Type II Power Topp-Leone Daggum Distribution With Application In Reliability

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Abstract

In this paper, we introduce a new continuous probability distribution named as type II power Topp-Leone Dagum distribution using the type II power Topp-Leone generated family studied by Rashad et al., [17]. We have obtained some reliability measures like reliability function, hazard rate function, reversed hazard rate function, mean waiting time, mean past life time, mean deviation, second failure rate function and mean residual life function. We have derived some statistical properties of the new probability distribution including mean, variance, moments, moment generating function, characteristics function, cumulant generating function, incomplete moments, inverted moments, central moments, conditional moments, probability weighted moments and order statistics. For the probability proposed new probability distribution. we have obtained some income inequality measures like Lorenz curve, Bonferroni index, Zenga index and Generalized entropy. The maximum likelihood estimation method is used to estimate the parameters of the probability distribution. Finally, the proposed generalized model is applied to life time data sets to evaluate the model performance.

Keywords: Dagum distribution, Reliability function, Hazard rate function, Generalized entropy, Lorenz curve, Maximum likelihood method.

I. Introduction

The life time distributions play a vital role in several research areas such as biological sciences, medical sciences, environmental sciences, actuarial science, engineering, finance and among others. The popular classical probability distributions do not provide greater flexibility for life time data set, because the classical distribution have one or two parameters only. In this situation, generalized family distribution commonly played a vital role many statistical research areas. The main advantage of generalized family is obtained by adding one more parameters through the classical probability distribution which gives more flexibility for generating a new probability distribution. In this current scenario generating family of probability distributions is attractive to many statisticians. The generating family of distributions have been investigated by many authors. Here, we list some generating family like Marshall-Olkin-G (MO-G) family introduced by Marshall and Olkin [16], Exponentiated-G (E-G) family introduced by Gupta et al., [13], Quadratic rank transmuted-G (QRTM-G) family introduced by Shaw and Buckley [18], gamma-G (G-G) family introduced by Zografos and Balakrishnan [21], Kumaraswamy-G (Kw-G) family introduced by Cordeiro and de Castro [6], Topp-Leone-G (TL-G) family introduced by Ali Al-Shomrani [3], Exponentiated extended-G (EE-G) family introduced by Elgarhy et al., [12] and odd Dagum-G (OD-G) family introduced by Afify and Alizadeh [1].

Camilo Dagum introduced a Dagum distribution in 1977 for closely fitting empirical income and wealth data. The Dagum distribution is classified into two types named type I specification (type I Dagum) and type II specification (type II Dagum), where type I specification deals with three parameters while type two specification deals with four parameters. This Dagum distribution has been extensively used in different areas like income and wealth data, meteorological data, reliability and survival analysis. The Dagum distribution is alternative to heavy tailed distributions such as generalized beta, Pareto and lognormal. The Dagum distribution is also known as the inverse Burr XII distribution, especially in the actuarial literature. Domma [9] studied characteristic of Dagum distribution that its hazard function can be monotonically decreasing, an upside-down bathtub, or bathtub. This behavior attracted many of authors to study the model in various fields. In fact Domma, et al., [10, 11] studied Dagum distribution with a reliability point of view and used to analyze survival data. Kleiber and Kotz [14] and Kleiber [15] provided an exhaustive review on the origin of the Dagum distribution and its applications. Recently, Domma et al.,[10] studied about Dagum distribution for estimated parameters with censored samples. We have focused the type I Dagum distribution in this research paper.

The probability density function (pdf) and cumulative distribution function (cdf) of Dagum distribution are given respectively by

$$f(x;\sigma,\theta,\beta) = \sigma\theta\beta\, x^{-\theta-1}\left(1+\sigma x^{-\theta}\right)^{-\beta-1} with\ x > 0, \sigma > 0, \theta > 0\ and\ \beta > 0. \tag{1}$$

and

$$F(x;\sigma,\theta,\beta) = \left(1+\sigma x^{-\theta}\right)^{-\beta} with\ x > 0, \sigma > 0, \theta > 0\ and\ \beta > 0. \tag{2}$$

where σ is scale parameter, θ and β are shape parameters. It is noted that if σ=1 the Dagum distribution becomes Burr III distribution and if θ=1, the Dagum distribution becomes Log-Logistic or Fisk distribution.

In this paper, we introduce a new generalization of type II power Topp-Leone Dagum distribution using the type II power Topp-Leone generated family studied by Rashad et al., [17]. This generated family introduces two new additional parameters and provides flexibility.

The contents of this paper are organized as follows: In Section 2, adopt type II power Topp-Leone family proposed new generating probability distribution. In Section 3, we discuss some reliability measures like reliability function, hazard rate function, reversed hazard rate function, cumulative hazard function, second failure rate function, mean waiting time, mean residual life function, mean past life time and average deviation. We have derived some statistical properties of new probability distribution such as moments, moment generating function, characteristic function, cumulant generating function, inverted r^{th} moments, central moments, conditional moments, probability weighted moments, order statistics are given in Section 3. In Section 4, some income inequality measures like Lorenz and Bonferroni curve, Zenga index and Generalized entropy are presented. In Section 5 estimation of the parameters of the type II power Topp-Leone Dagum distribution is consider maximum likelihood estimation method. The real life time data set is used for fitting type II power Topp-Leone Dagum distribution. The results are given in Section 6. Finally, we conclude the article in Section 7.

II. Type II Power Topp-Leone Family

The type II power Topp-Leone family is introduced by Rashad et al., [17]. The probability density function (pdf) and cumulative distribution function (cdf) of type II power Topp-Leone family of distribution are respectively defined by

$$f(x;\alpha,\tau,\xi) = 2\alpha\tau g(x;\xi)\left[1-G(x;\xi)\right]^{\alpha\tau-1}\left[2-\left[1-G(x;\xi)\right]^{\tau}\right]^{\alpha-1}\left[1-\left[1-G(x;\xi)\right]^{\tau}\right], x\epsilon R \tag{3}$$

and

$$F(x;\alpha,\tau,\xi) = 1-\left[1-G(x;\xi)\right]^{\alpha\tau}\left[2-\left[1-G(x;\xi)\right]^{\tau}\right]^{\alpha}, x\epsilon R \tag{4}$$

where α>0, $\tau > 0$, $g(x;\xi)$ and $G(x;\xi)$ are probability density function and cumulative distribution function of any baseline distribution with parameter vector ξ . The type II power Topp-Leone family of distributions is the generalization of the type II Topp-Leone-G family. It is very important note that for TIIPTL-G, if $\tau = 0$ the type II power Topp-Leone family becomes a type II Topp-Leone family of distribution. Some of motivations behind the type II power Topp-Leone family of distribution are to create different types of shapes for probability density function and hazard rate function to increase the flexibility for generating of type II power Topp-Leone distributions, skewed distribution transformed from the symmetrical distribution, build heavy tailed distribution and type II power Topp-Leone family provide better fits compare than other general families of distribution with baseline distribution.

I. Type II Power Topp-Leone Dagum Distribution

A random variable X is said to have type II power Topp-Leone Dagum distribution if the probability density function and cumulative distribution function are respectively is given by

$$f(x;\alpha,\tau,\sigma,\theta,\beta) = 2\alpha\tau\sigma\theta\beta\, x^{-\theta-1}\left(1+\sigma x^{-\theta}\right)^{-\beta-1}\left[1-(1+\sigma x^{-\theta})^{-\beta}\right]^{\alpha\tau-1} \times \left[2-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\tau}\right]^{\alpha-1}\left[1-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\tau}\right], x\epsilon R \quad (5)$$

where, $\sigma > 0, \theta > 0, \beta > 0, \alpha > 0$ *and* $\tau > 0.$

Note that, $(x-y)^r = \sum_{p=0}^{\infty}\binom{r}{p}(-1)^p x^{r-p} y^p.$

This binomial expansion is used to simply the probability density function of type II power Topp-Leone Dagum distribution. After some simplifications we get pdf for type II power Topp-Leone Dagum distribution and is given by

$$f(x;\alpha,\tau,\sigma,\theta,\beta) = 2\alpha\tau\sigma\theta\beta\, x^{-\theta-1}\left(1+\sigma x^{-\theta}\right)^{-\beta(p+s-v+1)-1} \quad (6)$$

where, $\psi = \binom{\alpha\tau-1}{p}\binom{\alpha-1}{q}\binom{\tau q}{s}\binom{1}{t}\binom{\tau t}{v}(-1)^{p+q+s+t+v}(2)^{\alpha-1-q}$

and the cdf is given by

$$F(x;\alpha,\tau,\sigma,\theta,\beta) = 1-\left[1-(1+\sigma x^{-\theta})^{-\beta}\right]^{\alpha\tau}\left[2-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\tau}\right]^{\alpha}, x\epsilon R \quad (7)$$

where α, τ are parameters of type II power Topp-Leone family, σ is scale parameter of Dagum distribution and θ, β are shape parameters of Dagum distribution. The following figures 1 to 4 shows the shape of pdf and cdf for different values of the parameters of type II power Topp-Leone Dagum distribution.

