An Imperfect Production System with Rework and Disruption for Delaying items Considering Shortage

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Abstract

In this article, we have developed an imperfect production system under the following four different cases. The first one is imperfect production with rework, and without disruption, second one is imperfect production with rework and disruption, third one is imperfect production with rework, disruption,and without backlogging, fourth one is imperfect production with rework, disruption and backlogging. For these all cases, we have optimized the regular production time, rework production time, backlog time and backlog quantity. In this article we have considered the demand rate, production rate defective item's production rate and deteriorating rate are all constant. We have also considered the production rate is always grater than the demand rate. The model is illustrated by numerically and analyzed by graphically.

Keywords: Imperfect production, Disruption, Deterioration, Backlogging, rework.
AMS Subject Classification: 90B05, 90B30, 90B50

1 Introduction

Serving advanced quality products and to make availability service can attract to customers and keep them coming back. However in real situation production process are not always perfect. Due to economic and environmental issues imperfect quality items are reworked to become serviceable again. Moreover shortages of an item are one of the crucial occurrence for any business organization which affects their profit seriously. Shortages may be divided into two categories one is natural shortages and other one is artificial shortages. The natural shortages of items may occur due to the following reasons: (1) the lack of coordination between various team members of production management, (2) The production of items is less than the demand, (3) Disruption like labor strike, political issues, and machines breakdown situation, etc. The artificial shortages of items are made by business organizations through advertising before the launching of new items or overstocking of popular items.

In this article we have developed a mathematical model for four various cases, first one is imperfect production with rework, and without disruption, second one is imperfect production with rework and disruption, third one is imperfect production with rework, disruption,and without backlogging, fourth one is imperfect production with rework, disruption and backlogging considering production and demand rate of product are constant.

Ghare *et al.* (1963) presented the classical inventory model for deteriorating items in which they consider the deterioration and demand rate are constant without shortages. Chakarbarti *et al.* (1997) introduced an EOQ model with linearly increasing demand allowing shortages in all cycle, and they found both the recorder number and time interval between two successive reorders. Furthermore all the shortage interval are determined in an optimal manner so as minimized the average total system cost. Mandel *et al.* (2017) investigated an inventory model for seasonal products using preservation technology investment to reduce the deterioration rate by considering stock dependent demand rate and allowing with shortage. Dye (2007) analyzed a pricing and ordering policy for deteriorating items allowing with time dependent backlogging rate in demand. He *et al.* (2010) suggested a production inventory model for a deteriorating items considering with disruption in production. In this study they considered a fact that some products may deteriorate during their shortage period. Khedlekar *et al.* (2014) presented a disrupted production inventory model by considering constant production rate and variable demand rate. A flexible optimal decisions have been developed by authors but due to disruption shortage is not considered by them. Furthermore in 2018 Khedlekar *et al.* (2018) developed a production inventory model by considering disruption during the production period allowing with shortage. For this, they considered, time dependent demand rate, incorporating shortages at the end of each production cycle but they have not considered the imperfect production.

In general, it is found that items always deteriorate continuously concerning time. Wee (1997), formulated a replenishment policy for time varying deteriorating rate of items by considering price dependent linear function of price allowing with shortages. After this in 2003, Papchristos and Skouri (2003), extended the model of wee (1997) by incorporating demand rate is a convex function of price and backlogging rate is a time dependent function. Dye (2013) studied the effect of preservation technology investment on a non instantaneous inventory system but he did not consider the shortages of items. Bhunia *et al.* (2009) developed a economical production model by using the application of TGA for deteriorating items. In this study they considered the price dependent demand function allowing with partial backlogging but they did not studied by considering imperfect production and production disruption. Dave *et al.* (1981) presented an EOQ model for deteriorating items, in which the demand rate is taken as linear function of time with constant deterioration rate of on hand inventory.

Goyal *et al.* (2001) and Baker *et al.* (2012) depicted a detailed review over the last 20 years. Rosenblatt *et al.* (1986) studied the effect of an imperfect production process on the optimal production time by assuming that the production system may be deteriorate during the production process and may be produce some defective items but they did not incorporate the rework of defective items. Shah *et al.* (2013) developed an inventory model for non instantaneous time-varying deteriorating items in which they considered demand rate is a function of retail price and advertisement cost of items. Due to increased competition, growth in population and globalisation. Recently there are two facts arise about production management, the first one is due to the pressure, the production of items run continuously, and the second one is the involvement of various stages in production system. Therefore the probability of defective item's production rate increases and consequently, the production management has to motivated for the rework process.

Hayak *et al.* (2001) studied the effect of imperfect quality items on finite production inventory model, when regular production stops and imperfect quality items are assumed to be rewoked at a stationary rate. Further they also incorporated the shortage and and backordered but they did not consider the production disruption in their inventory system. Chiu(2011) minimized the total production cost, delivery cost for their EMQ inventory model and they also incorporated rework process of defective items and multiple shipments. Samanta *et al.* (2004) developed a continuous production control inventory model for deteriorating items with shortages. They formulated the optimal average system cost, stock level, backlog level, when the deterioration rate is very small. He *et al.* (2010) developed a production inventory model for deteriorating items considering multiple market demand, where each market has constant demand rate with different selling cycles. They also provided a method to find the solution of optimal production plan and replenishment time. Khedlekar and Shukla *et al.* (2013) developed an inventory model for deteriorating items, considering the demand as a parametric dependent linear function of time and price both. Further they also examined the coefficient of time-parameter and coefficient of price-parameter, simultaneously and proved that time is dominating variable over price in terms of earning more profit. They also introduced the concept for logarithm demand.

