Product Of n Independent Maxwell Random Variables

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Abstract

We derive the exact probability density functions (pdf) of a product of n independent Maxwell distributed random variables. The distribution functions are derived by using an inverse Mellin transform technique from statistics, and are given in terms of a special function of mathematical physics, the Meijer G-function.

Keywords: Product Distribution, Maxwell Distribution, Mellin transform technique, Meijer G-function, probability density function.

1 Introduction

Engineering, Physics, Economics, Order statistics, Classification, Ranking, Selection, Number theory, Genetics, Biology, Medicine, Hydrology, Psychology, these all applied problems depend on the distribution of product of random variables[1][2].

As an example of use of the product of random variables in physics, Sornette [27] mentions:

"…To mimic system size limitation, Takayasu, Sato, and Takayasu introduced a threshold x_c …and found a stretched exponential truncating the power-law pdf beyond x_c. Frisch and Sornette recently developed a theory of extreme deviations generalizing the central limit theorem which, when applied to multiplication of random variables, predicts the generic presence of stretched exponential pdfs. The problem thus boils down to determining the tail of the pdf for a product of random variables …"

Several authors have studied the product distributions for independent random variables come from the same family or different families, see [21] for t and Rayleigh families, [4] for Pareto and Kumaraswamy families, [6] for the t and Bessel families, and [22] for the independent generalized gamma-ratio family. In this paper, we find analytically the probability distributions of the product $\prod_{i=1}^{n} X_i$, when X_i is a Maxwell random variable with probability density function (p.d.f)

$$f_{X_i}(x_i) = \sqrt{\frac{2}{\pi}} \frac{x_i^2}{b_i^3} e^{\frac{-x_i^2}{2(b_i)^2}}, \quad x_i \geq 0. \tag{1}$$

The functions are derived by using an inverse Mellin transform technique from statistics and given in terms of the Meijer G-function.

2 Basic Definitions

2.1 Mellin integral transform

The Mellin integral transform of $f(x)$ is defined only for $x \geq 0$, as:

$$M\{f(x)/s\} = E[x^{s-1}] = \int_0^\infty x^{s-1} f(x)dx \tag{2}$$

The inverse transform is:

$$f(x) = \frac{1}{2j\pi} \int_{c-j\infty}^{c+j\infty} x^{-s} M\{f(x)/s\} ds \tag{3}$$

The path of integration is any line parallel to the imaginary axis and lying within the strip of analyticity of $M\{f(x)/s\}$.

The Mellin integral transform of the density function $f(x)$ of the product $X = X_1.X_2 \ldots X_n$ of n independent random variables X_i with the density function $f_{X_i}(x_i)$ is defined as:

$$M\{f_X(x)/s\} = \prod_{i=1}^n M\{f_{X_i}(x_i)/s\} \tag{4}$$

Using the inverse transform formula we obtain the density function of the product distribution as:

$$f_X(x) = \frac{1}{2j\pi} \int_{c-j\infty}^{c+j\infty} x^{-s} \prod_{i=1}^n M\{f_{X_i}(x_i)/s\} ds \tag{5}$$

2.2 Meijer G-function

The Meijer G-function is defined by the contour integral:

$$G_{pq}^{mn}\left(z\Big|\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\right) = \frac{1}{2j\pi} \int_{c-j\infty}^{c+j\infty} z^{-s} \frac{\prod_{i=1}^m \Gamma(s+b_i) \prod_{i=1}^n \Gamma(1-a_i-s)}{\prod_{i=n+1}^p \Gamma(s+a_i) \prod_{i=m+1}^q \Gamma(1-b_i-s)} ds \tag{6}$$

where $z,\ \{a_i\}_i, and \{b_i\}_i$ are in general, complex-valued. The contour is chosen so that it separates the poles of the gamma products in the numerator. The Meijer G-function has been implemented in some commercial mathematics software packages.

