{00} & p_{01} \\ p_{10} & p_{11} \end{array} \right\| \end{array}$$

where, 0 refers to no reneging and 1 refers to the occurrence of reneging. Thus, the notation p_{ij} (i and j can either be 0 or 1) represents the probability of transitioning from present state to next possible state due to the reneging between the two consecutive transition marks.

Also, $p_{00} + p_{01} = 1$ and $p_{10} + p_{11} = 1$. Thus, the reneging in two consecutive transition marks is correlated. In case of no correlation: $p_{00} = p_{10}$ and $p_{01} = p_{11}$.

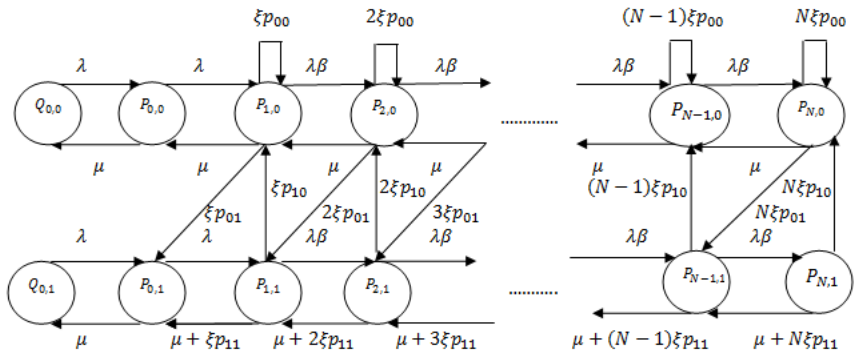

Figure 1: State-transition diagram of the model

3 Mathematical model

Defining the probabilities:

$Q_{0,r}(t)$ is the probability that at time t queue length is zero, the server is idle, and r is an indicator whether a patient has reneged or not at the previous transition mark (r=0 refers that a patient has not reneged and r=1 refers that a patient has reneged at previous transition mark).

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$P_{n,r}(t)$ is the probability that at time t queue length is $1 \le n \le N$, the server is not idle, and r is an indicator whether a patient has reneged or not at the previous transition mark (r=0 refers that a patient has not reneged and r=1 refers that a patient has reneged at previous transition mark).

$p_{i,j}$ = the probability of transitioning from present state to next possible state due to the reneging between the two consecutive transition marks, where both i and j can either be 0 or 1. 0 refers to no reneging and 1 refers to the occurrence of reneging at the considered transition mark.

The differential equations of the model are:

$$\frac{d}{dt} Q_{0,0}(t) = -\lambda Q_{0,0}(t) + \mu P_{0,0}(t) \quad (1)$$

$$\frac{d}{dt} P_{0,0}(t) = -(\lambda + \mu) P_{0,0}(t) + \mu P_{1,0} + \lambda Q_{0,0}(t) \quad (2)$$

$$\frac{d}{dt}P_{1,0}(t) = -(\lambda\beta + \mu + n\xi)P_{1,0}(t) + \mu P_{2,0}(t) + \lambda P_{0,0}(t)$$
$$+\xi[p_{00}P_{1,0}(t) + p_{10}P_{1,1}(t)] \quad (3)$$
$$\frac{d}{dt}P_{n,0}(t) = -(\lambda\beta + \mu + n\xi)P_{n,0}(t) + \mu P_{n+1,0}(t) + \lambda\beta P_{n-1,0}(t)$$
$$+n\xi[p_{00}P_{n,0}(t) + p_{10}P_{n,1}(t)], 1 < n < N \quad (4)$$
$$\frac{d}{dt}P_{N,0}(t) = -(\mu + N\xi)P_{N,0}(t) + \lambda\beta P_{N-1,0}(t)$$
$$+N\xi[p_{00}P_{N,0}(t) + p_{10}P_{N,1}(t)] \quad (5)$$
$$\frac{d}{dt}Q_{0,1}(t) = -\lambda Q_{0,1}(t) + \mu P_{0,1}(t) \quad (6)$$
$$\frac{d}{dt}P_{0,1}(t) = -(\lambda + \mu)P_{0,1}(t) + \mu P_{1,1} + \lambda Q_{0,1}(t) + \xi[p_{11}P_{1,1}(t)$$
$$+p_{01}P_{1,0}(t)] \quad (7)$$
$$\frac{d}{dt}P_{1,1}(t) = -(\lambda\beta + \mu + n\xi)P_{1,1}(t) + \mu P_{2,1}(t) + \lambda P_{0,1}(t)$$
$$+2\xi[p_{01}P_{2,0}(t) + p_{11}P_{2,1}(t)] \quad (8)$$
$$\frac{d}{dt}P_{n,1}(t) = -(\lambda\beta + \mu + n\xi)P_{n,1}(t) + \mu P_{n+1,1}(t) + \lambda\beta P_{n-1,1}(t)$$
$$+(n+1)\xi[p_{01}P_{n+1,0}(t) + p_{11}P_{n+1,1}(t)], 1 < n < N \quad (9)$$
$$\frac{d}{dt}P_{N,1}(t) = -(\mu + N\xi)P_{N,1}(t) + \lambda\beta P_{N-1,1}(t) \quad (10)$$

4 Steady-state analysis of the model

In this section, we derive the steady state solution of the model using matrix-decomposition method.