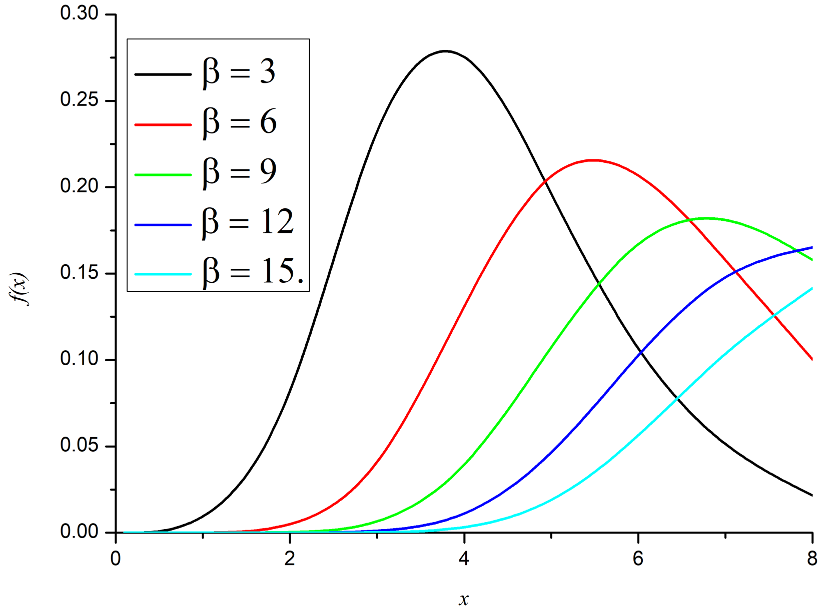

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Figure 1: *Pdfs of type II power Topp-Leone Dagum distribution for fixed value of* $\alpha = 4, \tau = 1, \sigma = 6, \theta = 2$ *and different values of* β.

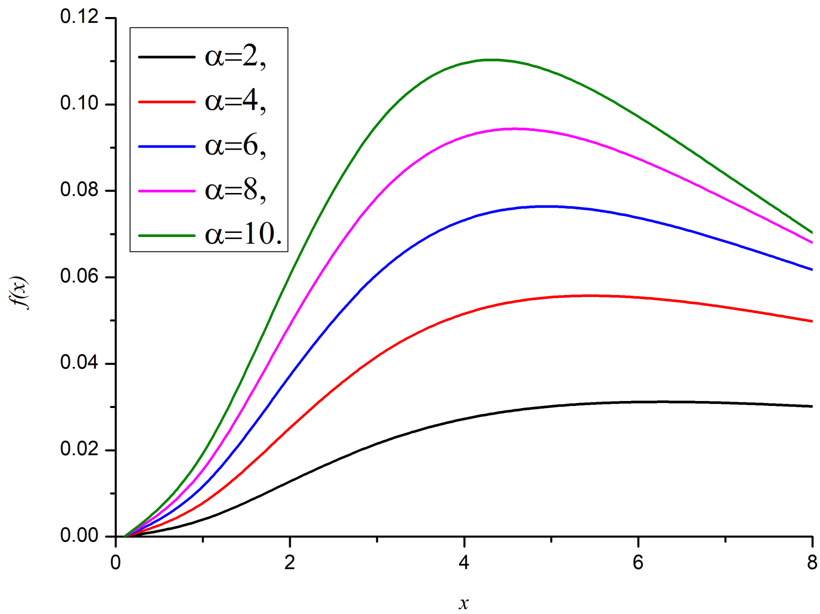

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Figure 2: *Pdfs of type II power Topp-Leone Dagum distribution for fixed value of* $\tau = 0.5, \sigma = 2, \theta = 1, \beta = 4$ *and different values of* α.

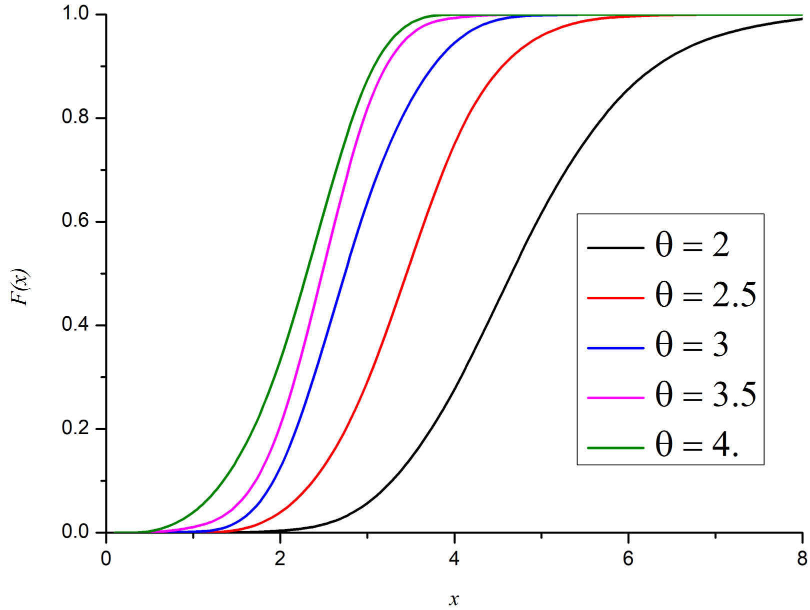

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Figure 3: *Cdfs of type II power Topp-Leone Dagum distribution for fixed value of $\alpha = 2, \tau = 4, \sigma = 6, \beta = 7$ and different values of θ.*

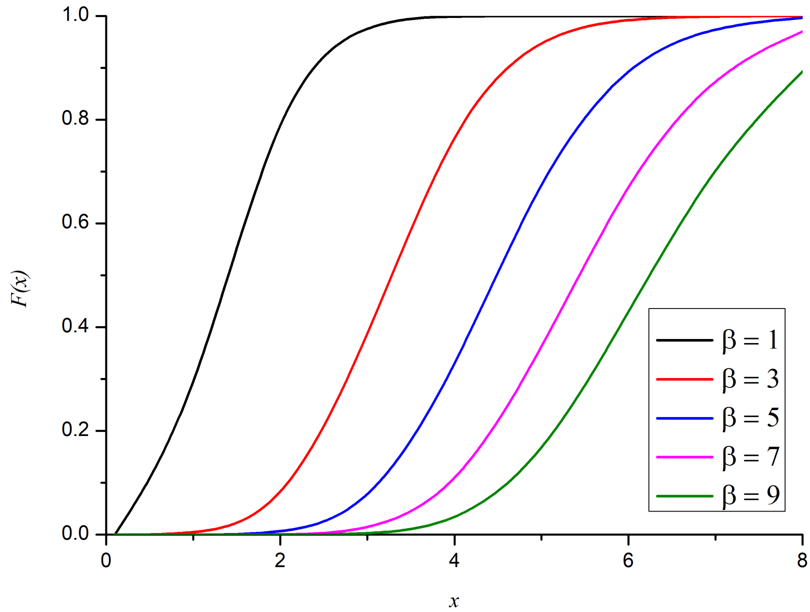

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Figure 4: *Cdfs of type II power Topp-Leone Dagum distribution for fixed value of $\alpha = 4, \tau = 2.5, \sigma = 8, \theta = 2$ and different values of β.*

III. Reliability Measures

I. Reliability function

The reliability function of type II power Topp-Leone Dagum distribution is given by

$$R(x) = 1 - \left[1 - \left(1 - (1+\sigma x^{-\theta})^{-\beta}\right)^{\alpha\tau} \left(2 - (1 - (1+\sigma x^{-\theta})^{-\beta})^{\tau}\right)^{\alpha}\right] \tag{8}$$

