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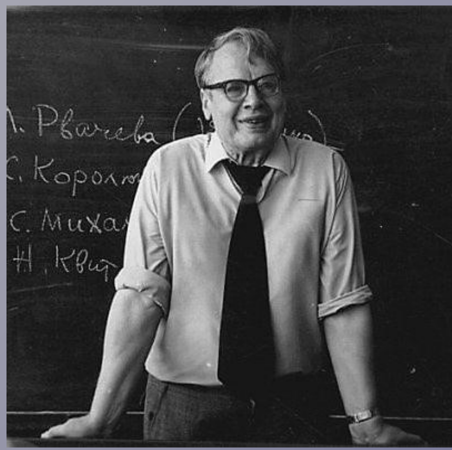

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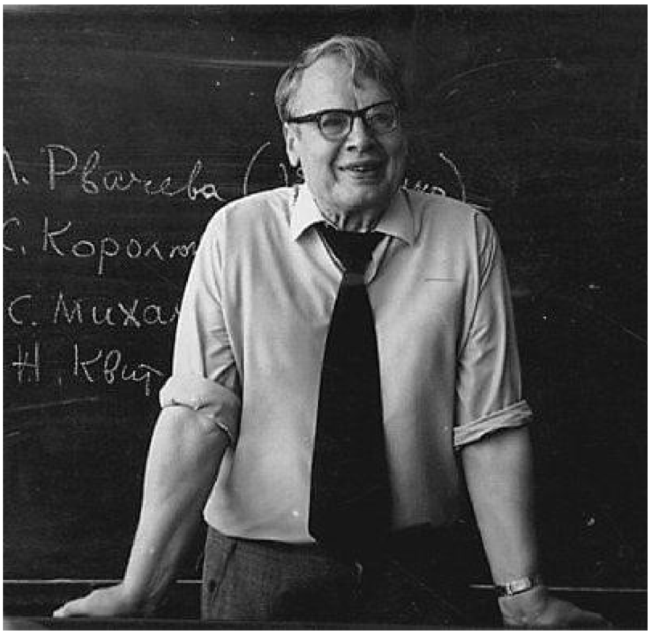

RELIABILITY:

THEORY & APPLICATIONS

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Priority is given to descriptions of case studies.
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An Innovative Methodology for Evaluation of Reliability Indices of Electric Traction System

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•
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Abstract

Evaluation of reliability is most important when we have to check the availability of supply in any electric power system. The basic reliability index which is of importance is failure rate, repair time and unavailability of the supply in any electric power system. In this paper evaluation of various basic reliability indices for the electric traction system is done. Electric traction system is very important as it is used for operation of passenger trains and freight trains across a large rail network throughout the world. As the traction system is very important therefore reliability evaluation of its various parameters are essential for proper and uninterrupted working of the whole electric traction system.

Keywords: Electric Traction System, Reliability, Failure Rate, Repair Rate.

I. Introduction

Reliability evaluation of a system or component or element is very important in order to predict its availability and other relevant indices. Reliability is the parameter which tells about the availability of the system under proper working conditions for a given period of time. A Markov cut-set composite approach to the reliability evaluation of transmission and distribution systems involving dependent failures was proposed by Singh et al. [1]. The reliability indices have been determined at any point of composite system by conditional probability approach by Billinton et al. [2]. Wojczynski et al. [3] discussed distribution system simulation studies which investigate the effect of interruption duration distributions and cost curve shapes on interruption cost estimates. New indices to reflect the integration of probabilistic models and fuzzy concepts was proposed by Verma et al. [4]. Zheng et al. [5] developed a model for a single unit and derived expression for availability of a component accounting tolerable repair time. Distributions of reliability indices resulting from two sampling techniques are presented and analyzed along with those from MCS by Jirutitijaroen and Singh [6]. Dzobe et al. [7] investigated the use of probability distribution function in reliability worth analysis of electric power system. Bae and Kim [8] presented an analytical technique to evaluate the reliability of customers in a micro grid including distribution generations. Reliability network equivalent approach to distribution system reliability assessment is proposed by Billinton and Wang [9].

Customer and energy based indices consideration for reliability enhancement of distribution system using Improved Teaching Learning based optimization is discussed [10]. An Innovative Self-Adaptive Multi-Population Jaya Algorithm based Technique for Evaluation and Improvement of Reliability Indices of Electrical Power Distribution System, Tiwary et al. [11].

Jirutitijaroen et al. [12] developed a comparison of simulation methods for power system reliability indexes and their distribution. Determination of reliability indices for distribution system using a state transition sampling technique accounting random down time omission, Tiwary et al. [13]. Tiwary et al. [14] proposed a methodology based on Inspection-Repair-Based Availability Optimization of Distribution System Using Bare Bones Particle Swarm Optimization. Bootstrapping based technique for evaluating reliability indices of RBTS distribution system neglecting random down time was evaluated [15].

Volkanavski et al. [16] proposed application of fault tree analysis for assessment of the power system reliability. Li et al. [17] studies the impact of covered overhead conductors on distribution reliability and safety. Reliability enhancement of distribution system using Teaching Learning based optimization considering customer and energy based indices was obtained in Tiwary et al. [18]. Self-Adaptive Multi-Population Jaya Algorithm based Reactive Power Reserve Optimization Considering Voltage Stability Margin Constraints was obtained in Tiwary et al. [19]. A smooth bootstrapping based technique for evaluating distribution system reliability indices neglecting random interruption duration is developed [20]. Tiwary et al. [21] have developed an inspection maintenance based availability optimization methodology for feeder section using particle swarm optimization. The impact of covered overhead conductors on distribution reliability and safety is discussed [22]. Tiwary et al. [23] has discussed a methodology for reliability evaluation of an electrical power distribution system, which is radial in nature. Sarantakos et al. [24] introduced a method to include component condition and substation reliability into distribution system reconfiguration. Tiwary et al. [25] has discussed a methodology for evaluation of customer orientated indices and reliability of a meshed power distribution system. Reliability evaluation of engineering system is discussed [26]. Battu et al. [27] discussed a method for reliability compliant distribution system planning using Monte Carlo simulation. Application of non-parametric bootstrap technique for evaluating MTTF and reliability of a complex network with non-identical component failure laws is discussed [28]. Tiwary and Tiwary [29] have developed an innovative methodology for evaluation of customer orientated indices and reliability study of electrical feeder system.

In this paper basic reliability indices, failure rate, repair rate and unavailability of the electric traction system is evaluated. Electric traction system is very important as it is used for operation of passenger trains and freight trains across a large rail network throughout the world. As the traction system is very important therefore reliability evaluation of its various parameters are essential for proper and uninterrupted working of the whole electric traction system. Reliability block diagram which is a diagrammatic method for showing how different components are connected in a system is designed for the traction system considered and various indices related to reliability are obtained.

II. Reliability block diagram representation of electric traction system

Reliability block diagram which is a diagrammatic method for showing how different components are connected in a system is obtained for the electric traction system. Electrical traction system is that system that uses electrical power for traction system i.e. for railways, trams, trolleys, etc. The track electrification means to the type of source which is used while powering the electric locomotive systems. The two main types of electric traction systems that exist are as Direct Current (DC) electrification system and Alternating Current (AC) electrification system. The reliability block diagram of Direct Current (DC) electrification system is shown in Fig. 1. It consist of source, overhead wire, pantograph, motor control and motor as its important parts. Each and every component of the system is connected in series manner.

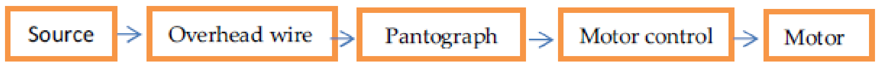

Fig. 1. Reliability block diagram of Direct Current (DC) electrification system

The reliability block diagram of Alternating Current (AC) electrification system is shown in Fig. 2. It consist of source, overhead wire, pantograph, transformer, rectifier, motor control and motor as its important parts. It can be seen from Fig. 2 that each and every component is connected in series.

Fig. 2. Reliability block diagram of Alternating Current (AC) electrification system

III. Evaluation of reliability and its various indices of electric traction system

The system is having a constant failure rate and therefore the reliability of the system having constant failure rate is evaluated by using the following relation.

$$\mathrm{R}(t) = e^{-\lambda t} \tag{1}$$

Where R(t) represents the reliability of each and every component. λ represents the failure rate per year and t represents time period which is taken as one year.

The mean time to failure (MTTF) can be obtained as follows:

$$MTTF = \frac{1}{\lambda} \tag{2}$$

A series system is that system in which one component fails, the complete system will fail and for working of the whole system it is mandatory that all the component are in working condition. If one assumes time independent reliability $r_1, r_2 \ldots r_n$, then reliability of series system is given as:

$$R_s = \prod_{i=1}^{n} r_i \tag{3}$$

In series configuration combined failure rate is calculated as follows.

$$\lambda_{Total} = \sum \lambda \tag{4}$$

Unavailability of series configuration is calculated by using following relation.

$$U_{Total} = \sum \lambda r \tag{5}$$

Total repair rate of the components connected in series manner is obtained as follows.

$$r_{Total} = \frac{U}{\lambda} \tag{6}$$

IV. Result and Discussion

Table 1 shows the initial data such as failure rate per year and repair time in hours for the components of direct current (DC) electrification traction system. There are four components in the DC electrification traction system which are overhead wire, pantograph, motor control and motor and shown as components c1, c2, c3 and c4 respectively.

Table 1: Initial data for different components of the direct current electrification traction system.

component	c1	c2	c3	c4
Failure rate/year	0.04	0.03	0.005	0.004
Repair time (hrs.)	3	4	5	6

Table 2 shows the initial data such as failure rate per year and repair time in hours for the components of alternating current (AC) electrification traction system. There are six components in the AC electrification traction system which are overhead wire, pantograph, transformer, rectifier, motor control and motor and shown as components c1, c2, c3, c4, c5 and c6 respectively.

Table 2: Initial data for different components of the alternating current electrification traction system.

component	c1	c2	c3	c4	c5	c6
Failure rate/year	0.04	0.03	0.002	0.003	0.005	0.004
Repair time (hrs.)	3	4	6	4	5	6

Table 3 provides the evaluated reliability of each component of DC electrification traction system as 0.9608, 0.9704, 0.9950 and 0.9960 respectively. The overall reliability of the DC electrification traction system obtained is as 0.9240.

Table 4 provides the evaluated reliability of each component c1, c2, c3, c4, c5 and c6 of AC electrification traction system as 0.9608, 0.9704, 0.9980, 0.9950, 0.9950 and 0.9960 respectively. The overall reliability of the AC electrification traction system obtained is as 0.9175.

Table 3 Evaluated Reliability of each component of DC electrification traction system

Component	Reliability
c1	0.9608
c2	0.9704
c3	0.9950
c4	0.9960

Table 4 Evaluated Reliability of each component of AC electrification traction system

Component	Reliability
c1	0.9608
c2	0.9704
c3	0.9980
c4	0.9950
c5	0.9950
c6	0.9960

Table 5 and Table 6 provide the evaluated mean time to failure of DC electrification traction system and evaluated mean time to failure of AC electrification traction system respectively.

Table 5 Evaluated mean time to failure of DC electrification traction system

Component	Evaluated mean time to failure
c1	25
c2	33.33
c3	200
c4	250

Table 6 Evaluated mean time to failure of AC electrification traction system

Component	Evaluated mean time to failure
c1	25
c2	33.33
c3	500
c4	333.33
c5	200
c6	250

Evaluated unavailability for each and every component of the DC electrification traction system is shown in Table 7. The evaluated unavailability obtained are 0.12, 0.12, 0.025 and 0.024 respectively. The evaluated unavailability of each component of the AC electrification traction system are 0.12, 0.12, 0.012, 0.012, 0.025 and 0.024 respectively as shown in Table 8.

Table 7 Evaluated unavailability for each and every component of the DC electrification traction system

component	c1	c2	c3	c4
Unavailability	0.12	0.12	0.025	0.024

Table 8 Evaluated unavailability for each and every component of the AC electrification traction system

component	c1	c2	c3	c4	c5	c6
Unavailability	0.12	0.12	0.012	0.012	0.025	0.024

Table 9 and Table 10 provide the component level evaluated failure rate, repair rate and unavailability for each and every component of the DC electrification traction system and AC electrification traction system respectively.

Table 9 Component level evaluated failure rate, repair rate and unavailability for each and every component of the DC electrification traction system.

Component Level	C1	C2	C3	C4
Failure rate	0.04	0.07	0.075	0.079
Repair rate	3	3.4286	3.5333	3.6582
Unavailability	0.12	0.24	0.265	0.289

Table 10 Component level evaluated failure rate, repair rate and unavailability for each and every component of the AC electrification traction system.

Component Level	C1	C2	C3	C4	C5	C6
Failure rate	0.04	0.07	0.072	0.075	0.08	0.084
Repair rate	3	3.4286	3.5	3.52	3.6125	3.7262
Unavailability	0.12	0.24	0.252	0.264	0.289	0.313

Fig. 3 and Fig. 4 provide the magnitude of evaluated reliability of each component of DC electrification traction system and AC electrification traction system respectively. Magnitude of evaluated mean time to failure of DC electrification traction system and that of AC electrification traction system is shown in Fig. 5 and Fig. 6 respectively.

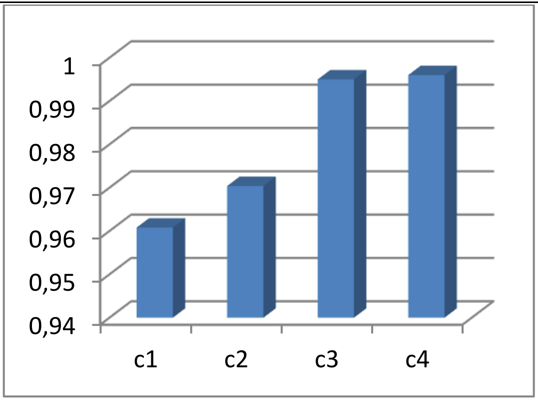

Fig. 3 Magnitude of evaluated reliability of each component of DC electrification traction system

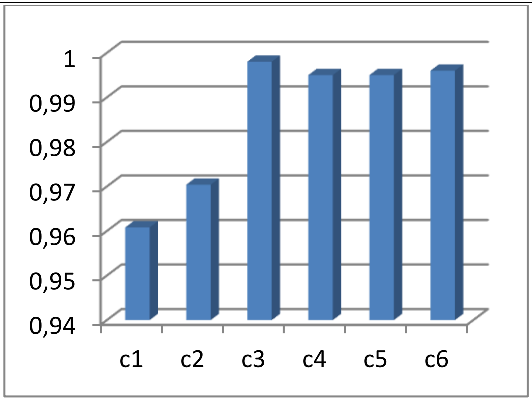

Fig. 4 Magnitude of evaluated reliability of each component of AC electrification traction system

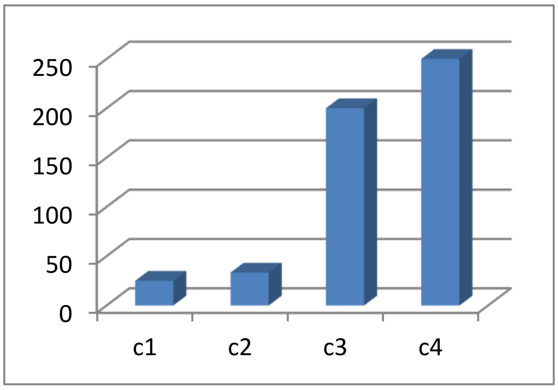

Fig. 5 Magnitude of evaluated mean time to failure of DC electrification traction system

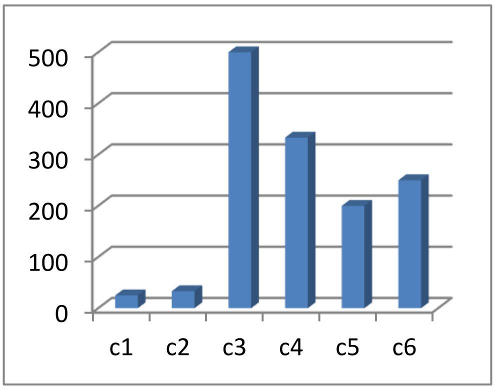

Fig. 6 Magnitude of evaluated mean time to failure of AC electrification traction system

Fig. 7 and Fig. 8 provide the magnitude of evaluated unavailability for each and every component of the DC electrification traction system and AC electrification traction system respectively.

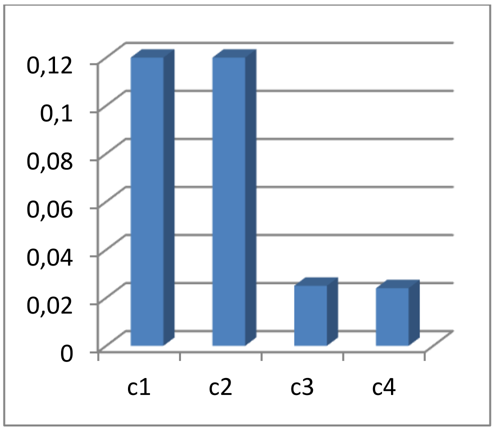

Fig. 7 Magnitude of evaluated unavailability for each and every component of the DC electrification traction system

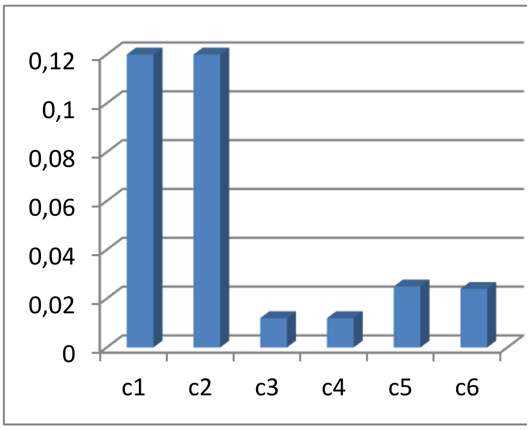

Fig. 8 Magnitude of evaluated unavailability for each and every component of the AC electrification traction system

Magnitude of component level evaluated failure rate for each and every component of the DC electrification traction system, magnitude of component level evaluated failure rate for each and every component of the AC electrification traction system, magnitude of component level evaluated repair rate for each and every component of the DC electrification traction system, magnitude of component level evaluated repair rate for each and every component of the AC electrification traction system are provided in Fig. 9, Fig. 10, Fig. 11 and Fig. 12 respectively.

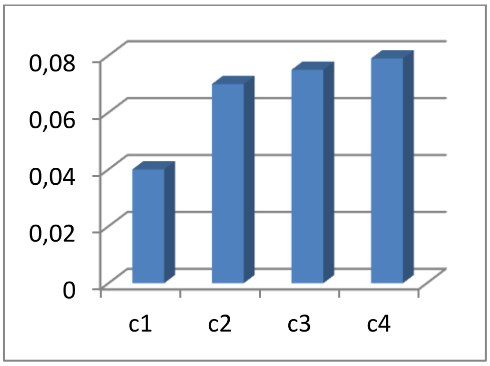

Fig. 9 Magnitude of component level evaluated failure rate for each and every component of the DC electrification traction system.

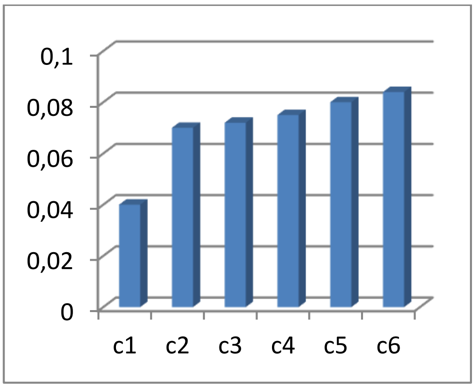

Fig. 10 Magnitude of component level evaluated failure rate for each and every component of the AC electrification traction system.

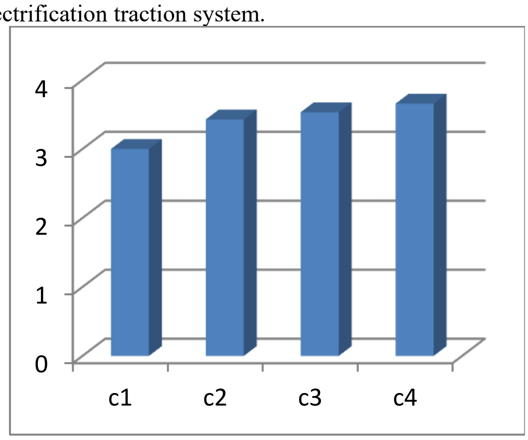

Fig. 11 Magnitude of component level evaluated repair rate for each and every component of the DC electrification traction system.

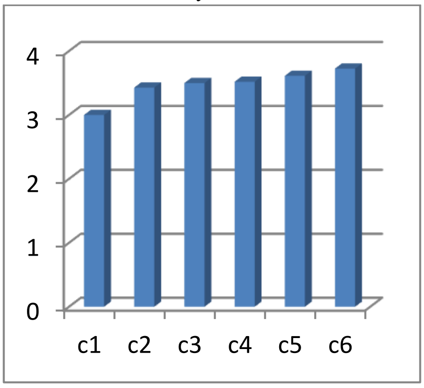

Fig. 12 Magnitude of component level evaluated repair rate for each and every component of the AC electrification traction system.

Fig. 13 and Fig. 14 provides magnitude of component level evaluated unavailability for each and every component of the DC electrification traction system and magnitude of component level evaluated unavailability for each and every component of the AC electrification traction system respectively.

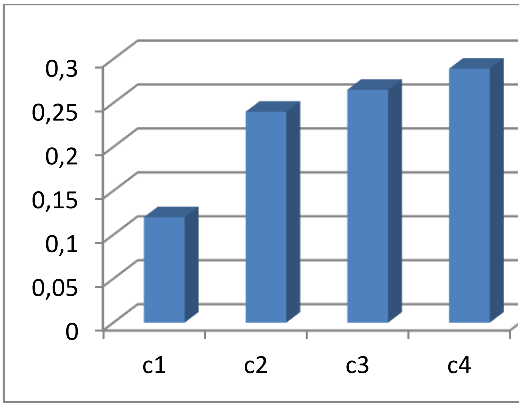

Fig. 13 Magnitude of component level evaluated unavailability for each and every component of the DC electrification traction system.

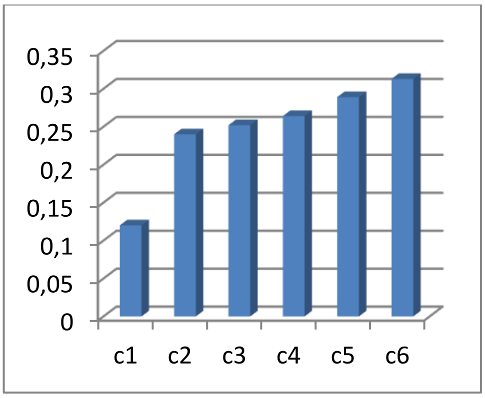

Fig. 14 Magnitude of component level evaluated unavailability for each and every component of the AC electrification traction system.

V. Conclusion

Identifying various values of reliability is most important when we have to check the availability of supply in any system. Electric traction system is very important as it is used for operation of passenger trains and freight trains across a large rail network throughout the world. Reliability of each component of DC electrification traction system and AC electrification traction system is obtained. Mean time to failure and unavailability for each and every component of the DC electrification traction system and AC electrification traction system is also calculated. This paper has also evaluated the basic reliability indices such as failure rate, repair rate and unavailability at the component level for the DC electrification traction system and AC electrification traction system respectively.

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Probabilistic analysis of a multi-state warm standby k-out-of-n: G system in a series configuration using copula linguists

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Abstract

This paper discusses the reliability analysis of repairable complex system comprising of two subsystems in series configuration together with the controllers. The two subsystems, consisting of three undistinguishable units in a parallel arrangement and functioning under 1-out-of-3: G operational policy. Controllers control both the subsystems and can be unstable, and the malfunction result in the controller prevents system operation. The system may have an unforeseeable catastrophic failure due to which the system may not perform its function once the situation arises. The failure rate of the units is constant, and the exponential distribution is assumed to obey. The two forms of repair namely general repair and Goumbel-Hougard copula repair are used to restore the existing failed units of the system. The supplementary variable technique with Laplace transformation is used to evaluate the output of the system. Using Stochastic theory, differential equations are derived to obtain essential features of reliability such as availability of the system, reliability of the system, MTTF, and profit analysis. Graphs were drawn to highlight the behavior of the results. Tables and figures display the findings and suggest that copula repair is a more efficient repair policy for the improved performance of repairable systems. It brings a different aspect to the research world to adopt multi-dimensional repair in the form of the copula. Besides, the findings of the model are useful for system engineers and maintenance managers.

Keywords: k-out-of-n: G/F system configuration; availability; reliability; MTTF; Controller; catastrophic failure; Gumbel-Hougaard family copula distribution.

I. Introduction

In the design of complex engineering systems, specifically in the manufacturing sector, the research community is lacking in the prospect of developing new frameworks. The architecture of the model must be such that it can execute the task effectively and meet high levels of availability and reliability. Every enhancement in system reliability is always associated with the cost of the system; the improvement in reliability is defensible to the degree that the cost of the system unapproachability exceeds the cost of the basic service offered. Reliability may be increased by the procurement and installation of new paraphernalia or the repair of existing facilities. In addition to the financial component of retaining the status of every industry, customer loyalty is often a crucial prerequisite.

The system reliability and its measures has a crucial role to play in preserving status and customer satisfaction. There are many mechanisms to increase system efficiency and redundancy to boost system efficiency and benefit gained. Any equivalent or non-identical components that are retained in standby mode assist system operations as required after the main unit/component has failed. As per the available reliability theory literature, in particular, three types of standby units viz. cold standby, mild standby, and hot standby have been tested by many scholars in the past. Moreover, redundancy is very cost-effective in ensuring a certain degree of efficiency of the system. Therefore, to increase the stability and efficiency of the k-out-of-n system configuration in which at least k components out of n have to run for the system to be operational play a critical role. To explore some examples of such type of configured structure, a telecommunications system with four transmitters can be modeled as a 2-out-of-4: G system. An extensive bus with six tires four is enabled to perform tasks for time is a 4-out-of-6: G system. Overwhelmingly k-out-of-n system plays a crucial role in system reliability theory for the proper operation of the system. The k-out-of-n-type warm standby method has found various applications in the field of reliability, including redundant system inspection, network architecture, power generation, and transmission networks, etc.

Extensive attempts have been made over the last decades by many scholars, including Kullstam (1981), Zhao (1994), Coit (2001), Park and Pham (2012), Wu and Guan (2005), Xing et al. (2012), and Ram et al. (2013) to establish strategies for solving k-out-of-n types of systems and computing availability, MTBF, and MTTR and other probabilistic measures for repairable systems. They have researched the performance of complex repairable systems employing k-out-of-n: G/F, operational schemes. Following the performance assessment of complex repairable systems, Zuo and Tian (2006) measured the performance of a series-parallel system under varying operating policy conditions. Malinowski (2016) has established a network of inflow points, transit-only nodes, and outflow points. In their network, arcs were regulated, and components were repairable with constant failure and repair rates. The efficiency of this network's performance is determined by the ratio of the total demand met at all the outflow points to the total demand needed at these points. Levitin et al. (2013) looked at mixed-designed series-parallel systems through reliability measures assuming random failure propagation time. The exact reliability formula for consecutive repairable k-out-of-n-type operative systems was showed by Liang et al. (2010). Sharma and Kumar (2017) measured availability and other efficiency measurements of the successive k-out-of- n machining system using standby with multiple working vacations. Eryilmaz (2007, 2009, 2010) has developed formulas for consecutive k-out-of-n: F system using lifetime distribution, reliability, and properties of the k-out-of-n system with arbitrarily dependent components and mixture representations for the protection of successive- k systems. A system with (M+N) units under k-out-of-(M+N): G scheme in which the M units were inactive warm standby mode has been analyzed by Zhang (2006). Kumar and Gupta (2007) evaluated the reliability characteristics of a 1-out-of-2 warm standby system comprising of the main unit with a supporting unit, including a repair facility. Cha et al. (2014) suggested a competing risk model reliability analysis by considering two types of failures partial failure and complete failure as a catastrophic failure phenomenon. They compared deterioration with catastrophic failure and showed that a catastrophic failure is more troubling as the system may not accomplish its function after a catastrophic failure happens. Levitin and Dai (2012) analyzed a multi-state sliding window system with multiple failures mode. Every element can have separate states for minor failed, major failures, and complete failed states. The reliability of the whole system also relies on component reliability, which is assembled in the system.

The controller is a device that controls the output variables and operating conditions imposed with the given dynamical systems. Several researchers around the globe have published their findings on the reliability of complex repairable systems using controllers. It can be used in engineering systems particularly in electronics to control a circuit, in computers as a peripheral unit, in software design to create an interface between models and views, game controllers, etc.

Controllers can also be used in other systems such as linguistics (control the verb), aviation (control the air traffic), biomedical, economic, and socio-economic systems. Digital computers are an integral part of complex engineering systems to control the variables like as to control centrifugal force for controlling speed, to control the furnace temperature, thermostat controller to control room temperature, etc. Ogata (2009) introduced and explained the idea of controllers for modern engineering systems. Authors such as Singh et al. (2013) investigated a system consisting of two subsystems in a series configuration with controllers in which the first subsystem functions under k-out-of-n: G, policy and the second subsystem has three similar units in parallel arrangements. The study under different failure rates and two forms of repairs was carried out. Computation of availability projected that multi-repair would result in better execution of the system. In all the research papers listed above, all the authors discussed several failures and a single method of repair. They fail to note if we have more than one form of repair between two adjacent states that could be possible in a variety of complex systems. If this is feasible, we can test the reliability characteristics using Goumbel-Hougard Copula repair distribution for a completely failed condition. Copulas allow one to isolate the dependency structure in a distribution where two or more variable quantities are involved. Copulas have been introduced by Nelson (2006). To quote some similar work posed by some authors including Ibrahim et al. (2017), Jia et al. (2017), Kumar et al. (2017), and Singh et al. (2020b) examined the reliability measurements of systems comprising subsystems in series configurations and k-out-of-n: G/ F policy with implications of a joint probability distribution. Monika et al. (2019) tested a complex repairable system via a switch and human failure and copula approach. Singh et al. (2020a) investigated a repairable network system of three server-based computer labs under a 2-out-of-3: G scheme. Raghav et al. (2020) studied a dynamic system with two subsystems in a series configuration with imperfect switching devices with copula linguistic approach implications and concluded that copula repair predicts better performance over the general repair. Rawal et al. (2013) analyzed a model of the internet data center including a redundant server with the main mail server trickling different types of failure and two types of repair employing copula distribution. Confirming the various operating choices in the system, some critical analysis was carried out to determine the various reliability features of the system. A repairable warm standby k-out-of-n: G and 2-out-of-4: G systems in series under catastrophic failure and a switching device was recently studied by Poonia et al. (2020) using copula repair. This model was built by taking n-k+1 states into account in the first subsystem in such a way that it formed a finite series during solution unlike as done in the past. Via this article, the scientific community is advised by the authors to carry out multi-dimension repairs in the form of copulas, since they have excellent results over the general repair.

2. Model description and notations

2.1 System description

Refer to the literature discussed in the introduction, none of the authors studied any system consisting of the k-out-of-n: G form of operating strategy with controllers under catastrophic failure. In order to close the difference, we examined the reliability of a repairable warm standby system in series configurations with two subsystems (namely subsystem-1 & 2). Each subsystem is having three similar units in a parallel configuration and follow 1-out-of-3: good working strategy. Units in both subsystems are connected to the controller for the proper functionality of the system, which could be unstable at the time of need and the switching time is instantaneous. Also, the system could face unexpected catastrophic failures during service. There are four types of possible states for the system operation: perfect state, minor failed state, major failed state, and completely failed states. The failure rates of the functional and standby units of each subsystem are constant in nature, but they follow exponential distributions. The repair system is fitted with two distributions general and

Goumbel-Hougard copula distribution. The rate of repair of each device in subsystem-1 and subsystem-2 is regarded as the same but different from each subsystem.

The article is formulated in the following way: Through Section-1, we studied the related work that can be retrieved in various articles. Section-2 of the manuscript deliberates system description, assumptions, and state description. Section-3 consists of system configuration and transition diagram. Section-4 presents mathematical modeling using differential equations. The empirical results for the different output measures of the system are simulated by considering a few specific cases listed in section-5. With the help of graphs, the concluding remarks on our findings with interpretations are provided in Section 6. MAPLE (software) is used to obtain both explicit expressions and numerical evaluations for reliability physiognomies.

2.2. Assumptions

In this article, we consider the following assumptions:

1. Subsystem-1 / subsystem-2 operates effectively until one or more units, are in good working order i.e. "1-out-of-3: G " policy.
2. Both the subsystems have a control unit that is unstable in the system, and the controller's function is as long as the controller fails, the whole system fails immediately."
3. An unforeseeable catastrophic failure of the system could occur at any time (t).
4. The system has four states: Good, minor partially failed, major partially failed, and utterly failed.
5. If the unit has been restored, it is again operational in both the subsystems. No failure was reported due to machine repair.
6. The repairman is available full time and ready to restore minor and major faults.
7. A repair person is available to full time and may repair partially or fully failed units.
8. Partially failed states are restored by employing general repair, while the Gumbel-Hougaard copula can be activated to reinstate the system in case of a complete failure.

2.3. Notations

s , t Laplace transform / Time scale variable

λ_1 / μ_1 Failure rate of each unit in subsystem-1/subsystem-2.

$\lambda_{c_1} / \lambda_{c_2}$ The failure rate of controllers in subsystem-1/subsystem-2.

λ_{c_T} Failure rate related to the catastrophic failure mode.

$\phi_1(x)/\psi_1(y)$ Repair rate of one unit in subsystem-1/subsystem-2.

$\phi_2(x)/\psi_2(y)$ Repair rate of two units in subsystem-1/subsystem-2.

$P_0(t)$ The state transition probability that the system is in S_i state at an instant for $i=0$.

$\bar{P}(s)$ Laplace transformation of the state transition probability $P(t)$.

$P_i(x,t)$ The probability that the system is in state S_i $i=1$ to 8 and the system is under repair with elapsed repair time is x,t x repaired variable and t is time variable.

$E_p(t)$ Expected profit in the interval $\left[0,t\right)$.

K_1, K_2 Revenue generated and service cost per unit time respectively.

$\mu_0(x)$ An expression of the joint probability from failed state S_i to good state S_0 according to Gumbel-Hougaard family copula is given as;

$$\mu_0(x) = C_\theta\{u_1(x), u_2(x)\} = \exp\left[x^\theta + \{\log\phi(x)\}^\theta\right]^{1/\theta} \quad \text{where } u_1(x) = \phi(x), \quad u_2(x) = e^x$$

Here θ is the parameter $1 < \theta < \infty$.

3. System configuration, transition diagram, and state description

The system configuration is shown in Fig 1 (a) while the state transition diagram in Fig 1 (b). The state description of the model highlights that initially all the units in both the subsystems are functioning perfectly and it in a state of S_0. After one unit has failed in either subsystem, it switches to S_1 or S_4 which are regarded as minor partially failed states. If two units have failed in any subsystem, they will be passed to S_2 or S_5 that are the major partially failed states. In both cases, to restore the system we use general repair. States S_3, S_6, S_7, and S_8 are complete failed states due to failure of all the three units in subsystem-1 or2, or due to failure controllers or catastrophic failure. In these complete failed states, a multidimensional repair in the form of the copula is used to restore the system.

Table 1 State Description

State	Description
S_0	This is a perfect state and all units of subsystem-1 and subsystem-2 are in good working condition.
S_1, S_4	The indicated state is deteriorated and deemed to be a minor failed state but is in operational mode after the failure of anyone unit in subsystem-1/2. The remaining two units are well-functioning. The system is being restored through general repair.
S_2, S_5	The indicated state is deteriorated and deemed to be a major failed state but is in operational mode after the failure of any two units in subsystem-1/2. The remaining unit is well-functioning. The system is being restored through general repair.
S_3, S_6 S_7, S_8	The states suggest that the system is in complete failure mode and is being revived using the copula distribution of the Gumbel-Hougaard family.

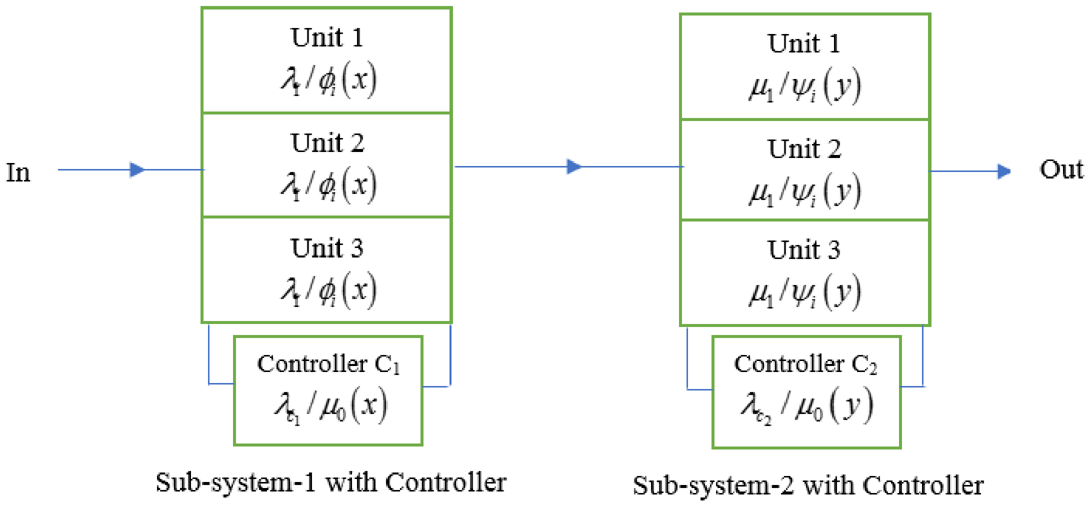

Figure 1 (a) System configuration

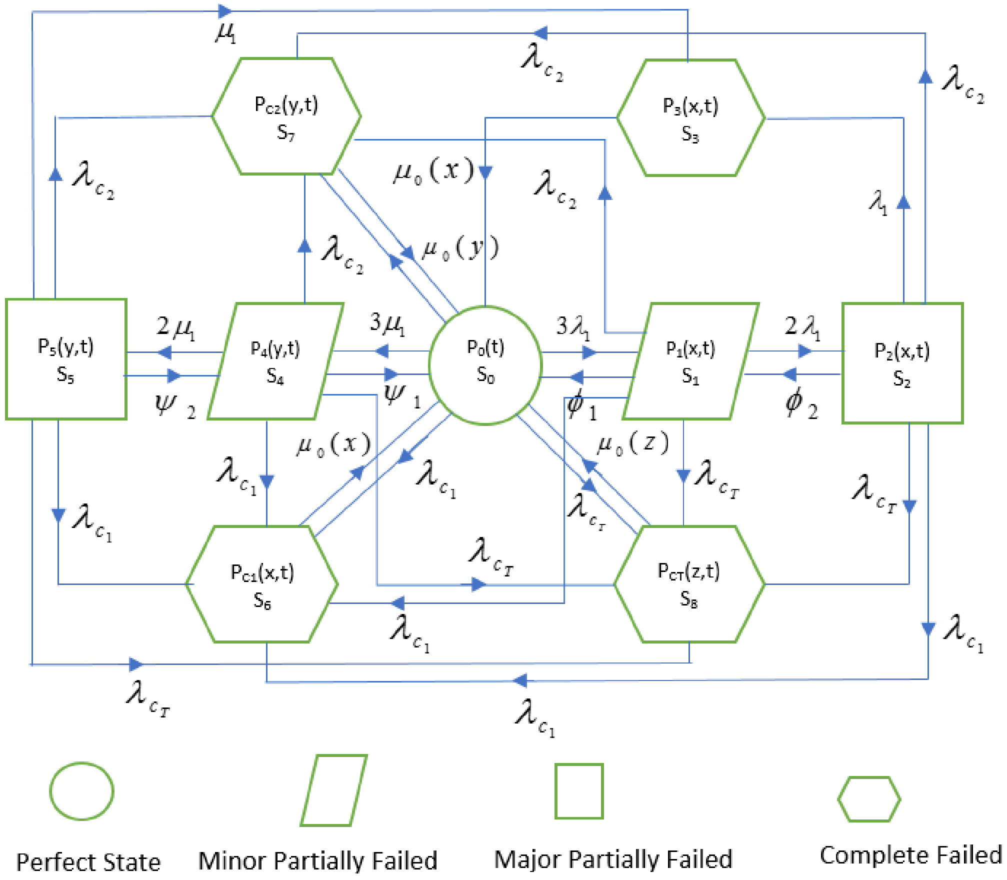

Figure 1 (b) State transition diagram of the model

4. Preparation of mathematical model

Based on stochastic theory arguments, one can easily develop the set of differentials equations associated with the existing mathematical model for the above-mentioned state transition diagram.

$$\left[\frac{\partial}{\partial t}+3\lambda_1+3\mu_1+\lambda_{c_1}+\lambda_{c_2}+\lambda_{c_T}\right]P_0(t)=\left[\int_0^\infty \phi_1(x)P_1(x,t)dx+\int_0^\infty \psi_1(y)P_4(y,t)dy\right.$$
$$+\int_0^\infty \mu_0(x)P_3(x,t)dx+\int_0^\infty \mu_0(x)P_{c_1}(x,t)dx$$
$$\left.+\int_0^\infty \mu_0(y)P_{c_2}(y,t)dy+\int_0^\infty \mu_0(z)P_{c_T}(z,t)dz\right] \tag{1}$$

$$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+2\lambda_1+\lambda_{c_1}+\lambda_{c_2}+\lambda_{c_T}+\phi_1(x)\right]P_1(x,t)=0 \tag{2}$$

$$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+\lambda_1+\lambda_{c_1}+\lambda_{c_2}+\lambda_{c_T}+\phi_2(x)\right]P_2(x,t)=0 \tag{3}$$

$$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+\mu_0(x)\right]P_3(x,t)=0 \tag{4}$$

$$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial y}+2\mu_1+\lambda_{c_1}+\lambda_{c_2}+\lambda_{c_T}+\psi_1(y)\right]P_4(y,t)=0 \tag{5}$$

$$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial y}+\mu_1+\lambda_{c_1}+\lambda_{c_2}+\lambda_{c_T}+\psi_2(y)\right]P_5(y,t)=0 \tag{6}$$

$$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+\mu_0(x)\right]P_{c_1}(x,t)=0 \tag{7}$$

$$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial y}+\mu_0(y)\right]P_{c_2}(y,t)=0 \tag{8}$$

$$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial z}+\mu_0(z)\right]P_{c_T}(z,t)=0 \tag{9}$$

Boundary conditions

$$P_1(0,t)=3\lambda_1 P_0(t) \tag{10}$$

$$P_2(0,t)=2\lambda_1 P_1(0,t)=6\lambda_1^2 P_0(t) \tag{11}$$

$$P_4(0,t)=3\mu_1 P_0(t) \tag{12}$$

$$P_5(0,t)=2\mu_1 P_4(0,t)=6\mu_1^2 P_0(t) \tag{13}$$

$$P_3(0,t)=\lambda_1 P_2(0,t)+\mu_1 P_5(0,t)=6\left(\lambda_1^3+\mu_1^3\right)P_0(t) \tag{14}$$

$$P_{c_1}(0,t)=\lambda_{c_1}\left[P_0(t)+P_1(0,t)+P_2(0,t)+P_4(0,t)+P_5(0,t)\right] \tag{15}$$

$$P_{c_2}(0,t)=\lambda_{c_2}\left[P_0(t)+P_1(0,t)+P_2(0,t)+P_4(0,t)+P_5(0,t)\right] \tag{16}$$

$$P_{c_T}(0,t)=\lambda_{c_T}\left[P_0(t)+P_1(0,t)+P_2(0,t)+P_4(0,t)+P_5(0,t)\right] \tag{17}$$

Initial conditions

$P_0(0)=1$, and other state probabilities are zero at $t=0$ (18)

Solution of the model

Taking Laplace transformation of equations (1) to (17) and using equation (18), we obtain

$$\left[s+3\lambda_1+3\mu_1+\lambda_{c_1}+\lambda_{c_2}+\lambda_{c_T}\right]\bar{P}_0(s)=1+\int_0^\infty \phi_1(x)\bar{P}_1(x,s)dx+\int_0^\infty \psi_1(y)\bar{P}_4(y,s)dy$$
$$+\int_0^\infty \mu_0(x)\bar{P}_3(x,s)dx+\int_0^\infty \mu_0(x)\bar{P}_{c_1}(x,s)dx$$
$$+\int_0^\infty \mu_0(y)\bar{P}_{c_2}(y,s)dy+\int_0^\infty \mu_0(z)\bar{P}_{c_T}(z,s)dz] \tag{19}$$

$$\left[s+\frac{\partial}{\partial x}+2\lambda_1+\lambda_{c_1}+\lambda_{c_2}+\lambda_{c_T}+\phi_1(x)\right]\bar{P}_1(x,s)=0 \tag{20}$$

$$\left[s+\frac{\partial}{\partial x}+\lambda_1+\lambda_{c_1}+\lambda_{c_2}+\lambda_{c_T}+\phi_2(x)\right]\bar{P}_2(x,s)=0 \tag{21}$$

$$\left[s+\frac{\partial}{\partial x}+\mu_0(x)\right]\bar{P}_3(x,s)=0 \tag{22}$$

$$\left[s+\frac{\partial}{\partial y}+2\mu_1+\lambda_{c_1}+\lambda_{c_2}+\lambda_{c_T}+\psi_1(y)\right]\bar{P}_4(y,s)=0 \tag{23}$$

$$\left[s+\frac{\partial}{\partial y}+\mu_1+\lambda_{c_1}+\lambda_{c_2}+\lambda_{c_T}+\psi_2(y)\right]\bar{P}_5(y,s)=0 \tag{24}$$

$$\left[s+\frac{\partial}{\partial x}+\mu_0(x)\right]\bar{P}_{c_1}(x,s)=0 \tag{25}$$

$$\left[s+\frac{\partial}{\partial y}+\mu_0(y)\right]\bar{P}_{c_2}(y,s)=0 \tag{26}$$

$$\left[s+\frac{\partial}{\partial z}+\mu_0(z)\right]\bar{P}_{c_T}(z,s)=0 \tag{27}$$

Boundary conditions

$$\bar{P}_1(0,s)=3\lambda_1\bar{P}_0(s) \tag{28}$$

$$\bar{P}_2(0,s)=2\lambda_1\bar{P}_1(0,s)=6\lambda_1^2\bar{P}_0(s) \tag{29}$$

$$\bar{P}_4(0,s)=3\mu_1\bar{P}_0(s) \tag{30}$$

$$\bar{P}_5(0,s)=2\mu_1\bar{P}_4(0,s)=6\mu_1^2\bar{P}_0(s) \tag{31}$$

$$\bar{P}_3(0,s)=\lambda_1\bar{P}_2(0,s)+\mu_1\bar{P}_5(0,s)=6\left(\lambda_1^3+\mu_1^3\right)\bar{P}_0(s) \tag{32}$$

$$\bar{P}_{c_1}(0,s)=\lambda_{c_1}\left[1+3(\lambda_1+\mu_1)+6\left(\lambda_1^2+\mu_1^2\right)\right]\bar{P}_0(s) \tag{33}$$

$$\bar{P}_{c_2}(0,s)=\lambda_{c_2}\left[1+3(\lambda_1+\mu_1)+6\left(\lambda_1^2+\mu_1^2\right)\right]\bar{P}_0(s) \tag{34}$$

$$\bar{P}_{c_T}(0,s)=\lambda_{c_T}\left[1+3(\lambda_1+\mu_1)+6\left(\lambda_1^2+\mu_1^2\right)\right]\bar{P}_0(s) \tag{35}$$

Now solving all the equations with the boundary conditions, one may get

$$\bar{P}_0(s)=\frac{1}{D(s)} \tag{36}$$

$$\bar{P}_1(s)=\frac{3\lambda_1}{D(s)}\frac{1}{\left(s+2\lambda_1+\lambda_{c_1}+\lambda_{c_2}+\lambda_{c_T}\right)} \tag{37}$$

$$\bar{P}_2(s)=\frac{6\lambda_1^2}{D(s)}\frac{1}{\left(s+\lambda_1+\lambda_{c_1}+\lambda_{c_2}+\lambda_{c_T}\right)} \tag{38}$$

$$\bar{P}_3(s)=\frac{6\left(\lambda_1^3+\mu_1^3\right)}{D(s)}\frac{1}{s} \tag{39}$$

$$\bar{P}_4(s)=\frac{3\mu_1}{D(s)}\frac{1}{\left(s+2\mu_1+\lambda_{c_1}+\lambda_{c_2}+\lambda_{c_T}\right)} \tag{40}$$

$$\bar{P}_5(s)=\frac{6\mu_1^2}{D(s)}\frac{1}{\left(s+\mu_1+\lambda_{c_1}+\lambda_{c_2}+\lambda_{c_T}\right)} \tag{41}$$

$$\bar{P}_{c_1}(s)=\frac{\lambda_{c_1}}{D(s)}\left[1+6\left(\lambda_1^2+\mu_1^2\right)+3\lambda_1+3\mu_1\right]\frac{1}{s}=\frac{\lambda_{c_1}}{D(s)}\frac{U}{s} \tag{42}$$

$$\bar{P}_{c_2}(s)=\frac{\lambda_{c_2}}{D(s)}\left[1+6\left(\lambda_1^2+\mu_1^2\right)+3\lambda_1+3\mu_1\right]\frac{1}{s}=\frac{\lambda_{c_2}}{D(s)}\frac{U}{s} \tag{43}$$

$$\bar{P}_{c_T}(s)=\frac{\lambda_{c_T}}{D(s)}\left[1+6\left(\lambda_1^2+\mu_1^2\right)+3\lambda_1+3\mu_1\right]\frac{1}{s}=\frac{\lambda_{c_T}}{D(s)}\frac{U}{s} \tag{44}$$

where $D(s)=s+3\lambda_1+3\mu_1+\lambda_{c_1}+\lambda_{c_2}+\lambda_{c_T}-3\lambda_1P-3\mu_1R-6\left(\lambda_1^3+\mu_1^3\right)T-\left(\lambda_{c_1}+\lambda_{c_2}+\lambda_{c_T}\right)UT$

$$P=\bar{S}_{\phi_1}\left(s+2\lambda_1+\lambda_{c_1}+\lambda_{c_2}+\lambda_{c_T}\right)=\frac{\phi_1}{s+2\lambda_1+\lambda_{c_1}+\lambda_{c_2}+\lambda_{c_T}+\phi_1}$$

$$Q=\bar{S}_{\phi_2}\left(s+\lambda_1+\lambda_{c_1}+\lambda_{c_2}+\lambda_{c_T}\right)=\frac{\phi_2}{s+\lambda_1+\lambda_{c_1}+\lambda_{c_2}+\lambda_{c_T}+\phi_2}$$

$$R=\bar{S}_{\psi_1}\left(s+2\mu_1+\lambda_{c_1}+\lambda_{c_2}+\lambda_{c_T}\right)=\frac{\psi_1}{s+2\mu_1+\lambda_{c_1}+\lambda_{c_2}+\lambda_{c_T}+\psi_1}$$

$$S=\bar{S}_{\psi_2}\left(s+\mu_1+\lambda_{c_1}+\lambda_{c_2}+\lambda_{c_T}\right)=\frac{\psi_2}{s+\mu_1+\lambda_{c_1}+\lambda_{c_2}+\lambda_{c_T}+\psi_2}$$

$$T = \bar{S}_{\mu_0}(s) = \frac{\mu_0}{s+\mu_0} \text{ and } U = 1 + 6\left(\lambda_1^2 + \mu_1^2\right) + 3\lambda_1 + 3\mu_1$$

Sum of Laplace transformations of the state transitions, where the system is in operational mode and failed state at any time, is as follows

$$\bar{P}_{up}(s) = \bar{P}_0(s) + \bar{P}_1(s) + \bar{P}_2(s) + \bar{P}_4(s) + \bar{P}_5(s) \tag{45}$$

$$\bar{P}_{down}(s) = 1 - \bar{P}_{up}(s) \tag{46}$$

5. Evaluation of Reliability Characteristics

5.1 Availability of the system

If the system performs with lowered efficiency i.e. it is in partial failure mode then the system is restored through general distribution, but in case of complete failure, repair follows a multivariate distribution namely Gumbel-Hougaard copula distribution, which uses the followings path.

$$\bar{S}_{\mu_0}(s) = \bar{S}_{\exp\left[x^\theta + \{\log\phi(x)\}^\theta\right]^{1/\theta}}(s) = \frac{\exp\left[x^\theta + \{\log\phi(x)\}^\theta\right]^{1/\theta}}{s + \exp\left[x^\theta + \{\log\phi(x)\}^\theta\right]^{1/\theta}} \text{ and } \bar{S}_\phi(s) = \frac{\phi}{s+\phi}.$$

We consider both general distribution and copula distribution while evaluating $\bar{P}_{up}(s)$. Taking the values of failure rates as $\lambda_1 = 0.02, \mu_1 = 0.03, \lambda_{c1} = 0.021, \lambda_{c2} = 0.022, \lambda_{cT} = 0.025$, $\theta = 1$ $x = 1$ $\phi_i = \psi_i = 1(i = 1,2)$ in equations (45). computing inverse Laplace transform, with Maple 17 software one can obtain the following availability expression of the system. Here we have considered the following particular cases:

(a) When both the subsystems have switching device, we obtain,

$$\begin{aligned} P_{up}(t) = {} & 0.030148e^{-2.8040t} + 0.024319e^{-1.2900t} - 0.003139e^{-1.1309t} - 0.011268e^{-1.0955t} \\ & - 0.021207e^{-1.0481t} - 0.029779e^{-1.0383t} + 1.007386e^{-0.0093t} - 0.001840e^{-1.0880t} \\ & + 0.005382e^{-1.0980t} \end{aligned} \tag{47}$$

(b) When subsystem-2 does not have a switching device i.e. $\lambda_{s_2} = 0$, we obtain,

$$\begin{aligned} P_{up}(t) = {} & -0.001895e^{-1.0660t} + 0.005182e^{-1.0760t} + 0.020739e^{-2.7765t} + 0.027661e^{-1.2728t} \\ & - 0.003152e^{-1.1089t} - 0.011093e^{-1.0734t} - 0.021381e^{-1.0261t} - 0.030907e^{-1.0162t} \\ & + 1.014845e^{-0.0104t} \end{aligned} \tag{48}$$

(c) When both subsystems 1 and 2 do not have a switching device i.e. $\lambda_{s_1} = \lambda_{s_2} = 0$, we obtain,

$$\begin{aligned} P_{up}(t) = {} & -0.001948e^{-1.0450t} + 0.005009e^{-1.0550t} + 0.011482e^{-2.7501t} + 0.031176e^{-1.2564t} \\ & - 0.003163e^{-1.0879t} - 0.010942e^{-1.0522t} - 0.021539e^{-1.0051t} - 0.032022e^{-0.9951t} \\ & + 1.021947e^{-0.0114t} \end{aligned} \tag{49}$$

Similar expressions can be obtained for availability under general repair by taking $\mu_0 = 1$. Put the values of t as $t = 0, 5, 10, 15, 20, 25, 30, 35, 40, 45$ and 50 units. The variation in availability under general repair and copula repair can be seen in table-2 and corresponding figure-2.

Table 2 Variation in availability for various t under copula and general repair.

Time (t)	Copula Repair			General Repair		
	(a)	(b)	(c)	(a)	(b)	(c)
0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
5	0.9610	0.9628	0.9648	0.9217	0.9357	0.9498
10	0.9173	0.9141	0.9116	0.8813	0.8895	0.8981
15	0.8754	0.8676	0.8610	0.8426	0.8456	0.8490
20	0.8354	0.8234	0.8132	0.8057	0.8038	0.8027
25	0.7972	0.7815	0.7680	0.7704	0.7640	0.7588
30	0.7607	0.7417	0.7254	0.7367	0.7263	0.7174
35	0.7260	0.7040	0.6851	0.7044	0.6903	0.6782
40	0.6928	0.6681	0.6471	0.6735	0.6562	0.6411
45	0.6611	0.6341	0.6112	0.6440	0.6238	0.6061
50	0.6309	0.6018	0.5772	0.6158	0.5929	0.5730

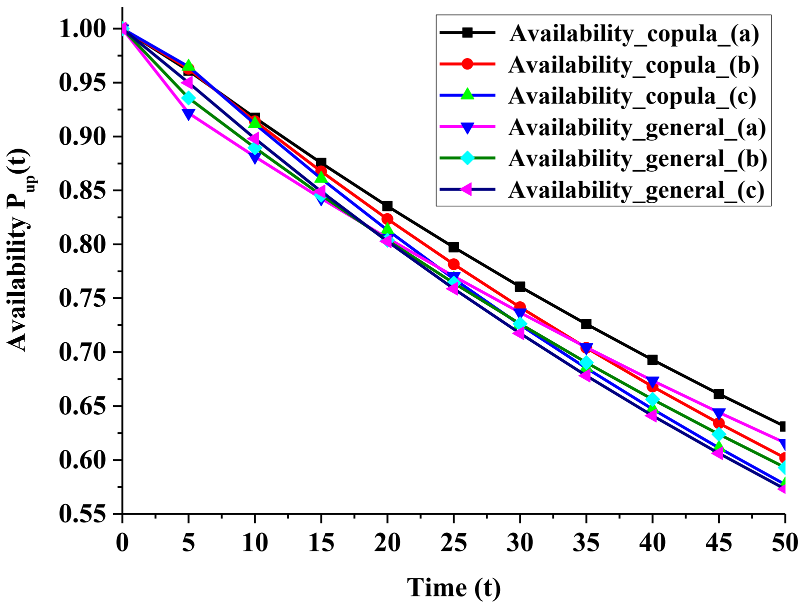

Figure 2 Variation in availability for various t under copula and general repair.

