Quadratic Fractional Bi-level Fuzzy Probabilistic Programming Problem When b_i Follows Exponential Distribution

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Abstract

Some of the actual life decisions are made in decentralized manner under uncertainty. This paper formulates a quadratic fractional bi-level (QFBL) programming problem with probabilistic constraints in both first (leader) and second level (follower) having two parameter exponential random variables with known probability distributions and fuzziness is considered as triangular and trapezoidal fuzzy number. These fuzzy numbers of the membership functions related with the proportional probability density function has been used to introduce a defuzzification approach for finding the crisp values of fuzzy numbers. In the proposed model the problem is first converted into an equivalent deterministic quadratic fractional fuzzy bi level programming model by applying chance constrained programming technique. Secondly, in the suggested model, each objective function of the bi-level quadratic fractional programming problem has its own non-linear membership function. The fuzzy goal programming (FGP) approach is used to find a compromise solution for the BLQFP problem. Finally, to demonstrate the applicability and performance of the proposed approach an illustrative numerical example is given.

Keywords: Bi-level programming, Quadratic programming, Fractional programming, Two parameter exponential distribution, Fuzzy chance constrained programming, Fuzzy goal programming.

I. Introduction

In actual life decision-making situations, decision-makers are frequently confronted with different kinds of vagueness, the most important of which are randomness and fuzziness. There are two common approaches to dealing with such uncertainties: probability theoretical approach and fuzzy set theoretical approach. Chance constrained programming is a well-defined approach described by Charnes and Cooper [1] for dealing with problems involving probabilistic data (CCP). Gardening, capability planning, banking, forestry, army, manufacture control and arrangement, sports, broadcastings, transport, and eco-friendly management planning are just a few of the fields where it is commonly used.

Quadratic programming (QP) is a technique of solving certain mathematical optimization problems involving quadratic function. Specially a quadratic programming minimized or maximized a multivariate quadratic function subject to linear constraint on the variables.

Quadratic programming is a type of nonlinear programming of the several. A wide range of applications of quadratic programming is portfolio selection, electrical energy growth, agriculture, and harvest selection. Probabilistic quadratic programming is applicable for financial and risk management, and various important related literatures are found in this direction, which are mentioned below.

Mc Carl *et al*. [2] presented some of the approaches during which QP are often used. Interval parameters are used to represent the cost coefficients, constraint coefficients, and right-hand sides in an interval quadratic programming problem proposed by Liu and Wang [6]. A fuzzy quadratic programming problem was introduced in [7] where fuzzy data represents the cost coefficients, constraint coefficients, and right-hand side values. Kausar and Adhami [18] have given a fuzzy goal programming approach for solving chance constrained bi-Level multi-objective quadratic fractional programming problems. Ammar [9] proposed a multi-objective quadratic programming problem based on the fuzzy random coefficient matrix's goals and constraints where the decision vector interpreted as a fuzzy vector. The problem of fuzzy quadratic programming was introduced by Liu [10] in which convex fuzzy numbers represent the cost coefficients, constraint coefficients, and constraints parameters on the right side. He [11] also specified a stream water quality control solution process. Qin and Huang [12] suggested an inexact chance constrained quadratic programming model. Nasseri [13] outlined a fuzzy quadratic programming problem with trapezoidal and/or triangular fuzzy numbers for the cost coefficients, constraint coefficients, and right-hand parameter values. Guo and Huang [14] developed an incorrect fuzzy-stochastic quadratic programming approach to efficiently distribute waste to a municipal solid waste management scheme while accounting for the nonlinear objective function and various parameter uncertainties in the constraints. Bi-level multi-objective stochastic linear fractional programming with general form of distribution has been developed by Kausar and Adhami [17]

In this paper, we are constructing a different approach for solving quadratic fractional bi level programming problems with probabilistic constraints having two parameter exponential distributed fuzzy random variables with known probability distributions. The probabilistic problem is changed into an equivalent deterministic model. Under the quadratic fractional programming outline, both uncertainty and fuzziness are considered. Poularikas [16] proposes a defuzzification technique for determining the crisp values of fuzzy numbers using the Mellin transformation.

I. Probabilistic Fuzzy Quadratic Fractional Bi-Level Programming Problem

In some circumstances, quadratic fractional programming with a quadratic fractional objective function and few linear constraints including fuzziness and randomness is called a probabilistic fuzzy quadratic fractional programming problem. When a number of the input parameters of QFP are described by stochastic and fuzzy parameters, the problem is treated as a probabilistic fuzzy quadratic fractional programming problem. A general probabilistic fuzzy quadratic fractional bi-level programming problem is presented as follows

$$\underset{X_1}{\mathrm{Max}}\, \tilde{f}_1(x_1,x_2)=\frac{f_{11}}{f_{12}}=\frac{\tilde{c}_{i1}x_j+\frac{1}{2}x^T\tilde{d}_{i1}x_j\ +\alpha_{i1}}{\tilde{c}_{i2}x_j+\frac{1}{2}x^T\tilde{d}_{i2}x_j\ +\beta_{i2}} \tag{1.1}$$

