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A MAP/PH/1 queue with Setup time, Bernoulli vacation, Reneging, Balking, Bernoulli feedback, Breakdown and repair

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Abstract

A single server classical queueing model with Markovian Arrival Process(MAP), phase-type(PH) distributed service time and rest of the random variables are distributed exponentially is investigated. By making use of matrix analytic method, the resultant QBD process is examined in the stationary state. The practical applicability, objectives and the uniqueness of our model have been provided. The busy period analysis has been done and the distribution function for the waiting time has also been obtained. Some performance measures are enlisted. At last, some graphical and numerical exemplifications are furnished.

Keywords: Markovian Arrival Process, Setup process, Phase type distribution, Feedback, Vacation, Balking of customers, Renege of customers, Breakdown, Repair

I. Introduction

As far as the theory of point processes is concerned, the Markovian Arrival Process(MAP) is one of the most adaptable modelling tools. With an objective to formulate the incoming processes which may not be compulsorily renewal processes, a different thought notably, Versatile Markovian Point Processes(VMPP) has been introduced by Neuts [20]. The new terminologies, specifically Batch MAP and MAP had been coined by Lucantoni *et al.* [16] for the purpose of easy understanding of VMPP. The concept of MAP has been extensively discussed by Chakravarthy [3] in the *"Encyclopaedia of Operations Research and Management Science"*. The parameter matrices (D_0, D_1) characterizes the MAP and these matrices are of dimension m. In particular, the change overs which are related to no arrivals are taken care by D_0, whereas the change overs related to arrivals are taken care by D_1.

The generator matrix of the resultant CTMC is given as $D = D_0 + D_1$. The invariant probability matrix of the MAP which is a particular style of semi-Markov process is as follows:

$$\int_0^t e^{D_0 x} D_1 dx = [I - e^{D_0 t}](-D_0)^{-1} D_1.$$

Suppose π indicates probability vector for the matrix $D = D_0 + D_1$ in the stable state with the condition that $\pi D = 0$ and $\pi e = 1$. Then, $\lambda = \pi D_1 e_m$ provides the average count of arrival for each section of time in the steady state form of the MAP and is named as the fundamental rate. The PH-distributions and QBD process have been intensively examined by Latouche *et al.* [13].

The two researchers who have discussed about different types of vacations namely, single and multiple for a queueing model are Levy and Yechiali [14]. Keilson and Servi [10] have initiated the notion of Bernoulli vacation. A queueing system with multi-server, exponentially distributed vacation times had been examined by Levy and Yechiali [15]. By making use of one of the analysing techniques namely, the partial generating function, the size of the system had been computed by them. Takács [24] was the first who introduced the concept of Bernoulli feedback. He has derived distribution for queue size for a stationary process.

Chakravarthy and Agnihothri [4] have analysed a non-Markovian queueing system in which service times are PH-distributed with back up server. The phase type nature of the duration of time in which the server is busy and the sojourn time(system and queue) have been shown by them. A cost model has also been developed by them for the purpose of finding the amicable threshold values. Chang *et al.* [5] have done an analysis of non-Markovian system where the arrivals come in groups with setup times and finite buffer by employing embedded Markov-chain technique. They have also derived the stationary distributions for the length of the waiting line at various instants.

Rajadurai *et al.* [23] have studied a non-Markovian retrial model where arrivals occur in groups with unreliable server, two stages of service and vacation. The probability generating function for the system size at different states have been obtained by them. Jain *et al.* [7] have examined the non-Markovian system where arrivals come in groups with breakdown, feedback and setup. By employing supplementary variable technique, they have established invariant distribution function of the queue length. They have also determined the staying time in the waiting line.

A Markovian queueing system with single server, feedback, vacation and impatient customers has been examined by Marichamy *et al.* [19]. They have employed probability generating function technique to study the invariant probability distribution. A multiserver Markovian retrial queueing system with impatient customers has been examined by Luh *et al.* [17] by using Matrix-Analytic Method. They have provided an analytic solution for their model. They have also employed eigen vector approach for analyzing their system.

Rakesh Kumar *et al.* [11] have done an analysis in transient and steady state for a queueing model with balking, reneging and two heterogeneous servers. They have provided various performance measures and have discussed about some particular cases. Bouchentouf *et al.* [2] have done an economic analysis of a batch arrival multiserver Markovian queueing model with feedback, multiple vacation and impatient customers. They have derived the steady state solution by using probability generating functions. A Markovian queueing system with multiserver and impatient customers along with the provision of additional removable servers has been examined by Jain *et al.* [6]. They have obtained equilibrium queue size distribution by employing recursive approach.

Ke *et al.* [9] have investigated the multiserver Markovian retrial system with balking and vacation. By using the Matrix-Geometric Method, the formulae for evaluating invariant probabilities and rate matrix have been derived by them. They have constructed cost function and have performed optimization tasks by employing various numerical methods. Rakesh Kumar *et al.* [12] have done a transient analysis of a queueing model with correlated inputs and reneging. They have studied the model with the aid of Runge-Kutta method. Bouchentouf *et al.* [1] have analysed a model with single server, feedback, multiple vacation and balking. They have obtained steady state probabilities and have developed the model for cost analysis. A non-Markovian retrial model with feedback, Bernoulli vacation and unreliable server has been investigated by Pavai Madheswari *et al.* [18]. The ergodicity condition for their model has been obtained by them. They have also obtained joint distribution function for various states of the server, system and orbit size. We have utilized the matrix-analytic method for our discussion and it has been introduced by Neuts[21]. The logarithmic reduction algorithm has been utilized to compute the rate matrix and it was described by Latouche *et al.* [13].

Consider a nationalized bank which has more than one serving counter. We may consider any one of those counters. Suppose the server in the counter deals with money transaction in the following ways(phases).

- Demand Draft(DD)
- Challan
- withdrawal/deposit forms

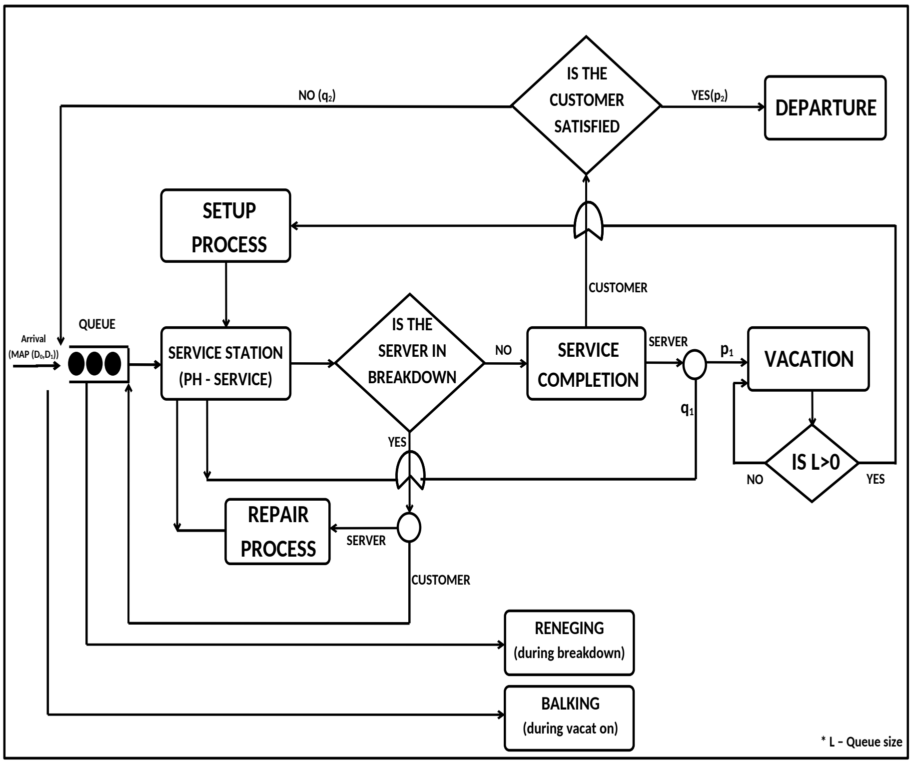

Figure 1: *Schematic representation of our model*

The arriving customer may demand money transaction in any of these ways. At the time of customer's entry, suppose the server is available, then the customer get the service at once. Otherwise, the customer joins the waiting line. Before each transaction, the server will perform some preparatory work(like refreshing the computer, selecting respective computer page for different modes of transaction, etc.). After offering the service, the server can either go for vacation(like attending telephone calls, cross checking the transaction amount, discussing with the adjacent servers, etc.) or may continue to serve the subsequent customers. Similarly, after receiving service, if the customer is not satisfied(like incorrect beneficiary name in the demand draft, deposited/withdrawn extra amount, etc.,), then the customer joins the queue to get the service again. Otherwise, they exit the bank permanently. During the busy period, the server may experience breakdown(like loss of internet connection, internal technical errors, virus attack to the system, etc.). After being repaired, the server will start to provide service to the customer who faced service interruption and is waiting in the anterior end of the queue. In the course of breakdown period, the customer in the queue may depart that particular counter(reneging). Moreover, in the course of vacation period of the server, the incoming customer may balk that particular counter. Our model has been formulated so that it will be on a par with the above circumstance.

The rest of our work is organized as follows: the description of our system is provided in Section II. Section III is devoted to the mathematical formulation of our model. The invariant analysis of our model has been presented in Section IV. The analysis of the active period of our system has been done in Section V. The analysis of the sojourn period of our model has been done in Section VI. Section VII contains a few performance measures of the system. Finally, in Section VIII, some illustrative examples are furnished via., tabular and graphical work.

II. Model Description

A queueing system with single server where the customers reach the system as specified by the MAP whose parameters matrices of dimension n are D_0 and D_1 has been considered. The duration of the service offered by the server is considered to be PH-distributed with notation (α, T) which is of order m, where $T^0 + Te = 0$. At the end of providing service, the server may choose to undergo vacation with p_1 as its probability or commence service to the succeeding customer with q_1 as its probability, where $p_1 + q_1 = 1$. The server always choose to avail vacation provided the system size is zero. The setup process begins at the completion of vacation period with the constraint that there must be a minimum of single customer in the space for the customer's to wait. Or else, the server carry on with his vacation upto a minimal of single customer waiting in the system for service while coming back from vacation. After the completion of setup process, the server commences service to the customer. Similarly, to the end of service completion, suppose a customer is fulfilled, he exits the service station forever with p_2 as its probability. Otherwise, the customer joins the anterior part of the waiting line with q_2 as its probability to acquire the service afresh. During the busy period, the server may get breakdown. As a result, the customer who is obtaining service at that time has to join the anterior end of the waiting line. At the completion of repair process, the server commences service to the customer, if any in the waiting line. Or else, the server undergoes vacation. During the customer's arrival, if the server is in vacation, then the customer may balk the system with b as its probability. Further, in the course of breakdown period of the server, the customer in the waiting line may renege due to their impatience. The vacation times, setup times, breakdown times, repair times and the reneging times are all supposed to follow exponential distribution with parameters η, τ, σ, δ abd r respectively.

