Decomposable Semi-Regenerative Processes: Review of Theory and Applications to Queueing and Reliability Systems

Vladimir Rykov[1,2]

•

[1]Gubkin Russian State University of Oil and Gas (Gubkin University),
65 Leninsky Prospekt, Moscow, 119991, Russia;
[2] Kharkevich Institute for transmission information problems RAS,
Bolshoy Karetny, 19 build.1, Moscow, 127051
vladimir_rykov@mail.ru

Abstract

A review of the Smith's regeneration idea development is proposed. As a generalization of this idea the main definitions and results of decomposable semi-regenerative processes are reminded. Their applications for investigation of various queueing and reliability systems are considered.

Keywords: Decomposable semi-regenerative processes, Stochastic models, time-dependent and steady state probabilities.

1. Introduction, Motivation, and Abbreviations

The main idea of this paper is to give a review of the Smith's *regeneration notion* development. Definition and main results of *decomposable semi-regenerative processes* (DSRPr) will be under our attention. Applications of these processes to investigation of some real-world queueing and reliability systems makes up an essential part of the paper.

As a generalization of classical independence the regeneration idea has been proposed by W. Smith in 1955. The regenerative approach allows not only to calculate the *regenerative process* (RPr) state *time-dependent probabilities* (t.d.p.'s) in terms of its state probabilities at the separate *regeneration period* (RP), but also to prove its stationary regime existence and find the *steady state probabilities* (s.s.p.'s).

However, if the process behavior in the separate regeneration period is enough complex and its distribution can not be analytically represented, the more detailed investigation of the process could be obtained with the help of *semi-regenerative processes* (SRPr's), which joins the regeneration approach with the Markov type dependency. The next step in the generalization of the regeneration idea consists in finding some new regeneration points of time into regeneration period and construction of so called *embedded regenerative process* (ERPr). This idea can be developed for construction of *decomposable semi-regenerative processes* (DSRPr's).

These ideas have been applied for investigation of several stochastic systems such as: priority queueing systems (QS's), one-server QS with recurrent input and service processes, polling systems, complex hierarchical systems, etc. In this paper we remind the results of some of these investigations. Recently the ERPr has been used for investigation of a double-redundant system with general distributions of its component life- and repair times and of a k-out-of-n system. These models also will be in our focus.

Along the paper the following general notations are used:

- $\mathbf{P}\{\cdot\}$, $\mathbf{E}[\cdot]$ — symbols of probability and expectation, symbols $\mathbf{P}_i\{\cdot\}$, $\mathbf{E}_i[\cdot]$ are used for conditional probability and expectation, given initial state of the process is i;
- the vectors are marked with arrows and are understood as column vectors, and transposition of vectors and matrices is indicated by a prime;
- the representatives of any sequence of independent identically distributed random variables A_i $(i = 1, 2, \dots)$ are denoted by appropriate letter without indexes A and their common cumulative distribution functions (c.d.f.'s) are denoted by the same letter with an argument $A(x) = \mathbf{P}\{A \leq x\}$;
- The *moment generating functions* (MGF's) of r.v.'s (the *Laplace-Stiltjes transforms* (LST's) of their c.d.f.'s) are denoted by appropriate small letters with tilde $\tilde{a}(s) = \mathbf{E}\left[e^{-sA}\right] = \int_0^\infty e^{-sx} A(dx)$.

The paper is organized as follows. In the next section the main ideas of the regenerative approach and its developments will be reminded. All other sections are devoted to applications of above methods to investigations of the real world systems which have been considered early and recently that have been proposed without proofs. The paper ends with Conclusion where some further possible investigations are pointed out.

In the paper the following abbreviations are used.

- RPr — regenerative process,
- SRPr — semi-regenerative process,
- ERPr — embedded semi-regenerative process,
- DSRPr — decomposable semi-regenerative process,
- RT — regeneration time,
- SRT — semi-regeneration time,
- ERT — embedded regeneration time,
- RP —regeneration period,
- SRP —semi-regeneration period,
- ESRP — embedded semi-regeneration period,
- RS — regeneration state,
- SRS — semi-regeneration state,
- ESRS — embedded semi-regeneration state,
- SMM — semi-Markov matrix,
- ESMM — embedded semi-Markov matrix,
- RF — regeneration function,
- ERF — embedded regeneration function,
- MRM — Markov renewal matrix,
- EMRM — embedded Markov renewal matrix,
- ERK — embedded renewal kernel,
- QS — queueing system,

- NoC — number of calls,
- PP — poling process,
- IP — idle period,
- BP — busy period,
- MGF — moment generating ff-unction
- LT — Laplace transform,
- LST — Laplace-Stiltjes transform,
- i.i.d. — independent identically distributed,
- r.v. —random variable,
- c.d.f. — cumulative distribution function,
- t.d.p. — time-dependent probability,
- s.s.p. — steady state probability

2. On regenerative approach

Here the main results of regenerative and semi-regenerative processes are reminded. The so-called decomposable semi-regenerative processes are also under our attention. We omit very known strong definitions and represent the main results, which will be used in the paper.

2.1. Regenerative process

As a generalization of classical independence the *regeneration idea* has been proposed by W. Smith in 1955 [1]. Consider a stochastic process $X = \{X(t) : \ t \geq 0\}$ with filtration $\mathcal{F}_t^X$. The process X is called the regenerative one if there exist a sequence of points of time, *regeneration times (RT)* S_n, in which the process forgets its past,

$$\mathbf{P}\{X(S_n + t) \in \Gamma \,|\mathcal{F}_{S_n}^X\} = \mathbf{P}\{X(S_n + t) \in \Gamma\} = \mathbf{P}\{X(S_1 + t) \in \Gamma\}.$$

The intervals $[S_{n-1}, S_n)$ and their length $T_n = S_n - S_{n-1}$ is called *regeneration periods* (RP). Note, that the functional elements $W_n = \{X(S_n + t), \, t \leq T_n\}$ are independent. They are called *regeneration cycles* (RC).

The regenerative approach allows to calculate the *regenerative process* (RPr) state probabilities $\pi(t; \Gamma) = \mathbf{P}\{X(t) \in \Gamma\}$ in terms of its state probabilities at the *separate regeneration period* $\pi^{(1)}(t, \Gamma) = \mathbf{P}\{X(S_{n-1} + t) \in \Gamma, \ t < T_n\}$ in the form

$$\pi(t; \Gamma) = \int\limits_0^t H(du)\pi^{(1)}(t - u, \Gamma), \tag{1}$$

where $F(t) = \mathbf{P}\{T_n \leq t\}$ and

$$H(t) = \mathbf{E}\left[\sum_{n\geq 1} 1_{\{S_n \leq t\}}\right] = \sum_{n\geq 1} \mathbf{P}\{S_n \leq t\} = \sum_{n\geq 1} F^{*n}(t).$$

is the *renewal function* (RF) of the process, where symbol "$\star$" means a convolution, The RF satisfies the Winner-Hopf equation

$$H(t) = F(t) + F \star H(t) \equiv F(t) + \int\limits_0^t F(du)H(t - u). \tag{2}$$

This approach allows not only to obtain the representation (1), but also to prove the existence of the stationary probabilities and give its close form representation in terms of the process distribution at separate regeneration periods. Namely it holds

$$\pi(\Gamma) = \lim_{t\to\infty} \pi(t;\Gamma) = \frac{1}{\mathbf{E}[T]} \int_0^\infty \pi^{(1)}(t,\Gamma)dt. \tag{3}$$

2.2. Semi–regenerative process

As a generalization to the Markov's dependency, an idea of *semi-Markov chains* has been proposed by E. Cinlar (1969) [2] and J Jakod (1971) [3]. This idea leaded further to the constructions of *semi-Markov processes* (SMPr) (see, for example [3], [4]). The joining of these notions with the regeneration idea leaded to introduction of *semi-regenerative processes* (SRPr's), which firstly appeared under different titles: as *semi-Markov processes with additional trajectories* in 1966 (Klimov [5]), *regenerative processes with several types of regeneration poins* in 1971 (Rykov & Yastrebenetsky [6]) before it became the name SRPr due to E. Nummeline [7].

The difference of the SRPr from the RPr consists in the assumption that in its semi-regeneration points of time S_n the future of the process does not depend of its past but depends on its present state belonging to some set E of *regeneration states* (RS's),

$$\mathbf{P}\{X(S_n+t)\in\Gamma\,|\mathcal{F}^X_{S_n}\} = \mathbf{P}\{X(S_n+t)\in\Gamma\,|X(S_n)\} = \mathbf{P}\{X(S_1+t)\in\Gamma\,|X(S_1)\}.$$

The main characteristic of the SRP is its *semi-Markov matrix* (SMM) $Q(t) = [Q_{ij}(t)]_{ij\in E}$ with components

$$Q_{ij}(t) = \mathbf{P}\{X(S_n)=j,\ T_n\le t,\ |\ X(S_{n-1})=i\}.$$

For SRPr with denumerable RS's E that starts in RT with an initial distribution $\alpha=\{\alpha_i,\ i\in E\}$ the formula (1) takes the form

$$\pi(t;\Gamma) = \sum_{i\in E}\alpha_i\left[\delta_{ij}\pi_j^{(1)}(t) + \int_0^t H_{ij}(du)\pi_j^{(1)}(t-u,\Gamma)\right]. \tag{4}$$

Here $\pi_j^{(1)}(t,\Gamma) = \mathbf{P}\{X(S_{n+1}+t)\in\Gamma,\ t<T_n\,|\,X(S_n+0)=j\}$ is the process state probability distribution on a separate semi-regeneration period (SRP) of type j, and $H(t)=[H_{ij}(t)]$ is its *Markov renewal matrix* (MRM) with

$$H_{ij}(t) = \mathbf{E}\left[\sum_{n\ge1} 1_{\{S_n\le t,\ X(S_n)=j\}}\,|\,X(0)=i\right] = \left[\sum_{n\ge1} Q^{*n}(t)\right]_{ij}.$$

these functions satisfy the Winner-Hopf equation, in which symbol "$\star$" means the matrix-functional convolution,

$$H(t) = Q(t) + Q\star H(t). \tag{5}$$

The corresponding limit theorem takes the form

$$\pi(\Gamma) = \lim_{t\to\infty}\pi(t,\Gamma) = \frac{1}{m}\int_0^\infty \sum_{j\in E}\bar\alpha_j\pi_j^{(1)}(t,\Gamma)dt, \tag{6}$$

where $\bar\alpha = \{\bar\alpha_i\,(i\in E)\}$ represents the invariant probabilities of the embedded Markov chain $Y=\{Y_n = X(S_n),\ n=1,2,\dots\}$, and $m=\sum_{i\in E}\bar\alpha_i\mathbf{E}_i[T]$ is the expected stationary regeneration period.

2.3. Decomposable semi-regenerative process

If the process behavior at the separate regeneration period T_n is too complex and its distribution can not be analytically represented, sometimes it is possible to find some embedded regeneration points of time $S_k^{(1)}$ $(k = 1, 2, \dots)$ into this period, in which the process forgets its past up to the present state $X^{(1)}(S_k^{(1)})$ conditionally to its behavior at the regeneration period T_n. This subset of the process state is called *embedded semi-regeneration states* (ESRS's) and denoted by $E^{(1)}$. This process is called an *embedded regeneration process* (ERPr). Spreading out this procedure for all regeneration periods of the main process leads to construction of *decomposable semi-regenerative process* (DSRPr). This procedure can be extended to several embedding levels. The strong definitions and details can be found in V. Rykov (1975) [8] (see also [9], and [10]).

While analyzing the DSRPr, the role of the ordinary MRM plays the *embedded Markov renewal matrix (EMRM) $H(t) = [H_{ij}(t)]$*, which is given by its components

$$H_{ij}^{(1)}(t) = \mathbf{E}_i \left[\sum_{k \geq 0} 1_{\{[0,t),j\}} \left(S_k^{(1)}, X(S_k^{(1)}) \right) 1_{\{S_k^{(1)} < T\}} \right].$$

This matrix study depends on the type of the embedded regeneration points construction. There are different scenarios of their construction. If they arise as $\min[S_k^{(1)}, T]$, then unlike equations (2), and (5) for EMRM holds the following equation,

$$H^{(1)}(t) = Q^{(1)}(t) + H^{(1)} \star Q^{(1)}(t) - Q(t). \tag{7}$$

Here $Q(t)$ and $Q^{(1)}(t)$ are the SMM of the external and internal embedded periods and the symbol $\star$ denotes as before the matrix-functional convolution.

In most practical situations, both internal and external regeneration points of time coincide with the times of the regeneration states destinations. At that the external regeneration points of time are the moments, when the process exits the subset of the embedded regeneration states. In this case, the transition matrix for the embedded regeneration points of time $Q^{(1)}(t)$ is a sub-matrix of the matrix $Q(t)$ with components from the subset of the embedded states $E^{(1)}$ and therefore it is a degenerative one. In this case the equation for EMRM has the form

$$H^{(1)}(t) = Q^{(1)}(t) + Q^{(1)} \star H^{(1)}(t).$$

Its solution is

$$H^{(1)}(t) = (I - Q^{(1)}(t))^{-1} Q^{(1)}(t) = \sum_{n \geq 1} Q^{(1)*n}(t).$$

It is bounded for all t and approaches the expected number of visits to subset of states $E^{(1)}$ when $t \to \infty$. Naturally when the both scenarios are applicable, the solutions of the last equation coincide with the solution of equation (7).

Similarly to (4) different characteristics of the DSRPr of the first level can be expressed in terms of its corresponding characteristics of the second level. Particularly, for the one-dimensional distributions $\pi_i^{(1)}(t, \Gamma)$ the following representation holds

$$\pi_i^{(1)}(t, \Gamma) = H_i^{(1)} \star \pi^{(2)}(t, \Gamma).$$

Here

$$\pi_i^{(2)}(t, \Gamma) = \mathbf{P} \left\{ X \left(S_{k-1}^{(1)} + t \right) \in \Gamma, \ t < T_k^{(1)} \mid X^{(1)}(S_{k-1}^{(1)}) = i \right\}$$

are the process state probabilities in ERPr and EMRM $H^{(1)}(t)$ satisfies the equation (7). These relations make possible to recover the process distribution by its distribution on a separate minimal periods of embedded regeneration. The limit theorem for SRPr's allows to calculate its stationary distributions, and the system of embedded regeneration periods make possible to calculate them in terms of distributions on smallest regeneration period.

In the next sections several applications of the DSRPr's for investigation of some real word QS's and reliability models are considered.

