A Copula Approach of Reliability Analysis for Hybrid Systems

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Abstract

Copula is a powerful tool to describe dependence among variables and has gained significant attention in many fields of research. In this paper, we study modeling the lifetimes of two fundamental hybrid systems: series-parallel and parallel-series by copula functions to reflect the effect on dependence of working components in the systems. We consider Farlie-Gumbel-Morgenstern (FGM) and Clayton copula functions to represent dependent structures in parallel and series configuration, respectively. A flexible lifetime distribution, the extended exponential distribution, is applied to the components of system. The explicit expressions of the reliability and mean time to failure (MTTF) of both hybrid systems are obtained. For the purpose of illustration, the effect of different degrees on dependence among components is analyzed and presented for various parameter settings in the lifetime distribution.

Keywords: series-parallel system, parallel-series system, copula, extended exponential distribution, reliability analysis

1 Introduction

In a system consisting of several components, most researches of reliability analysis focused on a system with all components being either in series or parallel configuration only. In many real situations, however, it is often seen a "hybrid" setup in which the working components are connected in a way of joining together with both series and parallel. For example, air supply systems generally are modular designed, where the power system consists of a number of semiconductor units combined in a series or hybrid circuit [1, 2]. The hybrid structure on its power transmission path makes hybrid electric vehicles possess the major features of both series and parallel systems, and more plentiful operation modes. Hence such design of vehicles has drawn many interests from many automotive companies [3].

We mainly focus on reliability analysis of two fundamental types of hybrid systems: the series-parallel and parallel-series systems with three components shown in Figure 1 since other more complex-system is some kind of composite of the two systems.

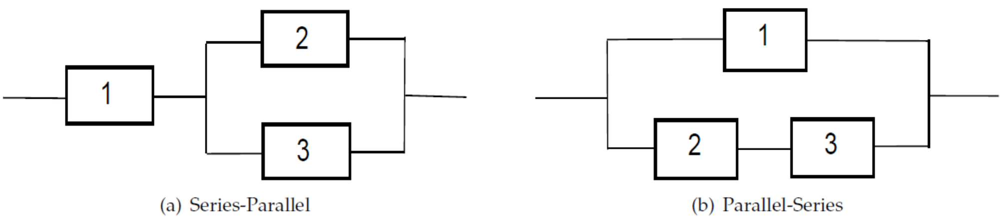


Figure 1: Hybrid Systems of Three Components

In many current literature focusing on system reliability for series or parallel conformation, each component in the system operating or failing is usually assumed to be independent of all the others [4, 5, 6, 7]. However, this assumption may not in line with the actual behavior of system life. In a common environment, components assembled into a system may be subject to the same stress or share the same load. For example, when an aero-engine is operating, the same loads on the blades mounted on one disk to make them rotate together at the same speed [8]. To relax the independent assumption, [9] presented a dependent time failure rate for non-independent components in series systems. In a coherent system including series and parallel systems, [10] and [11] described a multivariate distribution on the lifetimes of dependent components. Due to the difficulty to specify a joint distribution on the dependence situation, an alternative and convenient approach using copula has attracted much attention recently in correlation analysis. Copulas are the mechanisms that allow one to isolate the dependent structure from a multivariate distribution, and different copula functions represent different dependence between variables. The joint distribution of lifetime can be built by modeling dependence among components through a copula function, and hence it makes more convenient and flexible in applications [12]. A number of researchers have focused on employing copulas for modeling dependence in the context of system reliability (see [13, 14, 15, 16, 17], just name a few). Considering the dependence among lifetimes of components depicted by the Clayton copula, [18] investigated a multiple Type-I censored life test of series systems. Using a Gaussian copula to capture the effects of dependence structures on the coherent system reliability, [19] applied and extended the framework to multistate system. In addition, for addressing model uncertainty approximation in a product design space and integrating the model uncertainty into reliability-based design optimization, [20] proposed a copula-based bias modeling approach for reliability and demonstrated by two vehicle design problems.

In this article, we study the system reliability of the two hybrid systems in use of copula function to formalize dependence among components and is organized as follows. Section 2 presents copula functions in modeling the dependent structure of components in the hybrid systems, and Section 3 concentrates on the reliability analysis for the series-parallel and parallel-series systems, respectively. In Section 4, we present numerical and graphical illustrations for the comparisons of system reliability under various dependent situations formalized by copula functions. Lastly we conclude the article with a brief discussion in Section 5.

2 Model by Copula Function

A copula function can capture non-linear association among random variables comprising a vector. The Sklar's Lemma [21] in the case of bivariate variables states: given a bivariate distribution function $H(\cdot,\cdot)$ with two marginals $F(\cdot)$ and $G(\cdot)$, there exists a copula function $C(\cdot,\cdot)$ such that $H(x,y) = C(F(x), G(y))$ for all (x,y) in the domain. In reliability analysis of a system, the dependence structure among the components described by a copula is illustrated as follows. For the hybrid systems displayed in Figure 1, we denote $C_s(\cdot,\cdot)$ and $C_p(\cdot,\cdot)$ as the copulas for the two components in the series and parallel systems, respectively. To make notation simple, let T_i be the lifetime of component i whose density, distribution and reliability/survival functions $f_i(\cdot), F_i(\cdot), \overline{F}_i(\cdot) = 1 - F_i(\cdot)$, and the joint distribution and reliability/survival functions of component i and j are $F_{ij}(\cdot,\cdot), \overline{F}_{ij}(\cdot,\cdot), i,j = 1,2,3$. First, consider the series-parallel system in Figure 1(a), it is obvious that the system life is $T_{sp} = \min(T_1, T_p)$ with the sub-parallel system life $T_p = \max(T_2, T_3)$, and then the system reliability is

$$\begin{aligned} P(T_{sp} > t) &= P(T_1 > t, T_p > t) = P(T_1 > t) + P(T_p > t) - [1 - P(T_1 \le t, T_p \le t)] \\ &= \overline{F}_1(t) - F_p(t) + C_s(F_1(t), F_p(t)) \\ &= \overline{F}_1(t) - C_p(F_2(t), F_3(t)) + C_s(F_1(t), F_p(t)), t > 0. \end{aligned} \tag{1}$$

where

$$F_p(t) = P(T_p \le t) = P(T_2 \le t, T_3 \le t) = C_p(F_2(t), F_3(t)). \tag{2}$$