III. The Generator Matrix

In this section, the generator matrix of the system under study is constructed. Our work starts with the definition of the desired successive notations.

Notations:

* *N(t)*: Size of the system at epoch t
* I_n: $n \times n$ identity matrix
* **0**: The zero matrix of needed dimension
* e_r: $r \times 1$ vector with all its entities to be 1
* e=e_{3n+mn}
* e_1=e_{2n}
* $e_1(1)$: $2n \times 1$ vector in which initial n entries are 1 and rest of the entries are zero
* $e_2(2)$: $2n \times 1$ vector with $n+1$ to $2n$ entries to be 1 and leftover entries to be zero
* $e(1)$: $(3n + mn) \times 1$ vector with first n entries to be 1 and leftover entries to be zero
* $e(2)$: $(3n + mn) \times 1$ vector with $n+1$ to $2n$ entries to be 1 and leftover entries to be zero
* $e(3)$: $(3n + mn) \times 1$ vector with $2n+1$ to $2n + mn$ entries to be 1 and leftover entries to be zero
* $e(4)$: $(3n + mn) \times 1$ vector with $2n + mn + 1$ to $3n + mn$ entries to be 1 and rest of the entries to be zero
* $\otimes$: Symbol for Kronecker multiplication

* $\oplus$: Symbol for Kronecker addition
* *Y(t)* - Server's nature at instant t,
  where

$$Y(t) = \begin{cases} 0, & \text{server undergoes vacation} \\ 1, & \text{server is in setup process} \\ 2, & \text{server is offering service} \\ 3, & \text{server is in breakdown} \end{cases}$$

* $S(t)$: Service phase of the server at epoch t
* M(t): Phase of the MAP at epoch t
* λ: Fundamental rate of arrival and is mentioned by $\lambda = \beta D_1 \mathbf{e}$ in which β is the probability vector of the matrix $D = D_0 + D_1$ in the steady state
* γ: The rate at which the server offers service, where $\gamma = [\alpha(-T)^{-1}\boldsymbol{e}]^{-1}$

Clearly, $\{(N(t),Y(t),S(t),M(t)) : t \geq 0\}$ is a Continuous Time Markov Chain (CTMC) with succeeding state space:

$$\mathbf{\Omega} = U(0) \cup \bigcup_{j \geq 1} U(j)$$

where

$$U(0) = \{(0,0,k) : 1 \leq k \leq n\} \cup \{(0,3,k) : 1 \leq k \leq n\}$$

and

$$U(j) = \{(j,i,k) : i = 0,1; 1 \leq k \leq n\} \cup \{(j,2,l,k) : 1 \leq l \leq m, 1 \leq k \leq n\} \cup \{(j,3,k) : 1 \leq k \leq n\}$$

The generator matrix of our Markov chain is as below:

$$Q = \begin{bmatrix} B_{00} & B_{01} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ B_{10} & C_1 & C_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{0} & C_2 & C_1 & C_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{0} & C_2 & C_1 & C_0 & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & C_2 & C_1 & C_0 & \mathbf{0} & \cdots \\ \cdots & \cdots & \cdots & \ddots & \ddots & \ddots & \cdots & \cdots \end{bmatrix}$$

where

$$B_{00} = \begin{bmatrix} D_0 + bD_1 & \mathbf{0} \\ \delta I_n & D_0 - \delta I_n \end{bmatrix}, B_{01} = \begin{bmatrix} (1-b)D_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & D_1 \end{bmatrix},$$

$$B_{10} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ p_2 T^0 \otimes I_n & \mathbf{0} \\ \mathbf{0} & rI_n \end{bmatrix}, \; C_2 = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ p_1 p_2 T^0 \otimes I_n & \mathbf{0} & q_1 p_2 T^0 \alpha \otimes I_n & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & rI_n \end{bmatrix},$$

$$C_0 = \begin{bmatrix} (1-b)D_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & D_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_m \otimes D_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & D_1 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} D_0 - \eta I_n + bD_1 & \eta I_n & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & D_0 - \tau I_n & \alpha \otimes \tau I_n & \mathbf{0} \\ p_1 q_2 T^0 \otimes I_n & \mathbf{0} & (q_1 q_2 T^0 \alpha + T) \oplus (D_0 - \sigma I_n) & e_m \otimes \sigma I_n \\ \mathbf{0} & \mathbf{0} & \alpha \otimes \delta I_n & D_0 - (r+\delta) I_n \end{bmatrix}.$$

IV. System Analysis

I. Stability Condition

Define $C = C_0 + C_1 + C_2$ which results that C is a generator matrix and hence, we could compute it's invariant vector which is indicated by Ψ and it abides

$$\Psi C = \mathbf{0}; \quad \Psi \mathbf{e} = 1$$

where $\Psi = (\psi_0, \psi_1, \psi_2, \psi_3)$.
The vector Ψ, partitioned as $\Psi = (\psi_0, \psi_1, \psi_2, \psi_3)$ is determined by solving the successive equations:

$$\begin{aligned}
&\psi_0[D - \eta I_n] + \psi_2[p_1 T^0 \otimes I_n] = \mathbf{0}, \\
&\psi_0[\eta I_n] + \psi_1[D - \tau I_n] = \mathbf{0}, \\
&\psi_1[\alpha \otimes \tau I_n] + \psi_2[(q_1 T^0 \alpha + T) \oplus (D - \sigma I_n)] + \psi_3[\alpha \otimes \delta I_n] = \mathbf{0}, \\
&\psi_2[e_m \otimes \sigma I_n)] + \psi_3[D - \delta I_n] = \mathbf{0}
\end{aligned}$$

subject to

$$\psi_0 e_n + \psi_1 e_n + \psi_2 e_{mn} + \psi_3 e_n = 1.$$

The necessary and sufficient condition stablility is $\Psi A_0 \mathbf{e} < \Psi A_2 \mathbf{e}$
i.e.,

$$\psi_0[(1-b)D_1 e_n] + \psi_1[D_1 e_n] + \psi_2[e_m \otimes D_1 e_n] + \psi_3[D_1 e_n] < \psi_2[p_2 T^0 \otimes e_n] + \psi_3[r e_n].$$

II. The Invariant Probability Vector

In the steady state, let the probability vector of the generator Q be specified by $\mathbf{x}$ and it is of infinitesimal dimension.
This probability vector is further subdivided in the following fashion: $\mathbf{x} = (\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, ...)$, where the dimension of $\mathbf{x}_0$ and $\mathbf{x}_i$ are $2n$ and $3n + mn$ respectively, for $i \geq 1$.
Since $\mathbf{x}$ is an invariant vector of Q, the subsequent constraints will be abide by it:

$$\mathbf{x}Q = \mathbf{0}, \quad \mathbf{x}\mathbf{e} = 1.$$

Once the stableness is attained, the steady-state probability vector $\mathbf{x}$ may be determined by solving the subsequent equations.

$$\mathbf{x}_{i+1} = \mathbf{x}_1 R^i, \qquad i \geq 1$$

where R is the least non-negative solution of the equation

$$R^2 C_2 + R C_1 + C_0 = 0$$

and the remaining vectors namely, $\mathbf{x}_0$ and $\mathbf{x}_1$ can be determined by solving the subsequent equations:

$$\mathbf{x}_0 B_{00} + \mathbf{x}_1 B_{10} = \mathbf{0},$$

$$\mathbf{x}_0 B_{01} + \mathbf{x}_1 [C_1 + R C_2] = \mathbf{0}$$

with the normalizing condition

$$\mathbf{x}_0 \mathbf{e}_{2n} + \mathbf{x}_1 [I - R]^{-1} \mathbf{e}_{3n+mn} = 1.$$

The rate matrix R may be computed by using "Logarithmic Reduction Algorithm" given by Latouche *et al.* [13].

V. Busy Period Analysis

The time duration between the advent of the customer to the system no customers and the epoch at which the system size becomes zero for the first time is defined as the active period. Thus, the first passage time from level 1 to 0 and the active period are the same.

Likewise, the first return time to level 0 with minimum one visit to a state in any other level may be defined as the busy cycle. Initially, the idea of the fundamental period is proposed to analyze the active period. The first passage time from the level i to $i-1$,($i \geq 2$) may be defined as the fundamental period for the QBD process. A distinct argumentation has to be carried out for the boundary states viz., $i = 0,$ and 1.

NOTATIONS:

* $G_{jj\prime}(k,x)$ - The probability of the QBD process entering the level $i-1$ by performing precisely k changeovers to the left and also by entering the state $(i, j\prime)$ with the constraint that it started in the state (i, j) at instant $t = 0$.
* $\tilde{G}_{jj\prime}(z,s) = \sum_{k=1}^{\infty} z^k \int_0^{\infty} e^{-sx} dG_{jj\prime}(k,x) : |z| \leq 1,\ Re(s) \geq 0$
* $\tilde{G}(z,s) -$ The matrix $(\tilde{G}_{jj\prime}(z,s))$
* $G = (G_{jj\prime}) = \tilde{G}(1,0)$- The matrix which takes care of the first passage times for the states other than the boundary state.
* $G_{jj\prime}^{(1,0)}(k,x)$ - The probability of the QBD process get into the level 0 by doing exactly k change overs to the left with the condition that it commenced in the level 1 at instant $t = 0$.
* $G_{jj\prime}^{(0,0)}(k,x)$ - The first return time to the level 0.
* $\mathbb{E}_{1j}$, $\mathbb{E}_{2j}$ - The expected first passage time and the expected number of customers who acquired service in the interval of first passage time between the levels i and $i-1$ respectively, with the constraint that the process is in the state (i, j) at the instant $t = 0$.
* $\vec{\mathbb{E}}_1$, $\vec{\mathbb{E}}_2$ - The column vectors with $\mathbb{E}_{1j}$ and $\mathbb{E}_{2j}$ as their entries respectively.
* $\vec{\mathbb{E}}_1^{(1,0)}$, $\vec{\mathbb{E}}_2^{(1,0)}$ - The vectors providing the expected first passage time from level 1 to level 0 and the expected number of service completion in that interval respectively.
* $\vec{\mathbb{E}}_1^{(0,0)}$, $\vec{\mathbb{E}}_2^{(0,0)}$ - The vectors providing the expected first return time to level 0 and the expected number of service completion in that interval respectively.

It is evident that the matrix $\tilde{G}(z,s)$ abides the subsequent equation:

$$\tilde{G}(z,s) = z[sI - A_1]^{-1}A_2 + [sI - A_1]^{-1}A_0\tilde{G}^2(z,s)$$

If the rate matrix R is obtained, the determination of the matrix G may be done by utilizing the successive result

$$\text{“}G = -[A_1 + RA_2]^{-1}A_2\text{”}.$$

Likewise, the matrix G may be determined by using the logarithmic reduction algorithm(Latouche *et al.* [13]).