Recently, Pervin *et al.* (2018) introduced an integrated EPQ inventory model with time and price dependent demand. They also considered a production rate is a linear function of time and based on customers demand. To reduce the deterioration rate, they applied the preservation technology and optimized the investment cost. In real life, due to the imperfect production, many production systems have encountered considerable losses in their production. Our model is associated with the EPQ model with disruption articulated by Parlor *et al.* (1991) In this paper, we develops a deteriorated imperfect production inventory model with and without disruption. We considere rework on defective items at the end of the regular production. We also provided some useful results to find the optimal regular production rate, production time and rework time. In this paper, we incorporated the shortage at the end of planning horizon because after the planning horizon there is a possibility of some disruption before starting the next production run. For development of model we assumed that at the beginning of each cycle, the manufacturer decide the optimal production rate, production time and rework time with and without disruption so that the production quantity may meets both demand and deterioration. Furthermore, we also assumed that all units of product sold out at the end of each cycle and inventory level decreases to zero.

The development of this paper is organized as follows. In section 2 the notation and assumption are given which are used throughout in this study. In section 3 we have given development of mathematical EPQ model for imperfect production system with rework in four various cases. Case I develops a simple EPQ model which optimizes the regular and rework production time. Case 2 develops two situations which are, first one is when the system gets disrupted but not backlog and second one is a system is disrupted allowing with backlogging. Case 3 develops the deteriorated imperfect production and disrupted system with rework. Case 4 develops the deteriorated imperfect production and disrupted system with rework, allowing with shortage. We optimize the regular production time and rework production time and we also optimize the time of placing the order and order quantity for placing the order when a shortage occurs. In section 4 we have given a three numerical examples and managerial implications which illustrated the proposed model. In section 5 we have concluded the study.

2 Model Notations and Assumptions

2.1 Notations:

The following notations are used to develop the model.

1. The finite planning time interval is H,
2. The finite production rate per unit time is P,
3. The demand per unit time is d,
4. Te rate of deterioration per unit time is θ, which is constant,
5. The production time without disruption is T_P,
6. The production time with disruption is T_d,
7. New production time after system get disrupted T_d^P,
8. The production rate per unit time of defective items is ν,
9. Rework rate per unit time of defective items is P_r,
10. At a t on hand inventory level of defective items is $I_d(t)$,
11. The time of placing the order for replenishment, when shortages occur is T_r,
12. The Economical order quantity is Q_r for placing the order when shortages occurs.

2.2 Assumptions

The following assumption are used to develop the proposed model.

- Shortages are allowed and backlogged,
- Recovered imperfect quality items are considered as good quality items.
- It is assumed that defective items are fully reworkable into good items and there is no scrap items,
- The production rate, demand rate, defective item's production rate, rework rate is same in every cycle,
- The demand rate per unit time is deterministic and finite,
- The imperfect items' production rate is always grater than the sum of demand rate and defective items' production rate.

3 The Mathematical Model

In this section, the mathematical model is developed for four various cases, which are sequentially as follows:

3.1 Imperfect Production and Rework without Disruption

The as per assumption imperfect production rate is P, which is always grater then the sum of demand rate d and the defective items' production rate ν. Let $I_1(t)$ be a regular production inventory level till the time T_P. After that the regular production is stopped and rework process starts at a rate P_r per unit time. Let $I_2(t)$ be the inventory level of perfect items till the time t_1. After the time t_1, rework process is also stopped. During the time interval $[t_1, H]$ inventory level decreases due to demand and deterioration and at $t = H$ inventory level become zero.

The inventory level of the items' at time t over the interval [0, H], is governed by the following differential equations.

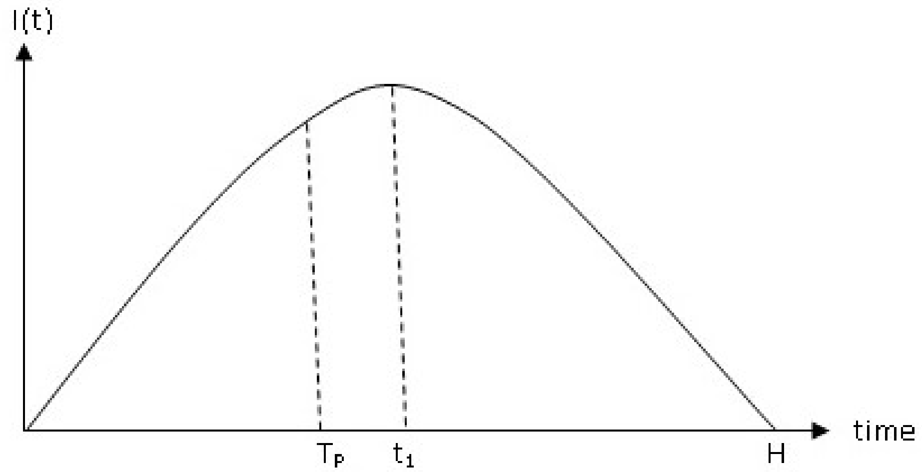

Figure 1: Production system without disruption

$$\frac{dI_1(t)}{dt} + \theta I_1(t) = P - \nu - d, 0 \le t \le T_P, I_1(0) = 0, \tag{1}$$
$$\frac{dI_2(t)}{dt} + \theta I_2(t) = P_r - d, T_P \le t \le t_1, I_1(T_P) = I_2(T_P), \tag{2}$$
$$\frac{dI_3(t)}{dt} + \theta I_3(t) = -d, t_1 \le t \le H, I_3(H) = 0 \tag{3}$$

Using the boundary conditions, the solution of above differential equations are

$$I_1(t) = \left(\frac{P-D-\nu}{\theta}\right)(1-e^{-\theta T}), 0 \le t \le T_P \tag{4}$$

$$I_2(t) = \frac{P_r - d}{\theta}(1-e^{\theta(T_P-t)}) + \frac{P-d-\nu}{\theta}(e^{\theta(T_P-t)}-1), T_P \le t \le t_1 \tag{5}$$

$$I_3(t) = \frac{d}{\theta}(e^{(\theta(H-t))}-1), t_1 \le t \le H. \tag{6}$$

Proposition 4.1 *If ν is the defective item rate per unit time, P_r is the rework rate per unit time and T_P is the regular production rate per time, then the rework time ie t_1, where*

$$t_1 = \left(\frac{\nu+P_r}{P_r}\right)T_P \tag{7}$$

Proof. Let ν be the defective items' production rate in time interval $[0, T_P]$, so the total defective items are νT_P and let P_r be the rework rate, so that the total reworkable items are $P_r(t_1 - T_P)$. In this model we assume that all defective items are reworkable, i e.