Likewise, for the the parallel-series system with three components as shown in Figure 1(b), the system life becomes $T_{ps} = \max(T_1, T_s)$ with the sub-series system life $T_s = \min(T_2, T_3)$, and its reliability becomes

$$P(T_{ps} > t) = 1 - P(\max(T_1, T_s) \le t) = 1 - P(T_1 \le t, T_s \le t)$$
$$= 1 - C_p(F_1(t), F_s(t)), t > 0 \quad (3)$$

where

$$F_s(t) = 1 - P(T_2 > t, T_3 > t) = 1 - [P(T_2 > t) + P(T_3 > t) - 1 + P(T_2 \le t, T_3 \le t)]$$
$$= F_2(t) + F_3(t) - C_s(F_2(t), F_3(t)). \quad (4)$$

In reliability, many classic families of distributions such as gamma, Weibull and log-normal, have been studied and applied quite extensively in the literature. Due to the characterizing memoryless property and many other nice properties, exponential distribution has found essential applications in many applied areas such as reliability/survival analysis, operations research and life tests. However, the exponential distribution has a constant failure. In practical situations, observed lifetime data often display varying shapes in the failure rate, and it is desirable that the assumed lifetime distribution has considerable flexibility to capture such characteristics and shapes. Recently, [22] applied an extended exponential distribution to describe dependent components in series and parallel systems. The extended exponential distribution has the distribution, reliability and density functions as follows

$$F(t) = \frac{1-e^{-\lambda t}}{1-\bar{\alpha}e^{-\lambda t}}, \bar{F}(t) = \frac{\alpha e^{-\lambda t}}{1-\bar{\alpha}e^{-\lambda t}}, f(t) = \frac{\alpha\lambda e^{-\lambda t}}{(1-\bar{\alpha}e^{-\lambda t})^2}, \alpha, \lambda > 0, \bar{\alpha} = 1-\alpha, t > 0, \quad (5)$$

with α, λ being the shape and scale parameters, and the distribution is denoted by $EE(\alpha, \lambda)$. Obviously $\alpha = 1$ leads to an exponential distribution with scale λ.

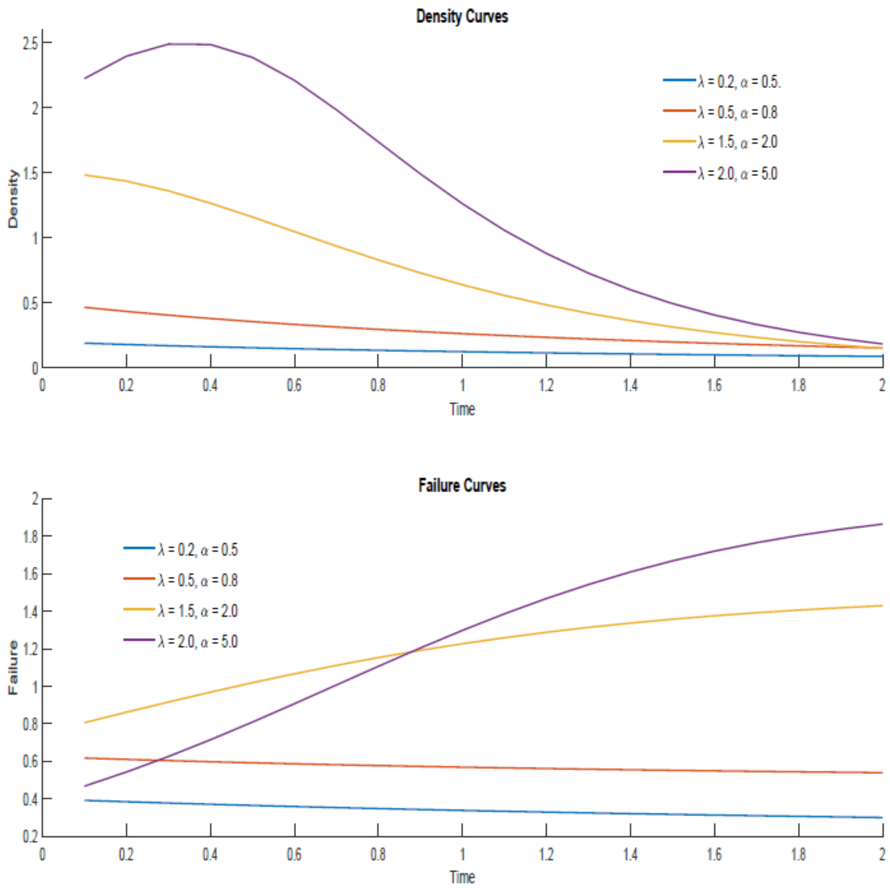


Figure 2: Density and Failure Functions of Extended Exponential Distribution

The density and failure curves of the extended exponential distribution for various settings of parameter values are shown in Figure 2, where one can see that the failures are decreasing over time for $0 < \alpha \leq 1$ while they are increasing for $\alpha > 1$. This extended form of distribution results in a flexible failure function depending on the value of α, called a tilt parameter. More interesting properties and applications of the distriubtion were introduced and discussed recently in [23, 24, 25, 26, 27].

3 Reliability of Hybrid Systems

The reliability analysis of the two hybrid systems is presented in this section. Besides real-time reliability, the mean time to failure (MTTF) is another important measure to explore the effect of dependent components in the system, and so we exhibit the MTTFs $E(T_{sp})$ and $E(T_{ps})$ for the series-parallel and parallel-series systems under various dependent degrees among components.