3. M/GI/1/∞ Queueing System

We apply the DSRPr to investigation of the main processes in $M|G|1|\infty$ QS. Many authors deals with this system. A.Ya. Khinchin [11] found the s.s.p.'s of this system in 1932. D. Kendall [12] in 1953 studied this system with the help of method of embedded Markov chains. G. Klimov [5] uses the method of probabilistic interpretation of generating functions (GF). In next section we propose the results, given by V. Rykov in [10].

3.1. Number of calls as a regenerative process

Consider a $M|GI|1|\infty$ QS with Poisson input $L(t)$ and recurrent service process, where service times are i.i.d. r.v.'s B_n with common c.d.f. $B(t) = \mathbf{P}\{B_n \le t\}$. Denote by $X = \{X(t),\, t \ge 0\}$ the *number of calls* (NoC) process in the system. Evidently, it is DSRPr, and its RP's R consists of *idle period* (IP) Δ and *busy period* (BP) Π, $R = \Delta + \Pi$ (see figure 1), while its RT's are $S_n = \sum_{1\le i\le n} R_i$.

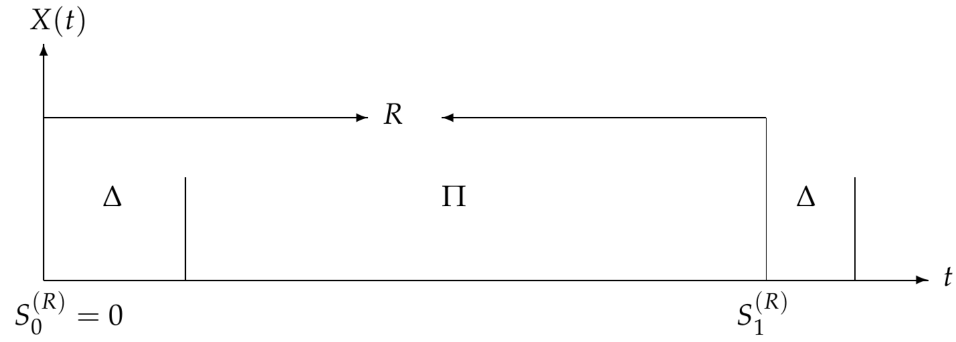

Figure 1: *The structure of a regeneration period*

The RP in turns consists of

- service time B_1 of the first arrived in free system call and
- random number $L(B_1)$ BP's, generated by this call.

Therefore, the BP satisfies to the following stochastic equation

$$\Pi = B_1 + \sum_{0\le i\le L(B_1)} \Pi_i, \tag{8}$$

and thus its MGF satisfied to the Kendall equation

$$\pi(s) = \beta(s + \lambda - \lambda\pi(s)). \tag{9}$$

while the MGF of the RP equals

$$\tau(s) = \mathbf{E}\left[e^{-sR}\right] = \mathbf{E}\left[e^{-s(\Delta+\Pi)}\right] = \frac{\lambda\pi(s)}{\lambda+s}. \tag{10}$$

If the system is idle in initial time, then the process behavior $X(t)$ in any time t can be represented in terms of its behavior at separate RP $X_R(t)$ as follows

$$X(t) = \sum_{n\ge 0} 1_{\{S_n \le t < S_{n+1}\}} X_R(t - S_n).$$

In terms of MGF Laplace transform (LT) of NoC $p(s,z)$ the last expression takes the form

$$\begin{aligned} p(s,z) &\equiv \int_0^\infty e^{-st}\mathbf{E}\left[z^{X(t)}\right] = \int_0^\infty e^{-su} \sum_{n\ge 1} d\mathbf{P}\{S_n \le u\} \int_u^\infty e^{-s(t-u)}\mathbf{E}\left[z^{X_R(t-u)}\right] dt = \\ &= p_R(s,z)\frac{s+\lambda}{s+\lambda-\lambda\pi(s)}. \end{aligned} \tag{11}$$

Here $p_R(s,z) = \int\limits_0^\infty e^{-st}\mathbf{E}\left[z^{X_R(t)}\right]$, and

$$\int\limits_0^\infty e^{-st}\sum_{n\geq 1} d\mathbf{P}\{S_n \leq t\} \equiv \int\limits_0^\infty e^{-st} dH(t) = \tilde{h}(s) = \frac{1}{1-\tau(s)} = \frac{s+\lambda}{s+\lambda-\lambda\pi(s)}.$$

is the Laplace-Stilties Transform (LST) of the RF $H(t)$, generated by RT's S_n.

3.2. NoC on a separate RP

The process $X_R(t)$ behavior on a separate RP in terms of its behavior on separate BP $X_\Pi(t)$ has the form

$$X_R(t) = \begin{cases} 0 & \text{for} \quad t < \Delta, \\ X_\Pi(t-\Delta) & \text{for} \quad \Delta \leq t < \Pi. \end{cases}$$

From here one can find the LT of the MGF of the process $X_R(t)$

$$p_R(s,z) = \int\limits_0^\infty e^{-st}\left[e^{-\lambda t} + \int\limits_0^t \lambda e^{\lambda v}\mathbf{E}z^{X_\Pi(t-v)}dv\right]dt = \frac{1+\lambda p_\Pi(1,s,z)}{s+\lambda}.$$

Jointly with (11) this gives a well known formula for LT MGF of NoC process for $M|GI|1|\infty$ QS in terms of appropriate characteristic at separate BP,

$$p(s,z) = p_R(s,z)\frac{s+\lambda}{s+\lambda-\lambda\pi(s)} = \frac{1+\lambda p_\Pi(1,s,z)}{s+\lambda-\lambda\pi(s)}. \qquad (12)$$

For the NoC process on a separate BP, opening with only one call, when $S_0^{(\Pi)} = 0$, $X(S_0^{(\Pi)}) = 1$, it holds

$$X_\Pi(t) = X(S_{n-1}^{(\Pi)}) + L(t - S_{n-1}^{(\Pi)}), \quad \text{for} \quad S_{n-1}^{(\Pi)} \leq t < S_n^{(\Pi)},$$

where $L(t)$ is an input Poisson process and a sequence of embedded RT's $S_n^{(\Pi)}$ is given recursively

$$S_1^{(\Pi)} = B_1, \;\; S_{n+1}^{(\Pi)} = S_n^{(\Pi)} + 1_{\{X(S_n^{(\Pi)})>0\}}B_n.$$

For the LT of NoC MGF on a separate BP $p_\Pi(1,s,z)$ by calculation with the help of conditional expectation formula we get

$$p_\Pi(1,s,z) = \int\limits_0^\infty e^{-st}\mathbf{E}\left[z^{X_\Pi(t)}\right] = [z + h^\Pi(1,s,z)]\frac{1-\beta(s+\lambda-\lambda z)}{s+\lambda-\lambda z}. \qquad (13)$$

Here $h^{(\Pi)}(1,s,z)$ denotes LT of ERF, generated by the sequence $S_n^{(\Pi)}$, jointly with the MGF of the process in these times

$$h^{(\Pi)}(1,s,z) = \mathbf{E}\left[\sum_{n\geq 1} e^{-sS_n^\Pi}z^{X(S_n^\Pi)}1_{\{X(S_n^\Pi)>0\}}\right].$$

Due to the expression for NoC at the end of service times

$$X(S_n^\Pi) = \begin{cases} X(S_{n-1}^\Pi) - 1 + L(S_n^\Pi - S_{n-1}^\Pi) & \text{for} \quad X(S_{n-1}^\Pi) \geq 1, \\ 0 & \text{for} \quad X(S_{n-1}^\Pi) = 0. \end{cases}$$

This function satisfies to the equation (details of its derivations see in [10])

$$h^{(\Pi)}(1,s,z) = \beta(s+\lambda-\lambda z) + z^{-1}h^{(\Pi)}(1,s,z)\beta(s+\lambda-\lambda z) - \pi(s).$$

Solution of this equation is

$$h(1,s,z) = z\frac{\beta(s+\lambda-\lambda z)-\pi(s)}{z-\beta(s+\lambda-\lambda z)}, \tag{14}$$

and its substitution in (13) for LT MGF of NoC at separate BP gives

$$p_\Pi(1,s,z) = z\frac{\pi(s)-z}{\beta(s+\lambda-\lambda z)-z} \times \frac{1-\beta(s+\lambda-\lambda z)}{s+\lambda-\lambda z}. \tag{15}$$

Thus, the last expression jointly with (12) allow to find the LT of the non-stationary MGF of the NoC process.

3.3. Stationary regime

For MGF of NoC process in stationary regime from (12) and taking into account that from (9) it follows that $\pi_1 = -\pi'(0) = b_1/(1-\rho)$ we obtain the very known Pollachek-Khinchin formula for stationary queue,

$$P(z) = \lim_{s\to 0} sp(s,z) = (1-\rho)\frac{(1-z)\beta(\lambda-\lambda z)}{\beta(\lambda-\lambda z)-z}.$$

4. Priority queueing systems $M_r/GI_r/1/\infty$

Priority QS arise in many applications. Such systems studied by many authors and by different methods: Klimov (1966) [5], Jaiswell (1968) [13], Gnedenko and all (1973) [14]), Klimov, Mishkoi (1979) [15]. The DSRPr method firstly has been applied for such systems investigation by V. Rykov in [8]. Here we shortly remind these results.

4.1. System description.

Consider a single-server QS $M_r/GI_r/1/\infty$ with r independent Poisson inputs $\vec{L}(t) = (L_1(t), \ldots L_r(t))$ intensities λ_k $(k = \overline{1,r}$, with common intensity $\Lambda = \sum_{1\le k\le r}\lambda_k$. Service times are i.i.d. r.v. B_k $(k = \overline{1,r})$ with common for each type of calls c.d.f. $B_k(t) = \mathbf{P}\{B_k \le t\}$. The calls are served with priority discipline in such a manner that the calls of k-th type has a priority before calls of the $(k+1)-st$, $k = \overline{1,r-1}$. There are different types of priorities:

- head-of-the-line;
- preemptive. In this case there are several sub-cases:
  - preemptive resume priority,
  - preemptive repeat priority (with new independent realization of interrupted service times),
  - preemptive repeat priority (with the same realization of first represented service time),
  - preemptive loss priority.

Denote by

- $\vec{x} = (x_1, x_2, \ldots, x_r)$ vector, k-th component of which means the number k-th type calls in the system;
- E the set of the system states;
- $\vec{X}(t) = (X_1(t), \ldots, X_r(t))$ is the NoC process.

It is evident that under given assumption the process $\vec{X}(t)$ is an DSRPr, and its main RP is the same as for the $M/GI/1/\infty$ QS and consists from an idle Δ and a busy Π periods (see figure 1). Denote by

$$p(s,\vec{z}) = \int_0^\infty e^{-st}\mathbf{E}\left[\vec{z}^{\vec{X}(t)}\right] dt$$

LT of the process $\vec{X}(t)$ MGF. Because the RP structure for the system $M_r/GI_r/1/\infty$ is the same as for the system $M/GI/1/\infty$, LT of NoC process MGF coincide with analogous for the system $M/GI/1/\infty$ (12),

$$p(s,\vec{z}) = p_R(s,\vec{z})\frac{s+\lambda}{s+\lambda-\lambda\pi(s)} = \frac{1+\lambda p_\Pi(1,s,\vec{z})}{s+\lambda-\lambda\pi(s)}. \tag{16}$$

However now $\Pi = \Pi_r$ is a BP, during which all calls are served. This period consists of *k-periods* Π_k, during which all calls of k-th and above priority are served and *k-cycles* during which only one call k-th type and all calls higher priority are served.

4.2. Structure of NoC process in $M_r/GI_r/1/\infty$ QS.

The structure of these periods is shown in the figures 2 and 3

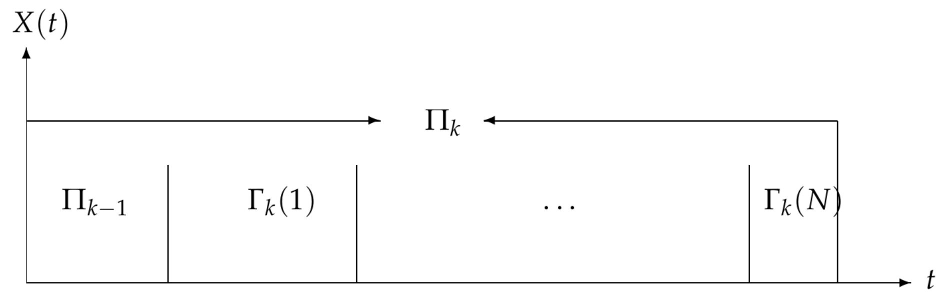

Figure 2: *The structure of k-period*

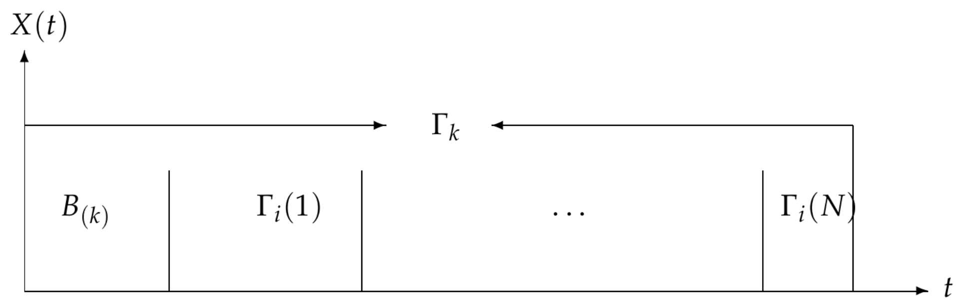

Figure 3: *The structure of k-cycle*

Thus, for calculation of appropriate LT of the MGF $p_\Pi(1,s,\vec{z})$ NoC at separate BP of the system and the MGF $\pi(s)$ of BP introduce the following notations

- $$\sigma_k = \sum_{1\le i\le k}\lambda_i, \quad v_k = \sum_{k\le i\le r}(\lambda_i - \lambda_i z_i), \quad V_k = s + v_k = \sigma_k(1-\pi_k(s+v_k));$$
- the LT MGF of NoC process at separate k-period

$$p_k(\vec{z},s) = \int_0^\infty e^{-st}\mathbf{E}\left[\vec{z}^{\vec{X}(t)} 1_{\Pi_k\le t}\right] dt;$$

- the LT MGF of NoC process at separate k-cycle

$$p^{\gamma_k}(\vec{z},s) = \int_0^\infty e^{-st}\mathbf{E}\left[\vec{z}^{\vec{X}(t)}1_{\Gamma_k \le t}\right] dt;$$

- the MGF of k-period by $\pi_k(s) = \mathbf{E}\left[e^{-s\Pi_k}\right]$;
- the MGF of k-cycle by $\gamma_k(s) = \mathbf{E}\left[e^{-s\Gamma_k}\right]$.