3 Product of n Independent Maxwell Random Variables

Theorem 1: Suppose $X_i,\ i = 1,..,n$ are independent random variables distributed according to (1). Then for $x > 0$ the probability density function *p.d.f.* of $X = \prod_{i=1}^n X_i$ can be expressed as:

$$f_X(x) = 2(\sqrt{\tfrac{2}{\pi}})^n \frac{1}{\prod_{i=1}^n b_i} G_{0n}^{n0}\left(x^2 2^{-n} \prod_{i=1}^n b_i^{-2} \Big|_{1, \ldots, 1}\right) \tag{7}$$

ProofConsider a product of n independent random variables

$$X = \prod_{i=1}^n X_i \tag{8}$$

where X_i is a Maxwell distributed random variable with probability density function according to (1), The Mellin integral transform of $f_{X_i}(x_i)$ is:

$$\begin{aligned} M\{f_{X_i}(x_i)/s\} &= \int_0^\infty x_i^{s-1} f_{X_i}(x_i) dx_i \\ &= \frac{1}{b_i^3}\sqrt{\frac{2}{\pi}} \int_0^\infty x_i^{s+1} e^{\frac{-x_i^2}{2(b_i)^2}} dx_i \\ &= \sqrt{\frac{2}{\pi}} 2^{s/2} \frac{1}{b_i} (b_i^{-2})^{-\frac{s}{2}} \Gamma(1+s/2) \end{aligned} \tag{9}$$

Where we have used the definition of the gamma function

$$\Gamma(t) = \int_0^\infty x^{t-1} e^{-x} \tag{10}$$

The Mellin integral transform of $f_X(x)$

$$\begin{aligned} M\{f_X(x)/s\} &= \prod_{i=1}^n M\{f_{X_i}(x_i)/s\} \\ &= \prod_{i=1}^n [\sqrt{\frac{2}{\pi}} 2^{s/2} \frac{1}{b_i} (b_i^{-2})^{-\frac{s}{2}} \Gamma(1+s/2)] \end{aligned}$$

We can find the pdf of X as the inverse Mellin transform

$$\begin{aligned} f_X(x) &= \frac{1}{2j\pi}\int_{c-j\infty}^{c+j\infty} x^{-s}[\prod_{i=1}^{n}[\sqrt{\frac{2}{\pi}}2^{s/2}\frac{1}{b_i}(b_i^{-2})^{-\frac{s}{2}}\Gamma(1+s/2)]]ds \\ &= \frac{1}{2j\pi}\int_{c-j\infty}^{c+j\infty}(x^2)^{-\frac{s}{2}}(\sqrt{\frac{2}{\pi}})^n(2^{-n})^{-\frac{s}{2}}\frac{1}{\prod_{i=1}^{n}b_i}(\prod_{i=1}^{n}(b_i)^{-2})^{-\frac{s}{2}}\prod_{i=1}^{n}\Gamma(1+\frac{s}{2})2\frac{ds}{2} \quad (11) \\ &= 2(\sqrt{\frac{2}{\pi}})^n\frac{1}{\prod_{i=1}^{n}b_i}\frac{1}{2j\pi}\int_{c-j\infty}^{c+j\infty}(x^2 2^{-n}\prod_{i=1}^{n}b_i^{-2})^{-\frac{s}{2}}\prod_{i=1}^{n}\Gamma(1+\frac{s}{2})\frac{ds}{2} \end{aligned}$$

Finally using the definition of the Meijer G-function we get

$$f_X(x) = 2(\sqrt{\frac{2}{\pi}})^n\frac{1}{\prod_{i=1}^{n}b_i}G_{0n}^{n0}\left(x^2 2^{-n}\prod_{i=1}^{n}b_i^{-2}\Big|_{1,\dots,1}\right) \quad (12)$$