5.2 Reliability of the system

Reliability is the probabilistic measure of a non-repairable system. Taking all repair rates to zero and obtain the inverse Laplace transform in (45), we get the reliability of the system after taking the failure rates as $\lambda_1 = 0.02, \mu_1 = 0.03, \lambda_{c1} = 0.021, \lambda_{c2} = 0.022, \lambda_{cT} = 0.025$. Now consider the same cases as availability, we have

(a) When both the subsystems have switching device, we obtain,

$$\begin{aligned} R_i(t) &= 0.049568e^{-0.0880t} + 12.572026e^{-0.1280t} + 0.141705e^{-1.1309t} + 2.158779e^{-0.1080t} \\ &\quad - 4.384342\ 10^{-41} e^{-1.4681t}\left(3.1754\ 10^{41}\cosh(1.3332t) + 3.1887\ 10^{41}\cosh(1.3332t)\right) \end{aligned} \tag{50}$$

Similar expressions for the reliability of the system can be obtained in the other two cases. For different values of time-variable $t = 0, 5, 10, 15, 20, 25, 30, 35, 40, 45$ and 50 units of time, one may get different values of reliability $R(t)$ for all the three cases as shown in table-3 and figure-3.

Table 3 Variation in reliability corresponding to the different cases

Time (t)	(a)	(b)	(c)
0	1.0000	1.0000	1.0000
5	0.6798	0.7589	0.8429
10	0.4189	0.5220	0.6440
15	0.2466	0.3431	0.4701
20	0.1419	0.2203	0.3354
25	0.0807	0.1399	0.2365
30	0.0456	0.0883	0.1659
35	0.0257	0.0557	0.1161
40	0.0145	0.0351	0.0813
45	0.0082	0.0221	0.0570
50	0.0046	0.0140	0.0401

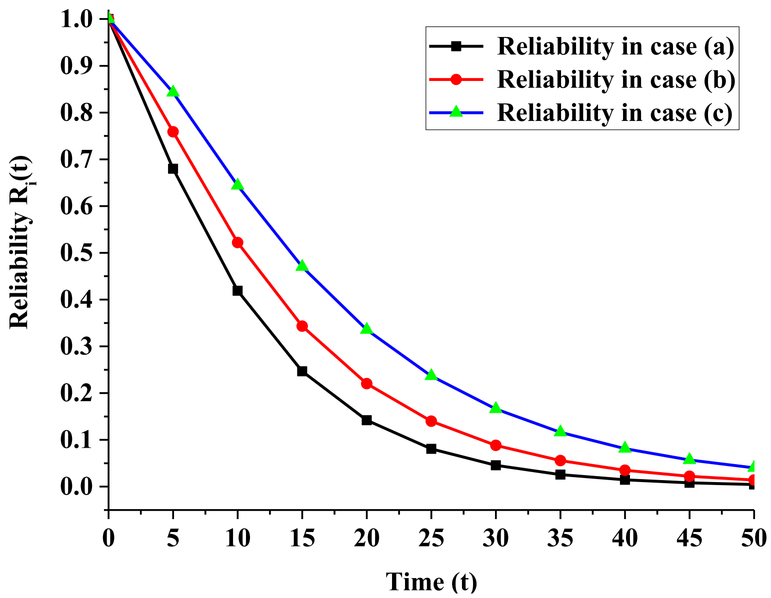

Figure 3 Reliability for various time (t)

5.3 Mean Time to Failure

Taking all repair rate to zero and the limit as s tends to zero in (45) for the exponential distribution; we can obtain the MTTF as:

$$MTTF = \frac{1}{\lambda}\left[1 + \frac{3\lambda_1}{2\lambda_1 + \mu} + \frac{6\lambda_1^2}{\lambda_1 + \mu} + \frac{3\mu_1}{2\mu_1 + \mu} + \frac{6\mu_1^2}{\mu_1 + \mu}\right] \tag{51}$$

where $\lambda = 3\lambda_1 + 3\mu_1 + \lambda_{c_1} + \lambda_{c_2} + \lambda_{c_T}$ and $\mu = \lambda_{c_1} + \lambda_{c_2} + \lambda_{c_T}$

Now taking the values of different parameters a $\lambda_1 = 0.02, \mu_1 = 0.03, \lambda_{c_1} = 0.021, \lambda_{c_2} = 0.022$ and $\lambda_{c_T} = 0.025$ and varying $\lambda_1, \mu_1, \lambda_{c_1}, \lambda_{c_2}$ and λ_{c_T} one by one respectively as $0.01, 0.02, 0.03, 0.04, 0.05, 0.06, 0.07, 0.08, 0.09, 0.10$ in (56), the variation of MTTF, with respect to failure rates can be obtained as given in table-4 and figure-4.

Table 4 Computation of MTTF corresponding to the failure rates

Failure Rate	MTTF				
	λ_1	μ_1	λ_{c_1}	λ_{c_1}	λ_{c_T}
0.01	11.2065	12.2242	11.9856	12.1127	12.5101
0.02	10.7387	11.5194	10.8417	10.9466	11.2732
0.03	10.1469	10.7387	9.8897	9.9776	10.2506
0.04	9.5608	10.0101	9.0855	9.1602	9.3916
0.05	9.0201	9.3626	8.3997	8.4619	8.6605
0.06	8.5337	8.7955	7.8031	7.8589	8.0309
0.07	8.1002	8.3002	7.2842	7.3331	7.4836
0.08	7.7143	7.8666	6.8277	6.8709	7.0035
0.09	7.3704	7.4852	6.4231	6.4615	6.5793
0.10	7.0628	7.1480	6.0623	6.0966	6.2018

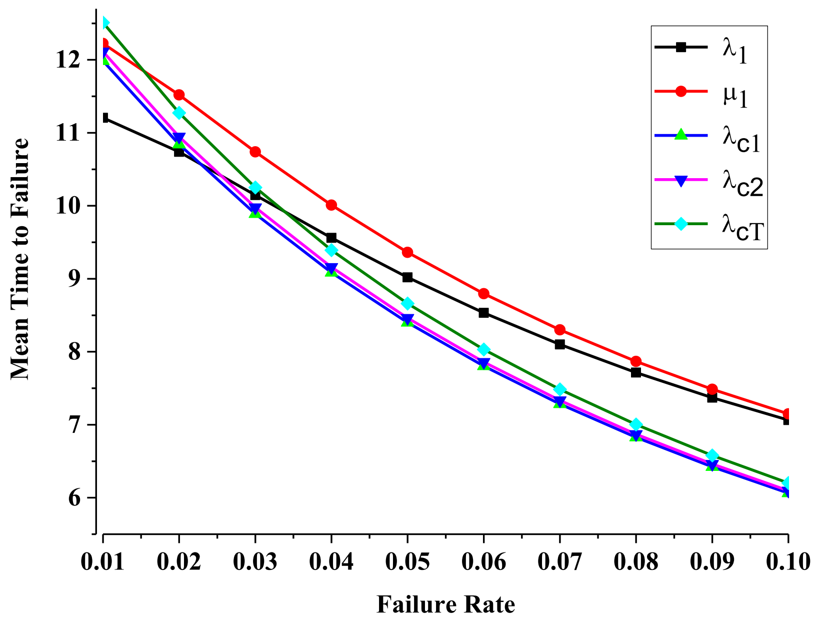

Figure 4 MTTF as a function of failure rates

5.4 Sensitivity Analysis of the system

The model's sensitivity analysis shows how the variance in the mathematical model's output can be attributed to various causes of uncertainty in its input or input variation by considering other inputs as constants. Sensitivity can be attained by taking the partial differentiation of the mean time to failure with respect to the failure rates of the system. Setting the parameters as $\lambda_1 = 0.02$, $\mu_1 = 0.03, \lambda_{c1} = 0.021, \lambda_{c2} = 0.022, \lambda_{cT} = 0.025$ in the partial differentiation of equation (51) obtained using maple, we will get the sensitivity of the system as shown in table-5 and figure-5.

Table 5 Computation of sensitivity with regard to the failure rates

Failure Rate	$\frac{\partial(MTTF)}{\partial\lambda_1}$	$\frac{\partial(MTTF)}{\partial\mu_1}$	$\frac{\partial(MTTF)}{\partial\lambda_{C1}}$	$\frac{\partial(MTTF)}{\partial\lambda_{C2}}$	$\frac{\partial(MTTF)}{\partial\lambda_{CT}}$
0.01	-31.0467	-56.2651	-125.8712	-128.4334	-136.6005
0.02	-56.4646	-77.9323	-103.9328	-105.8611	-111.9739
0.03	-59.9943	-76.3944	-87.2011	-88.6883	-93.3809
0.04	-56.6471	-68.9347	-74.1542	-75.3247	-79.0038
0.05	-51.3805	-60.6203	-63.7889	-64.7261	-67.6625
0.06	-45.9257	-52.9484	-55.4214	-56.1832	-58.5629
0.07	-40.8758	-46.2792	-48.5727	-49.1999	-51.1544
0.08	-36.3955	-40.6038	-42.8988	-43.4211	-45.0452
0.09	-32.4899	-35.8047	-38.1477	-38.5870	-39.9506
0.10	-29.1073	-31.7451	-34.1312	-34.5041	-35.6596

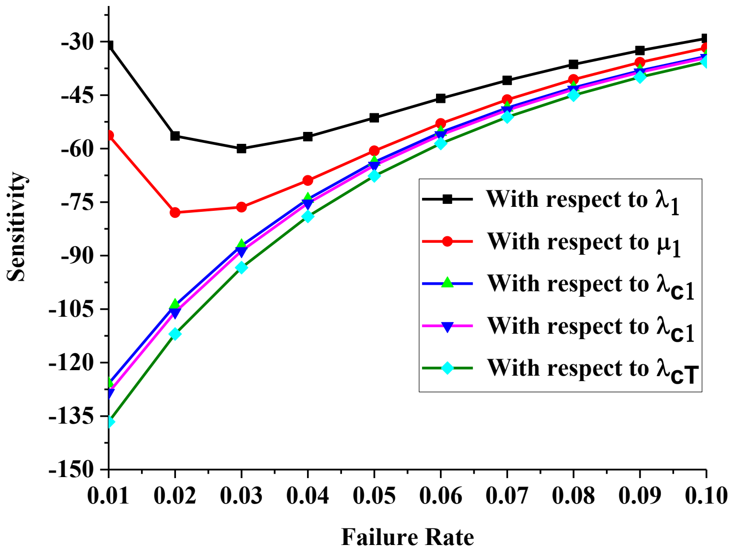

Figure 5 Sensitivity with respect to failure rates

5.5 Cost Analysis of the system

Incurred profit as the system follows copula repair and general repair has been calculated by assuming the same failure and repair rate as per section 5.1. Let us assume the service facility to be open at all times, then the estimated profit to be realized in the interval $[0,t)$ is

$$E_p(t) = K_1 \int_0^t P_{up}(t)dt - K_2 t \tag{52}$$

Where K_1 and K_2 are the revenue generation and service cost in unit time, respectively? Thus

$$\begin{aligned} E_p(t) = K_1 \{ & -0.010751e^{-2.8040t} - 0.018850e^{-1.2900t} + 0.020233e^{-1.0481t} - 107.640872e^{-0.0093t} \\ & + 0.002776e^{-1.130912t} + 0.010286e^{-1.0954t} + 0.028680e^{-1.0383t} - 0.004901e^{-1.0980t} \\ & + 0.001691e^{-1.0880t} + 107.611709 \} - K_2 t \end{aligned} \tag{53}$$

A similar expression can be obtained in case of general repair. Let $K_1 = 1$ $K_2 = 0.1, 0.2, 0.3, 0.4$ and 0.5 us varying $t = 0, 5, 10, 15, 20, 25, 30, 35, 40, 45$ and 50 units of time in Eq. (52), the expected profit under copula repair and general repair can be seen in table-6 and 7 and corresponding diagrams -6 and 7.

Table-6: Expected profit computation in copula repair policy

Time (t)	K_2=0.1	K_2=0.2	K_2=0.3	K_2=0.4	K_2=0.5
0	0.00	0.00	0.00	0.00	0.00
5	4.39	3.89	3.39	2.89	2.39
10	8.58	7.58	6.58	5.58	4.58
15	12.56	11.06	9.56	8.06	6.56
20	16.34	14.34	12.34	10.34	8.34
25	19.92	17.42	14.92	12.42	9.92
30	23.32	20.32	17.32	14.32	11.32
35	26.53	23.03	19.53	16.03	12.53
40	29.58	25.58	21.58	17.58	13.58
45	32.46	27.96	23.46	18.96	14.46
50	35.19	30.19	25.19	20.19	15.19

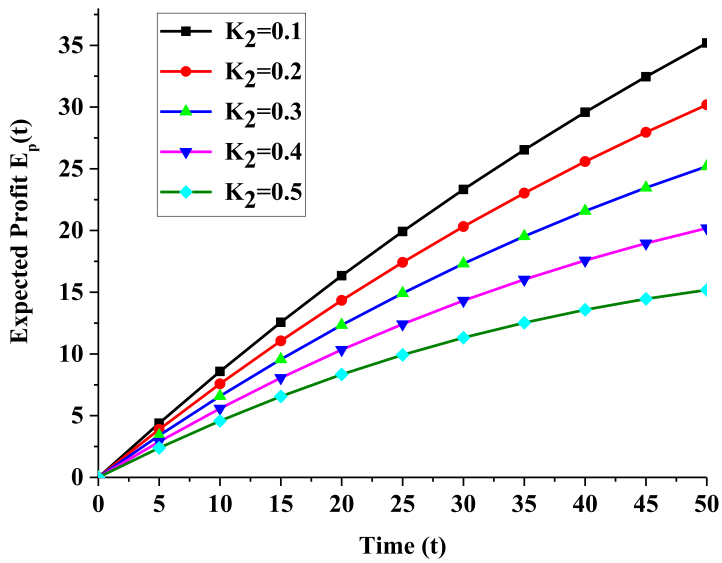

Figure-6: Computation of expected profit in copula repair policy

Table-7: Expected profit computation in general repair policy

Time (t)	K_2=0.1	K_2=0.2	K_2=0.3	K_2=0.4	K_2=0.5
0	0.00	0.00	0.00	0.00	0.00
5	4.23	3.73	3.23	2.73	2.23
10	8.24	7.24	6.24	5.24	4.24
15	12.05	10.55	9.05	7.55	6.05
20	15.67	13.67	11.67	9.67	7.67
25	19.11	16.61	14.11	11.61	9.11
30	22.38	19.38	16.38	13.38	10.38
35	25.48	21.98	18.48	14.98	11.48
40	28.42	24.42	20.42	16.42	12.42
45	31.22	26.72	22.22	17.72	13.22
50	33.86	28.86	23.86	18.86	13.86

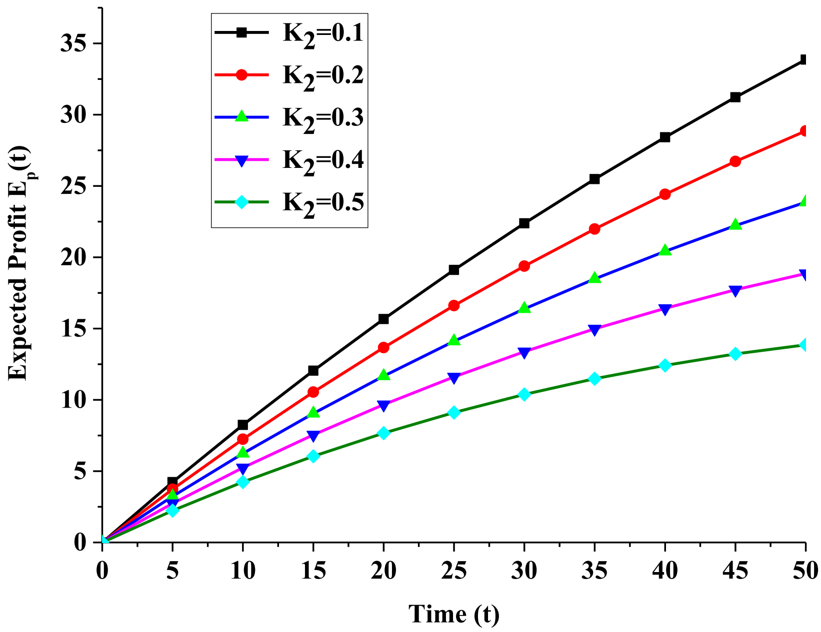

Figure-7: Computation of expected profit in copula repair policy

6. Result discussion and conclusion

This paper analyzes the probabilistic measures of a repairable system consisting of two subsystems in series arrangement with controllers under catastrophic failure. Each subsystem consists of three replica units in a parallel configuration and operates under 1-out-of-3: G strategy. A study of the model with the support of supplementary variables confirms that copula repair is a better and more effective repair policy. The following decisions can be made based on the analysis carried out in this paper:

Table-2 and figure-2 include the variation in the availability of the system in three possible situations under copula repair and general repair when failure rates are set at different time-related values. It can be easily shown that the availability decreases as time t increases in all situations, but it is better in the case of copula repair with controllers. The availability is low with general maintenance and without controllers. Moreover, not only the availability highlights the need for multivariate repair in the form of copulas but also the necessity of controllers.

Table-3 and figure-3 show evidence for the reliability of the system at various time values. The graph revealed a steep decrease in reliability from the top to the bottom in a succinct time in all three situations, depending on the failure rate of units. Furthermore, it can be found that the corresponding values of availability are higher than the reliability, which underlines the need for systematic repair for all dynamic systems for healthier outcomes.

The MTTF of the system concerning variation in $\lambda_1, \mu_1, \lambda_{s_1}, \lambda_{s_2}$, and λ_{c_T} indicated in table-4 and corresponding figure-4. It can be seen that the MTTF of the system reduces with rising values of all the parameters. The MTTF was observed to be the largest in the case of μ_1. Thus, MTTF of the system in all possible scenarios is decreasing as failure rates $\lambda_1, \mu_1, \lambda_{s_1}, \lambda_{s_2}$, and λ_{c_T} increase from 0.01 to 0.10.

Careful observations in table-5 and accompanying figure-5 demonstrate the sensitivity of the system and it is very important to note that sensitivity improves with a rise in failure rate values.

A critical analysis from table 6 (under copula repair) and 7 (under general repair) and figures 6 and 7 indicate that the estimated profit increases as the service cost K_2 decreases, while revenue cost per unit time is set at K_1=1. The estimated predicted profit is maximum for K_2= 0.1 while the minimum profit for K_2=0.5. One may observe that as service cost reduces, benefit swells with the variation of time. In comparison, copula repair is a more efficient repair approach for greater performance of repairable systems, since earnings are higher in the case of copula repair.

The model developed in this paper was found to be highly advantageous in proper maintenance analysis, decision, and evaluation of performance. As far as future studies are concerned, we may increase the number of units in both the subsystems. Furthermore, the optimum reliability and availability of the system can be determined.

Conflicts of Interest: Authors hereby state that there are no conflicts of interest is in this manuscript. This is soul work of authors which has not submitted in any other journal. The manuscript follows ethically guidelines of submissions for the Journal.

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Digitalization of Information Specified on the Grid

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Abstract

Digitalization is the process of implementing digital transmission systems at the level of primary networks, switching and control facilities that ensure the transmission and distribution of information flows in digital form at the level of secondary networks, which makes production more flexible, competitive, and profitable. First point considered here is an introduction of mathematical equivalent to the concept of a pixel, used when replacing the original information with its step-by-step approximation and estimate of its accuracy. Second point is the study of special knots: extreme knots or saddle knots on the grid and construction of level lines around them: ellipses or hyperbolas. This construction is connected with some meteorological problems and is based on the concept of positive definite quadratic form. Third point is an estimation of the average number of Poisson flow points in several cells of a square lattice in different problems of earth sciences. It is solved by introduction of relative error of the estimate.

Keywords: pixel, step–by–step approximation, positive definite quadratic form, relative error.

1. Introduction

Currently, the program of digitalization of information is being widely developed. Digitalization in a broad sense is the process of implementing digital transmission systems at the level of primary networks, switching and control facilities that ensure the transmission and distribution of information flows in digital form at the level of secondary networks, which makes production more flexible, competitive, and therefore more profitable. This trend allows us to take a fresh look at the already established methods in the processing of information and requires an assessment the accuracy of information conversion in various aspects and ensure the further development of information technology.

Such a statement of the question leads, among other things, to estimates of accuracy when replacing the original information with its digital expression. In turn, the accuracy estimates significantly affect the correctness of the actions of technical systems when receiving external information, and therefore the reliability of their operation. This requires the need to build mathematical equivalents to the concepts used in the process of digitalization. In particular, there is a need to take a fresh look at the concept of a pixel used when replacing the original information with its step-by-step approximation. As a result, it becomes necessary to study the transition from the original signals to the signals specified on a certain lattice. In turn, the reference to the signals given on the grid raises new questions for those tasks where such information is used.

Here, this problem is explored for some meteorology questions. It turns out that in these problems of meteorology, an important aspect of the analysis of such information is the study of singular points on the grid. A detailed study of special points (extreme points or saddle points) allows us to set and to solve the problems of forecasting meteorological systems in a new way.

These questions are the subject of research in this paper. Special attention is paid here to the accuracy of reproducing information from its discrete images and to the possibility of compressing information for its further use. For this purpose, it is possible to use such classical mathematical constructions as multidimensional Taylor series, positive definite quadratic forms, and etc.

Last problem is in estimating the mean number of Poisson flow points in some domain consisting of few cells in square grid. Its solution is based on a concept of relative error and on properties of Poisson distribution. It is connected with earth sciences problems, for example, with calculation of a number of rare animals.

2. The mathematical equivalent of the term "pixel"

In computer science, the term "pixel" (Engl. "pixel" is short for pictures element) [1] is the smallest logical two-dimensional element of a digital image in raster graphics, or a physical element of the matrix of displays that form the image. It is an indivisible object of rectangular (or round) shape, characterized by a certain color.

Signals that transmit sound or time-varying images are currently being digitized to make it convenient to transmit them from one point to another. At the same time, to determine and evaluate the quality of the transmitted information, it is desirable to construct a mathematical equivalent of the concept of "pixel".

For this purpose, it is natural to use a step-by-step approximation of the functions representing the transmitted signals. The quality of such an approximation increases with a decrease in the sampling step (in time and/or coordinate). Given the significance of this dependence of the approximation quality on the sampling step, it is natural to express this dependence mathematically.

Let's start solving the problem by analysing the stepwise approximation of the function $f(t)$, given on the half–interval [0,1). Let's assume that this function is continuously differentiable and for some positive F the equality holds

$$\sup_{0\le t\le 1}|f'(t)| = F. \tag{1}$$

We divide the semi-interval [0,1) into n parts by points $i/n,\ i = 0,1,\dots,n-1$. On the half–interval, $S_i = \left[\frac{i}{n}, \frac{i+1}{n}\right)$ we approximate the function $f(t)$ by constant $f\left(\frac{i}{n}+\frac{1}{2n}\right)$, by constructing a stepwise approximation in this way $\widehat{f}\,(t)$. Using the decomposition of the function $f(t)$ into a Taylor series with a Lagrange residual term in the neighbourhood of the radius $\frac{1}{2n}$ of the point $\frac{i}{n}+\frac{1}{2n}$, $i = 0,1,\dots,n-1$, we obtain the following inequality

$$\sup_{0\le t<1}\left|f(t) - \widehat{f}\,(t)\right| \le \frac{F}{2n}. \tag{2}$$

Let us now consider the continuously differentiable function of the m–dimensional argument $f(t_1, t_2, \dots, t_m)$, $(t_1, \dots, t_m) \in [0,1)^m$. Suppose that there are such positive numbers $F_1, F_2, \dots, F_m$, that the relations are satisfied

$$\sup_{0\le t_1,t_2,\dots,t_m<1}\left|\frac{\partial f(t_1,t_2,\dots,t_m)}{\partial t_k}\right| = F_k,\ \ k = 1,2,\dots,m. \tag{3}$$

We divide the direct product $[0,1)^m$ of half–intervals into direct products of half–intervals of the form $S_{i_1} \times S_{i_2} \times \ldots \times S_{i_m}$, $i_1, i_2, \ldots, i_m = 0,1\ldots, n-1$. Let's construct a stepwise approximation of $\hat{f}\,(t_1, t_2, \ldots, t_m)$ functions $f(t_1, t_2, \ldots, t_m)$, assuming it to be equal

$$f\left(\frac{i_1}{n} + \frac{1}{2n}, \frac{i_2}{n} + \frac{1}{2n}, \ldots, \frac{i_m}{n} + \frac{1}{2n}\right)$$

in a direct product $S_{i_1} \times S_{i_2} \times \ldots \times S_{i_m}$, $i_1, i_2, \ldots, i_m = 0,1, \ldots, n-1$.

Using the decomposition of the function $f(t_1, t_2, \ldots, t_m)$ in Taylor m–dimensional series (see, for example, [2]) with a Lagrange-shaped residual term in the set $S_{i_1} \times S_{i_2} \times \ldots \times S_{i_m}$, $i_1, i_2, \ldots, i_m = 0,1\ldots, n-1$ we get the inequality

$$\sup_{0 \le t_1, t_2, \ldots, t_m < 1} \left| f(t_1, t_2, \ldots, t_m) - \hat{f}\,(t_1, t_2, \ldots, t_m) \right| \le \frac{1}{2n} \sum_{k=1}^{m} F_k. \tag{4}$$

Each step in the $\hat{f}$ approximation of the f function can be called a pixel. Moreover, with a decrease in the linear pixel size $\frac{1}{n}$, the accuracy of such an approximation increases in accordance with the formula (4). Note that the accuracy estimates of the step approximation depending on the linear pixel dimensions are expressed in a uniform metric.

3. Geometric interpretation of meteorological information in square grid nodes

The most important element of the structure of the pressure field at an altitude of 5 km above the Far East is a stable and extensive hollow. Its intensity and geographical localization largely determine the nature of atmospheric circulation and weather [3], [4]. When studying this hollow, the nodes of the square grid on the map are identified, the node in which the pressure field takes the minimum value and the pressure level isolines are built.

However, it is quite difficult to algorithmize and analytically investigate such a construction. Therefore, in this paper, an attempt is made to investigate the function given at the lattice nodes in a small neighbourhood of the minimum point, assuming that the step of the square lattice is sufficiently small. Assuming that the smooth function given at the lattice nodes reaches a minimum at the point of the lattice node, one can calculate the coefficients of the Taylor series.

The first-order coefficients are zero, and the second-order coefficients define a positive-definite quadratic shape, whose level lines are ellipses. Calculations show that the orientation of the major and minor axes of such an ellipse and their ratio largely determine the nature of atmospheric circulation. But since the second coefficients of the Taylor series are determined by the values of the function at the nodes of the square lattice, it is necessary to investigate the algorithm for determining them and its accuracy depending on the length of the lattice step.

Let the thrice continuously differentiable function $f(x, y)$ be defined on the rectangle $D = \{0 \le x \le Nh,\ 0 \le y \le Mh\}$ and measured in points (ih, jh), $i = 0, \ldots N$, $j = 0, \ldots, M$. Everywhere else, we assume that the value of the lattice step h is sufficiently small. It is known that at the point (kh, lh), the function f reaches the global minimum. Moreover, the point (kh, lh) is internal in the discrete set $\{(ih, jh),\ i = 0, \ldots N,\ j = 0, \ldots, M\}$, $0 < k < N$, $0 < l < M$. Imagine the decomposition of the function $f(x, y)$ into a Taylor series with a Lagrange residual term under the condition $|x - kh| \le h$, $|y - lh| \le h$:

$$f(x, y) = f(kh, lh) + \frac{1}{2}[A(x - kh)^2 + B(y - lh)^2 + 2C(x - kh)(y - lh)] + O(h^3),$$

$$A = f_{x,x}(kh, lh),\ B = f_{y,y}(kh, lh),\ C = f_{x,y}(kh, lh).$$

Since the point (kh, lh) is the point of the global minimum of the function $f(x, y)$, then the quadratic form $A(x - kh)^2 + B(y - lh)^2 + 2C(x - kh)(lh)$ is positive definite and hence the inequalities are satisfied $A + B > 0,\ AB > C^2$.

We construct finite-difference estimates of partial derivatives A, B, C:

$$a = \frac{f((k+1)h,lh) - 2f(kh,lh) + f((k-1)h,lh))}{h^2} = A + O(h),$$

$$b = \frac{f(kh,(l+1)h) - 2f(kh,lh) + f(kh,(l-1)h)}{h^2} = B + O(h),$$

$$c = \frac{f((k+1)h,(l+1)h) - f((k+1)h,lh) - f(kh,(l+1)h) + f(kh,lh)}{h^2} = C + O(h).$$

Then the function f may be approximated by the function $\widehat{f}$ with an accuracy of $O(h^3)$ in variables $X = \frac{x-kh}{h},\ Y = \frac{y-lh}{h}$:

$$\widehat{f}(x, y) = f(kh, lh) + \frac{1}{2}(aX^2 + bY^2 + 2cXY),\ a + b > 0,\ ab > c^2$$

and so the quadratic form $aX^2 + bY^2 + 2kxy$ is also positive definite.

We reduce this quadratic form to a diagonal form (see, for example, [5]), for which we construct its matrix $A = \begin{pmatrix} a & c \\ c & b \end{pmatrix}$ and write out the characteristic equation $(a - \lambda)(b - \lambda) - c^2 = 0$. The roots of this equation are

$$\lambda_{\pm} = \frac{a+b}{2} \pm \sqrt{\frac{(a+b)^2}{4} - ab + c^2} > 0,$$

the eigenvalues of the matrix A, and its orthonormal eigenvectors $\vec{n}_{\pm}$ satisfy the linear equations $A\vec{n}_{\pm} = \lambda_{\pm}\vec{n}_{\pm}$.

Let's move to the coordinate system (u_+, u_-) with an orthonormal basis $\vec{n}_+\vec{n}_-$. In this coordinate system, the quadratic form is $aX^2 + bY^2 + 2xy$ is represented by the sum of squares $\lambda_+u_+^2 + \lambda_-u_-^2$. The level lines of this square shape are ellipses of the form $\lambda_+u_+^2 + \lambda_-u_-^2 = const > 0$. Denote $k = \sqrt{\frac{\lambda_+}{\lambda_-}}$, then to construct the specified ellipses, the circles given by the equation $u_+^2 + u_-^2 = const$ should be compressed along the u_+ axis by k times. The coefficient k may be interpreted as the ratio of the major and minor axes of an ellipse whose level lines are defined by a quadratic shape $aX^2 + bY^2 + 2cXY$.

It is interesting to note that if $f_x = f_y = 0$ and the condition $ab < c^2$, is satisfied, then it is not difficult to establish that $\lambda_+ > 0,\ \lambda_- < 0$ and hence the quadratic form $aX^2 + bY^2 + 2cXY$ in the variables u_+, u_- has the form $\lambda_+u_+^2 + \lambda_-u_-^2$ and is an alternating sign, and its level lines are hyperbolas.

4. Error in estimating the mean number of Poisson flow points

Specialists in the field of earth sciences have the task of estimating the error of the mean number of Poisson flow points from observations in different cells of a square grid. Let the study area be divided into m cells, and the number of points in them in the area k is $n_k,\ k = 1, \dots, m$. It is natural to assume that the random variables $n_1, \dots, n_m$ are independent and have Poisson distributions with the parameters $\lambda_1, \dots, \lambda_m$. Using the properties of the Poisson distribution, it is easy to establish that the random sum $N = \sum_{k=1}^{m} n_k$ has a Poisson distribution with the parameter $\Lambda = \sum_{k=1}^{m} \lambda_k$ and consequently $E(N) = \Lambda,\ Var(N) = \Lambda$.

Using the known properties of the mathematical expectation and the variance of the Poisson distribution, we proceed to estimate the relative error. To do this, consider the random variable $\frac{N}{E(N)} = \frac{N}{\Lambda}$. Variance of this random variable $Var\left(\frac{N}{\Lambda}\right) = \frac{1}{\Lambda}$ and so the relative error of such an estimate satisfies the relation $\sqrt{Var\left(\frac{N}{\Lambda}\right)} = \frac{1}{\sqrt{\Lambda}}$. Therefore, the relative error decreases with the growth of Λ.

This result does not depend on the nonuniformity of the distribution density of the Poisson flow of points, and therefore does not depend on the parameters $\lambda_1, \dots, \lambda_m$. It can be considered by choosing the efficiency indicator of a complex system like a relative error. This result is based on the well-known models of Poisson point flows in the theory of random sets, which are used in the earth sciences [6].

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Costs of Maintenance Service Policy: a New Approach on Constant Stress Partially Accelerated Life Test for Generalized Inverted Exponential Distribution

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Abstract

In this paper, we describe how to analyze and propose the accelerated life test plans for the development of the excellence and reliability of the product. We focus on estimating the costs of maintenance service policy because it has a very significant position to assist any manufacturing organization for sale and available its equipment and maintenance cost-effective. The constant-stress partially accelerated life test is assumed when the lifetime of test units follows Generalized Inverted Exponential distribution under the progressive censoring scheme. The maximum likelihood estimates, Fisher Information matrix, and the asymptotic variance and covariance matrix are obtained. The confidence intervals of the estimators are also obtained. Furthermore, a simulation study is conducted to check the accuracy of the findings.

Keywords: Life Testing, Constant-stress, Maintenance service policy, Progressive censoring, Generalized Inverted Exponential distribution, Simulation Study.

I. Introduction

In current scenario due to rapid and frequent technological changes the demand of manufacturing designs has been improving day by day due to which it is quit challenging and complex to obtain information about the lifetime of items or products under normal usage when the product of high reliability is tested because some commonly used life tests provide no or very few failures at the end of the test. So in such situation accelerated life testing (ALT) may be applied as one of the solution in which the product or material is tested under higher than usual used conditions to obtain the information quickly on the life distribution or performance of a product. These conditions are referred as stresses may be in the form of temperature, voltage, force etc. Generally there are three types of life test methods in accelerated life testing design – First is constant stress ALT, second is step stress ALT and third is progressive stress ALT. In the present research we are focusing only on constant stress accelerated life testing in which we may have fixed stress levels applied for different groups of tested items. It refers that every item is subjected to only one stress level until the item fails or the test is stopped for other reasons. ALT can be divided into two categories: complete (all failure data are available) or censored (some of the failure data are missing).

The data obtained from ALT cannot be extrapolated to use condition because in accelerated life testing (ALT), the mathematical model relating to the lifetime of an item and stress is known or can be assumed.

But, in some cases these relationships are not known and cannot be assumed, So a partially accelerated life test (PALT) can be used in such cases in which the test items are run at both normal and higher than normal stress conditions. The constant stress partially accelerated life test (CSPALT) and step stress partially accelerated life test (SSPALT) are the two commonly used methods in PALT. The products cann't be tested at either usual or higher than usual condition until and unless the test is terminated in CSPALT. On the other hand in SSPALT as an approach to accelerate failures which increases the load applied to the products in a specified discrete sequence. A sample of test items is first to run at use condition and if it does not fail for a specified time, then it is run at accelerated condition until prespecified numbers of failures are obtained or a prespecified time has reached.

In many cases when life data are analyzed, an experiment can be out of control due to many reasons like components of a system may break accidentally and all the units in the sample may not fail. This type of data is called censored or incomplete data. Due to different types of censoring, censored data can be divided into Type I censored (or time censored) data and Type II censored (or failure-censored) data. These two censoring schemes do not allow for units to be removed from the test at points other than the final termination point. Although, the removal of items or components from the test during testing is possible in the progressive type censoring scheme. In such types of situation, the multiple censoring schemes are the best choice for an engineer or reliability Weibull distribution with constant stress under the type-I censoring scheme. Anwar and Islam [4] analyzed the constant stress PALT plan for Gompertz distribution under type I censoring.

Zhang and Fang [5] dealt with an estimation of acceleration factor when the lifetime of units follows Exponential distribution under CSPALT based on type-I censored data. A new approach of constructing the exact lower and upper confidence limits is proposed by them for the acceleration factor. Sadia and Islam [6] presented a study on CSPALT plans when the lifetime of units follows Rayleigh distribution based on type-II censored data. Shi and Shi [7] dealt with a study on CSPALT using the masked series system when the lifetime of components follows Complementary Exponential distribution based on progressive type-II censoring. Ismail [8] discussed a study on CSPALT for Weibull distribution based on a hybrid censoring scheme. He makes a statistical inference by using two methods; maximum likelihood and percentile bootstrap method. Nassar and Elharoun [9] presented an inference on CSPALT for Exponentiated Weibull distribution in the case of multiply censored data. Hassan et al. [10] showed a study on CSPALT for inverted Weibull distribution in the multiply censoring scheme. Cheng and Wang [11] estimated parameters under multiply censoring scheme when the lifetime of items follows Burr XII distribution. Currently, Alam et al. [12] tackled CSPALT based on a multiply censoring scheme when the lifetime of failure units follows the Exponentiated Exponential failure model. Currently, Alam et al. [13] presented a study on ALT when the lifetime of test units follows Burr type-X for Type-II censoring and Progressive censoring, respectively. Alam et al. [14] also presented a study on maintenance service policy under SSPALT when the lifetime of test units follows the Power function failure model with progressive censoring.

The current study based on maintenance service policy problem under CSPALT for Progressive censoring when the lifetime of test units follows Generalized Inverted Exponential distribution.
The information (lifetime data) is censored when the accurate failure time of an item is unknown. Many types of censoring schemes are available, such as left, right, interval, Type-I, Type-II, hybrid, progressive, progressive Type-I, and progressive Type-II censoring, etc. We consider only the progressive Type-II censoring scheme in this paper. The Type-I and Type-II censoring schemes are mainly common and popular schemes in lifetime theory. The only major drawback in both Type-I and Type-II censoring schemes is that the experimenter cannot withdraw live items during the experiment. A newly censoring scheme which is a generalization of classical Type-II censoring scheme comes in light. It gives permission to draw item or items during the experiment.

For literature about this scheme, the authors refer to the book by Balakrishnan and Aggrawalla [15], and an article by Balakrishnan [16]. The progressive Type-II censoring is explained as follows:
The lifetime of n units are $X_1, X_2, ..., X_n$, and these test units are put on the testing. Also, suppose that $X_i, i = 1,2,..., n$ are independent and identically distributed (i.i.d) with cumulative distribution function $F(x)$ and probability distribution function $f(x)$. Before the experiment, an integer m $(m < n)$ is resolved, and the progressive Type-II censoring scheme $(R_1, R_2, ..., R_m, R_i > 0)$ and $n = m + \sum_{i=1}^{m} R_i$ is specified. Now, ith failure is observed, and after the failure, R_i functioning items are randomly removed from the test during the lifetime testing experiment. $X_{i:m:n}, i = 1,2,...n$ and m are the totally observed lifetimes, which are observed samples for the progressively Type-II censoring scheme. $x_{1:m:n} < x_{2:m:n} < ... < x_{m:m:n}$ are the observed values of the progressively Type-II right censored samples.

The paper is organized as follows; The model description, test procedure, and basic assumptions for CSPALT are given in section 2. The point estimation, interval estimation, Fisher information matrix, and confidence intervals are presented in section 3.. The estimating costs of maintenance service policy under Generalized Inverted Exponential distribution are presented in section 4. A simulation study using Monte-Carlo technique is proposed in section 5. Finally, the conclusions are made in last sections.

II. Model Description and Test Procedure

I. Model Description

In life testing theory, the one parameter (negative) Exponential distribution plays an important role because of its simplicity and it prefers to any other one parameter distribution. A generalized case of this distribution is presented by Gupta and Kundu [17] and known as Generalized Exponential distribution. A shape parameter is introduced by him. Lin et al. [18] introduced another extension of Exponential distribution, and this extension is known as Inverted Exponential distribution. They obtained the maximum likelihood estimator, confidence limits and also presented a comparison of this distribution with that of inverted Gaussian and Log-normal distributions using a maintenance data set. Bayes estimators of the parameter and risk functions under special loss functions are obtained by Dey [19]. A new distribution is presented by Abouammoh and Alshingiti [20] which is known as Generalized Inverted Exponential Distribution (GIED). Nadarajah and Kotz [21] noted that this distribution is original from the Exponentiated Frechet distribution. Due to the convenient structure of the distribution function, the GIED can be used in different applications, for example, in accelerated life testing, horse racing, queue theory, modeling wind speeds, etc.
The probability density function (pdf) for GIED is given as

$$f(t, \mu, \eta) = \frac{\eta\mu}{t^2} e^{-\mu/t} (1 - e^{-\mu/t})^{\eta-1}, \; t \geq 0, \; \mu, \eta > 0 \tag{1}$$

where, η and μ are shape and scale parameters, respectively.
The curve of the above equation (1) is shown in figure 1.

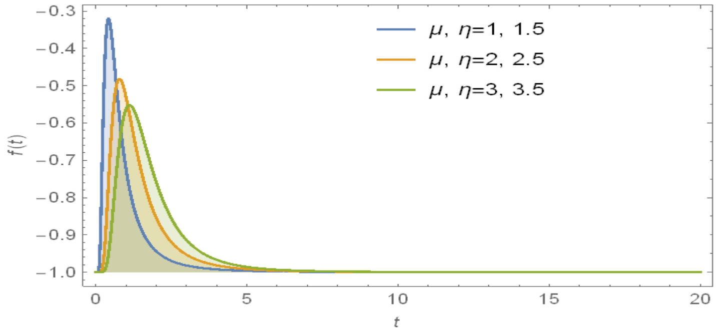

Figure1. *Probability density function curve of GIED*

The cumulative density function (cdf) for GIED is given as

$$F(t,\mu,\eta)=1-(1-e^{-\mu/t})^{\eta},\ t\geq 0\ ,\mu,\eta>0 \tag{2}$$

The curve of the above equation (2) is shown in figure 2.

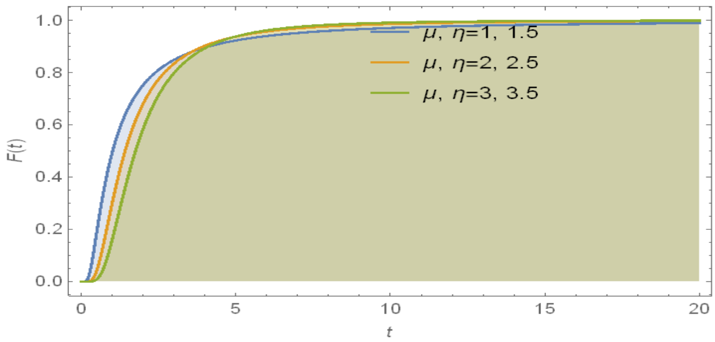

Figure2. *Cumulative density function curve of GIED*

The reliability function for GIED is given as

$$R(t,\mu,\eta)=(1-e^{-\mu/t})^{\eta}$$

The curve of the above expression is shown in figure 3.

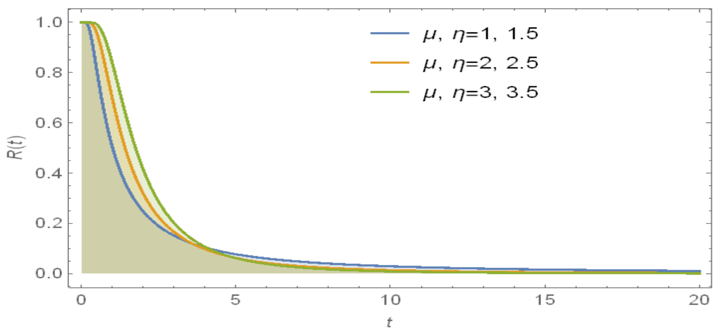

Figure3. *Reliability function curve of GIED*

The hazard function for GIED is given as

$$H(t,\mu,\eta)=\frac{\eta\mu}{t^{2}}e^{-\mu/t}(1-e^{-\mu/t})^{-1},$$

The curve of the above expression is shown in figure 4.

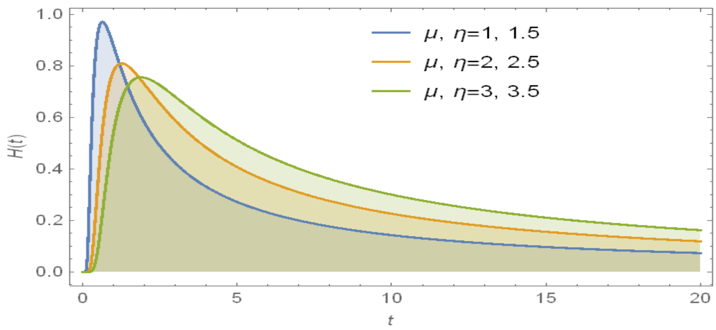

Figure4. *Hazard function curve of GIED*

Abouammoh and Alshingiti [20] and Nadarajah and Kotz [21] studied many interesting and useful properties of GIED. The hazard function of GIED depends on the shape parameter, and it can be increasing, or decreasing but not constant. If the shape parameter is greater than 4, this distribution has a unmoral and right-skewed density function Moreover, this distribution provides a better fit than Gamma, Weibull, Generalized Exponential, and Inverted Exponential distributions. The reliability estimation in the context of this distribution with progressively Type-II censoring scheme is studied by Krishna and Kumar [22].

II. Test Procedure

The test procedure of CSPALT based on progressive Type-II censored data assuming the lifetime item has GIED is described as follows

The pdf under normal condition is given as follows

$$f_1(t_i)=\frac{\eta\mu}{t^2}e^{-\mu/t_i}(1-e^{-\mu/t_i})^{\eta-1},\ \ t_i\geq 0,\ \mu,\eta>0, i=1,2,...,m_1 \tag{3}$$

The cdf under normal condition is given as follows

$$F_1(t_i)=1-(1-e^{-\mu/t_i})^{\eta},\ t_i\geq 0\ ,\mu,\eta>0 \tag{4}$$

where, t_i is the observed lifetime of an item i, that is tested at normal condition.

The pdf and cdf of the lifetime $Y=\beta^{-1}T$, under accelerated condition are given in following equations, (5) and (6)

$$f_2(y_j)=\frac{\beta\eta\mu}{(\beta y_j)^2}e^{-\mu/\beta y_j}(1-e^{-\mu/\beta y_j})^{\eta-1},\ \ y_j\geq 0,\ \mu,\eta>0 \tag{5}$$

$$F_2(y_j)=(1-e^{-\mu/\beta y_j})^{\eta},\ \ y_j\geq 0,\ \mu,\eta>0, j=1,2,...,m_2 \tag{6}$$

where, y_j is the observed lifetime of an item j, that is tested at the accelerated condition and $(\beta>1)$ is the acceleration factor.

III. Basic Assumptions

The necessary assumptions for CSPALT are given as

- The lifetimes of items $T_i\ \ i=1,2,...,m_1$ are independent and identically distributed (i.i.d.) random variable with pdf provided in equation (3), which is allocated to normal condition.
- The lifetimes of items $T_i\ \ i=1,2,...,m_1$ are independent and identically distributed (i.i.d.) random variable with pdf provided in equation (3), which is allocated to normal condition.
- T_i and Y_j are mutually independent also.

- m_1 and m_2 are the total number of items at normal and accelerated conditions, respectively.

$m = m_1 + m_2$ =Total number of items.

III. Estimation Procedure

The point and interval estimation are presented in this section.

I. Point Estimation

Let $X_1, X_2, ..., X_n$ are the lifetime of n independent units which put on test. These units are independently and identically distributed (i.i.d.) as GIED distribution with probability density function, which is presented in equation (1). The m completely ordered lifetimes are denoted by

$$x_{1:m:n} < x_{2:m:n} < ... < x_{J:m:n} < x_{n_1} < x_{J+1:m:n} < ... < x_{m:m:n}$$

Here, J denoted the number of failed units in normal conditions.

Hence, the likelihood function for GIED with progressively Type-II censored data under CS-PALT is given as:

$$L(x_i, \mu, \eta, \beta) = \prod_{i=1}^{J} f_1(x_i)[1 - F_1(x_i)]^{R_i} \times \prod_{i=J+1}^{m} f_2(x_i)[1 - F_2(x_i)]^{R_i} \tag{7}$$

After putting values from equations (3), (4), (5) and (6), we get the following log likelihood function, which is given as

$$\begin{aligned} \ln L(x_i, \mu, \eta, \beta) = & \sum_{i=1}^{J} \ln(\mu\eta) + \sum_{i=1}^{J} \ln(x_i^{-2}) + (\eta - 1)\left[\sum_{i=1}^{J} \ln(1 - e^{-\mu/x_i}) + \sum_{i=J+1}^{m} \ln(1 - e^{-\mu/\beta x_i})\right] + \\ & - \mu\left[\sum_{i=1}^{J} x_i^{-1} + \sum_{i=J+1}^{m} (\beta x_i)^{-1}\right] + \sum_{i=J+1}^{m} \ln(\beta\mu\eta) + \sum_{i=J+1}^{m} \ln(\beta x_i)^{-2} \\ & + \eta\left[\sum_{i=1}^{J} R_i \ln(1 - e^{-\mu/x_i}) + \sum_{i=J+1}^{m} R_i \ln(1 - e^{-\mu/\beta x_i})\right] \end{aligned} \tag{8}$$

where, $\ln L = L(x_i, \mu, \eta, \beta) = l$

The Maximum likelihood (ML) estimates of η, μ , and acceleration factor β are obtained from the following non-linear equations (9), (10) and (11).

$$\begin{aligned} \frac{\partial l}{\partial \mu} = & \sum_{i=1}^{J} \frac{1}{\mu} + (\eta - 1)\left[\sum_{i=1}^{J} \frac{e^{-\mu/x_i}}{x_i(1 - e^{-\mu/x_i})} + \sum_{i=J+1}^{m} \frac{e^{-\mu/\beta x_i}}{\beta x_i(1 - e^{-\mu/\beta x_i})}\right] - \left[\sum_{i=1}^{J} x_i^{-1} + \sum_{i=J+1}^{m} (\beta x_i)^{-1}\right] \\ & + \sum_{i=J+1}^{m} \frac{1}{\mu} + \sum_{i=J+1}^{m} \ln(\beta x_i)^{-2} + \eta\left[\sum_{i=1}^{J} R_i \frac{e^{-\mu/x_i}}{x_i(1 - e^{-\mu/x_i})} + \sum_{i=J+1}^{m} R_i \frac{e^{-\mu/\beta x_i}}{\beta x_i(1 - e^{-\mu/\beta x_i})}\right] \end{aligned} \tag{9}$$

$$\begin{aligned} \frac{\partial l}{\partial \eta} = & \left[\sum_{i=1}^{J} \ln(1 - e^{-\mu/x_i}) + \sum_{i=J+1}^{m} \ln(1 - e^{-\mu/\beta x_i})\right] + \sum_{i=1}^{J} \frac{1}{\eta} \\ & + \sum_{i=J+1}^{m} \frac{1}{\eta} + \sum_{i=J+1}^{m} \ln(\beta x_i)^{-2} + \left[\sum_{i=1}^{J} R_i n(1 - e^{-\mu/x_i}) + \sum_{i=J+1}^{m} R_i \ln(1 - e^{-\mu/\beta x_i})\right] \end{aligned} \tag{10}$$

$$\begin{aligned} \frac{\partial l}{\partial \beta} = & -2\sum_{i=J+1}^{m} \beta^{-1} + (\eta - 1)\mu\beta^{-2} \sum_{i=J+1}^{m} x_i^{-1} \frac{e^{-\mu/\beta x_i}}{(1 - e^{-\mu/\beta x_i})} + \eta\mu\beta^{-2} \sum_{i=J+1}^{m} R_i x_i^{-1} \frac{e^{-\mu/\beta x_i}}{(1 - e^{-\mu/\beta x_i})} \\ & + \mu \sum_{i=J+1}^{m} \beta^{-2} x_i^{-1} + \sum_{i=J+1}^{m} \beta^{-1} \end{aligned} \tag{11}$$

The solution of the above three non-linear equations is impossible manually. So an iterative technique (Newton-Raphson method) is applied to solve these equations.

II. Interval Estimation

The Fisher information matrix under progressive Type-II censoring scheme is given as

$$I=\begin{bmatrix} -\dfrac{\partial^2 l}{\partial\mu^2} & -\dfrac{\partial^2 l}{\partial\mu\partial\eta} & -\dfrac{\partial^2 l}{\partial\mu\partial\beta} \\ -\dfrac{\partial^2 l}{\partial\eta\partial\mu} & -\dfrac{\partial^2 l}{\partial\eta^2} & -\dfrac{\partial^2 l}{\partial\eta\partial\beta} \\ -\dfrac{\partial^2 l}{\partial\beta\partial\mu} & -\dfrac{\partial^2 l}{\partial\beta\partial\eta} & -\dfrac{\partial^2 l}{\partial\beta^2} \end{bmatrix} \tag{12}$$

$$-\frac{\partial^2 l}{\partial\mu^2}=\sum_{i=1}^{J}\frac{1}{\mu^2}+(\eta-1)\left[\sum_{i=1}^{J}\frac{e^{-\mu/x_i}}{(1-e^{-\mu/x_i})x_i^2}\left(1+\frac{e^{-\mu/x_i}}{1-e^{-\mu/x_i}}\right)+\sum_{i=J+1}^{m}\frac{e^{-\mu/\beta x_i}}{(\beta x_i)^2(1-e^{-\mu/\beta x_i})}\left(1+\frac{e^{-\mu/\beta x_i}}{1-e^{-\mu/\beta x_i}}\right)\right]$$

$$+\sum_{i=J+1}^{m}\frac{1}{\mu^2}-\sum_{i=J+1}^{m}\beta^{-2}x_i^{-1}+\eta\left[\begin{array}{l}\displaystyle\sum_{i=1}^{J}R_i\frac{e^{-\mu/x_i}}{(1-e^{-\mu/x_i})x_i^2}\left(1+\frac{e^{-\mu/x_i}}{1-e^{-\mu/x_i}}\right)+\\ \displaystyle\sum_{i=J+1}^{m}R_i\frac{e^{-\mu/\beta x_i}}{(\beta x_i)^2(1-e^{-\mu/\beta x_i})}\left(1+\frac{e^{-\mu/\beta x_i}}{1-e^{-\mu/\beta x_i}}\right)\end{array}\right]$$

$$-\frac{\partial^2 l}{\partial\eta^2}=\sum_{i=1}^{J}\frac{1}{\mu^2}+\sum_{i=J+1}^{m}\frac{1}{\mu^2}$$

$$-\frac{\partial^2 l}{\partial\beta^2}=-2\sum_{i=J+1}^{m}\beta^{-2}-(\eta-1)\sum_{i=J+1}^{m}\left[\frac{\mu x_i^{-1}\beta^{-2}e^{-\mu/\beta x_i}}{1-e^{-\mu/\beta x_i}}\left(\frac{\mu}{x_i\beta^2}+\frac{\mu x_i^{-1}\beta^{-2}e^{-\mu/\beta x_i}}{1-e^{-\mu/\beta x_i}}\right)-\frac{2\beta^{-3}\mu x_i^{-1}e^{-\mu/\beta x_i}}{1-e^{-\mu/\beta x_i}}\right]$$

$$-\eta\sum_{i=J+1}^{m}R_i\left[\frac{\mu x_i^{-1}\beta^{-2}e^{-\mu/\beta x_i}}{1-e^{-\mu/\beta x_i}}\left(\frac{\mu}{x_i\beta^2}+\frac{\mu x_i^{-1}\beta^{-2}e^{-\mu/\beta x_i}}{1-e^{-\mu/\beta x_i}}\right)-\frac{2\beta^{-3}\mu x_i^{-1}e^{-\mu/\beta x_i}}{1-e^{-\mu/\beta x_i}}\right]$$

$$+2\mu\sum_{i=J+1}^{m}\beta^{-3}x_i^{-1}+\sum_{i=J+1}^{m}\beta^{-2}$$

$$-\frac{\partial^2 l}{\partial\mu\partial\eta}=-\left[\sum_{i=1}^{J}\frac{e^{-\mu/x_i}}{x_i(1-e^{-\mu/x_i})}+\sum_{i=J+1}^{m}\frac{e^{-\mu/\beta x_i}}{\beta x_i(1-\beta x)}\right]+-\left[\sum_{i=1}^{J}R_i\frac{e^{-\mu/x_i}}{x_i(1-e^{-\mu/x_i})}+\sum_{i=J+1}^{m}R_i\frac{e^{-\mu/\beta x_i}}{\beta x_i(1-\beta x)}\right]$$

$$-\frac{\partial^2 l}{\partial\mu\partial\beta}=-(\eta-1)\left[\sum_{i=J+1}^{m}\frac{e^{-\mu/\beta x_i}}{\beta x_i(1-e^{-\mu/\beta x_i})}\left(\frac{\mu}{\beta^2 x_i}-\beta^{-1}+\frac{\mu x_i^{-1}\beta^{-2}e^{-\mu/\beta x_i}}{1-e^{-\mu/\beta x_i}}\right)\right]$$

$$-\eta\left[\sum_{i=1}^{J}R_i\frac{e^{-\mu/\beta x_i}}{\beta x_i(1-e^{-\mu/\beta x_i})}\left(\frac{\mu}{\beta^2 x_i}-\beta^{-1}+\frac{\mu x_i^{-1}\beta^{-2}e^{-\mu/\beta x_i}}{1-e^{-\mu/\beta x_i}}\right)\right]-\sum_{i=J+1}^{m}\beta^{-2}x_i^{-1}$$

$$-\frac{\partial^2 l}{\partial\eta\partial\beta}=\sum_{i=J+1}^{m}\frac{e^{-\mu/\beta x_i}\mu x_i^{-1}\beta^{-2}}{1-e^{-\mu/\beta x_i}}+\sum_{i=J+1}^{m}R_i\frac{e^{-\mu/\beta x_i}\mu x_i^{-1}\beta^{-2}}{1-e^{-\mu/\beta x_i}}$$

Now, the variance-covariance matrix under progressive Type-II censoring scheme is the inverse of the Fisher Information matrix and it is given as

$$\Sigma=I^{-1} \tag{14}$$

The ML estimates of distribution parameters and β are asymptotically normally distributed and consistent in large samples.