where for given X_1, X_2

$$\underset{X_2}{\mathrm{Max}}\, \tilde{f}_2(x_1,x_2)=\frac{f_{21}}{f_{22}}=\frac{\tilde{c}_{i1}x_j+\frac{1}{2}x^T\tilde{d}_{i1}x_j\ +\alpha_{i1}}{\tilde{c}_{i2}x_j+\frac{1}{2}x^T\tilde{d}_{i2}x_j\ +\beta_{i2}} \tag{1.2}$$

$$\mathrm{X}\in\mathrm{G}=\left\{X\in R^n \middle| P\left(\sum_{j=1}^{n}\tilde{a}_{ij}x_j\le b_i\right)\ge 1-\lambda_i,\qquad i=1,2,\ldots..m\right\} \tag{1.3}$$

$$x_j\ge 0,\quad j=1,2,\ldots,n \tag{1.4}$$

where $0<\lambda_i<1$ and $\tilde{b}_i$ (i=1,2,…,m) represent two parameter exponential distribution fuzzy random variables; $\tilde{c}_{i1},\tilde{c}_{i2},\tilde{d}_{i1},\tilde{d}_{i2}$ (i=1,2,…,m) and $\tilde{a}_{ij}$ are considered as fuzzy number and $\lambda_i\,\epsilon[0,1]$.

The decision vector $X_1 = (x_{11}, x_{12}, \dots, x_{1n_1})$ is controlled by the leader and decision vector $X_2 = (x_{21}, x_{22}, \dots, x_{2n_2})$ is controlled by the follower; $X_1 \cup X_2 = X = (x_1, x_2, \dots, x_n) \in R^n$ with $n_1 + n_2 = n$.

II. Some Preliminaries

In the model formulation the triangular and trapezoidal membership functions are discussed in this section. We can also introduce the Mellin transform to find the expected value of a random variable's function by using proportional probability density function related with membership functions of fuzzy numbers.

Definition 2.1 (triangular fuzzy number): A triangular fuzzy number is one that is represented by the triplet $\tilde{A} = (a_1, a_2, a_3)$ and has a piecewise linear membership function $\mu_{\tilde{A}}(x)$ is given by

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{x - a_1}{a_2 - a_1}, & a_1 \le x \le a_2 \\ \dfrac{a_3 - x}{a_3 - a_2}, & a_2 \le x \le a_3 \\ 0, & otherwise \end{cases} \tag{2.1}$$

Definition 2.2 (trapezoidal fuzzy number): A fuzzy number represented by the quadruplet $\tilde{A} = (a_1, a_2, a_3, a_4)$ and has a piecewise linear membership function $\mu_{\tilde{A}}(x)$ is given by

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{x - a_1}{a_2 - a_1}, & a_1 \le x \le a_2 \\ 1, & a_2 \le x \le a_3 \\ \frac{a_4 - x}{a_4 - a_3}, & a_3 \le x \le a_4 \\ 0, & otherwise \end{cases} \tag{2.2}$$

I. Defuzzification with Probability Density Function and Membership Function

Assume that $F(\mathbb{R})$ represent the sum of all fuzzy numbers. In$(\mathbb{R})$ the triangular and trapezoidal fuzzy numbers are (a_1, a_2, a_3) and (a_1, a_2, a_3, a_4) respectively. Now the method associated with a probability density function for the membership function of $\tilde{A}$ is defined as follows ([15], [4]).

Proportional probability distribution: describe a probability density function $f_1 = c\mu_{\tilde{A}}(x)$ associated with $\tilde{A}$, where the constant c is obtained by using the property of probability density function, where $\int_{-\infty}^{\infty} f_1(x)dx = 1$ and $\int_{-\infty}^{\infty} c\mu_{\tilde{A}}(x)dx = 1$.

II. Mellin Transform

The Mellin transform ([15], [4]) is used to find this expected value since any probability density function with finite support is associated with expected value.

Definition 2.3: The Mellin transform $M_X(t)$ of a probability density function $f(x)$, where x denote the positive, is given as

$$M_X(t) = \int_0^{\infty} x^{t-1} f(x)\, dx \tag{2.3}$$

Now in terms of expected values we find the Mellin transform. Remember that the expected value of any function $g(X)$ of the random variable X whose probability density function is $f(x)$, is given as

$$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f(x) dx \tag{2.4}$$

Thus, it follows that $M_X(t) = \mathbb{E}[X^{t-1}] = \int_0^{\infty} x^{t-1} f(x)\, dx$.

Hence, $\mathbb{E}[X^t] = M_X(t+1)$. Thus the expected value of random variable X is $\mathbb{E}[X] = M_X(2)$.

For example, if the triangular and trapezoidal fuzzy numbers are $\tilde{A}_1 = (a_1, a_2, a_3)$ and $\tilde{A}_2 = (a_1, a_2, a_3, a_4)$ respectively, then their crisp values are determined by finding expected values using the probability density function that corresponding to the membership functions of the given fuzzy number.