Corollary 1: Suppose $X_i,\ i=1,..,n$ are independent random variables distributed according to (1). Then for $t>0$ the cumulative distribution function *c.d.f.* of $X=\prod_{i=1}^{n}X_i$ can be expressed as:

$$F_X(t) = 2(\sqrt{\frac{2}{n}})^n\frac{1}{\prod_{i=1}^{n}b_i}\frac{t}{2}G_{1n+1}^{n1}\left(t^2 2^{-n}\prod_{i=1}^{n}b_i^{-2}\Big|_{1,\dots,1,-\frac{1}{2}}^{\frac{1}{2}}\right) \quad (13)$$

Proof The cumulative distribution function $F_X(t)=\int_0^t f_X(x)dx$ is obtained by integrating (7) with respect to x inside the contour integral by using:

$$\int_0^t x^{-s}dx = \frac{t^{1-s}}{1-s} = t^{1-s}\frac{1}{2}(\frac{2}{1-s}) = t^{1-s}\frac{1}{2}(\frac{1}{2}-\frac{s}{2})^{-1} \quad (14)$$

And

$$\frac{1}{2}-\frac{s}{2} = \frac{(\frac{1}{2}-\frac{s}{2})\Gamma(\frac{1}{2}-\frac{s}{2})}{\Gamma(\frac{1}{2}-\frac{s}{2})}$$

Then we get

$$\int_0^t x^{-s}dx = t^{1-s}\frac{1}{2}\frac{\Gamma(\frac{1}{2}-\frac{s}{2})}{\Gamma(\frac{3}{2}-\frac{s}{2})}$$

Let $\beta_n = 2(\sqrt{\frac{2}{n}})^n\frac{1}{\prod_{i=1}^{n}b_i}$

$$F_X(t) = t\beta_n\frac{1}{2j\pi}\int_{c-j\infty}^{c+j\infty}\frac{1}{2}(t^2)^{-\frac{s}{2}}(2^{-n}\prod_{i=1}^{n}b_i^{-2})^{-s/2}\frac{\Gamma(\frac{1}{2}-\frac{s}{2})\prod_{i=1}^{n}(\Gamma(1+\frac{s}{2}))}{\Gamma(\frac{3}{2}-\frac{s}{2})} \quad (15)$$

Finally using the definition of the Meijer G-function we obtain

$$F_X(t) = 2(\sqrt{\frac{2}{n}})^n\frac{1}{\prod_{i=1}^{n}b_i}\frac{t}{2}G_{1n+1}^{n1}\left(t^2 2^{-n}\prod_{i=1}^{n}b_i^{-2}\Big|_{1,\dots,1,-\frac{1}{2}}^{\frac{1}{2}}\right)$$

Corollary 2: Suppose $X_i,\ i=1,..,n$ are independent random variables distributed according to (1). Then for $r>0, \alpha>0$ the moment of order r of $X=\prod_{i=1}^{n}X_i$ can be expressed as:

$$E[X^r] = 2(\sqrt{\frac{2}{\pi}})^n\frac{\alpha^{r+1}}{\prod_{i=1}^{n}b_i}\frac{1}{2}G_{1n+1}^{n1}\left(\alpha^2 2^{-n}\prod_{i=1}^{n}b_i^{-2}\Big|_{1,\dots,1,-\frac{r}{2}-\frac{1}{2}}^{\frac{1}{2}-\frac{r}{2}}\right) \quad (16)$$

Proof

$$\begin{aligned} E[X^r] &= \int_{-\infty}^{+\infty}x^r f_X(x)dx \\ &= \int_{\alpha}^{+\infty}x^r f_X(x)dx \\ &= \beta_n\frac{1}{2j\pi}\int_{c-j\infty}^{c+j\infty}\int_{\alpha}^{\infty}x^{r-s}(2^{-n}\prod_{i=1}^{n}b_i^{-2})^{-s/2}\prod_{i=1}^{n}\Gamma(1+\frac{s}{2})dx\frac{ds}{2} \end{aligned}$$