4.1 Steady-state equations

Let us define the steady-state probabilities as follows: $Q_{0,i} = \lim_{t\to\infty} Q_{0,i}(t)$, i= 0,1 and $P_{n,i} = \lim_{t\to\infty} P_{n,i}(t)$, n=0,1,2,... and i= 0,1

From the equations (1)-(10) we have the steady-state equations as follows:

$$0 = -\lambda Q_{0,0} + \mu P_{0,0} \quad (11)$$
$$0 = -(\lambda + \mu)P_{0,0} + \mu P_{1,0} + \lambda Q_{0,0} \quad (12)$$
$$0 = -(\lambda\beta + \mu + n\xi)P_{1,0} + \mu P_{2,0} + \lambda P_{0,0} + \xi[p_{00}P_{1,0} + p_{10}P_{1,1}] \quad (13)$$
$$0 = -(\lambda\beta + \mu + n\xi)P_{n,0} + \mu P_{n+1,0} + \lambda\beta P_{n-1,0}$$
$$+n\xi[p_{00}P_{n,0} + p_{10}P_{n,1}], \quad 1 < n < N \quad (14)$$
$$0 = -(\mu + N\xi)P_{N,0} + \lambda\beta P_{N-1,0} + N\xi[p_{00}P_{N,0} + p_{10}P_{N,1}] \quad (15)$$
$$0 = -\lambda Q_{0,1} + \mu P_{0,1} \quad (16)$$
$$0 = -(\lambda + \mu)P_{0,1} + \mu P_{1,1} + \lambda Q_{0,1} + \xi[p_{11}P_{1,1} + p_{01}P_{1,0}] \quad (17)$$
$$0 = -(\lambda\beta + \mu + n\xi)P_{1,1} + \mu P_{2,1} + \lambda P_{0,1} + 2\xi[p_{01}P_{2,0} + p_{11}P_{2,1}] \quad (18)$$
$$0 = -(\lambda\beta + \mu + n\xi)P_{n,1} + \mu P_{n+1,1} + \lambda\beta P_{n-1,1}$$
$$+(n+1)\xi[p_{01}P_{n+1,0} + p_{11}P_{n+1,1}], \quad 1 < n < N \quad (19)$$
$$0 = -(\mu + N\xi)P_{N,1} + \lambda\beta P_{N-1,1} \quad (20)$$

4.2 Steady-state solution

We use a matrix-decomposition method to get the steady-state probabilities in a recursive manner. Let $P = (P_{0,0}, P_0, P_{0,1}, P_1)$ be the vectors of the steady-state probabilities, where $P_0 = (P_{1,0}, P_{2,0}, \ldots\ldots P_{N,0})$ and $P_1 = (P_{1,1}, P_{2,1}, \ldots\ldots P_{N,1})$. Also, from equations (11) and (16) we can express $Q_{0,0}$ and $Q_{0,0}$ as

$$Q_{0,0} = \frac{\mu}{\lambda}P_{0,0} \quad (21)$$
$$Q_{0,1} = \frac{\mu}{\lambda}P_{0,1} \quad (22)$$

respectively. Substituting the values of $Q_{0,0}$ and $Q_{0,1}$ in equations (11) and (16), we can re-write the set of steady-state differential equations as:

$$0 = -\lambda P_{0,0} + \mu P_{1,0} \tag{23}$$
$$0 = -(\lambda\beta + \mu + n\xi)P_{1,0} + \mu P_{2,0} + \lambda P_{0,0} + \xi[p_{00}P_{1,0} + p_{10}P_{1,1}] \tag{24}$$
$$0 = -(\lambda\beta + \mu + n\xi)P_{n,0} + \mu P_{n+1,0} + \lambda\beta P_{n-1,0}$$
$$+ n\xi[p_{00}P_{n,0} + p_{10}P_{n,1}], \quad 1 < n < N \tag{25}$$
$$0 = -(\mu + N\xi)P_{N,0} + \lambda\beta P_{N-1,0} + N\xi[p_{00}P_{N,0} + p_{10}P_{N,1}] \tag{26}$$
$$0 = -\lambda P_{0,1} + \mu P_{1,1} + \xi[p_{11}P_{1,1} + p_{01}P_{1,0}] \tag{27}$$
$$0 = -(\lambda\beta + \mu + n\xi)P_{1,1} + \mu P_{2,1} + \lambda P_{0,1} + 2\xi[p_{01}P_{2,0} + p_{11}P_{2,1}] \tag{28}$$
$$0 = -(\lambda\beta + \mu + n\xi)P_{n,1} + \mu P_{n+1,1} + \lambda\beta P_{n-1,1}$$
$$+(n+1)\xi[p_{01}P_{n+1,0} + p_{11}P_{n+1,1}], \quad 1 < n < N \tag{29}$$
$$0 = -(\mu + N\xi)P_{N,1} + \lambda\beta P_{N-1,1} \tag{30}$$

Thus, the steady-state equations from (23)-(30) can be expressed in matrix form as

$$PQ = 0 \tag{31}$$

Where 0 is the column vector of zeros, and

$$Q = \begin{pmatrix} -\lambda & A_{12} & 0 & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ 0 & A_{32} & -\lambda & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{pmatrix} \tag{32}$$

is a $(2N+2) \times (2N+2)$ square matrix.