Thus, the probability density function corresponding to triangular fuzzy number $\tilde{A}_1 = (a_1, a_2, a_3)$ is given as

$$f\tilde{A}_1(x) = c_1 \mu \tilde{A}_1(x) \tag{2.5}$$

where $\mu\tilde{A}_1(x)$ is defined as

$$\mu_{\tilde{A}_1}(x) = \begin{cases} \dfrac{x-a_1}{a_2-a_1}, & a_1 \le x \le a_2 \\ \dfrac{a_3-x}{a_3-a_2}, & a_2 \le x \le a_3 \\ 0, & otherwise \end{cases} \tag{2.6}$$

Now c_1 is calculated as

$$\int_{-\infty}^{\infty} f_{\tilde{A}_1}(x)dx = 1 \tag{2.7}$$

that is

$$\int_{-\infty}^{\infty} c_1 \mu_{\tilde{A}_1}(x)dx = 1 \tag{2.8}$$

that is

$$c_1 \int_{a_1}^{a_2} \frac{x-a_1}{a_2-a_1} dx + c_1 \int_{a_2}^{a_3} \frac{a_3-x}{a_3-a_2} dx = 1 \tag{2.9}$$

On integration, we get

$$c_1 = \frac{2}{a_3-a_1} \tag{2.10}$$

The proportional probability function corresponding to triangular fuzzy number $\widetilde{A}$ is given by

$$f_{X_{\tilde{A}_1}}(x) = \begin{cases} \dfrac{2(x-a_1)}{(a_2-a_1)(a_3-a_1)}, & a_1 \le x \le a_2 \\ \dfrac{2(a_3-x)}{(a_3-a_2)(a_3-a_1)}, & a_2 \le x \le a_3 \\ 0, & otherwise \end{cases} \tag{2.11}$$

Graphically it is presented in Figure 1

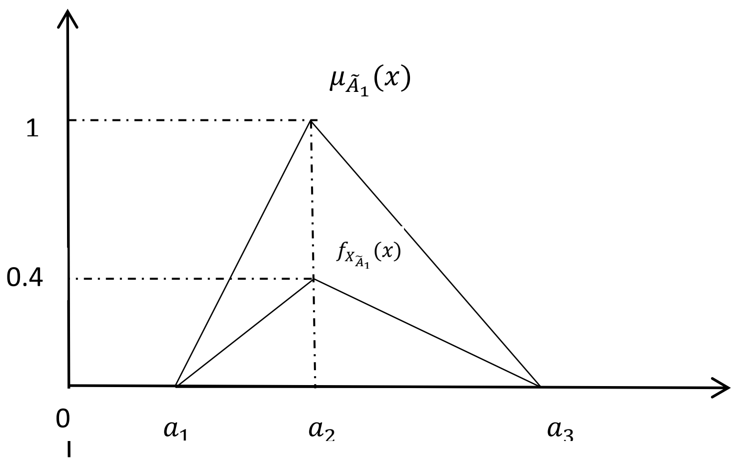

Figure 5.1: Proportional probability density function of triangular fuzzy number

By using the Mellin transform, we obtain

$$M_X(t) = \int_0^{\infty} x^{t-1} f_{X_{\tilde{A}_1}}(x)\,dx = \int_{a_1}^{a_2} x^{t-1} \frac{2(x-a_1)}{(a_2-a_1)(a_3-a_1)} dx + \int_{a_2}^{a_3} x^{t-1} \frac{2(x-a_1)}{(a_2-a_1)(a_3-a_1)} dx \tag{2.12}$$

On integration, we obtain

$$M_{X_{\tilde{A}_1}}(t) = \frac{2}{(a_3-a_1)t(t+1)} \left[\frac{a_3(a_3^t - a_2^t)}{(a_3-a_2)} - \frac{a_1(a_2^t - a_1^t)}{(a_2-a_1)} \right] \tag{2.13}$$

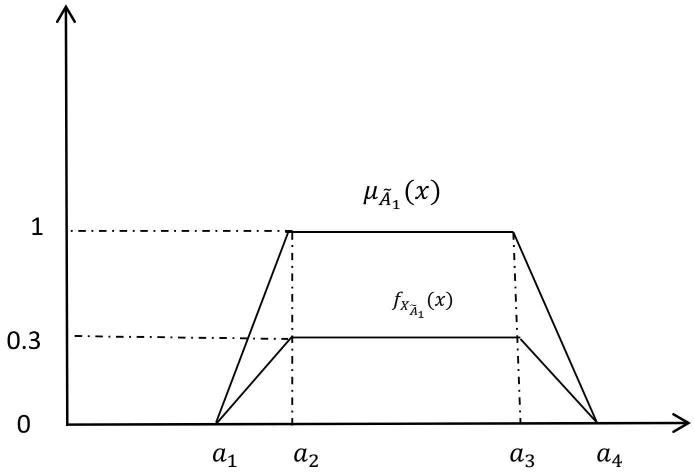

Figure 5.2: Proportional probability density function of trapezoidal fuzzy number

Thus the random variable $X_{\tilde{A}_1}$ has mean $(\mu_{X_{\tilde{A}_1}})$ and variance $\left(\sigma^2_{X_{\tilde{A}_1}}\right)$ can be obtained as

$$\mu_{X_{\tilde{A}_1}} = \mathbb{E}[X_{\tilde{A}_1}] = M_{X_{\tilde{A}_1}}(2) = \frac{a_1 + a_2 + a_3}{3} \tag{2.14}$$

$$\sigma^2_{X_{\tilde{A}_1}} = M_{X_{\tilde{A}_1}}(3) - \left[M_{X_{\tilde{A}_1}}\right]^2 = \frac{a_1^2 + a_2^2 + a_3^2 - a_1a_2 - a_2a_3 - a_3a_1}{18} \tag{2.15}$$