We have

$$\int_{\alpha}^{\infty}x^{-s+r}dx = \frac{\alpha^{1+r-s}}{s-r-1} \quad (17)$$

Then

$$E[X^r] = \beta_n\frac{1}{2j\pi}\int_{c-j\infty}^{c+j\infty}\frac{\alpha^{1+r}}{s-r-1}(2^{-n}\prod_{i=1}^{n}b_i^{-2}\alpha^2)^{-s/2}\prod_{i=1}^{n}\Gamma(1+\frac{s}{2})\frac{ds}{2} \quad (18)$$

Also we have

$$\frac{1}{s-r-1}=-\frac{1}{2}\frac{\Gamma(\frac{-s}{2}+\frac{r}{2}+\frac{1}{2})}{\Gamma(\frac{-s}{2}+\frac{r}{2}+\frac{3}{2})} \tag{19}$$

Finally using (19) and the definition of the Meijer G-function we obtain

$$\begin{aligned} E[X^r] &= \beta_n \frac{1}{2j\pi}\alpha^{r+1}(-\tfrac{1}{2})\int_{c-j\infty}^{c+j\infty} (2^{-n}\prod_{i=1}^n b_i^{-2}\alpha^2)^{-\frac{s}{2}}\frac{\Gamma(-\frac{s}{2}+\frac{r}{2}+\frac{1}{2})}{\Gamma(-\frac{s}{2}+\frac{r}{2}+\frac{3}{2})}\prod_{i=1}^n \Gamma(1+\tfrac{s}{2}) \\ &= 2(\sqrt{\tfrac{2}{\pi}})^n \frac{\alpha^{r+1}}{\prod_{i=1}^n b_i}\frac{1}{2} G_{1n+1}^{n1}\left(\alpha^2 2^{-n}\prod_{i=1}^n b_i^{-2}\middle|_{1,\dots,1,-\frac{r}{2}-\frac{1}{2}}^{\frac{1}{2}-\frac{r}{2}}\right) \end{aligned} \tag{20}$$

Corollary 3: Suppose $X_i,\ i=1,..,n$ are independent random variables distributed according to (1). Then for $\alpha>0$ the expected value of $X=\prod_{i=1}^n X_i$ can be expressed as: For $r=1$

$$E[X]=2(\sqrt{\tfrac{2}{\pi}})^n \frac{\alpha^2}{\prod_{i=1}^n b_i}\frac{1}{2}G_{1n+1}^{n1}\left(\alpha^2 2^{-n}\prod_{i=1}^n b_i^{-2}\middle|_{1,\dots,1,-1}^{0}\right) \tag{21}$$

Corollary 4: Suppose $X_i,\ i=1,..,n$ are independent random variables distributed according to (1). Then for $\alpha>0$ the expected value of $X=\prod_{i=1}^n X_i$ can be expressed as:

$$\begin{aligned} \sigma^2 &= 2(\sqrt{\tfrac{2}{\pi}})^n \frac{\alpha^3}{\prod_{i=1}^n b_i}\frac{1}{2}G_{1n+1}^{n1}\left(\alpha^2 2^{-n}\prod_{i=1}^n b_i^{-2}\middle|_{1,\dots,1,-\frac{3}{2}}^{-\frac{1}{2}}\right) \\ &\quad -[2(\sqrt{\tfrac{2}{\pi}})^n \frac{\alpha^2}{\prod_{i=1}^n b_i}\frac{1}{2}G_{1n+1}^{n1}\left(\alpha^2 2^{-n}\prod_{i=1}^n b_i^{-2}\middle|_{1,\dots,1,-1}^{0}\right)]^2 \end{aligned} \tag{22}$$