The succeeding equations which are fulfilled by $\tilde{G}^{(1,0)}(z,s)$ and $\tilde{G}^{(0,0)}(z,s)$ are for the boundary states viz., 1 and 0 respectively.

$$\tilde{G}^{(1,0)}(z,s) = z[sI - A_1]^{-1}B_{10} + [sI - A_1]^{-1}A_0\tilde{G}(z,s)\tilde{G}^{(1,0)}(z,s)$$

$$\tilde{G}^{(0,0)}(z,s) = [sI - B_{00}]^{-1}B_{01}\tilde{G}^{(1,0)}(z,s).$$

Since the matrices G, $\tilde{G}^{(1,0)}(1,0)$ and $\tilde{G}^{(0,0)}(1,0)$ are all stochastic, the subsequent moments may be readily computed. At $z = 1$ and $s = 0$,

$$\begin{aligned}
\vec{\mathbb{E}}_1 &= -\frac{\partial}{\partial s}\{\tilde{G}(z,s)\} = -[A_0(G+I)+A_1]^{-1}e, \\
\vec{\mathbb{E}}_2 &= \frac{\partial}{\partial z}\{\tilde{G}(z,s)\} = -[A_0(G+I)+A_1]^{-1}A_2 e, \\
\vec{\mathbb{E}}_1^{(1,0)} &= -\frac{\partial}{\partial s}\{\tilde{G}^{(1,0)}(z,s)\} = -[A_1 + A_0 G]^{-1}[A_0\vec{\mathbb{E}}_1 + e], \\
\vec{\mathbb{E}}_2^{(1,0)} &= \frac{\partial}{\partial z}\{\tilde{G}^{(1,0)}(z,s)\} = -[A_1 + A_0 G]^{-1}[B_{10}e + A_0\vec{\mathbb{E}}_2], \\
\vec{\mathbb{E}}_1^{(0,0)} &= -\frac{\partial}{\partial s}\{\tilde{G}^{(0,0)}(z,s)\} = -B_{00}^{-1}[e + B_{01}\vec{\mathbb{E}}_1^{(1,0)}], \\
\vec{\mathbb{E}}_2^{(0,0)} &= \frac{\partial}{\partial z}\{\tilde{G}^{(0,0)}(z,s)\} = -B_{00}^{-1}B_{01}\vec{\mathbb{E}}_2^{(1,0)}.
\end{aligned}$$

VI. Waiting time analysis

With the aid of analysis of first passage time, the distribution function for the waiting time of an arriving customer has been derived in this section.

Let W(t), where $t \geq 0$ be a vector of dimension $1 \times m$ which indicates the waiting time distribution of an arriving tagged customer in the queue. While taking a multi-server model with Bernoulli vacation under study, we could see that $W(0+) = 0$, because each arriving customer has to hold up for the completion of either vacation period or service period. Let $(*) \cup \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \cdots\}$ indicates the state space of an absorbing CTMC. The service for the arriving tagged customer will commence from their arrival into the absorbing state $(*)$. For this absorbing Markov chain, the transition matrix is as follows:

$$\tilde{Q} = \begin{bmatrix}
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\
H_0 & F_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\
H_1 & F_{10} & F & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\
\mathbf{0} & \mathbf{0} & F_2 & F & \mathbf{0} & \mathbf{0} & \cdots \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & F_2 & F & \mathbf{0} & \cdots \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & F_2 & F & \cdots \\
\cdots & \cdots & \cdots & \ddots & \ddots & \ddots & \cdots
\end{bmatrix}$$

where

$$H_0 = \begin{bmatrix} \eta \\ \delta \end{bmatrix}, F_0 = \begin{bmatrix} -\eta & \mathbf{0} \\ \mathbf{0} & -\delta \end{bmatrix}, H_1 = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ q_1 p_2 T^0 \\ \mathbf{0} \end{bmatrix}, F_{10} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ p_1 p_2 T^0 & \mathbf{0} \\ \mathbf{0} & r \end{bmatrix},$$

$$F = \begin{bmatrix} -\eta & \eta & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\tau & \tau\alpha & \mathbf{0} \\ p_1 q_2 T^0 & \mathbf{0} & q_1 q_2 T^0 \alpha + T - \sigma I_m & \sigma e_m \\ \mathbf{0} & \mathbf{0} & \delta\alpha & -(\delta + r) \end{bmatrix}, F_2 = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ p_1 p_2 T^0 & \mathbf{0} & q_1 p_2 T^0 \alpha & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & r \end{bmatrix}.$$

With an objective to derive the arriving tagged customer's waiting time distribution $W(t)$, where $(t \geq 0)$, we begin with the process of finding the system size probability vector in the steady state at the arrival instant and it is indicated by $\mathbf{z}(0) = (\mathbf{z}_0(0), \mathbf{z}_1(0), \mathbf{z}_2(0), ...)$. As the arrival process obeys the Markovian property, the system size probability vector in the steady state at the arrival epoch is as follows:

$$\mathbf{z}_0(0) = \mathbf{x}_0[I_2 \otimes \frac{D_1 e_n}{\lambda},]$$

$$\mathbf{z}_i(0) = \mathbf{x}_i[I_{3+m} \otimes \frac{D_1 e_n}{\lambda}], \text{for } i \geq 1$$

where λ indicates the fundamental arrival rate of the MAP.
Define $\mathbf{z}(t) = (\mathbf{z}_*(t), \mathbf{z}_0(t), \mathbf{z}_1(t), \cdots)$,
where
$\mathbf{z}_i(t),\ i \geq 1$ - $1 \times (3+m)$ vector,
$\mathbf{z}_0(t)$ - a 1×2 vector
and their components provide the probability of the CTMC whose generator matrix is $\tilde{Q}$ being in the respective state of level i at epoch t. Since $\mathbf{z}_*(t)$ specifies the probability of the tagged customer being in the absorbing state at epoch t, we get $W(t) = \mathbf{z}_*(t)$, where $t \geq 0$.
The differential equation $\acute{\mathbf{z}}(t) = \mathbf{z}(t)\tilde{Q}$, where $t \geq 0$ reduces to

$$\acute{\mathbf{z}}_*(t) = \mathbf{z}_1(t)H_1$$

$$\acute{\mathbf{z}}_0(t) = \mathbf{z}_0(t)F_0 + \mathbf{z}_1(t)F_{10}$$

$$\acute{\mathbf{z}}_i(t) = \mathbf{z}_i(t)F + \mathbf{z}_{i+1}(t)F_2,\ i \geq 1$$

where $\acute{}$ specifies the derivative with respect to t.

Let us compute the Laplace Stieltjes Transform(LST) for W(t) with the aid of technique indicated by Neuts *et al.* [21]. By commencing the process at the state i with $\mathbf{z}_i(0)$, $i \geq 1$ as initial probability vector, the row vector $\omega(s)$ specifies the LST of the first passage time to level 1. As indicated in [21], we get,

$$\omega(s) = \sum_{i=1}^{\infty} \mathbf{z}_i(0)[(sI - F)^{-1}F_2]^{i-1}. \tag{1}$$

Let the LST of the absorbing time to the state $(*)$ with the constraint that the process commences at level $i = 0, 1, 2$ be specified by $\phi(i, s)$. Just as in [21], we have

$$\phi(0, s) = [sI - F_0]^{-1}H_0, \tag{2}$$

$$\phi(1, s) = [sI - F]^{-1}F_{10}\phi(0, s) + [sI - F]^{-1}H_1. \tag{3}$$

Thus, we may simply observe that the LST for the distribution of waiting time is as below:

$$\bar{W}(s) = \mathbf{z}_0(0)\ \phi(0, s) + \omega(s)\ \phi(1, s). \tag{4}$$

I. Average waiting time

The average waiting time is provided as

$$EW = -\mathbf{z}_0(0)\acute{\phi}(0, 0) - \acute{\omega}(0)e_{3+m} - \omega(0)\acute{\phi}(1, 0)e_m. \tag{5}$$

The expected time to enter into the absorbing state $(*)$ given that the system is in the level $i = 0$ is provided by the foremost term of the preceding equation. In the same way, if the system is in level $i \geq 1$, then the mean time for accessing the state $(*)$ is provided by the end two terms of the preceding equation.
By differentiating (2) and (3), and setting $s = 0$, we obtain,

$$\acute{\phi}(0, 0) = (-1)[-F_0]^{-2}H_0, \tag{6}$$

$$\acute{\phi}(1, 0) = (-1)[-F]^{-2}F_{10}\phi(0, 0) + [-F]^{-1}F_{10}\acute{\phi}(0, 0) - [-F]^{-2}H_1. \tag{7}$$

With the help of (6) along with the vector $\mathbf{z}(0) = (\mathbf{z}_0(0), \mathbf{z}_1(0), \mathbf{z}_2(0), \cdots)$, we may readily compute the first term of (5). From (1), we get

$$\omega(0) = \sum_{i=1}^{\infty} \mathbf{z}_i(0)V^{i-1}, \tag{8}$$

where $V = [-F]^{-1}F_2$. Since the matrix V is stochastic, we get

$$\omega(0)e_{3+m} = 1 - \mathbf{z}_0(0)e_2. \tag{9}$$

With the help of (7) and (9) along with the vector $\mathbf{z}(0) = (\mathbf{z}_0(0), \mathbf{z}_1(0), \mathbf{z}_2(0), \cdots)$, we may readily compute the final term of (5).
By differentiating (1) and making $s = 0$, we get

$$\acute{\omega}(0) = (-1)\sum_{i=1}^{\infty} \mathbf{z}_{1+i}(0) \sum_{j=0}^{i-1} V^j [-F]^{-1} V^{i-j}. \tag{10}$$

As V is stochastic, we get

$$(-1)\acute{\omega}(0)e_{3+m} = (-1)\sum_{i=1}^{\infty} \mathbf{z}_{1+i}(0) \sum_{j=0}^{i-1} V^j [-F]^{-1} e_{3+m}. \tag{11}$$

With the help of the method mentioned in Kao *et al.* [8]. and Neuts *et al.* [22], let us evaluate the value of $(-1)\acute{\omega}(0)e_{3+m}$. We begin with the construction of a matrix V_2 which is such that V_2 is stochastic, generalized inverse of $I - V$ and $I - V + V_2$ is non-singular. The matrix V_2 can be assumed to be $V_2 = e_{1+m_1+m_2+m_1m_2}v_0$ in which v_0 is the stationary probability vector of V. Further, with the help of the property $VV_2 = V_2V = V_2$, we have

$$\sum_{j=0}^{i-1} V^j (I - V + V_2) = I - V^i + iV_2, \quad \text{for} \quad i \geq 1. \tag{12}$$

By using (14) in (13), we obtain,

$$(-1)\acute{\omega}(0)e_{3+m} = \{\mathbf{x}_1[I-R]^{-1}[I_{3+m} \otimes \frac{D_1e_n}{\lambda}] - \omega(0) + \mathbf{x}_1R[I-R]^{-2}[I_{3+m} \otimes \frac{D_1e_n}{\lambda}]V_2\} \times [I - V + V_2]^{-1}[-F]^{-1}e_{3+m}. \tag{13}$$

Hence, all the terms of (5) have been found out and so we may readily obtain the average period of waiting.