With these notations using the considered above structure of embedded k-periods and k-cycles with the help of DSRP methods in [8] (see also [10]) the following recursive relation for the MGF LT of NoC of $M_r/GI_r/1/\infty$-priority system QS has been obtained

$$\sigma p_\Pi(\vec{z},s) = \sum_{1\le i\le r} \frac{\lambda_i z_i + \sigma_{i-1}\pi_{i-1}(s+v_i) - \sigma_i\pi_i(s+v_{i+1})}{z_i - \gamma_i(s+v_i)} p^{\gamma_i}(\vec{z},s). \tag{17}$$

Here the LT of NoC process MGF on a separate k-cycle $p^{\gamma_i}(\vec{z},s)$ for different type of priorities satisfies to the recursive relations

$$p^{\gamma_k}(\vec{z},s) = z_k\frac{1-\beta_k(s+v_1)}{s+v_1} + \sum_{1\le i\le k-1} \frac{\beta_k(V_i) - \beta_k(V_{i+1})}{z_i - \gamma_i(s+v_i)} p^{\gamma_i}(\vec{z},s). \tag{18}$$

This provides an algorithm for calculating of all basic characteristics of the system. For details see [8] and [10].

5. Polling Systems

Next important application of DSRP are the polling systems that have a wide sphere of applications. There is vast bibliography on this topic including monographs of Takagi (1986) [16], Borst (1996) [17], Vishnevsky, Semenova (2012) [18] and reviews of Takagi (1997) [19], Vishnevsky, Semenova (2006, 2021) [20], [21]. A general description of the polling model one can find in Fricker & Jabi (1994) [22], where also a stability conditions for the system were presented. Most of polling systems investigations deal with the system stationary regime at point of times when server attends users. Here by following [23] we show the possibility of the DSRP theory to be applied for the poling process (PP) investigation in continuous time.

5.1. The system description

Following to Fricker and Jabi (1994) [22], consider the following model (see Fig 4).

There are r users and a single server. Calls from k-th user (*k-calls*) form Poisson input of intensity λ_k. Therefore, $\vec{L}(t) = (L_1(t), \dots L_k(t))$ is the vector flow with summary intensity $\Lambda = \sum_{1\le k\le r}\lambda_k$. Service times are supposed to be i.i.d. r.v. B_k $(k = \overline{1,r})$ with common for k-calls c.d.f. $B_k(t) = \mathbf{P}\{B_k \le t\}$. Beside for switching from i-th user to the j-th one some random times C_{ij} with c.d.f. $C_{ij}(t) = \mathbf{P}\{C_{ij} \le t\}$ are needed. Note that for up-to-date telecommunication systems the service time has the same order as the switching time, therefore it is important to take into account the switching times. Thus, in order to optimize the system behavior it is useful to introduce some delay for service in such manner that the service of k-calls begins only after their number attains some level, say l_k.

The service consists of several, say $n \ge r$, *stages* that are determined by the *polling table* — function f,

$$f : \{1,2,\dots,n\} \to \{1,2,\dots,r\},$$

where $f(j) = k$ denotes that at the j-th stage the k-th user is served and the full round over all users are accomplished during n stages that composes a *cycle* of service. Note that for given

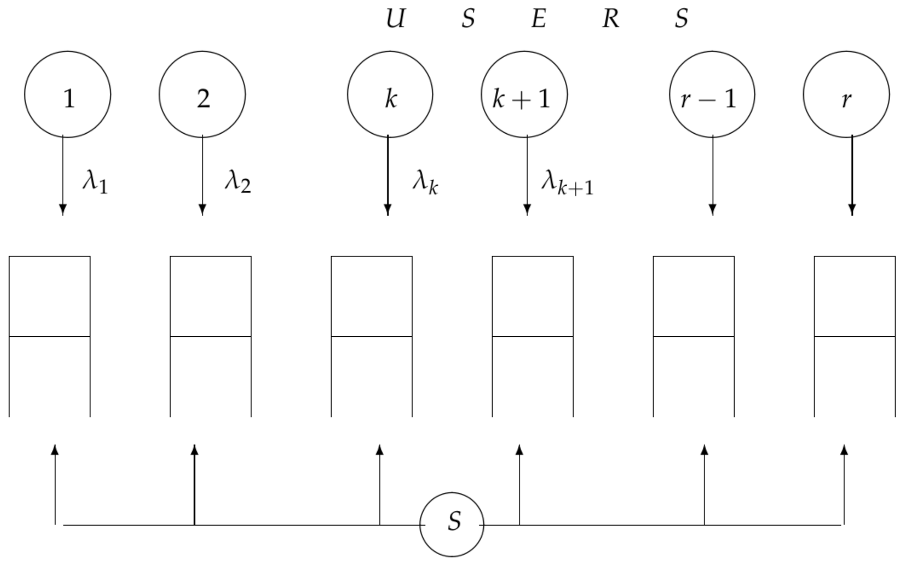

Figure 4: *The Polling model*

polling table the switching time C_j means switching times $C_{f(j),f(j+1)}$ from $f(j)$-th user to $f(j+1)$ one with c.d.f. $C_j(t)$.

There are different service disciplines $\delta(j)$ that could be used at different stages. Some of them are:

- *l-limited service discipline*, for which it is served fix, say l_k number of $f(j) = k$-calls, especially only one call if $l_k = 1$;
- *gated service discipline*, for which all $f(j)$-calls that are present at the very beginning of the stage are served during it;
- *exhaustive service discipline* for which the service of $f(j)$-th user is continued until the queue become empty.

It is supposed that
Assumption 1. All r.v.'s have at least *two finite moments*.
Assumption 2. the stability conditions for the system are fulfilled.

5.2. Structure of the Polling Process

For the Polling System investigation denote by $\vec{x} = (x_1, x_2, \ldots, x_r)$ the states of the polling system, where x_k is a number of k-calls, by E the set of all states, and consider a random process $\vec{X}(t) = \{(X_1(t), \ldots, X_r(t)) : \ t \geq 0\}$ with the states space E to which we will refer as a *polling process* (PP). It is evident that under given assumption the process $\vec{X}(t)$ is a DSRPr, and the structure of its main RP's are the same as for the $M|GI|1|\infty$ QS (see figure 1). However, the structure of its embedded busy periods now are different and in the following figures are represented (see Fig's 5, 6, 7). The BP consists of random number N service cycles G_j, each of which consists of n stages of $A_j \ \ (j = \overline{1,n})$ that also have enough complex structure, which depends on the service discipline at the stage.

5.3. Stochastic relations for Polling Process

Denote by $p(\vec{x}, t) = \mathbf{P}\{\vec{X}(t) = \vec{x}\}$ probability distribution of the PP $\vec{X}(t)$. From our assumptions it follows that the PP is usual multi-dimensional service process for $M_r/GI_r/1/\infty$ QS with r

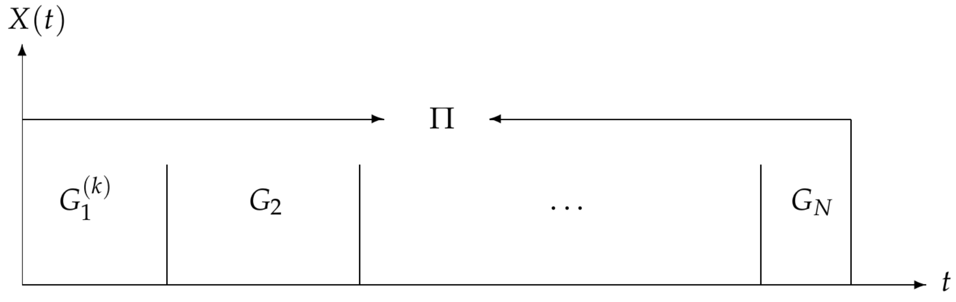

Figure 5: *The structure of a busy period*

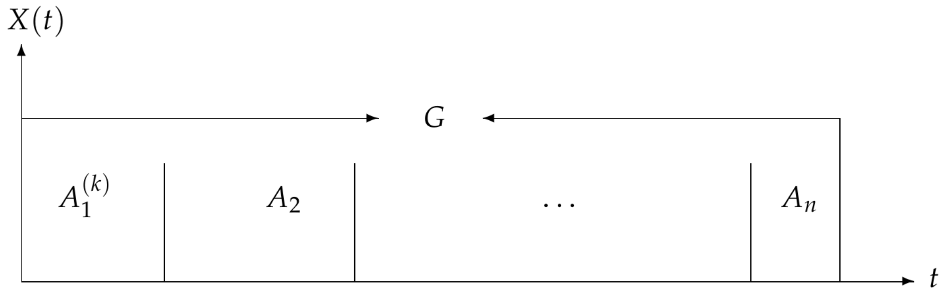

Figure 6: *The structure of a service cycle*

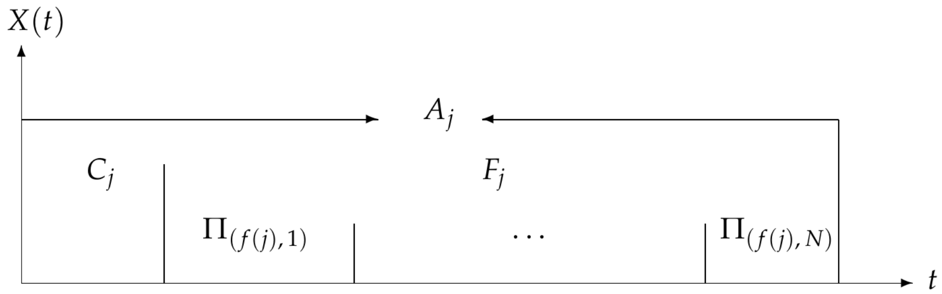

Figure 7: *The structure of a service stage*

types of calls, and therefore LT of the PP MGF accordingly to previous section has the form

$$p(s,\vec{z}) \equiv \int_0^\infty e^{-st}\mathbf{E}\left[\vec{z}^{\vec{X}(t)}\right] = \frac{1+\vec{\Lambda}'\vec{p}_\Pi(s,\vec{z})}{s+\Lambda-\vec{\Lambda}'\vec{\pi}(s)}, \tag{19}$$

where

$$\vec{\Lambda}'\vec{p}_\Pi(s,\vec{z}) = \sum_{1\le k\le r}\lambda_k p_\Pi(\vec{e},s,\vec{z}), \quad \vec{\Lambda}'\vec{\pi}(s) = \sum_{1\le k\le r}\lambda_k\pi_k(s).$$

Here

$$p_\Pi(\vec{e}_k,s,\vec{z}) = \int_0^\infty e^{-st}\mathbf{E}_{\vec{e}_k}\left[\vec{z}^{\vec{X}_\Pi(t)}\right]$$

is the LT MGF PP on a separate BP Π, opening with a single k-call, and $\pi_k(s) = \mathbf{E}_{e_k}\left[e^{-s\Pi}\right]$ is a MGF of a service cycle G, opening with a single k-call.

Therefore to find LT MGF PP $p(s,\vec{z})$ it is necessary to investigate appropriate functions at separate BP $p_\Pi(\vec{e}_k,s,\vec{z})$ and MGF $\pi_k(s)$ of the service cycle Π), opening with a single k-call and a number of calls in its end.

To calculate these functions consider the stochastic relations that associate the process behavior in any point of times and in points of times at separate embedded periods of regeneration, busy, cycles and stages of service which were denoted by

- $\vec{X}_R(t) = \{\vec{X}_R(t),\ t \le R\}$ is the PP at separate RP;
- $\vec{X}_\Pi(t) = \{\vec{X}_\Pi(t),\ t \le \Pi\}$ is the PP at separate BP;
- $\vec{X}_G(t) = \{\vec{X}_G(t),\ t \le G\}$ is the PP at separate Service Cycle; and
- $\vec{X}_j(t) = \{\vec{X}_j(t),\ t \le A_j\}$ is the PP at separate Service Stage.

Using the structure of the embedded RPr's the LT of the MGF of the PP in any time can be represented in terms of appropriate characteristics within the separate service stages. Using the process behavior at different separate stages the LT of MGF of the PP at them also can be calculated in closed form for any poling table and different service disciplines. The details can be found in [23].

6. $GI/GI/1/\infty$ Queueing System

The $GI/GI/1/\infty$ QS is a very interesting model both from theoretical and application point of views. The detailed study of this system one can find in the book of Cohen (1969) [24]. In this section we remind the results about application of DSRP method for investigation of one-server queueing system with recurrent input and generally distributed service time that has been done by Rykov (1983, 1984) [25], [26].

6.1. System description

Consider a $GI/GI/1/\infty$ QS with recurrent arrival and service processes. The system is a regenerative one, its RT's are the arrival times that find the system empty. For simplicity it is supposed that in the initial time $t = 0$ a new call arrive in the empty system Denote by

- A_n — inter-arrival times (i.i.d. r.v.),
- B_n — their service times (i.i.d. r.v.),
- R_k — k-th RP duration,
- Π_k — k-th BP duration,
- ν_k — number of calls, served during k-th BP,
- N_k — number of call, which open k-th RP,
- S_k — arrival times in empty system (RT's),
- $S^{(o)}_{n,k}$ — service completion times at separate k-th RP,
- $S^{(i)}_{n,k}$ — arrival times within separate k-th RP,

The above values are calculated recursively:

$$N_0 = 0, \quad N_k = N_{k-1} + \nu_k, \quad \nu_k = \min\{n:\ n > N_{k-1},\ S^{(o)}_{n,k} < S^{(i)}_{n,k}\},$$

$$S^{(o)}_{n,k} = \sum_{N_k < i \le n} B_i, \quad S^{(i)}_{n,k} = \sum_{N_k < i \le n} A_i.$$

$$\Pi_k = S^{(o)}_{\nu_k,k}, \quad R_k = S^{(i)}_{\nu_k,k}, \quad S_0 = 0, \quad S_k = S_[k-1] + R_k.$$

6.2. Main processes

The main processes of the $GI/GI/1/\infty$ QS are:

- Number of calls (NoC) in the system $X(t)$,
- Virtual waiting time (VWT) $V(t)$,
- Actual waiting time (AWT) W_n.

The first two processes are strongly (in sense of Smith) regenerative, which regeneration times S_k are the times of new calls arrivals in empty system. The discrete time AWT process is also regenerative sequence with respect to numbers N_k of calls arrived into empty system.