Proof. By definition the variance of X/Y is:

$$\sigma^2 = E[Z^2]-E[Z]^2 \tag{23}$$

Corollary 5: Suppose $X_i,\ i=1,..,n$ are independent random variables distributed according to (1). Then for $x>0$ the survival function of $X=\prod_{i=1}^n X_i$ can be expressed as:

$$S_X(x)=\begin{cases} 1 & \text{if} x\le 0 \\ 1-2(\sqrt{\tfrac{2}{n}})^n \frac{1}{\prod_{i=1}^n b_i}\frac{x}{2}G_{1n+1}^{n1}\left(x^2 2^{-n}\prod_{i=1}^n b_i^{-2}\middle|_{1,\dots,1,-\frac{1}{2}}^{\frac{1}{2}}\right) & \text{if} x>0 \end{cases} \tag{24}$$

Proof By definition of the survival function

$$S_X(x)=1-F_X(x) \tag{25}$$

Corollary 6: Suppose $X_i,\ i=1,..,n$ are independent random variables distributed according to (1). Then for $x>0$ the hazard function of $X=\prod_{i=1}^n X_i$ can be expressed as:

$$h_X(x)=\begin{cases} 0 & \text{if} x\le 0 \\ \dfrac{2(\sqrt{\frac{2}{\pi}})^n\frac{1}{\prod_{i=1}^n b_i}G_{0n}^{n0}\left(x^2 2^{-n}\prod_{i=1}^n b_i^{-2}\middle|_{1,\dots,1}\right)}{1-2(\sqrt{\frac{2}{n}})^n\frac{1}{\prod_{i=1}^n b_i}\frac{x}{2}G_{1n+1}^{n1}\left(x^2 2^{-n}\prod_{i=1}^n b_i^{-2}\middle|_{1,\dots,1,-\frac{1}{2}}^{\frac{1}{2}}\right)} & \text{if} x>0 \end{cases} \tag{26}$$

4 Examples and special cases

4.1 Product of two independent Maxwell random variables

1. **Probability density function:** Suppose $X_i,\ i=1,2$ are independent Maxwell random variables with scale parameters $b_1=1, b_2=2$ respectively, the probability density function of X is

$$f_X(x)=\begin{cases} 0 & if x\le 0 \\ \frac{2}{\pi}G_{02}^{20}\left(\frac{x^2}{16}\middle|_{1,1}\right) & if x>0 \end{cases} \tag{27}$$

2. **Cumulative distribution function:** Suppose $X_i,\ i = 1,2$ are independent Maxwell random variables with scale parameters $b_1 = 1, b_2 = 2$ respectively, the cumulative distribution function of X is

For $t > 0$

$$F_X(t) = \begin{pmatrix} 0 & if t \leq 0 \\ \frac{t}{\pi} G_{13}^{21}\left(\frac{t^2}{16}\Big|_{1,1,-\frac{1}{2}}^{\frac{1}{2}}\right) & if t > 0 \end{pmatrix} \tag{28}$$

3. **Moment of order "r":** Suppose $X_i,\ i = 1,2$ are independent Maxwell random variables with scale parameters $b_1 = 1, b_2 = 2$ respectively, the moment of order r of X is

For $\alpha > 0$

$$E[X^r] = -\frac{\alpha^{r+1}}{\pi} G_{13}^{21}\left(\frac{\alpha^2}{16}\Big|_{1,1,-\frac{r}{2}-\frac{1}{2}}^{\frac{1}{2}-\frac{r}{2}}\right) \tag{29}$$

4. **Expected value:** Suppose $X_i,\ i = 1,2$ are independent Maxwell random variables with scale parameters $b_1 = 1, b_2 = 2$ respectively, the Expected value of X is

$$E[X] = -\frac{\alpha^2}{\pi} G_{13}^{21}\left(\frac{\alpha^2}{16}\Big|_{1,1,-1}^{0}\right) \tag{30}$$

5. **Variance:** Suppose $X_i,\ i = 1,2$ are independent Maxwell random variables with scale parameters $b_1 = 1, b_2 = 2$ respectively, the Variance of X is

$$\begin{aligned} \sigma^2 &= -\frac{\alpha^3}{\pi} G_{13}^{21}\left(\frac{\alpha^2}{16}\Big|_{1,1,-\frac{3}{2}}^{-\frac{1}{2}}\right) \\ &\quad -[-\frac{\alpha^2}{\pi} G_{13}^{21}\left(\frac{\alpha^2}{16}\Big|_{1,1,-1}^{0}\right)]^2 \end{aligned} \tag{31}$$