II. Hazard rate function

The hazard rate function associated with type II power Topp-Leone Dagum distribution is given by

$$h(x) = \frac{\delta}{\eta} \tag{9}$$

where

$$\begin{aligned}\delta =& 2\alpha\tau\sigma\theta\beta\, x^{-\theta-1}\left(1+\sigma x^{-\theta}\right)^{-\beta-1} \left[1 - (1+\sigma x^{-\theta})^{-\beta}\right]^{\alpha\tau-1} \left[2 - \left(1 - (1+\sigma x^{-\theta})^{-\beta}\right)^{\tau}\right]^{\alpha-1} \\ &\times \left[1 - (1 - (1+\sigma x^{-\theta})^{-\beta})^{\tau}\right]\end{aligned}$$

$$\eta = 1 - \left[1 - \left(1 - (1+\sigma x^{-\theta})^{-\beta}\right)^{\alpha\tau} \left(2 - (1 - (1+\sigma x^{-\theta})^{-\beta})^{\tau}\right)^{\alpha}\right]$$

III. Reversed hazard rate function

The reversed hazard rate function of type II power Topp-Leone Dagum distribution is given by

$$r(x) = \frac{\delta}{\gamma} \tag{10}$$

where

$$\begin{aligned}\delta =& 2\alpha\tau\sigma\theta\beta\, x^{-\theta-1}\left(1+\sigma x^{-\theta}\right)^{-\beta-1} \left[1 - (1+\sigma x^{-\theta})^{-\beta}\right]^{\alpha\tau-1} \left[2 - \left(1 - (1+\sigma x^{-\theta})^{-\beta}\right)^{\tau}\right]^{\alpha-1} \\ &\times \left[1 - (1 - (1+\sigma x^{-\theta})^{-\beta})^{\tau}\right]\end{aligned}$$

$$\gamma = 1 - \left(1 - (1+\sigma x^{-\theta})^{-\beta}\right)^{\alpha\tau} \left(2 - (1 - (1+\sigma x^{-\theta})^{-\beta})^{\tau}\right)^{\alpha}$$

IV. Cumulative hazard function

The cumulative hazard function of type II power Topp-Leone Dagum distribution is given by

$$H(x) = -log\left[1 - \left(1 - (1+\sigma x^{-\theta})^{-\beta}\right)^{\alpha\tau} \left(2 - (1 - (1+\sigma x^{-\theta})^{-\beta})^{\tau}\right)^{\alpha}\right] \tag{11}$$

V. Second failure rate function

The second failure rate function of type II power Topp-Leone Dagum distribution is given by

$$h(x) = log\left[\frac{1 - \left[1 - (1 - (1+\sigma x^{-\theta})^{-\beta})^{\alpha\tau} \left(2 - (1 - (1+\sigma x^{-\theta})^{-\beta})^{\tau}\right)^{\alpha}\right]}{1 - \left[1 - (1 - (1+\sigma (x+1)^{-\theta})^{-\beta})^{\alpha\tau}\right] \left(2 - (1 - (1+\sigma (x+1)^{-\theta})^{-\beta})^{\tau}\right)^{\alpha}}\right] \tag{12}$$

VI. Mean waiting time

The mean waiting time is defined by

$$\varphi(x) = x - \left[\frac{1}{F(x)}\int_0^x xf(x)dx\right] \tag{13}$$

$$\varphi(x) = x - \left[\frac{1}{F(x)}\int_0^x x\left[\sum_{p,q,s,t,v=0}^{\infty}\psi 2\alpha\tau\sigma\theta\beta\, x^{-\theta-1}(1+\sigma x^{-\theta})^{-\beta(p+s-v+1)-1}\right]dx - x\right]$$

The mean waiting time of type II power Topp-Leone Dagum distribution is given by

$$\varphi(x) = x - \left[\frac{\sum_{p,q,s,t,v=0}^{\infty}\psi 2\alpha\tau\beta\sigma^{\frac{1}{\theta}} B\left(1-\frac{1}{\theta},\beta(p+s-v+1)+\frac{1}{\theta};y\right)}{1-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\alpha\tau}\left(2-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\tau}\right)^{\alpha}}\right] \tag{14}$$

VII. Mean residual life function

The mean residual life function plays a very important role in reliability and survival analysis. The mean residual life function of a life time random variable X is given by

$$\phi(x) = \frac{1}{s(x)}\int_x^{\infty} xf(x)dx - x \tag{15}$$

$$\phi(x) = \frac{1}{1-\left[1-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\alpha\tau}\left(2-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\tau}\right)^{\alpha}\right]} \times \int_0^{\infty} x\left[\sum_{p,q,s,t,v=0}^{\infty}\psi 2\alpha\tau\sigma\theta\beta\, x^{-\theta-1}(1+\sigma x^{-\theta})^{-\beta(p+s-v+1)-1}\right]dx - x$$

The mean residual life function type II power Topp-Leone Dagum distribution is given by

$$\phi(x) = \frac{\sum_{p,q,s,t,v=0}^{\infty}\psi 2\alpha\tau\beta\, B\left(1-\frac{1}{\theta},\beta(p+s-v+1)+\frac{1}{\theta}\right)}{1-\left[1-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\alpha\tau}\left(2-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\tau}\right)^{\alpha}\right]} - x \tag{16}$$

VIII. Mean past lifetime

The mean past lifetime of the component can be defined by

$$K(x) = E\left[x - X|X \le x\right] = \frac{\int_0^x F(t)dt}{F(x)} = x - \frac{\int_0^x tf(t)dt}{F(x)} \tag{17}$$

$$K(x) = x - \frac{\int_0^x t\left[\sum_{p,q,s,t,v=0}^{\infty}\psi 2\alpha\tau\sigma\theta\beta\, t^{-\theta-1}(1+\sigma t^{-\theta})^{-\beta(p+s-v+1)-1}\right]dt}{\left[1-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\alpha\tau}\left(2-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\tau}\right)^{\alpha}\right]}$$

The mean past life time of type II power Topp-Leone Dagum distribution is given by

$$K(x) = x - \frac{\sum_{p,q,s,t,v=0}^{\infty}\psi 2\alpha\tau\beta\, B\left(1-\frac{1}{\theta},\beta(p+s-v+1)+\frac{1}{\theta};y\right)}{\left[1-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\alpha\tau}\left(2-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\tau}\right)^{\alpha}\right]} \tag{18}$$

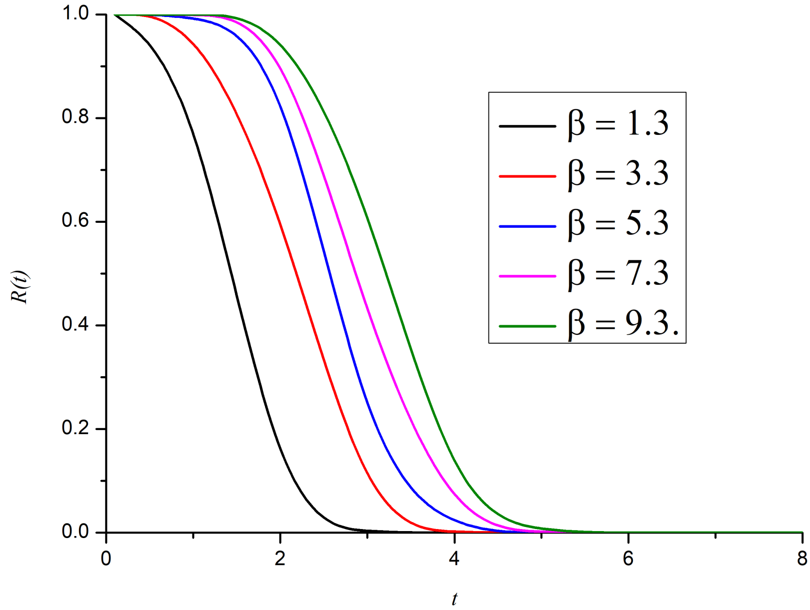

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Figure 5: *Reliability function of type II power Topp-Leone Dagum distribution for fixed value of $\alpha = 4.3, \tau = 2.2, \sigma = 6, \theta = 3$ and different values of β.*