$$\nu T_P = P_r(t_1 - T_P) \tag{8}$$

therefore, the rework time t_1 will be

$$t_1 = \left(\frac{\nu+P_r}{P_r}\right)T_P. \tag{9}$$

Theorem 4.2 *If T_P is the regular production time without disruption, then*

$$T_P = \frac{\frac{d}{\theta}(e^{\theta H}-1)}{P - d\frac{\nu+P_r}{P_r} + de^{\theta H\frac{\nu+P_r}{P_r}}} \tag{10}$$

Proof. Let T_P be the regular production time and $(t_1 - T_P)$ be the rework time. We know that the graph inventory is continuous at time t_1 shown in Fig.1

$$I_3(t_1) = I_1(T_P) + \nu T_P - d(t_1 - T_P),$$

by using this conditions given in the equations 4 and 6, we have

$$\frac{d}{\theta}(e^{\theta(H-t_1)}-1) = \left(\frac{P-D-\nu}{\theta}\right)(1-e^{-\theta T_P}) + \nu T_P - d(t_1 - T_P), \tag{11}$$

with the help of proposition 3.1 the above equation reduces into the following

$$\frac{d}{\theta}\left((e^{\theta H}-1) - \frac{\nu+P_r}{P_r}\theta e^{\theta H}T_P\right) = \left(\frac{P-d-\nu}{\theta}\right)\theta T_P + \left(\nu - d\left(\frac{\nu+P_r}{P_r}\right) + d\right)T_P. \tag{12}$$

so, the regular production time without disruption is

$$T_P = \frac{\frac{d}{\theta}(e^{\theta H}-1)}{P - d\frac{\nu+P_r}{P_r} + de^{\theta H\frac{\nu+P_r}{P_r}}} \tag{13}$$

From proposition 3.1 the rework time is t_1

$$t_1 = \left(\frac{\nu+P_r}{P_r}\right)\frac{\frac{d}{\theta}(e^{\theta H}-1)}{P - d\frac{\nu+P_r}{P_r} + de^{\theta H\frac{\nu+P_r}{P_r}}}T_P \tag{14}$$

Differentiating the equations 13 and 14 with respect to disruption rate θ, respectively, we have

$$\frac{dT_P}{d\theta} = \frac{(d\log\theta(e^{\theta H}-1) + de^{\theta H})}{\left(P - d\left(\frac{\nu+P_r}{P_r}\right) + de^{\theta H}\left(\frac{\nu+P_r}{P_r}\right)\right)} - \frac{\left(\frac{d}{\theta}\right)^2 e^{\theta H}(e^{\theta H}-1)\left(\frac{\nu+P_r}{P_r}\right)}{\left(P - d\left(\frac{\nu+P_r}{P_r}\right) + de^{\theta H}\left(\frac{\nu+P_r}{P_r}\right)\right)^2} \tag{15}$$

$$\frac{dt_1}{d\theta} = \left(\frac{\nu+P_r}{P_r}\right)\left[\frac{(d\log\theta(e^{\theta H}-1) + de^{\theta H})}{\left(P - d\left(\frac{\nu+P_r}{P_r}\right) + de^{\theta H}\left(\frac{\nu+P_r}{P_r}\right)\right)} - \frac{\left(\frac{d}{\theta}\right)^2 e^{\theta H}(e^{\theta H}-1)\left(\frac{\nu+P_r}{P_r}\right)}{\left(P - d\left(\frac{\nu+P_r}{P_r}\right) + de^{\theta H}\left(\frac{\nu+P_r}{P_r}\right)\right)^2}\right] \tag{16}$$

The above equations 15 and 16 manifests that the manufacturer has to produce more products when deterioration rate increases. Hence, if the deterioration rate decreases then the planning time interval and production cost decrease accordingly. Thus, we have the following corollary.

Corollary 4.3 *Assuming $\theta << 1$, then T_P and t_1 are havels in θ.*

3.2 Imperfect Production and Rework with Disruption

In this case, we assumed that $I_1(t)$ be the regular production inventory level till the time T_d. After this time the system get disrupted. Let $I_2(t)$ be a disrupted production inventory level with new production rate $P + \delta P$ where δP < 0. At time t= H-t_1 the production is stopped. The regular production and disrupted production has produced a defective items at a rate ν. Let $I_3(t)$ be a rework inventory level with rework rate P_r.

The inventory level of both (perfect and imperfect) items at time t over the interval [0, H], is governed by the following differential equations.

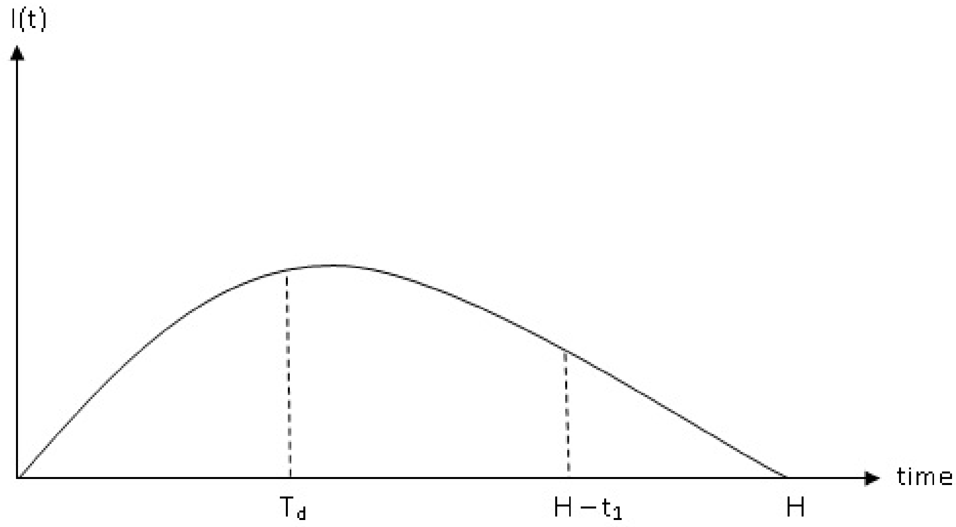

Figure 2: Inventory system with disruption

$$\frac{dI_1(t)}{dt} + \theta I_1(t) = P - \nu - d, 0 \le t \le T_d, I_1(0) = 0 \tag{17}$$
$$\frac{dI_2(t)}{dt} + \theta I_2(t) = P + \delta P - \nu - d, T_d < t < H - t_1 I_1(T_d) = I_2(T_d) \tag{18}$$
$$\frac{dI_3(t)}{dt} + \theta I_3(t) = P_r - d, H - t_1 < t < H, I_3(H) = 0 \tag{19}$$