So, the two-sided approximate $100(1-\alpha)\%$ confidence limits for distribution parameters and β are obtained in the following way:

$$\left.\begin{array}{c} L_{\hat{\eta}} = \hat{\eta} - z_{\gamma/2}\sigma(\hat{\eta}) and U_{\hat{\eta}} = \hat{\eta} + z_{\gamma/2}\sigma(\hat{\eta}) \\ L_{\hat{\mu}} = \hat{\mu} - z_{\gamma/2}\sigma(\hat{\mu}) and U_{\hat{\mu}} = \hat{\mu} + z_{\gamma/2}\sigma(\hat{\mu}) \\ L_{\hat{\beta}} = \hat{\beta} - z_{\gamma/2}\sigma(\hat{\beta}) and U_{\hat{\beta}} = \hat{\beta} + z_{\gamma/2}\sigma(\hat{\beta}) \end{array}\right\} \tag{15}$$

Where $z_{\gamma/2}$ is the $\left[100(1-\gamma)/2\right]^{th}$ standard normal percentile. $\sigma(*)$is the standard deviation for the ML estimates $\hat{\eta}, \hat{\theta}$and$\hat{\beta}$, it is calculating by taking the square root of the first diagonal element of the I^{-1}.

IV. Estimating Costs of Maintenance Service Policy under GIED

There are numbers of authors has explored the problem of maintenance service policy instance, some are as follows- Yiwei et al. [29] studied a cost-driven predictive maintenance policy for structural airframe maintenance. Maintenance policy is formally derived based on the trade-off between probabilities of occurrence of unscheduled and scheduled maintenance. Yiwei et al. [30] proposed predictive airframe maintenance strategies using model-based prognostics. According to them two predictive maintenance strategies based on the developed prognostic model and applied to fatigue damage propagation in fuselage panels. In the research of Lie et al. [31] a preventive maintenance policy is also proposed for the single-unit system failures which have sudden shocks and internal deterioration. The emphasize of the study was to minimize the expected cost per unit time defining the optimal preventive replacement interval, inspection interval, and the number of inspections. Another study was done by Sukhwa et al. [32] For designing and optimizing maintenance service policy. In another study Fabrian and Luis [33] has suggested a method to definite maintenance intervals to those of similar systems under development, and this method has been applied in an aircraft manufacturing company using the current operation database. Michail et al. [34] did one research and developed an aircraft maintenance planning optimization tool and its application to an aircraft component. In another important research by Shey-Huei et al. [35] has suggested the optimal preventive maintenance policy for multi-state systems.

The maintenance service policy ends when the arrangement period reaches time (usage level (H)). The system's renewal is not involved. The preventive and corrective maintenances are under this policy. At a constant interval of time (τ), the system should go for periodically preventive maintenance under this policy. At each failure within successive preventive maintenances, the system should go for minimally repaired. A complicated repairable system with a long life is perfect for this type of service arrangement.

The important Assumptions of the Maintenance Service Policy are:

- The successive failure and random actions are mutually independent.
- The successive failures are said to be known on the parameters of distributions.
- Only minimal repairs are conducted when the repairs were completed in maintenance.
- The Servicing activity held responsible to restores life to a bit.
- The repairs times are minor to compare to the item's life.
- The age renovation is stable even after each preventive maintenance.
- The unit amount of minimal repairs has a constant average between the unit amount of preventive maintenances and preventive maintenances.

The expected cost of maintenance service policy is the sum of the total sum of expected costs, all minimal repairs, and the expected costs of all planned preventive maintenance over the policy's period.

We can get the expected cost of maintenance service per unit time by dividing the expected total cost by the duration of service policy.

According to Rahman [30], the expected cost of maintenance service policy can be defined in the following steps

- Taking the equivalent length of the preventive maintenance period (τ), the expected cost of minimal repairs among preventive maintenance for GIED is given as
$$E(CT_{mr}) = CT_{mr}\left[\sum_{f=0}^{N-1}\int_{f\tau}^{(f+1)\tau} H(t-p\kappa)dt\right] = CT_{mr}\left[\sum_{f=0}^{N-1}\eta\ln(1-e^{-\mu/\tau})\right] \tag{16}$$
- The expected cost of preventive maintenance is given as
$$E(CT_{pm}) = N * CT_{pm} \tag{17}$$

Here, the arrangement is periodically maintained at Nth preventive maintenance.

- The total expected cost per unit time $CT(\tau, N)$ for GIED is given as

$$E(CT(\tau,N)) = \frac{E(CT_{mr}) + E(CT_{pm})}{H} = \frac{CT_{mr}\left[\sum_{f=0}^{N-1}\eta\ln(1-e^{-\mu/\tau})\right] + E(CT_{pm})}{H} \tag{18}$$

where $H = N \times \tau$

V. Simulation Study and Results

In this segment, we carry out a simulation study to check the performance of the estimators having GIED distribution using progressive Type-II censoring scheme. This study is prepared by Monte Carlo Simulation technique by R-Software. To test out the performance of estimators, the means square error (MSEs) and absolute relative bias (RAB) are estimated. The key steps for the study are

(i) The total sample m is divided into two parts, m_1 and m_2.
where $m_1 = m\pi$ and $m_2 = m(1-\pi)$

- Generate random samples of size $m_1(t_{1,1} < t_{2,2} < ... < t_{m_1,1})$ and $m_2(t_{2,1} < t_{2,2} < ... < t_{m_2,2})$ under normal and accelerated conditions, respectively, from GIED distribution by the inverse CDF method.
- Generate 1000 random samples of sizes 35, 70, and 105 and specify the following values.
Case (I) $(\mu = 0.9, \eta = 0.9, \beta = 2.2)$, Case (II) $(\mu = 0.7, \eta = 0.7, \beta = 2.5)$
Case (III) $(\mu = 0.6, \eta = 1.2, \beta = 2.2)$, Case (IV) $(\mu = 0.5, \eta = 0.9, \beta = 2.5)$
- The distribution parameters and acceleration factor are achieved for each sample and all set of parameters.
- By equation (15), for confidence levels $\alpha = 95\%, 99\%$, the two-sided confidence limits are obtained for parameters μ, η and β.
- The Newton Raphson technique is used to solve all non-linear equations.
- The above steps are replicated 1000 times with different values of parameters.
- From equations (16-18), the expected cost of maintenance service policy is estimated for preventive maintenance, total costs, minimal repairs, and expected cost rate, and the length of the maintenance service policy (H) is chosen as three years.
- At the usual cost $(CT_{pm} = 800)$, preventive maintenance be every four months $(\tau = 0.30)$. If there are failures linking two consecutive preventive maintenance, the minimal repairs will be completed at an average cost $(CT_{mr} = 650)$. Finally, the expected cost of preventive maintenance is 23360, $E(C_{pm}) = 23360$.

Table 1: The Biases and MSEs with different size of samples for progressive Type-II censoring

m	Parameters	Case I $(\mu = 0.9, \eta = 0.9, \beta = 2.2)$			Case II $(\mu = 0.7, \eta = 0.7, \beta = 2.5)$		
		Estimates	RAB	MSE	Estimates	RAB	MSE
35	μ	1.227	0.908	1.192	1.368	1.402	2.002
	η	0.409	0.504	1.063	0.208	0.608	1.155
	β	1.872	0.394	1.531	1.887	2.094	2.360
70	μ	1.098	0.611	0.969	1.109	1.318	1.559
	η	0.502	0.394	0.744	1.009	0.373	1.024
	β	1.998	0.576	0.902	2.665	1.670	2.006
105	μ	2.082	0.299	0.033	1.401	1.703	1.133
	η	0.280	0.155	0.214	0.218	0.099	0.715
	β	2.498	0.221	0.604	1.977	1.137	0.883

Table 2: The Biases and MSEs with different size of samples for progressive Type-II censoring

m	Parameters	Case III $(\mu = 0.6, \eta = 1.2, \beta = 2.2)$			Case IV $(\mu = 0.5, \eta = 0.9, \beta = 2.5)$		
		Estimates	RAB	MSE	Estimates	RAB	MSE
35	μ	1.809	1.767	2.092	2.001	0.969	2.004
	η	2.498	1.091	1.869	2.550	1.308	1.908
	β	1.005	1.351	1.531	1.990	1.782	2.400
70	μ	1.676	1.029	1.760	2.413	0.308	1.610
	η	2.012	0.762	1.444	1.610	0.810	1.042
	β	2.001	0.433	1.202	1.910	1.063	1.204
105	μ	1.302	0.650	1.033	2.915	0.344	0.772
	η	1.643	1.190	0.914	1.709	0.142	0.724
	β	0.985	0.125	8.104	2.809	0.771	1.771

Table 3: At Confidence Level $\alpha = 95\%, 99\%$, the Confidence Limits of Estimates at Various Size of Samples

m	Parameters	Case I : $(\mu = 0.9, \eta = 0.9, \beta = 2.2)$				σ	Case I I: $(\mu = 0.7, \eta = 0.7, \beta = 2.5)$				σ
		CI, $z = 1.96$		CI, $z = 2.58$			CI, $z = 1.96$		CI, $z = 2.58$		
		Lower Bound	Upper Bound	Lower Bound	Upper Bound		Lower Bound	Upper Bound	Lower Bound	Upper Bound	
35	μ	1.25	2.34	1.13	1.81	0.18	0.83	2.19	0.78	1.69	0.22
	η	1.42	2.51	0.97	1.60	0.11	1.40	2.54	1.09	1.66	0.48
	β	0.96	2.11	0.92	1.51	0.72	1.07	2.15	0.60	1.52	0.26
70	μ	1.04	2.05	0.85	1.34	0.10	1.15	2.03	1.16	1.62	0.19
	η	0.91	1.83	0.69	1.26	0.38	0.99	1.80	0.83	1.40	0.28
	β	0.99	1.54	0.93	1.29	0.59	0.81	1.47	0.65	1.45	0.40
105	μ	0.85	1.66	0.82	1.18	0.09	0.70	1.35	0.90	1.43	0.19
	η	0.69	1.32	0.77	1.10	0.28	0.77	1.23	0.81	1.16	0.36
	β	0.66	0.96	0.84	1.06	0.33	0.54	0.82	0.49	0.77	0.61

Table 4: At Confidence Level $\alpha = 95\%, 99\%$, the Confidence Limits of Estimates at Various Size of Samples

m	Parameters	Case III: $(\mu = 0.6, \eta = 1.2, \beta = 2.2)$				σ	Case I V: $(\mu = 0.5, \eta = 0.9, \beta = 2.5)$				σ
		CI, $z = 1.96$		CI, $z = 2.58$			CI, $z = 1.96$		CI, $z = 2.58$		
		Lower Bound	Upper Bound	Lower Bound	Upper Bound		Lower Bound	Upper Bound	Lower Bound	Upper Bound	
35	μ	1.04	2.11	0.91	1.90	0.09	0.78	2.16	0.97	1.91	0.11
	η	1.69	2.65	1.11	1.71	0.19	1.29	2.11	1.11	1.80	0.38
	β	1.73	2.43	0.85	1.63	0.32	1.09	1.88	0.78	1.63	0.31
70	μ	0.79	1.65	1.00	1.44	0.15	1.25	1.93	1.05	1.49	0.10
	η	0.71	1.56	0.61	1.03	0.40	0.96	1.52	0.72	1.29	0.20
	β	1.29	1.44	0.77	1.33	0.22	0.72	1.36	0.59	0.90	0.32
105	μ	0.55	1.09	0.65	0.90	0.14	0.49	0.85	0.62	0.87	0.16
	η	0.69	1.12	0.72	0.94	0.21	0.78	1.53	0.45	0.73	0.32
	β	0.79	0.99	0.85	1.26	0.28	0.76	0.99	0.66	1.16	0.22

Table 5: Expected cost rate, total cost, minimal repair time, and its confidence level for GIED

m	Minimal repair cost			Total cost			Cost rate		
	$E(CT_{mr})$	Lower Bound	Upper Bound	$E(CT_{total})$	Lower Bound	Upper Bound	$E(CT(\tau, N))$	Lower Bound	Upper Bound
Case –I $(\mu = 0.9, \eta = 0.9, \beta = 2.2)$									
35	274558.9	3872.8	8054.4	98192.3	3912.4	4562.3	96832.1	3421.6	49821.5
70	221852.0	5732.1	9023.7	84632.7	4132.5	7099.4	71093.7	5320.2	82313.4
105	19277.0	81432.4	9793.5	80981.4	2987.8	3983.3	65421.2	4542.7	69874.2
Case-II $(\mu = 0.7, \eta = 0.7, \beta = 2.5)$									
35	89198.3	45020.7	61345.8	62176.6	2970.9	36876.9	32790.6	2076.9	45786.9
70	71612.0	9112.4	11935.0	64830.3	4765.8	7876.8	25595.4	4765.2	7176.2
105	45423.8	4023.7	5839.5	59763.9	9762.3	10965.3	20954.2	4859.5	77654.5
Case-III $(\mu = 0.6, \eta = 1.2, \beta = 2.2)$									
35	53909.9	23754.7	3321.9	39654.9	3432.8	4876.8	16593.6	6543.9	9654.9
70	39976.4	44876.6	6654.4	35937.5	4325.8	7354.9	15987.8	6543.5	106543.9
105	41287.4	8565.5	106549.4	30654.1	5435.4	6543.1	29043.6	3876.3	5765.8

VI. Conclusion

This study proposed a partially accelerated life test plan under constant stress and estimating costs of maintenance service policy using the progressive Type-II censoring scheme for the Generalized Inverted Exponential distribution. The following assumptions are:

- As the sample size increases, the values of MSEs and RABs reduce and confidence intervals become narrower or the confidence interval size decreases. Thus, the MLEs have cheering statistical outcomes. We can also observe that the numerical outcomes and theoretical conclusions support each other, and our suppositions are also satisfied. (see, Table 1,2,3 and 4).

- The model parameters and costs of maintenance service policy have direct relationship for the Generalized Inverted Exponential distribution. (see Table 5)
- Also, maintenance service and sample sizes have inverse relationship. (see Table 5)

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Dus Transformation of Inverse Weibull Distribution: An Upside-Down Failure Rate Model

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Abstract

A new upside-down bathtub shaped failure rate distribution, DUS Inverse Weibull (DUS-IW) distribution is proposed and its properties are studied. The DUS-IW distribution has upside-down bathtub shaped and decreasing failure rate functions. Moments, moment generating function, characteristic function, quantiles, etc. are derived. Estimation of parameters of the distribution is performed via maximum likelihood method. Reliability of single component and multi component stress-strength models are derived. A simulation study is performed for validating the estimates of the model parameters. DUS-IW distribution is applied to two real data sets and found that DUS-IW distribution is a better fit than other well-known distributions.

Keywords: upside-down bathtub shaped failure rate, reliability, stress-strength

I. Introduction

The statistical analysis of lifetime data is of importance in various fields of applied science especially in reliability, biomedical, engineering, social sciences, etc. To explain reallife phenomenon, there are a lot of lifetime distributions. Among all of them, some frequently used distributions are Exponential, Gamma, Weibull, Lognormal, etc. Each has their own merits and demerits due to their flexibility of shapes and failure rates like increasing, decreasing or constant failure rate, depends on the nature of distributions. A comprehensive account of lifetime models and the methods for analyzing them are given in [15]. Weibull distribution is one of the most widely used distributions from the exponential family and is used in the reliability engineering, hydrological, energy studies, etc. There exist many variations of it using different transformations, such as Inverse Weibull (IW) distribution. Similar to Weibull distribution, IW distribution has its significance and role in real life phenomenon. The complementary Weibull and reciprocal Weibull (see [8] and [19]) are exactly the IW distribution. The IW distributionhas received some attention in the literature and is another lifetime probability distribution which can be used in the reliability engineering discipline. The IW distribution can be used to model a variety of failure characteristics such as infant mortality, useful life and wear-out periods. Extensive work has been doneon the IW distribution, see [12], [3], [4] and [16]. Along with these, IW distribution is used as an alternative to Weibull distribution to model wind speed data, [1].

Generalized Inverse Weibull distribution is another extension of Weibull distribution, [10]. Theoretical analysis of IW distribution is found to be quite interesting as well, [13].

In statistical literature, there are various methods to propose a new distribution by using some baseline distribution. A transformation gives a more accurate distribution with easy computation and interpretation as it contains no new parameters other than parameters involved in the baseline distribution. Exponential transformed Lindley distribution is applied to Yarn data, [18]. DUS transformation is a method which has been introducedto get a new distribution by using Exponential distribution as baseline distribution, [14], with application to survival data analysis. DUS transformation of Exponential distribution used for the problem of estimation of the parameter based on upper record values, [22]. An upside-down bathtub shaped failure rate model using DUS transformation on the Lomax distribution as baseline distribution is found to be a better fit than existing distributions, [7]. A new lifetime distribution based on the DUS transformation by using Weibull distribution as the baseline distribution is another choice of existing models, [11]. In this paper, an attempt has been made here to obtain a new distribution using the DUS transformation with IW distribution as baseline distribution, to study upside down bathtub data.

This paper is organized as follows. In Section II, the details of DUS Transformation are given. In section III, the probability density function, cumulative distribution function, survival function and failure rate function of the DUS transformation of IW distribution is given. Shapes of the probability density function andfailure rate function are discussed in this Section IV. Statistical properties including moments, moment generating function (mgf), characteristic function (cf) and quantile function of the proposed distribution are discussed in Section V. Stress-strength reliability evaluation is discussed in Section VI. Estimation of the parameters using method of maximum likelihood is discussed in SectionVII. The results are given in Section VIII which includes a simulation study that is conducted to validate the estimatesand a real data analysis which is used to illustrate the usefulness of the proposed distribution. Final conclusions and discussions are given in Section IX.

II. DUS Transformation

Let f(x) and F(x) be the probability density function (pdf) and cumulative distribution function (cdf) of some baseline distribution, then the pdf g(x) of the distribution obtained by DUS Transformation of the baseline distribution is given by

$$g(x) = \frac{1}{e-1} f(x) e^{F(x)}. \tag{1}$$

If the pdf g(x) of the distribution obtained by DUS Transformation is given by (1), then the corresponding cdf, survival function and failure rate function are given by

$$G(x) = \frac{1}{e-1}\left[e^{F(x)} - 1\right] \tag{2}$$

$$S(x) = \frac{1}{e-1}\left[e - e^{F(x)}\right] \tag{3}$$

and

$$h(x) = \frac{1}{e-e^{F(x)}} f(x) e^{F(x)} \tag{4}$$

respectively.

III. DUS-IW(α,β) Distribution

Let Y be a random variable from the two-parameter Weibull distribution with the shape parameter α and the scale parameter β. Its pdf is given by

$$f(y) = \frac{\alpha}{\beta}\left(\frac{y}{\beta}\right)^{\alpha-1} e^{-\left(\frac{y}{\beta}\right)^{\alpha}}, \text{ y>0, } \alpha\text{,β>0.}$$

By using the reciprocal of the random variable Y, i.e., X=1/Y, the distribution of X is called Inverse Weibull distribution with the shape parameter α and the scale parameter β, denoted by IW(α,β).

The pdf and the cdf of the IW distribution are given below

$$f(x) = \frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{-(\alpha+1)} e^{-\left(\frac{x}{\beta}\right)^{-\alpha}}, \text{ x>0, } \alpha, \beta\text{>0} \tag{5}$$

and

$$F(x) = e^{-\left(\frac{x}{\beta}\right)^{-\alpha}}, \text{x>0, } \alpha, \beta\text{>0} \tag{6}$$

respectively.

Let g(x) be the pdf obtained by DUS transformation (1), corresponding to the baseline pdf (5) and cdf (6), then

$$g(x) = \frac{1}{e-1}\frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{-(\alpha+1)} \exp\left\{-\left(\frac{x}{\beta}\right)^{-\alpha} + e^{-\left(\frac{x}{\beta}\right)^{-\alpha}}\right\}, \text{ x>0, } \alpha, \beta\text{>0.} \tag{7}$$

For simplicity, we call the distribution having pdf (7) as DUS transformation of IW(α,β) distribution and denote it as DUS-IW(α,β)distribution.

The cdf of DUS-IW(α,β) distribution is obtained using (2) and is given by,

$$G(x) = \frac{1}{e-1}\left\{e^{e^{-\left(\frac{x}{\beta}\right)^{-\alpha}}} - 1\right\}, \text{ x>0, } \alpha, \beta\text{>0.} \tag{8}$$

I. Survival function and Failure rate function

The survival function S(x), using (3), is given by,

$$S(x) = \frac{1}{e-1}\left\{e - e^{e^{-\left(\frac{x}{\beta}\right)^{-\alpha}}}\right\}, \text{ x>0, } \alpha, \beta\text{>0.} \tag{9}$$

The failure rate function of DUS-IW(α,β) distribution, using (4), is given by,

$$h(x) = \frac{1}{e - e^{e^{-\left(\frac{x}{\beta}\right)^{-\alpha}}}}\left(\frac{\alpha}{\beta}\right)\left(\frac{x}{\beta}\right)^{-(\alpha+1)} e^{\left\{-\left(\frac{x}{\beta}\right)^{-\alpha} + e^{-\left(\frac{x}{\beta}\right)^{-\alpha}}\right\}}, \text{ x>0, } \alpha, \beta\text{>0.} \tag{10}$$

IV. Shapes

It can be seen that the pdf of DUS-IW(α, β) distribution has the shape properties, specifically,

α < 1, β > 1 ⇒ g(x) is decreasing,
α < 1, β < 1 ⇒ g(x) is unimodal,
α > 1, β < 1 ⇒ g(x) is unimodal,
α > 1, β > 1 ⇒ g(x) is unimodal.

Mode of the distribution can be found as a solution of the equation,

$$\frac{d}{dx}\log g(x) = 0.$$

By substituting the pdf, we get

$$\frac{d}{dx}\log\left[\frac{1}{e-1}\frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{-(\alpha+1)} \exp\left\{-\left(\frac{x}{\beta}\right)^{-\alpha} + e^{-\left(\frac{x}{\beta}\right)^{-\alpha}}\right\}\right] = 0.$$

Simplifying the equation, we get,

$$\frac{-(\alpha+1)}{x} + \beta^{\alpha}\alpha x^{-(\alpha+1)}\left[1 + e^{-\left(\frac{x}{\beta}\right)^{-\alpha}}\right] = 0. \tag{11}$$

On solving (11) numerically, we get the mode of the distribution. The plots of pdf and failure rate function of DUS-IW(α,β) distribution for different values of α and β are shown the Figures 1 and 2 respectively.

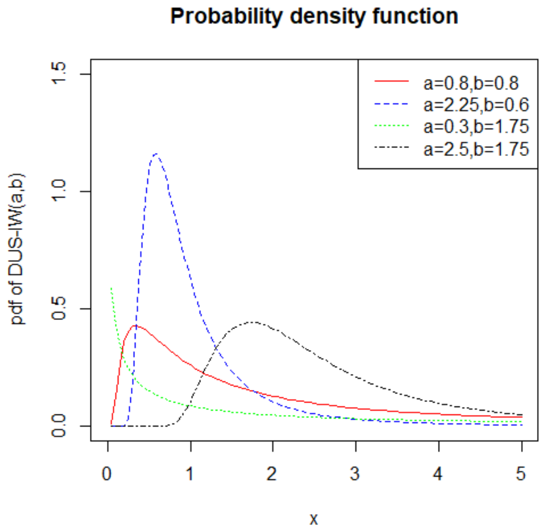

Figure 1: *Pdf of DUS-IW(α,β) for (0.8,0.8),(2.25,0.6), (0.3,1.75) and (2.5,1.75)*

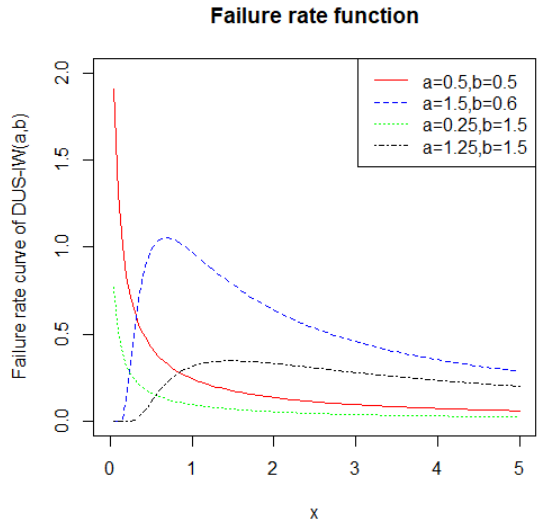

Figure 2: *Failure rate function of DUS-IW(α,β) for (0.5,0.5),(1.5,0.6),(0.25,1.5) and (1.25,1.5)*

Figure 2 shows the graph of failure rate function of DUS-IW(α, β) distribution for the parameter values (0.5,0.5), (1.5,0.6), (0.25,1.5) and (1.25,1.5). Failure rate function h(x) is monotonically decreasing for α <1 and upside-down for α >1.

V. Statistical Properties

I. Moments

Let X be a random variable with its pdf given by (7), then its rth raw moment is obtained by

$$\mu_r' = E(X^r)$$

$$= \int_0^\infty x^r \left[\frac{1}{e-1}\frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{-(\alpha+1)} e^{-\left(\frac{x}{\beta}\right)^{-\alpha}} e^{e^{-\left(\frac{x}{\beta}\right)^{-\alpha}}}\right] dx$$

$$= \frac{1}{e-1}\frac{\alpha}{\beta}\int_0^\infty x^r \left(\frac{x}{\beta}\right)^{-(\alpha+1)} e^{-\left(\frac{x}{\beta}\right)^{-\alpha}} \sum_{m=0}^{\infty}\frac{\left[e^{-\left(\frac{x}{\beta}\right)^{-\alpha}}\right]^m}{m!} dx$$

$$= \sum_{m=0}^{\infty}\frac{1}{e-1}\frac{\alpha}{\beta}\frac{\beta^{\alpha+1}}{m!}\int_0^\infty x^r\,(x)^{-(\alpha+1)} e^{-(m+1)\left(\frac{x}{\beta}\right)^{-\alpha}} dx.$$

Put$u = x^{-\alpha}$
Then, $du = -\alpha x^{-(\alpha+1)}dx$,
implies,

$$\mu_r' = \sum_{m=0}^{\infty} \frac{\beta^\alpha}{m!\,(e-1)} \int_0^\infty \left(u^{-\frac{1}{\alpha}}\right)^r e^{-(m+1)(\beta)^\alpha u} du$$
$$= \sum_{m=0}^{\infty} \frac{\beta^\alpha}{m!\,(e-1)} \int_0^\infty u^{-\frac{r}{\alpha}+1-1}\, e^{-(m+1)(\beta)^\alpha u} du$$
$$= \sum_{m=0}^{\infty} \frac{\beta^\alpha}{m!\,(e-1)} \frac{\Gamma(-\frac{r}{\alpha}+1)}{((m+1)\beta^\alpha)^{-\frac{r}{\alpha}+1}}$$
$$= \sum_{m=0}^{\infty} \frac{\Gamma(-\frac{r}{\alpha}+1)}{(m+1)!\,(e-1)((m+1)\beta^\alpha)^{-\frac{r}{\alpha}}}$$
$$= \sum_{m=0}^{\infty} \frac{\Gamma(-\frac{r}{\alpha}+1)(m+1)^{\frac{r}{\alpha}}\beta^r}{(m+1)!\,(e-1)}.$$

Therefore,

$$\mu_r' = \sum_{m=0}^{\infty} \frac{\Gamma(-\frac{r}{\alpha}+1)(m+1)^{\frac{r}{\alpha}}\beta^r}{(m+1)!(e-1)} \tag{12}$$

Hence, the r[th] raw moment.
Putting r=1 in (12), we get the 1[st] raw moment (mean) and is given by,

$$\mu_1' = E(X)$$
$$\mu_1' = \sum_{m=0}^{\infty} \frac{\Gamma(-\frac{1}{\alpha}+1)(m+1)^{\frac{1}{\alpha}}\beta}{(m+1)!(e-1)}. \tag{13}$$

Putting r=2 in (12), we get the 2[nd] raw moment and is given by,

$$\mu_2' = E(X^2)$$
$$\mu_2' = \sum_{m=0}^{\infty} \frac{\Gamma(-\frac{2}{\alpha}+1)(m+1)^{\frac{2}{\alpha}}\beta^2}{(m+1)!\,(e-1)}.$$

Then, the variance of the random variable X is given by,

$$V(X) = E(X^2) - \left(E(X)\right)^2$$
$$V(X) = \sum_{m=0}^{\infty} \frac{\Gamma(-\frac{2}{\alpha}+1)(m+1)^{\frac{2}{\alpha}}\beta^2}{(m+1)!(e-1)} - \left(\sum_{m=0}^{\infty} \frac{\Gamma(-\frac{1}{\alpha}+1)(m+1)^{\frac{1}{\alpha}}\beta}{(m+1)!(e-1)}\right)^2. \tag{14}$$

II. Moment Generating Function

The mgf of DUS-IW(α,β) distribution is,

$$M_X(t) = E(e^{tX}) = \int_0^\infty e^{tx}\, g(x)dx$$
$$= \int_0^\infty e^{tx} \left[\frac{1}{e-1}\frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{-(\alpha+1)} e^{-\left(\frac{x}{\beta}\right)^{-\alpha}} e^{e^{-\left(\frac{x}{\beta}\right)^{-\alpha}}}\right] dx$$
$$= \frac{1}{e-1}\frac{\alpha}{\beta}\beta^{(\alpha+1)} \int_0^\infty e^{tx}\,(x)^{-(\alpha+1)} e^{-\left(\frac{x}{\beta}\right)^{-\alpha}} \sum_{m=0}^{\infty} \frac{\left[e^{-\left(\frac{x}{\beta}\right)^{-\alpha}}\right]^m}{m!} dx$$
$$= \sum_{m=0}^{\infty} \frac{1}{e-1}\frac{\alpha\beta^\alpha}{m!} \int_0^\infty (x)^{-(\alpha+1)} e^{-(m+1)\left(\frac{x}{\beta}\right)^{-\alpha}} \sum_{n=0}^{\infty} \frac{[tx]^n}{n!} dx$$
$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{\alpha\beta^\alpha}{(e-1)m!\,n!} \int_0^\infty (x)^{-(\alpha+1)} e^{-(m+1)\left(\frac{x}{\beta}\right)^{-\alpha}} (tx)^n dx.$$

Put$u = x^{-\alpha}$
Then, $du = -\alpha x^{-(\alpha+1)}dx$
implies,

$$M_X(t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\beta^{\alpha}}{(e-1)m!\,n!}\int_0^{\infty}\left(\frac{1}{u}\right)^{\frac{n}{\alpha}} e^{-(m+1)(\beta)^{\alpha}u}(t)^n du$$
$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\beta^{\alpha}(t)^n}{(e-1)m!\,n!}\int_0^{\infty}(u)^{-\frac{n}{\alpha}+1-1}e^{-(m+1)(\beta)^{\alpha}u}du$$
$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(t)^n\Gamma(-\frac{n}{\alpha}+1)(m+1)^{\frac{n}{\alpha}}\beta^n}{(e-1)(m+1)!\,n!}.$$

Therefore,

$$M_X(t) = \textstyle\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(t)^n\Gamma(-\frac{n}{\alpha}+1)(m+1)^{\frac{n}{\alpha}}\beta^n}{(e-1)(m+1)!n!}. \quad (15)$$

III. Characteristic Function

The cf, $\phi_X(t)$ is given by,

$$\phi_X(t) = E(e^{itX}) = \int_0^{\infty} e^{itx}\, g(x).\, dx$$
$$= \int_0^{\infty} e^{itx}\left[\frac{1}{e-1}\frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{-(\alpha+1)} e^{-\left(\frac{x}{\beta}\right)^{-\alpha}} e^{e^{-\left(\frac{x}{\beta}\right)^{-\alpha}}}\right]dx$$
$$= \frac{1}{e-1}\frac{\alpha}{\beta}\beta^{(\alpha+1)}\int_0^{\infty} e^{itx}\,(x)^{-(\alpha+1)}e^{-\left(\frac{x}{\beta}\right)^{-\alpha}}\sum_{m=0}^{\infty}\frac{\left[e^{-\left(\frac{x}{\beta}\right)^{-\alpha}}\right]^m}{m!}dx$$
$$= \sum_{m=0}^{\infty}\frac{1}{e-1}\frac{\alpha\beta^{\alpha}}{m!}\int_0^{\infty}(x)^{-(\alpha+1)}e^{-(m+1)\left(\frac{x}{\beta}\right)^{-\alpha}}\sum_{n=0}^{\infty}\frac{[itx]^n}{n!}dx$$
$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\alpha\beta^{\alpha}}{(e-1)m!\,n!}\int_0^{\infty}(x)^{-(\alpha+1)}e^{-(m+1)\left(\frac{x}{\beta}\right)^{-\alpha}}(itx)^n dx$$

Put$u = x^{-\alpha}$.
Then, $du = -\alpha x^{-(\alpha+1)}dx$,
implies,

$$\phi_X(t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\beta^{\alpha}}{(e-1)m!\,n!}\int_0^{\infty}\left(\frac{1}{u}\right)^{\frac{n}{\alpha}} e^{-(m+1)(\beta)^{\alpha}u}(it)^n du$$
$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\beta^{\alpha}(it)^n}{(e-1)m!\,n!}\int_0^{\infty}(u)^{-\frac{n}{\alpha}+1-1}e^{-(m+1)(\beta)^{\alpha}u}du$$
$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(it)^n\Gamma(-\frac{n}{\alpha}+1)(m+1)^{\frac{n}{\alpha}}\beta^n}{(e-1)(m+1)!\,n!}.$$

Therefore,

$$\phi_X(t) = \textstyle\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(it)^n\Gamma(-\frac{n}{\alpha}+1)(m+1)^{\frac{n}{\alpha}}\beta^n}{(e-1)(m+1)!n!}. \quad (16)$$

IV. Quantile Function

The p^{th} quantile function, denoted by Q(p) of DUS-IW(α,β) distribution is obtained by solving F(Q(p))=p, where 0<p<1. That is,

$$\frac{1}{e-1}\left\{e^{e^{-\left(\frac{Q(p)}{\beta}\right)^{-\alpha}}}-1\right\} = p$$

and the p[th] quantile function is,

$$Q(p) = \frac{-\beta}{\{\log(\log(1+p(e-1)))\}^{\frac{1}{\alpha}}}. \tag{17}$$

Median (2[nd] quartile) of DUS-IW(α,β) distribution is obtained by substituting p = 1/2 in (17), i.e.,

$$Q_2 = \frac{-\beta}{\left\{\log(\log(1+\frac{1}{2}(e-1)))\right\}^{\frac{1}{\alpha}}}. \tag{18}$$

Setting p = 1/4 in (17), we get the 1[st] quartile of DUS-IW(α,β) distribution as follows,

$$Q_1 = \frac{-\beta}{\left\{\log(\log(1+\frac{1}{4}(e-1)))\right\}^{\frac{1}{\alpha}}}. \tag{19}$$

Setting p = 3/4 in (17), we get the 3[rd] quartile of DUS-IW(α,β)distribution as follows,

$$Q_3 = \frac{-\beta}{\left\{\log(\log(1+\frac{3}{4}(e-1)))\right\}^{\frac{1}{\alpha}}}. \tag{20}$$

A random sample X with DUS-IW(α,β) distribution can be simulated using

$$X = \frac{-\beta}{\{\log(\log(1+U(e-1)))\}^{\frac{1}{\alpha}}}, \text{ where } U \sim U(0,1) \tag{21}$$

VI. Stress-Strength Reliability

Stress-Strength model has a significant role in reliability engineering. The stress-strength reliability is defined as the probability that the random strength greater than the random stress of a component or system, [6]. A study on point estimation of the stress-strength reliability parameter for parallel system with independent and non-identical components can be seen in literature, [20]. In this section, reliability estimation of single component stress-strength model (SSS) and multicomponent stress-strength model (MSS) are considered.

I. Single Component Stress-Strength Reliability

Here we consider the reliability of SSS based on two independent random variables X and Y, where X represents the 'strength' and Y represents the 'stress'. Suppose X and Y have the DUS-IW(α, β_1) and DUS-IW (α, β_2) distributions respectively, then the system reliability R = P(Y < X) is

$$R = P(Y < X)$$
$$= \int_0^\infty g_X(x) G_Y(x).\,dx$$
$$= \int_0^\infty \frac{1}{e-1}\frac{\alpha}{\beta_1}\left(\frac{x}{\beta_1}\right)^{-(\alpha+1)} e^{-\left(\frac{x}{\beta_1}\right)^{-\alpha}} e^{e^{-\left(\frac{x}{\beta_1}\right)^{-\alpha}}} \frac{1}{e-1}\left\{e^{e^{-\left(\frac{x}{\beta_2}\right)^{-\alpha}}} - 1\right\} dx$$
$$= \frac{1}{(e-1)^2}\frac{\alpha}{\beta_1}\int_0^\infty \left(\frac{x}{\beta_1}\right)^{-(\alpha+1)} e^{-\left(\frac{x}{\beta_1}\right)^{-\alpha}} e^{e^{-\left(\frac{x}{\beta_1}\right)^{-\alpha}}} \left\{e^{e^{-\left(\frac{x}{\beta_2}\right)^{-\alpha}}} - 1\right\}.\,dx$$
$$= \frac{1}{(e-1)^2}\frac{\alpha}{\beta_1}\int_0^\infty \left(\frac{x}{\beta_1}\right)^{-(\alpha+1)} \left[\exp\left(-\left(\frac{x}{\beta_1}\right)^{-\alpha} + e^{-\left(\frac{x}{\beta_1}\right)^{-\alpha}} + e^{-\left(\frac{x}{\beta_2}\right)^{-\alpha}}\right) - \exp\left(-\left(\frac{x}{\beta_1}\right)^{-\alpha} + e^{-\left(\frac{x}{\beta_1}\right)^{-\alpha}}\right)\right] dx$$
$$= \frac{1}{(e-1)^2}\frac{\alpha}{\beta_1}\int_0^\infty \left(\frac{x}{\beta_1}\right)^{-(\alpha+1)} e^{-\left(\frac{x}{\beta_1}\right)^{-\alpha}} e^{e^{-\left(\frac{x}{\beta_1}\right)^{-\alpha}}} e^{e^{-\left(\frac{x}{\beta_2}\right)^{-\alpha}}} dx - \frac{1}{e-1}$$
$$= I_1 - \frac{1}{e-1}$$

where $I_1 = \frac{1}{(e-1)^2}\frac{\alpha}{\beta_1}\int_0^\infty \left(\frac{x}{\beta_1}\right)^{-(\alpha+1)} e^{-\left(\frac{x}{\beta_1}\right)^{-\alpha}} e^{e^{-\left(\frac{x}{\beta_1}\right)^{-\alpha}}} e^{e^{-\left(\frac{x}{\beta_2}\right)^{-\alpha}}} dx$

$$= \frac{1}{(e-1)^2}\frac{\alpha}{\beta_1}\int_0^\infty \left(\frac{x}{\beta_1}\right)^{-(\alpha+1)} e^{-\left(\frac{x}{\beta_1}\right)^{-\alpha}} \sum_{m=0}^{\infty}\frac{\left[e^{-\left(\frac{x}{\beta_1}\right)^{-\alpha}}\right]^m}{m!} \sum_{n=0}^{\infty}\frac{\left[e^{-\left(\frac{x}{\beta_2}\right)^{-\alpha}}\right]^n}{n!} dx$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{m!\,n!}\frac{1}{(e-1)^2}\frac{\alpha}{\beta_1}\int_0^{\infty}\left(\frac{x}{\beta_1}\right)^{-(\alpha+1)} e^{-(m+1)\left(\frac{x}{\beta_1}\right)^{-\alpha}} e^{-n\left(\frac{x}{\beta_2}\right)^{-\alpha}} dx$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{m!\,n!}\frac{1}{(e-1)^2}\frac{\alpha}{\beta_1}\int_0^{\infty}\left(\frac{x}{\beta_1}\right)^{-(\alpha+1)} e^{-(m+1)\left(\frac{x}{\beta_1}\right)^{-\alpha}} \sum_{p=0}^{\infty}\frac{(-1)^p n^p \left(\frac{x}{\beta_2}\right)^{-\alpha p}}{p!} dx$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(-1)^p n^p}{m!\,n!\,p!}\frac{1}{(e-1)^2}\frac{\alpha}{\beta_1}\int_0^{\infty}\left(\frac{x}{\beta_1}\right)^{-(\alpha+1)} e^{-(m+1)\left(\frac{x}{\beta_1}\right)^{-\alpha}} \left(\frac{x}{\beta_2}\right)^{-\alpha p} dx$$

Put$u = x^{-\alpha}$
Then, $du = -\alpha x^{-(\alpha+1)}dx$,
implies,

$$I_1 = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(-1)^p n^p}{m!\,n!\,p!}\frac{1}{(e-1)^2}\frac{\alpha}{\beta_1}\int_0^{\infty}\frac{{\beta_1}^{\alpha+1}{\beta_2}^{\alpha p}}{\alpha} e^{-(m+1){\beta_1}^{\alpha}u}.u^p.du$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(-1)^p n^p}{m!\,n!\,p!}\frac{1}{(e-1)^2}\frac{1}{\beta_1}{\beta_1}^{\alpha+1}{\beta_2}^{\alpha p}\int_0^{\infty} e^{-(m+1){\beta_1}^{\alpha}u}.u^p.du$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(-1)^p n^p}{m!\,n!\,p!}\frac{1}{(e-1)^2}{\beta_1}^{\alpha}{\beta_2}^{\alpha p}\frac{\Gamma(p+1)}{((m+1){\beta_1}^{\alpha})^{p+1}}$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(-1)^p n^p}{m!\,n!\,p!}\frac{1}{(e-1)^2}\frac{{\beta_1}^{\alpha}{\beta_2}^{\alpha p}}{{\beta_1}^{\alpha p}{\beta_1}^{\alpha}}\frac{\Gamma(p+1)}{((m+1))^{p+1}}$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(-1)^p n^p}{m!\,n!\,p!}\frac{1}{(e-1)^2}\left(\frac{\beta_2}{\beta_1}\right)^{\alpha p}\frac{\Gamma(p+1)}{((m+1))^{p+1}}$$

Therefore,

$$R = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(-1)^p n^p}{m!n!p!}\frac{1}{(e-1)^2}\left(\frac{\beta_2}{\beta_1}\right)^{\alpha p}\frac{\Gamma(p+1)}{((m+1))^{p+1}} - \frac{1}{e-1} \tag{22}$$

II. Multicomponent Stress-Strength Reliability

Here we consider the reliability of MSS based on the independent random variables $X_1, X_2, \ldots, X_k$ and Y, where $X_1, X_2, \ldots, X_k$ represents the k i.i.d. 'strength' components and Y represents the 'stress'. Suppose asystem with these X_i's functions if atleast s ($1 \le s \le k$) components operate. Let, F(.) be the distribution function of X and G(.) be the distribution function of Y. Then the system reliability $R_{s,k}$ = Prob(atleast s of the Xi's exceed Y).
i.e.,

$$R_{s,k} = \sum_{i=s}^{k}\binom{k}{i}\int_{-\infty}^{\infty}[1-F(x)]^i[F(x)]^{k-i}dG(x) \tag{23}$$

Various works have been done in MSS (see for example, [2], [17] and [21]). Suppose X_i's and Y have the DUS-IW(α, β_1) and DUS-IW(α, β_2) distributions respectively, then the reliability in MSS using (23) is,

$$R_{s,k} = \sum_{i=s}^{k}\binom{k}{i}\int_0^{\infty}\left[1-\frac{1}{e-1}\left\{e^{e^{-\left(\frac{x}{\beta_1}\right)^{-\alpha}}}-1\right\}\right]^i\left[\frac{1}{e-1}\left\{e^{e^{-\left(\frac{x}{\beta_1}\right)^{-\alpha}}}-1\right\}\right]^{k-i}$$
$$\frac{1}{e-1}\frac{\alpha}{\beta_2}\left(\frac{x}{\beta_2}\right)^{-(\alpha+1)} e^{-\left(\frac{x}{\beta_2}\right)^{-\alpha}} e^{e^{-\left(\frac{x}{\beta_2}\right)^{-\alpha}}} dx$$

$$= \sum_{i=s}^{k}\binom{k}{i}\int_0^{\infty}\left[1+\frac{1}{e-1}-\frac{1}{e-1}e^{e^{-\left(\frac{x}{\beta_1}\right)^{-\alpha}}}\right]^i\left[-\frac{1}{e-1}\left\{1-e^{e^{-\left(\frac{x}{\beta_1}\right)^{-\alpha}}}\right\}\right]^{k-i}$$
$$\frac{1}{e-1}\frac{\alpha}{\beta_2}\left(\frac{x}{\beta_2}\right)^{-(\alpha+1)} e^{-\left(\frac{x}{\beta_2}\right)^{-\alpha}} e^{e^{-\left(\frac{x}{\beta_2}\right)^{-\alpha}}} dx$$

$$= \sum_{i=s}^{k}\binom{k}{i}\int_0^{\infty}\left[\frac{e}{e-1}\left\{1-e^{e^{-\left(\frac{x}{\beta_1}\right)^{-\alpha}}-1}\right\}\right]^i\left[(-1)\frac{1}{e-1}\left\{1-e^{e^{-\left(\frac{x}{\beta_1}\right)^{-\alpha}}}\right\}\right]^{k-i}$$

$$\frac{1}{e-1}\frac{\alpha}{\beta_2}\left(\frac{x}{\beta_2}\right)^{-(\alpha+1)} e^{-\left(\frac{x}{\beta_2}\right)^{-\alpha}} e^{e^{-\left(\frac{x}{\beta_2}\right)^{-\alpha}}} dx$$

$$=\sum_{i=s}^{k}\binom{k}{i}\left(\frac{e}{e-1}\right)^{i}(-1)^{k-i}\left(\frac{1}{e-1}\right)^{k-i}\frac{1}{e-1}\frac{\alpha}{\beta_2}\int_0^\infty\left[1-e^{e^{-\left(\frac{x}{\beta_1}\right)^{-\alpha}}-1}\right]^{i}\left[1-e^{e^{-\left(\frac{x}{\beta_1}\right)^{-\alpha}}}\right]^{k-i}$$

$$\left(\frac{x}{\beta_2}\right)^{-(\alpha+1)} e^{-\left(\frac{x}{\beta_2}\right)^{-\alpha}} e^{e^{-\left(\frac{x}{\beta_2}\right)^{-\alpha}}} dx$$

Using binomial expansion $(1-x)^n=\sum_{k=0}^{n}\binom{n}{k}(-1)^k x^{n-k}$, we get,

$$=\sum_{i=s}^{k}\binom{k}{i}e^{i}(-1)^{k-i}\left(\frac{1}{e-1}\right)^{k+1}\frac{\alpha}{\beta_2}\int_0^\infty\sum_{j=0}^{i}\binom{i}{j}(-1)^j\left[e^{e^{-\left(\frac{x}{\beta_1}\right)^{-\alpha}}-1}\right]^{i-j}\sum_{p=0}^{k-i}\binom{k-i}{p}(-1)^p\left[e^{e^{-\left(\frac{x}{\beta_1}\right)^{-\alpha}}}\right]^{k-i-p}$$

$$\left(\frac{x}{\beta_2}\right)^{-(\alpha+1)} e^{-\left(\frac{x}{\beta_2}\right)^{-\alpha}}\sum_{m=0}^{\infty}\frac{\left[e^{-\left(\frac{x}{\beta_2}\right)^{-\alpha}}\right]^m}{m!}dx$$

$$=\sum_{i=s}^{k}\sum_{j=0}^{i}\sum_{p=0}^{k-i}\sum_{m=0}^{\infty}\binom{k}{i}\binom{i}{j}\binom{k-i}{p}\frac{e^{i}(-1)^{k-i}}{m!}\left(\frac{1}{e-1}\right)^{k+1}\frac{\alpha}{\beta_2}(-1)^{j+p}\int_0^\infty\left[e^{e^{-\left(\frac{x}{\beta_1}\right)^{-\alpha}}-1}\right]^{i-j}\left[e^{e^{-\left(\frac{x}{\beta_1}\right)^{-\alpha}}}\right]^{k-i-p}$$

$$\left(\frac{x}{\beta_2}\right)^{-(\alpha+1)} e^{-(m+1)\left(\frac{x}{\beta_2}\right)^{-\alpha}}dx$$

$$=\sum_{i=s}^{k}\sum_{j=0}^{i}\sum_{p=0}^{k-i}\sum_{m=0}^{\infty}\binom{k}{i}\binom{i}{j}\binom{k-i}{p}\frac{e^{i}\alpha{\beta_2}^{\alpha}}{m!\,(e-1)^{k+1}}(-1)^{k-i+j+p}e^{-i+j}\int_0^\infty\left[e^{e^{-\left(\frac{x}{\beta_1}\right)^{-\alpha}}}\right]^{k-j-p}$$

$$(x)^{-(\alpha+1)}e^{-(m+1)\left(\frac{x}{\beta_2}\right)^{-\alpha}}dx$$

$$=\sum_{i=s}^{k}\sum_{j=0}^{i}\sum_{p=0}^{k-i}\sum_{m=0}^{\infty}\binom{k}{i}\binom{i}{j}\binom{k-i}{p}\frac{e^{j}{\beta_2}^{\alpha}}{m!\,(e-1)^{k+1}}(-1)^{k-i+j+p}\int_0^\infty\alpha(x)^{-(\alpha+1)}$$

$$\sum_{n=0}^{\infty}\frac{\left[(k-j-p)e^{-\left(\frac{x}{\beta_1}\right)^{-\alpha}}\right]^n}{n!}e^{-(m+1)\left(\frac{x}{\beta_2}\right)^{-\alpha}}dx$$

$$=\sum_{i=s}^{k}\sum_{j=0}^{i}\sum_{p=0}^{k-i}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\binom{k}{i}\binom{i}{j}\binom{k-i}{p}\frac{e^{j}{\beta_2}^{\alpha}(k-j-p)^n}{m!\,(e-1)^{k+1}n!}(-1)^{k-i+j+p}\int_0^\infty\alpha(x)^{-(\alpha+1)}$$

$$e^{-n\left(\frac{x}{\beta_1}\right)^{-\alpha}}e^{-(m+1)\left(\frac{x}{\beta_2}\right)^{-\alpha}}dx.$$

Put $u=x^{-\alpha}$

Then, $du=-\alpha x^{-(\alpha+1)}dx$,

implies,

$$R_{s,k}=\sum_{i=s}^{k}\sum_{j=0}^{i}\sum_{p=0}^{k-i}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\binom{k}{i}\binom{i}{j}\binom{k-i}{p}\frac{e^{j}{\beta_2}^{\alpha}(k-j-p)^n}{m!\,(e-1)^{k+1}n!}(-1)^{k-i+j+p}\int_0^\infty e^{-n{\beta_1}^{\alpha}u}e^{-(m+1){\beta_2}^{\alpha}u}du$$

$$=\sum_{i=s}^{k}\sum_{j=0}^{i}\sum_{p=0}^{k-i}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\binom{k}{i}\binom{i}{j}\binom{k-i}{p}\frac{e^{j}{\beta_2}^{\alpha}(k-j-p)^n}{m!\,(e-1)^{k+1}n!}(-1)^{k-i+j+p}\int_0^\infty e^{-u\left(n{\beta_1}^{\alpha}+(m+1){\beta_2}^{\alpha}\right)}du$$

$$=\sum_{i=s}^{k}\sum_{j=0}^{i}\sum_{p=0}^{k-i}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\binom{k}{i}\binom{i}{j}\binom{k-i}{p}\frac{e^{j}{\beta_2}^{\alpha}(k-j-p)^n}{m!\,(e-1)^{k+1}n!}(-1)^{k-i+j+p}\left[\frac{e^{-u\left(n{\beta_1}^{\alpha}+(m+1){\beta_2}^{\alpha}\right)}}{-\left(n{\beta_1}^{\alpha}+(m+1){\beta_2}^{\alpha}\right)}\right]_0^\infty$$

$$=\sum_{i=s}^{k}\sum_{j=0}^{i}\sum_{p=0}^{k-i}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\binom{k}{i}\binom{i}{j}\binom{k-i}{p}\frac{e^{j}{\beta_2}^{\alpha}(k-j-p)^n(-1)^{k-i+j+p}}{m!\,(e-1)^{k+1}n!\,\left(n{\beta_1}^{\alpha}+(m+1){\beta_2}^{\alpha}\right)}.$$

Therefore,

$$R_{s,k}=\sum_{i=s}^{k}\sum_{j=0}^{i}\sum_{p=0}^{k-i}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\binom{k}{i}\binom{i}{j}\binom{k-i}{p}\frac{e^{j}{\beta_2}^{\alpha}(k-j-p)^n(-1)^{k-i+j+p}}{m!(e-1)^{k+1}n!\left(n{\beta_1}^{\alpha}+(m+1){\beta_2}^{\alpha}\right)}.\qquad(24)$$

VII. Maximum Likelihood Estimation

In this section, we discuss method of maximum likelihood for the estimation of parameters α and β. Let $X_1, X_2, \ldots, X_n$ be an observed random sample from DUS-IW(α,β) distribution with unknown parameters α and β. The maximum likelihood estimator(MLE)s of the parameters of the DUS-IW(α, β) distribution are derived as below. The likelihood function is

$$L(x) = \prod_{i=1}^{n} f(x_i, \alpha, \beta)$$

i.e.,

$$L(x) = \left(\frac{1}{e-1}\right)^n \frac{\alpha^n}{\beta^n} \prod_{i=1}^{n} \left(\frac{x_i}{\beta}\right)^{-(\alpha+1)} e^{-\sum_{i=1}^{n}\left(\frac{x_i}{\beta}\right)^{-\alpha}} e^{\sum_{i=1}^{n} e^{-\left(\frac{x_i}{\beta}\right)^{-\alpha}}}$$

So that the log-likelihood function becomes

$$\log L = -\,n\log(e-1) + n\log\alpha - n\log\beta - (\alpha+1)\sum_{i=1}^{n}[\log x_i - \log\beta] - \sum_{i=1}^{n}\left(\frac{x_i}{\beta}\right)^{-\alpha} + \sum_{i=1}^{n} e^{-\left(\frac{x_i}{\beta}\right)^{-\alpha}} \quad (25)$$

The partial derivatives of log L in (25) with respect to unknown parameters α and β are,

$$\frac{\partial \log L}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n}[\log x_i - \log\beta] + \sum_{i=1}^{n}\left(\frac{x_i}{\beta}\right)^{-\alpha} \log\sum_{i=1}^{n}\left(\frac{x_i}{\beta}\right) + \sum_{i=1}^{n} e^{-\left(\frac{x_i}{\beta}\right)^{-\alpha}} \left(\frac{x_i}{\beta}\right)^{-\alpha} \log\left(\frac{x_i}{\beta}\right) \quad (26)$$

$$\frac{\partial \log L}{\partial \beta} = -\frac{n}{\beta} + (\alpha+1)\frac{n}{\beta} - \alpha\sum_{i=1}^{n}(x_i)^{-\alpha}\beta^{\alpha-1} - \sum_{i=1}^{n} e^{-\left(\frac{x_i}{\beta}\right)^{-\alpha}} \alpha\sum_{i=1}^{n}(x_i)^{-\alpha}\beta^{\alpha-1} \quad (27)$$

Setting the left side of above two equations to zero, we get the likelihood equations as a system of two non-linear equations in α and β.

$$\frac{n}{\alpha} - \sum_{i=1}^{n}[\log x_i - \log\beta] + \sum_{i=1}^{n}\left(\frac{x_i}{\beta}\right)^{-\alpha} \log\sum_{i=1}^{n}\left(\frac{x_i}{\beta}\right) + \sum_{i=1}^{n} e^{-\left(\frac{x_i}{\beta}\right)^{-\alpha}} \left(\frac{x_i}{\beta}\right)^{-\alpha} \log\left(\frac{x_i}{\beta}\right) = 0 \quad (28)$$

$$-\frac{n}{\beta} + (\alpha+1)\frac{n}{\beta} - \alpha\sum_{i=1}^{n}(x_i)^{-\alpha}\beta^{\alpha-1} - \sum_{i=1}^{n} e^{-\left(\frac{x_i}{\beta}\right)^{-\alpha}} \alpha\sum_{i=1}^{n}(x_i)^{-\alpha}\beta^{\alpha-1} = 0 \quad (29)$$

Solving these systems, (28) and (29), in α and β gives the MLEs of α and β. These equations cannot be solved analytically and statistical software R can be used to solve them numerically, by taking initial value arbitrarily.

VIII. Results

I. Simulation Study

In statistics, simulation is used to assess the performance of the model. It is a numerical technique for conducting experiments on the computer. There are certain simulation techniques for generating and analyzing like Monte-Carlo simulation. With considered (21), here we take different combinations of parameters α and β with samples of sizes n=25, 50, 100,500 and 1000 and the samples are generated from the DUS-IW(α,β) model. The bias and the mean square error (MSE) of the parameter estimates are calculated using the equations,

$$\text{Bias} = \frac{1}{n}\sum_{i=1}^{n}(\hat{\varepsilon}_i - \varepsilon_i) \quad (30)$$

and

$$\text{MSE} = \frac{1}{n}\sum_{i=1}^{n}(\hat{\varepsilon}_i - \varepsilon_i)^2. \quad (31)$$

The simulation is conducted for three different cases using different true parameter values. The selected true parameter values are α=0.2 and β=0.8, α=1 and β=1.5 and α=1 and β=1 for the first, second, and third cases, respectively. As the sample size increases, MSE decreases for all selected parameter values as in Table 1, 2 and 3. Also, the bias is nearer to zero when the sample size increases. Thus, the estimates tend to the true parameter values as the sample size increases.