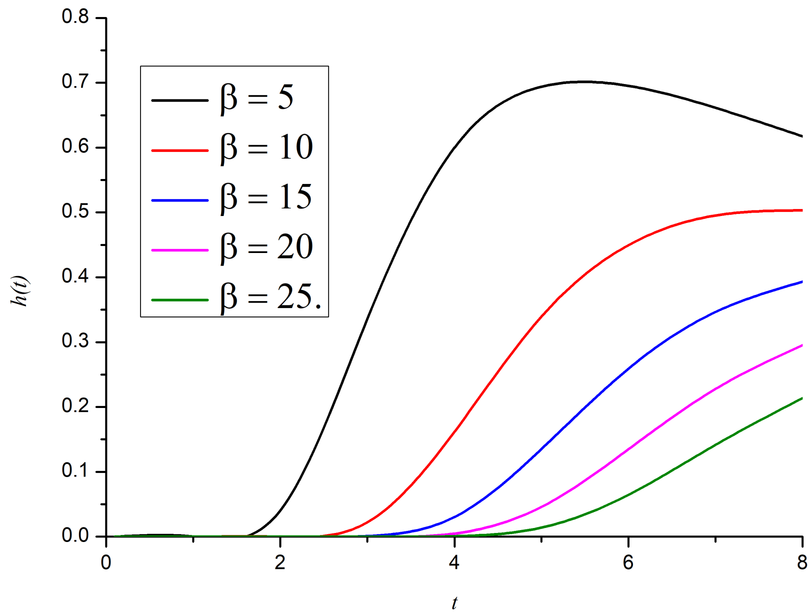

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Figure 6: *Hazard rate function of type II power Topp-Leone Dagum distribution for fixed value of $\alpha = 1, \tau = 3, \sigma = 4, \theta = 2$ and different values of β.*

IX. Mean deviation

The mean deviation is defined as

$$\pi(x) = 2\{\mu F(\mu) - \int_0^{\mu} x f(x)dx\} \tag{19}$$

$$\pi(x) = 2\{\mu F(\mu) - \int_0^{\mu} x \left[\sum_{p,q,s,t,v=0}^{\infty} \psi 2\alpha\tau\sigma\theta\beta\, x^{-\theta-1}(1+\sigma x^{-\theta})^{-\beta(p+s-v+1)} \right] dx\}$$

The mean deviation of type II power Topp-Leone Dagum distribution is given by

$$\pi(x) = 2\{\mu F(\mu) - \sum_{p,q,s,t,v=0}^{\infty} \psi 2\alpha\tau\beta\sigma^{\frac{1}{\theta}}\ B\left(1-\frac{1}{\theta}, \beta(p+s-v+1)+\frac{1}{\theta}\right)\} \tag{20}$$

IV. Statistical Properties

I. Moments

The r^{th} moment about the mean of a random variable X is given by

$$\mu_r^{'} = \int_{-\infty}^{\infty} x^r f(x)dx, for\ X\ is\ continuous. \tag{21}$$

$$\mu_r^{'} = \int_0^{\infty} x^r \left[\sum_{p,q,s,t,v=0}^{\infty} \psi 2\alpha\tau\sigma\theta\beta\, x^{-\theta-1}(1+\sigma x^{-\theta})^{-\beta(p+s-v+1)-1} \right] dx$$

$$= \sum_{p,q,s,t,v=0}^{\infty} \psi 2\alpha\tau\beta\sigma^{\frac{r}{\theta}} \int_0^{\infty} \frac{u^{-\frac{r}{\theta}}}{(1+u)^{\beta(p+s-v+1)+1}} du$$

The r^{th} moment of type II power Topp-Leone Dagum distribution is given by

$$\mu_r^{'} = \sum_{p,q,s,t,v=0}^{\infty} \psi 2\alpha\tau\beta\sigma^{\frac{r}{\theta}}\ B\left(1-\frac{r}{\theta}, \beta(p+s-v+1)+\frac{r}{\theta}\right). \ where\ r=1,2,3.., \tag{22}$$

In particular

$$E(X) = \sum_{p,q,s,t,v=0}^{\infty} \psi 2\alpha\tau\beta\sigma^{\frac{1}{\theta}}\ B\left(1-\frac{1}{\theta}, \beta(p+s-v+1)+\frac{1}{\theta}\right) \tag{23}$$

$$E(X^2) = \sum_{p,q,s,t,v=0}^{\infty} \psi 2\alpha\tau\beta\sigma^{\frac{2}{\theta}}\ B\left(1-\frac{2}{\theta}, \beta(p+s-v+1)+\frac{2}{\theta}\right)$$

$$E(X^3) = \sum_{p,q,s,t,v=0}^{\infty} \psi 2\alpha\tau\beta\sigma^{\frac{3}{\theta}}\ B\left(1-\frac{3}{\theta}, \beta(p+s-v+1)+\frac{3}{\theta}\right)$$

$$E(X^4) = \sum_{p,q,s,t,v=0}^{\infty} \psi 2\alpha\tau\beta\sigma^{\frac{4}{\theta}}\ B\left(1-\frac{4}{\theta}, \beta(p+s-v+1)+\frac{4}{\theta}\right)$$

The variance is given by

$$V(x) = \left[\sum_{p,q,s,t,v=0}^{\infty} \psi 2\alpha\tau\beta\sigma^{\frac{2}{\theta}}\ B\left(1-\frac{2}{\theta}, \beta(p+s-v+1)+\frac{2}{\theta}\right) \right] - \left[\sum_{p,q,s,t,v=0}^{\infty} \psi 2\alpha\tau\beta\sigma^{\frac{1}{\theta}}\ B\left(1-\frac{1}{\theta}, \beta(p+s-v+1)+\frac{1}{\theta}\right) \right]^2 \tag{24}$$

II. Moment generating function

The moment generating function of the random variable X is defined by

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x)dx, \text{ where } e^{tx} = \sum_{r=0}^{\infty} \frac{(tx)^r}{r!}$$

The moment generating function of type II power Topp-Leone Dagum distribution is given by

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \left[\sum_{p,q,s,t,v=0}^{\infty} \psi 2\alpha\tau\beta\sigma^{\frac{r}{\theta}} \; B\left(1 - \frac{r}{\theta}, \beta(p+s-v+1) + \frac{r}{\theta}\right) \right] \tag{25}$$

III. Characteristic function

The characteristic function of the random variable X is defined by

$$\Phi_X(t) = \int_{-\infty}^{\infty} e^{itx} f(x)dx, \text{ where } e^{itx} = \sum_{r=0}^{\infty} \frac{(itx)^r}{r!}; i^2 = -1$$

The characteristic function of type II power Topp-Leone Dagum distribution is given by

$$\Phi_X(t) = \sum_{r=0}^{\infty} \frac{(it)^r}{r!} \left[\sum_{p,q,s,t,v=0}^{\infty} \psi 2\alpha\tau\beta\sigma^{\frac{r}{\theta}} \; B\left(1 - \frac{r}{\theta}, \beta(p+s-v+1) + \frac{r}{\theta}\right) \right] \tag{26}$$

IV. Cumulant generating function

Cumulant generating function is defined by

$$K_X(t) = log\ M_X(t)$$

The cumulant generating function of type II power Topp-Leone Dagum distribution is given by

$$K_X(t) = log \left[\sum_{r=0}^{\infty} \frac{t^r}{r!} \left[\sum_{p,q,s,t,v=0}^{\infty} \psi 2\alpha\tau\beta\sigma^{\frac{r}{\theta}} B\left(1 - \frac{r}{\theta}, \beta(p+s-v+1) + \frac{r}{\theta}\right) \right] \right] \tag{27}$$

V. Incomplete r^{th} moment

Incomplete r^{th} moment is defined by

$$m_r(x) = \int_0^x x^r f(x)dx \tag{28}$$

$$m_r(x) = \int_0^x x^r \left[\sum_{p,q,s,t,v=0}^{\infty} \psi 2\alpha\tau\sigma\theta\beta \; x^{-\theta-1}(1+\sigma x^{-\theta})^{-\beta(p+s-v+1)-1} \right] dx$$

The incomplete r^{th} moment of type II power Topp-Leone Dagum distribution is given by

$$m_r(x) = \sum_{p,q,s,t,v=0}^{\infty} \psi 2\alpha\tau\beta\sigma^{\frac{r}{\theta}} B\left(1 - \frac{r}{\theta}, \beta(p+s-v+1) + \frac{r}{\theta}; y\right) \tag{29}$$