3.1 Independent Components

For independent components in the system, the applied copulas are $C_p(u,v) = C_s(u,v) = uv$. Then the reliability function for series-parallel system in (??) becomes

$$\begin{aligned}\overline{F}_{sp}(t) &= \overline{F}_1(t) - F_2(t)F_3(t) + F_1(t)F_p(t) = \overline{F}_1(t) - F_2(t)F_3(t) + F_1(t)F_2(t)F_3(t) \\ &= \overline{F}_1(t)[1 - F_2(t)F_3(t)]. \end{aligned} \tag{6}$$

Assuming that each component follows an identical $EE(\alpha,\lambda)$ in (5), we have $F_i(t) = F(t) = \frac{1-e^{-\lambda t}}{1-\bar{\alpha}e^{-\lambda t}}, i = 1,2,3$. To make notation simple in the presentation, we omit the time t in the distribution and reliability functions to designate $F = F(t), \overline{F} = \overline{F}(t)$, etc. Then the reliability for the series-parallel system becomes

$$\overline{F}_{sp} = \overline{F}(1 - F^2) = 2\overline{F}^2 - \overline{F}^3. \tag{7}$$

To compute the MTTF of systems, first let $\gamma(k) = \int_0^\infty \overline{F}^k(t)dt, k = 1,2,\cdots$. Taking substitution by $x = 1 - \bar{\alpha}e^{-\lambda t}$, the integrals are calculated as follows

$$\gamma(1) = \int_0^\infty \overline{F}(t)dt = \int_0^\infty \frac{\alpha e^{-\lambda t}}{1-\bar{\alpha}e^{-\lambda t}}dt = \frac{\alpha}{\bar{\alpha}\lambda}\int_\alpha^1 \frac{1}{x}dx = -\frac{\alpha}{\bar{\alpha}\lambda}\log\alpha, \tag{8}$$

$$\begin{aligned}\gamma(k) &= \int_0^\infty \overline{F}^k(t)dt = \int_0^\infty \left(\frac{\alpha e^{-\lambda t}}{1-\bar{\alpha}e^{-\lambda t}}\right)^k dt = \frac{\alpha^k}{\bar{\alpha}^k\lambda}\int_\alpha^1 \frac{(1-x)^{k-1}}{x^k}dx \\ &= \frac{\alpha^k}{\bar{\alpha}^k\lambda}\sum_{r=0}^{k-1}(-1)^r\binom{k-1}{r}\int_\alpha^1 x^{r-k}dx \\ &= \frac{\alpha^k}{\bar{\alpha}^k\lambda}\left[\sum_{r=0}^{k-2}(-1)^r\binom{k-1}{r}\frac{1-\alpha^{r-k+1}}{r-k+1} + (-1)^k\log\alpha\right], k = 2,3,\cdots \end{aligned} \tag{9}$$

Then the MTTF for the series-parallel system is

$$\begin{aligned}E(T_{sp}) &= \int_0^\infty \overline{F}_{sp}(t)dt = 2\int_0^\infty \overline{F}^2 dt - \int_0^\infty \overline{F}^3 dt = 2\gamma(2) - \gamma(3) \\ &= \frac{\alpha}{\bar{\alpha}^3\lambda}\left[\frac{1}{2}\alpha^2 - 2\alpha + \frac{3}{2} - \alpha(\alpha-2)\log\alpha\right] = \frac{g_{sp}(\alpha)}{2\lambda}. \end{aligned} \tag{10}$$

As a special case, when $\alpha = 1$, the MTTF is obtained through the L'Hopital's rule

$$\begin{aligned}\lim_{\alpha\to 1} g_{sp}(\alpha) &= \lim_{\alpha\to 1}\frac{\alpha(\alpha^2-4\alpha+3)-2\alpha^2(\alpha-2)\log\alpha}{(1-\alpha)^3} \\ &= \lim_{\alpha\to 1}\frac{\alpha^2-4\alpha+3-2(3\alpha^2-4\alpha)\log\alpha}{-3(1-\alpha)^2} \\ &= \lim_{\alpha\to 1}\frac{-2\alpha+2-(6\alpha-4)\log\alpha}{3(1-\alpha)} \\ &= \lim_{\alpha\to 1}\frac{-2-6\log\alpha-\frac{6\alpha-4}{\alpha}}{-3} = \frac{4}{3}. \end{aligned}$$

Hence $E(T_{sp}) = 2/(3\lambda)$ in the case of exponential lifetime for each component. Similarly, for the parallel-series system, the reliability function in (??) under independent components is

$$\begin{aligned}\overline{F}_{ps}(t) &= 1 - F_1(t)F_s(t) = 1 - F_1(t)[F_2(t) + F_3(t) - F_2(t)F_3(t)] \\ &= 1 - F_1(t)[1 - \overline{F}_2(t)\overline{F}_3(t)] = \overline{F}_1(t) + F_1(t)\overline{F}_2(t)\overline{F}_3(t). \end{aligned} \tag{11}$$

For the identical lifetime distribution $EE(\alpha,\lambda)$ in (5), the reliability becomes

$$\overline{F}_{ps} = \overline{F} + \overline{F}^2 F = \overline{F} + \overline{F}^2 - \overline{F}^3, \quad (12)$$

and the MTTF is

$$E(T_{ps}) = \int_0^\infty \overline{F}_{ps}(t)dt = \gamma(1) + \gamma(2) - \gamma(3)$$
$$= \frac{\alpha}{\overline{\alpha}^3\lambda}\left[-\frac{1}{2}\alpha^2 + \frac{1}{2} - (\alpha^2 - 3\alpha + 1)\log\alpha\right] = \frac{g_{sp}(\alpha)}{2\lambda}. \quad (13)$$

By the L'Hopital's rule, the MTTF when $\alpha = 1$ becomes

$$\lim_{\alpha\to1} g_{ps}(\alpha) = \lim_{\alpha\to1}\frac{-\alpha^3+\alpha-2\alpha(1-3\alpha+\alpha^2)\log\alpha}{(1-\alpha)^3}$$
$$= \lim_{\alpha\to1}\frac{-5\alpha^2+6\alpha-1-2(3\alpha^2-6\alpha+1)\log\alpha}{-3(1-\alpha)^2}$$
$$= \lim_{\alpha\to1}\frac{-10\alpha+6-2(6\alpha-6)\log\alpha-\frac{2(3\alpha^2-6\alpha+1)}{\alpha}}{6(1-\alpha)}$$
$$= \lim_{\alpha\to1}\frac{9-8\alpha-\frac{1}{\alpha}-(6\alpha-6)\log\alpha}{3(1-\alpha)}$$
$$= \lim_{\alpha\to1}\frac{-8+\frac{1}{\alpha^2}-6\log\alpha-\frac{-6+6\alpha}{\alpha}}{-3} = \frac{7}{3}.$$

Hence $E(T_{sp}) = 7/(6\lambda)$ for independent components with exponential lifetime in the system.