VII. Performance Measures

* Probability of server is on vacation:
  $P_{vacation} = \mathbf{x}_0e_1(1) + \mathbf{x}_1(I-R)^{-1}\ e(1)$
* Probability of server is in setup process:
  $P_{setup} = \mathbf{x}_1(I-R)^{-1}e(2)$
* Probability of server is busy:
  $P_{busy} = \mathbf{x}_1(I-R)^{-1}e(3)$
* Probability of server is in breakdown:
  $P_{breakdown} = \mathbf{x}_0e_1(2) + \mathbf{x}_1(I-R)^{-1}e(4)$
* The mean system:
  $E_{system} = \mathbf{x}_1(I-R)^{-2}e$
* The mean system size when the server is undergoing vacation:
  $E_v = \mathbf{x}_1(I-R)^{-2}e(1)$

* Expected system size during setup process:
  $E_s = \mathbf{x}_1(I - R)^{-2}e(2)$
* Average system size when the server is busy:
  $E_b = \mathbf{x}_1(I - R)^{-2}e(3)$
* Expected system size during breakdown:
  $E_{bd} = \mathbf{x}_1(I - R)^{-2}e(4)$

VIII. Numerical Illustrations

The comprehensive aim of this section is to explore the performance of our system through numerical and graphical exemplifications. For the arrival patterns, we took the following distinctive MAP representations so that their mean is 1, and Chakravarthy [3] suggested these values.

Erlang of order 2-(A-Erl):

$$D_0 = \begin{bmatrix} -2 & 2 \\ 0 & -2 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}$$

Exponential-(A-Exp):

$$D_0 = \begin{bmatrix} -1 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 1 \end{bmatrix}$$

Hyperexponential-(A-Hyp-Exp):

$$D_0 = \begin{bmatrix} -1.90 & 0 \\ 0 & -0.19 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 1.710 & 0.190 \\ 0.171 & 0.019 \end{bmatrix}$$

It is evident that they have zero correlation because of the renewal character of these three arrival processes.

MAP - Negative Correlation-(A-MAP-NC):

$$D_0 = \begin{bmatrix} -1.00222 & 1.00222 & 0 \\ 0 & -1.00222 & 0 \\ 0 & 0 & -225.75 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0.01002 & 0 & 0.99220 \\ 223.4925 & 0 & 2.2575 \end{bmatrix}$$

The successive PH - distributions have been taken for service times suggested by Chakravarthy [3] as well.

Erlang of order 2-(S-Erl):

$$\alpha = (1, 0), \qquad T = \begin{bmatrix} -2 & 2 \\ 0 & -2 \end{bmatrix}$$

Exponential-(S-Exp):

$$\alpha = (1), \qquad T = \begin{bmatrix} -1 \end{bmatrix}$$

Hyperexponential-(S-Hyp-Exp):

$$\alpha = (0.8, 0.2), \quad T = \begin{bmatrix} -2.8 & 0 \\ 0 & -0.28 \end{bmatrix}$$

Illustration 1

From the Tables 1-4, we study the impact of the repair rate δ against the probability of server being busy. Fix $\lambda = 1,\ \gamma = 6,\ \sigma = 3,\ \eta = 6,\ \tau = 5,\ r = 1,\ b = 0.6,\ p_2 = 0.5,\ q_2 = 0.5$.
For Bernoulli vacation(Bv): $p_1 = 0.5,\ q_1 = 0.5$.
For 1 - limited vacation(1-Lv): $p_1 = 1,\ q_1 = 0$.
From Tables 1-4, we derive the succeeding observations.

* As the repair rate(δ) maximizes, the probability of server being busy also increases for distinct feasible groupings of service and arrival times.
* While correlating the tabulated values for distinct arrival patterns, the probability of server being busy maximizes more rapidly for A-Hyp-Exp and gradually for A-MAP-NC. In the same way, the probability of server being busy increases gradually for S-Hyp-Exp and rapidly for S-Erl.
* Also, the probability of server being busy maximizes slowly for Bv and quickly for 1-Lv for distinct arrangements of arrival and service patterns.

Illustration 2

From the Tables 5-8, we study the impact of the service rate γ on the expected waiting time(EW). We fix $\lambda = 1,\ \delta = 1,\ \sigma = 3,\ \eta = 6,\ \tau = 5,\ r = 1,\ b = 0.6,\ p_2 = 0.5,\ q_2 = 0.5$.
For Bernoulli vacation(Bv): $p_1 = 0.5;\ q_1 = 0.5$.
For 1 - limited vacation(1-Lv): $p_1 = 1;\ q_1 = 0$.
From Tables 5-8, we get the subsequent interpretation.

* While raising the service rate, EW minimizes for distinct possible combinations of service and arrival patterns.
* While correlating the values of distinct arrival patterns, EW decreases more quickly in the case of A-Hyp-Exp whereas slowly for A-Erl. Similarly, EW decreases gradually for S-Hyp-Exp and more quickly in the case of S-Erl.
* Further, the average waiting time decreases rapidly for 1-Lv and slowly in the case of Bv.

Illustration 3

From the 2D graphs 2-13, we view the effect of the vacation rate η on average system size (E_{system}). Fix $\lambda = 1,\ \delta = 1,\ \sigma = 3,\ \gamma = 6,\ \tau = 5,\ r = 1,\ b = 0.6,\ p_2 = 0.5,\ q_2 = 0.5$.
For Bernoulli vacation(Bv): $p_1 = 0.5,\ q_1 = 0.5$.
For 1 - limited vacation(1 - Lv): $p_1 = 1,\ q_1 = 0$.

From Figures 2-13, we could see that while raising the vacation rate (η), the rate of decrement of E_{system} is high in the case of A-Hyp-Exp and low for A-Erl. Also, it is high in the case of S-Erl and low in the case of S-Hyp-Exp. Further, we may view that E_{system} decreases quickly in the case of 1 - Lv and slowly in the case of Bv.

Table 1: *Repair rate vs. Probability of server being busy - A-Exp*

δ	SERVICE					
	S-Exp		S-Erl		S-Hyp-Exp	
	Bv	1-Lv	Bv	1-Lv	Bv	1-Lv
2.0	0.16152	0.17895	0.18126	0.20038	0.10756	0.11959
3.0	0.17270	0.19495	0.19591	0.22133	0.11197	0.12588
4.0	0.17896	0.20409	0.20422	0.23354	0.11437	0.12930
5.0	0.18299	0.21001	0.20961	0.24155	0.11589	0.13147
6.0	0.18581	0.21417	0.21340	0.24721	0.11695	0.13296
7.0	0.18791	0.21725	0.21622	0.25142	0.11773	0.13406

Illustration 4

From the 2D graphs 14-25, we study the impact of the breakdown rate(σ) on the mean period of waiting(EW). Fix $\lambda = 1,\ \delta = 1,\ \eta = 6,\ \gamma = 6,\ \tau = 5,\ r = 1,\ b = 0.6,$ $p_2 = 0.5,\ q_2 = 0.5$.
For Bernoulli vacation(Bv): $p_1 = 0.5,\ q_1 = 0.5$.
For 1 - limited vacation(1 - Lv): $p_1 = 1,\ q_1 = 0$.

From Figures 14-25, we may view that while raising the breakdown rate (σ), the speed of increment of EW is maximum for A-Hyp-Exp and minimum for A-Erl. Also, it is high in the case of S-Erl and low in the case of S-Hyp-Exp. Further, we may view that EW increases quickly in the case of 1 - Lv and slowly in the case of Bv.

Illustration 5:

From the 3D graphs 26-37, we analyse the impact of the setup rate (τ) and the rate of service provided by the server(γ) on the probability of server is availing vacation($P_{vacation}$). Fix $\lambda = 1,\ \delta = 1,$ $\eta = 6,\ \sigma = 3,\ r = 1,\ b = 0.6,\ p_1 = 0.5,\ q_1 = 0.5,\ p_2 = 0.5,\ q_2 = 0.5$.

A quick view of Figures 26-37 reveals the fact that $P_{vacation}$ increases while maximizing both the setup rate and the rate of service offered by the server for various arrangement of arrival and service patterns. Further, it maximizes rapidly for A-MAP-NC and gradually for A-Hyp-Exp. In the same way, the rate of increment is high for S-Erl and low for S-Hyp-Exp.

Illustration 6:

From the 3D graphs 38-49, we observe the consequences of the customer's reneging rate(r) and the server's vacation rate(η) on the Average system size(E_{system}). Fix $\lambda = 1,\ \delta = 1,\ \gamma = 6,$ $\sigma = 3,\ \tau = 5,\ b = 0.6,\ p_1 = 0.5,\ q_1 = 0.5,\ p_2 = 0.5,\ q_2 = 0.5$.

A quick view of Figures 38-49 reveals the point that E_{system} reduces while maximizing both the reneging rate and the vacation rate of the customer and the server respectively for distinct groupings of arrival and service times. Further, it minimizes quickly for A-Hyp-Exp and slowly for A-Erl. In the same way, the rate of decrement of E_{system} is high in the case of S-Erl and low in the case of S-Hyp-Exp.