Using the boundary conditions, the solutions of differential equations are

$$I_1(t) = \left(\frac{P-D-\nu}{\theta}\right)(1 - e^{-\theta t}) \tag{20}$$
$$I_2(t) = \frac{P+\delta P-\nu}{\theta}\left(1 - e^{\theta(T_d - t)}\right) + \frac{P-D-\nu}{\theta}\left(e^{\theta(t_d - t)} - e^{-\theta t}\right) \tag{21}$$
$$I_3(t) = isexpressedbythefollowingpropositions \tag{22}$$

Proposition 4.4 *If ν is the defective items' production rate, P_r is the rework rate and H is the total length of cycle, then the rework time is t_1, where*

$$t_1 = \frac{\nu H}{P_r + \nu}. \tag{23}$$

Proof. Let ν be the defective items' production rate in the time interval $[0, H - t_1]$, so that the total defective items are $\nu(H - t_1)$ and let P_r be the rework rate in the time interval $[H - t_1, H]$, so that the total reworkable items are $P_r t_1$. As over assumptions all defective items are reworkable, i.e.

$$P_r t_1 = \nu(H - t_1) \tag{24}$$

therefore, the rework time t_1 is

$$t_1 = \frac{\nu H}{P_r + \nu} \tag{25}$$

Since the inventory is continuous at $H - t_1$, which is shown in Fig 2., i.e. $I_3(H - t_1) = I_2(H - t_1) + \nu(H - t_1)$

by using this conditions given in the equations 18 and 19, we have

$$I_3(t) = \nu(H - t_1)e^{-\theta t} + \frac{P_r - d}{\theta}(1 - e^{-\theta t}) + \frac{P - d - \nu}{\theta}(e^{\theta(t_d - t)} - e^{-\theta t})$$
$$+ \frac{P + \delta P - d - \nu}{\theta}(e^{\theta(H - t - t_1)} - e^{\theta(t_d - t)}) \tag{26}$$

So, the inventory level $I_3(t)$ becomes

$$I_3(t) = \nu(H - \frac{\nu H}{P_r + \nu})e^{-\theta t} + \frac{P_r - d}{\theta}(1 - e^{-\theta t}) + \frac{P - d - \nu}{\theta}(e^{\theta(t_d - t)} - e^{-\theta t})$$
$$+ \frac{P + \delta P - d - \nu}{\theta}(e^{\theta(H - t - \frac{\nu H}{P_r + \nu})} - e^{\theta(t_d - t)}) \tag{27}$$

Hence the inventory level $I_3(t)$ at the time H will be

$$I_3(H) = \nu(H - \frac{\nu H}{P - r + \nu})e^{-\theta H} + \frac{P_r - d}{\theta}(1 - e^{-\theta H}) + \frac{P - d - \nu}{\theta}(e^{\theta(t_d - t)} - e^{-\theta H})$$
$$+ \frac{P + \delta p - d - \nu}{\theta}(e^{\theta(H - \frac{\nu H}{P_r + \nu} - H)} - e^{(t_d - H)}) \tag{28}$$

Proposition 4.5 *If $I_3(H) \geq 0$, then the disrupted production rate follows the following inequalities.*

$$\delta P \geq -(P - d - \nu)r - \frac{\nu H(1 - \frac{\nu}{P_r + \nu})e^{-\theta H} + \frac{P_r - \nu}{\theta}(1 - e^{-\theta H}) + \frac{P - d - \nu}{\theta}(e^{\theta(t_d - H)} - e^{-\theta H})}{e^{\theta(\frac{-nuH}{P_r + \nu})} - e^{t_d - H}} \tag{29}$$

Proof. If $I_3(H) \geq 0$. that is

$$\delta P \geq -(P - d - \nu)r - \frac{\nu H(1 - \frac{\nu}{P_r + \nu})e^{-\theta H} + \frac{P_r - \nu}{\theta}(1 - e^{-\theta H}) + \frac{P - d - \nu}{\theta}(e^{\theta(t_d - H)} - e^{-\theta H})}{e^{\theta(\frac{-nuH}{P_r + \nu})} - e^{t_d - H}} \tag{30}$$

this means that the manufacturer can still satisfy the demand after disrupted production.

And if $I_3(H - t_1) = \nu(H - t_1)$ then we found the $I_3(t)$

$$I_3(t) = \frac{P - r - d}{\theta}\left(1 - e^{\theta(H - \frac{\nu H}{P_r + \nu} - t)}\right) + \nu H\left(1 - \frac{\nu}{P_r + \nu}\right)e^{\theta(H - \frac{\nu H}{P_r + \nu} - t)} \tag{31}$$

And if $I_2(H) < 0$. that is

$$-P \leq \delta P \leq -(P - d - \nu)r - \frac{\nu H(1 - \frac{\nu}{P_r + \nu})e^{-\theta H} + \frac{P_r - \nu}{\theta}(1 - e^{-\theta H}) + \frac{P - d - \nu}{\theta}(e^{\theta(t_d - H)} - e^{-\theta H})}{e^{\theta(\frac{-nuH}{P_r + \nu})} - e^{(t_d - H)}}, \tag{32}$$

Then the manufacturer will face shortage.