Each entry of the matrix **Q** is given below:

$$\mathbf{A_{12}} = \begin{pmatrix} \lambda & 0 & . & . & .0 \end{pmatrix}_{1\times N}, \mathbf{A_{14}} = \begin{pmatrix} 0 & 0 & . & . & .0 \end{pmatrix}_{1\times N}, A_{21} = \begin{pmatrix} \mu \\ 0 \\ . \\ . \\ . \\ 0 \end{pmatrix}_{N\times 1},$$

$$A_{22} = \begin{pmatrix} -(\lambda\beta + \mu + \xi - \xi p_{00}) & \lambda\beta & . & . & . & . & 0 \\ \mu & -(\lambda\beta + \mu + 2\xi - 2\xi p_{00}) & . & . & . & . & 0 \\ 0 & \mu & . & . & . & . & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & . & . & . & . & \lambda\beta \\ 0 & 0 & . & . & . & . & -(\mu + N\xi - N\xi p_{00}) \end{pmatrix}_{N\times N},$$

$$A_{23} = \begin{pmatrix} \xi p_{01} \\ 0 \\ . \\ . \\ . \\ 0 \end{pmatrix}_{N\times 1}, A_{24} = \begin{pmatrix} 0 & 0 & 0 & . & . & . & 0 & 0 \\ 2\xi p_{01} & 0 & 0 & . & . & . & 0 & 0 \\ 0 & 3\xi p_{01} & 0 & . & . & . & 0 & 0 \\ 0 & 0 & 0 & . & . & . & . & 0 \\ 0 & 0 & 0 & . & . & . & . & 0 \\ 0 & 0 & 0 & . & . & . & . & 0 \\ 0 & 0 & 0 & . & . & . & 0 & 0 \\ 0 & 0 & 0 & . & . & . & N\xi p_{01} & 0 \end{pmatrix}_{N\times N},$$

$$\mathbf{A_{32}} = \begin{pmatrix} 0 & 0 & . & . & .0 \end{pmatrix}_{1\times N}, \mathbf{A_{34}} = \begin{pmatrix} \lambda & 0 & . & . & .0 \end{pmatrix}_{1\times N}, A_{43} = \begin{pmatrix} \mu + \xi p_{11} \\ 0 \\ . \\ . \\ . \\ 0 \end{pmatrix}_{N\times 1},$$

$$A_{41} = \begin{pmatrix} 0 \\ 0 \\ . \\ . \\ . \\ 0 \end{pmatrix}_{N\times 1}, A_{42} = \begin{pmatrix} \xi p_{10} & 0 & 0 & . & . & . & 0 & 0 \\ & 0 & 2\xi p_{10} & . & . & . & 0 & 0 \\ 0 & 0 & 3\xi p_{01} & . & . & . & 0 & 0 \\ 0 & 0 & 0 & . & . & . & 0 & 0 \\ 0 & 0 & 0 & . & . & . & 0 & 0 \\ 0 & 0 & 0 & . & . & . & 0 & 0 \\ 0 & 0 & 0 & . & . & . & (N-1)\xi p_{10} & 0 \\ 0 & 0 & 0 & . & . & . & 0 & N\xi p_{10} \end{pmatrix}_{N\times N},$$

$$A_{44} = \begin{pmatrix} -(\lambda\beta+\mu+\xi) & \lambda\beta & 0 & . & . & . & 0 & 0 \\ (\mu+2\xi p_{11}) & -(\lambda\beta+\mu+2\xi) & \lambda\beta & . & . & . & 0 & 0 \\ 0 & (\mu+3\xi p_{11}) & \lambda\beta & . & . & . & 0 & 0 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & -(\lambda\beta+\mu+(N-1)\xi) & \lambda\beta \\ 0 & 0 & 0 & . & . & . & (\mu+N\xi p_{11}) & -(\mu+N\xi) \end{pmatrix}_{N\times N}$$

where A_{14} and A_{32} are row vectors having order N and all the elements are zero, A_{41} is a column vector of order N having all elements equal to zero. From equation (31) it follows that

$$-\lambda P_{0,0} + P_0 A_{21} = 0 \quad (33)$$
$$P_{0,0}A_{12} + P_0 A_{22} + P_1 A_{42} = 0 \quad (34)$$
$$P_0 A_{23} - \lambda P_{0,1} + P_1 A_{43} = 0 \quad (35)$$
$$P_0 A_{24} + P_{0,1}A_{34} + P_1 A_{44} = 0 \quad (36)$$

From equation (36), we get

$$P_1 = -(P_{0,1}\mathbf{A_{34}} + \mathbf{P}_0\mathbf{A_{24}})\mathbf{A_{44}^{-1}} \quad (37)$$

Substitute (37) in (35), and solve we get

$$P_0 = \frac{(\lambda+\mathbf{A_{34}A_{44}^{-1}A_{43}})P_{0,1}}{\mathbf{A_{23}}-\mathbf{A_{24}A_{44}^{-1}A_{43}}} = \mathbf{\Psi_1}P_{0,1} \quad (38)$$

where,

$$\mathbf{\Psi_1} = \frac{(\lambda+\mathbf{A_{34}A_{44}^{-1}A_{43}})}{\mathbf{A_{23}}-\mathbf{A_{24}A_{44}^{-1}A_{43}}}$$