6. **Survival function:** Suppose $X_i,\ i = 1,2$ are independent Maxwell random variables with scale parameters $b_1 = 1, b_2 = 2$ respectively, the Survival function of X is

$$S_X(t) = \begin{pmatrix} 1 & \text{if} t \leq 0 \\ 1 - \frac{t}{\pi} G_{13}^{21}\left(\frac{t^2}{16}\Big|_{1,1,-\frac{1}{2}}^{\frac{1}{2}}\right) & \text{if} t > 0 \end{pmatrix} \tag{32}$$

7. **Hazard function:** Suppose $X_i,\ i = 1,2$ are independent Maxwell random variables with scale parameters $b_1 = 1, b_2 = 2$ respectively, the Hazard function of X is

For $t > 0$

$$h_X(x) = \begin{pmatrix} 0 & if x \leq 0 \\ \dfrac{\frac{2}{\pi} G_{02}^{20}\left(\frac{x^2}{16}\Big|_{1,1}\right)}{1 - \frac{x}{\pi} G_{13}^{21}\left(\frac{x^2}{16}\Big|_{1,1,-\frac{1}{2}}^{\frac{1}{2}}\right)} & if x > 0 \end{pmatrix} \tag{33}$$

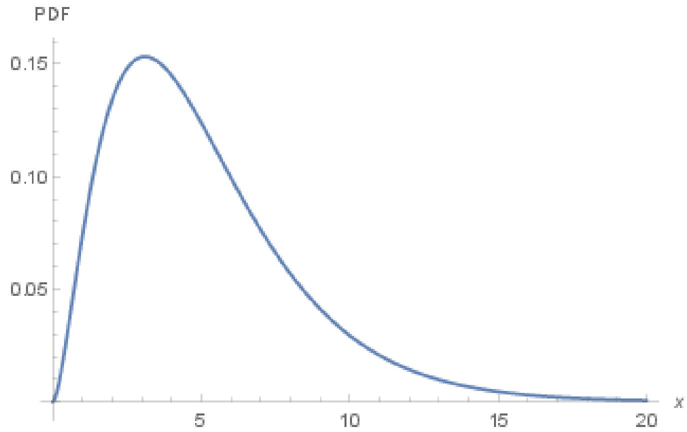

Figure 1: Plot of the probability density function for two independent Maxwell random variables for $b_1 = 1, b_2 = 2$.

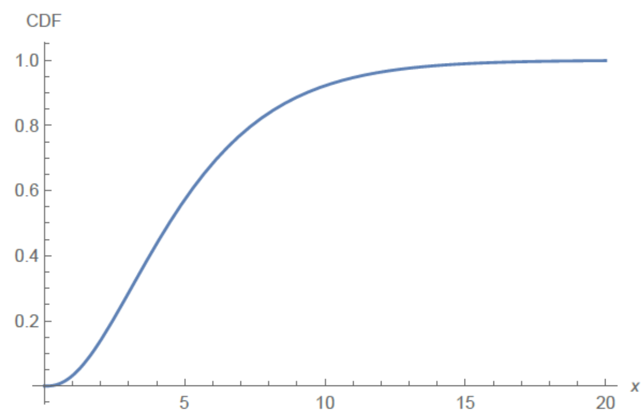

Figure 2: Plot of the cumulative distribution function for two independent Maxwell random variables for $b_1 = 1, b_2 = 2$.