Further, the probability density function corresponding to trapezoidal fuzzy number $\tilde{A}_2 = (a_1, a_2, a_3, a_4)$ is given as $f_{\tilde{A}_2}(x) = c_2\mu_{\tilde{A}_2}(x)$, where $\mu_{\tilde{A}_2}(x)$ is defined as

$$\mu_{\tilde{A}_2}(x) = \begin{cases} \dfrac{x - a_1}{a_2 - a_1}, & a_1 \le x \le a_2 \\ 1, & a_2 \le x \le a_3 \\ \dfrac{a_4 - x}{a_4 - a_3}, & a_3 \le x \le a_4 \\ 0, & otherwise \end{cases} \tag{2.16}$$

Now c_2 is calculated as

$$\int_{-\infty}^{\infty} f_{\tilde{A}_2}(x)dx = 1 \tag{2.17}$$

That is,

$$\int_{-\infty}^{\infty} c_2\mu_{\tilde{A}_1}(x)dx = 1 \tag{2.18}$$

That is,

$$c_2\int_{a_1}^{a_2} \frac{x - a_1}{a_2 - a_1}dx + c_2\int_{a_2}^{a_3} dx + c_2\int_{a_3}^{a_4} \frac{a_4 - x}{a_4 - a_3}dx = 1 \tag{2.19}$$

On integration, we get

$$c_2 = \frac{2}{a_4 + a_3 - a_1 - a_2} \tag{2.20}$$

The proportional probability density function corresponding to triangular fuzzy number $\tilde{A}_1$ is given by

$$f_{X_{\tilde{A}_2}}(x) = \begin{cases} \dfrac{2(x - a_1)}{(a_2 - a_1)(a_4 + a_3 - a_1 - a_2)}, & a_1 \le x \le a_2, \\ \dfrac{2}{(a_4 + a_3 - a_1 - a_2)}, & a_2 \le x \le a_3 \\ \dfrac{2(x - a_1)}{(a_2 - a_1)(a_4 + a_3 - a_1 - a_2)} & a_3 \le x \le a_4 \\ 0, & otherwise \end{cases} \tag{2.21}$$

Graphically it is shown in figure 5.2

Using the Mellin transform, we get

$$M_{\tilde{A}_2}(t) = \int_0^\infty x^{t-1} f_{X_{\tilde{A}_2}}(x)\,dx = \int_{a_1}^{a_2} x^{t-1} \frac{2(x-a_1)}{(a_2-a_1)(a_4+a_3-a_1-a_2)}dx + \int_{a_2}^{a_3} x^{t-1}\frac{2}{(a_4+a_3-a_1-a_2)}dx + \int_{a_3}^{a_4}\frac{2(a_4-x)}{(a_4-a_3)(a_4+a_3-a_1-a_2)}dx \quad (2.22)$$

On integration, we obtain

$$M_{\tilde{A}_2}(t) = \frac{2}{(a_4+a_3-a_1-a_2)t(t+1)}\left[\frac{(a_4^{t+1}-a_3^{t+1})}{(a_4-a_3)} - \frac{a_2^{t+1}-a_1^{t+1}}{(a_2-a_1)}\right] \quad (2.23)$$

Thus, the random variable $X_{\tilde{A}_2}$has mean ($\mu_{X_{\tilde{A}_2}}$) and variance ($\sigma^2_{X_{\tilde{A}_2}}$) can be obtained as

$$\mu_{X_{\tilde{A}_2}} = \mathbb{E}\left[X_{\tilde{A}_2}\right] = M_{X_{\tilde{A}_2}}(2) = \frac{1}{3}\left[(a_1+a_2+a_3+a_4) + \frac{(a_1a_2-a_3a_4)}{(a_4+a_3-a_2-a_1)}\right] \quad (2.24)$$

$$\sigma^2_{X_{\tilde{A}_2}} = M_{X_{\tilde{A}_1}}(3) - \left[M_{X_{\tilde{A}_2}}(2)\right]^2 = \frac{1}{6}\left[(a_1^2+a_2^2+a_3^2+a_4^2) + \frac{(a_1+a_2)(a_3^2+a_4^2)-(a_3+a_4)(a_1^2+a_2^2)}{(a_4+a_3-a_2-a_1)}\right] + (\mu_{X_{\tilde{A}_2}})^2 \quad (2.25)$$

III. Probabilistic Fuzzy Quadratic Programming Problem and Its Crisp Model

Let $\tilde{c}_j = (c_j^1, c_j^2, c_j^3), j = 1,2,\dots,n, \tilde{a}_{ij} = (a_{ij}^1, a_{ij}^2, a_{ij}^3), i = 1,2,\dots,m,\ j = 1,2,\dots,n$ and $\tilde{q}_{ij} = (q_{ij}^1, q_{ij}^2, q_{ij}^3)$, $i = 1,2,\dots,n, j = 1,2,\dots,n$ denote the triangular fuzzy numbers. By using the method of defuzzification the crisp values of these fuzzy numbers can be obtained with probability density function of given membership function given as follows

$$\hat{c}_j = \frac{c_j^1+c_j^2+c_j^3}{3},\quad j=1,2,\dots,n, \qquad \hat{a}_{ij} = \frac{a_{ij}^1+a_{ij}^2+a_{ij}^3}{3},\quad i = 1,2,\dots,m, j=1,2,\dots,n, \quad (2.26)$$

$$\hat{q}_{ij} = \frac{q_{ij}^1+q_{ij}^2+q_{ij}^3}{3},\quad i=1,2,\dots,m,\qquad j=1,2,\dots,n,$$

where the crisp value of the given fuzzy number $\tilde{c}_j$ is represented by $\hat{c}_j$ and so on.