Now, substitute the value of P_0 from (38) to (33), and solve we get

$$P_{0,0} = \frac{\mathbf{\Psi_1 A_{21}}}{\lambda}P_{0,1}. \quad (39)$$

Substituting the value of $\mathbf{P}_0$ from 38) in (37), on solving we get,

$$\mathbf{P}_1 = -(\mathbf{A_{34}} + \mathbf{\Psi_1 A_{24}})\mathbf{A_{44}^{-1}}P_{0,1} \quad (40)$$

Substituting the value of $P_{0,0}$ from equation 39 to (21), we have

$$Q_{0,0} = \frac{\mu}{\lambda}P_{0,0} = \frac{\mu}{\lambda^2}\mathbf{\Psi_1 A_{21}}P_{0,1} \quad (41)$$

We can obtain the unknown constant $P_{0,1}$ by using normalization equation:

$$Q_{0,0} + Q_{0,1} + \sum_{n=0}^{N}\sum_{1=0}^{1} P_{n,i} = Q_{0,0} + Q_{0,1} + P_{0,0} + P_0 e + P_{0,1} + P_1 e = 1 \quad (42)$$

where $\mathbf{e}$ is the unit column vector of dimension N.

Substituting the values from equations (22), (38)-(41) to (42) we get the explicit expression for $P_{0,1}$ as:

$$P_{0,1} = \frac{1}{\frac{\mu}{\lambda^2}\mathbf{\Psi_1 A_{21}}+\frac{\mu}{\lambda}+\frac{\mathbf{\Psi_1}}{\lambda}\mathbf{A_{21}}+\mathbf{\Psi_1 e}+1-(\mathbf{A_{34}}+\mathbf{\Psi_1 A_{24}})\mathbf{A_{44}^{-1}e}} \quad (43)$$

Thus, the steady-state probabilities $Q_{0,0}, Q_{0,1}, P_0, P_1$ and $P_{0,1}$ can be computed using equations (22), (38)-(41) respectively.

5 Transient performance analysis of the model

In this section, we perform the transient analysis of the model. To obtain the transient solution we use the Runge-Kutta method of fourth order. The $''ode45''$ function of MATLAB software is used to compute the transient numerical results.

5.1 Performance Measures

We study the following performance measures in the transient state:

1. Expected number of patients in queue ($L_q(t)$):
$$L_q(t) = \sum_{n=1}^{N} (n)[P_{n,0}(t) + P_{n,1}(t)] \tag{44}$$
2. Expected waiting time of patients in queue ($W_q(t)$):
$$W_q(t) = \frac{L_q(t)}{\mu[1-Q_{0,0}(t)-Q_{0,1}(t)-P_{0,0}(t)-P_{0,1}(t)]} \tag{45}$$

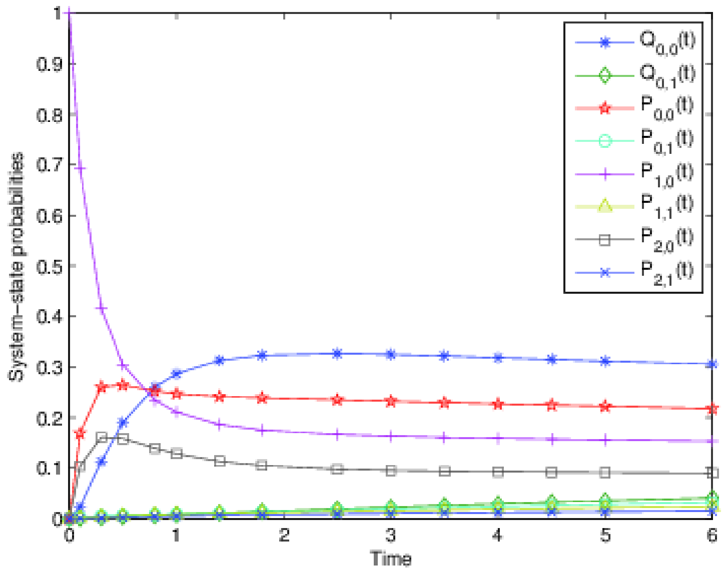

Figure 2: System-State probabilities vs Time

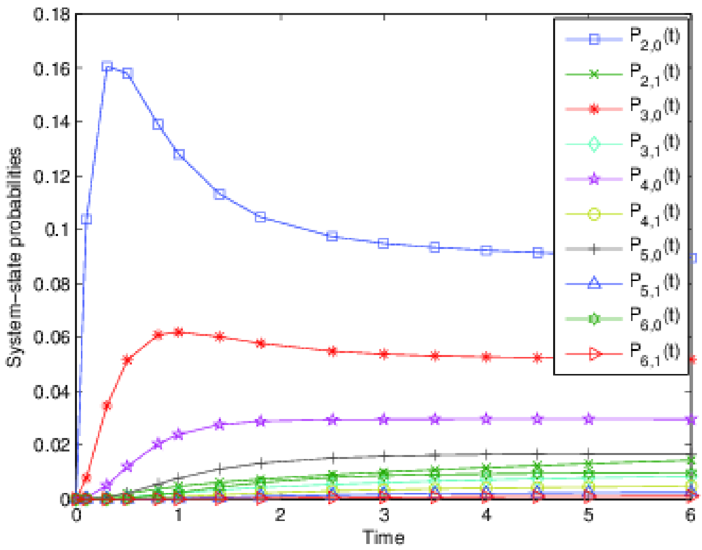

Figure 3: System-state probabilities vs Time (detail)