VI. Inverted moments

The r^{th} inverted moment is defined by

$$\mu_r^* = \int_{-\infty}^{\infty} x^{-r} f(x) dx \tag{30}$$

$$\mu_r^* = \int_0^{\infty} x^{-r} \left[\sum_{p,q,s,t,v=0}^{\infty} \psi 2\alpha\tau\sigma\theta\beta \; x^{-\theta-1}(1+\sigma x^{-\theta})^{-\beta(p+s-v+1)-1} \right] dx$$

The inverted r^{th} moment of type II power Topp-Leone Dagum distribution is given by

$$\mu_r^* = \sum_{p,q,s,t,v=0}^{\infty} \psi 2\alpha\tau\beta\sigma^{-\frac{r}{\theta}} B\left(1+\frac{r}{\theta}, \beta(p+s-v+1)-\frac{r}{\theta}\right) \tag{31}$$

The r^{th} inverted moment used to find harmonic mean. The harmonic mean of type II power Topp-Leone Dagum distribution is given by

$$\frac{1}{\mu_r^*} = \frac{1}{\left[\sum_{p,q,s,t,v=0}^{\infty} \psi 2\alpha\tau\beta\sigma^{-\frac{r}{\theta}} B\left(1+\frac{r}{\theta}, \beta(p+s-v+1)-\frac{r}{\theta}\right)\right]} \tag{32}$$

VII. Central moments

The r^{th} central moment is defined by

$$\mu_r = \int_{-\infty}^{\infty} (x-\mu_1^{'})^r f(x) dx = \sum_{m=0}^{r} \binom{r}{m} (-1)^m (\mu_1^{'})^m \mu_{r-m}^{'} \tag{33}$$

The r^{th} central moment of type II power Topp-Leone Dagum distribution is given by

$$\begin{aligned} \mu_r = \sum_{m=0}^{r} \binom{r}{m} (-1)^m \times \left[\sum_{p,q,s,t,v=0}^{\infty} \psi 2\alpha\tau\beta\sigma^{\frac{1}{\theta}} B\left(1-\frac{1}{\theta}, \beta(p+s-v+1)+\frac{1}{\theta}\right)\right]^m \\ \times \left[\sum_{p,q,s,t,v=0}^{\infty} \psi 2\alpha\tau\beta\sigma^{\frac{r-m}{\theta}} B\left(1-\frac{r-m}{\theta}, \beta(p+s-v+1)+\frac{r-m}{\theta}\right)\right] \end{aligned} \tag{34}$$

VIII. Conditional moments

The n^{th} conditional moment is defined by

$$E\big(X^n | X > x\big) = \frac{1}{S(x)} \int_x^{\infty} x^n f(x) dx \tag{35}$$

$$E\big(X^n | X > x\big) = \frac{1}{S(x)} \int_x^{\infty} x^n \left[\sum_{p,q,s,t,v=0}^{\infty} \psi 2\alpha\tau\sigma\theta\beta \; x^{-\theta-1}(1+\sigma x^{-\theta})^{-\beta(p+s-v+1)-1} \right] dx$$

where $S(x) = 1 - R(x).$

The n^{th} conditional moment of type II power Topp-Leone Dagum distribution is given by

$$E\big(X^n | X > x\big) = \frac{\sum_{p,q,s,t,v=0}^{\infty} \psi 2\alpha\tau\beta\sigma^{\frac{n}{\theta}} B\left(1-\frac{n}{\theta}, \beta(p+s-v+1)+\frac{n}{\theta}\right)}{1-\left[1-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\alpha\tau}\left(2-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\tau}\right)^{\alpha}\right]}. \; where \;\; n = 1,2,3.., \tag{36}$$

In particular

$$E(X|X>x)=\frac{\sum_{p,q,s,t,v=0}^{\infty}\psi 2\alpha\tau\beta\sigma^{\frac{1}{\theta}}B\left(1-\frac{1}{\theta},\beta(p+s-v+1)+\frac{1}{\theta}\right)}{1-\left[1-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\alpha\tau}\left(2-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\tau}\right)^{\alpha}\right]}$$

$$E(X^2|X>x)=\frac{\sum_{p,q,s,t,v=0}^{\infty}\psi 2\alpha\tau\beta\sigma^{\frac{2}{\theta}}B\left(1-\frac{2}{\theta},\beta(p+s-v+1)+\frac{2}{\theta}\right)}{1-\left[1-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\alpha\tau}\left(2-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\tau}\right)^{\alpha}\right]}$$

$$E(X^3|X>x)=\frac{\sum_{p,q,s,t,v=0}^{\infty}\psi 2\alpha\tau\beta\sigma^{\frac{3}{\theta}}B\left(1-\frac{3}{\theta},\beta(p+s-v+1)+\frac{3}{\theta}\right)}{1-\left[1-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\alpha\tau}\left(2-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\tau}\right)^{\alpha}\right]}$$

$$E(X^4|X>x)=\frac{\sum_{p,q,s,t,v=0}^{\infty}\psi 2\alpha\tau\beta\sigma^{\frac{4}{\theta}}B\left(1-\frac{4}{\theta},\beta(p+s-v+1)+\frac{4}{\theta}\right)}{1-\left[1-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\alpha\tau}\left(2-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\tau}\right)^{\alpha}\right]}$$

IX. Probability weighted moment

The probability weighted moment of the random variable X is defined by

$$\tau_{r,h}=E\left[X^r F(x)^h\right]=\int_{-\infty}^{\infty}x^r f(x)F(x)^h dx \tag{37}$$

$$\tau_{r,h}=\int_0^{\infty}x^r\left[\sum_{p,q,s,t,v=0}^{\infty}\psi 2\alpha\tau\sigma\theta\beta\, x^{-\theta-1}(1+\sigma x^{-\theta})^{-\beta(p+s-v+1)-1}\right]$$
$$\times\left[1-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\alpha\tau}\left(2-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\tau}\right)^{\alpha}\right]^h dx$$

Using the binomial series

$$(x-y)^r=\sum_{a=0}^{\infty}(-1)^a\binom{r}{a}x^{r-a}y^a,\ \ (1-y)^r=\sum_{a=0}^{\infty}\binom{r}{a}(-1)^a y^a$$

We have

$$\left[1-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\alpha\tau}\left(2-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\tau}\right)^{\alpha}\right]^h=$$
$$\sum_{a,b,c,d=0}^{\infty}(2)^{2(\alpha a-c)}(-1)^{a+b+c+d}\binom{h}{a}\binom{\alpha\tau a}{b}\binom{\alpha a}{c}\binom{\tau c}{d}(1+\sigma x^{-\theta})^{-\beta b-\beta d}$$

Therefore, we have

$$\tau_{r,h}=\int_0^{\infty}x^r\left[\sum_{p,q,s,t,v=0}^{\infty}\psi 2\alpha\tau\sigma\theta\beta\, x^{-\theta-1}(1+\sigma x^{-\theta})^{-\beta(p+s-v+1)-1}\right]$$
$$\times\left[\sum_{a,b,c,d=0}^{\infty}(2)^{\alpha h-c}(-1)^{a+b+c+d}\binom{h}{a}\binom{\alpha\tau a}{b}\binom{\alpha a}{c}\binom{\tau c}{d}(1+\sigma x^{-\theta})^{-\beta b-\beta d}\right]dx$$

$$=\sum_{p,q,s,t,v,a,b,c,d=0}^{\infty}\psi 2\alpha\tau\sigma\theta\beta(2)^{2(\alpha a-c)}\binom{h}{a}\binom{\alpha\tau a}{b}\binom{\alpha a}{c}\binom{\tau c}{d}\sigma^{\frac{r}{\theta}}\int_0^{\infty}\frac{u^{-\frac{r}{\theta}}}{(1+u)^{\beta(p+s-v+b+d+1)+1}}du$$

The probability weighted moment of type II power Topp-Leone Dagum distribution is given by

$$\tau_{r,h} = \sum_{p,q,s,t,v,a,b,c,d=0}^{\infty} \psi 2\alpha\tau\beta\sigma^{\frac{r}{\theta}} B\left(1-\frac{r}{\theta}, \beta(p+s-v+b+d+1)+\frac{r}{\theta}\right) \tag{38}$$

$$where \;\; \eta = (2)^{\alpha h - c}(-1)^{a+b+c+d}\binom{h}{a}\binom{\alpha\tau a}{b}\binom{\alpha a}{c}\binom{\tau c}{d}$$