Table 2: *Repair rate vs. Probability of server being busy - A-Erl*

δ	SERVICE					
	S-Exp		S-Erl		S-Hyp-Exp	
	Bv	1-Lv	Bv	1-Lv	Bv	1-Lv
2.0	0.15626	0.17639	0.17582	0.19840	0.103440	0.11650
3.0	0.16632	0.19163	0.18908	0.21864	0.107370	0.12228
4.0	0.17192	0.20028	0.19653	0.23038	0.109510	0.12541
5.0	0.17552	0.20587	0.20135	0.23805	0.110870	0.12739
6.0	0.17805	0.20978	0.20474	0.24347	0.111830	0.12876
7.0	0.17993	0.21269	0.20727	0.24750	0.112530	0.12977

Table 3: *Repair rate vs. Probability of server being busy - A-Hyp-Exp*

δ	SERVICE					
	S-Exp		S-Erl		S-Hyp-Exp	
	Bv	1-Lv	Bv	1-Lv	Bv	1-Lv
2.0	0.18240	0.18781	0.20235	0.20704	0.12405	0.13085
3.0	0.19805	0.20641	0.22252	0.23028	0.13007	0.13901
4.0	0.20676	0.21714	0.23403	0.24396	0.13318	0.14345
5.0	0.21229	0.22412	0.24145	0.25296	0.13506	0.14625
6.0	0.21610	0.22902	0.24664	0.25934	0.13632	0.14817
7.0	0.21890	0.23265	0.25046	0.26410	0.13723	0.14957

Table 4: *Repair rate vs. Probability of server being busy - A-MAP-NC*

δ	SERVICE					
	S-Exp		S-Erl		S-Hyp-Exp	
	Bv	1-Lv	Bv	1-Lv	Bv	1-Lv
2.0	0.15122	0.17400	0.17037	0.19642	0.099946	0.11412
3.0	0.16029	0.18861	0.18240	0.21606	0.103440	0.11950
4.0	0.16540	0.19691	0.18923	0.22745	0.105400	0.12244
5.0	0.16873	0.20229	0.19369	0.23490	0.106680	0.12432
6.0	0.17110	0.20607	0.19685	0.24017	0.107590	0.12563
7.0	0.17287	0.20887	0.19922	0.24409	0.108280	0.12659

Table 5: *Service rate vs. Average waiting time-A-Exp*

γ	SERVICE					
	S-Exp		S-Erl		S-Hyp-Exp	
	Bv	1-Lv	Bv	1-Lv	Bv	1-Lv
7.0	2.4993	8.3702	2.9533	11.000	1.4773	4.0849
8.0	2.0203	6.1915	2.3009	7.4783	1.2958	3.4923
9.0	1.7030	4.9497	1.8916	5.7077	1.1625	3.0774
10.0	1.4788	4.1511	1.6132	4.6482	1.06040	2.7703
11.0	1.3129	3.5965	1.4130	3.9461	0.97959	2.5337
12.0	1.1857	3.1900	1.2627	3.4485	0.91410	2.3457

Table 6: *Service rate vs. Average waiting time-A-Erl*

γ	SERVICE					
	S-Exp		S-Erl		S-Hyp-Exp	
	Bv	1-Lv	Bv	1-Lv	Bv	1-Lv
7.0	1.87310	6.2099	2.20200	8.1571	1.12500	3.0281
8.0	1.51780	4.5819	1.71990	5.5304	0.98996	2.5852
9.0	1.28300	3.6557	1.41810	4.2119	0.89072	2.2755
10.0	1.11750	3.0610	1.21320	3.4243	0.81472	2.0465
11.0	0.99508	2.6488	1.06610	2.9033	0.75462	1.8703
12.0	0.90133	2.3471	0.95583	2.5346	0.70591	1.7305

Table 7: *Service rate vs. Average waiting time-A-Hyp-Exp*

γ	SERVICE					
	S-Exp		S-Erl		S-Hyp-Exp	
	Bv	1-Lv	Bv	1-Lv	Bv	1-Lv
7.0	7.0763	25.198	8.5479	33.276	3.8226	12.076
8.0	5.5843	18.612	6.5041	22.603	3.2673	10.278
9.0	4.5926	14.841	5.2154	17.213	2.8615	9.0147
10.0	3.8918	12.403	4.3372	13.972	2.5526	8.0765
11.0	3.3742	10.703	3.7059	11.814	2.3102	7.3514
12.0	2.9785	9.4514	3.2334	10.279	2.1151	6.7739

Table 8: *Service rate vs. Average waiting time-A-MAP-NC*

γ	SERVICE					
	S-Exp		S-Erl		S-Hyp-Exp	
	Bv	1-Lv	Bv	1-Lv	Bv	1-Lv
7.0	2.4453	8.3500	2.8876	11.014	1.45970	4.0420
8.0	1.9814	6.1528	2.2527	7.4498	1.28680	3.4530
9.0	1.6761	4.9065	1.8574	5.6667	1.16010	3.0422
10.0	1.4616	4.1086	1.5902	4.6048	1.06330	2.7393
11.0	1.3035	3.5567	1.3989	3.9042	0.98690	2.5066
12.0	1.1826	3.1538	1.2559	3.4097	0.92503	2.3221