In figures 2 and 3, the variation in system-state probabilities of patients with respect to time is shown. It is seen that except the probability $P_{1,0}(t)$ all other probabilities initially starts from zero and asymptotically reach the steady-state. $P_{1,0}(t)$ initially starts from 1 due to the initial condition we considered i.e. $P_{1,0}(0) = 1$. The values of parameters are: $\lambda = 1.8, \mu = 2.5, \beta = 0.85, \xi = 0.2, N = 6, p_{00} = 0.8, p_{01} = 0.2, p_{10} = 0.7$ and $p_{11} = 0.3$.

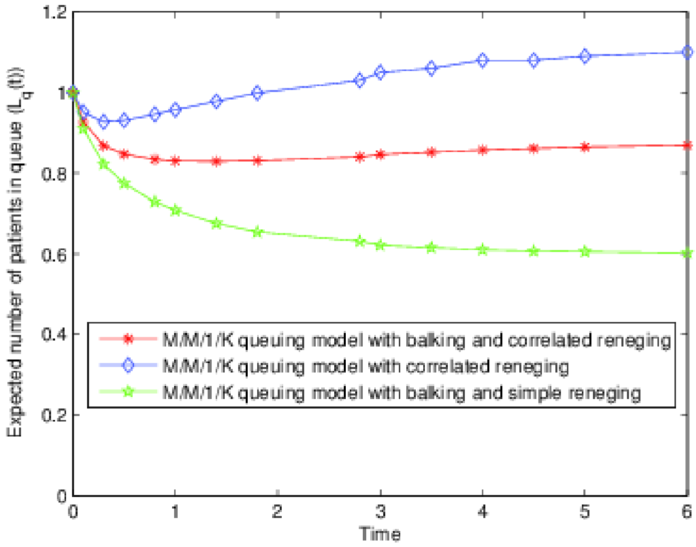

Figure 4: Expected number of patients in queue vs time

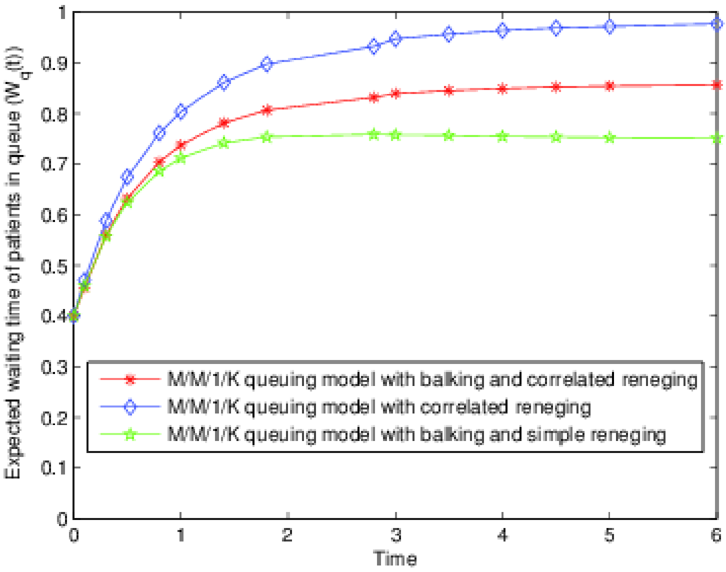

Figure 5: Expected waiting time of patients in queue vs time

Figures 4 and 5 show a comparative analysis of variation in expected number of patients in queue and expected waiting time of patients in queue respectively for the three queueing models. In figure 4 it is observed that the expected number of patients in queue for the M/M/1/K queuing model with balking and correlated reneging is higher than that of M/M/1/K queuing model with simple reneging and balking. This shows that in the cases of correlated reneging the expected number of patients in queue is actually higher than that of we considered so far.

Also, the expected number of patients in queue for M/M/1/K queuing model with correlated reneging is higher than that of M/M/1/K queuing model with balking and correlated reneging which shows the effect of balking. In figure 5 similar trend is observed for variation in expected waiting time of patients in queue. The values of the parameters are: $\lambda = 1.8, \mu = 2.5, \xi = 0.2, \beta = 0.85, N = 6, p_{0,0} = 0.8, p_{0,1} = 0.2, p_{1,0} = 0.7$, and $p_{1,1} = 0.3$ with initial condition $P_{1,0}(0) = 1$.