X. Order statistics

The pdf of the j^{th} order statistics for type II power Topp-Leone Dagum distribution $X_{(j)}$ is given by

$$\begin{aligned} f_{X_{(j)}}(x) =& \frac{n!}{(j-1)(n-j)!} \\ & \times \left[2\alpha\tau\sigma\theta\beta\, x^{-\theta-1}\left(1+\sigma x^{-\theta}\right)^{-\beta-1}\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\alpha\tau-1}\right. \\ & \left.\left(2-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\tau}\right)^{\alpha-1}\left(1-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\tau}\right)\right] \\ & \times \left[1-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\alpha\tau}\left(2-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\tau}\right)^{\alpha}\right]^{j-1} \\ & \times \left[1-\left(1-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\alpha\tau}\left(2-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\tau}\right)^{\alpha}\right)\right]^{n-1} \end{aligned} \tag{39}$$

The pdf of the smallest order statistics $X_{(1)}$ is given by

$$\begin{aligned} f_{X_{(1)}}(x) =& n\left[1-\left(1-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\alpha\tau}\left(2-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\tau}\right)^{\alpha}\right)\right]^{n-1} \\ & \times \left[2\alpha\tau\sigma\theta\beta\, x^{-\theta-1}\left(1+\sigma x^{-\theta}\right)^{-\beta-1}\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\alpha\tau-1}\right. \\ & \left.\left(2-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\tau}\right)^{\alpha-1}\left(1-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\tau}\right)\right] \end{aligned} \tag{40}$$

The pdf of the largest order statistics $X_{(n)}$ is given by

$$\begin{aligned} f_{X_{(n)}}(x) =& n\left[\left(1-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\alpha\tau}\left(2-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\tau}\right)^{\alpha}\right)\right]^{n-1} \\ & \times \left[2\alpha\tau\sigma\theta\beta\, x^{-\theta-1}\left(1+\sigma x^{-\theta}\right)^{-\beta-1}\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\alpha\tau-1}\right. \\ & \left.\left(2-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\tau}\right)^{\alpha-1}\left(1-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\tau}\right)\right] \end{aligned} \tag{41}$$

and the pdf of the median order statistics is given by

$$f_{m+1:n}(x) = \frac{(2m+1)}{m!m!}\left[F(x)\right]^m\left[1-F(x)\right]^m f_{(X)}(x)$$

$$\begin{aligned} f_{m+1:n}(x) =& \frac{(2m+1)}{m!m!}\left[1-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\alpha\tau}\left(2-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\tau}\right)^{\alpha}\right]^{m} \\ & \times \left[1-\left(1-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\alpha\tau}\left(2-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\tau}\right)^{\alpha}\right)\right]^{m} \\ & \times \left[2\alpha\tau\sigma\theta\beta\, x^{-\theta-1}\left(1+\sigma x^{-\theta}\right)^{-\beta-1}\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\alpha\tau-1}\right. \\ & \left.\left(2-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\tau}\right)^{\alpha-1}\left(1-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\tau}\right)\right] \end{aligned} \tag{42}$$

The joint distribution of the i^{th} and j^{th} order statistics for $1 \leq i < j \leq n$ is given by

$$f_{i:j;n}(x_i, x_j) = C\,[F(x_i)]^{i-1}\left[F(x_j) - F(x_i)\right]^{j-i-1}[1 - F(x_i)]^{n-j} f(x_i) f(x_j)$$

where

$$C = \frac{n!}{(i-1)!(j-i-1)!(n-j)!}$$

$$\begin{aligned} f_{i:j;n}(x_i, x_j) =& \frac{n!}{(i-1)!(j-i-1)!(n-j)!}\left[1 - W_{(i)}^{\alpha\tau}(2 - W_{(i)}^{\tau})^{\alpha}\right]^{i-1} \\ &\times \left[\left(1 - W_{(j)}^{\alpha\tau}(2 - W_{(j)}^{\tau})^{\alpha}\right) - \left(1 - W_{(j)}^{\alpha\tau}(2 - W_{(j)}^{\tau})^{\alpha}\right)\right]^{j-i-1} \\ &\times \left[1 - \left(1 - W_{(j)}^{\alpha\tau}\right)(2 - W_{(j)}^{\tau})^{\alpha}\right]^{n-j} \\ &\times \left[2\alpha\tau\sigma\theta\beta\, x_i^{-\theta-1}(1 + \sigma x^{-\theta})^{-\beta-1} W_{(i)}^{\alpha\tau-1}(2 - W_{(i)}^{\tau})^{\alpha-1}(1 - W_{(i)}^{\tau})\right] \\ &\times \left[2\alpha\tau\sigma\theta\beta\, x_i^{-\theta-1}(1 + \sigma x^{-\theta})^{-\beta-1} W_{(j)}^{\alpha\tau-1}(2 - W_{(j)}^{\tau})^{\alpha-1}(1 - W_{(j)}^{\tau})\right] \end{aligned} \tag{43}$$

$$where\ \ W_{(i)} = \left(1 - (1 + \sigma x_i^{-\theta})^{-\beta}\right), W_{(j)} = \left(1 - (1 + \sigma x_j^{-\theta})^{-\beta}\right)$$

The joint distribution of minimum and maximum of order statistics is given by

$$f_{1:n;n}(x_1, x_n) = n(n-1)\left[F(x_{(n)}) - F(x_{(1)})\right]^{n-2} f(x_1) f(x_n)$$

$$\begin{aligned} f_{1:n;n}(x_1, x_n) =& n(n-1)\left[\left(1 - W_{(n)}^{\alpha\tau}(2 - W_{(n)}^{\tau})^{\alpha}\right) - \left(1 - W_{(1)}^{\alpha\tau}(2 - W_{(1)}^{\tau})^{\alpha}\right)\right]^{n-2} \\ &\times \left[2\alpha\tau\sigma\theta\beta\, x_1^{-\theta-1}(1 + \sigma x_1^{-\theta})^{-\beta-1} W_{(1)}^{\alpha\tau-1}(2 - W_{(1)}^{\tau})^{\alpha-1}(1 - W_{(1)}^{\tau})\right] \\ &\times \left[2\alpha\tau\sigma\theta\beta\, x_n^{-\theta-1}(1 + \sigma x_n^{-\theta})^{-\beta-1} W_{(n)}^{\alpha\tau-1}(2 - W_{(n)}^{\tau})^{\alpha-1}(1 - W_{(n)}^{\tau})\right] \end{aligned} \tag{44}$$

$$where\ \ W_{(i)} = \left(1 - (1 + \sigma x_i^{-\theta})^{-\beta}\right), W_{(j)} = \left(1 - (1 + \sigma x_j^{-\theta})^{-\beta}\right)$$

V. Income inequality measures

I. Lorenz curve

The Lorenz curve is defined by

$$L(x) = \frac{1}{\mu}\int_0^x x f(x) dx \tag{45}$$

$$L(x) = \frac{1}{\mu}\int_0^x x\left[\sum_{p,q,s,t,v=0}^{\infty} \psi 2\alpha\tau\sigma\theta\beta\, x^{-\theta-1}(1 + \sigma x^{-\theta})^{-\beta(p+s-v+1)-1}\right] dx$$

The Lorenz curves of type II power Topp-Leone Dagum distribution is given by

$$L(x) = \frac{\sum_{p,q,s,t,v=0}^{\infty} \psi 2\alpha\tau\beta\sigma^{\frac{1}{\theta}} B\left(1 - \frac{1}{\theta}, \beta(p+s-v+1) + \frac{1}{\theta}; y\right)}{\sum_{p,q,s,t,v=0}^{\infty} \psi 2\alpha\tau\beta\sigma^{\frac{1}{\theta}} B\left(1 - \frac{1}{\theta}, \beta(p+s-v+1) + \frac{1}{\theta}\right)} \tag{46}$$

II. Bonferroni index

Bonferroni index is defined by

$$B(x) = \frac{L(x)}{F(x)} \tag{47}$$

The Bonferroni index of type II power Topp-Leone Dagum distribution is given by

$$B(x) = \frac{\omega}{\vartheta} \tag{48}$$

where

$$\omega = \sum_{p,q,s,t,v=0}^{\infty} \psi 2\alpha\tau\beta\sigma^{\frac{1}{\theta}} B\left(1 - \frac{1}{\theta}, \beta(p+s-v+1) + \frac{1}{\theta}; y\right)$$

$$\vartheta = \sum_{p,q,s,t,v=0}^{\infty} \psi 2\alpha\tau\beta\sigma^{\frac{1}{\theta}} B\left(1 - \frac{1}{\theta}, \beta(p+s-v+1) + \frac{1}{\theta}\right)$$
$$\times \left[1 - \left(1 - (1+\sigma x^{-\theta})^{-\beta}\right)^{\alpha\tau} \left(2 - \left(1 - (1+\sigma x^{-\theta})^{-\beta}\right)^{\tau}\right)^{\alpha}\right]$$