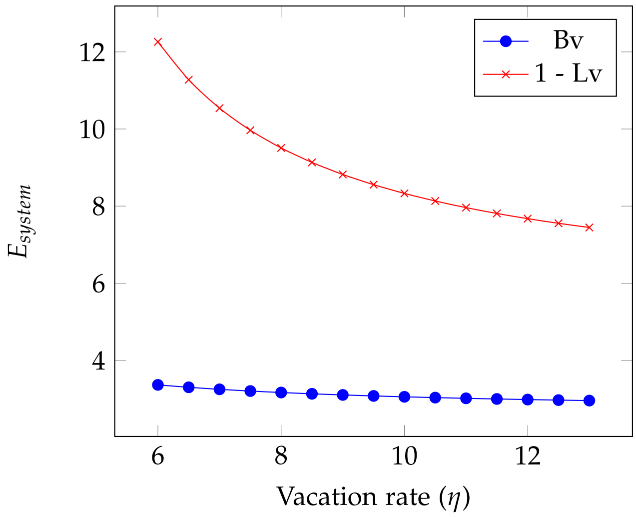

Figure 2: *Vacation rate vs. E_{system} - M/M/1*

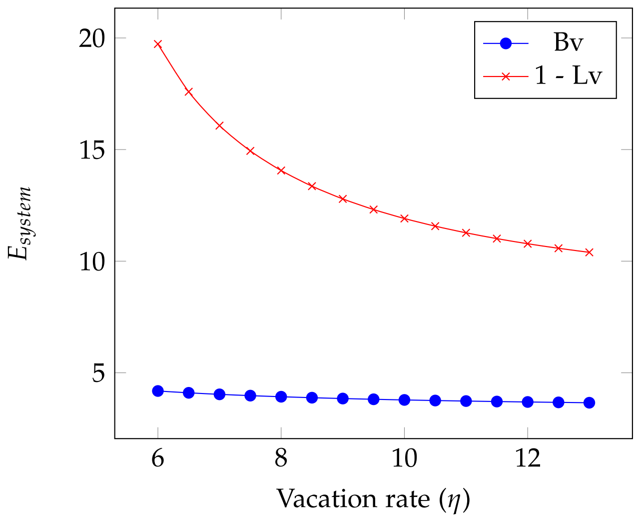

Figure 3: *Vacation rate vs. E_{system} - M/Ek/1*

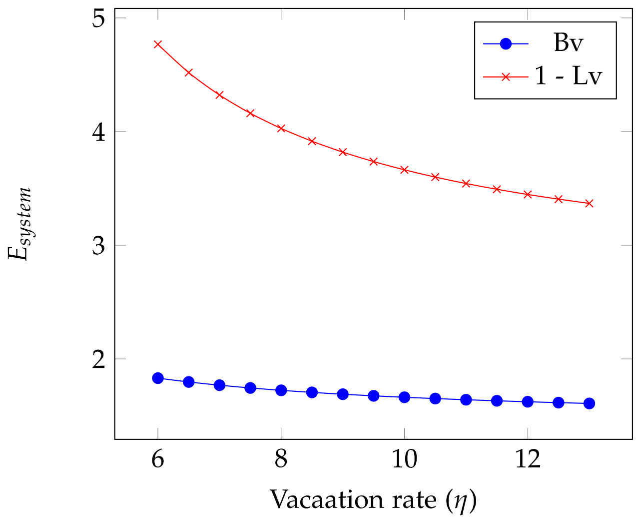

Figure 4: *Vacation rate vs. E_{system} - M/Hk/1*

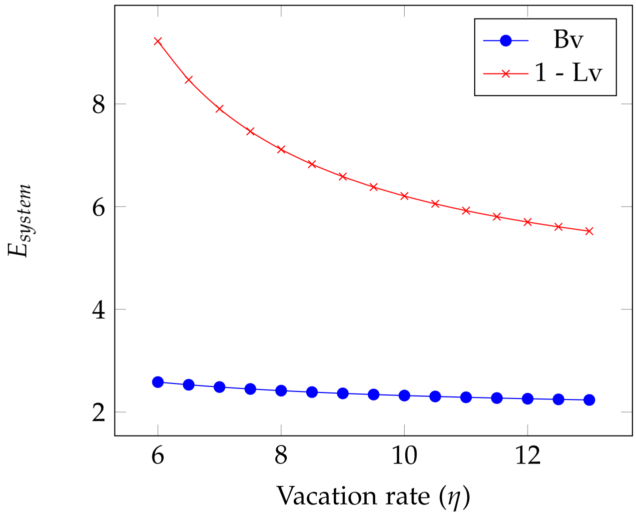

Figure 5: *Vacation rate vs. E_{system} - EK/M/1*

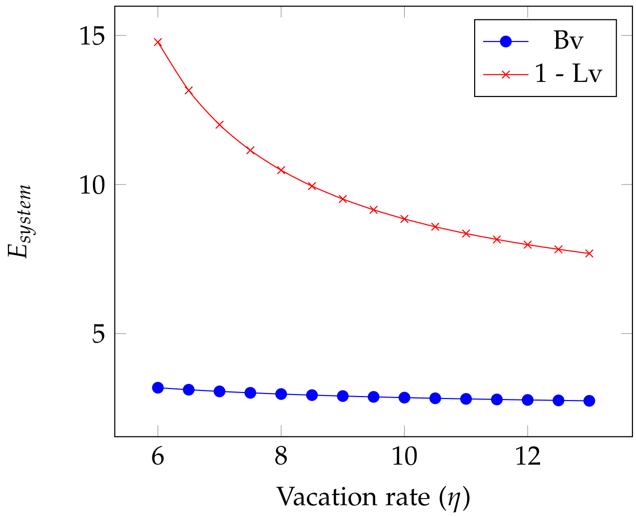

Figure 6: *Vacation rate vs. E_{system} - Ek/Ek/1*

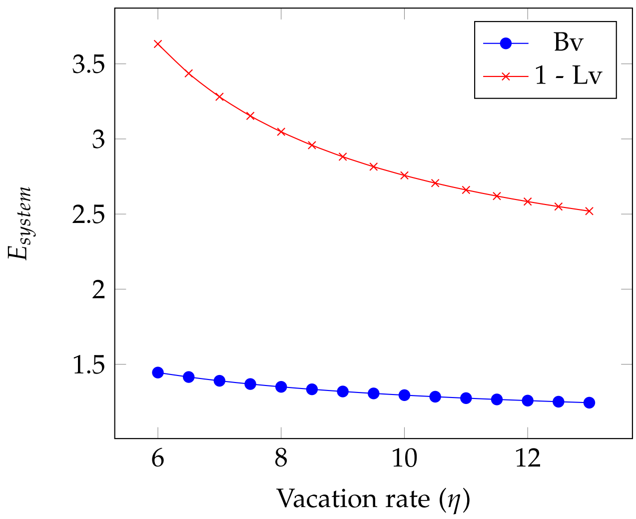

Figure 7: *Vacation rate vs. E_{system} - Ek/Hk/1*

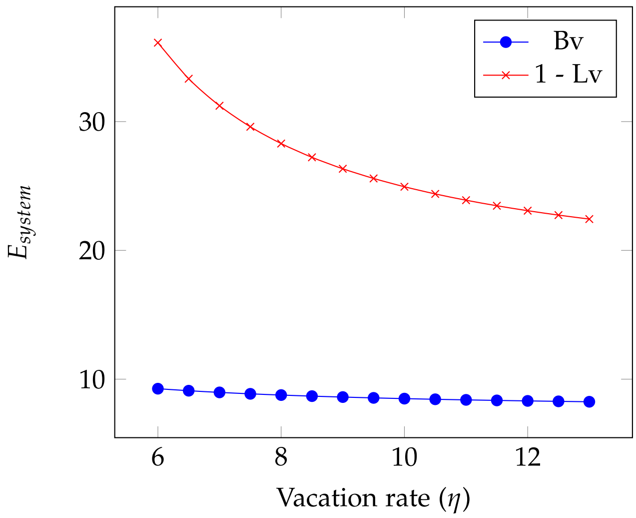

Figure 8: *Vacation rate vs.* E_{system} *- Hk/M/1*

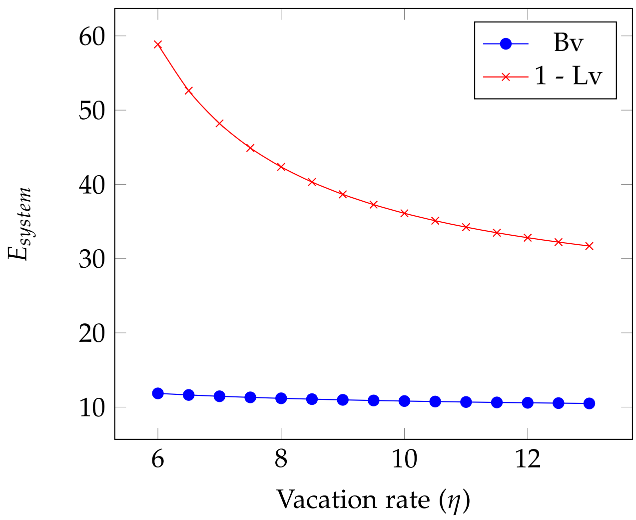

Figure 9: *Vacation rate vs.* E_{system} *- Hk/Ek/1*

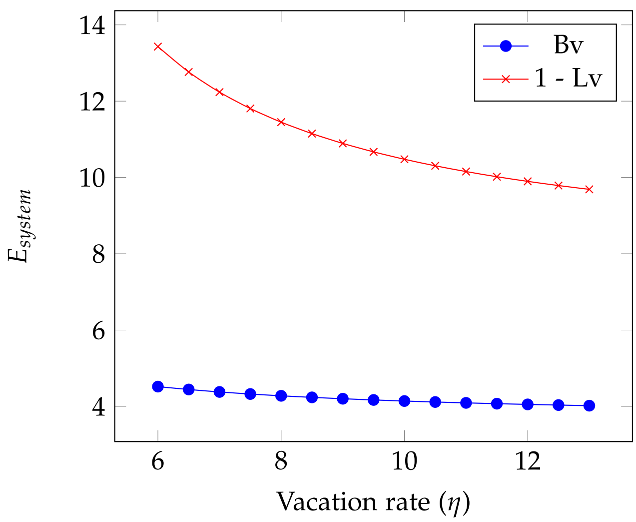

Figure 10: *Vacation rate vs.* E_{system} *- Hk/Hk/1*

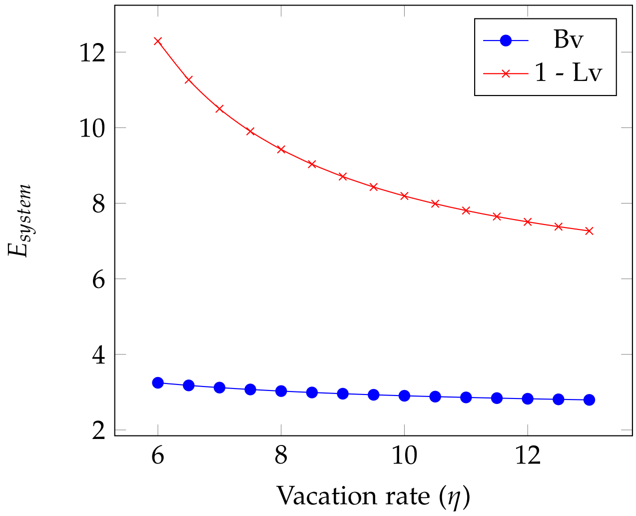

Figure 11: *Vacation rate vs. E_{system} - MAP-NC/M/1*

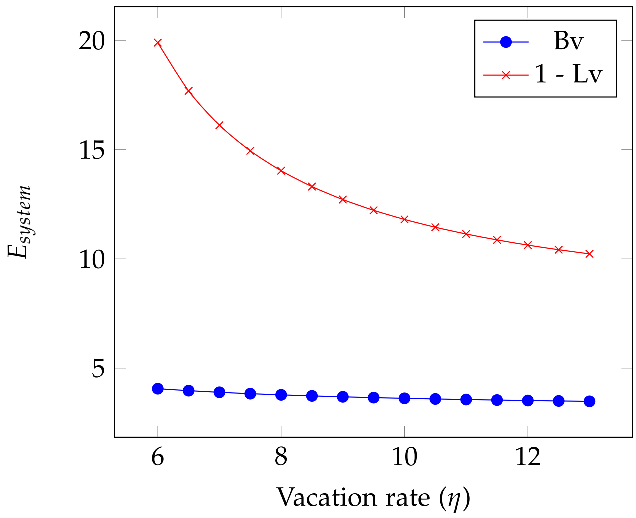

Figure 12: *Vacation rate vs. E_{system} - MAP-NC/Ek/1*

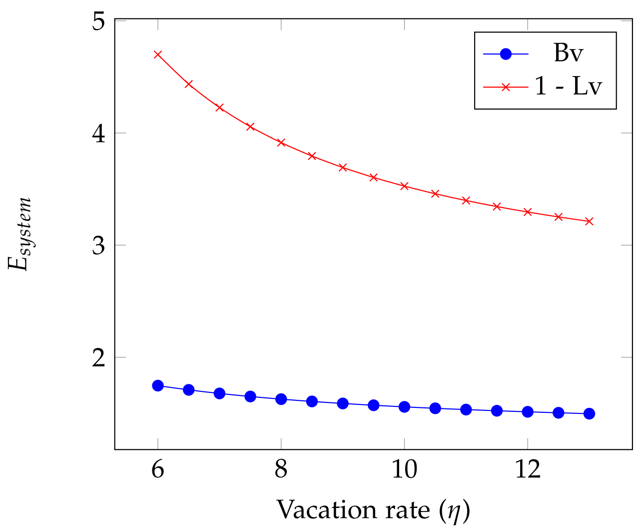

Figure 13: *Vacation rate vs. E_{system} - MAP-NC/Hk/1*

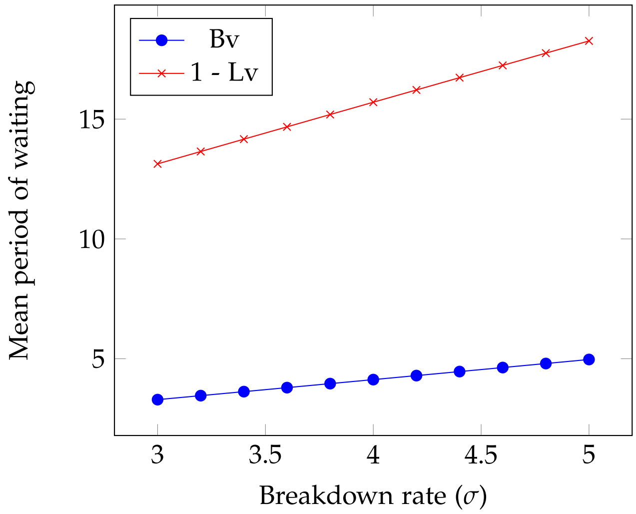

Figure 14: *Breakdown rate vs. Mean period of waiting - M/M/1*

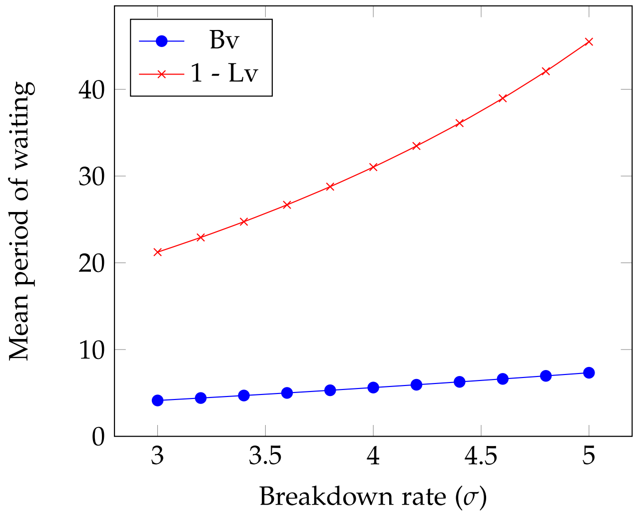

Figure 15: *Breakdown rate vs. Mean period of waiting - M/Ek/1*

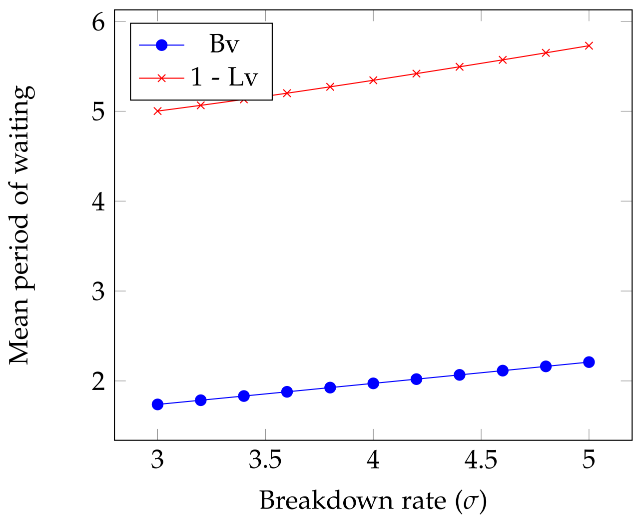

Figure 16: *Breakdown rate vs. Mean period of waiting - M/Hk/1*

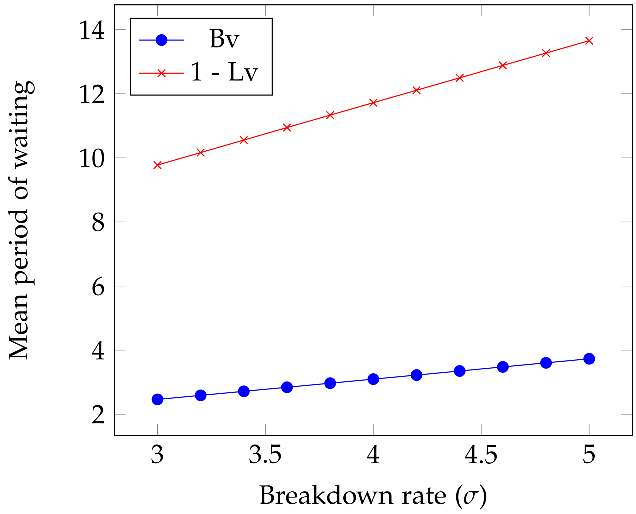

Figure 17: *Breakdown rate vs. Mean period of waiting - Ek/M/1*

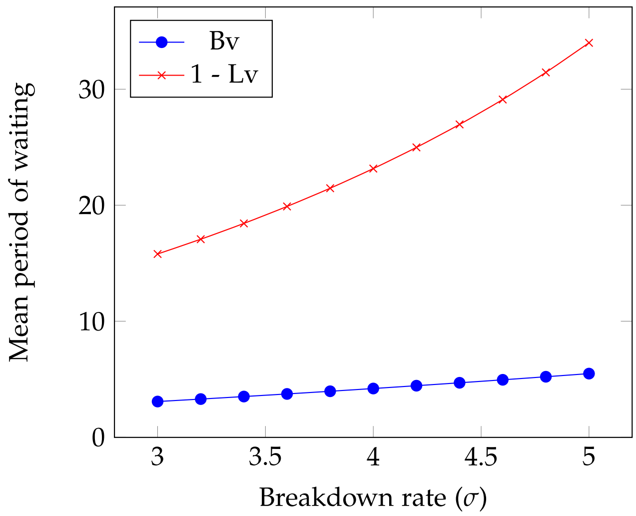

Figure 18: *Breakdown rate vs. Mean period of waiting - Ek/Ek/1*

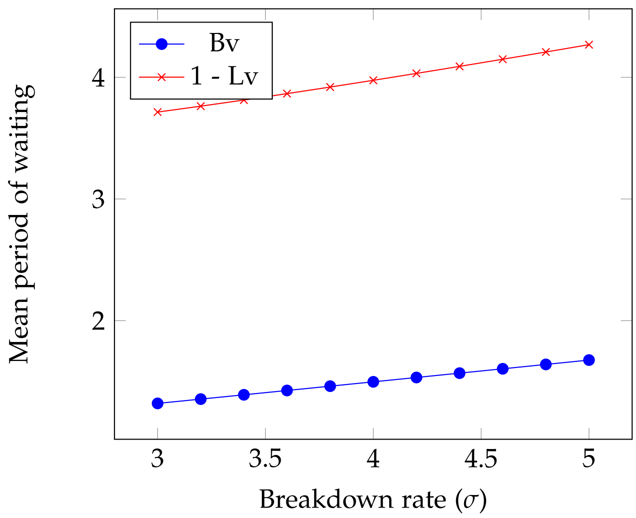

Figure 19: *Breakdown rate vs. Mean period of waiting - Ek/Hk/1*

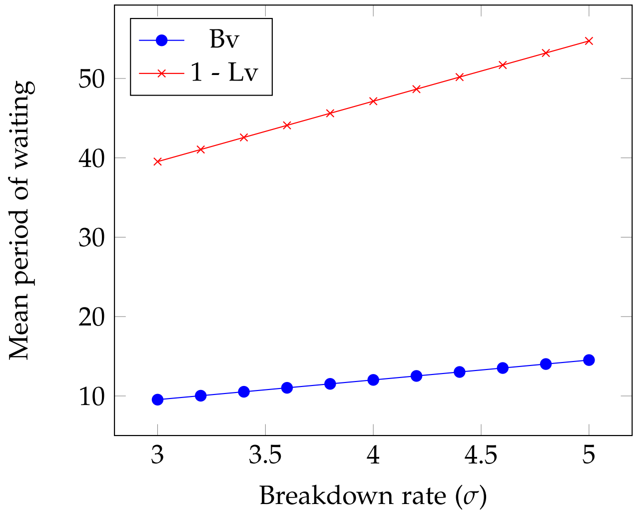

Figure 20: *Breakdown rate vs. Mean period of waiting - Hk/M/1*

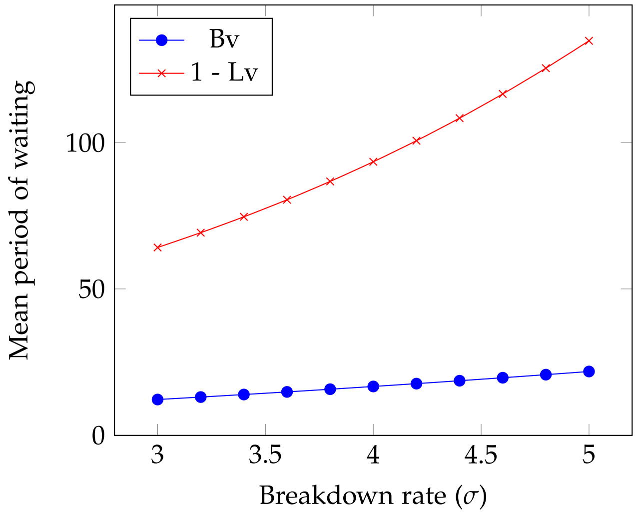

Figure 21: *Breakdown rate vs. Mean period of waiting - Hk/Ek/1*

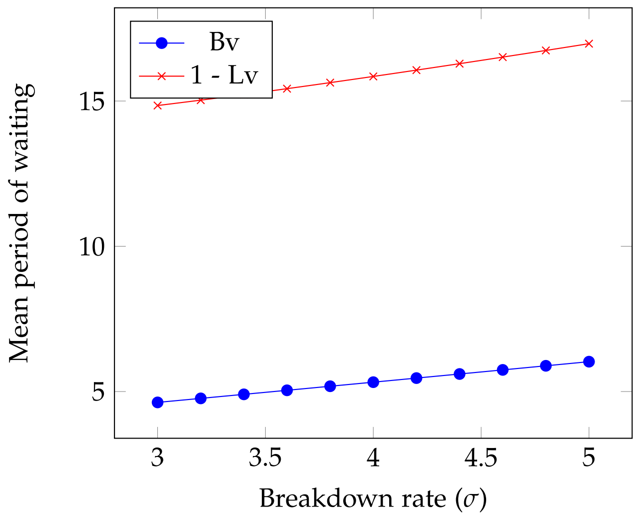

Figure 22: *Breakdown rate vs. Mean period of waiting - Hk/Hk/1*

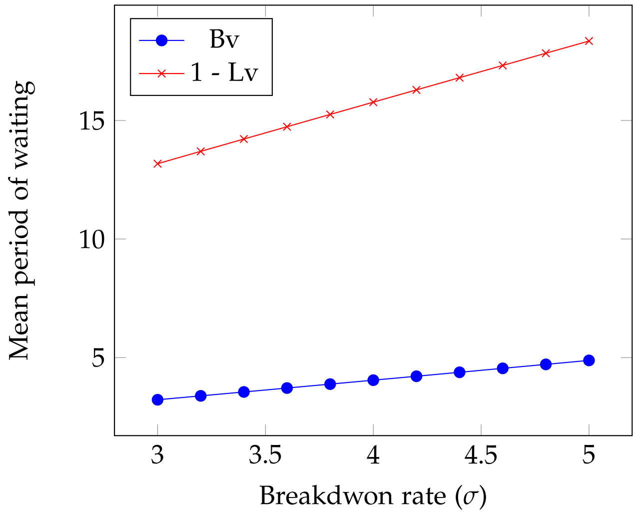

Figure 23: *Breakdown rate vs. Mean period of waiting - MAP-NC/M/1*