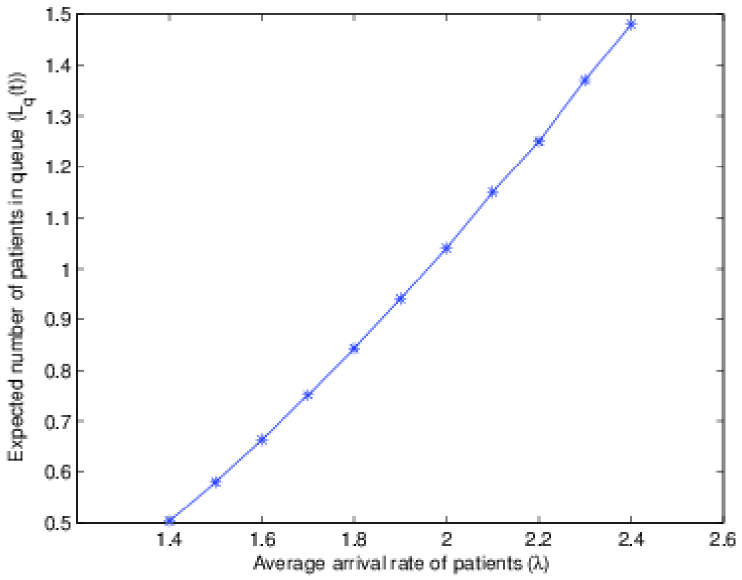

Figure 6: Variation in expected number of patients in queue with respect to average arrival rate of patients (λ)

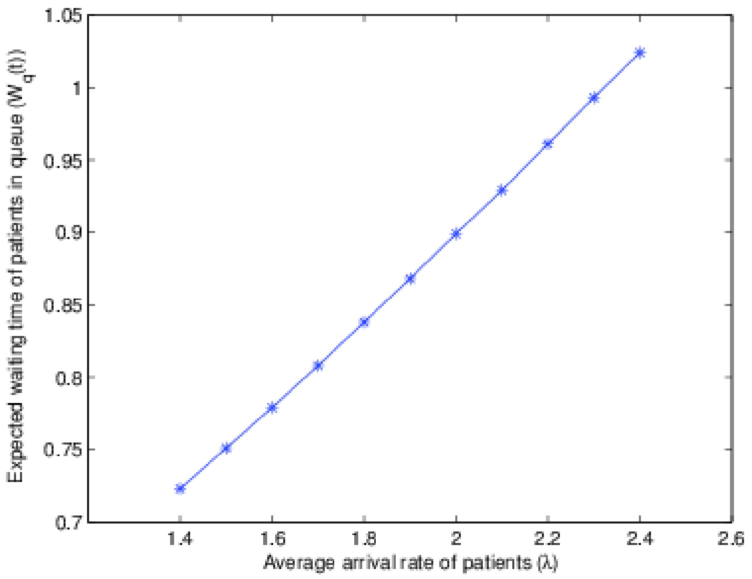

Figure 7: Variation in expected waiting time of patients in queue with respect to average arrival rate of patients (λ)

In figures 6 and 7 the effect of average arrival rate of patients on expected number of patients in queue and expected waiting time of patients in queue is observed respectively. One can easily see that with the increase in average arrival both the expected number of patients in queue and expected waiting time of patients in queue increase. The values of the parameters are: $\mu = 2.5, \xi = 0.2, \beta = 0.85, N = 6, t = 3, p_{0,0} = 0.8, p_{0,1} = 0.2, p_{1,0} = 0.7,$ and $p_{1,1} = 0.3$ with initial condition $P_{1,0}(0) = 1$.

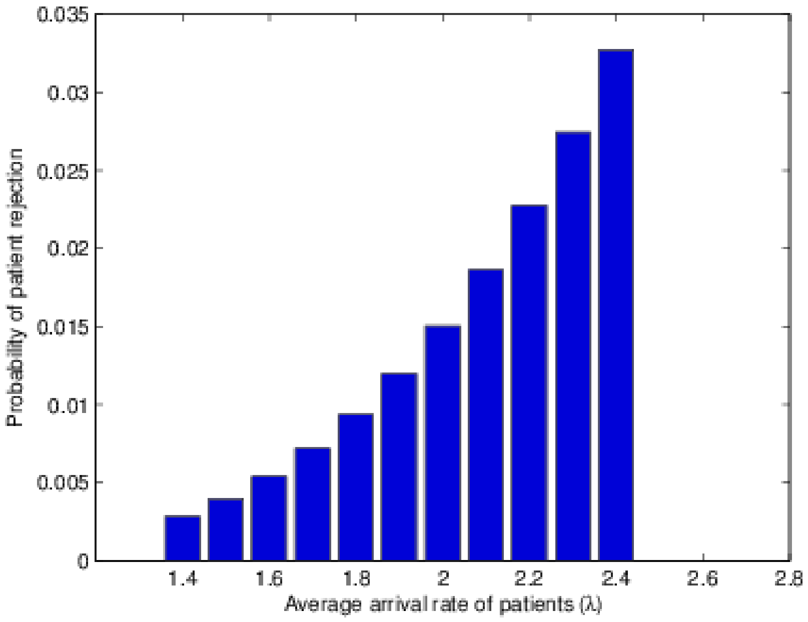

Figure 8: Probability of patient rejection vs Average arrival rate of patients (λ)