III. Generalized entropy

The generalized entropy is defined by

$$GE(w,\delta) = \frac{1}{\delta(\delta-1)\mu^{\delta}} \left[\int_0^{\infty} x^{\delta} f(x) dx\right] - 1 \tag{49}$$

where μ is the mean of distribution.

$$GE(w,\delta) = \frac{1}{\delta(\delta-1)\mu^{\delta}} \int_0^{\infty} x^{\delta} \left[\sum_{p,q,s,t,v=0}^{\infty} \psi 2\alpha\tau\sigma\theta\beta\, x^{-\theta-1}(1+\sigma x^{-\theta})^{-\beta(p+s-v+1)-1}\right] dx - 1$$

The Generalized entropy of type II power Topp-Leone Dagum distribution is given by

$$GE(w,\delta) = \frac{\sum_{p,q,s,t,v=0}^{\infty} \psi 2\alpha\tau\beta\sigma^{\frac{\delta}{\theta}} B\left(1 - \frac{\delta}{\theta}, \beta(p+s-v+1) + \frac{\delta}{\theta}\right)}{\delta(\delta-1)\left[\sum_{p,q,s,t,v=0}^{\infty} \psi 2\alpha\tau\beta\sigma^{\frac{1}{\theta}} B\left(1 - \frac{1}{\theta}, \beta(p+s-v+1) + \frac{1}{\theta}\right)\right]^{\delta}} - 1 \tag{50}$$

IV. Zenga index

Zenga index is defined by

$$Z = 1 - \frac{\bar{\mu}_{(x)}}{\mu^{+}_{(x)}} \tag{51}$$

where

$$\bar{\mu}_{(x)} = \frac{1}{F(x)} \int_0^{x} x f(x) dx$$
$$\mu^{+}_{(x)} = \frac{1}{1-F(x)} \int_0^{\infty} x f(x) dx$$

Consider,

$$\bar{\mu}_{(x)} = \frac{1}{F(x)} \int_0^{x} x f(x) dx$$

$$\bar{\mu_x} = \frac{1}{\left[1-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\alpha\tau}\left(2-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\tau}\right)^{\alpha}\right]} \times \int_0^x x\left[\sum_{p,q,s,t,v=0}^{\infty}\psi 2\alpha\tau\sigma\theta\beta x^{-\theta-1}(1+\sigma x^{-\theta})^{-\beta(p+s-v+1)-1}\right]dx$$

$$\bar{\mu_x} = \frac{\sum_{p,q,s,t,v=0}^{\infty}\psi 2\alpha\tau\beta\sigma^{\frac{1}{\theta}}B\left(1-\frac{1}{\theta},\beta(p+s-v+1)+\frac{1}{\theta};y\right)}{\left[1-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\alpha\tau}\left(2-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\tau}\right)^{\alpha}\right]}$$

Consider,

$$\mu_{(x)}^{+} = \frac{1}{1-F(x)}\int_0^{\infty} xf(x)dx$$

$$\mu_x^{+} = \frac{1}{\left[1-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\alpha\tau}\left(2-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\tau}\right)^{\alpha}\right]} \times \int_0^{\infty} x\left[\sum_{p,q,s,t,v=0}^{\infty}\psi 2\alpha\tau\sigma\theta\beta x^{-\theta-1}(1+\sigma x^{-\theta})^{-\beta(p+s-v+1)-1}\right]dx$$

$$\mu_x^{+} = \frac{\sum_{p,q,s,t,v=0}^{\infty}\psi 2\alpha\tau\beta\sigma^{\frac{1}{\theta}}B\left(1-\frac{1}{\theta},\beta(p+s-v+1)+\frac{1}{\theta}\right)}{\left[1-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\alpha\tau}\left(2-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\tau}\right)^{\alpha}\right]}$$

The Zenga index of type II power Topp-Leone Dagum distribution is given by

$$Z = 1-\frac{A}{B} \tag{52}$$

where

$$A = \sum_{p,q,s,t,v=0}^{\infty}\psi 2\alpha\tau\beta\sigma^{\frac{1}{\theta}}B\left(1-\frac{1}{\theta},\beta(p+s-v+1)+\frac{1}{\theta};y\right) \times \left[1-\left(1-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\alpha\tau}\left(2-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\tau}\right)^{\alpha}\right)\right]$$

$$B = \left[1-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\alpha\tau}\left(2-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\tau}\right)^{\alpha}\right] \sum_{p,q,s,t,v=0}^{\infty}\psi 2\alpha\tau\beta\sigma^{\frac{1}{\theta}}B\left(1-\frac{1}{\theta},\beta(p+s-v+1)+\frac{1}{\theta}\right)$$

VI. Parameter Estimation

Let $x_1, x_1, ..., x_n$ be a random sample from the type II power Topp-Leone Dagum distribution then the likelihood function is given by

$$L(\theta) = \prod_{i=1}^{n}\left[2\alpha\tau\sigma\theta\beta\, x^{-\theta-1}\left(1+\sigma x^{-\theta}\right)^{-\beta-1}\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\alpha\tau-1} \left(2-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\tau}\right)^{\alpha-1}\left(1-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\tau}\right)\right] \tag{53}$$

The log likelihood function is given by

$$\begin{aligned}L(\theta) =& nlog2 + nlog\alpha + nlog\tau + nlog\sigma + nlog\theta + nlog\beta + (-\theta-1)\sum_{i=1}^{n} logx_i \\ &+ (-\beta-1)\sum_{i=1}^{n} log(1+\sigma x_i^{-\theta}) + (\alpha\tau-1)\sum_{i=1}^{n} log\left(1-(1+\sigma x_i^{-\theta})^{-\beta}\right) \\ &+ (\alpha-1)\sum_{i=1}^{n} log\left(2-\left(1-(1+\sigma x_i^{-\theta})^{-\beta}\right)^{\tau}\right) + \sum_{i=1}^{n} log\left(1-\left(1-(1+\sigma x_i^{-\theta})^{-\beta}\right)^{\tau}\right)\end{aligned}$$

Taking the partial derivatives of the log-likelihood function with respect to parameters $\alpha, \tau, \sigma, \theta$ and β and then equate to zero.

$$\frac{\partial logL}{\partial\alpha} = 0, \frac{\partial logL}{\partial\tau} = 0, \frac{\partial logL}{\partial\sigma} = 0, \frac{\partial logL}{\partial\theta} = 0 \text{ and } \frac{\partial logL}{\partial\beta} = 0.$$

That is

$$\frac{\partial logL}{\partial\alpha} = \frac{n}{\alpha} + \alpha\sum_{i=1}^{n} log\left(1-(1+\sigma x_i^{-\theta})\right) + \sum_{i=1}^{n} log\left(2-\left(1-(1+\sigma x_i^{-\theta})^{-\beta}\right)^{\tau}\right) = 0 \quad (54)$$

$$\begin{aligned}\frac{\partial logL}{\partial\tau} =& \frac{n}{\tau} + \alpha\sum_{i=1}^{n} log\left(1-(1+\sigma x_i^{-\theta})^{-\beta}\right) \\ &- \sum_{i=1}^{n} \frac{(\alpha-1)\left(1-(1+\sigma x_i^{-\theta})^{-\beta}\right)^{\tau} log\left(1-(1+\sigma x_i^{-\theta})^{-\beta}\right)}{\left(2-\left(1-(1+\sigma x_i^{-\theta})^{-\beta}\right)^{\tau}\right)} \\ &- \sum_{i=1}^{n} \frac{\left(1-(1+\sigma x_i^{-\theta})^{-\beta}\right)^{\tau} log\left(1-(1+\sigma x_i^{-\theta})^{-\beta}\right)}{\left(1-\left(1-(1+\sigma x_i^{-\theta})^{-\beta}\right)^{\tau}\right)} = 0 \end{aligned} \quad (55)$$