The structure of RP's for this system is the same as for the system $M/GI/1/\infty$. However behavior of the main processes at RP's is enough complex, and for their investigation we'll use the construction ERP's. The structure of regeneration and busy periods at the figure 8 are presented.

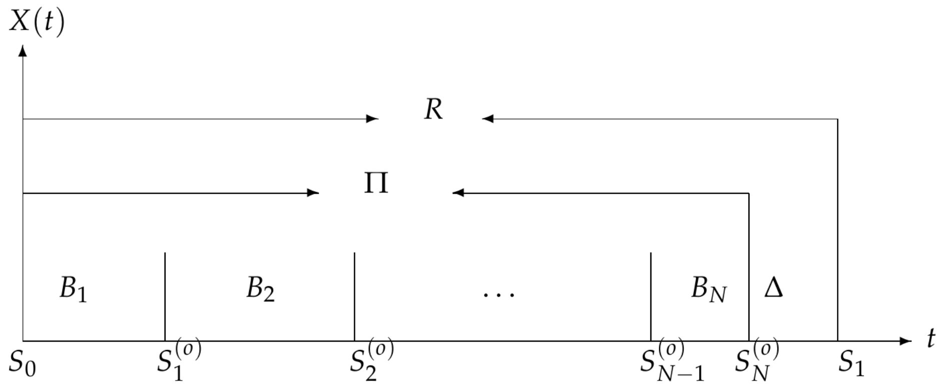

Figure 8: *Structure of the RP for the $GI/GI/1/\infty$-system*

Denote by

- $F(t) = \mathbf{P}\{R_n \le t\}$ RP c.d.f.;
- $f(s) = \mathbf{E}[e^{-sR_n}] = \int\limits_0^\infty e^{st} F(dt)$ the RP MGF;
- $P(z,t) = \mathbf{E}[z^{X(t)}]$ MGF of the N0C process X;
- $p(z,s) = \int\limits_0^\infty e^{-st} P(z,t)dt$ its LT;
- $P^{(R)}(z,t) = \mathbf{E}\left[z^{X(t)} 1\{t < R\}\right]$ process $X(t)$ MGF on a separate RP;
- $p(z,s) = \int\limits_0^\infty e^{-st} P^{(R)}(z,t)dt$ its LT;
- $V(x,t) = \mathbf{P}\{V(t) < x\}$ process $V(t)$ distribution;
- $v(x,s) = \int\limits_0^\infty e^{-st} V(x,t)dt$ its MGF;
- $V^{(R)}(x,t) = \mathbf{P}\{V(t) < x,\ t < R\}$ process $V(t)$ distribution on a separate RP;
- $v^{(R)}(x,s) = \int\limits_0^\infty e^{-st} V^{(R)}(x,t)dt$ its MGF within a separate RP;

- $H(t) = \sum\limits_{n\geq 1} \mathbf{P}\{S_n \leq t\} = \sum\limits_{n\geq 1} F^{(*n)}(t)$ the main processes RF.

According to the RPr's theory for LT the NoC and VWT processes it holds

$$\begin{aligned} p(z,s) &= p^{(R)}(z,s)(1+h(s)) = \frac{p^{(R)}(z,s)}{1-f(s)}; \\ v(x,s) &= s^{(R)}(x,s)(1+h(s)) = \frac{v^{(R)}(x,s)}{1-f(s)}; \end{aligned}$$

Analogous expressions take place for AWT process $\{W_n\}$ with respect to discrete RPr $\{N_k\}$, which discrete RF is

$$h_k = \mathbf{E}\left[\sum_{n\geq 0} 1_k(N_n)\right] = \sum_{n\geq 0} \mathbf{P}\{N_n = k\} = \sum_{n\geq 0} g^{*n}(k),$$

where $g(k) = \mathbf{P}\{N_1 = k\}$ is the distribution of the number of calls served at separate RP, and $g^{*n}(k)$ means the discrete n-th convolution of functions g at point k.

Therefore, to investigate the processes in any time point one should investigate them on a separate RP's. Because they are identically distributed it is enough to consider them at the first RP. To do that denote by $N = N_1 = \nu_1$ the number of calls served during the first RP and consider two-dimensional Random Walk $Y_n = (S_n^{(i)}, S_n^{(o)})$. For simplicity here and further the index of the RP number is omitted. It is evident that

$$N = \min\{n: \ S_n^{(o)} < S_n^{(i)}\}, \quad \Pi = S_N^{(o)}, \quad R = S_N^{(i)}.$$

6.3. Busy, idle periods and number of calls, served at BP

Denote by $(R_+^2, \mathcal{R}_+^2)$ the positive quadrant of an Euclid plane (t_1, t_2). Put

$$R_<^2 = \{(t_1, t_2): \ t_1 < t_2\}, \ R_\geq^2 = \{(t_1, t_2): \ t_1 \geq t_2\},$$

and introduce the sequence of measures

$$\begin{aligned} \pi_1(C) &= \mathbf{P}\{(A, B) \in C\} = \int\limits_{R_+^2} 1_C(t_1, t_2)A(dt_1)B(dt_2), \\ \pi_n(C) &= \mathbf{P}\{S_j^{(i)} < S_j^{(o)}: \ j = \overline{1, n-1}, \ (S_n^{(i)}, S_n^{(o)}) \in C\}. \end{aligned}$$

For $C \in R_\geq^2$ the value $\pi_n(C)$ is the probability of the event $\{N = n, \ (R, \Pi) \in C\}$,

$$\pi_n(C) = \mathbf{P}\{N = n, \ (R, \Pi) \in C\}.$$

So the parametric measure

$$\tilde{\pi}(z, C) = \sum_{n\geq 1} \pi_n(C) z^n$$

is the MGF of the number of calls served during RP jointly with RP and BP lengths. Many system characteristics can be expressed in terms of measure $\tilde{\pi}(z, C)$:

- For $C_x^1 = \{(t_1, t_2): \ t_2 - t_1 \geq x\}$ the value $1 - \tilde{\pi}(1, C_x^1)$ is server idle time distribution,

$$1 - \tilde{\pi}(1, C_x^1) = \mathbf{P}\{\Delta = R - \Pi \leq x\};$$

- For $C_x^2 = \{(t_1, t_2) \in R_\geq^2: \ t_1 < x\}$ the value $\tilde{\pi}(1, C_x^2)$ is the RP length distribution,

$$\tilde{\pi}(1, C_x^2) = \mathbf{P}\{R \leq x\};$$

- For $C_x^3 = \{(t_1, t_2) \in R^2_{\geq} : t_2 < x\}$ the value $\tilde{\pi}(1, C_x^3)$ is the BP length distribution

$$\tilde{\pi}(1, C_x^3) = \mathbf{P}\{\Pi \leq x\};$$

- For $C_x^0 = \{(t_1, t_2) \in R^2_{\geq} : t_2 > x > t_1\}$ one has

$$\tilde{\pi}(1, C_x^0) = \mathbf{P}\{\Pi \leq x < R\}.$$

In order to study the parametric measure $\tilde{\pi}(z, C)$ introduce a measure $q(c)$ and a kernel $Q(\omega, C)$ for $\omega \in R^2_+,\ C \in \mathcal{R}^2_+$ with $\mathbf{e}_1 = (1, 0),\ \mathbf{e}_2 = (0, 1)$ as follows:

$$\begin{aligned} q(C) &= \int\limits_{R^2_+} 1_C(t_1, t_2) A(dt_1) B(dt_2) = \pi_1(C); \\ Q(\omega, C) &= \int\limits_{R^2_+} 1_C(\omega + t_1\mathbf{e}_1,\ \omega + t_2\mathbf{e}_2) A(dt_1) B(dt_2). \end{aligned}$$

The following Theorem has been proved in Rykov (1983) [25].

Theorem 1. For $\rho = a^{-1}b < 1$ the function $\tilde{\pi}(z, C)$ is

(i) a probability measure on R^2_+ and a MGF with respect to z, i.e. $\tilde{\pi}(1, R^2_+) = 1$;

(ii) it satisfies to the equation

$$\tilde{\pi}(z, C) = z(q(C) + \tilde{\pi} o Q(C)), \tag{20}$$

where an operation $\tilde{\pi} o Q(C))$ means integration under the set $R^2_<$:

$$\tilde{\pi} o Q(C)) = \int\limits_{R^2_<} \tilde{\pi}(z, d\omega) Q(\omega, C).$$

■

It is possible to show that $Q(.,.)$ generates a continuous operator with the norm $\|Q\| = 1$, and so for $z < 1$ the equation (20) has a unique solution that could be represented in the form

$$\tilde{\pi}(z, C) = zqo \sum_{k \geq 0} z^k Q^{(ok)} = zqoG(z, C),$$

where

$$G(z, \omega, C) = \sum_{k \geq 0} z^k Q^{ok}(\omega, C). \tag{21}$$

Unfortunately, the condition $\|Q\| = 1$ does not allow directly to find the solution of this equation for $z = 1$ and does not guarantee its uniqueness. Nevertheless, the following theorem holds:

Theorem 2. (Rykov (1983) [25].) For $\rho = a^{-1}b < 1$ the kernel

$$G(z, \omega, C) = \mathbf{E}_\omega \left[\sum_{n \geq 0} z^n 1_{\{S_k^{(i)} < S_k^{(o)},\ k = \overline{1, n-1},\ (S_n^{(i)}, S_n^{(o)}) \in C\}} \right]$$

exists and is finite for any bounded sets $C \in \mathcal{R}^2_+$ and $|z| \leq 1$. Moreover, the representation (21) holds.

6.4. Investigation of the main processes on a separate BP

For investigation of VWT process on a separate BP consider the arrival epochs S_n^i as ERT's and introduce an ERK

$$U(\omega, C) = \mathbf{E}_\omega \left[\sum_{n<N} 1_C(V(S_n^{(i)}), S_n^{(i)}) \right],$$

that satisfies to the ERE

$$U(\omega, C) = Q^{(1)}(\omega, C) + U * Q^{(1)}(\omega, C) - Q(\omega, C).$$

Here Q and $Q^{(1)}$ are transition probability kernels of corresponding semi-Markov chains:

$$\begin{aligned} Q^{(1)}(\omega, C) &= \mathbf{E}_\omega 1_C(V(A), A) = \int_{R_+^2} 1_C(x + u - v, v) B(du) A(dv); \\ Q(\omega, C) &= \mathbf{E}_\omega 1_C(V(R), R) = \mathbf{P}\{(0, R) \in C\}. \end{aligned}$$

In the special case $\omega = 0$, ERK can be directly calculated

$$\begin{aligned} U(C) &= \mathbf{E}\left[\sum_{n<N} 1_C(V(S_n^i), S_n^i) \right] = \mathbf{E}\left[\sum_{n<N} 1_C(S_n^o - S_n^i), S_n^i) \right] = \\ &= \sum_{n\geq 0} \int_{R_<^2} \pi_n(du, dv) 1_C(v - u, u) = \int_{R_<^2} \tilde{\pi}(1, du, dv) 1_C(v - u, u). \end{aligned} \tag{22}$$

Therefore the DSRP theory leads to the following result.

Theorem 3. (Rykov (1984), [26]) C.d.f. $V^{(R)}(x, t)$ of VWT process for the $GI/GI/1/\infty$ QS on a separate RP is determined by the expression

$$\begin{aligned} V^{(R)}(x, t) &= (1 - A(t))(1 - B(x + t)) + \\ &+ \int\limits_{u<t, v<t-u+x} U(dv, du)(1 - A(t - u))(1 - B(t - u + x - v)), \end{aligned} \tag{23}$$

where ERK satisfies the expression (22).

Analogous argumentation allow to calculate the distribution of an AWT process on a separate RP. A little bit more complex argumentation that include ERT's of the second level is used for calculation of NoC distribution on a separate RP.

The above theorems proofs and detailed investigation of the $GI|GI|1|\infty$ QS with the DSRPr's methods one can find in Rykov (1983, 1984) [25], [26].

In further two sections some recent applications of DSRP will be proposed.

7. Reliability of a double redundant renewable system

Consider a homogeneous cold double redundant repairable system with generally distributed life- and repair times, which, according to modified Kendall's notations [12], will be denoted as $< GI_2/GI/1 >$. The system consists of two identical units which can be in two possible states: operational and failed. The system fails when both units are in a failed state. For the repairable system, different strategies of renovation are possible. In this section we consider a strategy when after the system failure it continues to operate in the previous regime, and after the repair of a failed unit, it returns into the state one, where a new system cycle begins, where one unit starts working while the other one being to repair (see figure 9).

7.1. The problem setup: assumption and notations

Denote by A_i $(i = 1, 2, \dots)$ lifetimes of the system units, by B_i $(i = 1, 2, \dots)$, their repair times, and suppose that all these r.v. are mutually independent and identically distributed for each sequence. Thus, denote by $A(t) = \mathbf{P}\{A_i \le t\}$ and $B(t) = \mathbf{P}\{B_i \le t\}$ the corresponding c.d.f. Suppose that the instantaneous failures and repairs are impossible and their mean times are finite:

$$A(0) = B(0) = 0, \quad a = \int_0^\infty (1 - A(x))dx < \infty, \quad b = \int_0^\infty (1 - B(x))dx < \infty,$$

and in the initial time $t = 0$ both units are in good state.

Denote by $E = \{i = 0, 1, 2\}$ the set of system states, where i means the number of failed units, and introduce a random process $X = \{X(t),\, t \ge 0\}$, where

$$X(t) = \text{number of failed units at time } t.$$

Denote by F the time between system failures, and by by F_1 the time to first system failure (see Figure 9). Their c.d.f. are: $F(t) = \mathbf{P}\{F \le t\}$ and $F_1(t) = \mathbf{P}\{F_1 \le t\}$. We are interesting in calculation of:

- the reliability function $R(t) = \mathbf{P}\{F > t] = 1 - F(t)$;
- the distribution of the time to the first system failure $F_1(t)$;
- the system t.d.p. $\pi_j(t) = \mathbf{P}\{X(t) = j\}$ $(j = 0, 1, 2)$;
- the s.s.p. $\pi_j = \lim\limits_{t\to\infty} \pi_j(t) \equiv \lim\limits_{t\to\infty} \mathbf{P}\{X(t) = j\},$ $(j = 0, 1, 2)$;
- the availability coefficient $K_{\text{av.}} = \pi_0 + \pi_1 = 1 - \pi_2$.