3.2 *Dependent Components*

First we consider a Farlie-Gumbel-Morgenstern (FGM) copula family which was adopted in [28, 29] to describe relationship of lifetimes for working components in both parallel and series systems. For the sub-parallel system (consisting components 2 & 3) in Figure 1(a) and sub-series system (consisting components 2 & 3) in Figure 1(b), we assume that the dependence structure of components is generated by FGM copulas, given by,

$$C_p(u,v) = uv + \theta_p uv(1-u)(1-v), C_s(u,v) = uv + \theta_s uv(1-u)(1-v), \quad (14)$$

where $-1 \le \theta_p, \theta_s \le 1$ are the parameters in the copulas used in parallel and series systems. Under the assumption that each component lifetime follows the identical $EE(\alpha,\lambda)$ in (5), i.e. $F_i(t) = F(t) = \frac{1-e^{-\lambda t}}{1-\overline{\alpha}e^{-\lambda t}}, i = 1,2,3$, the expression in (2) in series-parallel system becomes with omitting time t is $F_p = C_p(F,F) = F^2(1+\theta_p\overline{F}^2)$, and so the reliability function is

$$\overline{F}_{sp} = \overline{F} - C_p(F,F) + C_s(F,F_p) = \overline{F} - F^2(1+\theta_p\overline{F}^2) + FF_p(1+\theta_s\overline{F}\overline{F}_p)$$
$$= \overline{F} - F^2(1+\theta_p\overline{F}^2) + F^3(1+\theta_p\overline{F}^2)\left\{1+\theta_s\overline{F}[1-F^2(1+\theta_p\overline{F}^2)]\right\}$$
$$= \overline{F} - F^2 + F^3 - F^2\overline{F}^3\theta_p + F^3\overline{F}^2(2-\overline{F})\theta_s + F^3\overline{F}^3(1-2F^2)\theta_p\theta_s - F^5\overline{F}^5\theta_p^2\theta_s$$
$$= \overline{A}_1 + \overline{A}_2\theta_p + \overline{A}_3\theta_s + \overline{A}_4\theta_p\theta_s + \overline{A}_5\theta_p^2\theta_s, \quad (15)$$

where

$$\overline{A}_1 = 2\overline{F}^2 - \overline{F}^3, \overline{A}_2 = -\overline{F}^3 + 2\overline{F}^4 - \overline{F}^5, \overline{A}_3 = 2\overline{F}^2 - 7\overline{F}^3 + 9\overline{F}^4 - 5\overline{F}^5 + \overline{F}^6,$$
$$\overline{A}_4 = -\overline{F}^3 + 7\overline{F}^4 - 17\overline{F}^5 + 19\overline{F}^6 - 10\overline{F}^7 + 2\overline{F}^8,$$
$$\overline{A}_5 = -\overline{F}^5 + 5\overline{F}^6 - 10\overline{F}^7 + 10\overline{F}^8 - 5\overline{F}^9 + \overline{F}^{10}. \quad (16)$$

Thus, we have

$$E(T_{sp}) = \int_0^\infty \overline{F}_{sp}(t)dt = A_1 + A_2\theta_p + A_3\theta_s + A_4\theta_p\theta_s + A_5\theta_p^2\theta_s, \quad (17)$$

with

$$A_1 = E(\overline{A}_1) = 2\gamma(2) - \gamma(3) = \frac{\alpha}{\overline{\alpha}^3\lambda}\left[\frac{1}{2}\alpha^2 - 2\alpha + \frac{3}{2} - \alpha(\alpha-2)\log\alpha\right], \quad (18)$$

$$A_2 = E(\overline{A}_2) = -\gamma(3) + 2\gamma(4) - \gamma(5) = \frac{\alpha}{\overline{\alpha}^5\lambda}\left[\frac{1}{12}\alpha^4 - \frac{2}{3}\alpha^3 + \frac{2}{3}\alpha - \frac{1}{12} + \alpha^2\log\alpha\right], \quad (19)$$

$$A_3 = E(\overline{A}_3) = 2\gamma(2) - 7\gamma(3) + 9\gamma(4) - 5\gamma(5) + \gamma(6)$$
$$= \frac{\alpha}{\overline{\alpha}^6\lambda}\left[\frac{2}{15}\alpha^5 - \frac{11}{12}\alpha^4 + 3\alpha^3 - \frac{13}{3}\alpha^2 + \frac{5}{3}\alpha + \frac{9}{20} - \alpha(\alpha-2)\log\alpha\right], \quad (20)$$

$$A_4 = E(\overline{A}_4) = -\gamma(3) + 7\gamma(4) - 17\gamma(5) + 19\gamma(6) - 10\gamma(7) + 2\gamma(8)$$
$$= \frac{\alpha}{\overline{\alpha}^8\lambda}\left[\frac{1}{70}\alpha^7 - \frac{3}{20}\alpha^6 + \frac{13}{15}\alpha^5 - \frac{11}{12}\alpha^4 - \frac{13}{6}\alpha^3 + \frac{137}{60}\alpha^2 + \frac{1}{15}\alpha + \frac{1}{420}\right.$$
$$\left.-\alpha^2(\alpha^2 - 2\alpha - 1)\log\alpha\right], \quad (21)$$

$$A_5 = E(\overline{A}_5) = -\gamma(5) + 5\gamma(6) - 10\gamma(7) + 10\gamma(8) - 5\gamma(9) + \gamma(10)$$
$$= \frac{\alpha}{\overline{\alpha}^{10}\lambda}\left[-\frac{1}{630}\alpha^9 + \frac{1}{56}\alpha^8 - \frac{2}{21}\alpha^7 + \frac{1}{3}\alpha^6 - \alpha^5 + \frac{1}{5}\alpha^4 + \frac{2}{3}\alpha^3\right.$$

$$-\frac{1}{7}\alpha^2+\frac{1}{42}\alpha-\frac{1}{504}+\alpha^4\log\alpha\Big]. \quad (22)$$