Similarly if all the coefficients have trapezoidal fuzzy numbers such as, $\tilde{c}_j = c_j^1 + c_j^2 + c_j^3 + c_j^4, j = 1,2,\dots,n,\ \tilde{a}_{ij} = (a_{ij}^1 + a_{ij}^2 + a_{ij}^3 + a_{ij}^4), i = 1,2,\dots,m, j = 1,2,\dots,n$ and $\tilde{q}_{ij} = (q_{ij}^1, q_{ij}^2, q_{ij}^3, q_{ij}^4), i = 1,2,\dots,n, j = 1,2,\dots,n$, then the crisp values are given as

$$\hat{c}_j = \frac{1}{3}\left[(c_j^1+c_j^2+c_j^3+c_j^4) + \frac{(c_j^1c_j^2 - c_j^3c_j^4)}{(c_j^3+c_j^4-c_j^1-c_j^2)}\right],\ j=1,2,\dots,n$$

$$\hat{a}_{ij} = \frac{1}{3}\left[(a_{ij}^1+a_{ij}^2+a_{ij}^3+a_{ij}^4) + \frac{(a_{ij}^1a_{ij}^2 - a_{ij}^3a_{ij}^4)}{(a_{ij}^3+a_{ij}^4-a_{ij}^1-a_{ij}^2)}\right],\ i=1,2,\dots,m, j=1,2,\dots,n$$

$$\hat{q}_{ij} = \frac{1}{3}\left[(q_{ij}^1+q_{ij}^2+q_{ij}^3+q_{ij}^4) + \frac{(q_{ij}^1q_{ij}^2 - q_{ij}^3q_{ij}^4)}{(q_{ij}^3+q_{ij}^4-q_{ij}^1-q_{ij}^2)}\right],\ i=1,2,\dots,n, j=1,2,\dots,n \quad (2.27)$$

Thus the probabilistic quadratic fractional bi-level programming can be stated as follows

$$\underset{X_1}{\text{Max}}\, \tilde{f}_1(x_1,x_2) = \frac{f_{11}}{f_{12}} = \frac{\tilde{c}_{i1}x_j + \frac{1}{2}x^T\tilde{d}_{i1}x_j + \alpha_{i1}}{\tilde{c}_{i2}x_j + \frac{1}{2}x^T\tilde{d}_{i2}x_j + \beta_{i2}} \quad (2.28)$$

where for given X_1, X_2

$$\underset{X_2}{\text{Max}}\, \tilde{f}_2(x_1,x_2) = \frac{f_{21}}{f_{22}} = \frac{\tilde{c}_{i1}x_j + \frac{1}{2}x^T\tilde{d}_{i1}x_j + \alpha_{i1}}{\tilde{c}_{i2}x_j + \frac{1}{2}x^T\tilde{d}_{i2}x_j + \beta_{i2}} \quad (2.29)$$

$$\mathrm{X} \in \mathrm{G} = \left\{X \in R^n \middle| P(\textstyle\sum_{j=1}^n \tilde{a}_{ij}x_j \le b_i) \ge 1-\lambda_i,\quad i = 1,2,\dots..m\right\} \quad (2.30)$$

$$x_j \ge 0,\quad j = 1,2,\dots,n \quad (2.31)$$

where $0 < \lambda_i < 1, i = 1,2,\dots,m$

IV. Deterministic Model of the Probabilistic Quadratic Fractional Programming Problem

A technique for converting the quadratic fractional bi-level fuzzy probabilistic programming into its equivalent quadratic fractional bi-level fuzzy programming is discussed. We assume that $b_i (i = 1,2,\dots,m)$ in the model (2.28)-(2.31) are independent random variables following two-parameter exponential distribution [3] with parameters θ_i, σ_i where mean and variance of random variable b_i are given by:

$$E(b_i) = \theta_i + \sigma_i \quad i = 1,2,\dots,m \tag{2.32}$$

$$V(b_i) = \sigma_i^2 \quad i = 1,2,\dots,m \tag{2.33}$$

The probability density function of the i^{th} two-parameter exponential variable b_i is given by

$$f(b_i) = \frac{1}{\sigma_i} exp\left(\frac{-(b_i - \theta_i)}{\sigma_i}\right), \qquad i = 1,2,\dots,m \tag{2.34}$$

where $b_i \geq \theta_i, \sigma_i > 0$

To solve the problem (2.28)-(2.31), the deterministic form of the problem is established. Then from the chance-constraint (2.30), we have

$$\Pr\left(\sum_{j=1}^{n} a_{ij} x_j \leq b_i\right) \geq (1 - \gamma_i)$$

$$\Pr\left(b_i \geq \sum_{j=1}^{n} a_{ij} x_j\right) \geq (1 - \gamma_i)$$

$$\int_{\sum_{j=1}^{n} a_{ij} x_j}^{\infty} f(b_i) db_i \geq (1 - \gamma_i)$$

$$\int_{\sum_{j=1}^{n} a_{ij} x_j}^{\infty} \frac{1}{\sigma_i} exp\left(\frac{-(b_i - \theta_i)}{\sigma_i}\right) db_i \geq (1 - \gamma_i) \tag{2.35}$$