The following notations will be used next:

- a modified LT:

$$\tilde{a}_B(s) = \int_0^\infty e^{-sx} B(x)dA(x), \qquad \tilde{b}_A(s) = \int_0^\infty e^{-sx} A(x)dB(x); \tag{24}$$

- the modified mean values:

$$a_B = -\frac{d}{ds}\tilde{a}_B(s)\big|_{s=0} = \int_0^\infty xB(x)dA(x), \quad b_A = -\frac{d}{ds}\tilde{b}_A(s)\big|_{s=0} = \int_0^\infty xA(x)dB(x); \tag{25}$$

- the probabilities $\mathbf{P}\{B \le A\}$ and $\mathbf{P}\{B \ge A\}$ associated with these transformations through the relations:

$$p \equiv \mathbf{P}\{B \le A\} = \tilde{a}_B(0), \quad q = 1 - p \equiv \mathbf{P}\{B > A\} = \tilde{b}_A(0).$$

Note the property of transformations (24)

$$\tilde{a}_{1-B}(s) = \tilde{a}(s) - \tilde{a}_B(s), \tag{26}$$

7.2. Reliability Function

Process X is a regenerative one. A trajectory of this process is illustrated on Figure 9. Here, F means the time between system failures. The variable G specifies the length of a RP according to a defined renovation policy. The following lemma holds for the LSTs of the time F between failures and the time to the first failure F_1.

Lemma 1. The LST $\tilde{f}(s) = \mathbf{E}\left[e^{-sF}\right]$ of the time F between failures and the LST $\tilde{f}_1(s) = \mathbf{E}\left[e^{-sF_1}\right]$ of the time to the first failure F_1 are of the form:

$$\tilde{f}(s) = \frac{\tilde{a}(s) - \tilde{a}_B(s)}{1 - \tilde{a}_B(s)}, \qquad \tilde{f}_1(s) = \tilde{a}(s)\frac{\tilde{a}(s) - \tilde{a}_B(s)}{1 - \tilde{a}_B(s)}. \tag{27}$$

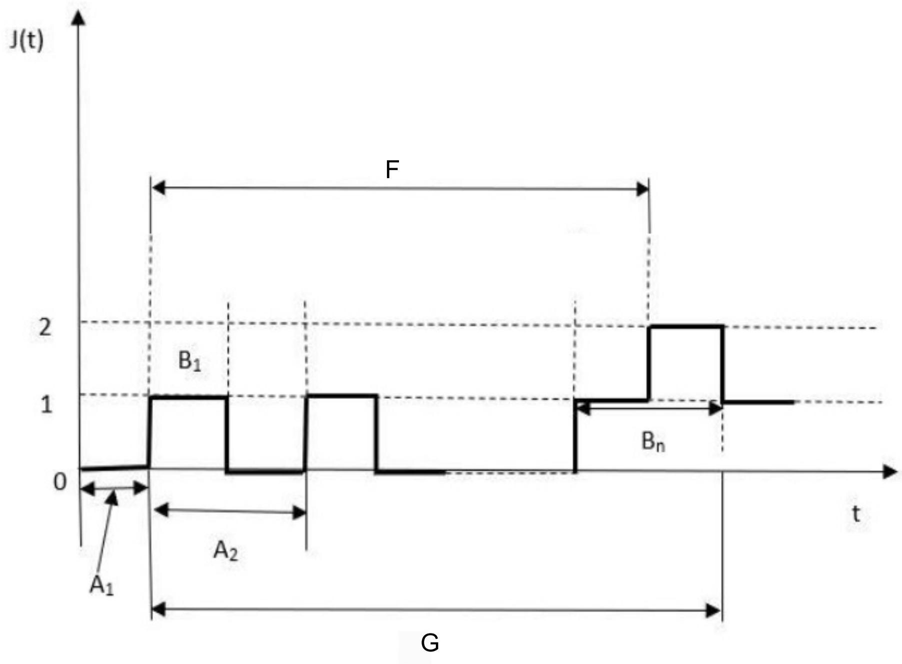

Figure 9: *Trajectory of the process X.*

Proof. From the Figure 9 can be seen that the system lifetime F satisfies the following stochastic equation:

$$F = \begin{cases} A + F & \text{if } B < A, \\ A & \text{if } B > A. \end{cases} \tag{28}$$

Applying the LT to this equation and taking into account notations (24), one can obtain

$$\tilde{f}(s) = \mathbf{E}\left[e^{-sF}\right] = \int_0^\infty e^{-st} dF(t) = \tilde{f}(s)\tilde{a}_B(s) + \tilde{a}_{1-B}(s). \tag{29}$$

From here the first relation in of (27) follows. The second one directly follows from the stochastic relation $F_1 = A + F$. ∎

The main result of this subsection is the following theorem:

Theorem 4. The LT $\tilde{R}(s)$ of the system reliability function $R(t) = 1 - F(t)$ is

$$\tilde{R}(s) = \frac{1 - \tilde{a}(s)}{s(1 - \tilde{a}_B(s))}. \tag{30}$$

Proof. Taking into account that the LT of any c.d.f. is connected with its LST by the relation $\tilde{F}(s) = s^{-1}\tilde{f}(s)$, the proof follows directly from (27). ∎

The expected system life time between failures $\mathbf{E}[F]$ and mean time to the first failure $\mathbf{E}[F_1]$ are obtained in the form:

$$\mathbf{E}[F] = \frac{a}{q}, \quad \mathbf{E}[F_1] = a + \frac{a}{q}.$$

7.3. Time dependent system state probabilities

For calculation of the system state t.d.p.'s the renewal theory is used. In our case, the process X is a RPr with a delay (see Figure 9) and its RT's are

$$S_0 = 0,\ S_1 = A_1,\ S_2 = S_1 + G_1,\ \ldots,\ S_{k+1} = S_k + G_k,\ \ldots.$$

Here, RP's G_i $(i = 1, 2, \dots)$ are the time intervals between two successive returns of the process X into state 1 after a system failure, when one of the system units begins to operate and the other one begins to repair. Thus, the process X state t.d.p.'s $\pi_j(t) = \mathbf{P}\{X(t) = j\}$ $(j = 0, 1, 2)$ can be represented in terms of its distribution on a separate RP $\pi_j^{(1)}(t) = \mathbf{P}[X(t) = j,\ t < G]$ $(j = 0, 1, 2)$ and the renewal function $H(t)$ as follows:

$$\pi_i(t) = \pi_i^{(1,1)}(t) + \int_0^t dH(u)\pi_j^{(1)}(t-u). \tag{31}$$

Here the distribution of the process X at the first RP is of the form

$$\pi_j^{(1,1)}(t) = \mathbb{P}[X(t) = j,\ t < A_1] = \delta_{j0}(1 - A(t)). \tag{32}$$

In terms of the c.d.f. $G(t) = \mathbf{P}\{G_i \le t\}$ of r.v.'s G_i, the RF $H(t)$ is determined as follows:

$$H(t) = \sum_{k\ge1} \mathbf{P}\left[\left(\sum_{1\le i\le k} G_i\right) \le t\right] = \sum_{k\ge1} G^{*k}(t). \tag{33}$$

Consider, first of all, the RP distribution.

Lemma 2. The LST of the RP is of the form:

$$\tilde{g}(s) = \mathbb{E}[e^{-sG}] = \frac{\tilde{b}_A(s)}{1 - \tilde{a}_B(s)}. \tag{34}$$

Proof. The RP is the time between two successive visits of the process to state 1 from the state 2, when two events begin simultaneously: operating of one unit and the repair the other one. Figure 9 shows that r.v. G satisfies the following stochastic equation:

$$G = \begin{cases} A + G & \text{if } A > B, \\ B & \text{if } A \le B. \end{cases} \tag{35}$$

Applying LST to this stochastic equation leads to the equation

$$\begin{aligned} \tilde{g}(s) &= \mathbb{E}\left[e^{-sG}\right] = \int_0^\infty e^{-st} dG(t) = \int_0^\infty dA(x)\left[B(x)e^{-sx}\tilde{g}(s) + \int_{y>x} e^{-sy} dB(y)\right] = \\ &= \tilde{g}(s)\int_0^\infty e^{-sx}B(x)dA(x) + \int_0^\infty e^{-sy}A(y)dB(y) = \tilde{g}(s)\tilde{a}_B(s) + \tilde{b}_A(s), \end{aligned} \tag{36}$$

which implies the expression (34) for the LST of the RP. ■

By differentiation due to properties of $\tilde{a}_B(0)$ and $\tilde{b}_A(0)$ one can obtain the mean length of the RP as:

$$\mathbf{E}[G] = \frac{a_B + b_A}{q}.$$

Lemma 3. The LST of the system RF is given by

$$\tilde{h}(s) = \frac{\tilde{b}_A(s)}{1 - (\tilde{a}_B(s) + \tilde{b}_A(s))}. \tag{37}$$

Proof. From the renewal theory, it is well known (and follows from (33)) that the LST of the RF $H(t)$ is defined as $\tilde{h}(s) = \tilde{g}(s)(1 - \tilde{g}(s))^{-1}$. Thus, substitution of the expression (34) for $\tilde{g}(s)$ into this one leads to (37). ■

Theorem 5. The LT's $\tilde{\pi}_j(s)$ of the process X state t.d.p's. $\pi_j(t)$ $(j = 0, 1, 2)$ are of the form:

$$\tilde{\pi}_j(s) = \delta_{j0}\frac{1 - \tilde{a}(s)}{s} + \frac{\tilde{b}_a(s)}{1 - (\tilde{a}_B(s) + \tilde{b}_A(s))}\tilde{\pi}_j^{(1)}(s), \ (j = 0, 1, 2), \tag{38}$$

where $\tilde{\pi}_j^{(1)}(s)$ $(j = 0, 1, 2)$ are the LTs of the t.d.p.'s $\pi_j^{(1)}(t)$ $(j = 0, 1, 2)$ in a separate RP. These probabilities will be calculated in he next subsection.

Proof. Applying LT to equation (31) and taking into account equation (32), one can obtain

$$\tilde{\pi}_j(s) = \delta_{j0}\frac{1 - \tilde{a}(s)}{s} + \tilde{h}(s)\tilde{\pi}_j^{(1)}(s). \tag{39}$$

A substitution into this equality of the expression (37) for $\tilde{h}(s)$ leads to (38). ∎

7.4. The state probabilities on a separate RP

Now, we calculate of the process state t.d.p.'s on a separate RP. The probability $\pi_2^{(1)}(t)$ can be calculated easy for the main level RP.

Lemma 4. The LT of the second t.d.p. $\pi_2^{(1)}(t)$ in the main RP is given by

$$\tilde{\pi}_2^{(1)}(s) = \frac{\tilde{a}(s) - (\tilde{a}_B(s) + \tilde{b}_A(s))}{s(1 - \tilde{a}_B(s))}. \tag{40}$$

Proof. Due to representation of the RP by formula (35), and as it is shown in Figure 9, the event $\{X(t) = 2, \ t < G\}$ occurs if and only if either the event $\{A_1 \le t \le B_1\}$ occurs or the events $\{t > u = A_1 > B_1\}$ and $\{X(t-u) = 2, \ t - u < G\}$ occur. Thus, it holds

$$\pi_2^{(1)}(t) = \mathbb{P}[J(t) = 2, \ t < G] = \mathbb{P}[A \le t < B] + \int_0^t dA(u)B(u)\pi_2^{(1)}(t-u).$$

From this equation it follows the LST

$$\tilde{\pi}_2^{(1)}(s) = \int_0^\infty e^{-st}A(t)(1 - B(t))dt + \tilde{a}_B(s)\pi_2^{(1)}(s),$$

and

$$\tilde{\pi}_2^{(1)}(s) = \frac{1}{1 - \tilde{a}_B(s)}\int_0^\infty e^{-st}A(t)(1 - B(t))dt.$$

Calculating the integral in the last expressions by partial integration we get (40). ∎

Since the calculation of the probabilities $\pi_j^{(1)}$ $(j = 0, 1)$ is not a trivial task, we intend to apply the theory of DSRP [6, 9, 10]. For this consider the process X as an ERPr at the time interval F, which ERT's are the random number $\nu = \min\{n : \ A_n < B_n\}$ of time epochs

$$S_1^{(1)} = A_1 1_{\{A_1 > B_1\}}, \ S_2^{(1)} = S_1^{(1)} + A_2 1_{\{A_1 > B_1, \ A_2 > B_2\}}, \dots$$

up to the time, when the event $\{A_n \le B_n\}$ happens for first time. It means that the time interval $G^{(1)}$ between ERT's has a distribution $G^{(1)}(t) = A(t)$ and these epochs lie within the time interval F, which is determined by the equation (28).

According to this theory, the process distribution within the basic RP (RP of the first level) $\pi_i^{(1)}(t)$, similar to the equation (31), can be presented in terms of distributions in ERP's (RP's of the second level) $\pi_j^{(2)}(t)$ and ERF $H^{(1)}(t)$ in the following way:

$$\pi_j^{(1)}(t) = \pi_j^{(2)}(t) + \int_0^\infty dH^{(1)}(u)\pi_j^{(2)}(t-u), \;\; j = 0, 1, \tag{41}$$

where

$$\pi_j^{(2)}(t) = \mathbf{P}\{X(t) = j, \; t < G^{(2)}\}, \;\; j = 0, 1$$

is the process t.d.p.'s on a separate RP of the second level, and the ERF $H^{(1)}(t)$ satisfies the equation

$$H^{(1)}(t) = A(t) + \int_0^t dH^{(1)}(u)A(t-u) - F(t), \tag{42}$$

where $F(t)$ is the CDF of the time between system failures determined by its LST (27).

Similar to the basic case, the solution of equations (41) and (42) can be represented in terms of their LTs and LSTs:

$$\tilde{\pi}_j^{(2)}(s) = \int_0^\infty e^{-st}\pi_j^{(2)}(t)dt, \quad \tilde{h}^{(1)}(s) = \int_0^\infty e^{-st}dH^{(1)}(t).$$

The next lemma specifies connections between process distributions in the first and in the second level regeneration cycles in terms of their LTs.