Note that when $\theta_p=\theta_s=0$ for the case of independent components, MTTF is the coefficient A_1 which is the same as the expression in (??). Likewise, for the parallel-series system with the expression in (??) being $F_s=2F-F^2(1+\theta_s\overline{F}^2)$, the reliability function is

$$\overline{F}_{ps}=1-C_p(F,F_s)=1-FF_s(1+\theta_p\overline{F}\overline{F}_s)=1-F^2[(1+\overline{F})-\theta_sF\overline{F}^2][1+\theta_p\overline{F}^3(1+\theta_sF^2)]$$

$$=\overline{B}_1+\overline{B}_2\theta_p+\overline{B}_3\theta_s+\overline{B}_4\theta_p\theta_s+\overline{B}_5\theta_p\theta_s^2, \quad (23)$$

where

$$\overline{B}_1=\overline{F}+\overline{F}^2-\overline{F}^3,\overline{B}_2=-\overline{F}^3+\overline{F}^4+\overline{F}^5-\overline{F}^6,$$
$$\overline{B}_3=\overline{F}^2-3\overline{F}^3+3\overline{F}^4-\overline{F}^5,\overline{B}_4=-\overline{F}^3+3\overline{F}^4-\overline{F}^5-5\overline{F}^6+6\overline{F}^7-2\overline{F}^8,$$
$$\overline{B}_5=\overline{F}^5-5\overline{F}^6+10\overline{F}^7-10\overline{F}^8+5\overline{F}^9-\overline{F}^{10}. \quad (24)$$

Thus, we have

$$E(T_{ps})=\int_0^\infty\overline{F}_{ps}(t)dt=B_1+B_2\theta_p+B_3\theta_s+B_4\theta_p\theta_s+B_5\theta_p\theta_s^2, \quad (25)$$

with

$$B_1=E(\overline{B}_1)=\gamma(1)+\gamma(2)-\gamma(3)=\frac{\alpha}{\overline{\alpha}^3\lambda}\Big[-\frac{1}{2}\alpha^2+\frac{1}{2}-(\alpha^2-3\alpha+1)\log\alpha\Big], \quad (26)$$

$$B_2=E(\overline{B}_2)=-\gamma(3)+\gamma(4)+\gamma(5)-\gamma(6)=\frac{\alpha}{\overline{\alpha}^6\lambda}\Big[-\frac{2}{15}\alpha^5+\frac{5}{4}\alpha^4-\frac{1}{3}\alpha^3-\frac{5}{3}\alpha^2$$
$$+\alpha-\frac{7}{60}-\alpha^2(2\alpha-1)\log\alpha\Big], \quad (27)$$

$$B_3=E(\overline{B}_3)=\gamma(2)-3\gamma(3)+3\gamma(4)-\gamma(5)$$
$$=\frac{\alpha}{\overline{\alpha}^5\lambda}\Big[-\frac{1}{12}\alpha^4+\frac{1}{2}\alpha^3-\frac{3}{2}\alpha^2+\frac{5}{6}\alpha+\frac{1}{4}+\alpha\log\alpha\Big], \quad (28)$$

$$B_4=E(\overline{B}_4)=-\gamma(3)+3\gamma(4)-\gamma(5)-5\gamma(6)+6\gamma(7)-2\gamma(8)$$
$$=\frac{\alpha}{\overline{\alpha}^8\lambda}\Big[-\frac{1}{70}\alpha^7+\frac{7}{60}\alpha^6-\frac{1}{3}\alpha^5+\frac{37}{12}\alpha^4-\frac{19}{6}\alpha^3-\frac{7}{60}\alpha^2+\frac{7}{15}\alpha-\frac{1}{28}$$
$$-\alpha^2(\alpha^2+2\alpha-1)\log\alpha], \quad (29)$$

$$B_5=E(\overline{B}_5)=\gamma(5)-5\gamma(6)+10\gamma(7)-10\gamma(8)+5\gamma(9)-\gamma(10)$$
$$=\frac{\alpha}{\overline{\alpha}^{10}\lambda}\Big[\frac{1}{630}\alpha^9-\frac{1}{56}\alpha^8+\frac{2}{21}\alpha^7-\frac{1}{3}\alpha^6+\alpha^5-\frac{1}{5}\alpha^4-\frac{2}{3}\alpha^3$$
$$+\frac{1}{7}\alpha^2-\frac{1}{42}\alpha+\frac{1}{504}-\alpha^4\log\alpha\Big]. \quad (30)$$

Note that the coefficient B_1 is the MTTF when $\theta_p=\theta_s=0$, corresponding to the case of independent components, whose expression is the same as in (??).

3.3 *Different Copulas*

For FGM copula family, the Kendall's tau, which measures the "concordance" of bivariate random variables, is $\tau_\theta=2\theta/9$, resulting in $\tau_\theta\in[-2/9,2/9]$ for $-1\le\theta\le1$. The limited range of dependence restricts the usefulness of this family for application [30]. Since there is usually a stronger correlation of lifetimes among the components in series systems than in parallel systems, it is not appropriate to use FGM copula for modeling an association between component lifetimes in a series system as presented in [28]. [18] used the Clayton copula, a member of Archimedean family, to correlate the lifetimes of components in a series system due to $\tau_\theta=\theta/(\theta+2)\in[0,1)$ for $\theta\ge0$, a larger range of dependence. Hence we assume that the dependence structures of components are generated, respectively, by FGM copula in the sub-parallel system (consisting of components 2 & 3) in Figure 1(a) and Clayton copula in the sub-series system (consisting of components 2 & 3) in Figure 1(b). The FGM and Clayton copulas are given by

$$C_f(u,v)=uv+\theta_fuv(1-u)(1-v),C_l(u,v)=(u^{-\theta_l}+v^{-\theta_l}-1)^{-\frac{1}{\theta_l}}, \quad (31)$$

where $-1\le\theta_f\le1$ and $\theta_l\ge0$ are the parameters in the copulas used in parallel and series systems. From the reliability expression in (??) for the series-parallel system, the system reliability is

$$\overline{F}_{sp}(t)=\overline{F}_1(t)-C_f(F_2(t),F_3(t))+C_l(F_1(t),F_p(t)),t>0, \quad (32)$$

with $F_p(t)=C_f(F_2(t),F_3(t))$. For the identical $EE(\alpha,\lambda)$, $F_i=F,i=1,2,3$ in (5), we have $F_p=C_f(F,F)=F^2(1+\theta_f\overline{F}^2)$ and $C_l(F,F_p)=(F^{-\theta_l}+F_p^{-\theta_l}-1)^{-1/\theta_l}$, and the reliability becomes

$$\overline{F}_{sp} = \overline{F} - C_f(F,F) + C_l(F,F_p) = \overline{F} - F^2(1+\theta_f\overline{F}^2) + C_l(F,F_p) = \overline{C}_1 + \overline{C}_2\theta_f, \quad (33)$$

where the coefficients

$$\overline{C}_1 = 3\overline{F} - \overline{F}^2 + C_l(F,F_p) - 1, \quad (34)$$

$$\overline{C}_2 = -\overline{F}^2 + 2\overline{F}^3 - \overline{F}^4. \quad (35)$$