After integrating the above equation, following result is obtained

$$\sum_{j=1}^{n} a_{ij} x_j \leq \theta_i - \sigma_i \ln(1 - \gamma_i) \tag{2.36}$$

Hence the quadratic fractional bi-level fuzzy probabilistic programming into its equivalent deterministic quadratic fractional bi-level fuzzy programming by using the derived methodology is given as follows

$$\underset{X_1}{\text{Max}}\, \tilde{f}_1(x_1, x_2) = \frac{\tilde{c}_{i1} x_j + \frac{1}{2} x^T \tilde{d}_{i1} x_j \ + \alpha_{i1}}{\tilde{c}_{i2} x_j + \frac{1}{2} x^T \tilde{d}_{i2} x_j \ + \beta_{i2}}$$

where for given X_1, X_2

$$\underset{X_2}{\text{Max}}\, \tilde{f}_2(x_1, x_2) = \frac{\tilde{c}_{i1} x_j + \frac{1}{2} x^T \tilde{d}_{i1} x_j \ + \alpha_{i1}}{\tilde{c}_{i2} x_j + \frac{1}{2} x^T \tilde{d}_{i2} x_j \ + \beta_{i2}}$$

Subject to

$$\sum_{j=1}^{n} \tilde{a}_{ij} x_j \leq \theta_i - \sigma_i \ln\,(1 - \gamma_i) \tag{2.37}$$

$$0 < \gamma_i < 1, \quad i = 1,2,\dots.,m$$
$$x_j > 0, \qquad j = 1,2,\dots.,m$$

III. Transformation of bi-level quadratic fractional programming problem to nonlinear programming problem

The objective function of each decision maker is transformed from fractional form to nonlinear form with the procedure described below in order to find the best solution for each level without considering other levels. Consider the most basic model of the quadratic fractional programming problem, which is defined as follows.

$$\underset{X_i}{\text{Max}}\, \tilde{f}_i = \frac{f_{i1}}{f_{i2}} = \frac{\tilde{c}_{i1}x_j + \frac{1}{2}x^T\tilde{d}_{i1}x_j \ + \alpha_{i1}}{\tilde{c}_{i2}x_j + \frac{1}{2}x^T\tilde{d}_{i2}x_j \ + \beta_{i2}}$$

Subject to

$$\sum_{j=1}^{n} a_{ij}x_j \leq \theta_i - \sigma_i \ln(1-\gamma_i)$$
$$0 < \gamma_i < 1, \quad i = 1,2,\ldots.,m$$
$$x_j > 0, \quad j = 1,2,\ldots.,n$$

To convert the above type of quadratic fractional programming problem into the nonlinear programming problem, take $\frac{1}{f_{i2}} = \frac{\grave{}1}{\tilde{c}_{i2}x_j + \frac{1}{2}x^T\tilde{d}_{i2}x_j + \beta_{i2}} = y_i \Rightarrow f_{i2}y_i = 1$

Thus, problem becomes as the following:

$$\underset{X_i}{\text{Max}}\, \tilde{f}_i = f_{i1}y_i = \left(\tilde{c}_{i1}x_j + \frac{1}{2}x^T\tilde{d}_{i1}x_j \ + \alpha_{i1}\right)y_i$$

Subject to

$$\left(\tilde{c}_{i2}x_j + \frac{1}{2}x^T\tilde{d}_{i2}x_j \ + \beta_{i2}\right)y_i = 1$$
$$\sum_{j=1}^{n} a_{ij}x_j \leq \theta_i - \sigma_i \ln(1-\gamma_i)$$
$$0 < \gamma_i < 1, \quad i = 1,2,\ldots.,m$$
$$x_j > 0, \quad j = 1,2,\ldots.,n$$

IV. Formulation of fuzzy goal programming approach for bi-level quadratic fractional programming problem

The decision makers of a bi-level quadratic fractional programming problem (BLQFPP) are fundamentally cooperative and make sequential decisions. By optimizing each decision maker individually for the given set of constraints for each objective functions f_i, $i = 1,2,\ldots m$ of the problem the fuzzy goal are taken as the maximum and minimum value of each decision maker. . Let $f_i^{U_i}, f_i^{L_i}$ is the maximum and minimum value of each objective function which are obtained by optimizing them as individually i.e. $f_i^{U_i} = \max_{x\in X} f_i$ with the solution set of decision variables as $(x_1^{U_i}, x_2^{U_i}, \ldots, x_n^{U_i})$
$f_i^{L_i} = \min_{x\in X} f_i$ with the solution set of decision variables as $(x_1^{L_i}, x_2^{L_i}, \ldots, x_n^{L_i})$
For $i = 1,2,\ldots,m$ now, for the given model, our objective was to maximize the objective function of each decision maker such that maximum value of objective function for first decision maker is $f_1^{U_1}$ at the point $(x_1^{U_1}, x_2^{U_1}, \ldots, x_n^{U_1})$, similarly maximum value of objective function for second decision maker is $f_2^{U_2}$ at the point $(x_1^{U_2}, x_2^{U_2}, \ldots, x_n^{U_2})$. It can be assumed reasonably that the value of $f_i \geq f_i^{U_i}$ are acceptable and all values less than $f_i^{L_i} = \min_{x\in X} f_i$ are absolutely unacceptable. Then the membership function $\mu_i f_i$ for the i^{th} fuzzy goal can be formulated as