3.3 The Imperfect Production and Rework and Disruption without Backlogging

In this case, we assume that $I_1(t)$ be the regular production inventory level till the time T_d, after that time the system get disrupted due to some appropriate problems. Let $I_2(t)$be a disrupted production inventory level with disrupted production rate $P + \delta P$ $(\delta P < 0)$ in the interval time $[T_d, T_d^P]$. Regular and disrupted production process have produced defective items at a rate ν. Afterward rework process starts on defective items at a rate P_r during the time $[t_1, T_d^P]$. Let $I_3(t)$ be the inventory level during the time interval $[t_1, T_d^P]$ and finally production stopped. Again costumer $I_4(t)$ represents the inventory level during the time interval $[t_1, H]$. The inventory at a time t is governed by the following differential equations

$$\frac{dI_1(t)}{dt} + \theta I_1(t) = P - d - \nu, 0 \leq t \leq T_d, I_1(0) = 0. \tag{33}$$

$$\frac{dI_2(t)}{dt} + \theta I_2(t) = P + \delta P - d - \nu, T_d \leq t \leq T_d^P, I_1(T_d) = I_2(T_d). \tag{34}$$

$$\frac{dI_3(t)}{dt} + \theta I_3(t) = P_r - d, T_d^P \leq t \leq t_1, I_3(t_1) = I_4(t_1). \tag{35}$$

$$\frac{dI_4(t)}{dt} + \theta I_4(t) = -d, t_1 \le t \le H, I_4(H) = 0. \quad (36)$$

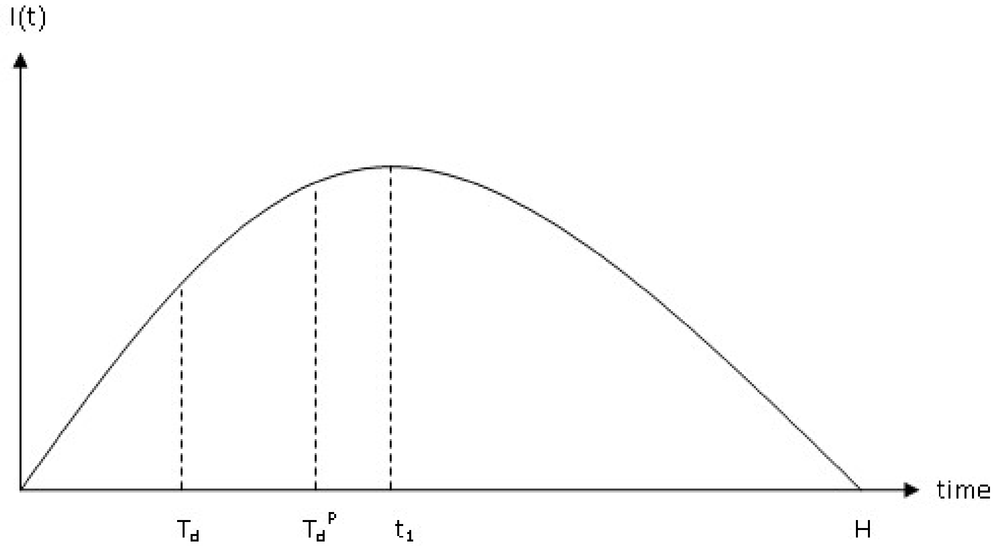

Figure 3:

Using the boundary conditions, the solution of above differential equations are

$$I_1(t) = \left(\frac{P-D-\nu}{\theta}\right)(1-e^{-\theta t}), 0 \le t \le T_d \quad (37)$$

$$I_2(t) = \frac{P+\delta P-d-\nu}{\theta}\left(1-e^{\theta(T_d-t)}\right) \quad (38)$$

$$+\frac{P-d-\nu}{\theta}(e^{\theta(T_d-t)}-e^{-\theta t}), T_d \le t \le T_d^P \quad (39)$$

$$I_3(t) = \frac{P_r-d}{\theta}\left(1-e^{\theta(T_d^P)}\right)+\frac{P+\delta P-d-\nu}{\theta}\left(e^{\theta(T_d^P)}-e^{\theta(T_d-t)}\right)$$

$$+\frac{P-d-\nu}{\theta}\left(e^{\theta(T_d-t)}-e^{-\theta t}\right), T_d^P \le t \le t_1 \quad (40)$$

$$I_4(t) = \frac{d}{\theta}\left(e^{\theta(H-t)}-1\right) t_1 \le t \le H \quad (41)$$

Proposition 4.6 *If ν is the defective items' production rate, P_r is the rework rate and T_P is the production time under disruption, then the rework time is given by t_1, where*

$$t_1 = \left(\frac{\nu+P_r}{P_r}\right)T_d^P \quad (42)$$

Proof. Let ν is the defective item production rate in the time interval $[0, T_d^P]$, so the total defective items are νT_d^P. Let P_r be the rework rate in the interval $[T_d^P, t_1]$, so the total reworkable items are $P_r(t_1 - T_d^P)$. In this case, all the defective items are reworkable, i.e.

$$\nu T_d^P = P_r(t_1 - T_d^P) \quad (43)$$

so, the rework time will be

$$t_1 = \left(\frac{\nu+P_r}{P_r}\right)t_d^P \quad (44)$$

Using equation 40 the total inventory after rework process

$$I_3(t) = \frac{P_r-d}{\theta}+\frac{d}{\theta}\left[e^{\theta(H-t)}-e^{\theta\left(\frac{\nu+P_r}{P_r}T_d^P-t\right)}\right]-\frac{P_r-d}{\theta}e^{\theta\left(\frac{\nu+P_r}{P_r}T_d^P-t\right)} \quad (45)$$

Theorem 4.7 *If T_d^P is the regular production time after disruption, then*

$$T_d^P = \frac{\frac{d}{\theta}(e^{\theta H}-1)+\delta P T_d}{\left(\frac{\nu+P_r}{P_r}(de^{\theta H}-1)+(P+\delta P-2d)\right)} \quad (46)$$

Proof. Since the inventory is continuous on t_1, which is shown in Fig.2, i.e.