Lemma 5. The LT's of the process t.d.p.'s of the first and in the second level RP's satisfy the relation

$$\tilde{\pi}_j^{(1)}(s) = \tilde{\pi}_j^{(2)}(s)\frac{\tilde{a}_B(s)}{1 - \tilde{a}_B(s)} \;\; j = 0, 1. \tag{43}$$

Proof. Applying LT to equation (41), we get

$$\tilde{\pi}_j^{(1)}(s) = (1 + \tilde{h}^{(1)}(s))\tilde{\pi}_j^{(2)}(s). \tag{44}$$

Due to (42), the LST $\tilde{h}^{(1)}(s)$ of the embedded renewal function $H^{(1)}(t)$ is of the form

$$\tilde{h}^{(1)}(s) = \tilde{a}(s) + \tilde{h}^{(1)}(s)\tilde{a}(s) - \tilde{f}(s),$$

which leads in turn to

$$\tilde{h}^{(1)}(s) = \frac{\tilde{a}(s) - \tilde{f}(s)}{1 - \tilde{a}(s)}.$$

Substitution of this relation into equation (44) and taking into account the expressions for $\tilde{f}(s)$ from (27) we get the result (40) that completes the proof. ■

We have to calculate now only the $\tilde{\pi}_j^{(2)}(s) \;\; (j = 0, 1)$.

Lemma 6. In notations (24), the LT's of the second level system state probabilities are:

$$\begin{aligned} \tilde{\pi}_0^{(2)}(s) &= \frac{1}{s}(\tilde{a}(s) - (\tilde{a}_B(s) + \tilde{b}_A(s))); \\ \tilde{\pi}_1^{(2)}(s) &= \frac{1}{s}[1 - (\tilde{a}(s) + \tilde{b}(s)) + (\tilde{a}_B(s) + \tilde{b}_A(s))]. \end{aligned} \tag{45}$$

Proof. For probabilities $\tilde{\pi}_j^{(2)}(t) \;\; (j = 0, 1)$ from Figure 9, it follows that

- The event $\{X(t) = 0, \; t < G^{(1)}$ occurs if and only if $\{B \le t < A\}$;
- The event $\{X(t) = 1, \; t < G^{(1)}\}$ occurs if and only if $\{t < B \le A\}$, or if $\{t < A \le B\}$.

Hence, the respective probabilities are

$$\begin{aligned}\pi_0^{(2)}(t) &= \mathbf{P}\{B < t < A\} = B(t)(1 - A(t)),\\ \pi_1^{(2)}(t) &= \mathbf{P}\{t < B < A\} + \mathbf{P}\{t < A < B\}.\end{aligned} \quad (46)$$

Calculation LT's of these expressions by partial integration in terms of notations (24) leads to (45), that ends the proof. The details of calculation one can find in [28]. ■

By combining all these results in [28], the process t.d.p. have been found. They are too cumbersome and are omitted here. We show here only the system s.s.p.'s. By using a Tauber theorem

$$\pi_j = \lim_{t\to\infty} \pi_j(t) = \lim_{s\to 0} s\tilde{\pi}_j(s) \quad (47)$$

in [28] for the process s.s.p.'s the following results have been obtained.

Theorem 6. The system state stationary probabilities are:

$$\pi_0 = 1 - \frac{b}{a_B + b_A}, \quad \pi_1 = \frac{a+b}{a_B + b_A} - 1, \quad \pi_2 = 1 - \frac{a}{a_B + b_A}. \quad (48)$$

For a Markov model $< M_2|M|1 >$, when $A(t) = 1 - e^{-\alpha t}$, $B(t) = 1 - e^{-\beta t}$ this result coincides with those calculated by direct approach using Birth and Death process for the Markov case.

8. k-out-of-n system

In this section we apply the theory of DSRP to study of k-out-of-n : F model, which has applications in many real-world phenomena. There are many papers devoted to investigations of this model. A detailed review of previous investigations of the model one can find in [29]. Some special applications of this model to engineering problems in oil and gs industry one can find in [35] and [36].

8.1. Stating the problem. Notations

Consider k-out-of-n : F system, which can be considered as a reparable n-components reliability system in parallel that fails when k of its components fail. It is supposed that the life times of the systems' components are i.i.d. r.v.'s with common exponential distribution of parameter α. Failed components are repaired by a single facility. Repair times are i.i.d. r.v. B_i $(i = 1, 2, \dots)$ with the common c.d.f. $B(t) = \mathbf{P}\{B_i \le t\}$.

For the system study introduce the following notations:

- $E = \{0, 1, \dots k\}$ is the system set of states, where j means number of failed components and k is the system failure state;
- $\lambda_i = (n - i)\alpha$ intensity of one of components failure, when the system is in the state i;
- define the random process $X = \{X(t),\ t \ge 0\}$ by the relation

$$X(t) = j, \text{ if in time epoch } t \text{ the system is in the state } j \in E;$$

- system (and the process) t.d.s. probabilities $\pi_j(t) = \mathbf{P}\{X(t) = j\}$;
- the process s.s.p. $\pi_j = \lim_{t\to\infty} \mathbf{P}\{X(t) = j\}$;
- the time T to the system failure, $T = \inf\{t : X(t) = k\}$;
- reliability function $R(t) = \mathbf{P}\{T > t\}$.

For repairable k-out-of-$n : F$ system there exist at least two possible scenarios of the system repair after its fail:

- **Partial repair regime**, when after the system failure it continues to work in previous regime and after the repair of repaired component it passes to the state $k-1$;
- **Full repair regime**, when after the system failure the repair of whole system begins, after which the system becomes as a new one, and comes to the state 0. After each system failure the full repair duration are i.i.d. r.v. G_i $(i = 1, 2, \dots)$ with the common c.d.f. $G(t) = \mathbf{P}\{G_i \le t\}$.

Suppose that in the very beginning all system components are in good (UP) state, which means that the initial state of the process is zero, $X(0) = 0$. It is also supposed that immediate components failures and repairs are impossible and mean failure and repair times are finite,

$$B(0) = G(0) = 0, \quad \int_0^\infty (1 - B(t))dt < \infty, \quad \int_0^\infty (1 - G(t))dt < \infty. \tag{49}$$

Further in the section the system t.d.p. $\pi_j(t)$, reliability function $R(t)$, and the system s.s.p. π_j are calculated.

8.2. Partial repair regime.

Consider firstly partial repair regime, when after the system failure the repair of previously failed component is continued and after its end the system goes to the state $k-1$. In this case we will consider the process X as a semi-regenerative one (see figure 10). Its regeneration times S_n of the type j are the times of repair end when system occurs in state j, $X(S_n + 0) = j$, SRP's are $T_n = S_n - S_{n-1}$, and the RS's is $E_1 = \{j : (j = \overline{0, k-1})\}$.

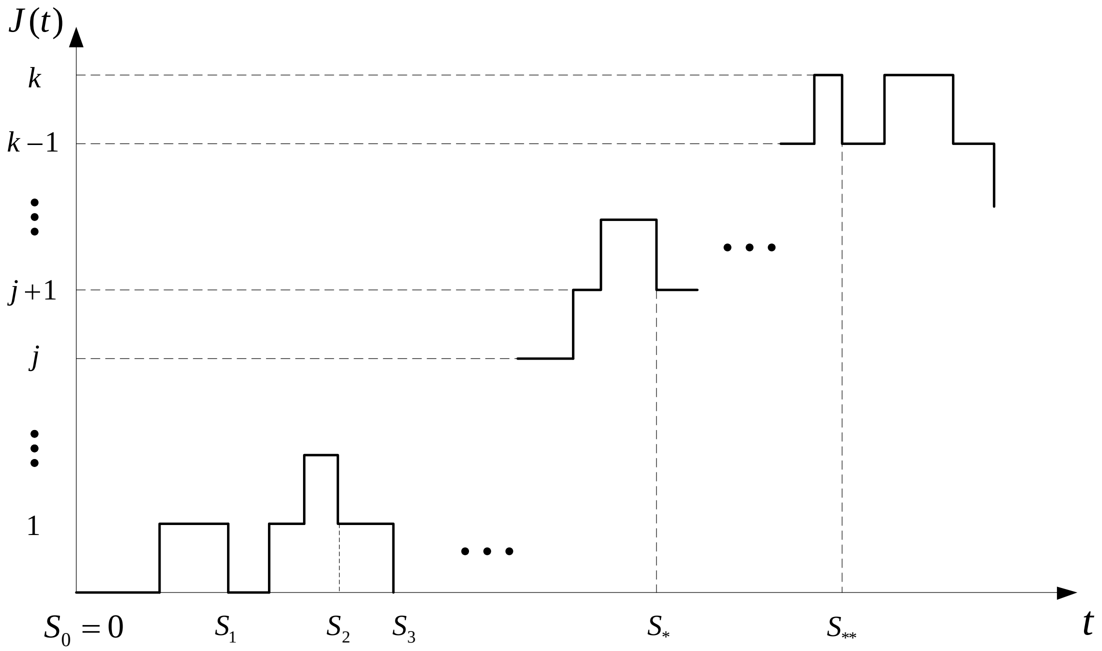

Figure 10: *Trajectory of the process J for system with partial repair*

The SMP X behavior is determined by its SMM $Q(t) = [Q_{ij}(t)]_{ij \in E_1}$ with transition probabilities

$$Q_{ij}(t) = \mathbf{P}\{X(S_n + 0) = j \;\; T_n \le t \,|X(S_{n-1} + 0) = i\}.$$

For its calculation denote by

$$p_{ij}(t) = \binom{n-i}{j-i}(1-e^{-\alpha t})^{j-i}e^{-(n-j)\alpha t}$$

the probability that during time t the process passes from the state i to the state j. Let

$$P_{ik}(t) = \sum_{j\geq k} p_{ij}(t) = 1 - \sum_{i\leq j\leq k-1} p_{ij}(t)$$

be the probability that starting from state i, the process leaves the subset of states E_1 during time t. Note that the last probability is the lifetime c.d.f. of the non-reparable k-out-of-$n : F$ system starting from the state i. To simplify further calculations, we represent these probabilities by Newton's binomial formula by using for simplicity the substitution $\lambda_i = (n-i)\alpha$ as

$$p_{ij}(t) = \binom{n-i}{j-i} e^{-\lambda_j t} \sum_{m=0}^{j-i} (-1)^m \binom{j-i}{m} e^{-\alpha m t}. \tag{50}$$

and

$$P_{ik}(t) = 1 - \sum_{i\leq j\leq k-1} \binom{n-i}{j-i} e^{-\lambda_j t} \sum_{m=0}^{j-i} (-1)^m \binom{j-i}{m} e^{-\alpha m t}. \tag{51}$$

Using these notations for the SMM in [29] the following lemma has been proved.

Lemma 7. The differentials of the process X SMMs components are:

$$\begin{aligned}
Q_{0j}(dt) &= \int_0^t \lambda_0 e^{-\lambda_0 u}\, du \; p_{1j+1}(t-u)B(dt-u), \;\; j=\overline{0,k-2}; \\
Q_{0k-1}(dt) &= \int_0^t \lambda_0 e^{-\lambda_0 u}\, du \; P_{1k}(t-u)B(dt-u); \\
Q_{ij}(dt) &= p_{ij+1}(t)B(dt), \;\; (i=\overline{1,k-2},\; j=\overline{i-1,k-2}); \\
Q_{ik-1}(dt) &= P_{ik}(t)B(dt).
\end{aligned} \tag{52}$$

Their LST $\tilde{q}_{ij}(s) = \int_0^\infty e^{-st}Q_{ij}(dt)$ are:

$$\begin{aligned}
\tilde{q}_{0j}(s) &= \frac{\lambda_0}{s+\lambda_0}\binom{n-1}{j}\sum_{m=0}^{j}(-1)^m\binom{j}{m}\tilde{b}(s+\lambda_{j+1-m}), \;\; j=\overline{0,k-2}; \\
\tilde{q}_{0k-1}(s) &= \frac{\lambda_0}{s+\lambda_0}\sum_{j\geq k}\binom{n-1}{j-1}\sum_{m=0}^{j-1}(-1)^m\binom{j-1}{m}\tilde{b}(s+\lambda_{j-m}); \\
\tilde{q}_{ij}(s) &= \binom{n-i}{j-i+1}\sum_{m=0}^{j-i+1}(-1)^m\binom{j-i+1}{m}\tilde{b}(s+\lambda_{j+1-m}); \\
\tilde{q}_{ik-1}(s) &= \sum_{j\geq k}\binom{n-i}{j-i}\sum_{m=0}^{j-i}(-1)^m\binom{j-i}{m}\tilde{b}(s+\lambda_{j-m}),
\end{aligned} \tag{53}$$

Remind now that the state t.d.p.'s of the process X are determined not only by its SMM but also by the initial distribution $\vec{\alpha}^{(0)} = \{\alpha_i^{(0)} : \; i \in E\}$. Thus, denoting by $\vec{\pi}(t) = \{\pi_j(t) : \; j \in E\}$ the vector of the process state probabilities, where

$$\pi_j(t) = \mathbf{P}\{X(t) = j\} \;\; (j \in E),$$

and by $\Pi(t) = [\pi_{ij}(t)]_{ij\in E}$ the probability transition matrix of the process X, where

$$\pi_{ij}(t) = \mathbf{P}\{X(t) = j \,|\, X(0) = i)\} \;\; (i,j \in E)$$

in matrix form it means that $\vec{\pi}(t) = \vec{\alpha}^{(0)}\Pi(t)$.