Table 1: *Simulation study at α=0.2 and β=0.8*

n	Estimated value of Parameters	Bias	MSE
25	$\hat{\alpha}$=0.2112962	0.01129618	0.001374719
	$\hat{\beta}$=1.871778	1.071778	13.29915
50	$\hat{\alpha}$=0.204941	0.004941044	0.0005514951
	$\hat{\beta}$=1.185226	0.3852255	2.116492
100	$\hat{\alpha}$=0.2020091	0.002009123	0.0002558455
	$\hat{\beta}$=0.9837115	0.1837115	0.4104516
500	$\hat{\alpha}$=0.1988177	-0.001182251	$4.310696\text{x}10^{-05}$
	$\hat{\beta}$=0.8201508	0.02015084	0.04473991
1000	$\hat{\alpha}$=0.1982047	-0.0017953	$2.387253\text{x}10^{-05}$
	$\hat{\beta}$=0.8006457	0.0006456743	0.01871488

Table 2: *Simulation study at α=1 and β=1.5*

n	Estimated value of Parameters	Bias	MSE
25	$\hat{\alpha}$=1.056977	0.0569774	0.03443659
	$\hat{\beta}$=1.594545	0.09454492	0.1449253
50	$\hat{\alpha}$=1.024739	0.02473883	0.01378851
	$\hat{\beta}$=1.536913	0.03691277	0.06391387
100	$\hat{\alpha}$=1.010164	0.01016389	0.006394628
	$\hat{\beta}$=1.522311	0.02231105	0.03083641
500	$\hat{\alpha}$=0.9940904	-0.005909631	0.001077642
	$\hat{\beta}$=1.499751	-0.0002490474	0.005798989
1000	$\hat{\alpha}$=0.9910253	-0.008974667	0.0005967668
	$\hat{\beta}$=1.496801	-0.003199132	0.002582339

Table 3: *Simulation study at α=1 and β=1*

n	Estimated value of Parameters	Bias	MSE
25	$\hat{\alpha}$= 1.056977	0.05697739	0.03443657
	$\hat{\beta}$= 1.06303	0.06302988	0.06441129
50	$\hat{\alpha}$=1.024739	0.02473881	0.0137885
	$\hat{\beta}$= 1.024608	0.02460846	0.02840616
100	$\hat{\alpha}$= 1.010164	0.0101639	0.00639462
	$\hat{\beta}$= 1.014874	0.01487402	0.0137051
500	$\hat{\alpha}$= 0.9940904	-0.005909626	0.001077644
	$\hat{\beta}$= 0.9998339	-0.0001661259	0.002577332
1000	$\hat{\alpha}$= 0.9910254	-0.008974635	0.0005967686
	$\hat{\beta}$= 0.9978672	-0.002132811	0.001147696

Hence, DUS-IW(α, β) distribution possesses least bias and MSE values as the sample size increases.

II. Data Analysis

In this section, we illustrate the use of DUS-IW(α, β) distribution using two real data sets. We fit DUS-IW(α, β) distribution to these two data sets and compare the results with IW distribution, DUS-Lomax (DUS-L) distribution, Lomax distribution, Gompertz Lomax (GL) distribution, DUS-Exponential(DUS-E) distribution and Inverse Lindley (IL) distribution. The first data-sets, considered here, represent the survival times of two groups of patients suffering from head and neck cancer disease. The data here considered is of the patients belonging to one group who were treated using a combined radiotherapy and chemotherapy (CT + RT) ([9]). Another data is of 46 observations reported on active repair times (hours) for an airborne communication transceiver ([5]). The data sets are given below:

Table 4: *Survival times of patients treated using RT+CT:*

12.2	23.56	23.74	25.87	31.98	37	41.35	47.38	55.46	58.36
63.47	68.46	78.26	74.47	81.43	84	92	94	110	112
119	127	130	133	140	146	155	159	173	179
194	195	209	249	281	319	339	432	469	519
633	725	817	1776						

Table 5: *Repair Time:*

0.2	0.3	0.5	0.5	0.5	0.5	0.6	0.6	0.7	0.7
0.7	0.8	0.8	1.0	1.0	1.0	1.0	1.1	1.3	1.5
1.5	1.5	1.5	2.0	2.0	2.2	2.5	2.7	3.0	3.0
3.3	3.3	4.0	4.0	4.5	4.7	5.0	5.4	5.4	7.0
7.5	8.8	9.0	10.3	22.0	24.5				

Using the R software, the analysis is carried out. The tables 6 and 7 gives the estimates of the model parameters, AIC (Akaike information criterion) and the BIC (Bayesian information criterion) values, where,

$$AIC = -2l + 2k \tag{32}$$

$$BIC = -2l + k\log n \tag{33}$$

where l denotes the log-likelihood function, k is the number of parameters and n is the sample size. Also, using the Kolmogorov-Smirnov(K-S) test, the perfection of competing models is tested.
The K-S statistic is given by,

$$KS = max\left\{\frac{i}{m} - z_i, z_i - \frac{i-1}{m}\right\} \tag{34}$$

where, z_i is the cumulative distribution of x_i, x_i's being the ordered observations and m is the number of classes. Both the KS-statistic and p-value are given in tables 6 and 7 as well.

Table 6: *Estimates of the parameters, AIC, BIC and KS statistic of the fitted model in data set 1 (Table 4):*

Model	Estimates	AIC	BIC	KS-statistic	p-value
DUS-IW	$\hat{\alpha}$=1.119 $\hat{\beta}$=57.556	561.83	565.40	0.087	0.868
IW	$\hat{\alpha}$= 1.013 $\hat{\beta}$=76.227	563.14	566.71	0.093	0.811
DUS-L	$\hat{\alpha}$=3.165 $\hat{\beta}$=0.003	563.81	567.38	0.093	0.806
Lomax	$\hat{\alpha}$=4.40 $\hat{\beta}$=0.001	564.91	568.48	0.104	0.695
GL	$\hat{\theta}$=0.0185 $\hat{\alpha}$=0.467 $\hat{\beta}$=0.719 $\hat{\gamma}$=1.99	571.54	578.68	0.129	0.414
DUS-E	$\hat{\theta}$=0.006	569.82	571.60	0.198	0.021
IL	$\hat{\theta}$=77.68	561.16	562.94	29.01	5.551×10^{-16}

Table 6 and Table 7 show that, DUS-IW(α, β) has lowest AIC, BIC, KS-Statistic, and largest p-value based on KS-Statistic. The second lowest AIC, BIC, KS-Statistic and second largest p-value are obtained by the IW distribution. The proposed distribution, DUS-IW(α, β) can be used when failure rate pattern of lifetime distribution is upside-down shaped. In Data set 1 and 2 it seems that DUS-IW(α,β) is more appropriate than other distributions (IW distribution, DUS-L distribution, Lomax distribution, GL distribution, DUS-E distribution and IL distribution). So DUS-IW(α, β) is better alternative in the situations in which upside-down distributions arises.

Table 7: *Estimates of the parameters, AIC, BIC and KS statistic of the fitted model in data set 2 (Table 5):*

Model	Estimates	AIC	BIC	KS-statistic	p-value
DUS-IW	$\hat{\alpha}$=1.109 $\hat{\beta}$=0.857	204.68	208.34	0.078	0.942
IW	$\hat{\alpha}$=1.013 $\hat{\beta}$=1.130	205.38	209.04	0.081	0.926
DUS-L	$\hat{\alpha}$=2.610 $\hat{\beta}$=0.227	209.40	213.06	0.118	0.548
Lomax	$\hat{\alpha}$=3.549 $\hat{\beta}$=0.108	209.91	213.57	0.127	0.446
GL	$\hat{\theta}$=1.776 $\hat{\alpha}$=1.165 $\hat{\beta}$=0.189 $\hat{\gamma}$=0.245	213.96	221.27	0.129	0.432
DUS-E	$\hat{\theta}$=0.344	217.31	219.14	0.211	0.033
IL	$\hat{\theta}$=1.577	204.34	206.17	0.883	2.2×10^{-16}

IX. Discussion

A new distribution, DUS-IW(α, β) distribution, is proposed and its properties are studied. The DUS-IW(α, β) has an upside-down shaped and decreasing failure rate functions. We derived the moments, moment generating function, characteristic function, quantiles, etc. of the proposed distribution. Estimation of parameters of the distribution is performed via maximum likelihood method. Reliability of single component and multi component stress-strength models are derived. A simulation study is performed to validate the estimates of the model parameters. DUS-IW(α, β) distribution is applied to two real data sets and shown that DUS-IW(α, β) distribution is a better fit than other well-known distributions. Thus DUS-IW(α, β) distribution can be used in real data analysis as a better alternative to the existing distributions.

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Double Sampling Based Parameter Estimation in Big Data and Application in Control Charts

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Abstract

Double sampling technique and control charts are used for predicting about unknown parameters of the big population and developing algorithms for imposing control over growth factor. This sampling procedure has two approaches like sub-sample and independent sample. Aim is to estimate mean file-size by both and to find out which approach is better in big data setup. Comparative mathematical tools used herein are mean squared error, confidence interval, relative confidence interval length measure and control charts of digital file-size for monitoring. Estimation strategies are proposed and confidence intervals are computed over multiple points of time. At each time, it was found that confidence intervals are catching the true values. First kind of approach (as case I) of double sampling found better than the second. A new simulation strategy is proposed who is observed efficient for comparison purpose. Single-valued simulated confidence intervals are obtained using the new simulation strategy and found covering the truth in its range. As an application of outcomes, control charts are developed to monitor the parametric growth over long duration. Upper and Lower control limits are drawn for business managers to keep a watch on digital file-size estimates whether their growth under control? Outcomes may be extended for reliability evaluation under discrete time domain. The content herein is a piece of thought, idea and analysis developed by deriving motivation from past references to handle big data using double sampling. Findings of the study can be used for developing software based monitoring system using process control charts for managers.

Keywords: Big Big-Data, Double Sampling, Sub-Sample, Estimation, Control Limits, Control Charts, Process Control, Social Media Portal, Mean squared error, Simulation, Confidence Interval (CI).

I. Introduction

Due to emergence of digital technologies and appearance of social media platforms worldwide, people are habitual for ease and comforts contained therein. While user registration, participation and content-communication through these platforms, the digital data is facing challenges in terms of drastic growth in volume, velocity and variety. Momentum of data over time domain has got immense speed to occupy memory space at servers/data centers. Often users do not remove their long past garbage data from the social media account. Space allocation to users is unlimited due to inherent competition in IT business. No service provider wants reduction in user database. Therefore, forecasting (or prediction) require about possible expansion of digital space over continuous time. A manager of Data center is interested to know how much investment cost needed for enhancing capacity of storing units relating to social media portal. Fig.1 to Fig. 3 reveal scenario of expanding digital space over time t_1, t_2 and t_3 ($t_1 < t_2 < t_3$).

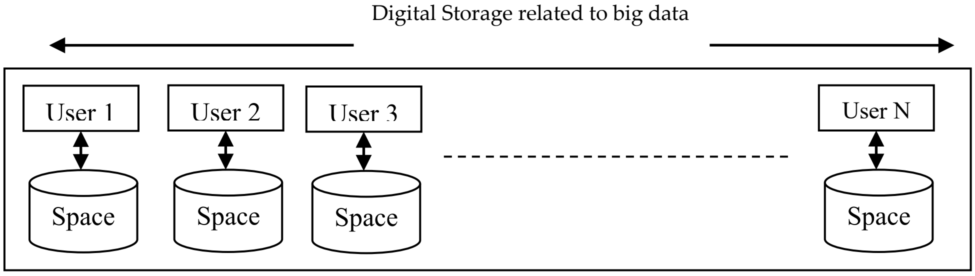


Figure 1: *The digital model of data storage at time* t_1

The figure 1 is allocation of default memory space at the time of user-registration on a portal at time t_1.

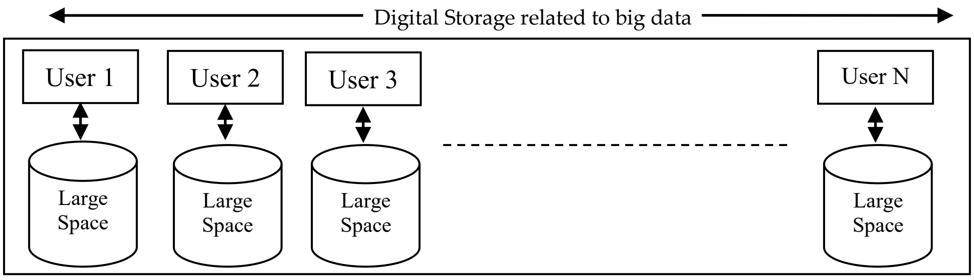


Figure 2: *Digital model of data storage at time* t_2

After t_1 and before t_2 $(t_2 > t_1)$, figure 2 shows increment in default allocated digital space. While at time instant t_3 $(t_3 > t_2 > t_1)$. the default space demand got extra ordinary longevity.

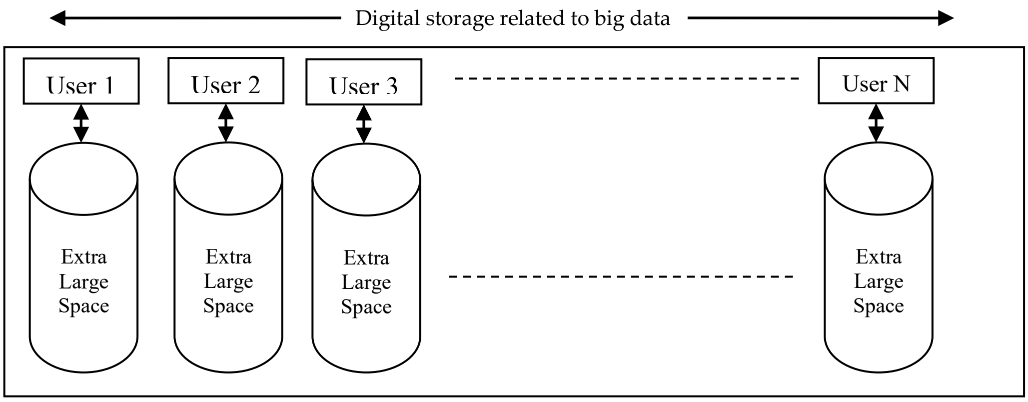


Figure 3: *The digital model of data storage at time* t_3

Alert system requires for constant monitoring of the storage space who can convey managers of IT-business for further planning and cost investments. It could be developed by the joint efforts of sampling methodologies available in literature and process control charts over time frame.

Double sampling scheme is a tool for estimating population parameters where first sample provides guess(low cost) estimate of parameters of the support variable while second sample provides precise estimates of main variable of interest. This scheme has two variants like (a) second sample as sub-sample of first (b) second sample as independent. Fig 4 shows the scheme diagrammatic layout of double sampling.

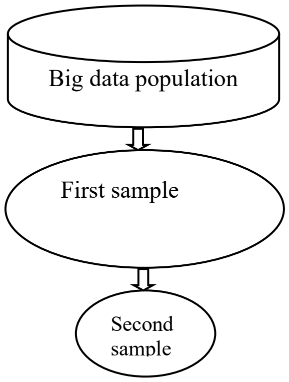


Figure 4.1: *Double sampling Scheme (at j^{th} occasion t_i) under case I*

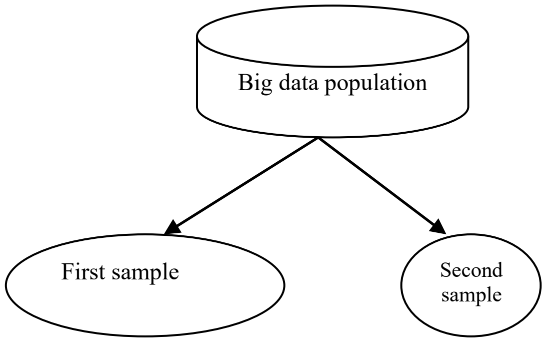


Figure 4.2: *Double sampling Scheme (at i^{th} occasion t_i) under case II*

Using scheme of figure 4.1,and 4.2, one can obtain the sample based prediction about average file-size floating in portal of social media communication model described in Figure 5 where two-way communication exist among large voluminous group of registered users.

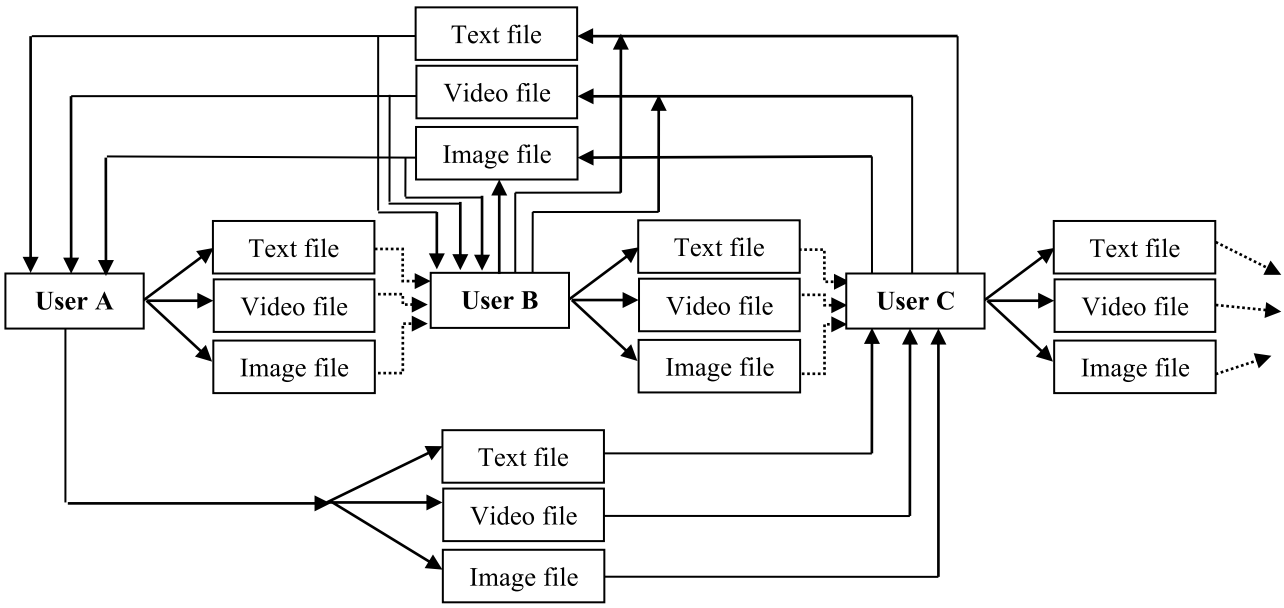


Figure 5. Portal based Social Media communication model and floating files

Statistical methods are used in process control and size measure, which exhibit the extent of conformity of the situation under specifications determined by the relevant authority. It is one of aspects affecting decision making based on specifications set at early level and continued until the completion [1]. Big data take into account digital streams with observations on file-size generated sequentially over time. Among many different purposes, one common task is to collect and analyze big data and to monitor the longitudinal performance of the related processes. Big data assume different forms of data-streams gathered through complex engineering systems like sequences of satellite images, climate data, website transaction logs, credit cards, etc which have complicated data structure and complex storage mechanism. Statistical process control charts [28] could be utilized as an important tool a for monitoring and decision making [2].

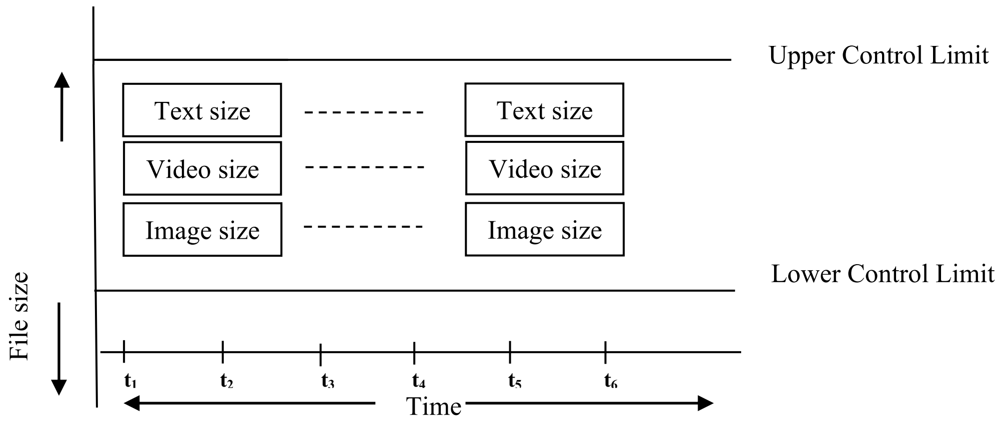


Figure 6: *Process control chart of file size variable*

II. Literature Review

Parameter estimation problem in big data setup is an opportunity which enhanced by contributions to the higher education sectors in terms of developing indicators for decision making [3][4]. Broad aspect of big data including special features and characteristics was narrated [6] with consolidated description by integrating definitions from practitioners and academics. It was focused on analytics related to unstructured data, whose share is 95% of the big data. While dealing with big data and sampling methodologies, scientists can derive machine learning algorithms by the implementation of supervised statistical data mining. Analysis techniques can be used for a single machine scheduling problem in light of hidden patterns using optimized scheduling sequence [7]. Analysis depends on the ability of data scientists to make sense and develop insight into huge data volume. Way of developing an actionable idea is known as data exploration who brings out hidden facts and so is challenging task.

Data requires first a small view to have insights [8] for further course of actions. Sampling methodologies can play vital role by creating preliminary insight into big data. Sampling frame is collection of all units of the population that can use in big data for sample selection. Random sample of registered users on portal can be drawn to obtain estimates of individuals and such precisions are the same as of those when frame consisting of a list of individuals is taken into account [9]. Online social network portals generate large population of individuals where unbiased sampling could be used as a tool for prediction. Convergence properties of the random walk were studied using sampling of Facebook data [10]. Estimation of the influence of an event connected through social media is a problem to handle with because social media is widely exploited as communication platform for relevant and irrelevant information. It opens avenue for mathematical formulation and characterization using casual inference [11]. Avenues in web-based large scale social networks are to explore concise and coherent methods for summarization and to draw valid conclusions. Sampling methodologies on social media network have key role as knowledge discovery tools while a suitable methodology can carry out with efficient decision [12]. Big data may suffer due to incompleteness of required values, and so, need to be replaced by neighboring computations. Digital media platforms often have distinctive temporal patterns that can be exploited for computations in situations involving incomplete information. Iterative type methods suggest to estimate the parameters in the underlying point process and assign weights to the unknown events with direct calculable score function [13].

The ratio based chain-type exponential estimator for finite population mean under double sampling with the auxiliary variable support can provide an efficient estimation methodology [14] applicable in big data. Empirical study as a tool could be used for the betterment of outcome. Extension of exponential-type estimator under double sampling was an outcome [15] with comparative efficiency of the past. In practice, one can find multiple variants affecting the main variable of interest. The multivariate exponential type estimators could be used in setup of double sampling [16] who may efficient performer than single support variant. In presence of non-response and with the help of fractional raw moments, estimate of population parameter could be more efficient using such inputs [17].

The Gaussian process is useful for prediction and can be applied over big data. It is useful where the additive model works well and response depends upon a small number of features [18]. International groups of customers relating to business deals constitute big data and, therefore, exist like open problem for business manager to deal with. Data of sale of cars available on the internet can be analyzed [19] just to have an insight for perception, preferences, preparedness and internet experience of potential car customers while making a purchase/sale decision. A coordination among manufacturer, supplier, retailer is essential to manage forward and reverse supply chain in business. It relates to price-fixing, storing capacity, profit sharing and decision making in a closed-loop supply chain in order to maintain the smooth functioning. Responsibility-sharing is prime factor that affects outcome and profit. Such require optimal selling price, optimal time, wholesale price, sharing percentage and optimal return rate in such a manner that objective function be maximize [20]. The stratified double sampling scheme can be used for estimating finite population mean in presence of more than one support variables using regression time estimators [21]. The basic idea used is to use the ranks of two support variables and extension of the same idea in double sampling setup is due to [22].

III. Motivation and Problem Undertaken

Mean estimation strategies exist in literature [23] for estimating average file-size of text, video and image type communication among registered users on asocial media platform. According to sampling theory [25-27],while the population mean of support variable is unknown, the double sampling is used. This motivates for developing generalization of the aspect [23] for real world situation. In big data, it is difficult to find out parametric information of support variable priorly known because of data-volume, velocity and variety. This motivates for use of double sampling in early literature. Moreover, simulation method with this scheme still not explored. This paper considers the scenario of absence of support variable parametric information and presents new estimation strategies along with a new simulation procedure useful in comparative analysis. As an application, control charts are developed to monitor parametric changes in big data over multiple time occasions. Reliability could be examined through discrete-time domain over a long period. The mathematical support is derived from [23], [24-26] and [27].

I. Assumptions

1. Let at time t_j, the j^{th} registered user on a web portal generates data values $\left\{(T_i)_{t_j}\right\}$ as text, $\left\{(V_i)_{t_j}\right\}$ as video and $\left\{(I_i)_{t_j}\right\}$ as image (reveals variety) $i = 1,2,3, \dots N, j = 1,2,3, \dots M$, where M is the total time points of observations (reveals velocity), N is total registered users, large in numbers, on a web portal (reveals volume).

2. Symbol T is used for text, V for video, and I for image.

IV. Parameters

$$\left((\bar{T}.)_{t_j}\right)_{text} = N^{-1}\left[\sum_{i=1}^{N}(T_i)_{t_j}\right] \tag{1}$$

$$\left((\bar{V}.)_{t_j}\right)_{video} = N^{-1}\left[\sum_{i=1}^{N}(V_i)_{t_j}\right] \tag{2}$$

$$\left((\bar{I}.)_{t_j}\right)_{image} = N^{-1}\left[\sum_{i=1}^{N}(I_i)_{t_j}\right] \tag{3}$$

Symbols $C_T^{(j)}$, $C_V^{(j)}$, $C_I^{(j)}$ are coefficients of variation shown in (7), (8), (9) and $\rho_{TV}, \rho_{VI}, \rho_{IT}$ are correlation coefficients of respective pair of populations. Also $\rho_{ab} = \rho_{ba}$ holds for any two pair of variables a and b. Whatever follows hereunder, the t_1 used for time occasion j=1; t_j for j^{th} time occasion. Some other symbols in use are:

$$\left(S_T^{2(j)}\right)_{text} = \frac{1}{N-1}\sum_{i=1}^{N}\left[(T_i)_{t_j} - (\bar{T}.)_{t_j}\right]^2 = S_T^{2(j)} \quad \text{at } t_j \tag{4}$$

$$\left(S_V^{2(j)}\right)_{video} = \frac{1}{N-1}\sum_{i=1}^{N}\left[(V_i)_{t_j} - (\bar{V}.)_{t_j}\right]^2 = S_V^{2(j)} \quad \text{at } t_j \tag{5}$$

$$\left(S_I^{2(j)}\right)_{image} = \frac{1}{N-1}\sum_{i=1}^{N}\left[(I_i)_{t_j} - (\bar{I}.)_{t_j}\right]^2 = S_I^{2(j)} \quad \text{at } t_j \tag{6}$$

The (4), (5), (6) show the variability factor of T, V and I with respect to their means (1), (2), (3). Moreover equations (7), (8), (9) are derived as ratio of (4), (5), (6) with (1), (2), (3).

$$\left(C_T^{(j)}\right)_{text} = \left[S_T^{(j)}/(\bar{T}.)_{t_j}\right] = C_T^{(j)} \quad \text{at } t_j \tag{7}$$

$$\left(C_V^{(j)}\right)_{video} = \left[S_V^{(j)}/(\bar{V}.)_{t_j}\right] = C_T^{(j)} \quad \text{at } t_j \tag{8}$$

$$\left(C_I^{(j)}\right)_{image} = \left[S_I^{(j)}/(\bar{I}.)_{t_j}\right] = C_T^{(j)} \quad \text{at } t_j \tag{9}$$

V. Sample Selection using Double Sampling

Consider figure 4.1 and figure 4.2 where a primary large sample $n^{'}$ $(n^{'} < N)$is drawn by random sampling without replacement and second sample of size $n^{''}(n^{''} < n^{'})$ is drawn in either of the following manners:

Case I: second sample as a sub-sample.

Case II: second sample as independent to primary sample.

Mean of primary sample $n^{'}$ at j^{th} point of time are:.

$$\left(\bar{T}.^{'(j)}\right)_{text} = \left(n^{'}\right)^{-1}\left[\sum_{k=1}^{n^{'}}\left(T_k^{'(j)}\right)\right] = \left(\bar{T}.^{'(j)}\right) \quad \text{at } t_j \tag{10}$$

$$\left(\bar{V}.^{'(j)}\right)_{video} = \left(n^{'}\right)^{-1}\left[\sum_{k=1}^{n^{'}}\left(V_k^{'(j)}\right)\right] = \left(\bar{V}.^{'(j)}\right) \quad \text{at } t_j \tag{11}$$

$$\left(\bar{I}.^{'(j)}\right)_{image} = \left(n^{'}\right)^{-1}\left[\sum_{k=1}^{n^{'}}\left(I_k^{'(j)}\right)\right] = \left(\bar{I}.^{'(j)}\right) \quad \text{at } t_j \tag{12}$$

The (10),(11) and (12) help to have an idea about the unknown (1), (2), (3) in rough manner with low cost and time effort. Means based on $n^{'}$ are not very accurate because role of primary sample $n^{'}$ is just to provide a guess of (1), (2), (3). To get precise estimate, second sample $n^{''}$ $\left(n^{''} < n^{'}\right)$ is drawn and appropriate methodologies are used on accurate data of $n^{''}$ to obtain sample means as under :

$$\left(\overline{T}.^{''(j)}\right)_{text} = \left(n^{''}\right)^{-1}\left[\sum_{k=1}^{n^{''}}\left(T_k^{''(j)}\right)\right] \qquad = \left(\overline{T}.^{''(j)}\right) \text{ at } t_j \tag{13}$$

$$\left(\overline{V}.^{''(j)}\right)_{video} = \left(n^{''}\right)^{-1}\left[\sum_{k=1}^{n^{''}}\left(V_k^{''(j)}\right)\right] \qquad = \left(\overline{V}.^{''(j)}\right) \text{ at } t_j \tag{14}$$

$$\left(\overline{I}.^{''(j)}\right)_{image} = \left(n^{''}\right)^{-1}\left[\sum_{k=1}^{n^{''}}\left(I_k^{''(j)}\right)\right] \qquad = \left(\overline{I}.^{''(j)}\right) \text{ at } t_j \tag{15}$$

The (13), (14), (15) are used to estimate unknown (1), (2), (3). Moreover, sample mean squares on $n^{''}$ are:

$$\left(S_T^{2''(j)}\right)_{text} = \frac{1}{n^{''}-1}\sum_{i=1}^{n^{''}}\left[\left(T_k^{''(j)}\right) - \left(\overline{T}.^{''(j)}\right)\right]^2 \qquad = S_T^{2''(j)} \text{ at } t_j \tag{16}$$

$$\left(S_V^{2''(j)}\right)_{video} = \frac{1}{n^{''}-1}\sum_{i=1}^{n^{''}}\left[\left(V_k^{''(j)}\right) - \left(\overline{V}.^{''(j)}\right)\right]^2 \qquad = S_V^{2''(j)} \qquad \text{at } t_j \tag{17}$$

$$\left(S_I^{2''(j)}\right)_{image} = \frac{1}{n^{''}-1}\sum_{i=1}^{n^{''}}\left[\left(I_k^{''(j)}\right) - \left(\overline{I}.^{''(j)}\right)\right]^2 \qquad = S_I^{2''(j)} \text{ at } t_j \tag{18}$$

Sample coefficients of variations are:

$$\left(C{''}_T^{(j)}\right)_{text} = \left[S_T^{2''(j)} / \left(\overline{T}.^{''(j)}\right)\right] \qquad = \left(C{''}_T^{(j)}\right) \text{ at } t_j \tag{19}$$

$$\left(C{''}_V^{(j)}\right)_{video} = \left[S_V^{2''(j)} / \left(\overline{V}.^{''(j)}\right)\right] \qquad = \left(C{''}_V^{(j)}\right) \text{ at } t_j \tag{20}$$

$$\left(C{''}_I^{(j)}\right)_{image} = \left[S_I^{2''(j)} / \left(\overline{I}.^{''(j)}\right)\right] \qquad = \left(C{''}_I^{(j)}\right) \text{ at } t_j \tag{21}$$

The symbols $\rho_{T,V}$, $\rho_{V,I}$, $\rho_{I,T}$ are used for correlation coefficient in population while $\rho^{''}{}_{T,V}$, $\rho^{''}{}_{V,I}$, $\rho^{''}{}_{I,T}$ are used for the same purpose but on data of $n^{''}$ called as sample estimate of correlation coefficient.

VI. Double Sampling based Methods of Estimation

Let E_1 , E_2 , E_3 are double sampling based estimation strategies for estimating unknown big-data population parameters (1), (2), (3) respectively. At j[th] point of time t_j , they are as under:

$$\left(E_1^{(j)}\right)_{text} = \left[\overline{T}.^{''(j)}\left(\left(\overline{V}.^{'}\right)_{t_j} / \overline{V}.^{''(j)}\right)\right] \tag{22}$$

$$\left(E_2^{(j)}\right)_{video} = \left[\overline{V}.^{''(j)}\left(\left(\overline{I}.^{'}\right)_{t_j} / \overline{I}.^{''(j)}\right)\right] \tag{23}$$

$$\left(E_3^{(j)}\right)_{image} = \left[\overline{I}.^{''(j)}\left(\left(\overline{T}.^{'}\right)_{t_j} / \overline{T}.^{''(j)}\right)\right] \tag{24}$$

Equations (22), (23), (24) are in accordance with references [24][25][26][27] extended for set-up of big data. These are logically formulated such that in (22) the T is variable of main interest while V is support variable correlated to T since larger T provides increment in V. Therefore, with the help of V, a better estimate of T could be obtained. Similar logical justifications are used for formulation of (23) and (24). The E_1 , E_2 , E_3 are biased for estimation of (1), (2), (3) because $E[E_m] \neq (T.)_{text}$ or $\left((\overline{V}.)_{t_j}\right)_{video}$ or $(\overline{I}.)_{image}$ for m=1,2,3 where $E[.]$ denotes expectation of estimate.

The general form of mean squared error (MSE) is describe below for $\hat{\theta}$ to be an estimator of true value θ.

$M(\hat{\theta}) = E\left[\hat{\theta} - \theta\right]^2$ where $\hat{\theta}$ corresponds to $E_i^{(j)}$,i=1,2,3 sequentially while θ are (1), (2), (3) respectively.

Mean squared error under ,Case I ,are ($see\ references$ [24][25] [26][27]):

$$\left[MSE\left(E_1^{(j)}\right)_{text}\right]_I = \left[(\bar{T}.)_{t_j}^2\right]\left[(V_{20})_T^{(j)} + \left((V_{02})_T^{(j)} - (V_{02}^{'})_T^{(j)}\right) - 2\left\{(V_{11})_T^{(j)} - (V_{11}^{'})_T^{(j)}\right\}\right] \tag{25}$$

$$\left[MSE\left(E_2^{(j)}\right)_{video}\right]_I = \left[(\bar{V}.)_{t_j}^2\right]\left[(V_{20})_V^{(j)} + \left((V_{02})_V^{(j)} - (V_{02}^{'})_V^{(j)}\right) - 2\left\{(V_{11})_V^{(j)} - (V_{11}^{'})_v^{(j)}\right\}\right] \tag{26}$$

$$\left[MSE\left(E_3^{(j)}\right)_{image}\right]_I = \left[(\bar{I}.)_{t_j}^2\right]\left[(V_{20})_I^{(j)} + \left((V_{02})_I^{(j)} - (V_{02}^{'})_I^{(j)}\right) - 2\left\{(V_{11})_I^{(j)} - (V_{11}^{'})_I^{(j)}\right\}\right] \tag{27}$$

Mean squared errors, under case II, are ($see\ references$ [24][25] [26][27]):

$$\left[MSE\left(E_1^{(j)}\right)_{text}\right]_{II} = \left[(\bar{T}.)_{t_j}^2\right]\left[(V_{20})_T^{(j)} + \left((V_{02})_T^{(j)} + (V_{02}^{'})_T^{(j)}\right) - 2(V_{11})_T^{(j)}\right] \tag{28}$$

$$\left[MSE\left(E_2^{(j)}\right)_{video}\right]_{II} = \left[(\bar{V}.)_{t_j}^2\right]\left[(V_{20})_V^{(j)} + \left((V_{02})_V^{(j)} + (V_{02}^{'})_V^{(j)}\right) - 2(V_{11})_V^{(j)}\right] \tag{29}$$

$$\left[MSE\left(E_3^{(j)}\right)_{image}\right]_{II} = \left[(\bar{I}.)_{t_j}^2\right]\left[(V_{20})_I^{(j)} + \left((V_{02})_I^{(j)} + (V_{02}^{'})_I^{(j)}\right) - 2(V_{11})_I^{(j)}\right] \tag{30}$$

Using expectation $E[.]$of sample mean, following are expressions up-to first order of approximations ($see\ references$ [24][25] [26][27]):

$$\left(V_{qm}^{(j)}\right)_T = E\left[\left\{\left(\bar{T}.^{''(j)}\right) - (\bar{T}.)_{t_j}\right\}^q \left\{\left(\bar{V}.^{''(j)}\right) - (\bar{V}.)_{t_j}\right\}^m\right]$$

$$= \rho_{TV}^r \frac{N-n}{Nn}\left[\left(C_T^{(j)}\right)_{text}\right]^q \left[\left(C_V^{(j)}\right)_{video}\right]^m \tag{31}$$

$$\left(V_{qm}^{'(j)}\right)_T = E\left[\left\{\left(\bar{T}.^{''(j)}\right) - (\bar{T}.)_{t_j}\right\}^q \left\{\left(\bar{V}.^{'(j)}\right) - (\bar{V}.)_{t_j}\right\}^m\right]$$

$$= \rho_{TV}^r \frac{N-n^{'}}{Nn^{'}}\left[\left(C_T^{(j)}\right)_{text}\right]^q \left[\left(C_V^{(j)}\right)_{video}\right]^m \tag{32}$$

$$\left(V_{qm}^{(j)}\right)_V = E\left[\left\{\left(\bar{V}.^{''(j)}\right) - (\bar{V}.)_{t_j}\right\}^q \left\{\left(\bar{I}.^{''(j)}\right) - (\bar{I}.)_{t_j}\right\}^m\right]$$

$$= \rho_{VI}^r \frac{N-n}{Nn}\left[\left(C_V^{(j)}\right)_{video}\right]^q \left[\left(C_I^{(j)}\right)_{image}\right]^m \tag{33}$$

$$\left(V_{qm}^{'(j)}\right)_V = E\left[\left\{\left(\bar{V}.^{''(j)}\right) - (\bar{V}.)_{t_j}\right\}^q \left\{\left(\bar{I}.^{'(j)}\right) - (\bar{I}.)_{t_j}\right\}^m\right]$$

$$= \rho_{VI}^r \frac{N-n^{'}}{Nn^{'}}\left[\left(C_V^{(j)}\right)_{video}\right]^q \left[\left(C_I^{(j)}\right)_{image}\right]^m \tag{34}$$

$$\left(V_{qm}^{(j)}\right)_I = E\left[\left\{\left(\bar{I}.^{''(j)}\right) - (\bar{I}.)_{t_j}\right\}^q \left\{\left(\bar{T}.^{''(j)}\right) - (\bar{T}.)_{t_j}\right\}^m\right]$$

$$= \rho_{IT}^r \frac{N-n}{Nn}\left[\left(C_I^{(j)}\right)_{image}\right]^q \left[\left(C_T^{(j)}\right)_{text}\right]^m \tag{35}$$

$$\left(V'^{(j)}_{qm}\right)_I = E\left[\left\{\left(\bar{I}.^{''(j)}\right) - (\bar{I}.)_{t_j}\right\}^q \left\{\left(\bar{T}.^{'(j)}\right) - (\bar{T}.)_{t_j}\right\}^m\right]$$

$$= \rho_{IT}^r \frac{N-n^{'}}{Nn^{'}}\left[\left(C_I^{(j)}\right)_{images}\right]^q \left[\left(C_T^{(j)}\right)_{text}\right]^m \tag{36}$$

where $q = 0,1,2$ $m = 0,1,2$ and $r = 1$ if $q = p = 1$ else $r = 0$.

The pooled estimates, based on sample $n^{''}$, over M different time points (occasions) are :

$$\left[(E_1)_{text}\right]_{\mathrm{I}} = \sum_{j=1}^{M} W_{jT} \left(E_1^{(j)}\right)_{text} \quad , \ W_{jT} = \frac{1}{M} \tag{37}$$

$$\left[(E_2)_{video}\right]_{\mathrm{I}} = \sum_{j=1}^{M} W_{jV} \left(E_2^{(j)}\right)_{video} \quad , \ W_{jV} = \frac{1}{M} \tag{38}$$

$$\left[(E_3)_{image}\right]_{\mathrm{I}} = \sum_{j=1}^{M} W_{jI} \left(E_3^{(j)}\right)_{image} \quad , \ W_{jI} = \frac{1}{M} \tag{39}$$

The (37), (38), (39) are weighted average over M occasions of E_1 , E_2 , E_3 under case I and same is derived for case II in (40), (41), (42).

$$\left[(E_1)_{text}\right]_{\mathrm{II}} = \sum_{j=1}^{M} W_{jT} \left(E_1^{(j)}\right)_{text} \quad , \; W_{jT} = \frac{1}{M} \tag{40}$$

$$[(E_2)_{video}]_{\mathrm{II}} = \sum_{j=1}^{M} W_{jV} \left(E_2^{(j)}\right)_{video} \quad , \; W_{jV} = \frac{1}{M} \tag{41}$$

$$\left[(E_3)_{image}\right]_{\mathrm{II}} = \sum_{j=1}^{M} W_{jI} \left(E_3^{(j)}\right)_{image} \quad , \; W_{jI} = \frac{1}{M} \tag{42}$$

The pooled mean squared errors (MSE) on M points of time also have weighted sum shown in (43) to (48) for Case I and II.

$$[MSE(E_1)_{text}]_{\mathrm{I}} = \sum_{j=1}^{M} W_{jT}^2 \; MSE(E_1^{(j)})_{text} \tag{43}$$

$$[MSE(E_2)_{video}]_{\mathrm{I}} = \sum_{j=1}^{M} W_{jV}^2 \; MSE\,(E_2^{(j)})_{video} \tag{44}$$

$$\left[MSE(E_3)_{image}\right]_{\mathrm{I}} = \sum_{j=1}^{M} W_{jI}^2 \; MSE(E_3^{(j)})_{image} \tag{45}$$

$$[MSE(E_1)_{text}]_{\mathrm{II}} = \sum_{j=1}^{M} W_{jT}^2 \; MSE(E_1^{(j)})_{text} \tag{46}$$

$$[MSE(E_2)_{video}]_{\mathrm{II}} = \sum_{j=1}^{M} W_{jV}^2 \; MSE\,(E_2^{(j)})_{video} \tag{47}$$

$$\left[MSE(E_3)_{image}\right]_{\mathrm{II}} = \sum_{j=1}^{M} W_{jI}^2 \; MSE(E_3^{(j)})_{image} \tag{48}$$

The 95% confidence interval, in general, is defined for two estimated numbers $a^{'}, b^{'}$ in probability sense denoted as $P[.]$ like $\mathrm{P}\left[a^{'} < True\ Value\ < b^{'}\right] = 0.95$. It is explained as estimate $a^{'}, b^{'}$ obtained from sample, there is 95% chance that $a^{'}, b^{'}$ will catch (predict) the true value. More explicitly, the 95% confidence interval is computed as $P\left[sample\ mean \pm 1.96\sqrt{standard\ error}\right] = 0.95$ (see [25]).

For Case I, the confidence intervals (CI) are in (49), (50) and (51).

$$P\left[(E_1)_{text} - 1.96\sqrt{[MSE(E_1)_{text}]_{\mathrm{I}}}\;, \quad (E_1)_{text} + 1.96\sqrt{[MSE(E_1)_{text}]_{\mathrm{I}}}\,\right] = 0.95 \tag{49}$$

$$P\left[(E_2)_{video} - 1.96\sqrt{[MSE(E_2)_{video}]_{\mathrm{I}}}\;, \quad (E_2)_{video} + 1.96\sqrt{[MSE(E_2)_{video}]_{\mathrm{I}}}\,\right] = 0.9 \tag{50}$$

$$P\left[(E_3)_{image} - 1.96\sqrt{\left[MSE(E_3)_{image}\right]_{\mathrm{I}}}\,, \quad (E_3)_{image} + 1.96\sqrt{\left[MSE(E_3)_{image}\right]_{\mathrm{I}}}\;\right] = 0.95 \tag{51}$$

For second case II, CI are expressed in (52), (53) and (54).

$$P\left[(E_1)_{text} - 1.96\sqrt{[MSE(E_1)_{text}]_{\mathrm{II}}}\;, \quad (E_1)_{text} + 1.96\sqrt{[MSE(E_1)_{text}]_{\mathrm{II}}}\,\right] = 0.95 \tag{52}$$

$$P\left[(E_2)_{video} - 1.96\sqrt{[MSE(E_2)_{video}]_{\mathrm{II}}}\;, \quad (E_2)_{video} + 1.96\sqrt{[MSE(E_2)_{video}]_{\mathrm{II}}}\,\right] = 0.95 \tag{53}$$

$$P\left[(E_3)_{image} - 1.96\sqrt{\left[MSE(E_3)_{image}\right]_{\mathrm{II}}}\,, \quad (E_3)_{image} + 1.96\sqrt{\left[MSE(E_3)_{image}\right]_{\mathrm{II}}}\;\right] = 0.95 \tag{54}$$

I. Population Description

For calculation and comparison, in order to avoid complexity, a small population of size N=100 is considered whose detail is in annexure A. Descriptive statistics of the population as per (1), (2), (3), (4), (5), (6), (7), (8) are in table 1 calculated at six points of time t_1 to t_6.

Table 1: *Descriptive statistics of population at six points of time (users are the same)*

t_1, N=100	$[\bar{T}.]_{t_1}$ =74.14	$[\bar{V}.]_{t_1}$ =105.3	$[\bar{I}.]_{t_1}$ =145.07	$\rho_{T,V}{}^{(1)}$ =0.7
	$S_T^{2(1)}$ =1537.04	$S_V^{2(1)}$ =3756.03	$S_I^{2(1)}$ =6784.69	$\rho_{V,I}{}^{(1)}$ =0.8
	$C_T^{(1)}$ =0.53	$C_V^{(1)}$ =0.58	$C_I^{(1)}$ =0.57	$\rho_{I,T}{}^{(1)}$ =0.7
t_2, N=100	$[\bar{T}.]_{t_2}$ =67.7	$[\bar{V}.]_{t_2}$ =98.13	$[\bar{I}.]_{t_2}$ =226.18	$\rho_{T,V}{}^{(2)}$ =0.6
	$S_T^{2(2)}$ =1365.71	$S_V^{2(2)}$ =3501.81	$S_I^{2(2)}$ =16979.73	$\rho_{V,I}{}^{(2)}$ =0.7
	$C_T^{(2)}$ =0.55	$C_V^{(2)}$ =0.60	$C_I^{(2)}$ =0.58	$\rho_{I,T}{}^{(2)}$ =0.5
t_3, N=100	$[\bar{T}.]_{t_3}$ =125.92	$[\bar{V}.]_{t_3}$ =137.29	$[\bar{I}.]_{t_3}$ =362.74	$\rho_{T,V}{}^{(3)}$ =0.5
	$S_T^{2(3)}$ =4212.01	$S_V^{2(3)}$ =7083.59	$S_I^{2(3)}$ =42405.57	$\rho_{V,I}{}^{(3)}$ =0.8
	$C_T^{(3)}$ =0.52	$C_V^{(3)}$ =0.61	$C_I^{(3)}$ =0.57	$\rho_{I,T}{}^{(3)}$ =0.7
t_4, N=100	$[\bar{T}.]_{t_4}$ =110.79	$[\bar{V}.]_{t_4}$ =144.05	$[\bar{I}.]_{t_4}$ =142.45	$\rho_{T,V}{}^{(4)}$ =0.7
	$S_T^{2(4)}$ =2382.75	$S_V^{2(4)}$ =5670.83	$S_I^{2(4)}$ =7309.01	$\rho_{V,I}{}^{(4)}$ =0.8
	$C_T^{(4)}$ =0.44	$C_V^{(4)}$ =0.52	$C_I^{(4)}$ =0.60	$\rho_{I,T}{}^{(4)}$ =0.6
t_5 , N=100	$[\bar{T}.]_{t_5}$ =148.92	$[\bar{V}.]_{t_5}$ =236.51	$[\bar{I}.]_{t_5}$ =257.97	$\rho_{T,V}{}^{(5)}$ =0.5
	$S_T^{2(5)}$ =7393.63	$S_V^{2(5)}$ =15047.95	$S_I^{2(5)}$ =17480.67	$\rho_{V,I}{}^{(5)}$ =0.8
	$C_T^{(5)}$ =0.58	$C_V^{(5)}$ =0.52	$C_I^{(5)}$ =0.51	$\rho_{I,T}{}^{(5)}$ =0.5
	W_{5T} =0.167	W_{5V} =0.167	W_{5I} =0.167	
t_6 , N=100	$[\bar{T}.]_{t_6}$ =173.5	$[\bar{V}.]_{t_6}$ =308.78	$[\bar{I}.]_{t_6}$ =306.78	$\rho_{T,V}{}^{(6)}$ =0.7
	$S_T^{2(6)}$ =4997.55	$S_V^{2(6)}$ =29899.47	$S_I^{2(6)}$ =29761.89	$\rho_{V,I}{}^{(6)}$ =0.8
	$C_T^{(6)}$ =0.41	$C_V^{(6)}$ =0.56	$C_I^{(6)}$ =0.56	$\rho_{I,T}{}^{(6)}$ =0.6

Primary sample of size $n^{'} = 40$ is drawn from N=100 to calculate the mean size of unknown parameters $[\bar{T}.]_{t_j}$, $[\bar{V}.]_{t_j}$ and $\bar{I}._{t_j}$ over six points of time. This sample is used to have a guess value of the population parameter to use as supportive information. Calculation of sample means on $n^{'} = 40$ is in table 2.

Table 2: *Sample-based mean estimates at six occasions ($n^{'}$=40 primary sample)*

At time t_1(occasion one) $n' = 40$	$[\bar{T}.^{'}]_{t_1}$ =83.58	$[\bar{V}.^{'}]_{t_1}$ =114.85	$\bar{I}.^{'}{}_{t_1}$ =152.85
At time t_2(occasion two)$n' = 40$	$[\bar{T}.^{'}]_{t_2}$ =75.95	$[\bar{V}.^{'}]_{t_2}$ =107.12	$\bar{I}.^{'}{}_{t_2}$ =244.53
At time t_3(occasion third) $n' = 40$	$[\bar{T}.^{'}]_{t_3}$ =139.07	$[\bar{V}.^{'}]_{t_3}$ =153.5	$\bar{I}.^{'}{}_{t_3}$ =382.25
At time t_4(occasion four) $n' = 40$	$[\bar{T}.^{'}]_{t_4}$ =120.95	$[\bar{V}.^{'}]_{t_4}$ =150.9	$\bar{I}.^{'}{}_{t_4}$ =157.97
At time t_5(occasion five) $n' = 40$	$[\bar{T}.^{'}]_{t_5}$ =174.12	$[\bar{V}.^{'}]_{t_5}$ =274.93	$\bar{I}.^{'}{}_{t_5}$ =281.48
At time t_6(occasion six) $n' = 40$	$[\bar{T}.^{'}]_{t_6}$ =181.38	$[\bar{V}.^{'}]_{t_6}$ =362.38	$\bar{I}.^{'}{}_{t_6}$ =337.05

A second sample of size $n^{''} = 10$ is taken for estimation of means on variable of main interest over six points of time. Estimates on $n^{''}$are in table 3 for strategy under case I. Similarly, for strategy under case II, the calculations are in table 4. The pooled estimate of text-data, video-data and images-data using equation (22), (23), (24) are in table 5 and table 6 along with MSE calculation using (25) to (30).

Table 3: *Result of sample-based calculation at six occasions (n''=10, first sample) under Case I [eq. (13)-(21) and (25)-(27)]*

t_1 $n' = 10$	$[\bar{T}.'']_{t_1}$ =99.30	$[\bar{V}.'']_{t_1}$ =118.30	$[\bar{I}.'']_{t_1}$ =148.90	MSE_Text=265.51	$\rho_{TV}^{''(1)}$ =0.4
	$S_T^{2''(1)}$ =1220.90	$S_V^{2''(1)}$ =5572.46	$S_I^{2''(1)}$ =7921.43	MSE_Video=566.86	$\rho_{VI}^{''(1)}$ =0.4
	$C_T^{''(1)}$ =0.35	$C_V^{''(1)}$ =0.63	$C_I^{''(1)}$ =0.60	MSE_Image=768.16	$\rho_{IT}^{'(1)}$ =0.2
t_2 $n' = 10$	$[\bar{T}.'']_{t_2}$ =84.80	$[\bar{V}.'']_{t_2}$ =102.40	$[\bar{I}.'']_{t_2}$ =238.20	MSE_Text=424.44	$\rho_{TV}^{''(2)}$ =-0.3
	$S_T^{2''(2)}$ =1230.18	$S_V^{2''(2)}$ =4394.04	$S_I^{2''(2)}$ =20243.96	MSE_Video=490.99	$\rho_{VI}^{''(2)}$ =0.3
	$C_T^{''(2)}$ =0.41	$C_V^{''(2)}$ =0.65	$C_I^{''(2)}$ =0.60	MSE_Image=2348.19	$\rho_{IT}^{''(2)}$ =0.1
t_3 $n' = 10$	$[\bar{T}.'']_{t_3}$ =144.40	$[\bar{V}.'']_{t_3}$ =165.40	$[\bar{I}.'']_{t_3}$ =372.40	MSE_Text=1197.09	$\rho_{TV}^{''(3)}$ =-0.4
	$S_T^{2''(3)}$ =3185.16	$S_V^{2''(3)}$ =10899.60	$S_I^{2''(3)}$ =49400.49	MSE_Video=1106.98	$\rho_{VI}^{''(3)}$ =0.4
	$C_T^{''(3)}$ =0.39	$C_V^{''(3)}$ =0.63	$C_I^{''(3)}$ =0.60	MSE_Image=6106.08	$\rho_{IT}^{''(3)}$ =-0.0
t_4 $n' = 10$	$[\bar{T}.'']_{t_4}$ =129.80	$[\bar{V}.'']_{t_4}$ =132.20	$[\bar{I}.'']_{t_4}$ =147.10	MSE_Text=98.56	$\rho_{TV}^{''(4)}$ =0.6
	$S_T^{2''(4)}$ =1307.51	$S_V^{2''(4)}$ =2028.84	$S_I^{2''(4)}$ =8529.66	MSE_Video=345.68	$\rho_{VI}^{''(4)}$ =0.7
	$C_T^{''(4)}$ =0.28	$C_V^{''(4)}$ =0.34	$C_I^{''(4)}$ =0.63	MSE_Image=523.50	$\rho_{IT}^{''(4)}$ =0.6
t_5 $n' = 10$	$[\bar{T}.'']_{t_5}$ =193.20	$[\bar{V}.'']_{t_5}$ =307.40	$[\bar{I}.'']_{t_5}$ =323.30	MSE_Text=752.35	$\rho_{TV}^{''(5)}$ =0.4
	$S_T^{2''(5)}$ =9574.18	$S_V^{2''(5)}$ =8915.60	$S_I^{2''(5)}$ =13703.34	MSE_Video=520.64	$\rho_{VI}^{''(5)}$ =0.8
	$C_T^{''(5)}$ =0.51	$C_V^{''(5)}$ =0.31	$C_I^{''(5)}$ =0.36	MSE_Image=3214.32	$\rho_{IT}^{''(5)}$ =0.0
t_6 $n' = 10$	$\bar{T}.''_{t_6}$ =212.00	$[\bar{V}.'']_{t_6}$ =412.60	$[\bar{I}.'']_{t_6}$ =276.10	MSE_Text=478.96	$\rho_{TV}^{''(6)}$ =0.7
	$S_T^{2''(6)}$ =2686.44	$S_V^{2''(6)}$ =38532.04	$S_I^{2''(6)}$ =27937.66	MSE_Video=6424.23	$\rho_{VI}^{''(6)}$ =0.2
	$C_T^{''(6)}$ =0.24	$C_V^{''(6)}$ =0.48	$C_I^{''(6)}$ =0.61	MSE_Image=1875.89	$\rho_{IT}^{''(6)}$ =0.6

Table 4: *Result of sample-based calculation (n''=10, first sample) under Case II [eq. (13)-(21) and (28)-(30)]*

t_1 $n' = 10$	$[\bar{T}.'']_{t_1}$ =99.30	$[\bar{V}.'']_{t_1}$ =118.30	$[\bar{I}.'']_{t_1}$ =148.90	MSE_Text=355.53	$\rho_{TV}^{''(1)}$ =0.4
	$S_T^{2''(1)}$ =1220.90	$S_V^{2''(1)}$ =5572.46	$S_I^{2''(1)}$ =7921.43	MSE_Video=654.93	$\rho_{VI}^{''(1)}$ =0.4
	$C_T^{''(1)}$ =0.35	$C_V^{''(1)}$ =0.63	$C_I^{''(1)}$ =0.60	MSE_Image=820.38	$\rho_{IT}^{''(1)}$ =0.2
t_2 $n' = 10$	$[\bar{T}.'']_{t_2}$ =84.80	$[\bar{V}.'']_{t_2}$ =102.40	$[\bar{I}.'']_{t_2}$ =238.20	MSE_Text=532.38	$\rho_{TV}^{''(2)}$ =-0.3
	$S_T^{2''(2)}$ =1230.18	$S_V^{2''(2)}$ =4394.04	$S_I^{2''(2)}$ =20243.96	MSE_Video=566.21	$\rho_{VI}^{''(2)}$ =0.3
	$C_T^{''(2)}$ =0.41	$C_V^{''(2)}$ =0.65	$C_I^{''(2)}$ =0.60	MSE_Image=2599.03	$\rho_{IT}^{''(2)}$ =0.1
t_3 $n' = 10$	$[\bar{T}.'']_{t_3}$ =144.40	$[\bar{V}.'']_{t_3}$ =165.40	$[\bar{I}.'']_{t_3}$ =372.40	MSE_Text=1503.78	$\rho_{TV}^{''(3)}$ =-0.4
	$S_T^{2''(3)}$ =3185.16	$S_V^{2''(3)}$ =10899.60	$S_I^{2''(3)}$ =49400.49	MSE_Video=1278.36	$\rho_{VI}^{''(3)}$ =0.4
	$C_T^{''(3)}$ =0.39	$C_V^{''(3)}$ =0.63	$C_I^{''(3)}$ =0.60	MSE_Image=6755.85	$\rho_{IT}^{''(3)}$ =-0.0
t_4 $n' = 10$	$[\bar{T}.'']_{t_4}$ =129.80	$[\bar{V}.'']_{t_4}$ =132.20	$[\bar{I}.'']_{t_4}$ =147.10	MSE_Text=124.07	$\rho_{TV}^{''(4)}$ =0.6
	$S_T^{2''(4)}$ =1307.51	$S_V^{2''(4)}$ =2028.84	$S_I^{2''(4)}$ =8529.66	MSE_Video=481.63	$\rho_{VI}^{''(4)}$ =0.7
	$C_T^{''(4)}$ =0.28	$C_V^{''(4)}$ =0.34	$C_I^{''(4)}$ =0.63	MSE_Image=499.85	$\rho_{IT}^{''(4)}$ =0.6
t_5 $n' = 10$	$[\bar{T}.'']_{t_5}$ =193.20	$[\bar{V}.'']_{t_5}$ =307.40	$[\bar{I}.'']_{t_5}$ =323.30	MSE_Text=783.31	$\rho_{TV}^{''(5)}$ =0.4
	$S_T^{2''(5)}$ =9574.18	$S_V^{2''(5)}$ =8915.60	$S_I^{2''(5)}$ =13703.34	MSE_Video=650.11	$\rho_{VI}^{''(5)}$ =0.8
	$C_T^{''(5)}$ =0.51	$C_V^{''(5)}$ =0.31	$C_I^{''(5)}$ =0.36	MSE_Image=4012.67	$\rho_{IT}^{''(5)}$ =0.0
t_6 $n'' = 10$	$[\bar{T}.'']_{t_6}$ =212.00	$[\bar{V}.'']_{t_6}$ =412.60	$[\bar{I}.'']_{t_6}$ =276.10	MSE_Text=678.99	$\rho_{TV}^{''(6)}$ =0.7
	$S_T^{2''(6)}$ =2686.44	$S_V^{2''(6)}$ =38532.04	$S_I^{2''(6)}$ =27937.66	MSE_Video=7951.35	$\rho_{VI}^{''(6)}$ =0.2
	$C_T^{''(6)}$ =0.24	$C_V^{''(6)}$ =0.48	$C_I^{''(6)}$ =0.61	MSE_Image=1816.54	$\rho_{IT}^{''(6)}$ =0.6

Table 5: *Result of sample-based pooled calculation at six occasions under case I using one sample*

$\boldsymbol{n''=10}$	$\left[(E_1)_{text}\right]_{\mathrm{I}}$ =126.06	$\left[(E_2)_{video}\right]_{\mathrm{I}}$ =195.47	$\left[(E_3)_{image}\right]_{\mathrm{I}}$ =258.18
	$[MSE(E_1)_{text}]_{\mathrm{I}}$ =40.93	$[MSE(E_2)_{video}]_{\mathrm{I}}$ =44.61	$\left[MSE(E_3)_{image}\right]_{\mathrm{I}}$ =304.28
CI	(113.52-138.59)	(182.38-208.56)	(223.99-292.36)
True Value	116.82	171.62	240.19
Length (CI)	25.07	26.18	68.37

Table 6: *Result of sample-based pooled calculation at six occasions under case II using one sample*

$\boldsymbol{n''=10}$	$\left[(E_1)_{text}\right]_{\mathrm{II}}$ =126.06	$\left[(E_2)_{video}\right]_{\mathrm{II}}$ =195.47	$\left[(E_3)_{image}\right]_{\mathrm{II}}$ =258.18
	$[MSE(E_1)_{text}]_{\mathrm{II}}$ =47.85	$[MSE(E_2)_{video}]_{\mathrm{II}}$ =54.95	$\left[MSE(E_3)_{image}\right]_{\mathrm{II}}$ =339.36
CI	(112.50-139.61)	(181.21-209.99)	(222.04-294.31)
True Value	116.82	171.62	240.19
Length (CI)	27.11	28.78	72.27

Table 5 and table 6 contain one-sample combined estimates, pooled to six occasions, on the variable of main interest (T or V or I). The MSE under case I is smaller than Case II. The length of confidence intervals in case I is lower showing efficiency over case II.