Thus, the MTTF of the system

$$E(T_{sp}) = \int_0^\infty \overline{F}_{sp}(t)dt = C_1 + C_2\theta_f, \quad (36)$$

with

$$C_1 = E(\overline{C}_1) = 3\gamma(1) - \gamma(2) + I_c = \frac{\alpha}{\overline{\alpha}^2\lambda}[\alpha - 1 + (2\alpha-3)\log\alpha] + I_c, \quad (37)$$

$$C_2 = E(\overline{C}_2) = -\gamma(2) + 2\gamma(3) - \gamma(4) = \frac{\alpha}{\overline{\alpha}^4\lambda}\left[-\frac{1}{6}\alpha^3 + \alpha^2 - \frac{1}{2}\alpha - \frac{1}{3} - \alpha\log\alpha\right], \quad (38)$$

where the integral $I_c = \int_0^\infty [C_l(F,F_p) - 1]dt$ has no analytic form and a numerical method has to be applied for evaluation. Note that if $\theta_f = \theta_l = 0$, then $F_p = C_f(F,F) = F^2$ and $C_l(F,F_p) = F^3$, and thus the coefficient $C_1 = 3\gamma(1) - \gamma(2) + \int_0^\infty (F^3 - 1)dt = 3\gamma(1) - \gamma(2) + \int_0^\infty [-3\overline{F} + 3\overline{F}^2 - \overline{F}^3]dt = 2\gamma(2) - \gamma(3)$ is the same as A_1 in (18), corresponding to the MTTF in the case of independent components, shown in (??).

Likewise, for the parallel-series system with three components as shown in Figure 1(b), the system reliability in (??) becomes

$$\overline{F}_{ps}(t) = 1 - C_f(F_1(t), F_s(t)), t > 0, \quad (39)$$

where

$$F_s(t) = F_2(t) + F_3(t) - C_l(F_2(t), F_3(t)). \quad (40)$$

Under the identical $EE(\alpha,\lambda)$ lifetime, $F_s = 2F - C_l(F,F)$ with $C_l(F,F) = (2F^{-\theta_l} - 1)^{-1/\theta_l}$, the reliability function is

$$\overline{F}_{ps} = 1 - C_f(F,F_s) = 1 - FF_s(1+\theta_f\overline{F}\overline{F}_s)$$
$$= 1 - F[2F - C_l(F,F)][1+\theta_f\overline{F}(2\overline{F} - 1 + C_l(F,F))] = \overline{D}_1 + \overline{D}_2\theta_f, \quad (41)$$

where

$$\overline{D}_1 = 2(2\overline{F} - \overline{F}^2) + FC_l(F,F) - 1, \quad (42)$$

$$\overline{D}_2 = 2(\overline{F} - 4\overline{F}^2 + 5\overline{F}^3 - 2\overline{F}^4) + F\overline{F}C_l(F,F)[C_l(F,F) + 4\overline{F} - 3]. \quad (43)$$

It follows that

$$E(T_{ps}) = \int_0^\infty \overline{F}_{ps}(t)dt = D_1 + D_2\theta_f, \quad (44)$$

with

$$D_1 = E(\overline{D}_1) = 2[2\gamma(1) - \gamma(2)] + I_{d1} = \frac{\alpha}{\overline{\alpha}^2\lambda}[2\alpha - 2 + 2(\alpha-2)\log\alpha] + I_{d1}, \quad (45)$$

$$D_2 = E(\overline{D}_2) = 2[\gamma(1) - 4\gamma(2) + 5\gamma(3) - 2\gamma(4)] + I_{d2}$$
$$= \frac{\alpha}{\overline{\alpha}^4\lambda}\left[\frac{1}{3}\alpha^3 - \alpha^2 + 5\alpha - \frac{13}{3} - 2(\alpha+1)\log\alpha\right] + I_{d2}, \quad (46)$$

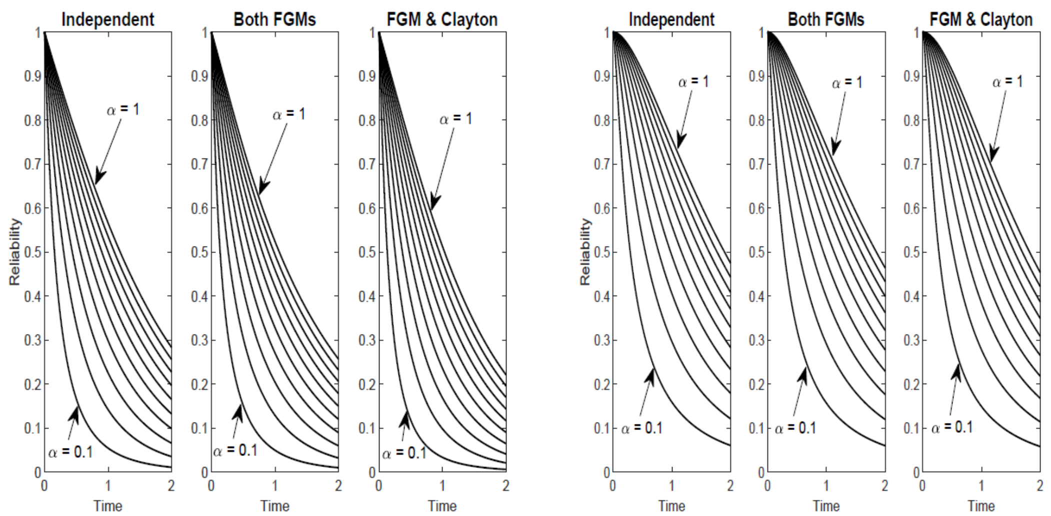
where the integrals $I_{d1} = \int_0^\infty [FC_l(F,F) - 1]dt, I_{d2} = \int_0^\infty F\overline{F}C_l(F,F)[C_l(F,F) + 4\overline{F} - 3]dt$ have no closed forms and they could be evaluated by a numerical method for computation. Specifically, $\theta_f = \theta_l = 0$ leads to $C_l(F,F) = F^2$, and then the coefficient $D_1 = 2[2\gamma(1) - \gamma(2)] + \int_0^\infty (F^3 - 1)dt = 2[2\gamma(1) - \gamma(2)] + \int_0^\infty [-3\overline{F} + 3\overline{F}^2 - \overline{F}^3]dt = \gamma(1) + \gamma(2) - \gamma(3)$ is the same as B_1 in (26), corresponding to the MTTF under the situation of independent components, as shown in (??).