On the other side, according to the theory of SRPr, the process transition probabilities $\Pi(t)$ in terms of appropriate transition probabilities $\Pi^{(1)}(t) = [\pi_{ij}^{(1)}(t)]_{ij\in E}$ where

$$\pi_{ij}^{(1)}(t) = \mathbf{P}\{X(S_{n-1}+t) = j,\ t \le T_n \mid X(S_{n-1}+0) = i)\}\ \ (i \in E_1,\ j \in E)$$

are the transition probabilities on separate RP's can be represented in the form

$$\Pi(t) = \Pi^{(1)}(t) + H \star \Pi^{(1)}(t). \tag{54}$$

here the MRM $H(t) = \left[H_{ij}(t)\right]_{ij\in E_1}$ with

$$H_{ij}(t) = \mathbf{E}\left[\sum_{n\ge 1} 1_{\{S_n \le t,\ J(S_n)=j\}} \mid X(0) = i\right]$$

satisfies to the equations

$$H(t) = Q(t) + Q \star H(t). \tag{55}$$

The above results show that the best way for the system state t.d.p. representation and the equations for MRM solution is its representation in terms of LS and LST. Therefore passing to LT

$$\tilde{\Pi}(s) = \int_0^\infty e^{-st}\Pi(t)dt, \quad \tilde{\Pi}^{(1)}(s) = \int_0^\infty e^{-st}\Pi^{(1)}(t)dt,$$

and LST

$$\tilde{q}(s) = \int_0^\infty e^{-st}Q(dt), \quad \tilde{h}(s) = \int_0^\infty e^{-st}H(dt)$$

from equations (54) and (55) one can obtain the following results:

$$\tilde{\Pi}(s) = \tilde{\Pi}^{(1)}(s) + \tilde{h}(s)\cdot\tilde{\Pi}^{(1)}(s) \tag{56}$$

and

$$\tilde{h}(s) = \tilde{q}(s) + \tilde{q}(s)\cdot\tilde{h}(s). \tag{57}$$

At least the process t.d.p. on separate RP's given in the following lemma (its proof can be found in [29]):

Lemma 8. The process state t.d.p. on a separate SRP are:

$$\begin{aligned}
\pi_{00}^{(1)}(t) &= e^{-\lambda_0 t};\\
\pi_{0j}^{(1)}(t) &= \int_0^t \lambda_0 e^{-\lambda_0 u} p_{1j}(t-u)(1-B(t-u))du\ \ (j=\overline{1,k-1});\\
\pi_{0k}^{(1)}(t) &= \int_0^t \lambda_0 e^{-\lambda_0 u} P_{1k}(t-u)(1-B(t-u))du;\\
\pi_{ij}^{(1)}(t) &= p_{ij}(t)(1-B(t))\ \ (1\le i\le j\le k-1);\\
\pi_{ik}^{(1)}(t) &= P_{ik}(t)(1-B(t))\ \ (i=\overline{1,k-1}).
\end{aligned} \tag{58}$$

Their LT $\tilde{\pi}_{ij}^{(1)}(s)$ are:

$$\begin{aligned}
\tilde{\pi}_{00}^{(1)}(s) &= \frac{1}{s+\lambda_0};\\
\tilde{\pi}_{0j}^{(1)}(s) &= \frac{\lambda_0}{s+\lambda_0}\binom{n-1}{j-1}\sum_{m=0}^{j-1}(-1)^m\binom{j-1}{m}\frac{1-\tilde{b}(s+\lambda_{j-m})}{s+\lambda_{j-m}} \quad (j=\overline{1,k-1});\\
\tilde{\pi}_{0k}^{(1)}(s) &= \frac{\lambda_0}{s+\lambda_0}\sum_{j\geq k}\binom{n-1}{j-1}\sum_{m=0}^{j-1}(-1)^m\binom{j-1}{m}\frac{1-\tilde{b}(s+\lambda_{j-m})}{s+\lambda_{j-m}};\\
\tilde{\pi}_{ij}^{(1)}(s) &= \binom{n-i}{j-i}\sum_{m=0}^{j-i}(-1)^m\binom{j-i}{m}\frac{1-\tilde{b}(s+\lambda_{j-m})}{s+\lambda_{j-m}}, \quad (1\leq i\leq j\leq k-1);\\
\tilde{\pi}_{ik}^{(1)}(s) &= \sum_{j\geq k}\binom{n-i}{j-i}\sum_{m=0}^{j-i}(-1)^m\binom{j-i}{m}\frac{1-\tilde{b}(s+\lambda_{j-m})}{s+\lambda_{j-m}}. \qquad (59)
\end{aligned}$$

By joining the above results the following theorem follows

Theorem 7. The LT of the process X state t.d.p. in matrix form given by the equality

$$\tilde{\Pi}(s) = (I-\tilde{q}(s))^{-1}\tilde{\Pi}^{(1)}(s), \qquad (60)$$

where components $\tilde{\pi}_{ij}^{(1)}(s)$ of the matrix $\tilde{\Pi}^{(1)}(s)$ are given by formulas (59) in lemma 8, and components $\tilde{q}_{lj}(s)$ of matrix $\tilde{q}(s)$ given by the formula (52) from lemma 7. ■

The s.s.p.'s of the process could be calculated by passing to limits as $t\to\infty$ in the last equality. But it would be preferable to use the limit theorem for transition probabilities of SRPr's.

Theorem 8. The stationary regime of the considered system under partial repair regime exists and its s.s.p.'s equal

$$\pi_j = \frac{1}{m}\sum_{0\leq l\leq j\wedge(k-1)}\alpha_l\tilde{\pi}_{lj}^{(1)}(0) \qquad (j=\overline{0,k}), \qquad (61)$$

where $m = \lambda_0^{-1}(\alpha_0+\lambda_0 b)$, $\tilde{\pi}_{ij}^{(1)}(0)$ can be found from the formulas (59) of lemma 8, and $\vec{\alpha} = \{\alpha_l:\ l\in E\}$ is the invariant probability measure of the embedded Markov chain that satisfies the system of equations

$$\vec{\alpha}' = \vec{\alpha}'\tilde{q}(0), \qquad \sum_{l\in E}\alpha_l = 1.$$

■

8.3. Full repair regime.

For study of the system behavior under the full repair regime we will consider the main process X as a regenerative one, whose regeneration points of time $S_n\ (n=1,2,\dots), S_0=0$ are the times when the system fully repaired after its failure. The regeneration periods $\Theta_n = S_n - S_{n-1}$ of the process X consist of two terms: the system life times (times to the system failure after its repair) F_n and the system repair times after its failure $G_n:\ \Theta_n = F_n + G_n$ (see figure 11). Denote by $F(t) = \mathbf{P}\{F_n\leq t\}$ and $\Gamma(t) = \mathbf{P}\{\Theta_n\leq t\}$ the common distribution function of r.v.'s F_n and $\Theta_n\ (n=1,2,\dots)$.

From the RPr's theory in sense of Smith it follows that the process t.d.p. $\pi_j(t) = \mathbf{P}\{X(t)=j\}$ can be represented in terms of corresponding process t.d.p. $\pi_j^{(\Theta)}(t) = \mathbf{P}\{X(t)=j,\ t<\Theta\}$ at separate RP Θ as follows

$$\pi_j(t) = \pi_j^{(\Theta)}(t) + \int_0^t \pi_j^{(\Theta)}(t-u)H(du). \qquad (62)$$

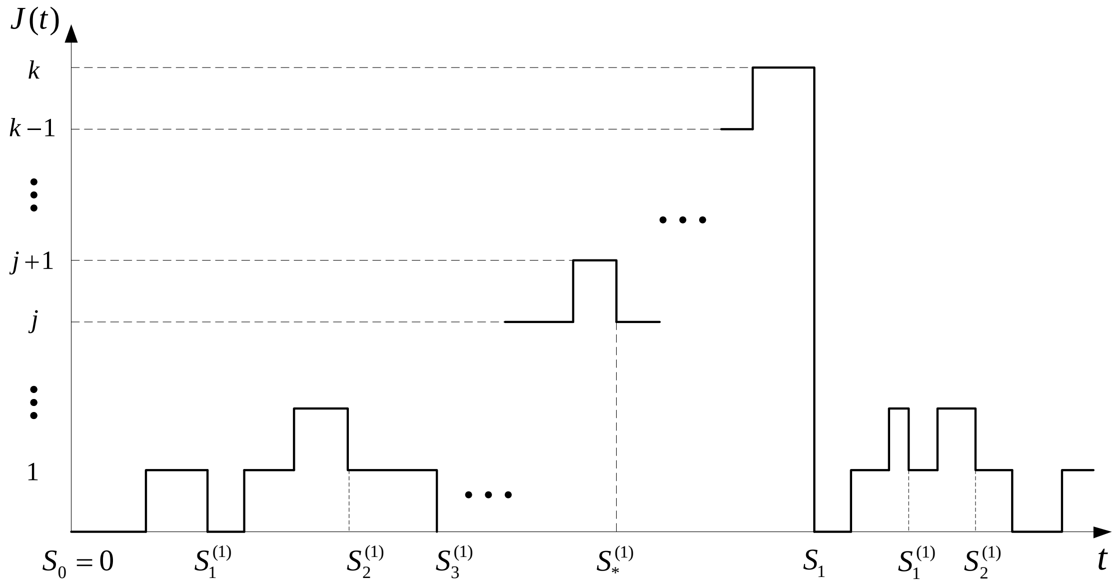

Figure 11: *The main process as a regenerative one.*

Here $H(t)$ is the appropriate RF, which is generated by the distribution $\Gamma(t) = \mathbf{P}\{\Theta_n \le t\}$ of r.v's. Θ_n, and can be calculated as

$$H(t) = \sum_{n\ge1} \mathbf{P}\{S_n \le t\} = \sum_{n\ge1} \Gamma^{(*n)}(t). \tag{63}$$

Its LST equals

$$\tilde{h}(s) = \int_0^\infty e^{-st} H(dt) = \frac{\tilde{\gamma}(s)}{1-\tilde{\gamma}(s)}, \tag{64}$$

where $\tilde{\gamma}(s)$ is the LST of the c.d.f. $\Gamma(t)$.

Thus, for the process analysis we need firstly to calculate the RP Θ_n distribution and the process distribution $\pi_j^{(\Theta)}(t)$ on them. Remind that the r.v. Θ_n is the sum of two independent r.v.'s F_n and G_n, the distributions of the second one is supposed to be known and the distribution of the first one will be done jointly with its LST later in lemma 10 (see Corollary 1 from it).

Let us turn now to calculation of the process t.d.p.'s $\pi_j^{(\Theta)}(t)$ on a separate RP. The process behavior on the separate RP's $\Theta_n = F_n + G_n$ can be divided into two parts: as a process behavior at separate system lifetime F_n and its behavior during the repair time G_n,

$$\{X(S_{n-1}+t) = j,\ t \le \theta_n\} = \begin{cases} X(S_{n-1}+t) = j & \text{for } t \le F_n \ \ (j \ne k), \\ k & \text{for } F_n < t \le \Theta_n. \end{cases} \tag{65}$$

Thus, the process t.d.p.'s on at a separate RP are given in the following lemma.

Lemma 9. The process state t.d.p.'s at the separate RP Θ in terms of according probabilities on separate system life times F are

$$\pi_j^{(\Theta)}(t) = (1-\delta_{jk})\pi_j^{(F)}(t) + \delta_{jk}\mathbf{P}\{F \le t < \Theta\} \tag{66}$$

and have LT

$$\tilde{\pi}_j^{(\Theta)}(s) = (1-\delta_{jk})\tilde{\pi}_j^{(F)}(s) + \delta_{jk}\tilde{f}(s)\frac{1-\tilde{g}(s)}{s}. \tag{67}$$

Proof. The proof follows directly from the relation (65) and details can be found in[29]. ■

The last equality shows that we need in the system lifetime F distribution and the process distribution on it. For this, we turn to the process X analysis on separate system lifetime. Since the process behavior on the system lifetimes F_n is rather complicated, the process X behavior within any of them will be considered as an ESRP $X_n^{(1)} = \{X_n^{(1)}(t): \ t \geq 0\}$ with

$$X_n^{(1)}(t) = X(S_{n-1} + t), \quad t \leq F_n.$$

Its ESRT $S_l^{(1)}$ of the type j are the times of any repair ends (inside a separate system lifetime F_n) that find the system in state j. They are the same as for the SRP in the subsection 8.2 except the fact that right now the process $X_n^{(1)}$ is considered on a separate system lifetime and therefore never occurs in state $k-1$ after the repair ends. Thus there are only $k-1$ ERS's, $E^{(1)} = \{0, 1, \ldots, k-2\}$.

To study the process behavior on separate system lifetime denote by

- $T_l^{(1)} = S_l^{(1)} - S_{l-1}^{(1)}, \ l = 1, 2, \ldots$ the intervals between ESRT's of the ESRP $X^{(1)}$ (the times between repair ends);
- $Q^{(1)}(t) = [Q_{ij}^{(1)}(t)]_{ij \in E^{(1)}}$ ESMM, which components are the process transition probabilities between ESRT,
$$Q_{ij}^{(1)}(t) = \mathbf{P}\{X^{(1)}(S_l^{(1)} + 0) = j, \ T_l^{(1)} \leq t \,|\, X^{(1)}(S_{l-1}^{(1)} + 0) = i\};$$
- $H^{(1)}(t) = [H_{ij}^{(1)}(t)]_{ij \in E^{(1)}}$ EMRM, which components are the conditional ERF's on a separate lifetime period
$$H_{ij}^{(1)}(t) = \mathbf{E}\left[\sum_{l \geq 1} 1_{\{X^{(1)}(S_l^{(1)}+0)=j, \ S_l^{(1)} \leq t\}} \,|\, X^{(1)}(S_0^{(1)}) = i\right].$$

We start with calculation of the ESMM $Q^{(1)}(t) = [Q_{ij}^{(1)}(t)]_{ij \in E^{(1)}}$ of the ESRP $X^{(1)}$. For this note that its components coincide with those from subsection 8.2 except the fact that now the process $X^{(1)}$ never falls in state $k-1$ after the repair end. Thus they are defined only for $j \leq k-2$ and in terms of notations (50, 51) are represented in differential forms in lemma 7 by formulas (52, 53). Now, since the set of ERS's is a proper subset of the process states, the ESMM is a degenerative matrix in contrast to the matrix of the previous section. Hence, we are ready to represent some useful characteristics of the model. Introduce:

- the vector-function $\vec{F}(dt) = [F_{ik}(dt)]$, components of which are the differentials of the c.d.f.'s of the absorbing state k destination time by the ESRP starting from state i $(i = \overline{0, k-2})$;
- the vector-function $\vec{Q}^{(1)}(t) = [Q_{ik}^{(1)}(t)]$, components of which are differentials f the c.d.f.'s of the absorbing state k destination time by the ESRPr starting from state i along a monotone trajectory.

The components of the last one analogous to lemma 7, satisfy to the expressions

$$\begin{aligned} Q_{0k}^{(1)}(dt) &= \int_0^t \lambda_0 e^{-\lambda_0 u}\, du \ P_{1k}(t-u) B(dt-u), \\ Q_{ik}^{(1)}(dt) &= P_{ik}(t) B(dt). \end{aligned} \tag{68}$$

Their LST $\tilde{q}_{ij}^{(1)}(s)$ are

$$\begin{aligned} \tilde{q}_{0k}^{(1)}(s) &= \frac{\lambda_0}{s+\lambda_0} \sum_{j \geq k-1} \binom{n-1}{j-1} \sum_{m=0}^{j-1} (-1)^m \binom{j-1}{m} \tilde{b}(s + \lambda_{j-m}); \\ \tilde{q}_{ik}^{(1)}(s) &= \sum_{j \geq k-1} \binom{n-i}{j-i} \sum_{m=0}^{j-i} (-1)^m \binom{j-i}{m} \tilde{b}(s + \lambda_{j-m}). \end{aligned} \tag{69}$$

For the vector $\vec{F}(t)$ the following representation holds.