II. Practically Difficulty

The confidence intervals (CI) in table 5 and table 6 are sample dependent therefore difficult to conclude uniquely. Reason behind is that one can draw many samples of size n'' from n' (total $n'_{C_{n''}}$) and many from N (total $N_{C_{n''}}$). Each time the average of sample estimate fluctuates and accordingly variation occur in predicted value of confidence intervals. Look at table 6 , $\left[(E_2)_{video}\right]_{\mathrm{II}}$ =195.47, CI = (181.21-209.99) where CI does not catch the true value 171.62 which is evidence of difficulty. To cope up this, a new simulation procedure is proposed in section 6.2 based on many samples who ultimately determines the single-value of lower and upper limits.

III. Simulation Procedure Algorithm for Double Sampling

In order to get single-value of limits of 95% confidence interval , a simulation procedure is proposed:

Step 1: Draw a primary random sample of size n'.

Step2: Draw second sample as under
Case I: as sub-sample of n'
Case II: as independent sample from N

Step 3: Compute lower limit (say 'a') and upper limit (say 'b') of confidence interval(CI) using each sub-sample (or independent sample) , where 95% confidence interval is $Prob.[a < true\ value < b] = 0.95$. It is like table 3 and table 4 form t_1 to t_6 using equations (49) to (54).

Step 4: Repeat step 2 and step 3 for k times (k=200).

Step 5: Compute the Less Than Type (LTT) and More Than Type (MTT) cumulative probabilities by constructing class-intervals for 'a' and 'b' separately for each CI.

Step 6: Plot data of step 5 of cumulative probabilities (on y-axis) over class-intervals (on x-axis) and draw two graphs. A perpendicular drawn from point of intersection of two graphs ,on the x-axis, determines single-point of simulated value of lower limit 'a' (and corresponding upper limit 'b') of confidence interval for unknown parameters to be predicted. Express outcomes in tabular presentation like tables 8,9,10 and table 11.

IV. Features of proposed simulation procedure for double sampling:

(a) It is based on k-samples, where K may be as large possible.

(b) It considers cumulative probabilities which is ratio of cumulative frequency to total frequency.

(c) It takes into account the perpendicular drawn from point of intersection of the cumulative probability curves which always remain unique for lower as well as upper limit.

(d) It eliminates problem discussed in section I.

V. Demonstration of Simulation procedure:

Out of k= 200 samples, after calculation of confidence intervals on each sample, let f_i be frequencies of class intervals $\propto_i - \propto_{i+1}$ relating to lower limit of CI , such that $\sum f_i = k = 200$ holds. Probabilities are $p_i = f_i/k$, $i = 1,2,3, \ldots$

Table 7. Demonstration of Simulation Procedure

Class Intervals (for lower limit 'a')	**Frequencies** (Occurrence of estimate 'a ')	**Probabilities**	**LTT (Step 5)**	**MTT (Step 5)**
$\propto_1-\propto_2$	f_1	$p_1 = f_1/k$	$C_1 = p_1$	$C_1' = 1$
$\propto_2-\propto_3$	f_2	$p_2 = f_2/k$	$C_2 = p_1 + p_2$	$C_2' = 1 - p_1$
$\propto_3-\propto_4$	f_3	$p_3 = f_3/k$	$C_3 = p_1 + p_2 + p_3$	$C_3' = 1 - p_1 - p_2$
$\propto_4-\propto_5$	f_4	$p_4 = f_4/k$	$C_4 = p_1 + p_2 + p_3 + p_4$	$C_4' = 1 - p_1 - p_2 - p_3$
$---$	.	.	.	.
$---$	.	.	.	.
$---$	.	.	.	.
Total	$\sum f_i = k = 200$	$\sum p_i = 1$		

Plot C_i and C_i' over class-interval on a graph to find point of intersection of two curves (step 6). Draw a perpendicular on X-axis from point of intersection which uniquely determine single-value of 'a'.

VI. Application of proposed Simulation Procedure

Figure 7-78 provide the simulated single-valued lower limit 'a' and simulated single-valued upper limit 'b' of confidence intervals as an application.
Graphs (fig. 7-42) are under case I on t_1 to t_6 . Note that SCI symbolized for simulated confidence interval, TD text dataset, VD video data and ID indicates image dataset in all hereunder:

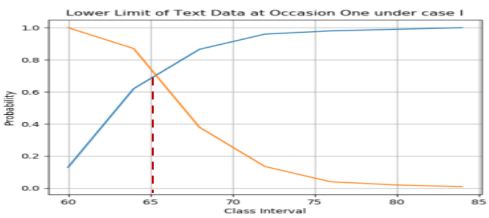


Figure 7: *At t_1, Case I, Lower limit of text data at (a=65.30)*

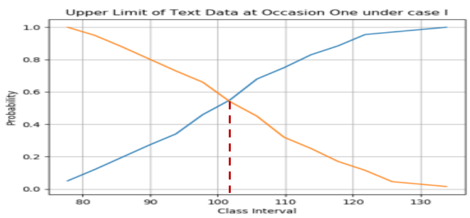
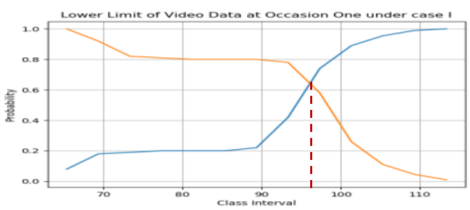


Figure 8: *At t_1, Case I, Lower limit of text data at (b=101.56)*

The figure 7 is calculates value a=65.30 (perpendicular from intersection point) which is the lower limit of simulated confidence interval of text dataset at $t_1$under case I. Similarly, figure 8 has upper limit of simulated confidence interval of text data at t_1 under case I whose perpendicular from point of intersection is b=101.56.



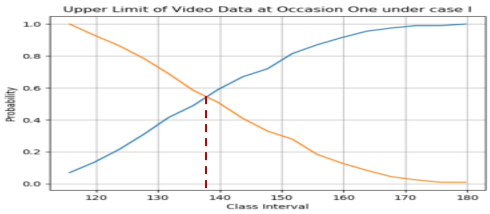
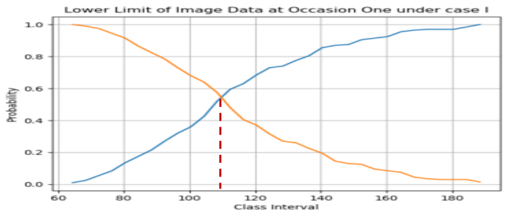


Figure 9: *At t_1, Case I, Lower limit of video data at (a=96.04)* **Figure10:** *At t_1, Case I, Upper limit of video data at (b=137.79)*

Figure 9 provides value a=96.04 (perpendicular from intersection point) as lower limit of simulated confidence interval of video dataset at t_1 under case I. Likewise, figure 10 shows upper limit of simulated confidence interval of video data at t_1 under case I where perpendicular from intersection point is at b=137.79.



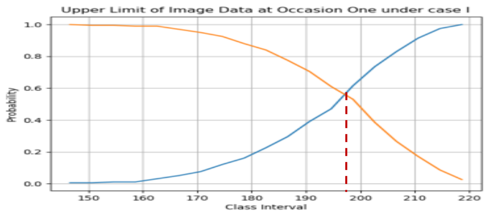
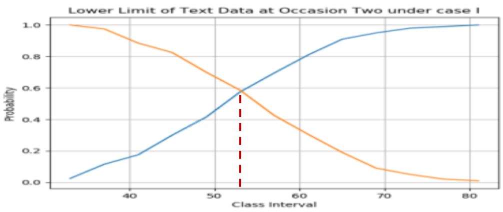


Figure 11: *At t_1, Case I, Lower limit of image data at (a=109.36)* **Figure 12:** *At t_1, Case I, Upper limit of image data at (b=197.01)*

Figure 11 reveals the value a=109.36 (perpendicular from intersection point) as lower limit of SCI of image dataset at t_1 under case I. Figure 12 displays upper limit of SCI of image data at occasion one, under case I which is b=197.01.



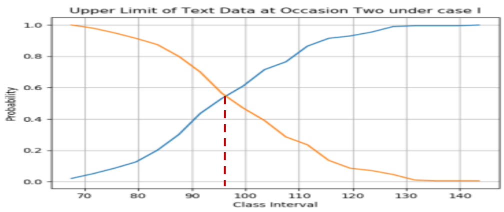
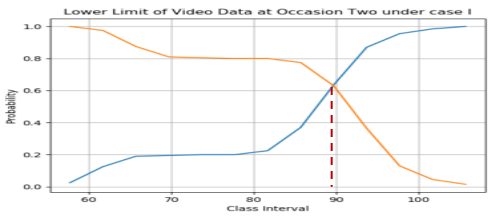


Figure 13: *At t_2, Case I, Lower limit of text data at (a=53.08)* **Figure 14:** *At t_2, Case I, Upper limit of text data at (b=96.30)*

Figure 13 is at time t_2 showing value a=53.08 , under case I , and figure 14 is similar for upper limit b=96.30 under case I at t_2



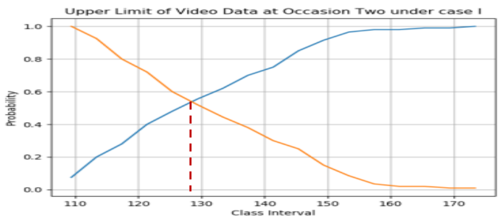
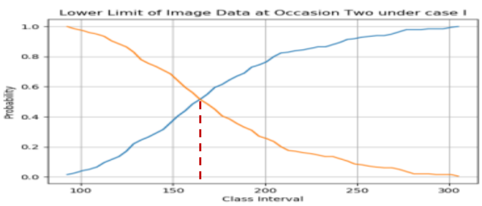


Figure 15: *At t_2, Case I, Lower limit of video data at (a=89.56)* **Figure 16:** *At t_2, Case I, Upper limit of video data at (b=128.33)*

Figure 15 and 16 reveal value a=89.56 as lower and b=128.33 as upper at t_2, for case I.



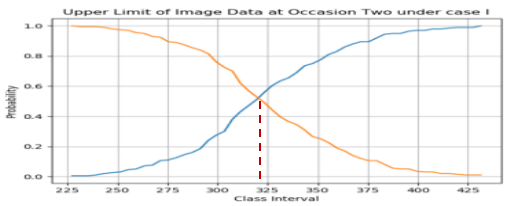
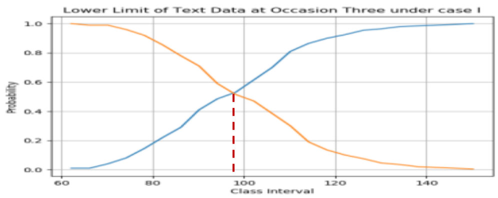


Figure 17: *At t_2, Case I, Lower limit of image data at (a=164.44)* **Figure 18:** *At t_2, Case I, Upper limit of image data at (b=320.34)*

Figure 17 and 18 reflect towards value a=164.44 and b=320.34 at t_2 case I.



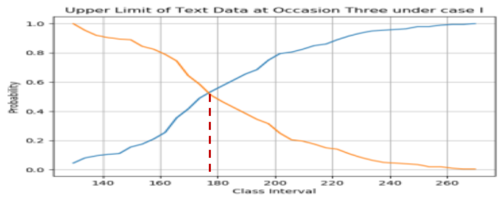
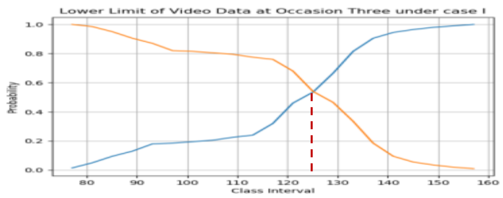


Figure 19: *At t_3, Case I, Lower limit of text data at (a=97.39)* **Figure 20:** *At t_3, Case I, Upper limit of text data at (b=176.44)*

Figure 19 and 20 reveal for a=97.39 and b=176.44 at t_2 case I.



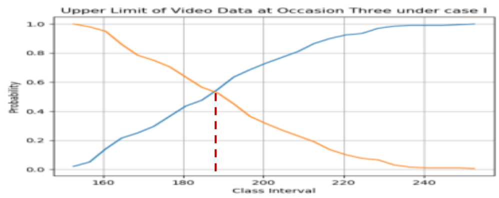
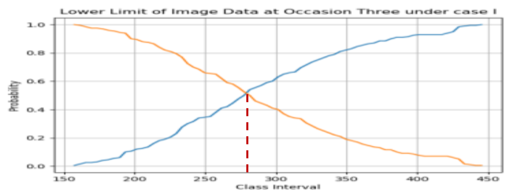


Figure 21: *At t_3, Case I, Lower limit of video data at (a=125.06)* **Figure 22:** *At t_3, Case I, Upper limit of video data at (b=187.50)*

The figure 21 and 22 have a=125.06, b=187.50.



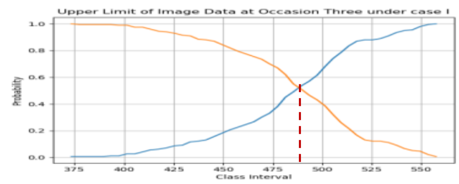


Figure 23: *At t_3, Case I, Lower limit of image data at (a=278.48)* **Figure 24:** *At t_3, Case I, Upper limit of image data at (b=487.83)*

Values a=278.48 and b=487.83 are in figure 23 to 24. Similar are in figure 25 to 78 under case I and case II for T, V and I over time t_1 to t_6. Figure caption from 25-78 are self explanatory and reveal auto interpretation as above

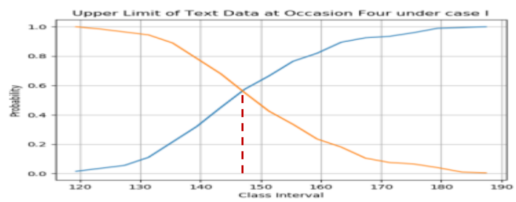
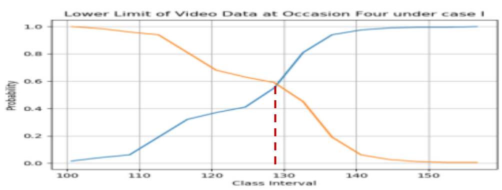


Figure 25: *At t_4, Case I, Lower limit of text data at (a=93.22)* **Figure 26:** *At t_4, Case I, Upper limit of text data at (b=146.75)*



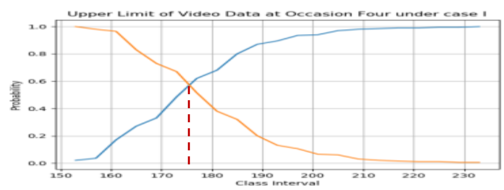
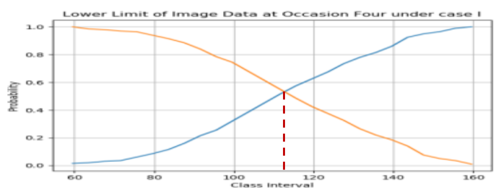


Figure 27: *At t_4, Case I, Lower limit of video data at (a=129.03)* **Figure 28:** *At t_4, Case I, Upper limit of video data at (b=175.43)*



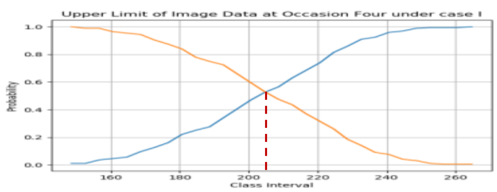
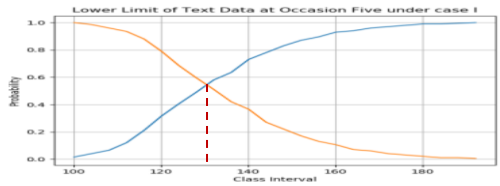


Figure 29: *At t_4, Case I, Lower limit of image data at (a=112.47)* **Figure 30:** *At t_4, Case I, Upper limit of image data at (b=204.60)*



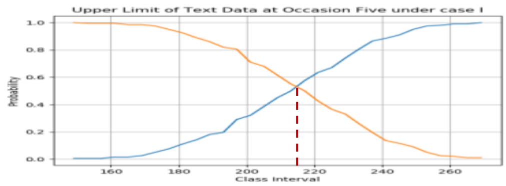
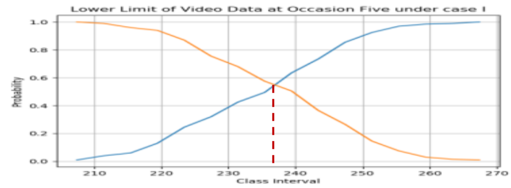


Figure 31: *At t_5, Case I, Lower limit of text data at (a=130.39)* **Figure 32:** *At t_5, Case I, Upper limit of text data at (b=214.44)*



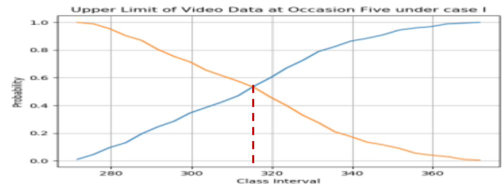


Figure 33: *At t_5, Case I, Lower limit of video data at (a=236.75)* **Figure 34:** *At t_5, Case I, Upper limit of video data at (b=314.93)*

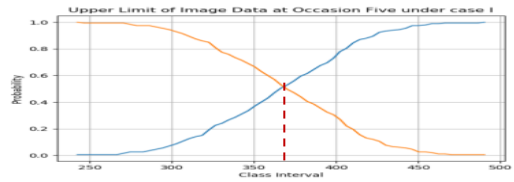


Figure 35: *At t_5, Case I, Lower limit of image data at (a=200.37)* **Figure 36:** *At t_5, Case I, Upper limit of image data at (b=368.03)*

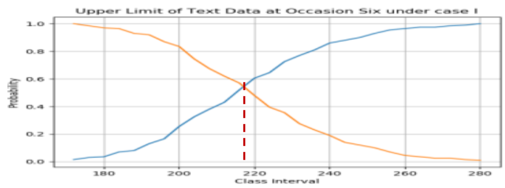
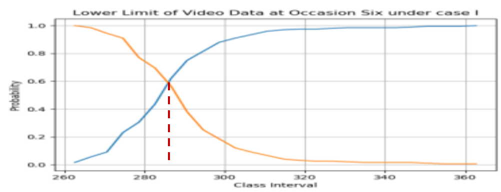


Figure 37: *At t_6, Case I, Lower limit of text data at (a=142.73)* **Figure 38:** *At t_6, Case I, Upper limit of text data at (b=216.89)*



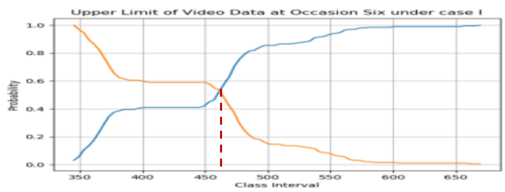
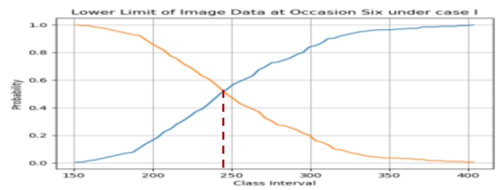


Figure 39: *At t_6, Case I, Lower limit of video data at (a=285.58)* **Figure 40:** *At t_6, Case I, Upper limit of video data at (b=461.31)*



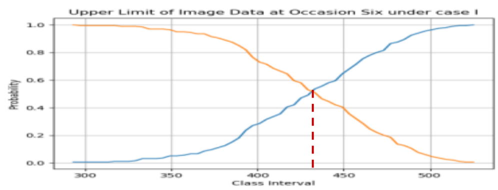


Figure 41: *At t_6, Case I, Lower limit of image data at (a=244.35)* **Figure 42:** *At t_6, Case I, Upper limit of image data at (b=431.30)*

The following Fig. 43-78 are showing time point wise t_1 to t_6 the simulated results under case II.

Figure 43: *At t_1, Case II, Lower limit of text data at (a=63.72)* **Figure 44:** *At t_1, Case II, Upper limit of text data at (b=103.49)*

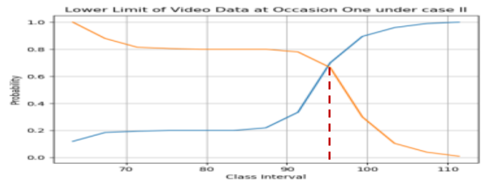


Figure 45: *At t_1, Case II, Lower limit of video data at (a=94.95)*

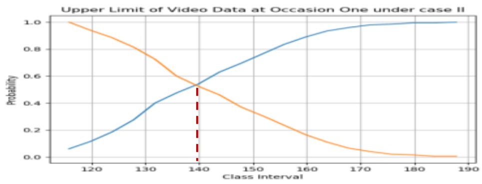


Figure 46: *At t_1, Case II, Upper limit of video data at (b=139.18)*

Figure 47: *At t_1, Case II, Lower limit of image data at (a=106.98)*

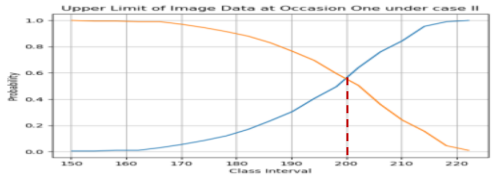


Figure 48: *At t_1, Case II, Upper limit of image data at (b=199.75)*

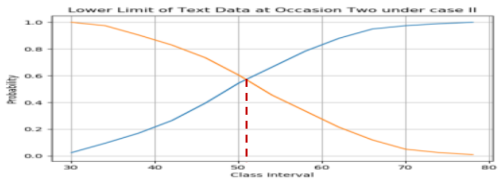


Figure 49: *At t_2, Case II, Lower limit of text data at (a=50.86)*

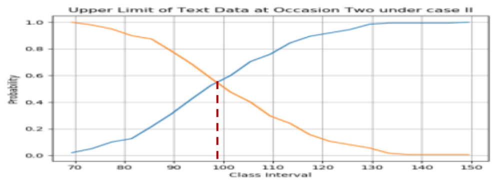


Figure 50: *At t_2, Case II, Upper limit of text data at (b=98.45)*

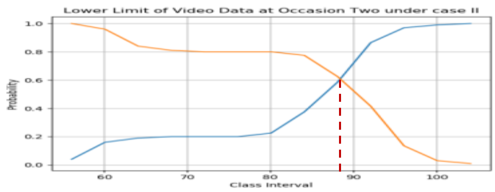


Figure 51: *At t_2, Case II, Lower limit of video data at (a=88.27)*

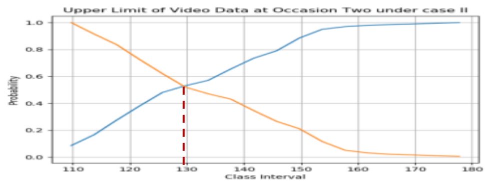


Figure 52: *At t_2, Case II, Upper limit of video data at (b=129.26)*

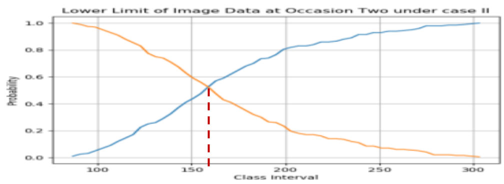


Figure 53: *At t_2, Case II, Lower limit of image data at (a=158.51)*

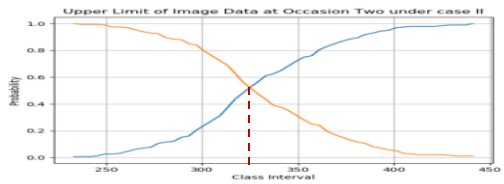


Figure 54: *At t_2, Case II, Upper limit of image data at (b=324.26)*

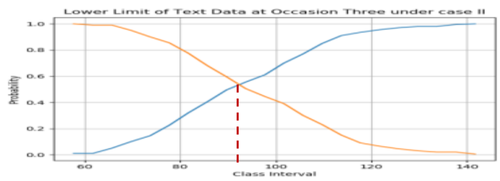


Figure 55: *At t_3, Case II, Lower limit of text data at (a=92.06)*

Figure 56: *At t_3, Case II, Upper limit of text data at (b=180.96)*

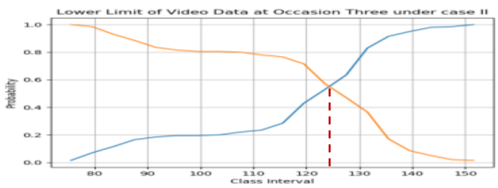


Figure 57: *At t_3, Case II, Lower limit of video data at (a=124.15)*

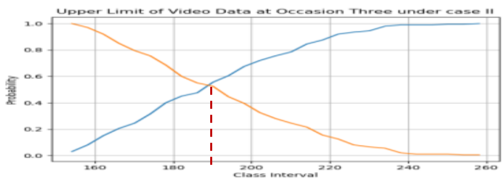
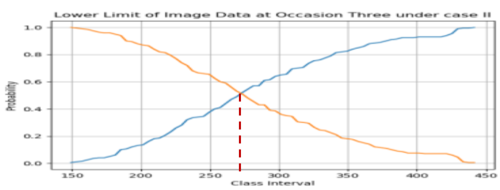


Figure 58: *At t_3, Case II, Upper limit of video data at (b=188.82)*



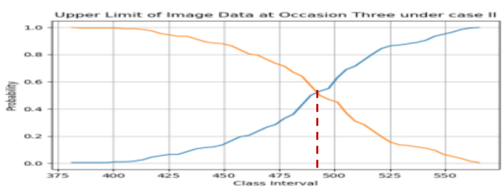


Figure 59: *At t_3, Case II, Lower limit of image data at (a=271.49)* **Figure 60:** *At t_3, Case II, Upper limit of image data at (b=491.64)*

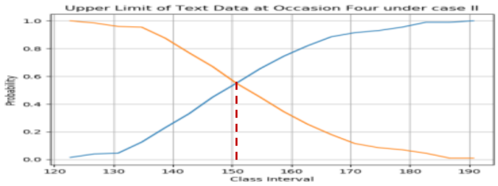
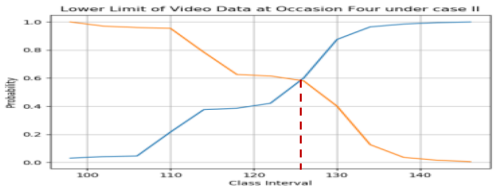


Figure 61: *At t_4, Case II, Lower limit of text data at (a=90.28)* **Figure 62:** *At t_4, Case II, Upper limit of text data at (b=150.54)*



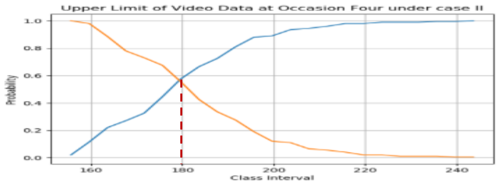
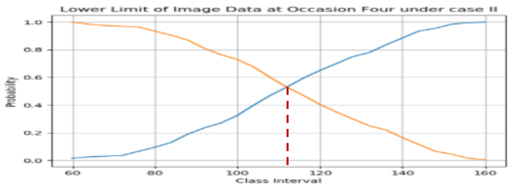


Figure 63: *At t_4, Case II, Lower limit of video data at (a=125.51)* **Figure 64:** *At t_4, Case II, Upper limit of video data at (b=178.96)*



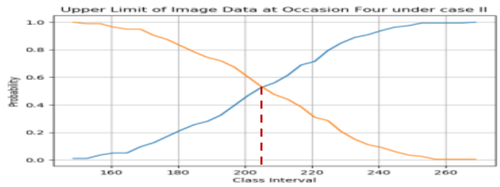
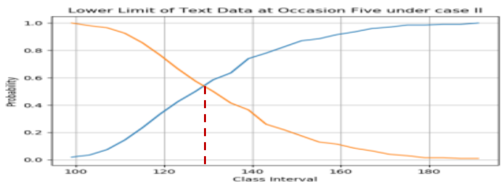


Figure 65: *At t_4, Case II, Lower limit of image data at (a=111.75)* **Figure 66:** *At t_4, Case II, Upper limit of image data at (b=205.07)*



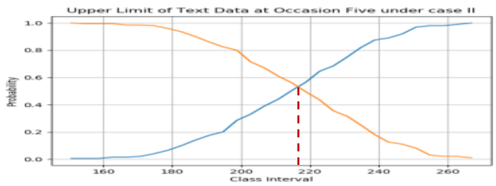
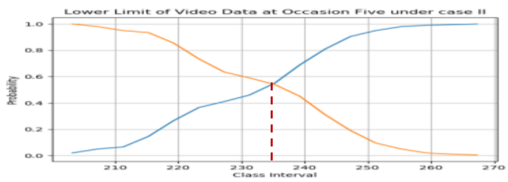


Figure 67: *At t_5, Case II, Lower limit of text data at (a=129.04)* **Figure 68:** *At t_5, Case II, Upper limit of text data at (b=216.24)*



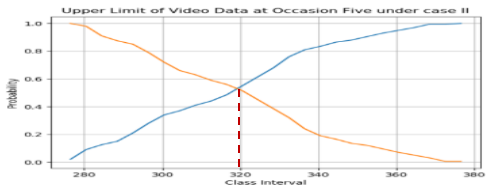
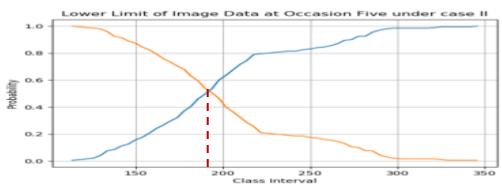


Figure 69: *At t_5, Case II, Lower limit of video data at (a=235.86)* **Figure 70:** *At t_5, Case II, Upper limit of video data at (b=318.93)*



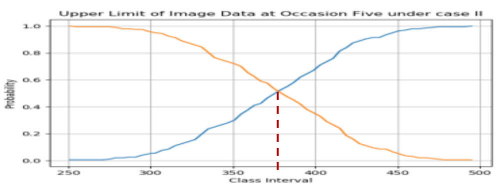


Figure 71: *At t_5, Case II, Lower limit of image data at (a=191.57)* **Figure 72:** *At t_5, Case II, Upper limit of image data at (b=376.60)*

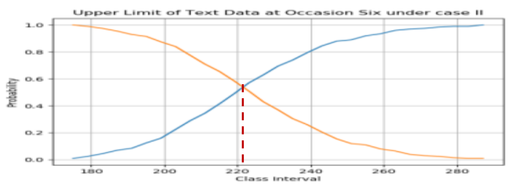


Figure 73: *At t_6, Case II, Lower limit of text data at (a=138.04)* **Figure 74:** *At t_6, Case II, Upper limit of text data at (b=221.40)*

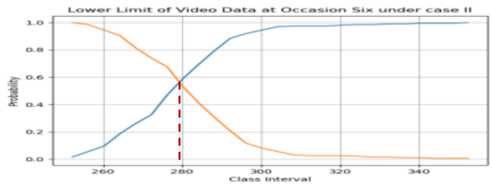


Figure 75: *At t_6, Case II, Lower limit of video data at (a=278.96)*

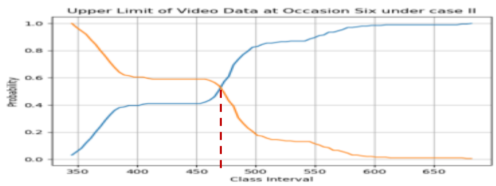


Figure 76: *At t_6, Case II, Upper limit of video data at (b=469.30)*

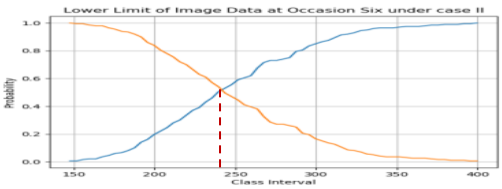


Figure 77: *At t_6, Case II, Lower limit of image data at (a=241.10)*

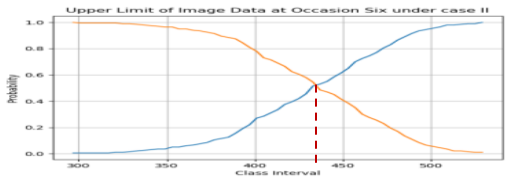


Figure 78: *At t_6, Case II, Upper limit of image data at (b=434.08)*

VII. Tabular Presentation for summarization (part of Step 5):

After simulation is over, outcomes of all above graphs are summarized in table 8 9, 10 and 11.

Table 8: *Summary of simulated CI over t_1 to t_6 under case I (based on figures 7-42)*

Time-Occasions	Dataset	Figures	Lower Limit	Figures	Upper Limit	True Value
t_1	T	Figure 7	a=65.30	Figure 8	b=101.56	74.14
	V	Figure 9	a=96.04	Figure 10	b=137.79	105.3
	I	Figure 11	a=109.36	Figure 12	b=197.01	145.07
t_2	T	Figure 13	a=53.08	Figure 14	b=96.30	67.7
	V	Figure 15	a=89.56	Figure 16	b=128.33	98.13
	I	Figure 17	a=164.44	Figure 18	b=320.34	226.18
t_3	T	Figure 19	a=97.39	Figure 20	b=176.44	125.92
	V	Figure 21	a=125.06	Figure 22	b=187.50	137.29
	I	Figure 23	a=278.48	Figure 24	b=487.83	362.74
t_4	T	Figure 25	a=93.22	Figure 26	b=146.75	110.79
	V	Figure 27	a=129.03	Figure 28	b=175.43	144.05
	I	Figure 29	a=112.47	Figure 30	b=204.60	142.45
t_5	T	Figure 31	a=130.39	Figure 32	b=214.44	148.92
	V	Figure 33	a=236.75	Figure 34	b=314.93	236.51
	I	Figure 35	a=200.37	Figure 36	b=368.03	257.97
t_6	T	Figure 37	a=142.73	Figure 38	b=216.89	173.5
	V	Figure 39	a=285.58	Figure 40	b=461.31	308.78
	I	Figure 41	a=244.35	Figure 42	b=431.30	306.78

Table 9: *Summary of simulated CI over t_1 to t_6 under case II (based on figures 43-78)*

Time-Occasions	Dataset	Figures	Lower Limit	Figures	Upper Limit	True Value
t_1	T	Figure 43	a=63.70	Figure 44	b=103.49	74.14
	V	Figure 45	a=94.95	Figure 46	b=139.18	105.3
	I	Figure 47	a=106.98	Figure 48	b=199.75	145.07
t_2	T	Figure 49	a=50.86	Figure 50	b=98.45	67.7
	V	Figure 51	a=88.27	Figure 52	b=129.26	98.13
	I	Figure 53	a=158.51	Figure 54	b=324.26	226.18
t_3	T	Figure 55	a=92.06	Figure 56	b=180.96	125.92
	V	Figure 57	a=124.15	Figure 58	b=188.82	137.29

	I	Figure 59	a=271.49	Figure 60	b=491.64	362.74
t_4	T	Figure 61	a=90.28	Figure 62	b=150.54	110.79
	V	Figure 63	a=125.51	Figure 64	b=178.96	144.05
	I	Figure 65	a=111.75	Figure 66	b=205.07	142.45
t_5	T	Figure 67	a=129.04	Figure 68	b=216.24	148.92
	V	Figure 69	a=235.86	Figure 70	b=318.93	236.51
	I	Figure 71	a=191.57	Figure 72	b=376.60	257.97
t_6	T	Figure 73	a=138.04	Figure 74	b=221.40	173.5
	V	Figure 75	a=278.96	Figure 76	b=469.30	308.78
	I	Figure 77	a=241.10	Figure 78	b=434.08	306.78

Table 8 and 9 reflect scenario of corresponding true values within the predicted range which is beauty of method.

Table 10: *Pooled simulated confidence interval average result over t_1 to t_6 undercase I [Using eq. (37)-(39)]*

Time-Occasions	Dataset	Lower Limit	Upper Limit	True Value	Length
t_1-t_6	T	a=96.67	b=158.17	116.82	61.5
	V	a=160.00	b=233.67	171.62	73.67
	I	a=184.50	b=334.50	240.19	150

Table 11: *Pooled simulated confidence interval average result over t_1 to t_6 under case II [Using eq (40).(41),(42)]*

Time-Occasions	Dataset	Lower Limit	Upper Limit	True Value	Length
t_1-t_6	T	a=93.67	b=161.33	116.82	67.66
	V	a=157.33	b=236.83	171.62	79.5
	I	a=179.67	b=338.17	240.19	158.5

VII. Discussion

In view to outcomes of table 8, 9, 10, 11 , one can observe in table 8, at t_1, the true values are 74.14 for text-data, 105.3 for video-data and 145.07 for image-data whereas simulated confidence intervals are$(65.30 - 101.56)_T$, $(69.04 - 137.79)_V$ and $(109.36 - 197.01)_I$ respectively. All true values are within the simulated confidence intervals. Similarly, at t_2, the true value are 67.7 for text-data, 98.13 for video-data and 226.18 for the image-data while simulated CI are $(53.08 - 96.30)_T$, $(89.56 - 128.33)_V$ and $(164.44 - 320.34)_I$. At t_3, true values are 125.92 for T, 137.29 for V and 362.74 for I while corresponding CI are $(97.39 - 176.44)_T$, $(125.06 - 187.50)_V$ and $(278.48 - 487.83)_I$. All true values found well within the simulated confidence intervals under case I.

Observing t_4, in table 8, simulated CI are$(93.22 - 146.75)_T$, $(129.03 - 175.43)_V$ and $(112.47 - 204.60)_I$ against true values 110.79, 144.05 and 142.45. These are also catching the truth. At t_5, true values are 148.92 for T, 236.51 for V and for I 257.97 while CI are $(130.39 - 214.44)_T$, $(236.75 - 314.93)_V$ and $(200.37 - 368)_I$ At t_6, the true are 173.05, 308.78 and 306.78 whereas the SCI are predicting accurately to true values being within the range $(142.73 - 216.89)_T$, $(285.58 - 461.31)_V$ and $(244.35 - 431.30)_I$ respectively for the case I.

Looking at estimation by Double sampling strategy under case II, true values are same as earlier but the confidence intervals, at t_1, are $(63.70 - 103.49)_T$, $(94.95 - 139.18)_V$ and $(106.98 - 199.75)_I$ showing all true values within the confidence intervals. Similarly, $at\ t_2$, the confidence interval are $(50.86 - 98.45)_T$, $(88.27 - 129.26)_V$ and $(158.51 - 324.26)_I$, at t_3, they are $(92.06 - 180.96)_T$, $(124.15 - 188.82)_V$ and $(271.49 - 491.64)_I$, at t_4, we have $(90.28 - 150.54)_T$, $(125.51 - 178.96)_V$ and $(11.75 - 205.07)_I$, at t_5, one can find as $(129.04 - 216.24)_T$, $(235.86 - 318.93)_V$

and $(191.57 - 376.60)_I$, and lastly, at t_6, their presence are as $(138.04 - 221.40)_T$, $(278.96 - 469.30)_V$ and $(241.10 - 434.08)_I$ respectively.

In the context to table 10, the single valued pooled simulated confidence intervals, under case I, are $(96.67 - 158.17)_T$, $(160.00 - 233.67)_V$ and $(184.50 - 334.50)_I$ with respect to average true values 116.82, 171.62 and 240.19. The length of simulate confidence intervals, at average level, are 61.5, 73.67, and 150 in sequence for T, V and I.

Likewise, table 11 contains same under case II who are $(93.67 - 161.33)_T$, $(157.33 - 236.83)_V$ and $(179.67 - 338.17)_I$ Lengths of confidence intervals, at average level ,are 67.66, 79.5, and 158.5 respectively.

VIII. Comparison and Efficiency

Define Relative CI Length Measure (RCILM) as

$$\mathbf{RCILM} = \left[\frac{(\text{CI length})_{\text{case II}}}{(\text{CI length})_{\text{case I}}}\right] \times 100$$

Table 12: *Relative CI Length Measure using table 10 and table 11 (under simulation)*

Dataset	RCILM
T	110.01%
V	107.91%
I	105.66%

Table 12.1: *Relative CI Length Measure using table 5 and table 6 (without simulation)*

Dataset	RCILM
T	108.13%
V	109.93%
I	105.70%

It is observed in table 12, and table12.1, the case I is having a smaller length of confidences than case II consistently in every type T, V and I.

IX. Developing Control Charts using simulated confidence intervals as tools for managerial decision

Control charts for managerial decision about web-portals, data-centers are one of applications of confidence interval. Consider the theory discussed in section 1 and in figure 6. The graphical trace of CI over $t_1$to t_6 displayed in figure 79 to 90, for the case I and case II , can be used.

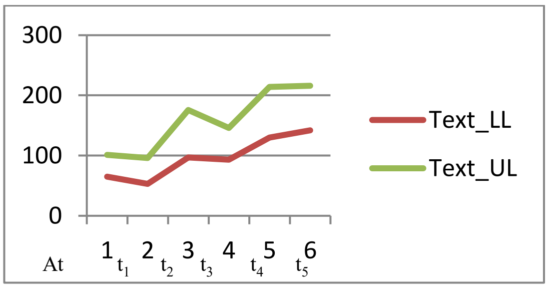


Figure 7: *CL of Text file-size measures under case I*

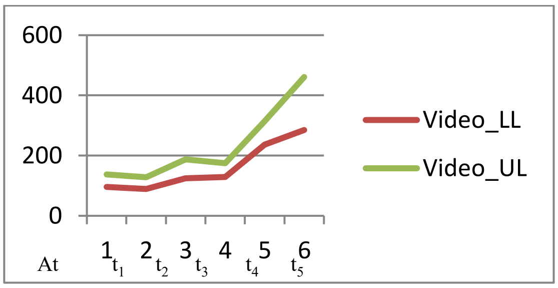


Figure 80: *CL of Video file-size measures under case I*

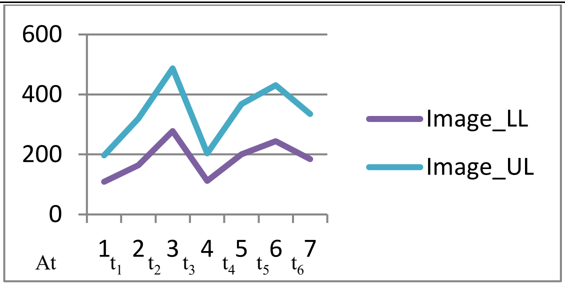


Figure 81: *CL of Image file-size measures under case I*

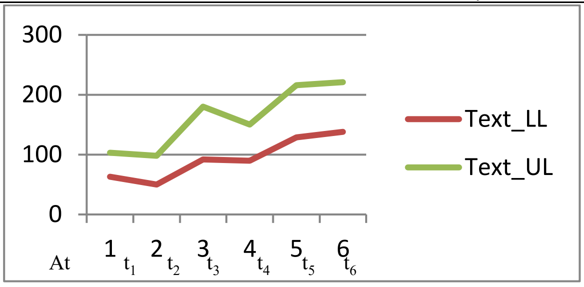


Figure 82: *CL of Text file-size measures under case II*

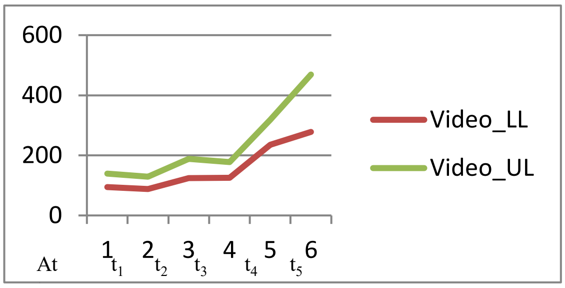


Figure 83: *CL of Video file-size measures under case II*

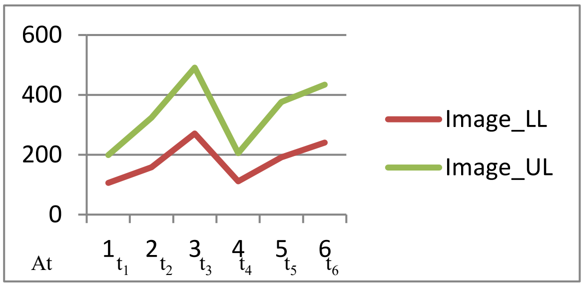


Figure 84: *CL of Image file-size measures under case II*

Fig. 79-81 reveal Upper Control Limit (UCL or UL) and Lower Control Limit (LCL or LL) of the file-size measures of text-data, video-data and image-data used in communication . Similarly, Fig. 82-84 are showing same application for case II and these are file-size production procss control charts. The simulated value 'a' is LCL(or LL) and simulated value 'b' is UCL(or UL) of the confidence intervals.

Such are helpful for decision making regarding control over size measure of communication files on social media web-portal and ,as a consequence, alert can issue for further infrastructure, resources required to achieve the goal of profit. For example, IT- industry (Servers/Data Center/ hardware/software), if alarmed well before about flowing digital file-size, who is growing fast in big data environment over time, better management can be thought of in timely manner. While file-size measure, if increases exponentially over time then investment in Data Centers urgently needed .

In view to fig. 85-90 , matter of importance is to watch whether same habits of communication of users are maintained ? If at n^{th} point of time t_n (n = 1,2,3 ....,∞), the Upper Control Limit (UCL) or Lower Control Limit (LCL) are crossed in control charts , there is significant evidance exist for change of habits of communication of file-size. At this juncture, the industry owner needs to review decision regarding up-gradation or framing new policy to share memory resources with others in order to maximize profit. Simulated confidence intervals play key role for developing such monitoring.

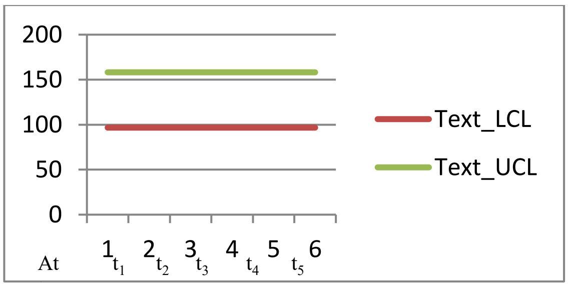


Figure 85: *CL of Text file-size under case I*

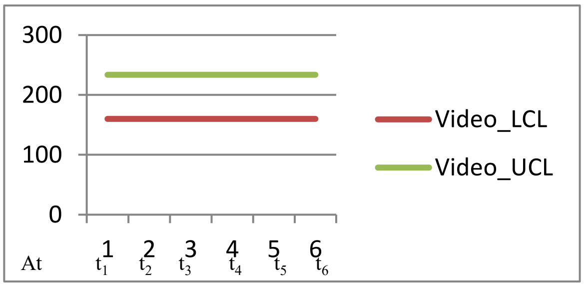


Figure 86: *CL of Video file-size under case I*

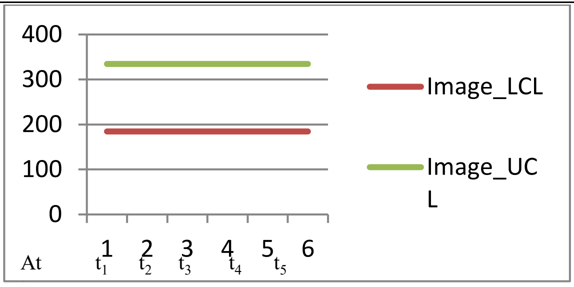


Figure 87: *CL of Image file-size under case I*

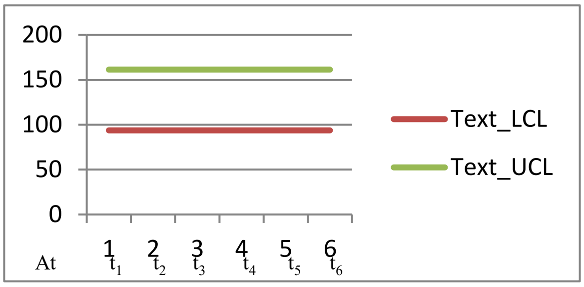


Figure 88: *CL of Text file-size under case II*

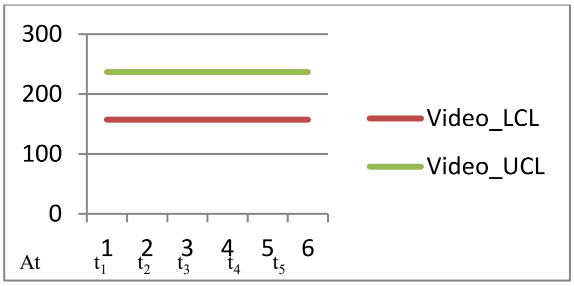


Figure 89: *CL of Video file-size under case II*

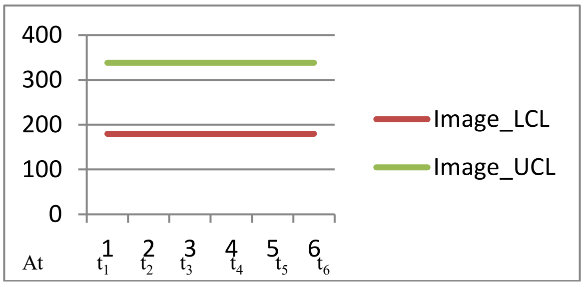


Figure 90: *CL of Image file-size under case II*

X. Conclusion

On recapitulation, the double sampling approach has been adopted in the content for estimating the population parameter in the setup of big data where volume, variety and velocity characteristics are present simultaneously. The idea of number of registered users on a social networking platform communicating through Text, Video and Image files has been considered over different time span. Estimate of average file size is focused whose growth needs to be monitored over time variations. Estimation strategies have been suggested in the setup of double sampling. When has two approaches as (a) sub-sample and (b) independent sample. Both have been compared and found that the proposed methods capture the true values of the population mean over several occasions (time frame). The merge setup of the average of all occasions also reveals that both strategies (case I and case II) cover the true values. A new simulation algorithm based on double sampling is suggested to obtain a single value estimate of 95% confidence interval whose estimate also predict about the true value. Efficiency comparison of two cases is made through the tool RCILM which shows case I better than case II. The study is useful for a managerial decision since the lower control limit and upper control limit growth can generate an alert for IT-business managers. Control charts predict for the future event to check whether the average values of file sizes at farther occasions are within the UCL or LCL or not. If the control limits violate then re-thinking about IT-business infrastructure may be originated to cope up the future challenges. If the control limits violates then re-thinking about IT-business infrastructure may be originated to cope up the future challenges.