$$I_4(t_1) = I_3(T_d^P) + \nu T_d^P - d(t_1 - T_d^P)$$

by using this condition respectively in the equation 40 and equation 41, we have

$$\frac{d}{\theta}\left(e^{\theta(H-t_1)-1}\right) = \frac{P+\delta P-d-\nu}{\theta}\left(1-e^{\theta(T_d-T_d^P)}\right) + \frac{P-d-\nu}{\theta}\left(e^{\theta(T_d-T_d^P)} - e^{-\theta T_d^P}\right)$$

$$+\nu T_d^P - d(t_1 - T_d^P) \quad (47)$$

with the help of above proposition above equation reduces into the following form,

$$\frac{\nu+P_r}{P_r} de^{\theta H} T_d^P + (P+\delta P-d-\nu)(T_d^P - T_d) + (P-d-\nu)T_d + \left(\nu - d\frac{\nu+P_r}{P_r} - d\right)T_d^P \quad (48)$$

$$\left(\frac{\nu+P_r}{P_r} de^{\theta H} + (P+\delta P-2d) - d) + dfrac\nu + P_r P_r\right) t_d^P = \frac{d}{\theta}(e^{\theta H}-1) + \delta P T_d \quad (49)$$

So, the regular production time during disruption

$$T_d^P = \frac{\frac{d}{\theta}(e^{\theta H}-1)+\delta P T_d}{\left(\frac{\nu+P_r}{P_r}(de^{\theta H}-1)+(P+\delta P-2d)\right)} \quad (50)$$

From proposition 3.6 the rework production time is

$$t_1 = \left(\frac{\nu+P_r}{P_r}\right)\frac{\frac{d}{\theta}(e^{\theta H}-1)+\delta P T_d}{\left(\frac{\nu+P_r}{P_r}(de^{\theta H}-1)+(P+\delta P-2d)\right)} \quad (51)$$

Differentiating equation 50 and equation 51 with respect to the the deterioration rate θ respectably, we have

$$\frac{dT_d^P}{d\theta} = \frac{K\left(\frac{d}{\theta}He^{\theta H}+\frac{d}{\theta^2}(1-e^{\theta H})\right)+M\left(d\left(\frac{\nu+P_r}{P_r}\right)He^{\theta H}\right)}{K^2} \quad (52)$$

$$\frac{dt_1}{d\theta} = \left(\frac{\nu+P_r}{P_r}\right)\frac{K\left(\frac{d}{\theta}He^{\theta H}+\frac{d}{\theta^2}(1-e^{\theta H})\right)+M\left(d\left(\frac{\nu+P_r}{P_r}\right)He^{\theta H}\right)}{K^2} \quad (53)$$

where

$$K = \frac{\nu+P_r}{P_r}(de^{\theta H}-1) + (P+\delta P-2d)) \quad (54)$$

$$M = \frac{d}{\theta}(e^{\theta H}-1) + \delta P T_d \quad (55)$$

the above equation 52 and equation 53 manifests that manufacturer has to produce more product when increases the deterioration rate. Thus, we have the following corollary.

Corollary 4.8 *Assuming $\theta << 1$. then the T_d^P and t_1 have positive growth with respect to θ.*

3.4 The Imperfect Production with Rework Disruption and Backlogging

In this case, We assumed $I_1(t)$ be a regular production inventory level till the time T_d, after that the production get disrupted consequently the regular production has been stopped and during the regular production defective items are produced at a rate ν. Let $I_2(t)$ be the inventory level during the rework time interval [T_d, t_1]. After stopping rework process, stored inventory fulfill the demand and at time Tr inventory reach to zero level. Let $I_4(t)$ be the inventory level when the production is restarted till the time $H - t_2$ at a disrupted production rate $P + \delta P$. During the interval $H - t_2$ production system also produced defective items at rate ν. Let $I_5(t)$ be the inventory level during the time interval $[H - t_2, H]$ when rework process is in progress at a rate P_r. At any time t, the inventory status is governed by the following equations

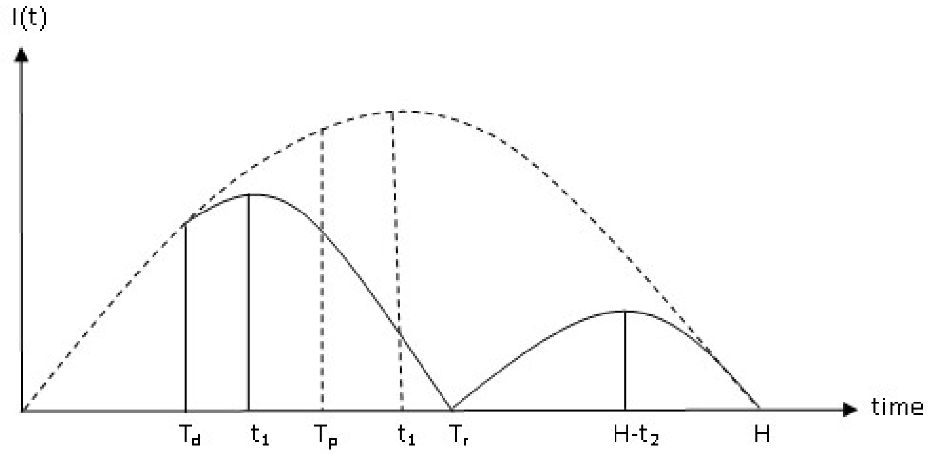

Figure 4:

$$\frac{dI_1(t)}{dt}+\theta I_1(t)=P-d-\nu, 0\le t\le T_d, I_1(0)=0 \tag{56}$$
$$\frac{dI_2(t)}{dt}+\theta I_2(t)=P_r-d, T_d\le t\le t_1, I_1(T_d)=I_2(T_d) \tag{57}$$
$$\frac{dI_3(t)}{dt}+\theta I_3(t)=-d, t_1\le t\le T_r, I_2(t_1)=I_3(t_1) \tag{58}$$
$$\frac{dI_4(t)}{dt}+\theta I_4(t)=P+\delta P-d-\nu, 0\le T_r\le H-t_2, I_4(T_r)=0 \tag{59}$$
$$\frac{dI_5(t)}{dt}+\theta I_5(t)=P_r-d, H-t_2\le t\le H, I_5(H)=0. \tag{60}$$