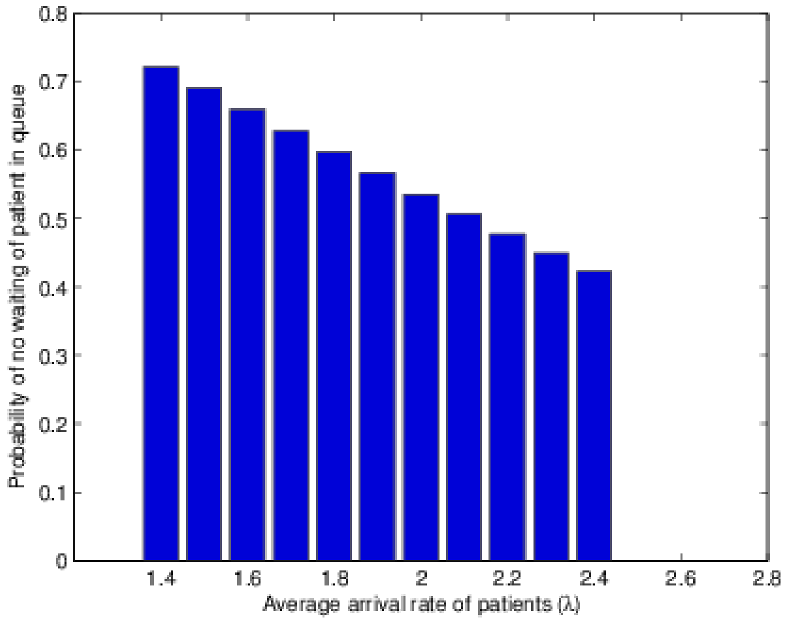

Figure 9: Probability of no waiting of pateint in queue vs Average arrival rate of patients (λ)

In figures 8 and 9 the effect of average arrival rate of patients on probability of patient rejection and probability of no waiting of patient in queue is observed respectively. One can easily see that with the increase in average arrival rate of patients, the probability of patient rejection increases whereas the probability of no waiting of patient in queue decreases. The values of the parameters are: $\mu = 2.5, \xi = 0.2, \beta = 0.85, N = 6, t = 3, p_{0,0} = 0.8, p_{0,1} = 0.2, p_{1,0} = 0.7$, and $p_{1,1} = 0.3$ with initial condition $P_{1,0}(0) = 1$.