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ANNEXURE A

Population N=100

ID	T	V	I	T	V	I	T	V	I	T	V	I	T	V	I	T	V	I
0	10	15	6	13	10	10	18	21	15	20	25	27	31	35	38	41	49	52
1	5	8	7	9	3	11	8	11	17	9	12	13	15	16	18	19	23	24
2	9	12	11	5	7	17	16	17	27	18	23	24	28	32	34	37	44	47
3	8	11	14	16	6	23	14	15	36	15	19	21	24	27	29	32	38	40
4	12	20	18	10	15	29	22	28	45	24	30	32	37	42	46	49	59	62
5	11	16	22	9	11	35	19	22	54	21	27	29	33	37	41	44	52	56
6	13	18	25	11	13	41	23	25	64	26	33	35	40	46	49	53	64	68
7	14	21	29	17	16	47	25	29	73	27	34	37	42	48	52	56	67	71
8	15	23	33	13	18	52	27	32	82	30	38	41	47	53	58	62	74	79
9	17	26	37	15	21	58	30	36	91	33	42	45	52	58	63	68	82	87
10	18	28	40	24	23	64	33	39	101	36	46	49	56	64	69	75	89	95
11	20	31	44	18	26	70	35	43	110	39	49	53	61	69	75	81	97	102
12	21	33	48	19	28	76	38	46	119	42	53	57	66	74	81	87	104	110
13	23	36	51	21	31	82	41	50	128	45	57	61	70	79	86	93	111	118
14	24	38	55	22	33	88	44	53	138	48	61	65	75	85	92	99	119	126
15	26	41	59	24	36	94	46	57	147	51	64	69	80	90	98	105	126	134
16	27	43	62	25	38	100	49	60	156	54	68	73	84	95	103	112	133	141
17	29	46	66	27	41	106	52	64	165	57	72	77	89	100	109	118	141	149
18	30	48	70	28	43	112	54	67	175	60	76	82	94	106	115	124	148	157
19	32	51	74	30	46	118	57	71	184	63	79	86	98	111	120	130	155	165
20	33	53	77	31	48	124	60	74	193	66	83	90	103	116	126	136	163	173
21	35	56	81	33	51	129	62	78	202	69	87	94	108	121	132	142	170	180
22	36	58	85	34	53	135	65	81	212	72	91	98	112	127	138	148	177	188
23	38	61	88	36	56	141	68	85	221	75	94	102	117	132	143	155	185	196
24	39	63	92	37	58	147	71	88	230	78	98	106	122	137	149	161	192	204
25	41	66	96	39	61	153	73	92	239	81	102	110	126	142	155	167	199	212
26	42	68	99	40	63	159	76	95	249	84	106	114	131	148	160	173	207	219
27	44	71	103	42	66	165	79	99	258	87	109	118	135	153	166	179	214	227
28	45	73	107	43	68	171	81	102	267	90	113	122	140	158	172	185	221	235
29	47	76	111	45	71	177	84	106	276	93	117	126	145	163	177	191	229	243
30	48	78	114	46	73	183	87	109	286	96	121	130	149	169	183	198	236	251
31	50	81	118	48	76	189	89	113	295	99	124	134	154	174	189	104	244	258
32	51	83	122	49	78	195	92	116	304	102	128	138	159	179	195	110	251	266
33	53	86	125	51	81	200	95	120	313	105	132	142	163	184	200	116	258	274
34	54	88	129	52	83	206	98	123	323	108	136	146	168	190	206	122	266	282
35	56	91	133	54	86	212	100	127	332	111	139	150	173	195	212	128	273	290
36	57	93	136	55	88	218	103	130	341	114	143	154	177	200	217	135	280	297
37	59	96	140	57	91	224	106	134	350	117	147	158	182	205	223	141	288	305
38	60	98	144	58	93	230	108	137	360	120	151	163	187	211	229	147	295	313
39	62	101	148	60	96	236	111	141	369	123	154	167	191	216	234	153	302	321
40	63	103	151	61	98	242	114	144	378	126	158	171	196	221	240	159	310	329
41	65	106	155	63	101	248	116	148	387	129	162	175	101	226	246	165	317	336
42	66	108	159	64	103	254	119	151	397	132	166	179	105	232	252	171	324	344
43	68	111	162	66	106	260	122	155	406	135	169	183	110	237	257	178	332	352
44	69	113	166	67	108	266	125	158	415	138	173	187	115	242	263	184	339	360
45	71	116	170	69	111	272	127	162	424	141	177	191	119	247	269	190	346	368
46	72	118	173	70	113	277	130	165	434	144	181	195	124	253	274	96	354	375
47	74	121	177	72	116	283	133	169	443	147	184	199	128	258	280	102	361	383
48	75	123	181	73	118	289	135	172	452	150	188	203	133	263	286	208	368	391
49	77	126	185	75	121	295	138	176	461	153	192	207	138	268	291	114	376	399
50	78	128	188	76	123	301	141	179	471	156	196	211	142	274	297	121	383	407
51	80	131	192	78	126	307	143	183	480	159	199	215	147	279	303	127	391	414
52	81	133	196	79	128	313	146	186	489	162	203	219	152	284	309	233	398	422
53	83	136	199	81	131	319	149	190	498	165	207	223	156	289	314	139	405	430
54	84	138	203	82	133	325	152	193	508	168	211	227	123	295	320	145	413	438
55	86	141	207	84	136	331	154	197	517	171	214	231	120	300	326	251	420	446
56	87	143	210	85	138	337	157	200	526	174	218	235	170	305	331	158	427	453
57	89	146	214	87	141	343	160	204	535	177	222	239	175	310	337	264	435	461
58	90	148	218	88	143	348	162	207	545	180	226	244	180	316	343	170	442	469
59	92	151	222	90	146	354	165	211	554	183	229	248	84	321	348	245	449	477
60	93	153	225	91	148	360	168	214	563	186	233	252	28	326	354	180	457	485
61	95	156	229	93	151	366	170	218	572	189	237	256	94	331	360	288	464	492

62	96	158	233	94	153	372	173	221	582	192	241	260	98	337	366	135	471	500
63	98	161	236	96	156	378	176	225	591	195	244	264	102	342	371	202	479	508
64	99	163	240	97	158	384	179	228	600	198	248	268	108	347	377	207	486	516
65	101	166	244	99	161	390	181	232	609	101	252	272	113	352	383	196	493	524
66	102	168	247	80	163	396	184	235	619	104	256	276	114	358	388	199	501	531
67	104	171	251	35	166	402	187	239	628	107	259	280	213	363	394	258	508	539
68	105	173	255	81	168	408	189	242	637	105	263	284	126	368	400	231	515	547
69	107	176	259	102	171	414	192	246	646	113	267	288	131	373	405	237	523	555
70	108	178	262	99	173	420	195	249	656	116	271	292	135	379	411	244	530	563
71	110	181	266	88	176	425	197	253	665	109	274	296	140	384	417	250	538	570
72	111	183	270	106	178	431	200	256	674	85	278	300	145	389	423	256	545	578
73	113	186	273	59	181	437	203	260	683	95	282	304	149	394	428	262	552	586
74	45	188	277	43	183	443	81	263	693	100	113	122	140	158	171	185	221	234
75	116	191	281	114	186	449	208	267	702	132	289	212	159	405	440	274	567	602
76	117	193	284	26	188	455	211	270	711	102	293	123	163	410	445	281	574	609
77	119	196	288	117	191	461	214	274	720	132	297	13	168	415	451	284	582	617
78	120	45	292	118	40	467	216	63	730	140	102	25	173	421	457	293	589	625
79	122	201	70	120	196	112	219	281	175	125	100	30	177	129	462	299	596	633
80	123	170	85	121	165	136	178	238	213	142	108	33	182	331	268	205	604	641
81	125	106	103	123	101	165	159	148	258	127	113	137	187	336	274	211	611	648
82	126	208	107	82	203	171	130	291	268	152	116	241	191	342	340	217	618	156
83	128	211	210	43	206	336	85	295	525	155	119	145	196	347	385	224	626	164
84	129	13	214	55	8	342	133	18	535	150	125	110	101	352	291	230	633	272
85	131	216	215	129	102	344	135	302	538	142	140	270	105	357	497	236	640	180
86	132	218	231	130	103	370	138	305	578	178	179	154	110	363	302	242	148	287
87	134	25	225	132	20	200	142	35	563	167	180	102	114	368	508	248	155	295
88	135	223	229	133	218	205	243	112	573	170	165	78	319	373	375	254	162	203
89	137	226	233	135	221	100	246	116	583	173	141	69	324	378	519	260	170	311
90	138	105	110	136	100	176	249	125	275	176	88	73	328	384	525	267	300	319
91	122	5	15	120	0	24	220	7	38	144	107	29	378	27	464	200	98	234
92	102	152	210	100	147	336	184	102	525	104	102	75	316	57	288	218	300	230
93	113	136	148	111	131	237	203	130	370	126	99	10	350	396	229	300	290	188
94	110	20	35	108	15	56	198	28	88	120	100	97	341	385	218	200	201	172
95	106	120	152	104	115	243	191	68	380	112	142	86	329	371	203	205	219	201
96	102	119	125	100	114	200	184	67	313	102	153	175	316	357	288	203	200	130
97	105	100	101	103	95	162	189	40	253	103	152	184	326	368	299	231	215	220
98	100	85	90	98	80	144	180	19	225	108	50	70	310	350	280	210	290	175
99	120	75	60	118	70	96	216	100	150	130	100	100	372	320	356	292	288	124
100	125	70	25	123	65	40	225	98	63	85	113	12	388	338	375	213	213	150

ANNEXURE B

CI of 200 Sample each size n=10 (Case I)

SNo	Text Lower	Text Upper	Video Lower	Video Upper	Image Lower	Image Upper
1	113.52	138.60	182.37	208.56	223.99	292.37
2	134.15	169.82	178.21	208.20	183.23	259.15
3	108.70	125.20	172.51	192.64	278.78	322.86
4	98.32	117.18	172.70	207.84	301.50	346.83
5	119.85	144.72	175.90	211.48	223.60	286.24
6	121.10	150.73	182.54	218.73	205.16	279.25
7	135.22	165.82	162.97	191.85	213.89	276.67
8	107.45	131.03	173.59	190.97	267.18	329.69
9	115.60	145.78	179.46	217.16	223.00	289.18
10	102.87	128.53	177.76	216.92	263.89	328.07
11	119.50	148.09	185.18	227.06	202.44	264.31
12	102.11	119.69	179.10	199.41	275.68	331.41
13	118.51	153.85	180.19	218.56	199.86	279.23
14	106.84	127.94	160.98	185.00	302.64	342.08
15	103.74	129.75	179.12	204.78	245.36	322.51
16	117.55	148.77	186.01	224.57	198.17	272.13
17	105.39	122.01	182.75	223.76	252.77	303.68
18	105.64	131.99	181.69	221.76	242.17	309.79
19	115.26	151.50	184.97	228.85	191.27	272.69
20	139.27	173.37	175.76	208.77	182.90	256.83
21	106.22	135.39	186.06	226.95	225.24	299.26
22	126.17	154.80	165.86	192.99	239.90	295.12
23	119.28	148.90	186.63	228.59	197.25	265.25
24	104.62	129.03	182.03	233.05	244.82	302.58
25	109.70	132.10	180.57	207.67	241.39	301.99
26	109.59	136.66	193.94	250.75	224.91	283.77
27	112.16	134.94	171.35	186.03	265.34	319.65
28	110.89	132.73	172.02	192.35	269.83	321.56
29	117.83	142.03	169.00	195.69	250.72	309.13
30	104.50	128.95	186.04	239.64	229.02	294.26
31	115.42	142.35	178.97	219.60	216.32	284.98
32	109.82	133.50	172.50	192.44	259.31	319.72
33	118.35	148.42	170.85	207.27	220.90	290.64
34	117.23	147.41	168.79	210.57	232.76	302.81
35	106.93	128.39	172.62	192.64	277.41	326.53
36	100.11	124.66	174.54	212.28	276.27	334.68
37	116.23	151.28	170.03	222.55	222.16	297.20
38	115.01	143.91	172.40	209.72	228.85	298.03
39	124.20	151.61	181.42	204.62	209.17	274.60
40	109.92	134.92	168.40	202.58	255.59	315.39
41	96.04	116.00	180.62	229.87	280.33	332.95
42	114.19	135.99	179.51	203.90	241.11	295.96
43	112.50	141.00	186.95	227.50	216.99	286.86
44	105.28	131.40	173.25	212.63	245.62	313.91
45	119.59	152.94	194.83	247.39	182.06	258.71
46	109.18	138.42	178.99	215.70	232.22	306.16
47	114.30	143.11	175.95	213.92	223.81	295.47
48	135.22	163.57	178.67	197.15	196.52	260.81
49	111.77	140.48	181.12	209.78	224.21	295.44
50	110.99	142.28	165.22	211.79	236.64	304.69
51	114.41	139.16	182.11	206.76	225.46	290.48
52	129.25	157.94	193.22	236.13	178.00	237.86
53	106.16	137.59	168.71	222.81	245.27	312.38
54	113.25	135.28	179.40	206.03	240.28	296.19
55	113.81	140.55	174.47	212.01	229.48	297.07
56	102.93	130.27	193.84	245.45	213.22	291.80
57	117.30	143.95	178.53	203.92	225.81	289.36
58	102.00	126.05	162.87	202.67	297.51	344.91
59	102.80	120.83	173.43	208.17	290.48	332.71
60	94.26	113.51	178.32	227.06	279.52	341.99
61	110.06	136.33	177.09	214.88	246.23	312.45
62	98.90	117.90	173.64	219.16	289.73	339.52

63	120.69	152.76	169.95	200.50	221.78	298.05
64	125.12	157.64	183.58	229.75	190.87	265.37
65	122.31	148.69	180.86	204.78	219.87	281.78
66	107.87	139.21	168.18	214.57	231.77	314.20
67	122.91	147.34	176.85	198.15	225.04	282.71
68	108.28	132.06	177.18	200.03	252.48	315.42
69	119.26	149.26	189.96	232.44	194.19	260.22
70	115.23	137.86	172.60	208.24	248.60	299.72
71	130.76	158.31	189.61	225.55	193.67	253.14
72	101.05	117.19	171.90	185.84	311.34	348.68
73	130.26	167.89	186.65	231.77	167.56	244.90
74	113.95	137.82	173.55	191.39	253.35	311.21
75	108.32	132.68	170.30	207.15	259.30	317.84
76	127.24	149.82	176.72	195.96	234.56	280.55
77	108.21	132.38	170.43	187.15	270.10	328.65
78	102.58	131.93	172.38	226.51	249.05	317.04
79	105.12	133.54	192.45	251.23	223.54	293.44
80	119.23	137.94	160.77	182.95	285.97	317.89
81	106.73	134.75	185.04	235.27	223.06	293.84
82	127.90	150.28	173.66	192.43	227.59	277.71
83	118.47	142.52	162.79	190.44	265.12	312.64
84	112.34	139.64	190.83	240.43	206.42	278.53
85	112.02	139.97	166.58	211.26	248.96	306.42
86	113.40	145.49	174.46	219.57	226.59	296.77
87	117.36	153.15	171.03	224.48	216.81	292.28
88	107.48	130.00	177.39	202.16	253.91	314.81
89	116.87	144.85	177.76	197.15	227.76	295.46
90	102.62	129.83	173.64	219.05	256.83	321.29
91	132.26	165.75	186.10	226.84	183.44	248.20
92	116.66	145.53	180.27	227.12	221.99	283.49
93	118.67	147.80	174.80	194.71	222.91	293.50
94	106.96	125.08	168.88	203.92	281.77	324.85
95	121.57	152.26	177.17	201.19	218.88	286.17
96	107.10	132.29	178.30	205.28	242.23	312.23
97	124.98	151.55	161.41	188.17	250.34	301.76
98	144.42	173.99	179.73	205.87	180.77	241.99
99	113.76	142.47	174.36	198.14	236.32	304.03
100	111.82	136.96	185.79	225.34	222.31	289.02
101	118.94	147.54	160.93	188.91	247.23	310.88
102	120.52	150.59	168.16	211.94	230.45	292.34
103	112.27	134.90	172.52	191.48	264.89	316.28
104	110.50	134.47	179.39	217.60	243.27	300.45
105	108.86	131.46	175.06	197.80	264.91	322.23
106	105.38	130.44	175.56	213.76	248.98	313.69
107	100.86	126.16	176.44	224.74	266.79	328.91
108	137.86	168.70	179.01	204.44	188.28	254.12
109	112.96	134.38	179.05	202.83	244.80	300.10
110	146.40	172.92	191.06	215.25	171.53	228.63
111	133.29	161.08	178.94	202.07	195.95	263.39
112	112.13	136.56	176.36	212.66	253.29	306.83
113	112.07	141.27	180.71	220.82	217.99	288.03
114	121.40	151.16	182.64	221.29	199.96	264.56
115	130.97	160.24	184.38	225.38	193.31	246.37
116	101.96	119.74	168.33	203.89	292.71	339.24
117	131.08	157.41	180.32	204.41	206.36	264.20
118	107.00	126.61	183.28	221.06	246.91	307.00
119	114.62	141.94	174.13	206.74	233.25	302.13
120	109.11	135.06	183.31	222.81	223.14	290.25
121	106.97	137.69	171.73	211.77	237.46	313.65
122	138.62	169.74	162.26	192.30	209.02	273.30
123	123.40	152.62	168.72	211.98	227.90	287.55
124	123.95	154.05	185.21	224.92	188.40	259.66
125	100.97	124.92	182.99	235.75	239.76	306.35
126	113.10	139.03	184.98	224.20	220.55	289.13
127	131.00	161.47	188.77	229.57	187.59	242.68

128	104.45	130.78	174.70	215.45	248.93	309.91
129	116.14	145.17	182.67	220.74	217.37	287.61
130	119.59	143.56	177.61	200.25	224.88	288.67
131	139.59	168.84	186.43	212.12	175.49	238.51
132	101.07	127.38	177.86	227.41	260.52	322.35
133	113.49	140.83	187.81	228.50	209.60	279.21
134	123.47	150.70	177.77	214.50	221.05	276.41
135	129.57	156.81	177.44	202.23	211.75	274.06
136	113.90	137.60	185.45	213.09	225.00	289.65
137	151.23	182.37	185.20	210.32	166.18	229.20
138	109.05	137.35	182.31	222.73	241.16	306.61
139	111.25	140.68	182.88	221.79	230.25	294.18
140	119.68	151.28	172.65	204.08	216.32	291.78
141	111.33	135.82	178.23	217.73	234.30	294.04
142	129.24	154.75	176.48	198.12	219.97	276.14
143	115.70	147.14	183.91	233.86	200.61	273.73
144	133.68	169.69	194.57	245.77	166.55	230.67
145	106.15	139.15	180.00	220.91	218.83	300.05
146	105.37	123.99	170.08	205.25	274.75	328.35
147	118.10	155.00	177.60	233.26	193.53	276.06
148	125.58	150.07	188.84	229.81	197.75	251.16
149	135.94	164.45	184.08	210.96	186.99	246.94
150	104.12	124.58	169.63	204.70	274.30	329.46
151	126.64	158.34	186.55	236.42	181.21	251.28
152	111.67	135.75	181.82	222.37	228.22	291.62
153	129.85	146.69	179.01	196.17	240.98	279.24
154	129.12	160.75	179.37	225.28	192.42	257.94
155	118.93	147.14	178.49	203.57	223.14	291.47
156	123.62	154.94	183.44	220.45	194.34	264.52
157	115.46	138.07	178.16	215.99	240.49	287.55
158	142.28	172.84	178.38	201.38	178.61	246.30
159	132.57	170.26	170.69	219.54	182.60	259.57
160	134.76	169.72	176.13	202.65	190.19	260.80
161	110.83	136.12	192.88	243.94	217.56	282.83
162	103.09	118.83	171.20	186.36	307.64	342.80
163	117.30	150.31	169.34	215.27	221.04	290.24
164	134.78	166.99	172.47	203.11	205.06	271.89
165	127.64	158.93	185.65	227.58	186.98	251.48
166	116.50	146.59	191.48	234.36	192.69	263.23
167	112.50	135.44	179.32	203.39	238.32	301.97
168	117.43	139.12	182.66	210.58	227.24	282.80
169	107.71	128.50	162.05	184.25	303.12	340.93
170	126.09	157.91	166.31	194.73	221.03	292.49
171	110.77	127.46	176.64	200.93	262.13	309.77
172	121.91	154.19	190.60	231.74	194.20	267.03
173	111.93	135.28	171.07	186.38	263.67	317.41
174	114.30	140.12	171.24	193.14	243.84	305.55
175	124.56	143.17	173.88	191.05	250.50	291.29
176	117.73	140.49	175.37	198.99	242.57	294.25
177	138.12	164.32	181.89	204.01	195.24	254.53
178	98.23	117.15	172.21	205.04	303.78	347.61
179	102.97	121.83	176.99	224.98	261.30	315.23
180	138.15	174.29	179.45	223.88	185.93	253.26
181	117.81	142.28	184.95	224.19	210.27	272.82
182	111.68	135.20	174.22	196.36	254.10	311.13
183	115.89	135.99	180.74	218.71	242.30	284.06
184	104.81	123.76	175.46	211.93	285.40	329.53
185	112.44	132.76	177.43	211.57	261.46	305.85
186	112.63	133.06	180.89	219.72	239.74	291.32
187	111.58	138.81	169.62	213.62	246.36	304.13
188	100.55	116.56	171.81	185.89	313.20	350.22
189	120.20	145.68	178.03	215.08	213.41	278.58
190	121.14	155.67	167.54	213.37	215.88	292.09
191	123.48	151.46	171.57	193.28	222.67	288.17
192	106.22	128.97	168.46	198.06	269.61	327.15

193	110.94	134.72	180.65	216.55	250.46	303.02
194	103.31	128.20	181.03	221.10	246.16	311.76
195	117.42	147.06	183.95	222.36	216.35	284.42
196	110.70	146.09	166.99	220.61	233.86	308.95
197	108.89	135.16	172.39	209.54	241.38	312.29
198	134.67	161.67	186.68	212.45	184.12	244.42
199	122.52	155.03	163.82	193.10	232.96	303.74
200	123.31	154.67	173.09	211.61	204.41	275.16

ANNEXURE C

CI of 200 Sample each size n=10 (Case II)

SNo	Text Lower	Text Upper	Video Lower	Video Upper	Image Lower	Image Upper
1	112.5025	139.6188	180.9357	209.9947	222.0762	294.2889
2	132.1565	171.8114	176.3829	210.0216	181.4341	260.9476
3	107.8549	126.0527	172.3538	192.8011	278.2256	323.4121
4	97.28737	118.2201	171.5393	209.0049	300.528	347.8061
5	119.0032	145.5677	174.4599	212.9213	221.1926	288.6421
6	119.6648	152.1567	180.9075	220.362	202.5174	281.8964
7	133.4725	167.5706	161.5407	193.2813	212.6052	277.958
8	106.3945	132.0859	173.0844	191.4717	264.9768	331.8918
9	114.2793	147.0991	177.7743	218.8427	220.1311	292.0491
10	101.6652	129.7376	175.9887	218.6873	261.7236	330.2351
11	117.9328	149.6625	183.0345	229.202	200.5343	266.216
12	101.3133	120.485	177.9547	200.5548	273.9451	333.1438
13	116.7251	155.6399	178.0277	220.7157	197.357	281.7325
14	105.6572	129.1162	160.3586	185.6207	302.185	342.5367
15	102.6085	130.8796	177.2882	206.6135	243.301	324.5688
16	116.2565	150.0644	183.7047	226.8764	195.5426	274.7563
17	104.1755	123.2273	181.0051	225.5034	251.9731	304.4761
18	104.5469	133.0791	179.7977	223.6472	239.8835	312.0763
19	113.4035	153.364	182.4263	231.3981	189.4269	274.5269
20	137.7428	174.8995	173.485	211.0486	181.0419	258.6833
21	104.8304	136.781	183.4532	229.554	222.7347	301.7651
22	124.682	156.2853	164.5653	194.2808	238.3067	296.7146
23	117.6546	150.5266	184.4309	230.7915	195.521	266.9823
24	103.3036	130.3428	179.9946	235.0848	242.992	304.4074
25	108.5566	133.2433	179.3298	208.9106	240.0204	303.3612
26	108.2785	137.9636	191.7557	252.9432	222.619	286.0615
27	111.4435	135.6584	170.8133	186.5734	262.7295	322.2601
28	109.8958	133.7237	171.5627	192.8094	268.3515	323.0353
29	117.0376	142.8189	167.4221	197.2665	248.2261	311.6299
30	103.1887	130.268	183.2596	242.4131	227.2967	295.9827
31	113.7117	144.0559	177.2317	221.3402	214.475	286.8212
32	108.6669	134.6575	172.1115	192.8282	257.6001	321.4268
33	116.9968	149.7776	169.2943	208.8191	218.1333	293.4088
34	115.7656	148.8779	166.8516	212.5117	230.6757	304.8981
35	105.8899	129.4305	172.0856	193.1804	275.9146	328.0331
36	98.78647	125.9776	173.0555	213.7639	274.4069	336.545
37	114.1647	153.3442	167.345	225.2402	219.7946	299.5632
38	113.3935	145.5338	170.5726	211.5429	226.6755	300.2043
39	123.1525	152.6558	180.143	205.89	206.9668	276.8079
40	108.9226	135.9182	167.0473	203.9361	252.8565	318.1285
41	94.78109	117.2568	178.463	232.0219	279.3064	333.9716
42	113.0547	137.1217	178.6081	204.7991	239.4949	297.5753
43	111.0507	142.4457	184.4639	229.9909	214.6993	289.1541
44	104.0987	132.5776	171.2642	214.6111	243.4342	316.0932
45	117.7712	154.7576	191.8136	250.4098	179.607	261.17
46	108.0273	139.5723	176.8206	217.8769	229.442	308.9327
47	113.0693	144.3429	174.0301	215.8397	221.1914	298.0877
48	134.4277	164.3694	176.9454	198.8787	193.7765	263.5526
49	110.2277	142.0148	179.8179	211.0823	222.3989	297.2529
50	109.4672	143.8032	163.0289	213.979	235.2198	306.1044
51	112.9665	140.5978	181.0044	207.8623	223.8664	292.0724
52	127.6045	159.583	191.2084	238.1448	176.3396	239.5216
53	104.4922	139.2542	165.9372	225.5766	243.4363	314.2153
54	112.1048	136.4315	178.4944	206.9279	239.0231	297.442
55	112.707	141.657	172.8391	213.6413	227.3741	299.1733
56	101.7639	131.4424	190.2886	248.9992	210.958	294.0564
57	116.2937	144.959	177.0841	205.3588	223.2077	291.9614
58	100.6352	127.4147	161.2607	204.2833	296.4775	345.9437
59	101.7613	121.8703	172.3789	209.2262	289.5168	333.6739
60	93.31854	114.4549	175.6672	229.7062	277.6419	343.8712
61	108.4735	137.9135	175.9495	216.0254	244.3709	314.3022
62	97.85368	118.9428	171.6796	221.1267	287.9315	341.3232

63	119.3528	154.0918	167.9264	202.5278	219.4674	300.3561
64	123.5925	159.1735	180.8238	232.5091	188.5559	267.6838
65	121.0287	149.9641	179.6038	206.0375	217.7891	283.8648
66	106.4229	140.6509	165.4635	217.2859	229.691	316.2784
67	121.6842	148.5709	176.536	198.4615	223.3503	284.4004
68	107.1758	133.173	176.0211	201.1831	250.487	317.4138
69	117.8529	150.6682	187.9882	234.4153	192.1432	262.2635
70	114.0986	138.9898	171.4805	209.3636	246.5689	301.7476
71	129.5237	159.5445	188.6509	226.5084	191.1456	255.6659
72	100.1975	118.0451	171.8944	185.8425	310.5186	349.5013
73	128.1539	170.001	183.8726	234.5459	165.4989	246.967
74	112.7568	139.0161	173.1246	191.8106	251.5786	312.9848
75	106.9005	134.0995	168.8298	208.6236	257.5159	319.6272
76	126.3709	150.6879	176.0532	196.6314	232.4573	282.6508
77	107.3033	133.2811	169.9509	187.6264	267.6271	331.1282
78	100.941	133.5688	169.1504	229.743	247.266	318.8223
79	103.3659	135.2866	189.7995	253.8758	221.5567	295.4243
80	118.2461	138.9255	160.0072	183.7118	285.9793	317.8764
81	105.0789	136.4004	182.3798	237.9373	221.1902	295.7084
82	127.3167	150.8695	173.0966	192.9996	225.2877	280.0133
83	117.3727	143.6146	161.816	191.4196	264.0154	313.7467
84	111.1262	140.8496	188.3426	242.918	203.8726	281.0866
85	110.4382	141.5568	164.5645	213.2762	247.9363	307.4482
86	111.5953	147.2998	171.925	222.1018	224.5882	298.7639
87	115.1705	155.3432	168.4218	227.0897	214.7593	294.3302
88	106.2111	131.2661	176.1948	203.3549	252.4467	316.2729
89	115.834	145.8789	176.3625	198.5479	225.2968	297.9281
90	101.4559	130.9953	171.5939	221.0973	253.754	324.3593
91	130.4779	167.5326	184.7005	228.2409	181.154	250.484
92	115.396	146.7969	178.6149	228.7789	218.7634	286.7163
93	117.9274	148.5433	173.338	196.1644	220.197	296.2196
94	105.9154	126.1265	167.785	205.0123	280.7667	325.8502
95	119.9302	153.8944	176.0191	202.3343	216.9271	288.1204
96	105.7466	133.6415	177.3202	206.2592	240.4651	313.994
97	123.4758	153.0476	160.4645	189.1168	249.5679	302.5239
98	143.2534	175.1605	178.4569	207.136	178.6118	244.1425
99	112.9867	143.2411	173.0715	199.4294	233.3486	306.9992
100	110.6316	138.1493	184.2426	226.8832	219.9981	291.3308
101	117.6532	148.8222	159.9025	189.9333	245.2808	312.8312
102	119.0967	152.0117	165.9236	214.1748	228.1585	294.6355
103	110.9365	136.2283	172.0827	191.9236	263.6667	317.5054
104	109.1383	135.8324	177.9972	218.9959	241.5299	302.183
105	108.0657	132.2497	174.1478	198.716	262.8316	324.3086
106	104.0505	131.7665	173.7124	215.6026	247.1579	315.5129
107	99.54875	127.4673	174.2892	226.8904	264.2541	331.4424
108	136.637	169.9269	177.7045	205.7468	185.7354	256.6656
109	111.8287	135.5027	178.0552	203.8318	243.1506	301.7452
110	145.5738	173.7484	189.6491	216.6588	169.0974	231.0597
111	132.2655	162.0995	177.8587	203.1535	193.9084	265.4285
112	110.5288	138.1531	175.1532	213.8681	251.8669	308.2521
113	110.9403	142.4005	178.2127	223.3101	215.333	290.6819
114	120.2631	152.2921	180.8275	223.1041	197.455	267.0683
115	129.6357	161.5753	182.8615	226.9011	190.9967	248.6889
116	101.0557	120.6438	167.1494	205.0728	291.5208	340.4316
117	129.9248	158.558	179.7832	204.9454	204.4899	266.0657
118	105.7336	127.8727	181.5886	222.7499	245.1186	308.7944
119	113.3288	143.2347	172.1463	208.7279	231.0965	304.2803
120	108.093	136.0751	181.241	224.8768	220.9618	292.4217
121	105.2444	139.4157	169.8732	213.6228	235.3795	315.7308
122	136.9557	171.4033	160.8921	193.6687	207.6163	274.7066
123	122.1504	153.8653	166.551	214.1491	225.4448	289.9981
124	122.8419	155.1536	182.8078	227.3192	185.8128	262.2422
125	99.70152	126.1885	179.9936	238.7465	237.9428	308.1689
126	111.5485	140.5842	183.4325	225.7504	218.3515	291.324
127	129.3029	163.1657	187.4728	230.8761	185.528	244.7415

128	103.3323	131.8956	173.1128	217.0381	247.1662	311.6779
129	114.8862	146.4198	180.9907	222.4157	214.7773	290.1951
130	118.8749	144.2761	176.5123	201.3529	223.1046	290.4441
131	138.3963	170.0363	184.5617	213.9877	173.3222	240.6787
132	99.62565	128.8244	175.7908	229.4773	258.4013	324.4739
133	111.9322	142.3953	185.9308	230.3726	207.8235	280.9824
134	122.3692	151.7961	176.5784	215.6949	218.2109	279.252
135	128.7016	157.6722	175.7999	203.8778	209.0222	276.7923
136	112.8307	138.6694	184.0605	214.4809	222.9509	291.697
137	149.7657	183.8338	183.7802	211.7355	164.2749	231.1092
138	107.8751	138.5291	180.4376	224.5936	238.6345	309.1309
139	109.8208	142.1098	181.2026	223.4656	227.8544	296.584
140	118.2247	152.7289	170.3822	206.3409	214.1352	293.9634
141	110.0752	137.0799	176.7099	219.2462	232.6888	295.6495
142	127.9007	156.0903	175.8884	198.7099	218.2205	277.8915
143	114.3328	148.5068	181.4102	236.3515	198.13	276.2131
144	131.7717	171.6004	192.3412	248.0014	164.0839	233.1391
145	104.5398	140.7624	177.6861	223.2159	216.4322	302.4524
146	104.47	124.8887	168.545	206.7871	272.9205	330.1836
147	116.2984	156.8052	174.386	236.4733	191.0548	278.5333
148	124.033	151.6164	187.5675	231.087	196.4229	252.4811
149	134.8634	165.529	182.7169	212.3166	184.3861	249.5506
150	102.8558	125.8455	168.0124	206.3133	272.8902	330.8742
151	125.2486	159.7288	183.7088	239.2591	179.211	253.2814
152	110.5752	136.8422	179.8302	224.3573	225.8394	293.9977
153	129.6543	146.8904	178.4509	196.7308	238.645	281.5744
154	127.8518	162.0119	177.0669	227.5873	190.1802	260.1814
155	117.8019	148.2678	177.4925	204.5629	220.7603	293.8481
156	122.4298	156.1313	181.6233	222.2668	191.6167	267.2493
157	114.07	139.4558	176.7261	217.4209	239.1659	288.8712
158	141.2664	173.853	176.8435	202.9079	176.3973	248.5132
159	130.4362	172.3929	168.117	222.1044	180.8877	261.2883
160	132.8644	171.6159	174.6374	204.1507	188.0881	262.8969
161	109.7245	137.229	190.5085	246.3098	215.027	285.3557
162	102.3437	119.5781	171.2071	186.3528	307.0306	343.4032
163	115.8665	151.7409	166.8963	217.7171	218.7386	292.5458
164	133.2373	168.5281	170.6996	204.8804	203.0442	273.9077
165	125.9633	160.6011	183.2076	230.0186	184.9568	253.5047
166	115.3864	147.7065	189.0317	236.8081	190.4789	265.4336
167	111.2776	136.6608	178.3354	204.3773	236.5786	303.7095
168	116.1472	140.4064	181.7183	211.5185	226.201	283.8409
169	106.7678	129.4426	161.3679	184.9314	301.7355	342.314
170	124.6367	159.3601	164.6671	196.3782	219.115	294.4027
171	109.7975	128.4284	175.9608	201.6023	261.4589	310.4398
172	120.6427	155.4623	188.467	233.8666	191.5926	269.6411
173	111.1079	136.0974	170.5417	186.9067	261.3276	319.7563
174	113.0445	141.3792	170.6827	193.6963	242.1039	307.2931
175	123.6469	144.0884	173.6048	191.3177	249.253	292.5336
176	116.3674	141.8538	174.9314	199.4319	241.5215	295.2986
177	137.1128	165.3278	180.5412	205.3599	193.0907	256.6776
178	97.34554	118.0329	170.8858	206.3621	302.1018	349.2931
179	101.7979	122.9972	174.4955	227.4812	259.8926	316.6379
180	136.2239	176.2208	177.4457	225.8892	183.9656	255.22
181	116.6495	143.4375	183.0356	226.1076	208.4037	274.6845
182	110.3211	136.5583	173.6371	196.9454	252.8043	312.4312
183	114.8908	136.9921	179.6202	219.8316	240.3142	286.0459
184	103.4087	125.1564	174.5705	212.8136	285.218	329.7199
185	111.5307	133.6676	176.3725	212.631	259.0846	308.2217
186	111.5881	134.1007	179.2584	221.352	237.8675	293.1888
187	110.1395	140.2489	167.3017	215.9429	244.8081	305.6851
188	99.8415	117.2698	171.8125	185.8856	312.1827	351.2323
189	119.1446	146.7326	176.3303	216.783	211.1309	280.8579
190	119.3813	157.4293	165.3169	215.5944	213.7207	294.2584
191	122.3186	152.6193	170.6861	194.1627	220.5516	290.2861
192	104.9061	130.2859	167.0852	199.4431	268.4693	328.2909

193	109.9692	135.6914	179.3392	217.8561	247.9055	305.5764
194	102.0151	129.4977	178.9765	223.1482	244.4347	313.4777
195	115.9212	148.5581	182.438	223.87	213.8521	286.9141
196	108.8774	147.9102	164.328	223.2739	231.0375	311.7662
197	107.4867	136.5637	170.6027	211.3267	239.1731	314.5028
198	133.9345	162.4037	185.2204	213.9091	181.8937	246.6481
199	120.6932	156.8539	162.5758	194.3481	231.302	305.4011
200	121.6709	156.3097	171.2976	213.3957	202.1978	277.3722

ANNEXURE D

200 Sample each size n=10 (estimate on value Case I))

S.No.	Mean(T)	Mean(V)	Mean (I)	MSE(T)	MSE(V)	MSE (I)
1	126.06	195.47	258.18	40.93	44.61	304.28
2	151.98	193.20	221.19	82.81	58.51	375.05
3	116.95	182.58	300.82	17.71	26.38	126.48
4	107.75	190.27	324.17	23.15	80.34	133.73
5	132.29	193.69	254.92	40.23	82.36	255.37
6	135.91	200.63	242.21	57.13	85.26	357.26
7	150.52	177.41	245.28	60.96	54.26	256.51
8	119.24	182.28	298.43	36.16	19.66	254.32
9	130.69	198.31	256.09	59.25	92.50	285.11
10	115.70	197.34	295.98	42.87	99.79	268.01
11	133.80	206.12	233.38	53.19	114.16	249.09
12	110.90	189.25	303.54	20.10	26.83	202.10
13	136.18	199.37	239.54	81.25	95.80	410.04
14	117.39	172.99	322.36	28.97	37.52	101.20
15	116.74	191.95	283.93	44.01	42.86	387.44
16	133.16	205.29	235.15	63.44	96.79	355.90
17	113.70	203.25	278.22	17.96	109.47	168.70
18	118.81	201.72	275.98	45.20	104.48	297.62
19	133.38	206.91	231.98	85.47	125.28	431.42
20	156.32	192.27	219.86	75.67	70.89	355.72
21	120.81	206.50	262.25	55.37	108.83	356.51
22	140.48	179.42	267.51	53.33	47.91	198.43
23	134.09	207.61	231.25	57.09	114.56	300.92
24	116.82	207.54	273.70	38.76	169.45	217.10
25	120.90	194.12	271.69	32.67	47.83	238.95
26	123.12	222.35	254.34	47.69	210.03	225.43
27	123.55	178.69	292.49	33.77	14.03	191.94
28	121.81	182.19	295.69	31.03	26.88	174.10
29	129.93	182.34	279.93	38.11	46.38	222.02
30	116.73	212.84	261.64	38.89	186.98	276.96
31	128.88	199.29	250.65	47.19	107.46	306.73
32	121.66	182.47	289.51	36.48	25.87	237.46
33	133.39	189.06	255.77	58.85	86.33	316.47
34	132.32	189.68	267.79	59.27	113.58	319.38
35	117.66	182.63	301.97	29.98	26.08	157.01
36	112.38	193.41	305.48	39.22	92.69	222.06
37	133.75	196.29	259.68	79.95	179.53	366.48
38	129.46	191.06	263.44	54.36	90.63	311.41
39	137.90	193.02	241.89	48.87	35.03	278.62
40	122.42	185.49	285.49	40.68	76.05	232.69
41	106.02	205.24	306.64	25.92	157.88	180.21
42	125.09	191.70	268.54	30.92	38.72	195.83
43	126.75	207.23	251.93	52.88	107.03	317.74
44	118.34	192.94	279.76	44.38	100.91	303.46
45	136.26	221.11	220.39	72.37	179.74	382.32
46	123.80	197.35	269.19	55.64	87.71	355.76
47	128.71	194.93	259.64	54.02	93.83	334.17
48	149.40	187.91	228.66	52.30	22.22	269.04
49	126.12	195.45	259.83	53.64	53.46	330.12
50	126.64	188.50	270.66	63.69	141.16	301.33
51	126.78	194.43	257.97	39.88	39.53	275.14
52	143.59	214.68	207.93	53.55	119.83	233.16
53	121.87	195.76	278.83	64.30	190.49	293.11
54	124.27	192.71	268.23	31.58	46.15	203.44
55	127.18	193.24	263.27	46.53	91.74	297.30
56	116.60	219.64	252.51	48.63	173.33	401.87
57	130.63	191.22	257.58	46.22	41.95	262.80
58	114.02	182.77	321.21	37.67	103.06	146.22
59	111.82	190.80	311.60	21.16	78.52	116.07
60	103.89	202.69	310.76	24.10	154.59	253.94
61	123.19	195.99	279.34	44.90	92.92	285.39
62	108.40	196.40	314.63	23.49	134.83	161.35

63	136.72	185.23	259.91	66.91	60.73	378.53
64	141.38	206.67	228.12	68.83	138.69	361.17
65	135.50	192.82	250.83	45.28	37.26	249.44
66	123.54	191.37	272.98	63.92	140.09	442.15
67	135.13	187.50	253.88	38.83	29.54	216.41
68	120.17	188.60	283.95	36.80	33.97	257.85
69	134.26	211.20	227.20	58.59	117.44	283.75
70	126.54	190.42	274.16	33.33	82.65	170.07
71	144.53	207.58	223.41	49.41	84.03	230.22
72	109.12	178.87	330.01	16.94	12.65	90.75
73	149.08	209.21	206.23	92.17	132.51	389.23
74	125.89	182.47	282.28	37.09	20.72	217.87
75	120.50	188.73	288.57	38.61	88.39	222.98
76	138.53	186.34	257.55	33.20	24.10	137.62
77	120.29	178.79	299.38	38.03	18.19	223.08
78	117.25	199.45	283.04	56.03	190.68	300.84
79	119.33	221.84	258.49	52.56	224.87	317.92
80	128.59	171.86	301.93	22.77	32.04	66.31
81	120.74	210.16	258.45	51.10	164.20	326.00
82	139.09	183.05	252.65	32.61	22.92	163.49
83	130.49	176.62	288.88	37.63	49.75	146.97
84	125.99	215.63	242.48	48.49	160.04	338.39
85	126.00	188.92	277.69	50.85	129.88	214.85
86	129.45	197.01	261.68	67.00	132.42	320.52
87	135.26	197.76	254.54	83.37	185.87	370.72
88	118.74	189.77	284.36	32.99	39.92	241.38
89	130.86	187.46	261.61	50.95	24.47	298.25
90	116.23	196.35	289.06	48.19	134.21	270.41
91	149.01	206.47	215.82	73.00	108.04	272.90
92	131.10	203.70	252.74	54.22	142.83	246.11
93	133.24	184.75	258.21	55.21	25.80	324.25
94	116.02	186.40	303.31	21.37	79.88	120.78
95	136.91	189.18	252.52	61.28	37.55	294.62
96	119.69	191.79	277.23	41.32	47.37	318.86
97	138.26	174.79	276.05	45.95	46.58	172.08
98	159.21	192.80	211.38	56.89	44.46	243.92
99	128.11	186.25	270.17	53.61	36.79	298.34
100	124.39	205.56	255.66	41.16	101.79	289.60
101	133.24	174.92	279.06	53.24	50.93	263.69
102	135.55	190.05	261.40	58.86	124.74	249.30
103	123.58	182.00	290.59	33.34	23.40	171.81
104	122.49	198.50	271.86	37.40	95.00	212.76
105	120.16	186.43	293.57	33.24	33.66	213.80
106	117.91	194.66	281.34	40.85	94.99	272.57
107	113.51	200.59	297.85	41.66	151.78	251.18
108	153.28	191.73	221.20	61.89	42.09	282.16
109	123.67	190.94	272.45	29.86	36.79	199.05
110	159.66	203.15	200.08	45.75	38.10	212.23
111	147.18	190.51	229.67	50.25	34.80	295.99
112	124.34	194.51	280.06	38.84	85.76	186.50
113	126.67	200.76	253.01	55.46	104.70	319.24
114	136.28	201.97	232.26	57.63	97.21	271.59
115	145.61	204.88	219.84	55.79	109.37	183.23
116	110.85	186.11	315.98	20.57	82.26	140.90
117	144.24	192.36	235.28	45.12	37.75	217.72
118	116.80	202.17	276.96	25.02	92.85	234.94
119	128.28	190.44	267.69	48.59	69.21	308.75
120	122.08	203.06	256.69	43.85	101.53	293.06
121	122.33	191.75	275.56	61.44	104.31	377.78
122	154.18	177.28	241.16	63.02	58.69	268.87
123	138.01	190.35	257.72	55.57	121.76	231.56
124	139.00	205.06	224.03	58.96	102.61	330.48
125	112.95	209.37	273.06	37.34	181.13	288.63
126	126.07	204.59	254.84	43.73	100.14	306.05
127	146.23	209.17	215.13	60.43	108.34	197.54

128	117.61	195.08	279.42	45.10	108.05	241.96
129	130.65	201.70	252.49	54.82	94.35	321.07
130	131.58	188.93	256.77	37.36	33.37	264.76
131	154.22	199.27	207.00	55.67	42.95	258.44
132	114.23	202.63	291.44	45.06	159.76	248.81
133	127.16	208.15	244.40	48.64	107.75	315.29
134	137.08	196.14	248.73	48.24	87.80	199.44
135	143.19	189.84	242.91	48.30	39.99	252.65
136	125.75	199.27	257.32	36.58	49.71	272.05
137	166.80	197.76	197.69	63.13	41.05	258.45
138	123.20	202.52	273.88	52.12	106.32	278.73
139	125.97	202.33	262.22	56.36	98.49	265.97
140	135.48	188.36	254.05	64.99	64.29	370.61
141	123.58	197.98	264.17	39.03	101.52	232.21
142	142.00	187.30	248.06	42.33	30.48	205.28
143	131.42	208.88	237.17	64.32	162.36	347.96
144	151.69	220.17	198.61	84.40	170.59	267.51
145	122.65	200.45	259.44	70.87	108.92	429.33
146	114.68	187.67	301.55	22.57	80.52	187.00
147	136.55	205.43	234.79	88.60	201.57	443.32
148	137.82	209.33	224.45	39.05	109.25	185.66
149	150.20	197.52	216.97	52.90	47.02	233.88
150	114.35	187.16	301.88	27.26	80.06	197.97
151	142.49	211.48	216.25	65.40	161.84	319.56
152	123.71	202.09	259.92	37.72	107.00	261.58
153	138.27	187.59	260.11	18.44	19.17	95.27
154	144.93	202.33	225.18	65.09	137.16	279.30
155	133.03	191.03	257.30	51.82	40.93	303.91
156	139.28	201.95	229.43	63.86	89.13	320.53
157	126.76	197.07	264.02	33.28	93.13	144.16
158	157.56	189.88	212.46	60.78	34.43	298.15
159	151.41	195.11	221.09	92.42	155.29	385.56
160	152.24	189.39	225.49	79.56	45.77	324.47
161	123.48	218.41	250.19	41.61	169.62	277.22
162	110.96	178.78	325.22	16.12	14.95	80.46
163	133.80	192.31	255.64	70.93	137.25	311.60
164	150.88	187.79	238.48	67.51	61.10	290.62
165	143.28	206.61	219.23	63.73	114.38	270.67
166	131.55	212.92	227.96	58.93	119.70	323.81
167	123.97	191.36	270.14	34.24	37.71	263.61
168	128.28	196.62	255.02	30.62	50.74	200.94
169	118.11	173.15	322.02	28.11	32.09	93.05
170	142.00	180.52	256.76	65.92	52.58	332.29
171	119.11	188.78	285.95	18.14	38.40	147.74
172	138.05	211.17	230.62	67.79	110.13	345.17
173	123.60	178.72	290.54	35.47	15.27	187.97
174	127.21	182.19	274.70	43.37	31.19	247.81
175	133.87	182.46	270.89	22.53	19.18	108.27
176	129.11	187.18	268.41	33.72	36.31	173.75
177	151.22	192.95	224.88	44.69	31.85	228.76
178	107.69	188.62	325.70	23.29	70.17	125.00
179	112.40	200.99	288.27	23.16	149.86	189.31
180	156.22	201.67	219.59	84.98	128.45	294.98
181	130.04	204.57	241.54	38.96	100.20	254.60
182	123.44	185.29	282.62	36.02	31.90	211.65
183	125.94	199.73	263.18	26.29	93.83	113.45
184	114.28	193.69	307.47	23.37	86.57	126.73
185	122.60	194.50	283.65	26.89	75.84	128.24
186	122.84	200.31	265.53	27.17	98.12	173.16
187	125.19	191.62	275.25	48.27	125.97	217.14
188	108.56	178.85	331.71	16.69	12.90	89.18
189	132.94	196.56	245.99	42.26	89.34	276.46
190	138.41	190.46	253.99	77.60	136.73	377.96
191	137.47	182.42	255.42	50.94	30.67	279.22
192	117.60	183.26	298.38	33.69	57.02	215.45

193	122.83	198.60	276.74	36.79	83.87	179.82
194	115.76	201.06	278.96	40.29	104.49	280.06
195	132.24	203.15	250.38	57.18	96.00	301.57
196	128.39	193.80	271.40	81.50	187.09	366.96
197	122.03	190.96	276.84	44.91	89.82	327.25
198	148.17	199.56	214.27	47.46	43.19	236.56
199	138.77	178.46	268.35	68.76	55.81	326.04
200	138.99	192.35	239.79	63.98	96.56	325.79

ANNEXURE E

200 Samples each of size n=10 (estimates value ($Case\ II$))

S.No.	Mean(T)	Mean(V)	Mean (I)	MSE(T)	MSE(V)	MSE (I)
1	126.06	195.47	258.18	47.85	54.95	339.36
2	151.98	193.20	221.19	102.33	73.64	411.44
3	116.95	182.58	300.82	21.55	27.21	132.88
4	107.75	190.27	324.17	28.52	91.35	145.46
5	132.29	193.69	254.92	45.92	96.27	296.06
6	135.91	200.63	242.21	68.70	101.30	410.05
7	150.52	177.41	245.28	75.66	65.56	277.94
8	119.24	182.28	298.43	42.95	22.00	291.39
9	130.69	198.31	256.09	70.10	109.76	336.59
10	115.70	197.34	295.98	51.28	118.65	305.46
11	133.80	206.12	233.38	65.52	138.71	280.75
12	110.90	189.25	303.54	23.92	33.24	228.06
13	136.18	199.37	239.54	98.55	118.59	463.30
14	117.39	172.99	322.36	35.81	41.53	105.96
15	116.74	191.95	283.93	52.01	55.96	429.80
16	133.16	205.29	235.15	74.38	121.29	408.35
17	113.70	203.25	278.22	23.62	128.86	179.39
18	118.81	201.72	275.98	52.98	125.13	339.17
19	133.38	206.91	231.98	103.92	156.07	471.29
20	156.32	192.27	219.86	89.85	91.83	392.30
21	120.81	206.50	262.25	66.43	138.31	406.46
22	140.48	179.42	267.51	65.00	57.46	222.01
23	134.09	207.61	231.25	70.32	139.87	332.33
24	116.82	207.54	273.70	47.58	197.50	245.46
25	120.90	194.12	271.69	39.66	56.94	261.09
26	123.12	222.35	254.34	57.35	243.64	261.93
27	123.55	178.69	292.49	38.16	16.16	230.63
28	121.81	182.19	295.69	36.95	29.38	194.60
29	129.93	182.34	279.93	43.25	57.96	261.61
30	116.73	212.84	261.64	47.72	227.71	307.02
31	128.88	199.29	250.65	59.92	126.61	340.61
32	121.66	182.47	289.51	43.96	27.93	265.11
33	133.39	189.06	255.77	69.93	101.66	368.75
34	132.32	189.68	267.79	71.35	135.68	358.51
35	117.66	182.63	301.97	36.06	28.96	176.77
36	112.38	193.41	305.48	48.12	107.84	251.27
37	133.75	196.29	259.68	99.90	218.13	414.09
38	129.46	191.06	263.44	67.22	109.24	351.84
39	137.90	193.02	241.89	56.65	43.14	317.43
40	122.42	185.49	285.49	47.43	88.56	277.26
41	106.02	205.24	306.64	32.87	186.68	194.47
42	125.09	191.70	268.54	37.69	44.64	219.53
43	126.75	207.23	251.93	64.14	134.89	360.76
44	118.34	192.94	279.76	52.78	122.28	343.56
45	136.26	221.11	220.39	89.02	223.44	432.93
46	123.80	197.35	269.19	64.76	109.70	411.21
47	128.71	194.93	259.64	63.65	113.76	384.80
48	149.40	187.91	228.66	58.34	31.31	316.84
49	126.12	195.45	259.83	65.76	63.61	364.63
50	126.64	188.50	270.66	76.72	168.93	326.99
51	126.78	194.43	257.97	49.69	46.94	302.74
52	143.59	214.68	207.93	66.55	143.37	259.79
53	121.87	195.76	278.83	78.64	231.47	326.01
54	124.27	192.71	268.23	38.51	52.61	222.09
55	127.18	193.24	263.27	54.54	108.34	335.48
56	116.60	219.64	252.51	57.32	224.32	449.38
57	130.63	191.22	257.58	53.47	52.03	307.62
58	114.02	182.77	321.21	46.67	120.45	159.24
59	111.82	190.80	311.60	26.32	88.36	126.89
60	103.89	202.69	310.76	29.07	190.04	285.45
61	123.19	195.99	279.34	56.40	104.52	318.25
62	108.40	196.40	314.63	28.94	159.11	185.51

63	136.72	185.23	259.91	78.53	77.91	425.80
64	141.38	206.67	228.12	82.39	173.85	407.46
65	135.50	192.82	250.83	54.49	45.47	284.13
66	123.54	191.37	272.98	76.24	174.77	487.91
67	135.13	187.50	253.88	47.04	31.28	242.55
68	120.17	188.60	283.95	43.98	41.20	291.49
69	134.26	211.20	227.20	70.08	140.27	319.97
70	126.54	190.42	274.16	40.32	93.39	198.14
71	144.53	207.58	223.41	58.65	93.27	270.91
72	109.12	178.87	330.01	20.73	12.66	98.89
73	149.08	209.21	206.23	113.96	167.10	431.92
74	125.89	182.47	282.28	44.87	22.72	245.39
75	120.50	188.73	288.57	48.14	103.05	251.06
76	138.53	186.34	257.55	38.48	27.56	163.95
77	120.29	178.79	299.38	43.92	20.33	262.42
78	117.25	199.45	283.04	69.28	238.93	333.21
79	119.33	221.84	258.49	66.31	267.19	355.09
80	128.59	171.86	301.93	27.83	36.57	66.21
81	120.74	210.16	258.45	63.84	200.87	361.37
82	139.09	183.05	252.65	36.10	25.78	194.90
83	130.49	176.62	288.88	44.81	57.03	160.95
84	125.99	215.63	242.48	57.49	193.83	387.99
85	126.00	188.92	277.69	63.02	154.42	230.48
86	129.45	197.01	261.68	82.96	163.85	358.06
87	135.26	197.76	254.54	105.02	223.99	412.04
88	118.74	189.77	284.36	40.85	48.01	265.11
89	130.86	187.46	261.61	58.75	32.03	343.30
90	116.23	196.35	289.06	56.78	159.48	324.42
91	149.01	206.47	215.82	89.35	123.37	312.80
92	131.10	203.70	252.74	64.17	163.76	300.50
93	133.24	184.75	258.21	61.00	33.91	376.11
94	116.02	186.40	303.31	26.58	90.19	132.27
95	136.91	189.18	252.52	75.07	45.07	329.84
96	119.69	191.79	277.23	50.64	54.50	351.84
97	138.26	174.79	276.05	56.91	53.43	182.50
98	159.21	192.80	211.38	66.25	53.53	279.46
99	128.11	186.25	270.17	59.57	45.21	353.00
100	124.39	205.56	255.66	49.28	118.32	331.13
101	133.24	174.92	279.06	63.22	58.69	296.95
102	135.55	190.05	261.40	70.50	151.51	287.59
103	123.58	182.00	290.59	41.63	25.62	188.63
104	122.49	198.50	271.86	46.37	109.39	239.41
105	120.16	186.43	293.57	38.06	39.28	245.95
106	117.91	194.66	281.34	49.99	114.20	304.07
107	113.51	200.59	297.85	50.72	180.06	293.78
108	153.28	191.73	221.20	72.12	51.17	327.41
109	123.67	190.94	272.45	36.47	43.24	223.43
110	159.66	203.15	200.08	51.66	47.48	249.85
111	147.18	190.51	229.67	57.92	41.64	332.88
112	124.34	194.51	280.06	49.66	97.54	206.90
113	126.67	200.76	253.01	64.41	132.35	369.47
114	136.28	201.97	232.26	66.76	116.31	315.36
115	145.61	204.88	219.84	66.39	126.22	216.60
116	110.85	186.11	315.98	24.97	93.59	155.68
117	144.24	192.36	235.28	53.35	41.20	246.74
118	116.80	202.17	276.96	31.90	110.26	263.86
119	128.28	190.44	267.69	58.20	87.09	348.54
120	122.08	203.06	256.69	50.96	123.91	332.32
121	122.33	191.75	275.56	75.99	124.56	420.16
122	154.18	177.28	241.16	77.22	69.91	292.92
123	138.01	190.35	257.72	65.46	147.44	271.18
124	139.00	205.06	224.03	67.94	128.93	380.14
125	112.95	209.37	273.06	45.66	224.64	320.94
126	126.07	204.59	254.84	54.86	116.54	346.53
127	146.23	209.17	215.13	74.62	122.60	228.18

128	117.61	195.08	279.42	53.09	125.56	270.84
129	130.65	201.70	252.49	64.71	111.67	370.15
130	131.58	188.93	256.77	41.99	40.16	295.10
131	154.22	199.27	207.00	65.15	56.35	295.25
132	114.23	202.63	291.44	55.48	187.57	284.10
133	127.16	208.15	244.40	60.39	128.53	348.31
134	137.08	196.14	248.73	56.35	99.57	242.48
135	143.19	189.84	242.91	54.62	51.30	298.89
136	125.75	199.27	257.32	43.45	60.22	307.56
137	166.80	197.76	197.69	75.53	50.86	290.69
138	123.20	202.52	273.88	61.15	126.88	323.42
139	125.97	202.33	262.22	67.85	116.24	307.41
140	135.48	188.36	254.05	77.48	84.15	414.71
141	123.58	197.98	264.17	47.46	117.75	257.97
142	142.00	187.30	248.06	51.71	33.89	231.72
143	131.42	208.88	237.17	76.00	196.44	396.77
144	151.69	220.17	198.61	103.23	201.61	310.33
145	122.65	200.45	259.44	85.39	134.90	481.54
146	114.68	187.67	301.55	27.13	95.17	213.39
147	136.55	205.43	234.79	106.78	250.86	498.00
148	137.82	209.33	224.45	49.51	123.25	204.51
149	150.20	197.52	216.97	61.20	57.02	276.34
150	114.35	187.16	301.88	34.39	95.46	218.80
151	142.49	211.48	216.25	77.37	200.82	357.04
152	123.71	202.09	259.92	44.90	129.03	302.32
153	138.27	187.59	260.11	19.33	21.75	119.93
154	144.93	202.33	225.18	75.94	166.10	318.89
155	133.03	191.03	257.30	60.40	47.69	347.63
156	139.28	201.95	229.43	73.91	107.50	372.26
157	126.76	197.07	264.02	41.94	107.77	160.78
158	157.56	189.88	212.46	69.10	44.21	338.45
159	151.41	195.11	221.09	114.56	189.68	420.67
160	152.24	189.39	225.49	97.73	56.68	364.19
161	123.48	218.41	250.19	49.23	202.64	321.88
162	110.96	178.78	325.22	19.33	14.93	86.09
163	133.80	192.31	255.64	83.75	168.08	354.51
164	150.88	187.79	238.48	81.05	76.03	326.79
165	143.28	206.61	219.23	78.08	142.60	305.79
166	131.55	212.92	227.96	67.98	148.54	365.62
167	123.97	191.36	270.14	41.93	44.13	293.27
168	128.28	196.62	255.02	38.30	57.79	216.21
169	118.11	173.15	322.02	33.46	36.13	107.16
170	142.00	180.52	256.76	78.46	65.44	368.87
171	119.11	188.78	285.95	22.59	42.79	156.13
172	138.05	211.17	230.62	78.90	134.13	396.42
173	123.60	178.72	290.54	40.64	17.43	222.17
174	127.21	182.19	274.70	52.25	34.47	276.55
175	133.87	182.46	270.89	27.19	20.42	121.90
176	129.11	187.18	268.41	42.27	39.06	188.20
177	151.22	192.95	224.88	51.81	40.09	263.13
178	107.69	188.62	325.70	27.85	81.90	144.93
179	112.40	200.99	288.27	29.25	182.70	209.55
180	156.22	201.67	219.59	104.11	152.72	330.41
181	130.04	204.57	241.54	46.70	120.73	285.89
182	123.44	185.29	282.62	44.80	35.35	231.37
183	125.94	199.73	263.18	31.79	105.23	136.10
184	114.28	193.69	307.47	30.78	95.18	128.88
185	122.60	194.50	283.65	31.89	85.56	157.13
186	122.84	200.31	265.53	32.98	115.31	199.16
187	125.19	191.62	275.25	59.00	153.97	241.18
188	108.56	178.85	331.71	19.77	12.89	99.23
189	132.94	196.56	245.99	49.53	106.49	316.40
190	138.41	190.46	253.99	94.21	164.50	422.11
191	137.47	182.42	255.42	59.75	35.87	316.46
192	117.60	183.26	298.38	41.92	68.14	232.89

193	122.83	198.60	276.74	43.06	96.55	216.44
194	115.76	201.06	278.96	49.15	126.97	310.22
195	132.24	203.15	250.38	69.32	111.71	347.38
196	128.39	193.80	271.40	99.15	226.12	424.12
197	122.03	190.96	276.84	55.02	107.93	369.28
198	148.17	199.56	214.27	52.74	53.56	272.88
199	138.77	178.46	268.35	85.09	65.69	357.32
200	138.99	192.35	239.79	78.08	115.33	367.76

The population dataset and Python programming code which we have used in this paper for calculate the results of each occasion is available at: https://abdulalim90.blogspot.com/

On a Reliability of Tree-Like Transportation Networks

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Abstract

The degree of reliability of the transportation tree-like networks is proposed to be estimated by the index of operational reliability, which is the relative volume of the product not delivered to the point for some time due to the failures of its elements. A method is proposed for calculating this index using a characteristic feature of the structure of transportation network – a tree-like structure and assumes the time invariance of failure and repair flows of its elements. On such structure, the Y-shaped structure-forming fragment is distinguished, the assessment of the reliability of which (in the accepted understanding) is carried out analytically using the concept of the state space. Each of Y-shaped fragment is virtually replaced by one fictitious element, the destruction parameter of which is calculated from the condition of equality of the volumes of the product undelivered to the network output during such a replacement. The calculation of the operational index is reduced to a step-by-step recurrent procedure using the results obtained in the previous step.