Using the boundary conditions the solution of the above differential equations are

$$I_1(t)=\frac{P-d-\nu}{\theta}(1-e^{-\theta t}), 0\le t\le T_d \tag{61}$$
$$I_2(t)=\frac{P_r-d}{\theta}(1-e^{\theta(T_d-t)})+\frac{P-d-\nu}{\theta}(e^{\theta(t_d-t)}-e^{-\theta t}), T_d\le t\le t_1 \tag{62}$$

$$I_3(t)=\frac{d}{\theta}(e^{\theta(t_1-t)}-1)+\frac{P_r-d}{\theta}(e^{\theta(t_1-t)}-e^{\theta(T_d-t)})$$
$$+\frac{p-d-\nu}{\theta}(e^{\theta(T_d-t)}-e^{-\theta t}), t_1\le t\le T_r \tag{63}$$

$$I_4(t)=\frac{P+\delta P-d-\nu}{\theta}(1-e^{\theta(T_r-t)}), 0\le T_r\le H-t_2, \tag{64}$$
$$I_5(t)=\frac{P_r-d}{\theta}(1-e^{\theta(H-t)}), H-t_2\le t\le H. \tag{65}$$

Proposition 4.9 *If ν is the defective items' production rate, P_r is the rework rate, T_d is the disruption time on regular process, then the rework time after regular process is t_1, where*

$$t_1=\frac{\nu+P_r}{P_r}T_d. \tag{66}$$

Proof. Let ν is the defective item rate on time interval $[0, T_d]$, so the total defective items are νT_d and lat P_r is the rework production rate, so the total reworkable items are $P_r(t_1-T_d$. In this model, all defective items are reworkable, i.e.

$$\nu t_d=P_r(t_1-T_d), \tag{67}$$

so, the rework production time after regular process

$$t_1=\frac{\nu+P_r}{P_r}T_d. \tag{68}$$

Proposition 4.10 *If ν is the defective item rate, P_r is the rework production rate, T_r is the total backlog time and H is the total time cycle, then the rework production time under backlog process is*

$$t_2 = \frac{\nu(H-T_r)}{\nu+P_r}. \tag{69}$$

Proof. Let ν is the defective item rate on time interval $[T_r, H-T_2]$, so, the total defective items are $\nu(H-t_2-T_r)$, and let P_r is the rework production rate, then the total reworkable items are $P_r t_2$. In this model, the total defective items are reworkable, i.e.

$$\nu(H-t_2-T_r) = P_r t_2 \tag{70}$$

$$\nu(H-T_r) = (\nu+p_r)t_2, \tag{71}$$

so, the rework production time under backlog process is

$$t_2 = \frac{\nu(H-T_r)}{\nu+P_r}. \tag{72}$$

Theorem 4.11 *If T_r is the time of placing when shortage occur, then*

$$T_r = \frac{1}{\theta}\log\left(\frac{P_r e^{\theta\left(\frac{\nu+P_r}{p_r}\right)T_d} + (P-P_r-\nu)e^{\theta T_d} - (P-d-\nu)}{d}\right) \tag{73}$$

Proof. Let T_r is the time of placing when shortage occur, and the inventory $I_3(t)$ fulfill the demand and reach to zero at the time T_r shown in Fig. 4. i.e. $I_3(T_r) = 0$.
using equation 63, we have

$$\frac{d}{\theta}\left(e^{\theta(t_1-T_r)}-1\right) + \frac{P_r-d}{\theta}\left(e^{\theta(t_1-T_r)} - e^{\theta(T_d-T_r)}\right) + \frac{P-d-\nu}{\theta}\left(e^{\theta(T_d-T_r)} - e^{-\theta T_r}\right) = 0. \tag{74}$$

$$\frac{d}{\theta}e^{\theta T_r} = \frac{P_r}{\theta}e^{\theta t_1} + \frac{P-P_r-\nu}{\theta}e^{\theta T_d} - \frac{P-d-\nu}{\theta} \tag{75}$$

$$T_r = \frac{1}{\theta}\log\left(\frac{P_r e^{\theta t_1} + (P-P_r-\nu)e^{\theta T_d} - (P-d-\nu)}{d}\right) \tag{76}$$

from proposition 3.9, the time of placing the order when shortage accrue is

$$T_r = \frac{1}{\theta}\log\left(\frac{P_r e^{\theta\left(\frac{\nu+P_r}{p_r}\right)T_d} + (P-P_r-\nu)e^{\theta T_d} - (P-d-\nu)}{d}\right) \tag{77}$$

from eq.3.9 on hand inventory is

$$I_4(t) = \frac{P+\delta P-d-\nu}{\theta}\left(1 - e^{\theta\left(\frac{1}{\theta}log\frac{P_r e^{\theta\left(\frac{\nu+P_r}{p_r}\right)T_d} + (P-P_r-\nu)e^{\theta T_d} - (P-d-\nu)}{d} - t\right)}\right). \tag{78}$$

Theorem 4.12 *If Q_r is the order quantity for placing the order when shortages occurs, then*

$$Q_r = \frac{P+\delta P-d-\nu}{\theta}\left(1 - e^{\theta\left(T_r - \nu\frac{\nu(H-T_r)}{\nu+P_r} - H\right)}\right) + \nu\left(H - T_r - \frac{\nu(H-T_r)}{\nu+P_r}\right) \tag{79}$$

Proof. We know that the quantity

$$Q_r = I_4(H-t_2) + \nu(H-t_2-T_r), \tag{80}$$

from equation(10) and corollary(3.3), shortage quantity Q_r will be

$$Q_r = \frac{P+\delta P-d-\nu}{\theta}\left(1 - e^{\theta\left(T_r - \nu\frac{\nu(h-T_r)}{\nu+P_r} - H\right)}\right) + \nu\left(H - T_r - \frac{\nu(h-T_r)}{\nu+P_r}\right) \tag{81}$$