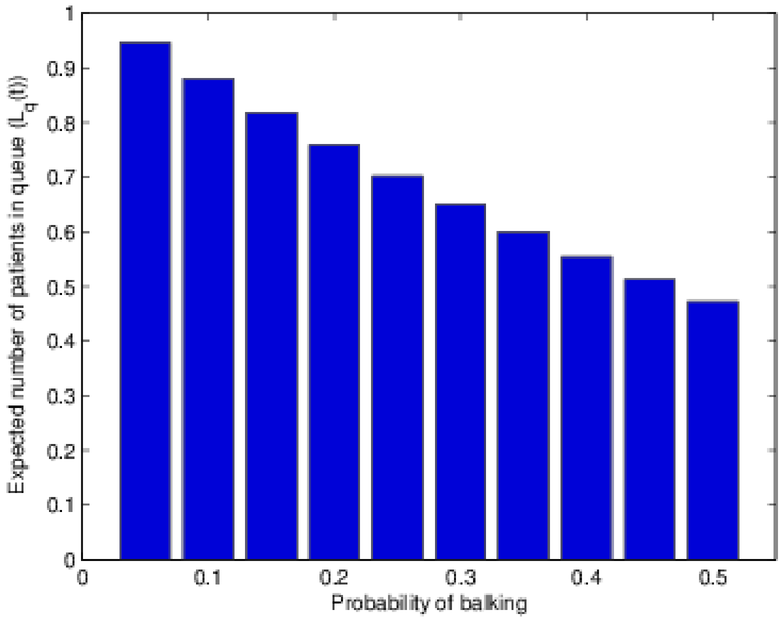

Figure 10: Variation in $L_q(t)$ w.r.t. Probability of balking

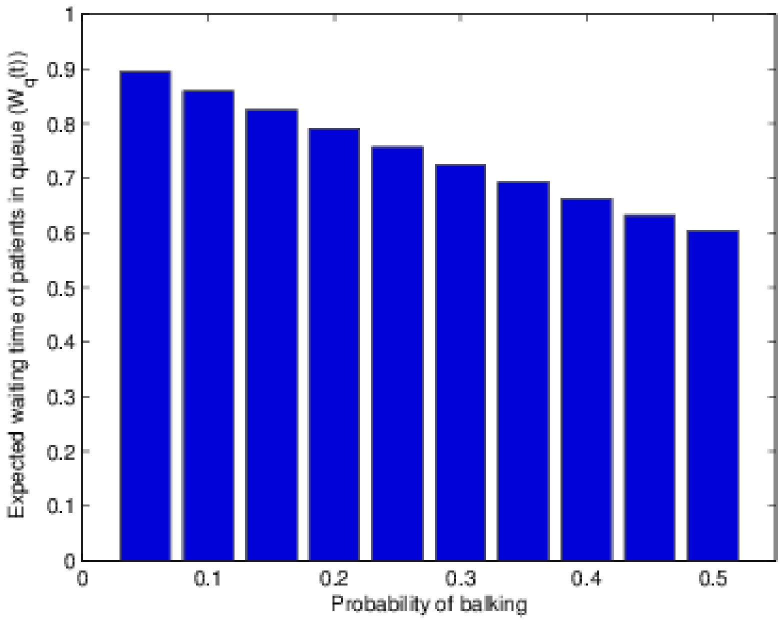

Figure 11: Variation in $W_q(t)$ w.r.t.Probability of balking

In figures 10 and 11 the effect of probability of balking on expected number of patients in queue and expected waiting time of patients in queue is observed respectively. It is seen that with the increase in probability of balking both the performance parameters decreases which is quite obvious. The values of the parameters are: $\lambda = 1.8, \mu = 2.5, \xi = 0.3, N = 6, p_{0,0} = 0.8, p_{0,1} = 0.2, p_{1,0} = 0.7$, and $p_{1,1} = 0.3$ with initial condition $P_{1,0}(0) = 1$ at t=3.

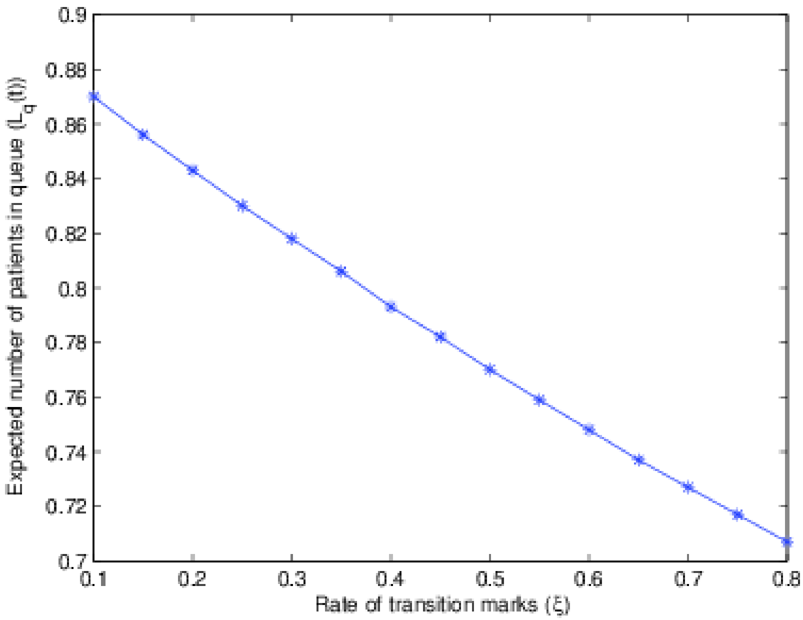

Figure 12: Variation in expected number of patients in queue
with respect to rate of transition marks

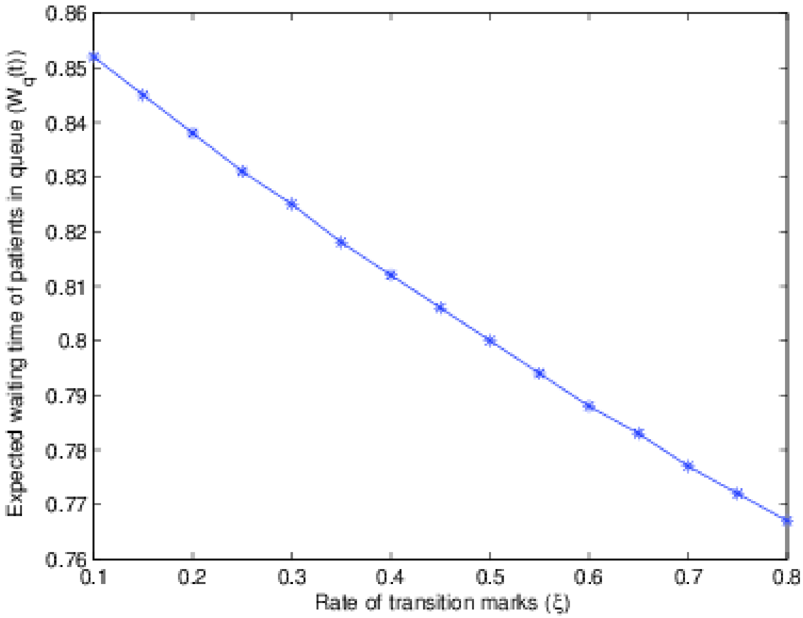

Figure 13: Variation in expected waiting time of patients in queue
with respect to rate of transition marks

In figures 12 and 13 the variation in expected number of patients in queue and expected waiting time of patients in queue with respect to change in rate of transition marks is observed respectively. It is seen that with the increase in rate of transition marks both the performance paramters decreases which is quite obvious. The values of the parameters are: $\lambda = 1.8, \mu = 2.5, \beta = 0.85, N = 6, p_{0,0} = 0.8, p_{0,1} = 0.2, p_{1,0} = 0.7$, and $p_{1,1} = 0.3$ with initial condition $P_{1,0}(0) = 1$ at t=3.

6 Conclusion

A queuing model with balking and correlated reneging is studied. The application of the model in health care queue management system is extensively explained and transient-state analysis is performed. The steady-state solution is obtained using matrix-decomposition method.

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