Key words: transportation network, product, operational reliability, Y-shaped fragment, failure-repair process, virtual equivalence.

It is difficult to point presently a field of production activity in which, to greater or lesser extent, transportation networks would not be used. With all their diversity, under the transportation network, in the general case, one can understand the aggregate of transport links (mains) along which the necessary movement of a certain "product" in space is carried out.

Transportation networks are classified according to various criteria, in particular, according to their topology. This article discusses a class of tree-like networks designed to transport a product entering at some network inlets to its only outlet, i.e. performing, figuratively speaking, an "aggregation" function (Fig. 1). Examples of such networks, under varying degrees of idealization, are: oil or gas transportation systems from production sites to the main pipeline; sewer networks of large cities; a network of approach lines to the gravity hump of a railway sorting yard of the station; assembly conveyors of various flow-line productions and much more. One of the properties of a transportation network, like any technical object, is the reliability of its functioning. In the "classical" reliability theory, an object is usually estimated using a set of quantitative indexes such as "survival function", "mean operating time between failures", and others [1]. For most technical objects, such indexes most often correspond to the everyday insights about the reliability of the object's functioning and are easily interpreted physically.

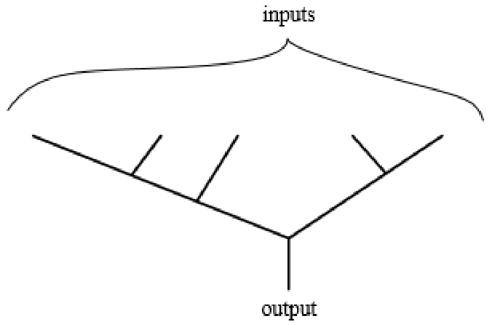

Fig. 1. Tree-like transportation network.

When it comes to transportation networks, these indexes sometimes lead to ambiguity in their interpretation and require additional explanations and clarifications [2-4]. Moreover, when analyzing the operation of transportation networks, it often turns out that such generally different concepts as "reliability" and "operational efficiency" are so interrelated that it is very difficult to separate them from each other. Therefore, the choice of a single quantitative index, to a certain extent, uniting these two concepts, as well as the development of an engineering methodology for its calculation, seems to be actual and important task. One of the possible solution to this problem may be the approach proposed in this article.

For the uniqueness of further understanding, we will specially stipulate the terminology and assumptions accepted in this work:

- topologically, the network is a simply connected labeled graph with several entrance nodes (inputs) and one outgoing (output);
- each transport link of the network will be considered as an "element";
- regardless of the physical character, what the network is intended to transport will be called "product";
- the intensity of the product flow arriving into *i*-th inlet of the network will be called "product flow rate";
- the movement of the product along each element of the network is unidirectional;
- in the process of functioning, any element of the network can be one of two possible states: "in operation" or "under repair";
- transition of any element from working state to repair and back occurs at random moment of continuous time;
- flows of failures and repairs of the *i*-th element are stationary [5] with parameters λ_i and μ_i, respectively.

Consider the simplest Y-shaped transportation network (Fig. 2a)); to Fig. 2b) we will return later.

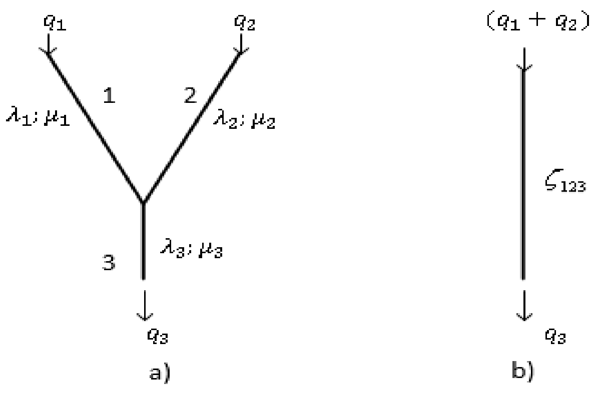

Fig. 2. Y-shaped transportation network a) and its equivalent b).

The network consists of three elements (1, 2 and 3); into inputs 1 and 2 come product value per time unit q_1 and q_2. The purpose of the network is to transport the incoming product to the output of 3nd element (q_3). In the process of functioning, the network elements can fail with rates λ_1, λ_2 and λ_3, respectively, be repaired (with rates μ_1, μ_2 and μ_3), and re-enter in operation. As a result, a part of the product ΔQ that arrived at the network inlets during the time *T* is not delivered to its output. Let us evaluate ΔQ, assuming that all listed technological parameters and parameters of the failure-restoration process are given.

We depict the graph of states for Y-shaped network shown in Fig. 2a). For this purpose, let us number and describe all possible states of the network (there are 8 of them) and assign each of them its stationary possibility p_i ($i = 0 \div 7$), namely:

0: (all three elements in operation) - p_0;

1: (1st element under repair, 2nd and 3rd in operation) - p_1;

2: (2nd element under repair, 1st and 3rd in operation) - p_2;

3: (3rd element under repair, 1st and 2nd in operation) - p_3;

4: (1st and 2nd elements under repair, 3rd in operation) - p_4;

5: (1st and 3rd elements under repair, 2nd in operation) - p_5;

6: (2nd and 3rd elements under repair, 1st in operation) - p_6;

7: (all three elements are under repair) - p_7.

Then, the state space graph for Y-shaped network will be shown in the following form (Fig. 3).

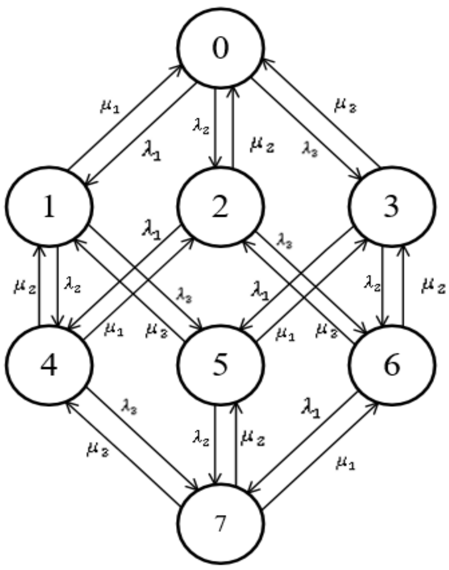

Fig. 3. State space graph for Y-shaped network.

Determination of the probability values p_i is realized in line with a common procedure [5] and reduces to solving of the matrix equation:

$$A \cdot \tilde{P} = \tilde{B}, \tag{1}$$

where (under preceding notations and taking account of the normalization condition $\sum_{i=0}^{7} p_i = 1$) A - is the square matrix of size $[8 \times 8]$ having the form:

$$A = \begin{bmatrix} (\lambda_1+\lambda_2+\lambda_3) & -\mu_1 & -\mu_2 & -\mu_3 & 0 & 0 & 0 & 0 \\ -\lambda_1 & (\lambda_2+\lambda_3+\mu_1) & 0 & 0 & -\mu_2 & -\mu_3 & 0 & 0 \\ -\lambda_2 & 0 & (\lambda_1+\lambda_3+\mu_2) & 0 & -\mu_1 & 0 & -\mu_3 & 0 \\ -\lambda_3 & 0 & 0 & (\lambda_1+\lambda_2+\mu_3) & 0 & -\mu_1 & -\mu_2 & 0 \\ 0 & -\lambda_2 & -\lambda_1 & 0 & (\lambda_3+\mu_1+\mu_2) & 0 & 0 & -\mu_3 \\ 0 & -\lambda_3 & 0 & -\lambda_1 & 0 & (\lambda_2+\mu_1+\mu_3) & 0 & -\mu_2 \\ 0 & 0 & -\lambda_3 & -\lambda & 0 & 0 & (\lambda_1+\mu_2+\mu_3) & -\mu_1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix},$$

B - is the row-matrix of free members [0 0 0 0 0 0 0 1], P - is the row-matrix of required stationary probabilities [p_0 p_1 p_2 p_3 p_4 p_5 p_6 p_7], and "~" – is the transposition symbol.

With every state i can associate a certain product volume ΔQ_i undelivered to the network output at time T due to the network elements failure. This volume is easily determined from Fig. 2a) and equals to: for the state $0 \rightarrow \Delta Q_0 = 0$; $1 \rightarrow \Delta Q_1 = q_1 T$; $2 \rightarrow \Delta Q_2 = q_2 T$; $3 \div 7 \rightarrow \Delta Q_3 \div \Delta Q_7 = (q_1 + q_2)T$.

The undelivered product volume ΔQ is calculated as expectation of the random variable ΔQ_i:

$$\Delta Q = \sum_{i=0}^{7} p_i \cdot \Delta Q_i . \tag{2}$$

We introduce a new index of operational reliability.

As a quantitative measure (index) of the operational reliability of Y-shaped transportation network (Fig. 2a)) γ *we will be considering the relative part of the product undelivered to the output due to the failure of network elements*, i.e. [6]:

$$\gamma = \frac{\Delta Q}{Q} = \frac{\sum_{i=0}^{7} p_i \cdot \Delta Q_i}{Q}, \tag{3}$$

where Q is the product volume incoming at the network inputs during the time T ; in the case under consideration $Q = (q_1 + q_2)T$.

As can be seen from (3) the value γ is normalized and can vary from 0 to 1. The value $\gamma = 0$ corresponds to a reliable, and $\gamma = 1$ to an unreliable network. Intermediate values γ characterize the degree 0f reliability of this object (in considered sense).

The index γ is more informative than generally accepted indexes of the "classical" reliability theory ("survival function", etc.) since its value depends not only on the technological parameters of the transportation network ("product flow rate" for inputs), but also from the location of the failed element in the object structure.

For this class of tasks such an index is applicable to characterize a reliability of not only three-element network considered above, but also networks containing any (usually n) number of elements. However, methods for calculating it for practical cases, when n can reach ten or even hundreds, causes significant difficulties: the number of possible network states is 2^n [5], and, for large n, the calculation of their stationary probabilities becomes very resource-expendable even for modern computing technologies. It is possible to use topological calculation methods [17-19, etc.], which directly from the state graph of a complex system allow us to formalize the reliability indicators.

However, for the considered class of problems of analyzing systems with network structures and a specific reliability indicator, the use of known topological methods is also largely resource intensive.

Overcome these difficulties allows an approach proposed further. This approach proceeds from the fact that the probabilities of simultaneous failure of two or more elements of Y-shaped network shown in Fig. 2a) are extremely low. It is important to note that this assumption is applies only to each element of Y-shaped fragment. The state graph of such system is shown in Fig. 4.

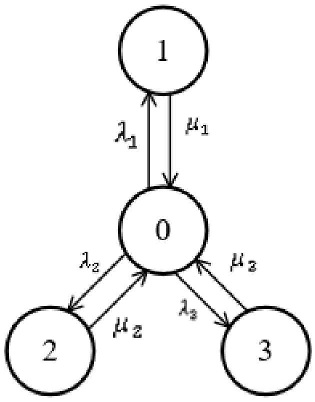

Fig. 4. State graph for Y-shaped network corresponding to the simplifying assumption.

The state numbers accepted in Fig. 4, their contents and notations of the assigned possibilities p_i ($i = 0 \div 3$) are explained below:

0: (all three elements in operation) - p_0;

1: (1st element under repair, 2nd and 3rd in operation) - p_1;

2: (2nd element under repair, 1st and 3rd in operation) - p_2;

3: (3rd element under repair, 1st and 2nd in operation) - p_3.

In this case the solution of equation (3) simplifies and leads to following expressions for stationary possibilities [7]:

$$p_0 = \frac{1}{1+\zeta_1+\zeta_2+\zeta_3}; \tag{4}$$

$$p_1 = \frac{\zeta_1}{1+\zeta_1+\zeta_2+\zeta_3}; \tag{5}$$

$$p_2 = \frac{\zeta_2}{1+\zeta_1+\zeta_2+\zeta_3}; \tag{6}$$

$$з_2 = \frac{\zeta_3}{1+\zeta_1+\zeta_2+\zeta_3}, \tag{7}$$

when dimensionless parameters are additionally introduced: $\zeta_1 = \lambda_1 / \mu_1$; $\zeta_2 = \lambda_2 / \mu_2$; $\zeta_3 = \lambda_3 / \mu_3$.

Each i-th state corresponds to the volume of the product undelivered to the network output that will be now (see Fig. 2a)): $0 \to \Delta Q_0 = 0$; $1 \to \Delta Q_1 = q_1 T$; $2 \to \Delta Q_2 = q_2 T$; $3 \to \Delta Q_3 = (q_1 + q_2)T$, and the expectation ΔQ is calculated as:

$$\Delta Q = \sum_{i=0}^{3} p_i \cdot \Delta Q_i = \frac{(\zeta_1+\zeta_3)q_1 + (\zeta_2+\zeta_3)q_2}{1+\zeta_1+\zeta_2+\zeta_3} \cdot T. \tag{8}$$

The results of performed numerical calculations given in [3] allow us to speak that a calculation error is relative and is acceptable when replacing the exact formula (2) with approximate (8) amounting to the tenth parts of a percent.

Let us carry out the equivalency operation symbolically shown in Fig. 2, i.e. let us replace the Y-shaped network with one element, choosing the parameter ζ_{123} that determines its "failure-repair" process from the condition of equality of expectations ΔQ corresponding to Fig. 2a) and Fig. 2b). Applying the technique described in [8], we get:

$$\zeta_{123} = \frac{(\zeta_1 + \zeta_3)q_1 + (\zeta_2 + \zeta_3)q_2}{(1+\zeta_2)q_1 + (1+\zeta_1)q_2}. \tag{9}$$

Considering that ζ_1 and ζ_2 are usually negligible in comparison with unity, we have with a high degree of accuracy finally:

$$\zeta_{123} = \frac{(\zeta_1 + \zeta_3)q_1 + (\zeta_2 + \zeta_3)q_2}{q_1 + q_2}. \tag{10}$$

It can be shown [3] that the value ζ_{123} calculated by (10) numerically coincides with the quantitative measure of the reliability of Y-shaped network (see (3)) proposed in this article. Thus, the system in Fig. 2a) can be formally replaced by one equivalent fictitious element (Fig. 2b)), the only parameter ζ_{123} of which is unambiguously expressed through the parameters of the original Y-shaped network.

The operation of equivalency can be applied not only to Y-shaped network, but also to any three-like transportation structure having an arbitrary number of elements. This possibility follows from the topological feature of such networks.

Consider again the network shown in Fig. 1. It is easy to see that this tree-like transportation network is a certain connection (composition) of Y-shaped fragments. In this sense, such fragment of the network can be considered as a structure-forming fragment. Taking into account that each such fragment can be virtually replaced by one equivalent fictitious element, the following procedure is proposed for quantifying the reliability (in sense considered here) of the network as a whole.

Determination of the operational reliability index is reduced to a recurrent step-by-step procedure for equivalent Y-shaped network fragments, at each stage of which the results of calculations at the previous step are used as input data. Each such step, starting from the inputs, leads to a new (virtual) network, in respect of which the procedure is repeated. The equivalence process ends when the original network is represented by only one fictitious Y-shaped fragment, the reliability index of which is determined in an elementary way (by analogy with formula (10).

It is most convenient to demonstrate the application of this algorithm with a specific numerical example. As such an example, consider the network shown in Fig. 1, enumerating its elements and designating the parameters used in further calculations (Fig. 5a)).

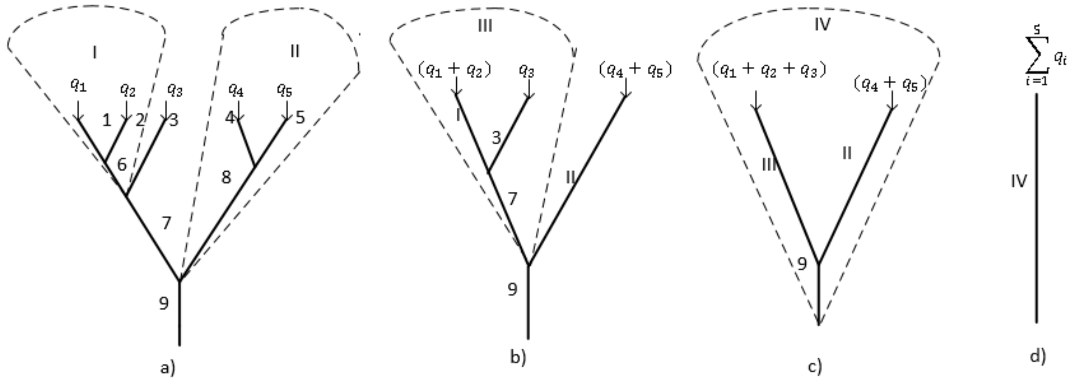

Fig. 5. Transportation network a) and its virtual transformations b), c), d).

We assume the values of the failure (λ_i) and repair (μ_i) rates for all elements of the original network proceeding only from convenience of considerations in calculations carrying out. Besides, we take into account that they mutual relations are typical for real objects [3]. These data are tabulated in Table 1.

Table 1. Input data for the numerical example.

Element number; i	1	2	3	4	5	6	7	8	9
λ_i ; proper units	0,5	0,7	0,8	0,2	0,9	0,4	0,2	0,3	0,1
μ_i; proper units	365	365	365	365	365	365	300	300	180
$\zeta_i \times 10^3$	1,37	1,92	2,19	0,54	2,47	1,09	0,67	1,0	0,56

In lower line of Table 1 the values of dimensionless parameter ζ_i characterizing the "failure-repair" process of *i*-th element of the considered network are given. In addition, we assign the product flow rates at network inputs (in proper units): $q_1 = 1{,}5$; $q_2 = 1{,}5$; $q_3 = 2{,}0$; $q_4 = 2{,}0$; $q_5 = 3{,}0$.

We proceed to the step-by-step procedure.

Step 1. From Fig. 5a) it can be seen that from the network elements adjacent to the inputs, two Y-shaped fragments can be distinguished: I, including elements 1, 2 and 6, and II, including elements 4, 5 and 8. In Fig. 5a) these fragments are conventionally enclosed in contours bounded by dashed lines. Let us carry out the equivalence of these fragments calculating, respectively, ζ_I and ζ_{II} according to the formula (10):

$$\zeta_I = \frac{(\zeta_1 + \zeta_6)q_1 + (\zeta_2 + \zeta_6)q_2}{q_1 + q_2} = 2{,}735 \cdot 10^{-3}, \tag{11}$$

$$\zeta_{II} = \frac{(\zeta_4 + \zeta_8)q_4 + (\zeta_5 + \zeta_8)q_8}{q_4 + q_5} = 2{,}698 \cdot 10^{-3} \tag{12}$$

This makes it possible to replace the original network with its virtual analogue (Fig. 5b)).

Step 2. On the network thus obtained, we carry out again the equivalence operation (contour III), and calculate ζ_{III} using the results (11) and (12):

$$\zeta_{III} = \frac{(\zeta_I + \zeta_7)(q_1 + q_2) + (\zeta_3 + \zeta_7)q_3}{q_1 + q_2 + q_3} = 3{,}187 \cdot 10^{-3}. \tag{13}$$

We turn to the network shown in Fig. 5c).

Step 3. Similarly, we equivalent the system of elements III, II and 9 (contour IV). We calculate ζ_{IV}:

$$\zeta_{IV} = \frac{(\zeta_{III} + \zeta_9)(q_1 + q_2 + q_3) + (\zeta_{II} + \zeta_9)(q_4 + q_5)}{q_1 + q_2 + q_3 + q_4 + q_5} = 3{,}502 \cdot 10^{-3}. \tag{14}$$

Step 4. We pass to one virtual element (Fig. 5d)) which replaces the entire original network.

However, as explained above, the value ζ_{IV} numerically coincides with quantitative measure of reliability (operational reliability index) γ for the original network. Physically, for this example, the result obtained indicates that, on average, about 0,35 % of the product volume entering the network inputs is not delivered to the output due to the unreliability of its elements.

Thus, the proposed method for determining the index of the operational reliability of the transportation network is solved in just four steps of the procedure of sequential virtual transformations (equivalence) of the object, at each of which simple intermediate calculations are performed. Let us note, by the way, that the solution of this problem by drawing up the Kolmogorov equations would lead to the need to determine the stationary probabilities of network states from a system of 512 interconnected algebraic equations.

In conclusion we note that the operational index γ can be useful in solving many practical problems, such as, for example, assessing a technical condition of the network at the current time, drawing up plans for medium- and long-term actions for the sequence of repairs and renovations of the network, developing alternate versions for its expansion and development (if necessary), and others. At the same time, considering the informational richness of index γ, at least some of this kind of tasks can be posed and solved as optimization ones [9].

Finally, one more remark. The beginning of the article, as a limitation of the method under discussion, the assumption about the time invariance of failure and repair flows of all transportation network elements is indicated. Meanwhile, there are cases when for some elements this assumption clearly contradicts reality, and, thus, makes the application of the developed method incorrect. This difficulty can be circumvented by approximate "stationarization" non-stationary flows of events. The methodology of such stationarization has be developed and published both for special cases: seasonally changing [10-13] or linearly increasing in time failure rate [14], and for the case when network element "ages" according to an arbitrary but well-known law [15-16].

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A New Ranking in Hexagonal Fuzzy number by Centroid of Centroids and Application in Fuzzy Critical Path

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Abstract

This paper intends to introduce a different ranking approach for obtaining the critical path of the fuzzy project network. In the network, each activity time duration is viewed by the fuzzy hexagonal number. This study proposes an advanced ranking approach by applying the centroid of the Hexagonal fuzzy number. The Hexagon is separated into two right angles and one polygon. By applying the right angle and polygon centroid formula, we can calculate the centroid of each plane and calculate the centroid of the centroid. It also focuses on the arithmetic operations in Hexagonal fuzzy numbers. The developed strategy has been described by a numerical illustration and is correlated with a few of the existing ranking approaches.

Keywords: Fuzzy critical path, fuzzy triangular number, ranking function, centroid, the centroid of centroid, hexagonal fuzzy number.

I. Introduction

Construction is an essential planning tool and organizes the implementation of a specific project. The network diagram plays a critical aspect in the completion period of the formative project. The Critical Path Method (CPM) is a successful approach to scheduling and controlling large management and construction projects. The Critical Path Method was developed at the beginning of the 1960s; with the support of the critical path, the decision-maker will follow an acceptable technique of maximizing the project period and the possible tools to achieve the project's earliest completion and quality.

The fuzzy set theory can always play significant role in dealing with the complexity of the activity's durations in a project network in this type of problem.

In 1965, Zadeh [8] recommended the fuzzy set concept to represent undefined terms. Jain [10, 11] recommended a ranking approach applying the notion of maximizing the fuzzy number of the order set in 1976.

Yager [12, 13] also suggested some ranking functions, where the hypothesis of normality or convexity is not assumed. In1981, FPERT [20] was suggested by Stefan Chanas and Kamburowski for project completion time estimation. They did not suggest FPERT as an alternative approach to probabilistic methods because there is no statistical correlation between them. Therefore, there was a need for developing the concept of ranking fuzzy numbers. Subsequently, Lee and Li [5] suggested fuzzy ranking depending on two distinctive factors: mean and distribution of fuzzy numbers in 1981. Cheng suggested a later coefficient of variance (CV index) in 1988 to enhance Lee and Li's concept [3]. In 2006, Abbasbandy et. al [17] introduced a ranking approach based on sign distance. Asady et. al [2] suggested a new ranking fuzzy number strategy by minimization of distance. There are some limitations to Asady's strategy. So, Abbasbandy and Hajjari [18] suggested the magnitude of fuzzy numbers in 2009 to enhance Asady's strategy. In 2013, Rajarajeswari and Sahaya Sudha [14, 15] suggested a new ranking function in linear fuzzy hexagonal numbers and applied it to the fuzzy linear programming problem. In 2015, Thamaraiselvi and Santhi [21] resolved the fuzzy transportation problem by applying the magnitude of Hexagonal fuzzy numbers. In 2016, Sudha and Revathi [1] proposed a new ranking on Hexagonal fuzzy numbers and utilized it to a fuzzy linear programming problem. Selvakumari and Sowmiya [19] proposed a methodology in 2017 for locating fuzzy critical paths utilizing Pascal's triangle graded mean integration when the period of every activity is expressed as a Linear Hexagonal fuzzy number. In 2017, Elumalai et al; introduced [6] to solve the fuzzy transportation problem by applying the Robust ranking approach. Rajendran et.al; proposed a new ranking in generalized hexagonal fuzzy numbers in 2018, and results compared with the magnitude of a fuzzy hexagonal number. In 2020, Leela-Apiradee et. al; introduced [9] Hexagonal fuzzy number cardinality, which is used to understand a technique for categorizing Hexagonal fuzzy numbers, and proposed a ranking approach Hexagonal fuzzy numbers, especially on their possible mean values. In 2020, Avishek et.al.[4] introduced a new ranking and defuzzification idea to transform a fuzzy hexagonal number into a crisp number determining its significance for solving decision-making problems. In 2020, Thirupathi et al. [22] Introduced a new raking approach depending on the fuzzy hexagonal number utilizing the centroid formula of triangle and rectangle and considering the distance from the origin to centroid centroided. It is considered as a ranking in Hexagonal fuzzy numbers.

2. Basic Definitions

2.1 Fuzzy Set [8]

"A fuzzy set $\tilde{A}$ is defined on the universal set of real numbers; $\tilde{R}$ is a fuzzy number with the following its membership function.

(i) $\mu_{\tilde{A}}(x)$:$\tilde{R}\rightarrow$ [0,1] is continuous
(ii) $\mu_{\tilde{A}}(x) = 0$ for all $x \in [-\infty, p] \cup [s, \infty]$
(iii) $\mu_{\tilde{A}}(x)$ is strictly increasing on $[p, q]$ and strictly decreasing on $[r, s]$
(iv) $\mu_{\tilde{A}}(x)$=1 for all $x \in [q, r]$ where $p \le q \le r \le s$."

2.2 Fuzzy Number [8]

"A Fuzzy set $\tilde{A}$ of the real line $\tilde{R}$ with membership function $\mu_{\tilde{A}}(x)$: $\tilde{R}\rightarrow$ [0,1] is called the fuzzy number, if

(i) $\tilde{A}$ must be a normal and convex fuzzy set.
(ii) The support of $\tilde{A}$ is finite."

2.3 Generalized Fuzzy Number [13]

"A fuzzy set $\tilde{A}$ is known as a generalized fuzzy number on a universal set of real numbers if its membership function has the following conditions:

(i) $\mu_{\tilde{A}}(x)$: $\tilde{R} \rightarrow [0,1]$ is continuous

(ii) $\mu_{\tilde{A}}(x) = 0$ for all $x \in [-\infty, p] \cup [s, \infty]$

(iii) $\mu_{\tilde{A}}(x)$is strictly increasing on $[p, q]$ and strictly decreasing on $[r, s]$

(iv) $\mu_{\tilde{A}}(x) = \omega$ for all $x \in [q, r]$ where $0<\omega\leq 1$."

2.4 Trapezoidal Fuzzy Number [14]

"A fuzzy number $\tilde{A}$ is a Trapezoidal fuzzy number denoted by (p, q, r, s), and its membership function is given below. Where $p \leq q \leq r \leq s$.

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{x-p}{p-q}, for\ p \leq x \leq q \\ 1, \quad for\ q \leq x \leq r \\ \frac{s-x}{s-r}, \ for\ r \leq x \leq s \\ 0, \quad otherwiise" \end{cases} \tag{1}$$

2.5 Generalized Trapezoidal Fuzzy Number [14]

"Generalized Fuzzy number $\tilde{A} = (p, q, r, s, \omega)$is said to be a generalized trapezoidal fuzzy number if its membership function is given by

$$\mu_{\tilde{A}}(x) = \begin{cases} \omega\left(\frac{x-p}{p-q}\right), for\ p \leq x \leq q \\ \omega, \quad for\ q \leq x \leq r \\ \omega\left(\frac{s-x}{s-r}\right), \ for\ r \leq x \leq s \\ 0, \quad otherwise" \end{cases} \tag{2}$$

Generalized Trapezoidal fuzzy number diagram represented in Figure 1.

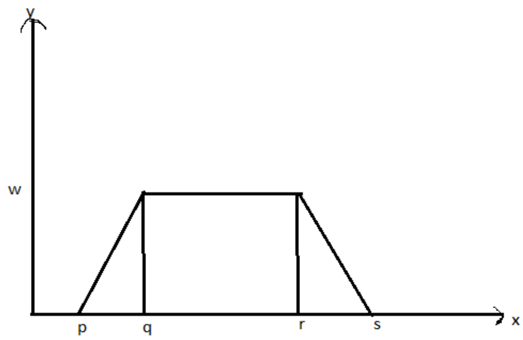

Figure 1: Graphical representation of GTFN

2.6 Hexagonal Fuzzy Number [14]

"A Fuzzy number $\tilde{A}_H$ is a Hexagonal fuzzy number represented by $\tilde{A}_H = (p, q, r, s, t, u)$where a, b, c, d, e, f are real numbers and its membership function $\mu_{\tilde{A}}(x)$is given below

$$\mu_{\tilde{A}_H}(x) = \begin{cases} \frac{1}{2}\left(\frac{x-p}{q-p}\right), \ for\ p \leq x \leq q \\ \frac{1}{2} + \frac{1}{2}\left(\frac{x-q}{r-q}\right), \ for\ q \leq x \leq r \\ 1, \quad for\ r \leq x \leq s \\ 1 - \frac{1}{2}\left(\frac{x-s}{t-s}\right), \ for\ s \leq x \leq t \\ \frac{1}{2}\left(\frac{u-x}{u-t}\right), \quad for\ t \leq x \leq u \\ 0, \quad otherwise" \end{cases}$$

2.7 Generalized Hexagonal Fuzzy Number [16]

"A generalized Hexagonal Fuzzy number represented by $\tilde{A}_H = (p, q, r, s, t, u, \omega)$ where a, b, c, d, e, f are real numbers and its membership function $\mu_{\tilde{A}_H}(x)$ is given by

$$\mu_{\tilde{A}_H}(x) = \begin{cases} \frac{1}{2}\left(\frac{x-p}{q-p}\right), & for\ p \le x \le q \\ \frac{1}{2} + \frac{\omega}{2}\left(\frac{x-q}{r-q}\right), & for\ q \le x \le r \\ \omega, & for\ r \le x \le s \\ 1 - \frac{\omega}{2}\left(\frac{x-s}{t-s}\right), & for\ s \le x \le t \\ \frac{1}{2}\left(\frac{u-x}{u-t}\right), & for\ t \le x \le u \\ 0, & otherwise" \end{cases}$$

Generalized Hexagonal Fuzzy Number diagram represented in Figure 2.

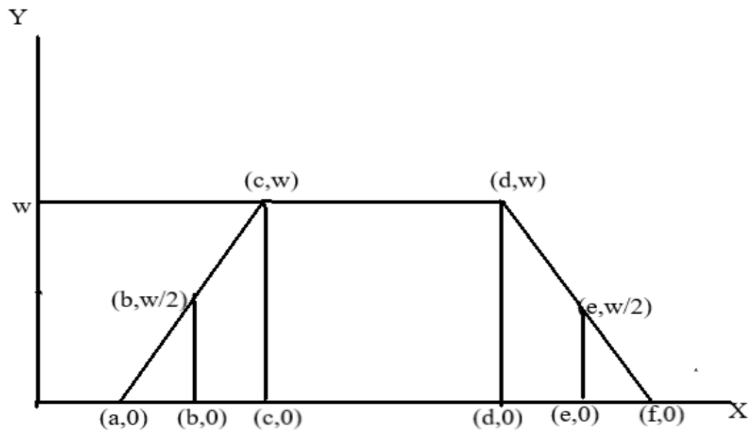

Figure2: *Generalized Hexagonal Fuzzy Number*

2.8 Ordering of Hexagonal Fuzzy Number [16]

"Let $\tilde{A}_H = (p_1, p_2, p_3, p_4, p_5, p_6)$ and $\tilde{B}_H = (q_1, q_2, q_3, q_4, q_5, q_6)$ be in fuzzy real number be the set of real Hexagonal fuzzy numbers

(i) $\tilde{A}_H \simeq \tilde{B}_H$ iff $p_i = q_i,\ i = 1,2,3,4,5,6$
(ii) $\tilde{A}_H \le \tilde{B}_H$ iff $p_i \le q_i,\ i = 1,2,3,4,5,6$
(iii) $\tilde{A}_H \ge \tilde{B}_H$ iff $p_i \ge q_i, i = 1,2,3,4,5,6$"

2.9 Ranking of Hexagonal Fuzzy Number [16]

An effective approach for comparing fuzzy numbers is to use a ranking function $\mathcal{R}: F(R) \longrightarrow R$, where $F(R)$ is a collection of fuzzy numbers that maps each fuzzy number into a real number, where a natural number order exists. For any two Hexagonal fuzzy numbers $\tilde{P}_H = (p_1, p_2, p_3, p_4, p_5, p_6)$ and $\tilde{Q}_H = (q_1, q_2, q_3, q_4, q_5, q_6)$ have the following comparison.

(i) $\tilde{P}_H = \tilde{Q}_H \Leftrightarrow R(\tilde{P}_H) = R(\tilde{Q}_H)$
(ii) $\tilde{P}_H \le \tilde{Q}_H \Leftrightarrow R(\tilde{P}_H) \le R(\tilde{Q}_H)$
(iii) $\tilde{P}_H \ge \mathrm{Q} \Leftrightarrow R(\tilde{P}_H) \ge R(\tilde{Q}_H)$

3. Proposal of a new ranking in Linear Hexagonal Fuzzy Number

We suggest a successful method for calculating the rank of Hexagonal fuzzy numbers. The Proposal ranking in the Hexagonal fuzzy number diagram is represented in Figure 3.

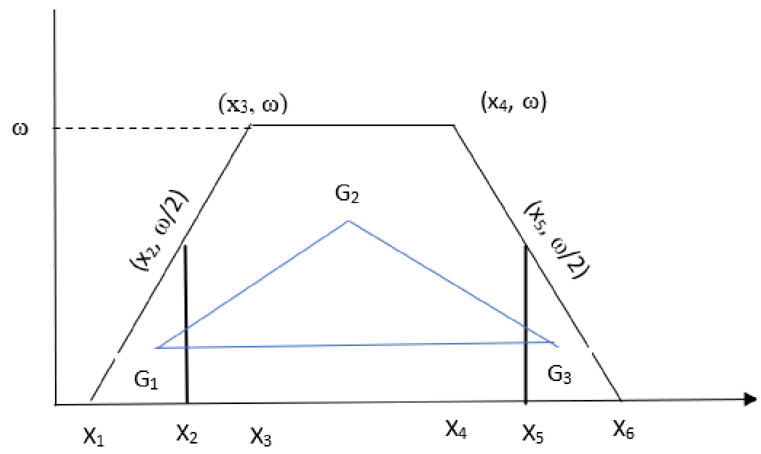

Figure3: *Proposed Ranking Method*

In Figure 3, the hexagonal is split into two right angles and one polygon. By applying the centroid formula of right angle and polygon, calculate the centroid of triangles and polygon, respectively. The circumcentre of the centroids of the fuzzy hexagonal number is taken into a balancing point of Hexagon in Figure 3. The circumcentre of the centroids of this three-plane figure is taken as the ranking of generalized Hexagonal fuzzy numbers. Let G_1, G_2, and G_3 be the centroid of the three plane figures.

G_1 specify the centroid of the right angle with vertices $(x_1,0)$, $\left(x_2,\frac{\omega}{2}\right)$, $(x_2,0)$.
G_2 specify the centroid of the triangle with vertices
$(x_2,0), \left(x_2,\frac{\omega}{2}\right), (x_3,\omega), (x_4,\omega), \left(x_5,\frac{\omega}{2}\right), (x_5,0)$
G_3 specify the centroid of the right angle with vertices $(x_5,0), \left(x_5,\frac{\omega}{2}\right), (x_6,0)$
The centroid of these three planes is;

$$G_1=\left(\frac{x_1+2x_2}{3},\frac{\omega}{6}\right), G_2=\left(\frac{2x_2+x_3+x_4+2x_5}{6},\frac{\omega}{2}\right), G_3=\left(\frac{2x_5+x_6}{3},\frac{\omega}{6}\right) \text{ respectively.}$$

The circumcentre of G1, G_2, and G_3 is

$$G_{\tilde{A}_H}=\left(\frac{2x_1+6x_2+x_3+x_4+6x_5+2x_6}{18},\frac{5\omega}{18}\right)$$

Therefore, the generalized Hexagonal fuzzy number $\tilde{A}_H=(x_1,x_2,x_3,x_4,x_5,x_6,\omega)$ new ranking function is:

$$R(\tilde{A}_H)=(\bar{x}_0\bar{y}_0)=\left(\frac{2x_1+6x_2+x_3+x_4+6x_5+2x_6}{18}\right)*\frac{5\omega}{18}$$

Here, we utilize 4 sets of Hexagonal fuzzy numbers. These are opted from ref [16] to analyze the suggested method with convinced current ranking methods. The sets and the outcome obtained by the suggested method are given in Table1.

Table1: *Ranking order obtained results by the suggested method*

Sets	Hexagonal Fuzzy Numbers	$\mathcal{R}(\tilde{F}_H)$	Conclusion
Example 1			
Set-1	$\tilde{A}$(1,2,3,4,5,6;0.8)	0.7777	$\tilde{D} < \tilde{C} < \tilde{B} < \tilde{A}$
	$\tilde{B}$(-1,0,2,4,5,6;0.8)	0.5679	
	$\tilde{C}$ (-2, -1,0,2,4,6;0.8)	0.3456	
	$\tilde{D}$(-2, -1,0,1,2,3;0.8)	0.1111	
Example2			
Set-2	$\tilde{A}$ (1,0,0.2,0.3,0.4,0.5;0.6)	0.017	$\tilde{C}$<$\tilde{D}$<$\tilde{B}$<$\tilde{A}$
	$\tilde{B}$(-1, -0.5,0,0.4,0.5,1;0.6)	0.003	
	$\tilde{C}$ (-1, -0.6,0.3,0.2,0.5,1;0.6)	-0.006	
	$\tilde{D}$(-1, -0.2, -0.1,0.2,0.3,1;0.6)	0.006	
Example 3			
Set-3	$\tilde{A}$(0.1,0.2,0.4,0.6,0.7,0.9;1)	0.12963	$\tilde{B}$ >$\tilde{A}$
	$\tilde{B}$(0.2,0.4,0.6,0.7,0.8,0.9;1)	0.16512	
Example 4			
Set-4	$\tilde{A}$(0.2,0.3,0.5,0.6,0.7,0.9;0.7)	0.1	Â >$\hat{B}$
	$\tilde{B}$(0.1,0.2,0.4,0.5,0.6,0.9;0.7)	0.08	

The Proposal ranking method is compared with some existing methods represented in Table 2.

Table2: *The comparison of different ranking methods*

Ranking method	Set1	Set2	Set3	Set4
Avishek method [4]	$\tilde{D} < \tilde{C} < \tilde{B} < \tilde{A}$	$\tilde{C}$<$\tilde{D}$<$\tilde{B}$<$\tilde{A}$	$\tilde{B}$>$\tilde{A}$	$\tilde{A}$> $\tilde{B}$
Nagoor method [7]	$\tilde{D} < \tilde{C} < \tilde{B} < \tilde{A}$	$\tilde{C}$<$\tilde{D}$<$\tilde{B}$<$\tilde{A}$	$\tilde{B}$>$\tilde{A}$	$\tilde{A}$>$\tilde{B}$
Rajendran method [16]	$\tilde{D} < \tilde{C} < \tilde{B} < \tilde{A}$	$\tilde{C}$<$\tilde{D}$<$\tilde{B}$<$\tilde{A}$	$\tilde{B}$>$\tilde{A}$	$\tilde{A}$>$\tilde{B}$
Proposal method	$\tilde{D} < \tilde{C} < \tilde{B} < \tilde{A}$	Ĉ <$\hat{D}$<$\hat{B}$ < Â	$\tilde{B}$>$\tilde{A}$	$\tilde{A}$>$\tilde{B}$

4 Fuzzy Critical by a new ranking in Hexagonal fuzzy number

4.1 Analytical Example

This section, come out with numerical application of the proposal fuzzy set CPM-based methodology on an activity network. We consider a network with a set of fuzzy events $\tilde{A}$= {1, 2, 3, 4, 5, 6, 7} and the fuzzy activity time represented as a Hexagonal fuzzy number for each activity in table3. (All durations in days) Furthermore, the project network diagram is represented in Figure 4.

Table 3: *Fuzzy project network information*

Activity	Hexagonal fuzzy numbers
1-2	(3,7,11,15,19,24)
1-3	(3,5,7,9,10,12)
2-4	(11,14,17,21,25,30)
3-4	(3,5,7,9,10,12)
2-5	(5,7,10,13,17,21)
3-6	(7,9,11,14,18,22)
4-7	(7,9,11,14,18,22)
5-7	(2,3,4,6,7,9
6-7	(5,7,8,11,14,17)

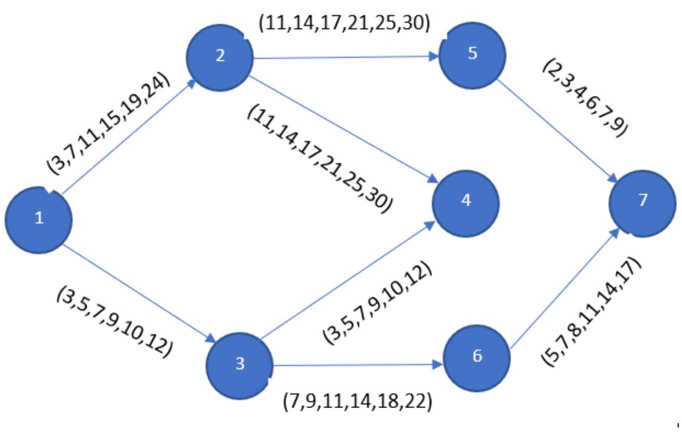

Figure 4: *Fuzzy project network*

Hexagonal fuzzy number transformed into an activity duration by proposal method. This activity duration taken as the time between the nodes and fuzzy critical path is calculating by applying the traditional method. The expected time between activities represented in Table4 and the related diagram is represented in Figure5.

Table 4: Expected time between activities

Activity	Hexagonal Fuzzy Numbers	Expected time
1-2	(3,7,11,15,19,24)	3.64
1-3	(3,5,7,9,10,12)	2.09
2-4	(11,14,17,21,25,30)	5.46
3-4	(3,5,7,9,10,12)	2.09
2-5	(5,7,10,13,17,21)	3.37
3-6	(7,9,11,14,18,22)	3.78
4-7	(7,9,11,14,18,22)	3.78
5-7	(2,3,4,6,7,9	1.41
6-7	(5,7,8,11,14,17)	2.91

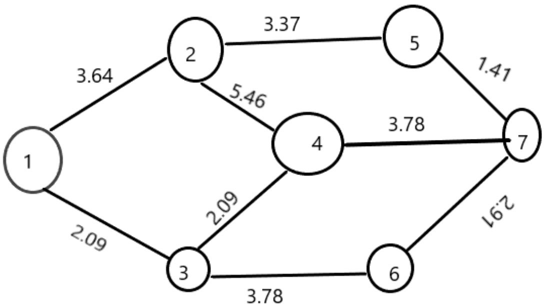

Figure 6.5. *Expected time between activities*

The possible paths of the project network's total duration time are represented in Table5.

Table 5. *Possible paths of project network total duration time*

Path	Expected Time
1-2-5-7	8.42
1-2-4-7	12.88
1-3-4-7	7.96
1-3-6-7	8.78

In table 5, the maximum value is 12.88.
Therefore, the project completion duration is 12.88, and the critical path is 1-2-4-7.

5 Comparison with Existing methods

Existing Method1[14]

In 2013, Rajarajeswari and Sahaya Sudha suggested a revised ranking in Hexagonal fuzzy numbers. Their new ranking function in Hexagonal fuzzy number is;

$$\mathcal{R}(\tilde{A}_H) = \frac{2f_1 + 3f_2 + 4f_3 + 4f_4 + 3f_5 + 2f_6}{18} \times \frac{5}{18}$$

Existing Method2 [21]

In 2015, Thamaraiselvi et. al. suggested the Magnitude of Hexagonal fuzzy numbers. The magnitude of Hexagonal fuzzy number is;

$$Mag(\tilde{A}_H) = \frac{2f_1 + 3f_2 + 4f_3 + 4f_4 + 3f_5 + 2f_6}{18}$$

Existing Method3 [1]

In 2016, Sahaya Sudha and Revathi introduced an improved ranking in generalized Hexagonal fuzzy numbers. Their new ranking function is;

$$\mathcal{R}(\tilde{A}_H) = \frac{2f_1 + 4f_2 + 9f_3 + 9f_4 + 4f_5 + 2f_6}{6} \times \frac{11\omega}{6}$$

Existing Method4 [16]

In 2017, Rajendran et.al., suggested a revised ranking in generalized Hexagonal fuzzy numbers. In their method, the new ranking function is;

$$\mathcal{R}(\tilde{A}_H) = \frac{2f_1 + 3f_2 + 4f_3 + 4f_4 + 3f_5 + 2f_6}{18} \times \frac{5\omega}{18}$$

Existing Method5 [22]

In 2020, Thirupathi et.al., suggested a revised ranking function in generalized Hexagonal fuzzy numbers. The revised ranking function is;

$$\mathcal{R}(\tilde{A}_H) = \frac{2f_1 + 4f_2 + 3f_3 + 3f_4 + 4f_5 + 2f_6}{18} \times \frac{13\omega}{36}$$

Existing Method6 [4]

In 2020, Avishek et.al., suggested a ranking in generalized Hexagonal fuzzy numbers. Their developed ranking function in Hexagonal fuzzy number is;

$$\mathcal{R}(\tilde{A}_H) = \frac{4f_1 + 10f_2 + 16f_3 + 16f_4 + 10f_5 + 4f_6}{12} \times \frac{5\omega}{6}$$

Table 6 represents project completion time estimates using the developed model and existing methodologies. Figure 5 presented the fuzzy critical path and project completion time of some recent methods as well as the proposal method.

Table 6: *Fuzzy Critical path compare with existing methods*

Method	Critical path	Project completion time
Rajendran et.al method	1-2-4-7	12.01
Magnitude of HFN	1-2-4-7	45.89
Rajarajeswari and Sahaya Sudha	1-2-4-7	12.73
Thirupathi et.al	1-2-4-7	16.62
Avishek et.al	1-2-4-7	128.74
Proposal method	1-2-4-7	12.88

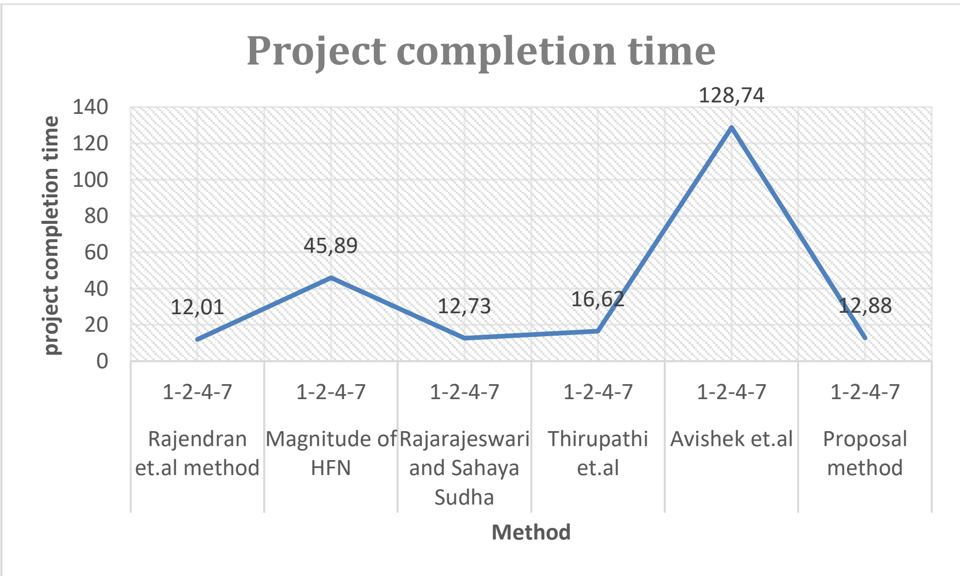

Figure 5: *Fuzzy critical path compared with existing methods graph*

4. Conclusion

In this paper, activity durations in the network represented by Hexagonal fuzzy numbers and suggested an advanced ranking function by Hexagonal fuzzy numbers with the centroid of centroid method. The new ranking function has been applied to calculate the critical path for the fuzzy project network. Numerous experiments have been conducted, and the results are correlated with some of the available methods. The attained results are similar to Rajendran, Rajarajeswari, and Tirupati methods, and completion time is less when compared to the magnitude of hexagonal fuzzy numbers and Avishek et al. method despite having the same critical path in all methods. The Avishek method gives the highest value in the comparison results and is not correlated with any existing method. So, the proposed method is better than the Avishek method. Moreover, the investigators can use the present concept on Hexagonal fuzzy numbers in numerous domains such as Engineering problems, Transportation problems, Neural networks, Cloud computing, image processing, mobile computing, etc. Further attention proceeded by constructing a new ranking function by various kinds of fuzzy numbers with project networks.

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Type II Power Topp-Leone Daggum Distribution With Application In Reliability

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Abstract

In this paper, we introduce a new continuous probability distribution named as type II power Topp-Leone Dagum distribution using the type II power Topp-Leone generated family studied by Rashad et al., [17]. We have obtained some reliability measures like reliability function, hazard rate function, reversed hazard rate function, mean waiting time, mean past life time, mean deviation, second failure rate function and mean residual life function. We have derived some statistical properties of the new probability distribution including mean, variance, moments, moment generating function, characteristics function, cumulant generating function, incomplete moments, inverted moments, central moments, conditional moments, probability weighted moments and order statistics. For the probability proposed new probability distribution. we have obtained some income inequality measures like Lorenz curve, Bonferroni index, Zenga index and Generalized entropy. The maximum likelihood estimation method is used to estimate the parameters of the probability distribution. Finally, the proposed generalized model is applied to life time data sets to evaluate the model performance.

Keywords: Dagum distribution, Reliability function, Hazard rate function, Generalized entropy, Lorenz curve, Maximum likelihood method.

I. Introduction

The life time distributions play a vital role in several research areas such as biological sciences, medical sciences, environmental sciences, actuarial science, engineering, finance and among others. The popular classical probability distributions do not provide greater flexibility for life time data set, because the classical distribution have one or two parameters only. In this situation, generalized family distribution commonly played a vital role many statistical research areas. The main advantage of generalized family is obtained by adding one more parameters through the classical probability distribution which gives more flexibility for generating a new probability distribution. In this current scenario generating family of probability distributions is attractive to many statisticians. The generating family of distributions have been investigated by many authors. Here, we list some generating family like Marshall-Olkin-G (MO-G) family introduced by Marshall and Olkin [16], Exponentiated-G (E-G) family introduced by Gupta et al., [13], Quadratic rank transmuted-G (QRTM-G) family introduced by Shaw and Buckley [18], gamma-G (G-G) family introduced by Zografos and Balakrishnan [21], Kumaraswamy-G (Kw-G) family introduced by Cordeiro and de Castro [6], Topp-Leone-G (TL-G) family introduced by Ali Al-Shomrani [3], Exponentiated extended-G (EE-G) family introduced by Elgarhy et al., [12] and odd Dagum-G (OD-G) family introduced by Afify and Alizadeh [1].

Camilo Dagum introduced a Dagum distribution in 1977 for closely fitting empirical income and wealth data. The Dagum distribution is classified into two types named type I specification (type I Dagum) and type II specification (type II Dagum), where type I specification deals with three parameters while type two specification deals with four parameters. This Dagum distribution has been extensively used in different areas like income and wealth data, meteorological data, reliability and survival analysis. The Dagum distribution is alternative to heavy tailed distributions such as generalized beta, Pareto and lognormal. The Dagum distribution is also known as the inverse Burr XII distribution, especially in the actuarial literature. Domma [9] studied characteristic of Dagum distribution that its hazard function can be monotonically decreasing, an upside-down bathtub, or bathtub. This behavior attracted many of authors to study the model in various fields. In fact Domma, et al., [10, 11] studied Dagum distribution with a reliability point of view and used to analyze survival data. Kleiber and Kotz [14] and Kleiber [15] provided an exhaustive review on the origin of the Dagum distribution and its applications. Recently, Domma et al.,[10] studied about Dagum distribution for estimated parameters with censored samples. We have focused the type I Dagum distribution in this research paper.

The probability density function (pdf) and cumulative distribution function (cdf) of Dagum distribution are given respectively by

$$f(x;\sigma,\theta,\beta) = \sigma\theta\beta\, x^{-\theta-1}\left(1+\sigma x^{-\theta}\right)^{-\beta-1} with\ x>0, \sigma>0, \theta>0\ and\ \beta>0. \tag{1}$$

and

$$F(x;\sigma,\theta,\beta) = \left(1+\sigma x^{-\theta}\right)^{-\beta} with\ x>0, \sigma>0, \theta>0\ and\ \beta>0. \tag{2}$$

where σ is scale parameter, θ and β are shape parameters. It is noted that if σ=1 the Dagum distribution becomes Burr III distribution and if θ=1, the Dagum distribution becomes Log-Logistic or Fisk distribution.

In this paper, we introduce a new generalization of type II power Topp-Leone Dagum distribution using the type II power Topp-Leone generated family studied by Rashad et al., [17]. This generated family introduces two new additional parameters and provides flexibility.

The contents of this paper are organized as follows: In Section 2, adopt type II power Topp-Leone family proposed new generating probability distribution. In Section 3, we discuss some reliability measures like reliability function, hazard rate function, reversed hazard rate function, cumulative hazard function, second failure rate function, mean waiting time, mean residual life function, mean past life time and average deviation. We have derived some statistical properties of new probability distribution such as moments, moment generating function, characteristic function, cumulant generating function, inverted r^{th} moments, central moments, conditional moments, probability weighted moments, order statistics are given in Section 3. In Section 4, some income inequality measures like Lorenz and Bonferroni curve, Zenga index and Generalized entropy are presented. In Section 5 estimation of the parameters of the type II power Topp-Leone Dagum distribution is consider maximum likelihood estimation method. The real life time data set is used for fitting type II power Topp-Leone Dagum distribution. The results are given in Section 6. Finally, we conclude the article in Section 7.