4 Numerical Examples of Various Cases

Example 5.1 ***for case I:*** *We use the following numbers as the base value of parameter P= 500, δP = -5, d= 50 θ = 0.01, H = 30, ν = 10%, P_r = 10, we obtained T_P = 3.37 and t_1 = 3.41.*

Example 5.2 ***for case III:****We used the following number as the based of parameter P= 500, t_d= 5, δP = -5, d= 50,θ= 0.01, ν = 10%, P_r =10, we obtained t_d^P =401784 and t_1 =4.2201.*

Example 5.3 ***for case IV:****We used the following number as the based of parameter P= 500, t_d= 5 , δP = -5, d= 50, θ= 0.01, ν = 10%, P_r =10, we obtained t_1 = 5.05, T_r = 17.97, Q_r = 859.2 and t_2 = 0.11.*

5.1 Sensitivity Analysis

We have analyzed analytically numerically and graphically we have received the following information about this model:

5.1.1 Case I:

1. Increment of deterioration rate increases the regular production rate (Fig. 5).
2. Increment of defective items' production rate decreases the regular production rate while increases the rework time (Data table 1).

5.1.2 Case III:

1. Increment of deterioration rate increases the disrupted regular production rate (Fig.6).
2. Increment of defective items' production rate, increases the disrupted regular production rate while increases the rework time.

5.1.3 Case IV:

1. Increment of deterioration rate decreases the backlog time T_r, whereas increases the backlog rework time t_2 (Fig. 7).
2. Increment of defective items' production rate, increases the rework production time t_1 and backlog rework time t_2, so backlog quantity Q_r also increase, but the backlog time T_r affected less.

Table 1: Effect of ν on T_P and t_1.

ν	T_P	t_1
0.1	3.3791	3.4129
0.15	3.3786	3.4292
0.20	3.3780	3.4456
0.25	3.3774	3.4619

Table 2: Effect of (θ_1)on various outputs in disrupted system for ($\theta_2 > 0$.)

ν	t_1	t_2	T_r	Q_r
0.05	5.52	0.052	1.4627	8538.16
0.1	5.55	0.1043	19.4628	8714.50
0.15	5.58	0.1557	19.4629	8889.91
0.20	5.61	0.2061	19.4630	9064.41

.

Table 3: Effect of ν on T_d^P and t_1

ν	T_d^P	t_1
0.1	4.2419	4.2843
0.15	4.2410	4.3046
0.20	4.2400	4.3249
0.25	4.2391	4.3415

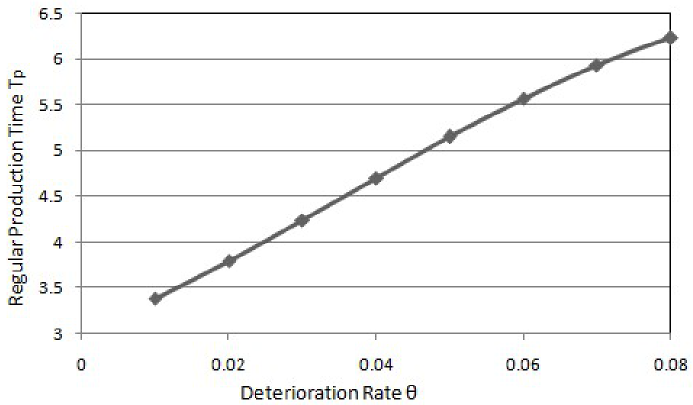

Figure 5: Effect of θ on T_P

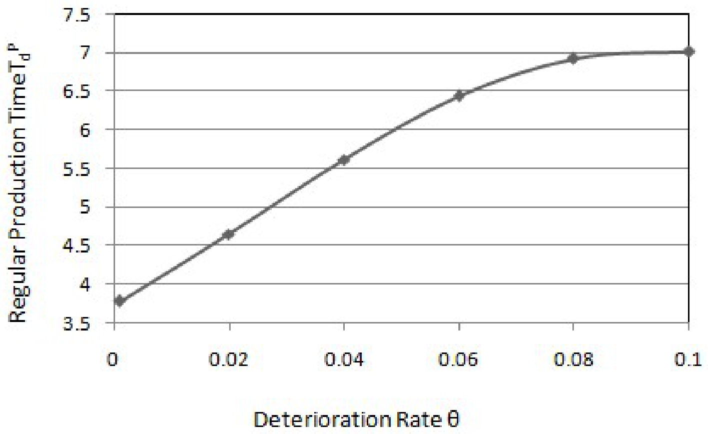

Figure 6: Effect of θ on T_d^P

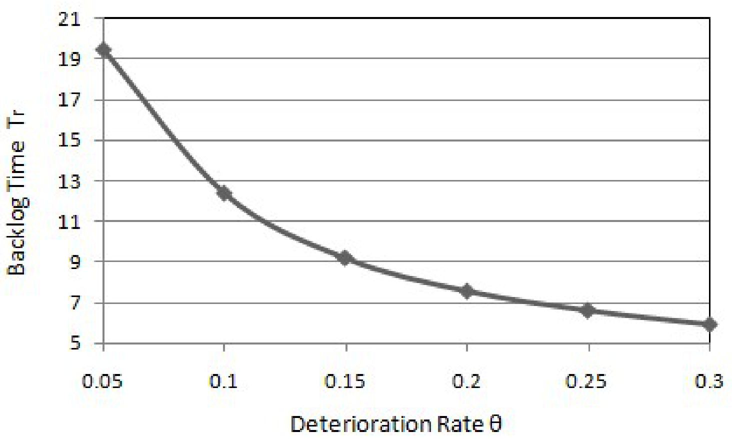

Figure 7: Effect of θ on T_r

6 Conclusion

In this article we have developed an imperfect production system considering with constant demand rate for deteriorating items. Further we have applied rework process on defective items. Rework process starts after stopping regular production. The model is developed in four different cases, which are (1) the imperfect production and rework, without disruption, (2) the imperfect production and rework with disruption, (3) the imperfect production and rework, without disruption and backlogging and (4) the imperfect production and rework, with disruption and backlogging. we have optimized t_1, T^p, T_d^P, T_r and Q_r for all these above cases. The model is also analyzed by graphically and verified by numerically.

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