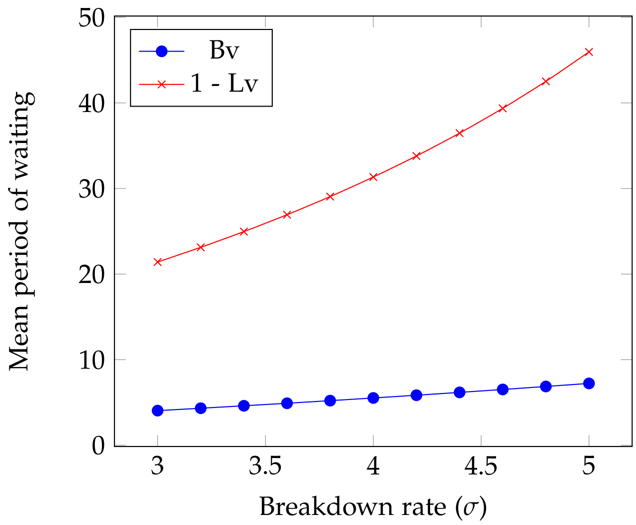

Figure 24: *Breakdown rate vs. Mean period of waiting - MAP-NC/Ek/1*

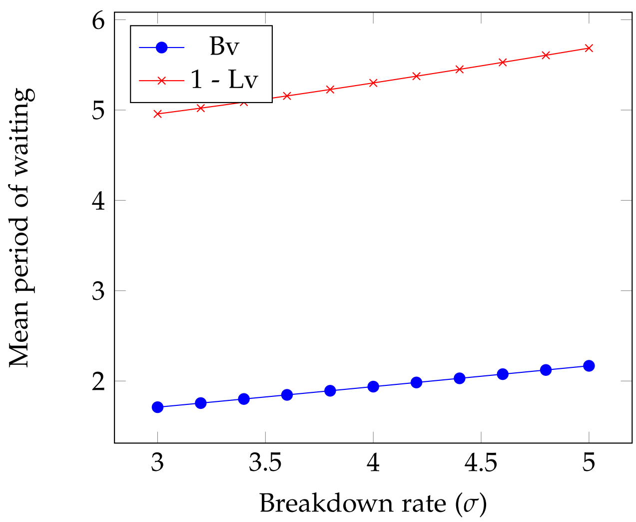

Figure 25: *Breakdown rate vs. Mean period of waiting - MAP-NC/Hk/1*

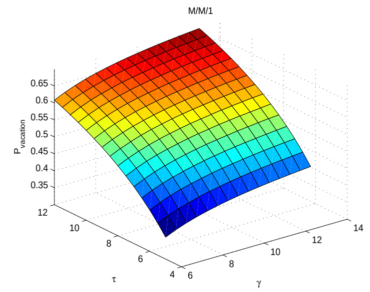

Figure 26: *(Setup and Service rate of the Server) vs.* $P_{vacation}$

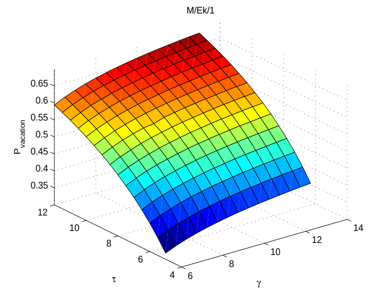

Figure 27: *(Setup and Service rate of the Server) vs.* $P_{vacation}$

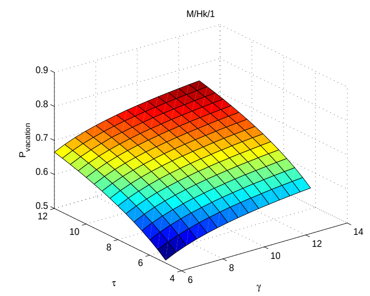

Figure 28: *(Setup and Service rate of the Server) vs.* $P_{vacation}$

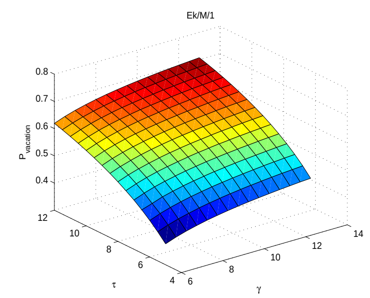

Figure 29: *(Setup and Service rate of the Server) vs.* $P_{vacation}$

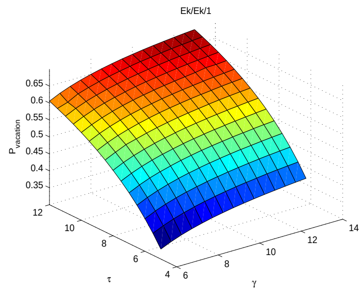

Figure 30: *(Setup and Service rate of the Server) vs.* $P_{vacation}$

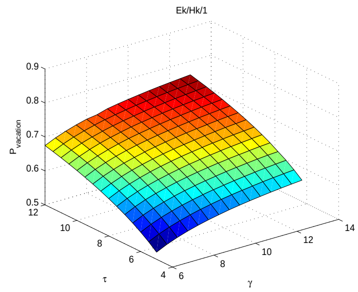

Figure 31: *(Setup and Service rate of the Server) vs.* $P_{vacation}$

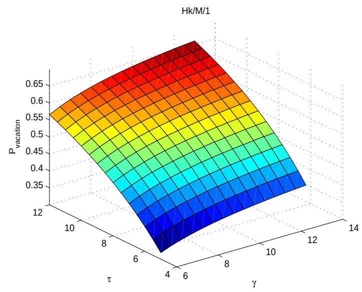

Figure 32: *(Setup and Service rate of the Server) vs.* $P_{vacation}$

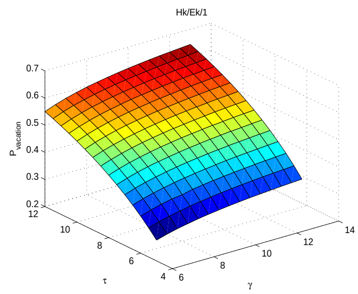

Figure 33: *(Setup and Service rate of the Server) vs.* $P_{vacation}$

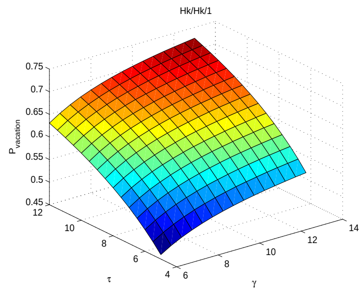

Figure 34: *(Setup and Service rate of the Server) vs.* $P_{vacation}$

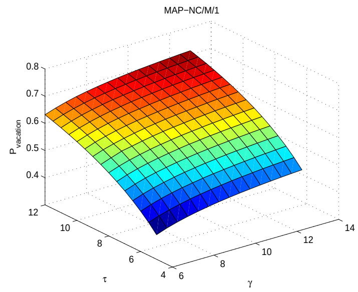

Figure 35: *(Setup and Service rate of the Server) vs.* $P_{vacation}$

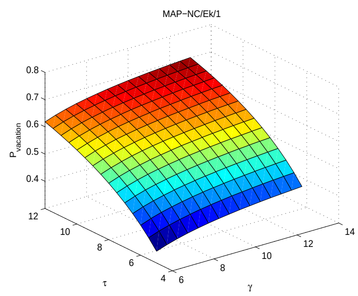

Figure 36: *(Setup and Service rate of the Server) vs.* $P_{vacation}$

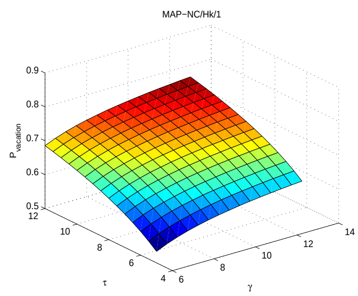

Figure 37: *(Setup and Service rate of the Server) vs.* $P_{vacation}$

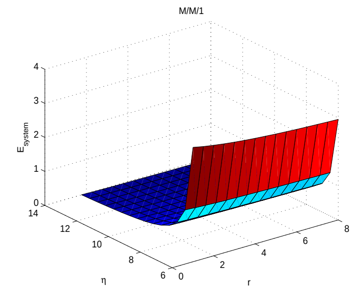

Figure 38: *(Vacation rate and Reneging rate of the server and customer resp.) vs.* E_{system}

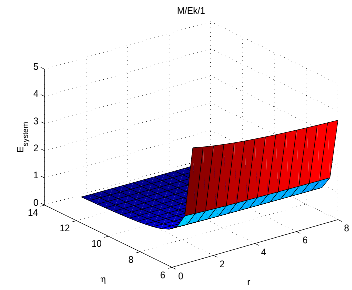

Figure 39: *(Vacation rate and Reneging rate of the server and customer resp.) vs.* E_{system}

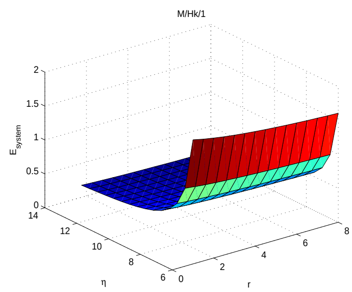

Figure 40: *(Vacation rate and Reneging rate of the server and customer resp.) vs.* E_{system}

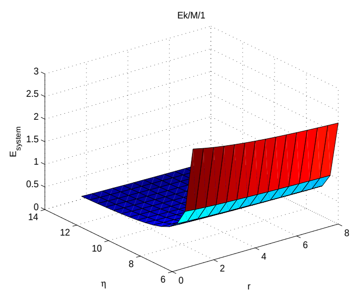

Figure 41: *(Vacation rate and Reneging rate of the server and customer resp.) vs.* E_{system}