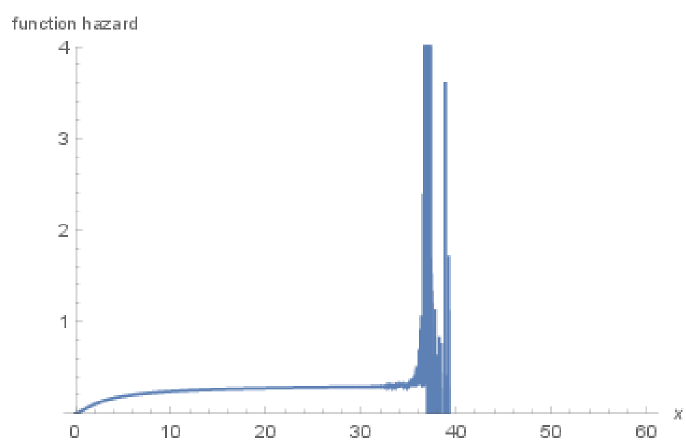

Figure 3: Plot of the hazard function for two independent Maxwell random variables for $b_1 = 1, b_2 = 2$.

5 Applications

The air molecules surrounding us are not all traveling at the same speed, even if the air is all at a single temperature. Some of the air molecules will be moving extremely fast, some will be moving with moderate speeds, and some of the air molecules will hardly be moving at all. Because of this, we can't ask questions like "What is the speed of an air molecule in a gas?" since a molecule in a gas could have any one of a huge number of possible speeds.

So instead of asking about any one particular gas molecule, we ask questions like, "What is the distribution of speeds in a gas at a certain temperature?" In the mid to late 1800s, James Clerk Maxwell and Ludwig Boltzmann figured out the answer to this question. Their result is referred to as the Maxwell-Boltzmann distribution, because it shows how the speeds of molecules are distributed for an ideal gas. The Maxwell-Boltzmann distribution is often represented with the following graph.

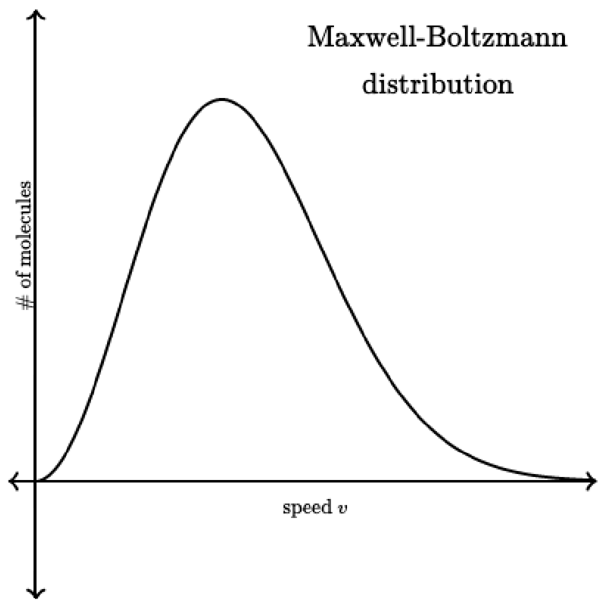

Figure 4: Maxwell-Boltzmann distribution

The y-axis of the Maxwell-Boltzmann graph can be thought of as giving the number of molecules per unit speed. So, if the graph is higher in a given region, it means that there are more gas molecules moving with those speeds.

Let take the following example: we are interested to find the distribution $X = X_1X_2X_3X_4X_5X_6X_7X_8X_9X_{10}$, where X_i are independent Maxwell random variables with scale parameters b_i,$b_1 = 1, b_2 = 2, b_3 = 3, b_4 = 4, b_5 = 5, b_6 = 6, b_7 = 7, b_8 = 8, b_9 = 9, b_{10} = 10$.