$$\mu_i f_i = \begin{cases} 1, & if\ f_i \geq f_i^{U_i} \\ \dfrac{f_i - f_i^{L_i}}{f_i^{U_i} - f_i^{L_i}} & if\ f_i^{L_i} \leq f_i \leq f_i^{U_i}\ \ i = 1,2, \ldots m \\ 0 & if\ \ f_i \leq f_i^{L_i} \end{cases}$$

Each decision maker seeks to maximize his or her own objective function when making a decision. When each decision maker's optimum solution is computed separately, it is considered as the best solution, and the associated value of the objective function is regarded as the aspiration level of the corresponding fuzzy goal. In fuzzy programming approach, the highest degree of membership is one Mohamed [5]. The flexible membership goal having the aspired level unity can be represented as follows:

$$\mu_{f_i}(f_i) + d_i^- - d_i^+ = 1, \quad i = 1,2, \ldots., m$$

Or equivalently as

$$\frac{f_i - f_i^{L_i}}{f_i^{U_i} - f_i^{L_i}} + d_i^- - d_i^+ = 1, \quad i = 1,2, \ldots., m,$$

where $d_i^-, d_i^+ \geq 0$, with $d_i^- \times d_i^+ = 0$, represent the under and over deviations, respectively, from the aspired levels Pramanik and Roy [8]

In the formulation of fuzzy goal programming, the under and over deviational variables in the achievement function for minimizing them depends up on the type of objective functions to be optimized. To reach the aspiration level in the proposed fuzzy goal programming approach, the sum of under deviational variables must be minimized. The proposed fuzzy goal programming model for BLQFP problem follows as:

$$MinZ = \sum_{i=1}^{m} w_i d_i^-$$

$$\frac{f_i - f_i^{L_i}}{f_i^{U_i} - f_i^{L_i}} + d_i^- - d_i^+ = 1$$

$$\left(\tilde{c}_{i2} x_j + \frac{1}{2} x^T \tilde{d}_{i2} x_j \ + \beta_{i2}\right) y_i = 1$$

$$\sum_{j=1}^{n} a_{ij} x_j \leq \theta_i - \sigma_i \ln{(1 - \gamma_i)}$$

$$0 < \gamma_i < 1, \quad i = 1,2, \ldots., m$$

$$x_j = x_j^* \quad\quad j = 1,2, \ldots, n$$

and $d_i^-, d_i^+ = 0, d_i^-, d_i^+ \geq 0, \forall i = 1,2, \ldots, k$

where Z is the fuzzy goal's achievement function which is made up of the weighted under-deviational variables. The numerical weights w_i represent the relative importance of achieving the aspired levels of the respective fuzzy goals. The values of w_i are determined as Mohamed [5]:

$$w_i = \frac{1}{f_i^{U_i} - f_i^{L_i}} \quad\quad i = 1,2, \ldots, m$$

V. Numerical Examples

To demonstrate the proposed FGP approach, consider the following BLQFP problem with probabilistic nature in the constraints.

[FLDM]

$$MaxF_1 = \frac{\tilde{6}x_1 + \tilde{3}x_2 + \widetilde{-1}x_1^2 + \widetilde{-1}x_2^2 + 6}{\tilde{1}x_1^2 + \tilde{1}x_2^2 + 4}$$

where x_2 solves

[SLDM]

$$MaxF_2 = \frac{\tilde{1}x_1 + \tilde{5}x_2 + \widetilde{-1}x_2^2 + 8}{\tilde{1}x_1^2 + \tilde{1}x_2 + 6}$$

Subject to

$$P(\tilde{1}x_1 + \tilde{1}x_2 \le b_1) \ge 0.99$$
$$P(\tilde{3}x_1 + \tilde{2}x_2 \le b_2) \ge 0.95$$
$$P(\tilde{2}x_1 + \tilde{1}x_2 \ge b_3) \ge 0.90$$
$$x_1, x_2 \ge 0 \tag{5.1}$$

Here, we assume that $b_i\ (i = 1,2,3)$ are random variables following two parameter exponential distributions with following parameters:

$E(b_1) = 161,\quad E(b_2) = 144,\quad E(b_3) = 106$
$Var(b_1) = 25,\quad Var(b_2) = 36,\quad Var(b_3) = 64$

Using (2.32) and (2.33), the parameters are calculated as follows:

$\theta_1 = 156,\quad \sigma_1 = 5,\quad \theta_2 = 138,\quad \sigma_2 = 6,\quad \theta_3 = 98\ , \sigma_3 = 8$