4 Illustrations

In this section, we present numerical examples to investigate the performance of reliability and MTTF for each hybrid system under the considered copula functions. Since the lifetimes of components usually appear to be concordance strong correlated in series and weak in parallel system, for the FGM copula applied in both series and parallel systems, we specify a larger positive value for the parameter θ_s and a smaller positive value θ_p to bring about their Kendall's tau values accordingly. Similarly, for the Clayton copula which describes dependence structure for the components in a series system, a larger positive value of θ_l seems appropriate. Therefore we specify parameter values $\theta_p = 0.5, \theta_s = 0.8$ for the case of both parallel and series systems whose components structures are modeled by FGM copulas, and $\theta_f = 0.5, \theta_l = 1.0$ for the case where the parallel and series systems are modeled by FGM and Clayton copulas, respectively. Additionally, with a fixed scale value λ or tilt value α for the extended exponential distribution, a set of parameter values α and λ are specified to investigate their effects on reliability and MTTF of the hybrid systems. Here we consider two settings: $\lambda = 0.5, \alpha = 0.1, 0.2, \cdots, 1.0$ and $\alpha = 0.5, \lambda = 0.1, 0.2, \cdots, 1.0$ for illustrating purpose.

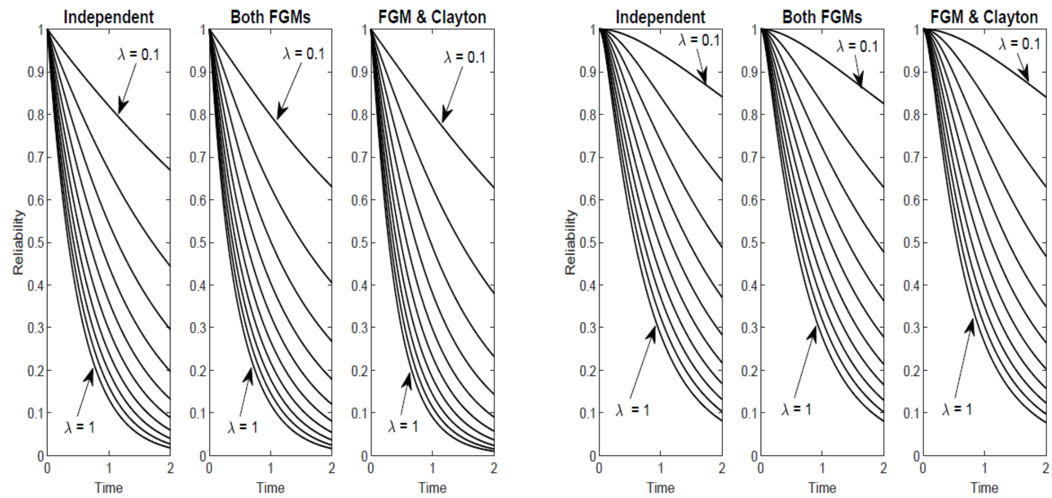
Reliability curves are displayed in Figures 3 for varying α with fixed λ and in Figure 4 for varying λ with fixed α, respectively. The main findings for both hybrid systems are: First, apparently, the reliability increases as α increases with fixed λ, while the reliability decreases as λ increases with fixed α. Secondly, for any pair of (α, λ): (i) as expected, the highest reliability is for the system with independent components, lower for the dependent components with both modeled by FGM copulas, and the lowest with the parallel modeled by FGM and the series modeled by Clayton copula. (ii) the higher reliability is for the parallel-series than the series-parallel system. Thirdly, it seems that there are larger changes of reliability curves with λ changing than with α changing. Lastly, among the three dependent structures, larger differences of reliability occur in the series-parallel while smaller do in the parallel-series system.

The MTTF curves for the hybrid systems with the two settings of (α, λ) above are shown in Figure 5 and 6, respectively. The main features are summarized as follows: (i) For both hybrid systems, the MTTF increases as α increases with fixed λ, while the MTTF decreases as λ increases with fixed α (this is in accordance with the fact that the MTTF is proportional to λ^{-1} in all cases). It seems, however, that the MTTF decreases much faster as λ increases than that MTTF increases as α increases. (ii) Similar to the situation for reliability, the MTTFs of the hybrid systems are highest for independent components, lower for dependent components with FGM copula modeling both systems, and lowest for dependent components with different copula modeling parallel systems by FGM copula and series by Clayton copula. (iii) The MTTFs for the series-parallel system (shown in Figure 5(a)) are lower than the ones for the parallel-series system (shown in Figure 5(b)) at any α value with fixed λ in either independent or dependent structure of components. The same situation is exhibited at various values of λ with fixed α as shown in Figures 6. (iv) Larger differences of MTTFs can be discerned for the series-parallel system across three dependent structures, and less for the parallel-series system. These are consistent with the exhibitions of the reliability curves in Figures 3 and 4.



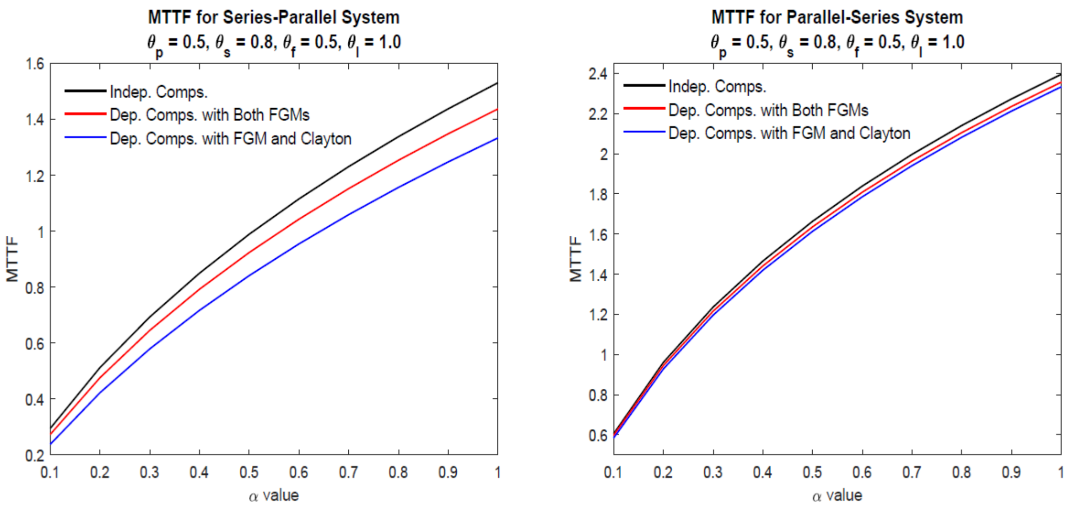
(a) Reliability of Series-Parallel System with $\lambda = 0.5$ (b) Reliability of Parallel-Series System with $\lambda = 0.5$