$$\begin{aligned}\frac{\partial logL}{\partial\sigma} =& \frac{n}{\sigma} + \sum_{i=1}^{n} \frac{(-\beta-1)x_i^{-\theta}}{(1+\sigma x_i^{-\theta})} - \sum_{i=1}^{n} \frac{(\alpha\tau-1)x_i^{-\theta}}{\left(1-(1+\sigma x_i^{-\theta})^{-\beta}\right)} \\ &- \sum_{i=1}^{n} \frac{(\alpha-1)\tau\left(1-(1+\sigma x_i^{-\theta})^{-\beta}\right)^{\tau-1} \beta(1+\sigma x_i^{-\theta})^{-\beta-1}x_i^{-\theta}}{\left(2-\left(1-(1+\sigma x_i^{-\theta})^{-\beta}\right)^{\tau}\right)} \\ &- \sum_{i=1}^{n} \frac{\tau\left(1-(1+\sigma x_i^{-\theta})^{-\beta}\right)^{\tau-1} \beta(1+\sigma x_i^{-\theta})^{-\beta-1}x_i^{-\theta}}{\left(1-\left(1-(1+\sigma x_i^{-\theta})^{-\beta}\right)^{\tau}\right)} = 0 \end{aligned} \quad (56)$$

$$\begin{aligned}\frac{\partial logL}{\partial\theta} =& \frac{n}{\theta} - \sum_{i=1}^{n} logx_i + \sum_{i=1}^{n} \frac{(-\beta-1)\sigma x_i^{-\theta}logx_i}{(1+\sigma x_i^{-\theta})} - \sum_{i=1}^{n} \frac{(\alpha\tau-1)\sigma x_i^{-\theta}logx_i}{\left(1-(1+\sigma x_i^{-\theta})\right)} \\ &- \sum_{i=1}^{n} \frac{(\alpha-1)\beta\tau\left(1-(1+\sigma x_i^{-\theta})^{-\beta}\right)^{\tau-1} (1+\sigma x_i^{-\theta})^{-\beta-1}\sigma x_i^{-\theta}logx_i}{\left(2-\left(1-(1+\sigma x_i^{-\theta})^{-\beta}\right)^{\tau}\right)} \\ &- \sum_{i=1}^{n} \frac{\beta\tau\left(1-(1+\sigma x_i^{-\theta})^{-\beta}\right)^{\tau-1} (1+\sigma x_i^{-\theta})^{-\beta-1}\sigma x_i^{-\theta}logx_i}{\left(1-\left(1-(1+\sigma x_i^{-\theta})^{-\beta}\right)^{\tau}\right)} = 0 \end{aligned} \quad (57)$$

and

$$\begin{aligned}\frac{\partial logL}{\partial \beta} =& \frac{n}{\beta} - \sum_{i=1}^{n} log\left(1+\sigma x_i^{-\theta}\right) \\ &+ \sum_{i=1}^{n} \frac{\tau(\alpha-1)\left(1-(1+\sigma x_i^{-\theta})^{-\beta}\right)^{\tau-1}(1+\sigma x_i^{-\theta})^{-\beta} log\left(1+\sigma x_i^{-\theta}\right)}{\left(2-\left(1-(1+\sigma x_i^{-\theta})^{-\beta}\right)^{\tau}\right)} \\ &+ \sum_{i=1}^{n} \frac{\tau\left(1-(1+\sigma x_i^{-\theta})^{-\beta}\right)^{\tau-1}(1+\sigma x_i^{-\theta})^{-\beta} log\left(1+\sigma x_i^{-\theta}\right)}{\left(1-\left(1-(1+\sigma x_i^{-\theta})^{-\beta}\right)^{\tau}\right)} = 0 \end{aligned} \tag{58}$$

The above mentioned five non-linear equations are very difficult to solve analytically. In this situation we can use to iteration techniques like Newton-Raphson, bisection and regular falsi method to compute numerical solution. However, we used *R* software for estimate the parameters of the proposed distribution.

VII. Applications

In this section, we consider two real data sets for type II power Topp Leone-Dagum distribution. This first data set represent the survival times (days) of 40 patients suffering from leukemia and is studied by Abouammoh et al., [2] and Bhatti et al., [5]. The second data set related to actuarial science data (Mortality death). This data describes 280 observations on the age of death (in years) of retired women with temporary disabilities who died during 2004 and which are incorporated in the Mexican insurance public system. This data set recently studied by Balakrishnan et al., [4] and Tahir et al., [19].

I. Data set 1: survival time data

The survival time data set is analysed using the *R* software. The following tables Table 1 to 3 explain about summary of statistics, estimated parameters values and statistical model selection for survival time data.

We compared statistical models namely type II power Topp-Leone Dagum distribution (TIIPTLDD) with Dagum distribution (DD), modified Burr III distribution (MBIIID), Burr III distribution (BIIID), log-logistic distribution (LLD), modified Frechet distribution (MFD) and Frechet distribution (FD). The statistical model selection based on the minimum value of statistic information theoretic criterion, such as Akaike information criterion (AIC), Bayesian information criterion (BIC), consistent Akaike information criterion (CAIC), and -2log-likelihood was carried out. The type II power Topp-Leone Dagum distribution provides better fit and flexibility compared to other competitive statistical models based on these statistics measures.

Table 1: *Summary of statistics*

n	Mean	Median	Minimum	Maximum	Q_3	Q_3
40	1137	1222.0	115.0	1852.0	802.5	1852.0

Table 2: *The value of estimated parameters*

Model	Estimated value of the parameters
TIIPTLD-D	α=47.1760, τ=6.7544, σ=51.2247, θ=0.4767, β=3.8028
D-D	α=124635.5, β=1.199734, γ=5.0000
MBIII-D	α=124637.2, β=1.7052, γ=121473.2
BIII-D	α=2503.6088, β=1.1982.
LL-D	β=0.2235.
MF-D	β=0.9013, θ=7111.323, λ=0.0021.
F-D	β=1.1984, θ=685.7135

Table 3: *Statistical model selection*

Model	-2LL	AIC	AICC	BIC
TIIPTLD-D	-614.164	624.164	625.9287	632.6084
D-D	651.2760	659.2761	660.4189	666.0316
MBIII-D	638.7774	644.773	645.444	649.844
BIII-D	651.2590	655.2589	655.5832	658.6367
LL-D	825.6310	827.6309	827.7362	829.3198
MF-BIII	701.9472	707.9472	708.6139	713.0138
F-D	651.2778	655.2778	655.6022	658.6556

II. Data set 2: Actuarial science data

The actuarial science data set carried out using the *R* software. The following Tables 4 to 6 explain about summary of statistics, estimated parameters values and statistical model selection for actuarial science data.

We compared statistical models namely type II power Topp-Leone Dagum distribution (TIIPTLDD) with Dagum distribution (D). The statistical model selection based on the minimum value of statistic information theoretic criterion, such as Akaike information criterion (AIC), Bayesian information criterion (BIC), consistent Akaike information criterion (CAIC) and 2log-likelihood. The type II power Topp-Leone Dagum distribution provides better fit and flexibility compare than Dagum distribution based on statistics measures.

Table 4: *Summary of statistics*

n	Mean	Median	Minimum	Maximum	Q_3	Q_3
280	47.79	49.00	22.00	86.00	40.00	55.25

Table 5: *The value of estimated parameters*

Model	Estimated value of the parameters
TIIPTLD-D	α=37.3031, τ=0.6554, σ=26.2491, θ=1.5575, β=25.4281
D-D	σ=1282.2665, β=3.9888, γ=2183.6861

Table 6: *Statistical model selection*

Model	-2LL	AIC	AICC	BIC
TIIPTLD-D	2113.836	2123.836	2124.055	2142.01
D-D	2203.86	2209.860	2209.947	2220.764

VIII. Conclusion

In this article, we introduced new generating probability distribution called type II power Topp-Leone Dagum distribution. Many of reliability measures are investigated including reliability function, hazard rate function, reversed hazard rate function, mean waiting time, mean past life time, mean deviation, second failure rate function and mean residual life function. We have obtained different statistical properties such as moments, moment generating function, characteristic function, cumulant generating function, inverted moments, central moments, conditional moments, probability weighted moments and order statistics. We derived some of income inequality measures like Lorenz curve, Bonferroni index, Zenga index and Generalized entropy for proposed new probability distribution. The parameters of proposed new probability distributions are estimated by method of maximum likelihood. Finally, we fitted the type II power Topp-Leone Dagum distribution for real life time data sets and showed that TIIPTLD-D provide better fit these two data set.

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