The coefficients of the objectives are taken as triangular fuzzy number with the values
$-\tilde{1} = (-1.5, -1, -0.5), \tilde{6} = (4,6,9), -\tilde{1} = (-1.1, -1, -0.15), \tilde{5} = (4.5,5,5.8),\ \tilde{1} = (0.95,1,1.05), \tilde{3} = (2,3,5), -\tilde{1}\ = (-1.04, -1,0.03), -\tilde{2}(-1.5, -2, -2.5), -\tilde{2} = (-1.8, -2, -2.2), \tilde{5} = (4.2,5,5.8), \tilde{1} = (0.8,1,1.2), \tilde{3} = (2,3,4)$

The coefficients of the probabilistic constraints are taken as trapezoidal fuzzy number with the values

$$\tilde{1} = (0.2,0.8,1,1.2), \tilde{1} = (0.4,1,1.6,2), \tilde{3} = (1,2,3,4), \tilde{2} = (0.5,1.5,2.5,3), \tilde{2} = (1,1.8,2.2,3), \tilde{1} = (0.5,1,1.5,2).$$

On the basis of the method of defuzzification with probability density function and CCP technique the above model (5.1) can be expressed as

[FLDM]

$$MaxF_1 = \frac{6.33x_1 + 3.33x_2 - x_1^2 - 0.75x_2^2 + 8}{0.8x_1^2 + 0.93x_2^2 + 4}$$

where x_2 solves

[SLDM]

$$MaxF_2 = \frac{x_1^2 + 5.1x_2 - 0.61x_1 + 10}{1.06x_1^2 + 1.2x_2 + 6}$$

Subject to

$$0.78x_1 + 1.24x_2 \le 156.05$$
$$2.67x_1 + 1.86x_2 \le 138.308$$
$$2x_1 + 1.25x_2 \le 98.41$$
$$x_1, x_2 \ge 0$$

The bi-level multi-objective quadratic fractional programming problem is transformed into the bi-level quadratic programming model as follows.

[FLDM]

$$MaxF_1 = (6.33x_1 + 3.33x_2 - x_1^2 - 0.75x_2^2 + 8)y_1$$

where x_2 solves

[SLDM]

$$MaxF_2 = (x_1^2 + 5.1x_2 - 0.61x_1 + 10)y_2$$

Subject to

$$(0.8x_1^2 + 0.93x_2^2 + 4)y_1 = 1$$
$$(1.06x_1^2 + 1.2x_2 + 6)y_2 = 1$$
$$0.78x_1 + 1.24x_2 \le 156.05$$

$2.67x_1 + 1.86x_2 \leq 138.3082x_1 + 1.25x_2 \leq 98.41$
$x_1, x_2 \geq 0$

We first obtain the value of $F_1^{max} = 2.945$, $F_2^{max} = 4.088, F_1^{min} = 2, F_2^{min} = 0.924$
and $f_1^U = 2.945$, $f_2^U = 4.088$, $f_1^L = 2, f_2^L = 0.924$
To solve FGP models to get $x_1 = x_1^*$. Thus the first level FGP model follows as:

$MinZ = 1.058d_1^-$
Subject to
$(6.33x_1 + 3.33x_2 - x_1^2 - 0.75x_2^2 + 8)y_1 + 0.9448d_1^- - 0.9448d_1^+ = 2.945$
$(0.8x_1^2 + 0.93x_2^2 + 4)y_1 = 1$
$(1.06x_1^2 + 1.2x_2 + 6)y_2 = 1$
$0.78x_1 + 1.24x_2 \leq 156.05$
$2.67x_1 + 1.86x_2 \leq 138.3082x_1 + 1.25x_2 \leq 98.41$
$x_1, x_2 \geq 0$
$d_1^+, d_1^- \geq 0$

Using Lingo software, the compromise solution of first level decision maker problem is obtained as; (x_1, x_2) = (0.954, 0.478). Then assuming that the FLDM set $x_1^* = 0.954$
$MinZ = 1.058d_1^- + 0.316d_2^-$
$(6.33x_1 + 3.33x_2 - x_1^2 - 0.75x_2^2 + 8)y_1 + 0.945d_1^- - 0.945d_1^+ = 2.945$
$(x_1^2 + 5.1x_2 - 0.61x_1 + 10)y_2 + 3.164d_2^- - 3.164d_2^+ = 4.0877$
$(0.8x_1^2 + 0.93x_2^2 + 4)y_1 = 1$
$(1.06x_1^2 + 1.2x_2 + 6)y_2 = 1$
$0.78x_1 + 1.24x_2 \leq 156.05$
$2.67x_1 + 1.86x_2 \leq 138.3082x_1 + 1.25x_2 \leq 98.41$
$x_1^* = 0.943$
$d_1^+, d_1^-, d_2^+, d_2^- \geq 0$

Using Lingo software, the compromise solution of the BLQFP problem is obtained as (x_1, x_2) = (0.954, 0.798) with the corresponding objective function.

VI. Discussion

In this paper, the technique for solving a Quadratic Fractional Bi-level Fuzzy Probabilistic Programming (QFBLFP) programming is outlined by considering the fact that the random variable follows two parameter exponential distributions and using a combination of probabilistic and fuzzy concepts. Firstly the probabilistic nature of the problem is transformed into an equivalent deterministic problem and then a fuzzy goal programming technique is used to solve the bi-level deterministic model.

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