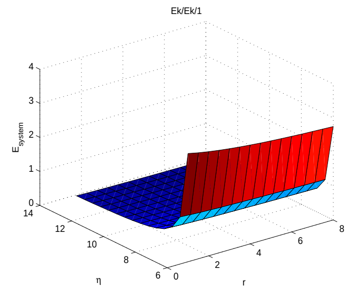

Figure 42: *(Vacation rate and Reneging rate of the server and customer resp.) vs.* E_{system}

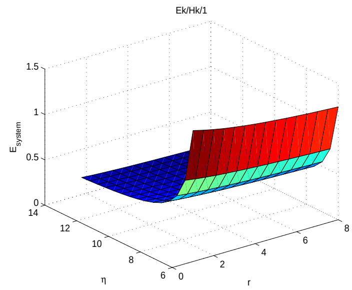

Figure 43: *(Vacation rate and Reneging rate of the server and customer resp.) vs.* E_{system}

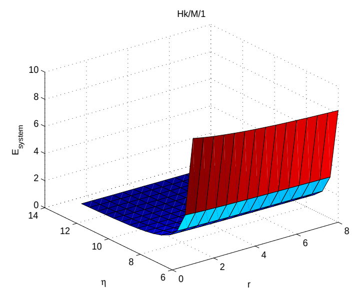

Figure 44: *(Vacation rate and Reneging rate of the server and customer resp.) vs.* E_{system}

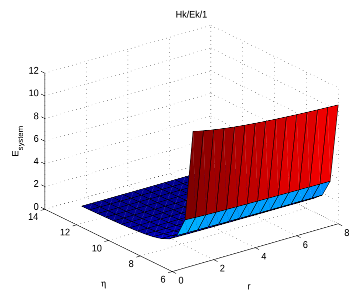

Figure 45: *(Vacation rate and Reneging rate of the server and customer resp.) vs.* E_{system}

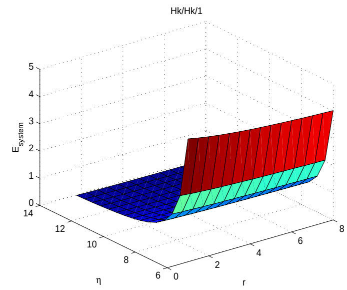

Figure 46: *(Vacation rate and Reneging rate of the server and customer resp.) vs.* E_{system}

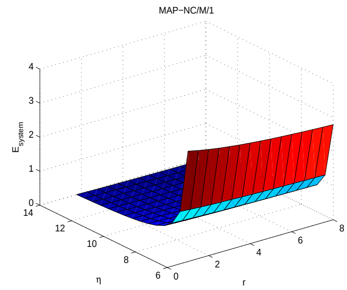

Figure 47: *(Vacation rate and Reneging rate of the server and customer resp.) vs.* E_{system}

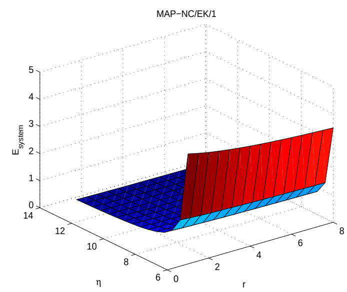

Figure 48: *(Vacation rate and Reneging rate of the server and customer resp.) vs.* E_{system}

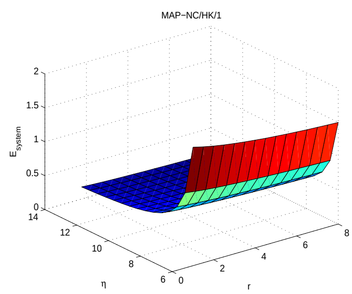

Figure 49: *(Vacation rate and Reneging rate of the server and customer resp.) vs.* E_{system}

IX. Conclusion

Our article deals with a classical queueing model with MAP arrival, single server, PH-service together with vacation, setup time, breakdown, repair, feedback, balking and reneging. The stability condition for our system has been obtained. In addition, the active period of model under study has been explored. The consequences of the vacation rate(η) and the breakdown rate(σ) upon average size of the system and expected waiting time respectively for two different types of vacation, namely 1-limited and Bernoulli vacation have been visualized with the aid of 2D graphs. Further, the impact of both the setup(τ) and service rate(γ) of the server upon probability that the server is undergoing vacation has been pictured with the support of 3D graphs. Also, the consequences of both reneging rate(r) and vacation rate (η) on the average size of the system has been pictured with the support of 3D graphs. In addition, one can perform the cost analysis for our model and can also extend the work by considering BMAP for arrival process.

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