Lemma 10. The vector $\vec{F}(t)$ satisfies to the equation

$$\vec{F}(dt) = \vec{Q}^{(1)}(dt) + Q^{(1)} \star \vec{F}(dt), \tag{70}$$

whose unique solution in terms of LST is

$$\tilde{\vec{f}}(s) = (I - \tilde{q}^{(1)}(s))^{-1}\tilde{\vec{q}}^{(1)}(s). \tag{71}$$

Proof. The first equation is obtained with the help of the complete probability formula and its solution in terms of LST is evident. ■

From this lemma some useful corollaries follow.

Corollary 1. The first component $F_{0k}(t)$ of vector $\vec{F}(t)$ is c.d.f. of time to the first (and between) failures for system starting from state 0. The first component $\tilde{f}_{0k}(s)$ of vector $\tilde{\vec{f}}(s)$ is the MGF of the respective times. For simplicity we will denote them without indexes as it was done before $\tilde{f}(s) \equiv \tilde{f}_{0k}(s)$. ■

Corollary 2. Since the process regeneration cycle Θ equals to the sum of two independent r.v.'s: time to the system failure and its repair time its MGF is

$$\tau(s) \equiv \mathbf{E}\left[e^{-s\Theta}\right] = \tilde{f}(s)\tilde{g}(s). \tag{72}$$

■

Now the expression (64) leads to the following corollary.

Corollary 3. The LST $\tilde{h}(s)$ of the RF $H(t)$ of the system operating in the full repair regime equals to

$$\tilde{h}(s) = \frac{\tilde{f}(s)\tilde{g}(s)}{1 - \tilde{f}(s)\tilde{g}(s)}. \tag{73}$$

■

Taking into account that

$$R(t) = 1 - F(t) = 1 - \int_0^t f(u)du$$

one can obtain the following corollary

Corollary 4. The LT of the reliability function of the system is

$$\tilde{R}(s) = \frac{1}{s}(1 - \tilde{f}(s)). \tag{74}$$

■

To study the process behavior during a separate system lifetime cycle we consider

- the matrix $\Pi^{(F)}(t) = [\Pi_{ij}^{(F)}(t)]_{ij\in E^{(1)}}$, which components are the transition probabilities of the process on a separate life-cycle,

$$\Pi_{ij}^{(F)}(t) = \mathbf{P}\{X_n^{(1)}(t) = j,\ t < F_n \mid X_n^{(1)}(0) = i\};$$

- the matrix $\Pi^{(1)}(t) = [\Pi_{ij}^{(1)}(t)]_{ij\in E^{(1)}}$, which components are the transition probabilities of the process on a separate ESRP (between successive repair ends),

$$\Pi_{ij}^{(1)}(t) = \mathbf{P}\{X_n^{(1)}(S_{l-1}^{(1)} + t) = j,\ t < T_l^{(1)} \mid X_n^{(1)}(S_{l-1}^{(1)} + 0) = i\}.$$

In terms of these notations and according to the DSRP theory the following representations take place (symbol $\star$ here means matrix-functional convolution)

$$\Pi^{(F)}(t) = \Pi^{(1)}(t) + H^{(1)} \star \Pi^{(1)}(t). \tag{75}$$

In terms of the LT of matrices $\Pi^{(F)}(t)$, $\Pi^{(1)}(t)$ and LST of matrix $H^{(1)}(t)$ this equation take the form

$$\tilde{\Pi}^{(F)}(s) = \tilde{\Pi}^{(1)}(s) + \tilde{h}^{(1)}(s) \cdot \tilde{\Pi}^{(1)}(s). \tag{76}$$

In our case

$$H^{(1)}(dt) = Q^{(1)}(dt) + Q^{(1)} * H^{(1)}(dt),$$

and therefore

$$\tilde{h}^{(1)}(s) = (I - \tilde{q}^{(1)}(s))^{-1}\tilde{q}^{(1)}(s).$$

From here it follows

$$I + \tilde{h}^{(1)}(s) = I + \sum_{l\geq 1} \tilde{q}^{(1)*l}(s) = (I - \tilde{q}^{(1)}(s))^{-1},$$

and

$$\tilde{\Pi}^{(F)}(s) = \tilde{\Pi}^{(1)}(s) + \tilde{h}^{(1)}(s) \cdot \tilde{\Pi}^{(1)}(s) = (I - \tilde{q}^{(1)}(s))^{-1} \cdot \tilde{\Pi}^{(1)}(s). \tag{77}$$

The system state transition probabilities on a separate embedded repair time $\Pi^{(1)}(t)$ coincide for the embedded set of states $E^{(1)}$ with the corresponding probabilities as in the section 8.2 and represented in the formula (58) in lemma 8. The above results can be summing as follows.

Theorem 9. The LT $\tilde{\pi}_j(s)$ of the process X starting from the state zero state t.d.p.'s equals to

$$\tilde{\pi}_j(s) = \frac{1}{1 - \tilde{f}(s)\tilde{g}(s)} \begin{cases} \tilde{\pi}_j^{(F)}(s) & \text{for } j = \overline{1, k-1}, \\ \tilde{f}(s)\frac{1-\tilde{g}(s)}{s} & \text{for } j = k, \end{cases} \tag{78}$$

where $\tilde{f}(s)$ is defined by the first component of the vector $\tilde{\tilde{f}}(s)$ from corollary 1, which is represented by formula (71). The LT of time-dependent process state probabilities on a separate process lifetime period $\tilde{\pi}_j^{(F)}(s)$ is the first row of matrix $\tilde{\Pi}^{(F)}(s)$, which can be calculated from (76, 59).

Remark 1. Since any main RP begins with the state 0, for calculation of the process state t.d.p.'s as well as s.s.p.'s we need only in probabilities with the initial zero state.

Remark 2. The representation of the final results in the initial system information in general are too cumbersome and it needs additional study for concrete situations.

From this theorem by using the Smith's key renewal theorem one can obtain the stationary process probabilities.

Theorem 10. The s.s.p. π_j of the process X, starting from any state, equals to

$$\pi_j = \frac{1}{\mathbf{E}[F] + \mathbf{E}[G]} \begin{cases} \tilde{\pi}_j^{(F)}(0) & \text{for } j = \overline{1, k-1}, \\ \mathbf{E}[G] & \text{for } j = k, \end{cases} \tag{79}$$

where $\mathbf{E}[F] = -\tilde{f}'(0)$ is the expected system lifetime. It can be found from the formula (71), and the values $\tilde{\pi}_j^{(F)}(0)$ are the components of first row of matrix $\tilde{\Pi}^{(F)}(0)$, which can be calculated from (76, 59).

9. Conclusion

A review of the Smith's regeneration idea development is proposed in this paper. It is shown that the DSRPr can be used as a useful method for study of different stochastic models. Several previous and two recent results of application this method are demonstrated. For all considered systems the state t.d.p.'s and the s.s.p.'s are represented in terms of respective probabilities on a separate ERP. Some other applications for complex hierarchical system investigations can be find in [30], [31], [32].

The proposed approach allows to obtain analytical expressions of main quality of service characteristics for various complex stochastic models. Presence of analytical results allows to propose more detailed analysis of such systems. Especially these results can be used for further investigation of output systems characteristics sensitivity to the shape of the input distributions that determine the system behavior. Some of these investigations can be found in series of our papers. Review of these one can see for example in [33] and [34].

10. Acknowledgments

The author would like to thank Professor Boyan Dimitrov who carefully read the text and gave very useful remarks which allowed to improve the manuscript, and Professor Dmitry Kozyrev for his help in various technical issues as well as unknown to me referee who helped me to improve the English text.

References

[1] W. Smith (1955). Regenerative stochastic processes. *Proc. of Royal Soc.*, Ser. A, v. 232, 1955.

[2] E. Cinlar (1969). On semi-Markov processes on arbitrary space. Proc. Camb. Philos, Math. Proc. Camb. Philos. Soc. 1969, 66, 381-392.

[3] J. Jacod (1971). Theoreme de renouvellement et classification pour les chaines semi-Markoviennes. Ann. inst. Henri Poincare, sect. B, 1971, **7**, 85–129.

[4] V.S. Korolyuk, A.F. Turbin (1976). Semi-Markov processes and their applications. Kiev: Naukova Dumka, 1976, 184p (in Russian).

[5] G.P. Klimov (1966). Stochastic service systems. M.: "Nauka", 1966 (In Russian).

[6] V.V. Rykov, M.A. Yastrebenetsky (1971). On regenerative processes with several types of regeeration points *Cybernetics*, No. 3 (1971), pp. 82-86. Kiev, 1971 (In Russian).

[7] E. Nummelin (1978). Uniform and ratio-limit theorems for Markov-renewal and semi-regenerative processes on a general state space. Ann. inst. Henri Poincare, sect. B, 1978, **14**, 119–143.

[8] V.V. Rykov (1975). Regenerative processes with embedded regeneration periods and their application for priority queueing systems investigation. *Cybernetics*, No. 6 (1975), pp. 105-111. Kiev, 1975 (In Russian).

[9] V.V. Rykov, S.Yu. Jolkoff (1981). Generalized regenerative processes with embedded regeneration periods and their applications. MOS. Ser. Optimization. V.12 (1981), No.4, P.575-591.

[10] V.V. Rykov (2011). Decomposable Semi-Regenerative Processes and their Applications. LAMBERT Academic Publishing, 2011, 84pp.

[11] A.Ya. Khinchin (1932). The mathematical theory of stationary queue. Math sbornik , v. 32 (1932), No. 4, pp. 73-84 (in Russian). See also. Works on mathematical theory of queues. M: FIZMATGIZ, 1963, pp. 149-163.

[12] D.G. Kendall. (1953) Stochastic processes occurring in the theory of queues and their analysis by the method of the embedded Markov chains. // Ann. Math. Stat. V. 24 (1853), pp. 338-354.

[13] N.K. Jaiswell (1968). Priority queues. N-Y, Lnd.: Academic Press, 1968.

[14] B.V. Gnedenko at all (1973). Priority queueing systems. M.: Publ. Hous of Moscow State Univ., 1973.

[15] G.P. Klimov, G.K. Miskoi (179). Priority queueing systems with orientation. Moscow: Publ. Hous of Moscow State Univ. 1979.

[16] H. Takagi (1986). Analysis of polling systems. MIT Press, 1986.

[17] S.M. Borst (1996). Polling Systems. Amsterdam: Stichting Mathematish Centrum, 1996.

[18] V. Vishnevsky. O. Semenova (2021). Polling Systems: Theory and Applications for Broadband Wireless Networks. London: Academic Publishing, 2012, 317p.

[19] H. Takagi (1997). Queueing analysis of polling systems: progress in 1990-1994. //In: Frontiers in Queueing. Ed.: J.H. Dshalalow. CRC, Boca Raton, Fl. 1997. P. 119-146.

[20] V.M. Vishnevskiy, O.V. Semenova (2006). Mathematical methods for investigation 0f polling system. // Authom. and Remote control, 2006 No.2, pp. 3-56 (in Russian).

[21] V.M. Vishnevskiy, O.V. Semenova (2021). Polling systems and their application to telecommunication networks // Mathematics. 2021. No 9. P. 117. https://www.mdpi.com/2227-7390/9/2/117

[22] C. Fisker, M.R. Jabi (1994). Monotonicity and stability of periodic polling systems. // Queueing Syst. Theory Appl. 1994. V. 15. N0. 3. P. 211-238.

[23] V. Rykov (2009). On analysis of periodic polling systems. // Autom. and Remote Control. Volume 70, Issue 6 (2009), pp. 997-1018. DOI: 10.1134/S0005117909060071

[24] J.W. Cohen (1969). The single server queue. North Holland Publ. Co. Amsterdam, 1969.

[25] V.V. Rykov (1983). Investigation of a general type single-server queueing system with the regenerative processes method. // I. Iznv. AN SSSR, Techn. cibern. No 6 (1983), pp. 13-20 (In Russian).

[26] V. Rykov (1984). Investigation of a general type single-server queueing system with the regenerative processes method. II. Study of the main processes at the regenerative period. // Izv. AN SSSR, Techn. cibern. No 1 (1984), pp. 126-132 (In Russian).

[27] V. Rykov (2020). On Reliability of a Double Redundant Renewable System. // Springer Nature Switzerland AG 2020 M. Gribaudo et al. (Eds.): ASMTA 2019, LNCS 12023, pp. 1-9, 2020. https://doi.org/10.1007/978-3-030-62885-7_3

[28] V. Rykov, D. Efrosinin, N. Stepanova and J. Sztrik (2020). On Reliability of a Double Redundant Renewable System with a Generally Distributed Life and Repair Times. // Mathematics 2020, 8, 278; doi:10.3390/math8020278 www.mdpi.com/journal/mathematics

[29] V. Rykov, N. Ivanova, D. Kozyrev (2021). Application of Decomposable Semi-regenerative Processes to the Study of k-out-of-n Systems / Submitted to Mathematics, MDPI.

[30] V.V. Rykov (1996). On decomposition of hierarchical computer communication networks. // In. Proceedings of The International Conference on Distributed Computer Communication Networks. Pp. 77-85. M.: Inst. for Information Transmission Problems RAS, 1996

[31] V.V. Rykov (1997a). Two approaches to the decomposition of compound hierarchical stochastic systems // Autom. and Remote Control. Volume 58, Issue 10 (1997), pp. 1628-1639.

[32] V. Rykov (1997b). Two approaches to the decomposition of compound hierarchical stochastic systems. Part 2. Aggregative systems // Autom. and Remote Control, Volume 58, Issue 12 (1997), pp. 1958-1965.

[33] V.V. Rykov, D.V. Kozyrev (2020). On sensitivity of stochastic models. Abstracts of talks given at the 4th international conference on stochastic methods // Theory Of Probability And Its Applications, Vol. 65, No. 1, pp. 156-157.

[34] V.V. Rykov, N.M. Ivanova, D.V. Kozyrev (2020). Sensitivity Analysis of a k-out-of-n:F System Characteristics to Shapes of Input Distribution // Lecture Notes in Computer Science, Vol.12563, Springer, 2020. pp. 485-496. DOI: 10.1007/978-3-030-66471-8_37.

[35] Vladimir V. Rykov, Mikhail G. Sukharev and Victor Yu. Itkin (2020). Investigations of the Potential Application of k-out-of-n Systems in Oil and Gas Industry Objects. // J. Mar. Sci. Eng. 2020, 8, 928; doi:10.3390/jmse8110928 www.mdpi.com/journal/jmse

[36] Vladimir Rykov, Olga Kochueva, and Mais Farkhadov (2021). Preventive Maintenance of a k-out-of-n System with Applications in Subsea Pipeline Monitoring // J. Mar. Sci. Eng. 2021, 9, 85; doi.org/10.3390/jmse9010085