Figure 3: Comparisons of Reliability in Hybrid Systems



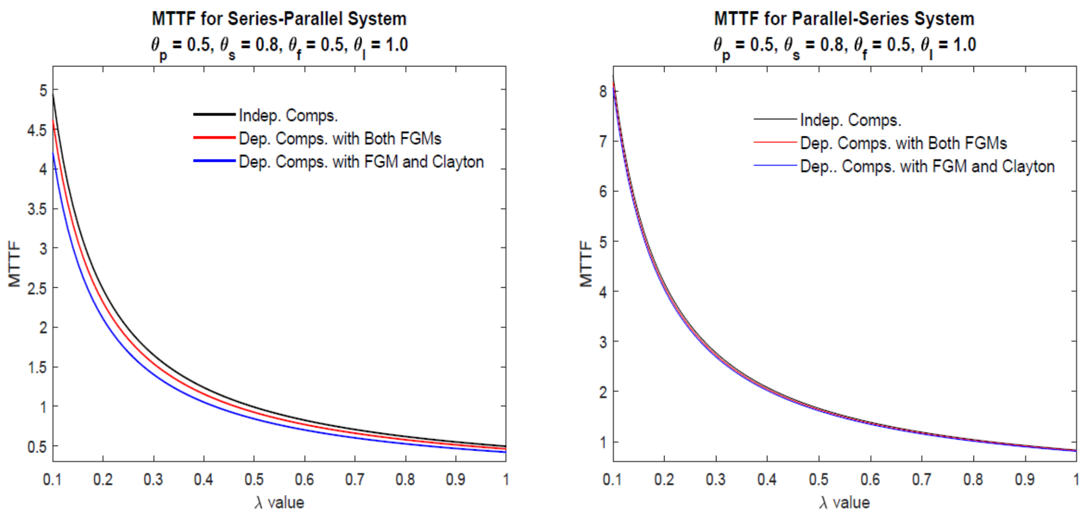
(a) Reliability of Series-Parallel System with $\alpha = 0.5$ (b) Reliability of Parallel-Series System with $\alpha = 0.5$

Figure 4: Comparisons of Reliability in Hybrid Systems



(a) MTTF of Series-Parallel System (b) MTTF of Parallel-Series System

Figure 5: Comparisons of MTTF for Hybrid Systems with $\lambda = 0.5$



(a) MTTF of Series-Parallel System (b) MTTF of Parallel-Series System

Figure 6: Comparisons of MTTF for Hybrid Systems with $\alpha = 0.5$

To further investigate the effect of α and λ values on the system's MTTF under three dependent structures, we display the change rates of MTTF in Tables 1 & 2, where we notice that, for both hybrid systems, the rates are obviously higher for the case of independent components than the ones in the dependent settings in which their changing rates are similar. Furthermore, the change rates are larger in parallel-series system than these in series-parallel system for all cases of dependent structure. These findings are consistent with the curvatures of MTTF displayed in Figures 5 & 6.

Table 1: MTTF Increasing Rate for α with $\lambda = 0.5$

α	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
				Series-Parallel System					
Independent	2.18	1.81	1.57	1.39	1.26	1.16	1.07	0.99	0.93
Dep. Both FGMs	2.03	1.69	1.47	1.31	1.19	1.09	1.01	0.95	0.89
Dep. FGM & Clayton	1.84	1.57	1.38	1.24	1.13	1.05	0.97	0.91	0.86
				Parallel-Parallel System					
Independent	3.55	2.75	2.29	1.98	1.75	1.58	1.44	1.32	1.23
Dep. Both FGMs	3.49	2.71	2.25	1.95	1.73	1.55	1.42	1.30	1.21
Dep. FGM & Clayton	3.44	2.68	2.24	1.94	1.72	1.55	1.41	1.30	1.21

Table 2: MTTF Decreasing Rate for λ with $\alpha = 0.5$

λ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
				Series-Parallel System					
Independent	24.73	8.24	4.12	2.47	1.65	1.18	0.88	0.69	0.55
Dep. Both FGMs	22.11	7.37	3.68	2.21	1.47	1.06	0.79	0.61	0.49
Dep. FGM & Clayton	21.03	7.01	3.50	2.10	1.40	1.00	0.75	0.58	0.47
				Parallel-Parallel System					
Independent	41.59	13.86	6.93	4.16	2.77	1.98	1.49	1.16	0.92
Dep. Both FGMs	40.38	13.46	6.73	4.04	2.69	1.92	1.44	1.12	0.90
Dep. FGM & Clayton	40.34	13.45	6.72	4.03	2.69	1.92	1.44	1.12	0.90

5 Conclusions

In this paper, we have studied reliability analysis of two fundamental hybrid systems: series-parallel and parallel-series, where the lifetimes of dependent components modeled by copula functions. The dependence in parallel and series structures were described by FGM and Clayton copulas to reflect different degree of association, respectively. With the flexible extended exponential lifetime distribution for the components in the system, we obtained the analytic forms of the reliability and mean time to failure (MTTF) of the hybrid systems. Under various parameter settings in the lifetime distribution, the illustrative examples demonstrated that there are higher reliability and longer MTTF for independent components, lower and shorter for FGM dependence in both series and parallel systems, and lowest and shortest for FGM dependence in parallel and Clayton in series. Lastly, in all cases considered, we observed that there are higher reliability and longer MTTF for the parallel-series than the series-parallel system.

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