So the speeds of molecules are distributed for an ideal gas with respect to the probability density function of X

$$f_X(x) = 2(\sqrt{\frac{2}{\pi}})^{10} \frac{1}{(12345678910)} G_{010}^{100}\left(x^2 2^{-10}(\frac{1}{13168189440000})\Big|_{1,1,1,1,1,1,1,1,1,1}\right) \tag{34}$$

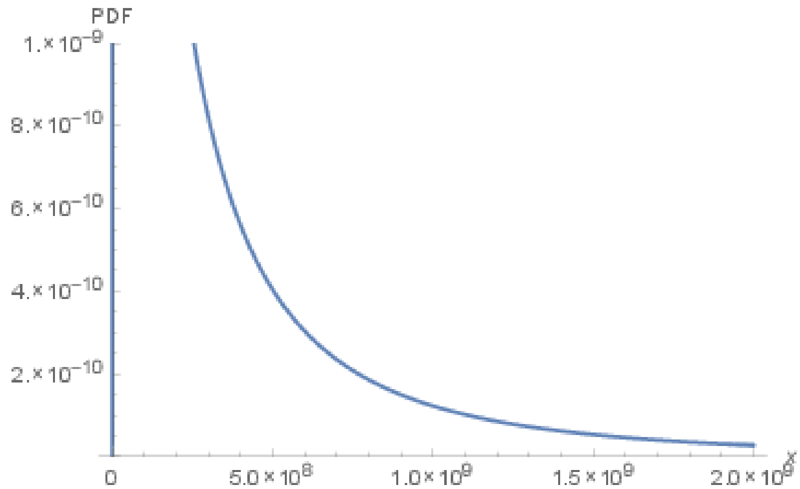

Figure 5: Plot of n=10 independent Maxwell rondom variables for $b_1 = 1, b_2 = 2, b_3 = 3, b_4 = 4, b_5 = 5, b_6 = 6, b_7 = 7, b_8 = 8, b_9 = 9, b_{10} = 10$.

6 Monte Carlo simulation:

Monte Carlo simulations are used to model the probability of different outcomes in a process that cannot easily be predicted due to the intervention of random variables. It is a technique used to understand the impact of risk and uncertainty in prediction and forecasting models.

A Monte Carlo simulation can be used to tackle a range of problems in virtually every field such as finance, engineering, supply chain, and science. It is also referred to as a multiple probability simulation.

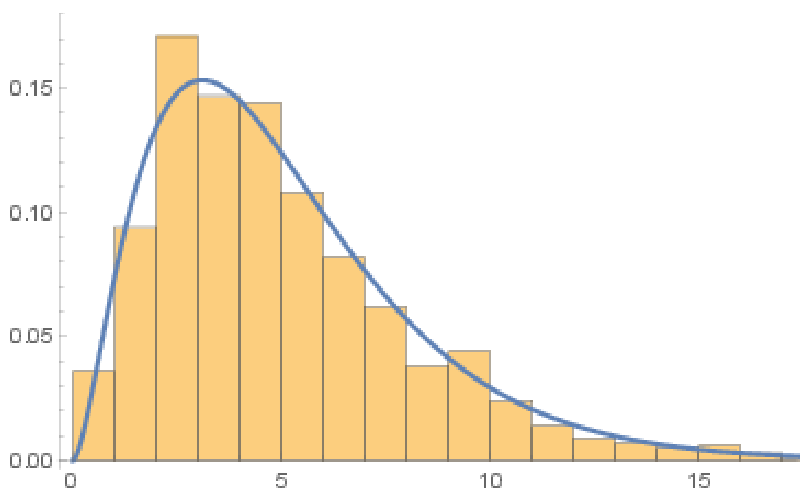

Figure 6: Monte Carlo simulation for the product of two independent maxwell random variables for scale parameters $b_1 = 1, b_2 = 2$.

7 Conclusion

This paper has derived the analytical expressions of the PDF, CDF, the moment of order r, the survival function, and the hazard function, for the distribution of $X = \prod_{i=1}^{n} X_i$ when X_i are Maxwell random variables distributed independently of each other, we have illustrated our results for $n = 2$ as a special case, then we have discussed an application of the distribution of product $X = \prod_{i=1}^{n} X_i$, finally, we have confirmed our result using Monte Carlo simulation.

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