II. Type II Power Topp-Leone Family

The type II power Topp-Leone family is introduced by Rashad et al., [17]. The probability density function (pdf) and cumulative distribution function (cdf) of type II power Topp-Leone family of distribution are respectively defined by

$$f(x;\alpha,\tau,\xi) = 2\alpha\tau g(x;\xi)\left[1-G(x;\xi)\right]^{\alpha\tau-1}\left[2-\left[1-G(x;\xi)\right]^{\tau}\right]^{\alpha-1}\left[1-\left[1-G(x;\xi)\right]^{\tau}\right], x\epsilon R \tag{3}$$

and

$$F(x;\alpha,\tau,\xi) = 1-\left[1-G(x;\xi)\right]^{\alpha\tau}\left[2-\left[1-G(x;\xi)\right]^{\tau}\right]^{\alpha}, x\epsilon R \tag{4}$$

where α>0, $\tau > 0$, $g(x;\xi)$ and $G(x;\xi)$ are probability density function and cumulative distribution function of any baseline distribution with parameter vector ξ . The type II power Topp-Leone family of distributions is the generalization of the type II Topp-Leone-G family. It is very important note that for TIIPTL-G, if $\tau = 0$ the type II power Topp-Leone family becomes a type II Topp-Leone family of distribution. Some of motivations behind the type II power Topp-Leone family of distribution are to create different types of shapes for probability density function and hazard rate function to increase the flexibility for generating of type II power Topp-Leone distributions, skewed distribution transformed from the symmetrical distribution, build heavy tailed distribution and type II power Topp-Leone family provide better fits compare than other general families of distribution with baseline distribution.

I. Type II Power Topp-Leone Dagum Distribution

A random variable X is said to have type II power Topp-Leone Dagum distribution if the probability density function and cumulative distribution function are respectively is given by

$$\begin{aligned} f(x;\alpha,\tau,\sigma,\theta,\beta) =& 2\alpha\tau\sigma\theta\beta\, x^{-\theta-1}\left(1+\sigma x^{-\theta}\right)^{-\beta-1}\left[1-(1+\sigma x^{-\theta})^{-\beta}\right]^{\alpha\tau-1} \\ &\times\left[2-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\tau}\right]^{\alpha-1}\left[1-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\tau}\right], x\epsilon R \end{aligned} \tag{5}$$

where, $\sigma > 0, \theta > 0, \beta > 0, \alpha > 0$ *and* $\tau > 0$.

Note that, $(x-y)^r = \sum_{p=0}^{\infty}\binom{r}{p}(-1)^p x^{r-p} y^p$.

This binomial expansion is used to simply the probability density function of type II power Topp-Leone Dagum distribution. After some simplifications we get pdf for type II power Topp-Leone Dagum distribution and is given by

$$f(x;\alpha,\tau,\sigma,\theta,\beta) = 2\alpha\tau\sigma\theta\beta\, x^{-\theta-1}\left(1+\sigma x^{-\theta}\right)^{-\beta(p+s-v+1)-1} \tag{6}$$

where, $\psi = \binom{\alpha\tau-1}{p}\binom{\alpha-1}{q}\binom{\tau q}{s}\binom{1}{t}\binom{\tau t}{v}(-1)^{p+q+s+t+v}(2)^{\alpha-1-q}$

and the cdf is given by

$$F(x;\alpha,\tau,\sigma,\theta,\beta) = 1-\left[1-(1+\sigma x^{-\theta})^{-\beta}\right]^{\alpha\tau}\left[2-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\tau}\right]^{\alpha}, x\epsilon R \tag{7}$$

where α, τ are parameters of type II power Topp-Leone family, σ is scale parameter of Dagum distribution and θ, β are shape parameters of Dagum distribution. The following figures 1 to 4 shows the shape of pdf and cdf for different values of the parameters of type II power Topp-Leone Dagum distribution.

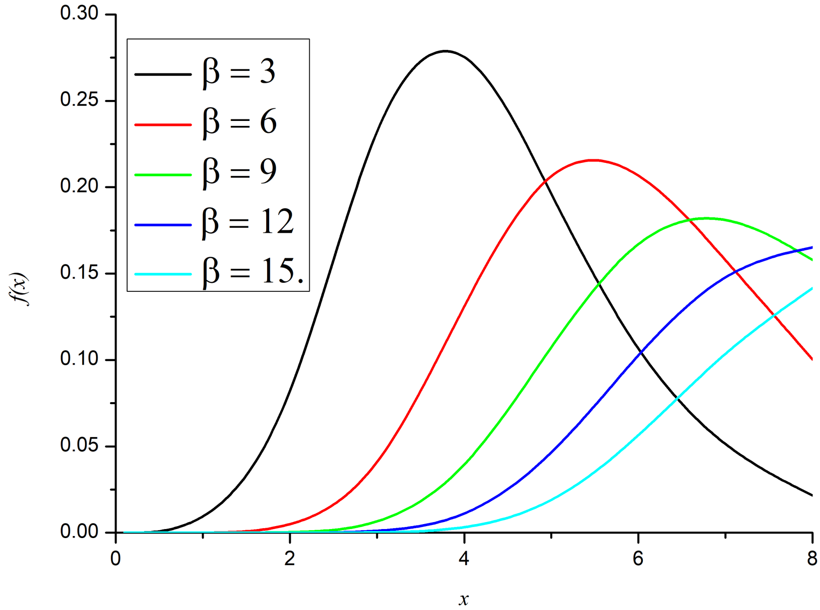

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Figure 1: *Pdfs of type II power Topp-Leone Dagum distribution for fixed value of $\alpha = 4, \tau = 1, \sigma = 6, \theta = 2$ and different values of β.*

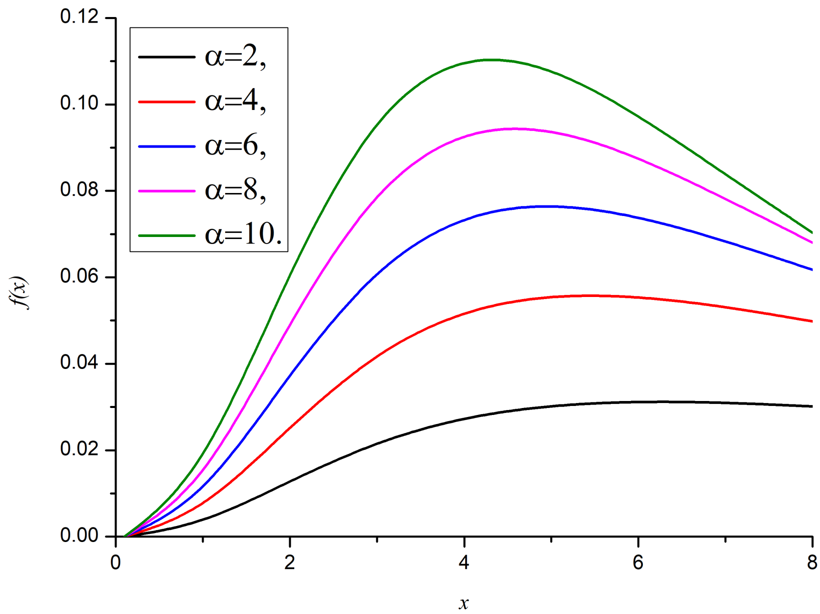

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Figure 2: *Pdfs of type II power Topp-Leone Dagum distribution for fixed value of $\tau = 0.5, \sigma = 2, \theta = 1, \beta = 4$ and different values of α.*

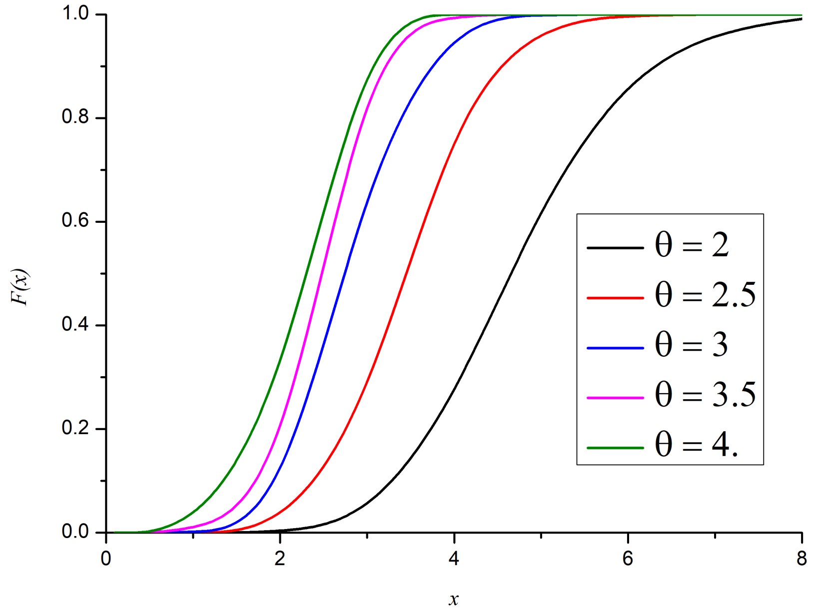

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Figure 3: *Cdfs of type II power Topp-Leone Dagum distribution for fixed value of* $\alpha = 2, \tau = 4, \sigma = 6, \beta = 7$ *and different values of* θ.

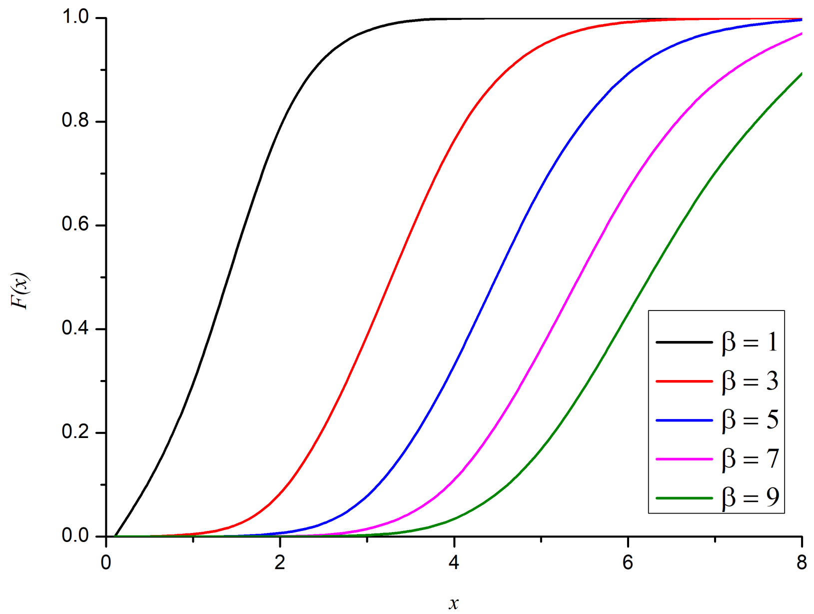

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Figure 4: *Cdfs of type II power Topp-Leone Dagum distribution for fixed value of* $\alpha = 4, \tau = 2.5, \sigma = 8, \theta = 2$ *and different values of* β.

III. Reliability Measures

I. Reliability function

The reliability function of type II power Topp-Leone Dagum distribution is given by

$$R(x) = 1 - \left[1 - \left(1 - (1 + \sigma x^{-\theta})^{-\beta}\right)^{\alpha\tau} \left(2 - (1 - (1 + \sigma x^{-\theta})^{-\beta})^{\tau}\right)^{\alpha}\right] \tag{8}$$

II. Hazard rate function

The hazard rate function associated with type II power Topp-Leone Dagum distribution is given by

$$h(x) = \frac{\delta}{\eta} \tag{9}$$

where

$$\begin{aligned}\delta =& 2\alpha\tau\sigma\theta\beta\, x^{-\theta-1}(1 + \sigma x^{-\theta})^{-\beta-1} \left[1 - (1 + \sigma x^{-\theta})^{-\beta}\right]^{\alpha\tau-1} \left[2 - (1 - (1 + \sigma x^{-\theta})^{-\beta})^{\tau}\right]^{\alpha-1} \\ &\times \left[1 - (1 - (1 + \sigma x^{-\theta})^{-\beta})^{\tau}\right]\end{aligned}$$

$$\eta = 1 - \left[1 - \left(1 - (1 + \sigma x^{-\theta})^{-\beta}\right)^{\alpha\tau} \left(2 - (1 - (1 + \sigma x^{-\theta})^{-\beta})^{\tau}\right)^{\alpha}\right]$$

III. Reversed hazard rate function

The reversed hazard rate function of type II power Topp-Leone Dagum distribution is given by

$$r(x) = \frac{\delta}{\gamma} \tag{10}$$

where

$$\begin{aligned}\delta =& 2\alpha\tau\sigma\theta\beta\, x^{-\theta-1}(1 + \sigma x^{-\theta})^{-\beta-1} \left[1 - (1 + \sigma x^{-\theta})^{-\beta}\right]^{\alpha\tau-1} \left[2 - (1 - (1 + \sigma x^{-\theta})^{-\beta})^{\tau}\right]^{\alpha-1} \\ &\times \left[1 - (1 - (1 + \sigma x^{-\theta})^{-\beta})^{\tau}\right]\end{aligned}$$

$$\gamma = 1 - \left(1 - (1 + \sigma x^{-\theta})^{-\beta}\right)^{\alpha\tau} \left(2 - (1 - (1 + \sigma x^{-\theta})^{-\beta})^{\tau}\right)^{\alpha}$$

IV. Cumulative hazard function

The cumulative hazard function of type II power Topp-Leone Dagum distribution is given by

$$H(x) = -log\left[1 - \left(1 - (1 + \sigma x^{-\theta})^{-\beta}\right)^{\alpha\tau} \left(2 - (1 - (1 + \sigma x^{-\theta})^{-\beta})^{\tau}\right)^{\alpha}\right] \tag{11}$$

V. Second failure rate function

The second failure rate function of type II power Topp-Leone Dagum distribution is given by

$$h(x) = log\left[\frac{1 - \left[1 - (1 - (1 + \sigma x^{-\theta})^{-\beta})^{\alpha\tau} \left(2 - (1 - (1 + \sigma x^{-\theta})^{-\beta})^{\tau}\right)^{\alpha}\right]}{1 - \left[1 - (1 - (1 + \sigma (x+1)^{-\theta})^{-\beta})^{\alpha\tau}\right] \left(2 - (1 - (1 + \sigma (x+1)^{-\theta})^{-\beta})^{\tau}\right)^{\alpha}}\right] \tag{12}$$

VI. Mean waiting time

The mean waiting time is defined by

$$\varphi(x) = x - \left[\frac{1}{F(x)} \int_0^x xf(x)dx\right] \tag{13}$$

$$\varphi(x) = x - \left[\frac{1}{F(x)} \int_0^x x \left[\sum_{p,q,s,t,v=0}^{\infty} \psi 2\alpha\tau\sigma\theta\beta \; x^{-\theta-1}(1+\sigma x^{-\theta})^{-\beta(p+s-v+1)-1}\right] dx - x\right]$$

The mean waiting time of type II power Topp-Leone Dagum distribution is given by

$$\varphi(x) = x - \left[\frac{\sum_{p,q,s,t,v=0}^{\infty} \psi 2\alpha\tau\beta\sigma^{\frac{1}{\theta}} B\left(1-\frac{1}{\theta}, \beta(p+s-v+1)+\frac{1}{\theta}; y\right)}{1-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\alpha\tau}\left(2-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\tau}\right)^{\alpha}}\right] \tag{14}$$

VII. Mean residual life function

The mean residual life function plays a very important role in reliability and survival analysis. The mean residual life function of a life time random variable X is given by

$$\phi(x) = \frac{1}{s(x)} \int_x^{\infty} xf(x)dx - x \tag{15}$$

$$\begin{aligned}\phi(x) = &\frac{1}{1-\left[1-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\alpha\tau}\left(2-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\tau}\right)^{\alpha}\right]}\\ &\times \int_0^{\infty} x \left[\sum_{p,q,s,t,v=0}^{\infty} \psi 2\alpha\tau\sigma\theta\beta \; x^{-\theta-1}(1+\sigma x^{-\theta})^{-\beta(p+s-v+1)-1}\right] dx - x\end{aligned}$$

The mean residual life function type II power Topp-Leone Dagum distribution is given by

$$\phi(x) = \frac{\sum_{p,q,s,t,v=0}^{\infty} \psi 2\alpha\tau\beta \; B\left(1-\frac{1}{\theta}, \beta(p+s-v+1)+\frac{1}{\theta}\right)}{1-\left[1-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\alpha\tau}\left(2-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\tau}\right)^{\alpha}\right]} - x \tag{16}$$

VIII. Mean past lifetime

The mean past lifetime of the component can be defined by

$$K(x) = E\left[x - X|X \leq x\right] = \frac{\int_0^x F(t)dt}{F(x)} = x - \frac{\int_0^x tf(t)dt}{F(x)} \tag{17}$$

$$K(x) = x - \frac{\int_0^x t\left[\sum_{p,q,s,t,v=0}^{\infty} \psi 2\alpha\tau\sigma\theta\beta \; t^{-\theta-1}(1+\sigma t^{-\theta})^{-\beta(p+s-v+1)-1}\right] dt}{\left[1-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\alpha\tau}\left(2-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\tau}\right)^{\alpha}\right]}$$

The mean past life time of type II power Topp-Leone Dagum distribution is given by

$$K(x) = x - \frac{\sum_{p,q,s,t,v=0}^{\infty} \psi 2\alpha\tau\beta \; B\left(1-\frac{1}{\theta}, \beta(p+s-v+1)+\frac{1}{\theta}; y\right)}{\left[1-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\alpha\tau}\left(2-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\tau}\right)^{\alpha}\right]} \tag{18}$$

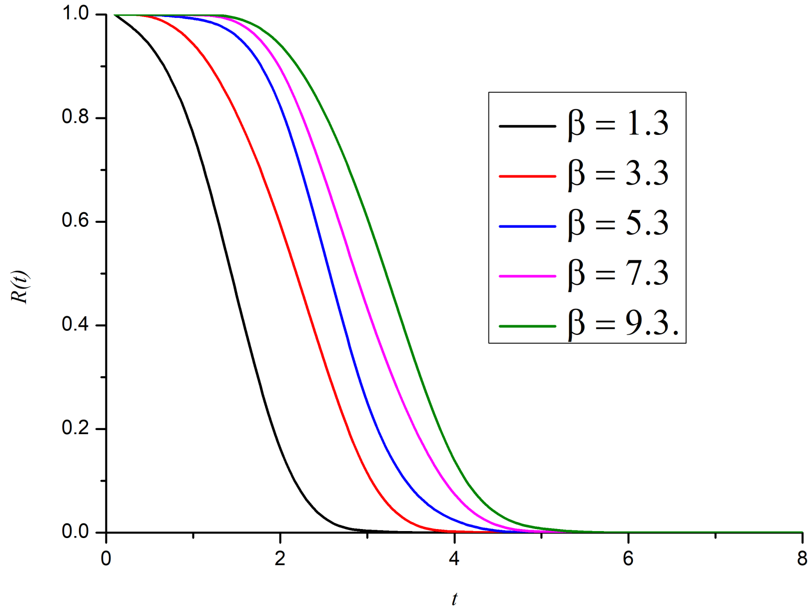

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Figure 5: *Reliability function of type II power Topp-Leone Dagum distribution for fixed value of* $\alpha = 4.3, \tau = 2.2, \sigma = 6, \theta = 3$ *and different values of* β.

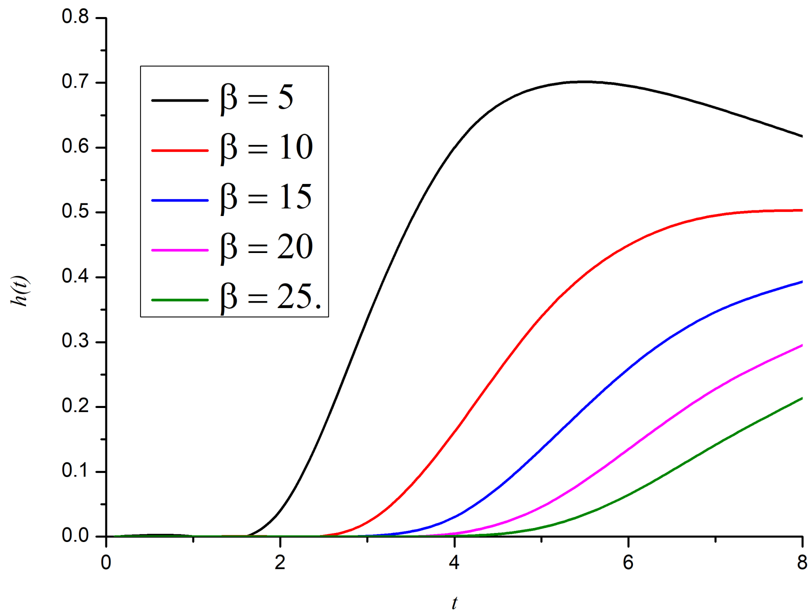

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Figure 6: *Hazard rate function of type II power Topp-Leone Dagum distribution for fixed value of* $\alpha = 1, \tau = 3, \sigma = 4, \theta = 2$ *and different values of* β.

IX. Mean deviation

The mean deviation is defined as

$$\pi(x) = 2\{\mu F(\mu) - \int_0^{\mu} x f(x) dx\} \tag{19}$$

$$\pi(x) = 2\{\mu F(\mu) - \int_0^{\mu} x \left[\sum_{p,q,s,t,v=0}^{\infty} \psi 2\alpha\tau\sigma\theta\beta \; x^{-\theta-1}(1+\sigma x^{-\theta})^{-\beta(p+s-v+1)} \right] dx\}$$

The mean deviation of type II power Topp-Leone Dagum distribution is given by

$$\pi(x) = 2\{\mu F(\mu) - \sum_{p,q,s,t,v=0}^{\infty} \psi 2\alpha\tau\beta\sigma^{\frac{1}{\theta}} \; B\left(1-\frac{1}{\theta}, \beta(p+s-v+1)+\frac{1}{\theta}\right)\} \tag{20}$$

IV. Statistical Properties

I. Moments

The r^{th} moment about the mean of a random variable X is given by

$$\mu_r^{'} = \int_{-\infty}^{\infty} x^r f(x) dx, for\ X\ is\ continuous. \tag{21}$$

$$\mu_r^{'} = \int_0^{\infty} x^r \left[\sum_{p,q,s,t,v=0}^{\infty} \psi 2\alpha\tau\sigma\theta\beta \; x^{-\theta-1}(1+\sigma x^{-\theta})^{-\beta(p+s-v+1)-1} \right] dx$$

$$= \sum_{p,q,s,t,v=0}^{\infty} \psi 2\alpha\tau\beta\sigma^{\frac{r}{\theta}} \int_0^{\infty} \frac{u^{-\frac{r}{\theta}}}{(1+u)^{\beta(p+s-v+1)+1}} du$$

The r^{th} moment of type II power Topp-Leone Dagum distribution is given by

$$\mu_r^{'} = \sum_{p,q,s,t,v=0}^{\infty} \psi 2\alpha\tau\beta\sigma^{\frac{r}{\theta}} \; B\left(1-\frac{r}{\theta}, \beta(p+s-v+1)+\frac{r}{\theta}\right). \ where\ r = 1,2,3.., \tag{22}$$

In particular

$$E(X) = \sum_{p,q,s,t,v=0}^{\infty} \psi 2\alpha\tau\beta\sigma^{\frac{1}{\theta}} \; B\left(1-\frac{1}{\theta}, \beta(p+s-v+1)+\frac{1}{\theta}\right) \tag{23}$$

$$E(X^2) = \sum_{p,q,s,t,v=0}^{\infty} \psi 2\alpha\tau\beta\sigma^{\frac{2}{\theta}} \; B\left(1-\frac{2}{\theta}, \beta(p+s-v+1)+\frac{2}{\theta}\right)$$

$$E(X^3) = \sum_{p,q,s,t,v=0}^{\infty} \psi 2\alpha\tau\beta\sigma^{\frac{3}{\theta}} \; B\left(1-\frac{3}{\theta}, \beta(p+s-v+1)+\frac{3}{\theta}\right)$$

$$E(X^4) = \sum_{p,q,s,t,v=0}^{\infty} \psi 2\alpha\tau\beta\sigma^{\frac{4}{\theta}} \; B\left(1-\frac{4}{\theta}, \beta(p+s-v+1)+\frac{4}{\theta}\right)$$

The variance is given by

$$V(x) = \left[\sum_{p,q,s,t,v=0}^{\infty} \psi 2\alpha\tau\beta\sigma^{\frac{2}{\theta}} \; B\left(1-\frac{2}{\theta}, \beta(p+s-v+1)+\frac{2}{\theta}\right) \right] - \left[\sum_{p,q,s,t,v=0}^{\infty} \psi 2\alpha\tau\beta\sigma^{\frac{1}{\theta}} \; B\left(1-\frac{1}{\theta}, \beta(p+s-v+1)+\frac{1}{\theta}\right) \right]^2 \tag{24}$$

II. Moment generating function

The moment generating function of the random variable X is defined by

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x)dx, \text{ where } e^{tx} = \sum_{r=0}^{\infty} \frac{(tx)^r}{r!}$$

The moment generating function of type II power Topp-Leone Dagum distribution is given by

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \left[\sum_{p,q,s,t,v=0}^{\infty} \psi 2\alpha\tau\beta\sigma^{\frac{r}{\theta}} \; B\left(1 - \frac{r}{\theta}, \beta(p+s-v+1) + \frac{r}{\theta}\right) \right] \tag{25}$$

III. Characteristic function

The characteristic function of the random variable X is defined by

$$\Phi_X(t) = \int_{-\infty}^{\infty} e^{itx} f(x)dx, \text{ where } e^{itx} = \sum_{r=0}^{\infty} \frac{(itx)^r}{r!}; i^2 = -1$$

The characteristic function of type II power Topp-Leone Dagum distribution is given by

$$\Phi_X(t) = \sum_{r=0}^{\infty} \frac{(it)^r}{r!} \left[\sum_{p,q,s,t,v=0}^{\infty} \psi 2\alpha\tau\beta\sigma^{\frac{r}{\theta}} \; B\left(1 - \frac{r}{\theta}, \beta(p+s-v+1) + \frac{r}{\theta}\right) \right] \tag{26}$$

IV. Cumulant generating function

Cumulant generating function is defined by

$$K_X(t) = log \; M_X(t)$$

The cumulant generating function of type II power Topp-Leone Dagum distribution is given by

$$K_X(t) = log \left[\sum_{r=0}^{\infty} \frac{t^r}{r!} \left[\sum_{p,q,s,t,v=0}^{\infty} \psi 2\alpha\tau\beta\sigma^{\frac{r}{\theta}} B\left(1 - \frac{r}{\theta}, \beta(p+s-v+1) + \frac{r}{\theta}\right) \right] \right] \tag{27}$$

V. Incomplete r^{th} moment

Incomplete r^{th} moment is defined by

$$m_r(x) = \int_0^x x^r f(x)dx \tag{28}$$

$$m_r(x) = \int_0^x x^r \left[\sum_{p,q,s,t,v=0}^{\infty} \psi 2\alpha\tau\sigma\theta\beta \; x^{-\theta-1}(1+\sigma x^{-\theta})^{-\beta(p+s-v+1)-1} \right] dx$$

The incomplete r^{th} moment of type II power Topp-Leone Dagum distribution is given by

$$m_r(x) = \sum_{p,q,s,t,v=0}^{\infty} \psi 2\alpha\tau\beta\sigma^{\frac{r}{\theta}} B\left(1 - \frac{r}{\theta}, \beta(p+s-v+1) + \frac{r}{\theta}; y\right) \tag{29}$$

VI. Inverted moments

The r^{th} inverted moment is defined by

$$\mu_r^* = \int_{-\infty}^{\infty} x^{-r} f(x) dx \tag{30}$$

$$\mu_r^* = \int_{0}^{\infty} x^{-r} \left[\sum_{p,q,s,t,v=0}^{\infty} \psi 2\alpha\tau\sigma\theta\beta \; x^{-\theta-1} (1+\sigma x^{-\theta})^{-\beta(p+s-v+1)-1} \right] dx$$

The inverted r^{th} moment of type II power Topp-Leone Dagum distribution is given by

$$\mu_r^* = \sum_{p,q,s,t,v=0}^{\infty} \psi 2\alpha\tau\beta\sigma^{-\frac{r}{\theta}} B\left(1+\frac{r}{\theta}, \beta(p+s-v+1)-\frac{r}{\theta}\right) \tag{31}$$

The r^{th} inverted moment used to find harmonic mean. The harmonic mean of type II power Topp-Leone Dagum distribution is given by

$$\frac{1}{\mu_r^*} = \frac{1}{\left[\sum_{p,q,s,t,v=0}^{\infty} \psi 2\alpha\tau\beta\sigma^{-\frac{r}{\theta}} B\left(1+\frac{r}{\theta}, \beta(p+s-v+1)-\frac{r}{\theta}\right)\right]} \tag{32}$$

VII. Central moments

The r^{th} central moment is defined by

$$\mu_r = \int_{-\infty}^{\infty} (x-\mu_1^{'})^r f(x) dx = \sum_{m=0}^{r} \binom{r}{m} (-1)^m (\mu_1^{'})^m \mu_{r-m}^{'} \tag{33}$$

The r^{th} central moment of type II power Topp-Leone Dagum distribution is given by

$$\begin{aligned}\mu_r = \sum_{m=0}^{r} \binom{r}{m} (-1)^m \times \left[\sum_{p,q,s,t,v=0}^{\infty} \psi 2\alpha\tau\beta\sigma^{\frac{1}{\theta}} B\left(1-\frac{1}{\theta}, \beta(p+s-v+1)+\frac{1}{\theta}\right)\right]^m \\ \times \left[\sum_{p,q,s,t,v=0}^{\infty} \psi 2\alpha\tau\beta\sigma^{\frac{r-m}{\theta}} B\left(1-\frac{r-m}{\theta}, \beta(p+s-v+1)+\frac{r-m}{\theta}\right)\right]\end{aligned} \tag{34}$$

VIII. Conditional moments

The n^{th} conditional moment is defined by

$$E\big(X^n|X>x\big) = \frac{1}{S(x)} \int_{x}^{\infty} x^n f(x) dx \tag{35}$$

$$E\big(X^n|X>x\big) = \frac{1}{S(x)} \int_{x}^{\infty} x^n \left[\sum_{p,q,s,t,v=0}^{\infty} \psi 2\alpha\tau\sigma\theta\beta \; x^{-\theta-1} (1+\sigma x^{-\theta})^{-\beta(p+s-v+1)-1} \right] dx$$

where $S(x) = 1 - R(x)$.
The n^{th} conditional moment of type II power Topp-Leone Dagum distribution is given by

$$E\big(X^n|X>x\big) = \frac{\sum_{p,q,s,t,v=0}^{\infty} \psi 2\alpha\tau\beta\sigma^{\frac{n}{\theta}} B\left(1-\frac{n}{\theta}, \beta(p+s-v+1)+\frac{n}{\theta}\right)}{1-\left[1-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\alpha\tau}\left(2-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\tau}\right)^{\alpha}\right]}. \; where \;\; n = 1,2,3.., \tag{36}$$

In particular

$$E(X|X>x)=\frac{\sum_{p,q,s,t,v=0}^{\infty}\psi 2\alpha\tau\beta\sigma^{\frac{1}{\theta}}B\left(1-\frac{1}{\theta},\beta(p+s-v+1)+\frac{1}{\theta}\right)}{1-\left[1-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\alpha\tau}\left(2-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\tau}\right)^{\alpha}\right]}$$

$$E(X^2|X>x)=\frac{\sum_{p,q,s,t,v=0}^{\infty}\psi 2\alpha\tau\beta\sigma^{\frac{2}{\theta}}B\left(1-\frac{2}{\theta},\beta(p+s-v+1)+\frac{2}{\theta}\right)}{1-\left[1-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\alpha\tau}\left(2-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\tau}\right)^{\alpha}\right]}$$

$$E(X^3|X>x)=\frac{\sum_{p,q,s,t,v=0}^{\infty}\psi 2\alpha\tau\beta\sigma^{\frac{3}{\theta}}B\left(1-\frac{3}{\theta},\beta(p+s-v+1)+\frac{3}{\theta}\right)}{1-\left[1-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\alpha\tau}\left(2-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\tau}\right)^{\alpha}\right]}$$

$$E(X^4|X>x)=\frac{\sum_{p,q,s,t,v=0}^{\infty}\psi 2\alpha\tau\beta\sigma^{\frac{4}{\theta}}B\left(1-\frac{4}{\theta},\beta(p+s-v+1)+\frac{4}{\theta}\right)}{1-\left[1-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\alpha\tau}\left(2-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\tau}\right)^{\alpha}\right]}$$

IX. Probability weighted moment

The probability weighted moment of the random variable X is defined by

$$\tau_{r,h}=E\left[X^rF(x)^h\right]=\int_{-\infty}^{\infty}x^rf(x)F(x)^hdx \tag{37}$$

$$\begin{aligned}\tau_{r,h}=&\int_0^{\infty}x^r\left[\sum_{p,q,s,t,v=0}^{\infty}\psi 2\alpha\tau\sigma\theta\beta\, x^{-\theta-1}(1+\sigma x^{-\theta})^{-\beta(p+s-v+1)-1}\right]\\&\times\left[1-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\alpha\tau}\left(2-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\tau}\right)^{\alpha}\right]^h dx\end{aligned}$$

Using the binomial series

$$(x-y)^r=\sum_{a=0}^{\infty}(-1)^a\binom{r}{a}x^{r-a}y^a,\ \ (1-y)^r=\sum_{a=0}^{\infty}\binom{r}{a}(-1)^ay^a$$

We have

$$\left[1-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\alpha\tau}\left(2-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\tau}\right)^{\alpha}\right]^h=$$
$$\sum_{a,b,c,d=0}^{\infty}(2)^{2(\alpha a-c)}(-1)^{a+b+c+d}\binom{h}{a}\binom{\alpha\tau a}{b}\binom{\alpha a}{c}\binom{\tau c}{d}(1+\sigma x^{-\theta})^{-\beta b-\beta d}$$

Therefore, we have

$$\begin{aligned}\tau_{r,h}=&\int_0^{\infty}x^r\left[\sum_{p,q,s,t,v=0}^{\infty}\psi 2\alpha\tau\sigma\theta\beta\, x^{-\theta-1}(1+\sigma x^{-\theta})^{-\beta(p+s-v+1)-1}\right]\\&\times\left[\sum_{a,b,c,d=0}^{\infty}(2)^{\alpha h-c}(-1)^{a+b+c+d}\binom{h}{a}\binom{\alpha\tau a}{b}\binom{\alpha a}{c}\binom{\tau c}{d}(1+\sigma x^{-\theta})^{-\beta b-\beta d}\right]dx\end{aligned}$$

$$=\sum_{p,q,s,t,v,a,b,c,d=0}^{\infty}\psi 2\alpha\tau\sigma\theta\beta(2)^{2(\alpha a-c)}\binom{h}{a}\binom{\alpha\tau a}{b}\binom{\alpha a}{c}\binom{\tau c}{d}\sigma^{\frac{r}{\theta}}\int_0^{\infty}\frac{u^{-\frac{r}{\theta}}}{(1+u)^{\beta(p+s-v+b+d+1)+1}}du$$

The probability weighted moment of type II power Topp-Leone Dagum distribution is given by

$$\tau_{r,h} = \sum_{p,q,s,t,v,a,b,c,d=0}^{\infty} \psi 2\alpha\tau\beta\sigma^{\frac{r}{\theta}} B\left(1-\frac{r}{\theta}, \beta(p+s-v+b+d+1)+\frac{r}{\theta}\right) \tag{38}$$

$$where \;\; \eta = (2)^{\alpha h-c}(-1)^{a+b+c+d}\binom{h}{a}\binom{\alpha\tau a}{b}\binom{\alpha a}{c}\binom{\tau c}{d}$$

X. Order statistics

The pdf of the j^{th} order statistics for type II power Topp-Leone Dagum distribution $X_{(j)}$ is given by

$$\begin{aligned} f_{X_{(j)}}(x) =& \frac{n!}{(j-1)(n-j)!} \\ &\times \Big[2\alpha\tau\sigma\theta\beta\, x^{-\theta-1}\left(1+\sigma x^{-\theta}\right)^{-\beta-1}\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\alpha\tau-1} \\ &\left(2-(1-(1+\sigma x^{-\theta})^{-\beta})^{\tau}\right)^{\alpha-1}\left(1-(1-(1+\sigma x^{-\theta})^{-\beta})^{\tau}\right)\Big] \\ &\times \left[1-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\alpha\tau}\left(2-(1-(1+\sigma x^{-\theta})^{-\beta})^{\tau}\right)^{\alpha}\right]^{j-1} \\ &\times \left[1-\left(1-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\alpha\tau}\left(2-(1-(1+\sigma x^{-\theta})^{-\beta})^{\tau}\right)^{\alpha}\right)\right]^{n-1} \end{aligned} \tag{39}$$

The pdf of the smallest order statistics $X_{(1)}$ is given by

$$\begin{aligned} f_{X_{(1)}}(x) =& n\left[1-\left(1-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\alpha\tau}\left(2-(1-(1+\sigma x^{-\theta})^{-\beta})^{\tau}\right)^{\alpha}\right)\right]^{n-1} \\ &\times \Big[2\alpha\tau\sigma\theta\beta\, x^{-\theta-1}\left(1+\sigma x^{-\theta}\right)^{-\beta-1}\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\alpha\tau-1} \\ &\left(2-(1-(1+\sigma x^{-\theta})^{-\beta})^{\tau}\right)^{\alpha-1}\left(1-(1-(1+\sigma x^{-\theta})^{-\beta})^{\tau}\right)\Big] \end{aligned} \tag{40}$$

The pdf of the largest order statistics $X_{(n)}$ is given by

$$\begin{aligned} f_{X_{(n)}}(x) =& n\left[\left(1-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\alpha\tau}\left(2-(1-(1+\sigma x^{-\theta})^{-\beta})^{\tau}\right)^{\alpha}\right)\right]^{n-1} \\ &\times \Big[2\alpha\tau\sigma\theta\beta\, x^{-\theta-1}\left(1+\sigma x^{-\theta}\right)^{-\beta-1}\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\alpha\tau-1} \\ &\left(2-(1-(1+\sigma x^{-\theta})^{-\beta})^{\tau}\right)^{\alpha-1}\left(1-(1-(1+\sigma x^{-\theta})^{-\beta})^{\tau}\right)\Big] \end{aligned} \tag{41}$$

and the pdf of the median order statistics is given by

$$f_{m+1:n}(x) = \frac{(2m+1)}{m!m!}\left[F(x)\right]^m\left[1-F(x)\right]^m f_{(X)}(x)$$

$$\begin{aligned} f_{m+1:n}(x) =& \frac{(2m+1)}{m!m!}\left[1-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\alpha\tau}\left(2-(1-(1+\sigma x^{-\theta})^{-\beta})^{\tau}\right)^{\alpha}\right]^{m} \\ &\times \left[1-\left(1-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\alpha\tau}\left(2-(1-(1+\sigma x^{-\theta})^{-\beta})^{\tau}\right)^{\alpha}\right)\right]^{m} \\ &\times \Big[2\alpha\tau\sigma\theta\beta\, x^{-\theta-1}\left(1+\sigma x^{-\theta}\right)^{-\beta-1}\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\alpha\tau-1} \\ &\left(2-(1-(1+\sigma x^{-\theta})^{-\beta})^{\tau}\right)^{\alpha-1}\left(1-(1-(1+\sigma x^{-\theta})^{-\beta})^{\tau}\right)\Big] \end{aligned} \tag{42}$$

The joint distribution of the i^{th} and j^{th} order statistics for $1 \le i < j \le n$ is given by

$$f_{i:j;n}(x_i, x_j) = C\,[F(x_i)]^{i-1}\left[F(x_j) - F(x_i)\right]^{j-i-1}[1 - F(x_i)]^{n-j} f(x_i) f(x_j)$$

where

$$C = \frac{n!}{(i-1)!(j-i-1)!(n-j)!}$$

$$\begin{aligned}
f_{i:j;n}(x_i, x_j) =& \frac{n!}{(i-1)!(j-i-1)!(n-j)!}\left[1 - W_{(i)}^{\alpha\tau}(2 - W_{(i)}^{\tau})^{\alpha}\right]^{i-1} \\
&\times \left[\left(1 - W_{(j)}^{\alpha\tau}(2 - W_{(j)}^{\tau})^{\alpha}\right) - \left(1 - W_{(j)}^{\alpha\tau}(2 - W_{(j)}^{\tau})^{\alpha}\right)\right]^{j-i-1} \\
&\times \left[1 - \left(1 - W_{(j)}^{\alpha\tau}\right)(2 - W_{(j)}^{\tau})^{\alpha}\right]^{n-j} \\
&\times \left[2\alpha\tau\sigma\theta\beta\; x_i^{-\theta-1}(1 + \sigma x^{-\theta})^{-\beta-1} W_{(i)}^{\alpha\tau-1}(2 - W_{(i)}^{\tau})^{\alpha-1}(1 - W_{(i)}^{\tau})\right] \\
&\times \left[2\alpha\tau\sigma\theta\beta\; x_i^{-\theta-1}(1 + \sigma x^{-\theta})^{-\beta-1} W_{(j)}^{\alpha\tau-1}(2 - W_{(j)}^{\tau})^{\alpha-1}(1 - W_{(j)}^{\tau})\right]
\end{aligned} \tag{43}$$

$$where\;\; W_{(i)} = \left(1 - (1 + \sigma x_i^{-\theta})^{-\beta}\right), W_{(j)} = \left(1 - (1 + \sigma x_j^{-\theta})^{-\beta}\right)$$

The joint distribution of minimum and maximum of order statistics is given by

$$f_{1:n;n}(x_1, x_n) = n(n-1)\left[F(x_{(n)}) - F(x_{(1)})\right]^{n-2} f(x_1) f(x_n)$$

$$\begin{aligned}
f_{1:n;n}(x_1, x_n) =& n(n-1)\left[\left(1 - W_{(n)}^{\alpha\tau}(2 - W_{(n)}^{\tau})^{\alpha}\right) - \left(1 - W_{(1)}^{\alpha\tau}(2 - W_{(1)}^{\tau})^{\alpha}\right)\right]^{n-2} \\
&\times \left[2\alpha\tau\sigma\theta\beta\; x_1^{-\theta-1}(1 + \sigma x_1^{-\theta})^{-\beta-1} W_{(1)}^{\alpha\tau-1}(2 - W_{(1)}^{\tau})^{\alpha-1}(1 - W_{(1)}^{\tau})\right] \\
&\times \left[2\alpha\tau\sigma\theta\beta\; x_n^{-\theta-1}(1 + \sigma x_n^{-\theta})^{-\beta-1} W_{(n)}^{\alpha\tau-1}(2 - W_{(n)}^{\tau})^{\alpha-1}(1 - W_{(n)}^{\tau})\right]
\end{aligned} \tag{44}$$

$$where\;\; W_{(i)} = \left(1 - (1 + \sigma x_i^{-\theta})^{-\beta}\right), W_{(j)} = \left(1 - (1 + \sigma x_j^{-\theta})^{-\beta}\right)$$

V. Income inequality measures

I. Lorenz curve

The Lorenz curve is defined by

$$L(x) = \frac{1}{\mu}\int_0^x x f(x) dx \tag{45}$$

$$L(x) = \frac{1}{\mu}\int_0^x x\left[\sum_{p,q,s,t,v=0}^{\infty} \psi 2\alpha\tau\sigma\theta\beta\; x^{-\theta-1}(1 + \sigma x^{-\theta})^{-\beta(p+s-v+1)-1}\right] dx$$

The Lorenz curves of type II power Topp-Leone Dagum distribution is given by

$$L(x) = \frac{\sum_{p,q,s,t,v=0}^{\infty} \psi 2\alpha\tau\beta\sigma^{\frac{1}{\theta}} B\left(1 - \frac{1}{\theta}, \beta(p + s - v + 1) + \frac{1}{\theta}; y\right)}{\sum_{p,q,s,t,v=0}^{\infty} \psi 2\alpha\tau\beta\sigma^{\frac{1}{\theta}} B\left(1 - \frac{1}{\theta}, \beta(p + s - v + 1) + \frac{1}{\theta}\right)} \tag{46}$$

II. Bonferroni index

Bonferroni index is defined by

$$B(x) = \frac{L(x)}{F(x)} \tag{47}$$

The Bonferroni index of type II power Topp-Leone Dagum distribution is given by

$$B(x) = \frac{\omega}{\vartheta} \tag{48}$$

where

$$\omega = \sum_{p,q,s,t,v=0}^{\infty} \psi 2\alpha\tau\beta\sigma^{\frac{1}{\theta}} B\left(1-\frac{1}{\theta}, \beta(p+s-v+1)+\frac{1}{\theta}; y\right)$$

$$\vartheta = \sum_{p,q,s,t,v=0}^{\infty} \psi 2\alpha\tau\beta\sigma^{\frac{1}{\theta}} B\left(1-\frac{1}{\theta}, \beta(p+s-v+1)+\frac{1}{\theta}\right)$$
$$\times \left[1-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\alpha\tau}\left(2-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\tau}\right)^{\alpha}\right]$$

III. Generalized entropy

The generalized entropy is defined by

$$GE(w,\delta) = \frac{1}{\delta(\delta-1)\mu^{\delta}}\left[\int_0^{\infty} x^{\delta} f(x)dx\right] - 1 \tag{49}$$

where μ is the mean of distribution.

$$GE(w,\delta) = \frac{1}{\delta(\delta-1)\mu^{\delta}}\int_0^{\infty} x^{\delta}\left[\sum_{p,q,s,t,v=0}^{\infty} \psi 2\alpha\tau\sigma\theta\beta\, x^{-\theta-1}(1+\sigma x^{-\theta})^{-\beta(p+s-v+1)-1}\right]dx - 1$$

The Generalized entropy of type II power Topp-Leone Dagum distribution is given by

$$GE(w,\delta) = \frac{\sum_{p,q,s,t,v=0}^{\infty} \psi 2\alpha\tau\beta\sigma^{\frac{\delta}{\theta}} B\left(1-\frac{\delta}{\theta}, \beta(p+s-v+1)+\frac{\delta}{\theta}\right)}{\delta(\delta-1)\left[\sum_{p,q,s,t,v=0}^{\infty} \psi 2\alpha\tau\beta\sigma^{\frac{1}{\theta}} B\left(1-\frac{1}{\theta}, \beta(p+s-v+1)+\frac{1}{\theta}\right)\right]^{\delta}} - 1 \tag{50}$$

IV. Zenga index

Zenga index is defined by

$$Z = 1 - \frac{\bar{\mu}_{(x)}}{\mu^{+}_{(x)}} \tag{51}$$

where

$$\bar{\mu}_{(x)} = \frac{1}{F(x)}\int_0^{x} xf(x)dx$$
$$\mu^{+}_{(x)} = \frac{1}{1-F(x)}\int_0^{\infty} xf(x)dx$$

Consider,

$$\bar{\mu}_{(x)} = \frac{1}{F(x)}\int_0^{x} xf(x)dx$$

$$\bar{\mu_x} = \frac{1}{\left[1-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\alpha\tau}\left(2-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\tau}\right)^{\alpha}\right]} \times \int_0^x x\left[\sum_{p,q,s,t,v=0}^{\infty}\psi 2\alpha\tau\sigma\theta\beta x^{-\theta-1}(1+\sigma x^{-\theta})^{-\beta(p+s-v+1)-1}\right]dx$$

$$\bar{\mu_x} = \frac{\sum_{p,q,s,t,v=0}^{\infty}\psi 2\alpha\tau\beta\sigma^{\frac{1}{\theta}}B\left(1-\frac{1}{\theta},\beta(p+s-v+1)+\frac{1}{\theta};y\right)}{\left[1-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\alpha\tau}\left(2-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\tau}\right)^{\alpha}\right]}$$

Consider,

$$\mu^{+}_{(x)} = \frac{1}{1-F(x)}\int_0^{\infty} xf(x)dx$$

$$\mu^{+}_{x} = \frac{1}{\left[1-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\alpha\tau}\left(2-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\tau}\right)^{\alpha}\right]} \times \int_0^{\infty} x\left[\sum_{p,q,s,t,v=0}^{\infty}\psi 2\alpha\tau\sigma\theta\beta x^{-\theta-1}(1+\sigma x^{-\theta})^{-\beta(p+s-v+1)-1}\right]dx$$

$$\mu^{+}_{x} = \frac{\sum_{p,q,s,t,v=0}^{\infty}\psi 2\alpha\tau\beta\sigma^{\frac{1}{\theta}}B\left(1-\frac{1}{\theta},\beta(p+s-v+1)+\frac{1}{\theta}\right)}{\left[1-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\alpha\tau}\left(2-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\tau}\right)^{\alpha}\right]}$$

The Zenga index of type II power Topp-Leone Dagum distribution is given by

$$Z = 1 - \frac{A}{B} \tag{52}$$

where

$$A = \sum_{p,q,s,t,v=0}^{\infty}\psi 2\alpha\tau\beta\sigma^{\frac{1}{\theta}}B\left(1-\frac{1}{\theta},\beta(p+s-v+1)+\frac{1}{\theta};y\right) \times \left[1-\left(1-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\alpha\tau}\left(2-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\tau}\right)^{\alpha}\right)\right]$$

$$B = \left[1-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\alpha\tau}\left(2-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\tau}\right)^{\alpha}\right] \sum_{p,q,s,t,v=0}^{\infty}\psi 2\alpha\tau\beta\sigma^{\frac{1}{\theta}}B\left(1-\frac{1}{\theta},\beta(p+s-v+1)+\frac{1}{\theta}\right)$$

VI. Parameter Estimation

Let $x_1, x_1, ..., x_n$ be a random sample from the type II power Topp-Leone Dagum distribution then the likelihood function is given by

$$L(\theta) = \prod_{i=1}^{n}\left[2\alpha\tau\sigma\theta\beta\, x^{-\theta-1}\left(1+\sigma x^{-\theta}\right)^{-\beta-1}\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\alpha\tau-1} \left(2-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\tau}\right)^{\alpha-1}\left(1-\left(1-(1+\sigma x^{-\theta})^{-\beta}\right)^{\tau}\right)\right] \tag{53}$$

The log likelihood function is given by

$$\begin{aligned}L(\theta) =& nlog2 + nlog\alpha + nlog\tau + nlog\sigma + nlog\theta + nlog\beta + (-\theta - 1)\sum_{i=1}^{n} logx_i \\ &+ (-\beta - 1)\sum_{i=1}^{n} log(1 + \sigma x_i^{-\theta}) + (\alpha\tau - 1)\sum_{i=1}^{n} log\left(1 - (1 + \sigma x_i^{-\theta})^{-\beta}\right) \\ &+ (\alpha - 1)\sum_{i=1}^{n} log\left(2 - \left(1 - (1 + \sigma x_i^{-\theta})^{-\beta}\right)^{\tau}\right) + \sum_{i=1}^{n} log\left(1 - \left(1 - (1 + \sigma x_i^{-\theta})^{-\beta}\right)^{\tau}\right)\end{aligned}$$

Taking the partial derivatives of the log-likelihood function with respect to parameters $\alpha, \tau, \sigma, \theta$ and β and then equate to zero.

$$\frac{\partial logL}{\partial \alpha} = 0, \frac{\partial logL}{\partial \tau} = 0, \frac{\partial logL}{\partial \sigma} = 0, \frac{\partial logL}{\partial \theta} = 0 \text{ and } \frac{\partial logL}{\partial \beta} = 0.$$

That is

$$\frac{\partial logL}{\partial \alpha} = \frac{n}{\alpha} + \alpha\sum_{i=1}^{n} log\left(1 - (1 + \sigma x_i^{-\theta})\right) + \sum_{i=1}^{n} log\left(2 - \left(1 - (1 + \sigma x_i^{-\theta})^{-\beta}\right)^{\tau}\right) = 0 \quad (54)$$

$$\begin{aligned}\frac{\partial logL}{\partial \tau} =& \frac{n}{\tau} + \alpha\sum_{i=1}^{n} log\left(1 - (1 + \sigma x_i^{-\theta})^{-\beta}\right) \\ &- \sum_{i=1}^{n} \frac{(\alpha - 1)\left(1 - (1 + \sigma x_i^{-\theta})^{-\beta}\right)^{\tau} log\left(1 - (1 + \sigma x_i^{-\theta})^{-\beta}\right)}{\left(2 - \left(1 - (1 + \sigma x_i^{-\theta})^{-\beta}\right)^{\tau}\right)} \\ &- \sum_{i=1}^{n} \frac{\left(1 - (1 + \sigma x_i^{-\theta})^{-\beta}\right)^{\tau} log\left(1 - (1 + \sigma x_i^{-\theta})^{-\beta}\right)}{\left(1 - \left(1 - (1 + \sigma x_i^{-\theta})^{-\beta}\right)^{\tau}\right)} = 0\end{aligned} \quad (55)$$

$$\begin{aligned}\frac{\partial logL}{\partial \sigma} =& \frac{n}{\sigma} + \sum_{i=1}^{n} \frac{(-\beta - 1)x_i^{-\theta}}{(1 + \sigma x_i^{-\theta})} - \sum_{i=1}^{n} \frac{(\alpha\tau - 1)x_i^{-\theta}}{\left(1 - (1 + \sigma x_i^{-\theta})^{-\beta}\right)} \\ &- \sum_{i=1}^{n} \frac{(\alpha - 1)\tau\left(1 - (1 + \sigma x_i^{-\theta})^{-\beta}\right)^{\tau - 1} \beta(1 + \sigma x_i^{-\theta})^{-\beta - 1} x_i^{-\theta}}{\left(2 - \left(1 - (1 + \sigma x_i^{-\theta})^{-\beta}\right)^{\tau}\right)} \\ &- \sum_{i=1}^{n} \frac{\tau\left(1 - (1 + \sigma x_i^{-\theta})^{-\beta}\right)^{\tau - 1} \beta(1 + \sigma x_i^{-\theta})^{-\beta - 1} x_i^{-\theta}}{\left(1 - \left(1 - (1 + \sigma x_i^{-\theta})^{-\beta}\right)^{\tau}\right)} = 0\end{aligned} \quad (56)$$

$$\begin{aligned}\frac{\partial logL}{\partial \theta} =& \frac{n}{\theta} - \sum_{i=1}^{n} logx_i + \sum_{i=1}^{n} \frac{(-\beta - 1)\sigma x_i^{-\theta} logx_i}{(1 + \sigma x_i^{-\theta})} - \sum_{i=1}^{n} \frac{(\alpha\tau - 1)\sigma x_i^{-\theta} logx_i}{\left(1 - (1 + \sigma x_i^{-\theta})\right)} \\ &- \sum_{i=1}^{n} \frac{(\alpha - 1)\beta\tau\left(1 - (1 + \sigma x_i^{-\theta})^{-\beta}\right)^{\tau - 1} (1 + \sigma x_i^{-\theta})^{-\beta - 1} \sigma x_i^{-\theta} logx_i}{\left(2 - \left(1 - (1 + \sigma x_i^{-\theta})^{-\beta}\right)^{\tau}\right)} \\ &- \sum_{i=1}^{n} \frac{\beta\tau\left(1 - (1 + \sigma x_i^{-\theta})^{-\beta}\right)^{\tau - 1} (1 + \sigma x_i^{-\theta})^{-\beta - 1} \sigma x_i^{-\theta} logx_i}{\left(1 - \left(1 - (1 + \sigma x_i^{-\theta})^{-\beta}\right)^{\tau}\right)} = 0\end{aligned} \quad (57)$$

and

$$\begin{aligned}\frac{\partial logL}{\partial\beta} =&\frac{n}{\beta}-\sum_{i=1}^{n} log\left(1+\sigma x_i^{-\theta}\right)\\&+\sum_{i=1}^{n}\frac{\tau(\alpha-1)\left(1-(1+\sigma x_i^{-\theta})^{-\beta}\right)^{\tau-1}(1+\sigma x_i^{-\theta})^{-\beta}log\left(1+\sigma x_i^{-\theta}\right)}{\left(2-\left(1-(1+\sigma x_i^{-\theta})^{-\beta}\right)^{\tau}\right)}\\&+\sum_{i=1}^{n}\frac{\tau\left(1-(1+\sigma x_i^{-\theta})^{-\beta}\right)^{\tau-1}(1+\sigma x_i^{-\theta})^{-\beta}log\left(1+\sigma x_i^{-\theta}\right)}{\left(1-\left(1-(1+\sigma x_i^{-\theta})^{-\beta}\right)^{\tau}\right)}=0\end{aligned} \tag{58}$$

The above mentioned five non-linear equations are very difficult to solve analytically. In this situation we can use to iteration techniques like Newton-Raphson, bisection and regular falsi method to compute numerical solution. However, we used *R* software for estimate the parameters of the proposed distribution.

VII. Applications

In this section, we consider two real data sets for type II power Topp Leone-Dagum distribution. This first data set represent the survival times (days) of 40 patients suffering from leukemia and is studied by Abouammoh et al., [2] and Bhatti et al., [5]. The second data set related to actuarial science data (Mortality death). This data describes 280 observations on the age of death (in years) of retired women with temporary disabilities who died during 2004 and which are incorporated in the Mexican insurance public system. This data set recently studied by Balakrishnan et al., [4] and Tahir et al., [19].

I. Data set 1: survival time data

The survival time data set is analysed using the *R* software. The following tables Table 1 to 3 explain about summary of statistics, estimated parameters values and statistical model selection for survival time data.

We compared statistical models namely type II power Topp-Leone Dagum distribution (TIIPTLDD) with Dagum distribution (DD), modified Burr III distribution (MBIIID), Burr III distribution (BIIID), log-logistic distribution (LLD), modified Frechet distribution (MFD) and Frechet distribution (FD). The statistical model selection based on the minimum value of statistic information theoretic criterion, such as Akaike information criterion (AIC), Bayesian information criterion (BIC), consistent Akaike information criterion (CAIC), and -2log-likelihood was carried out. The type II power Topp-Leone Dagum distribution provides better fit and flexibility compared to other competitive statistical models based on these statistics measures.

Table 1: *Summary of statistics*

n	Mean	Median	Minimum	Maximum	Q_3	Q_3
40	1137	1222.0	115.0	1852.0	802.5	1852.0

Table 2: *The value of estimated parameters*

Model	Estimated value of the parameters
TIIPTLD-D	α=47.1760, τ=6.7544, σ=51.2247, θ=0.4767, β=3.8028
D-D	α=124635.5, β=1.199734, γ=5.0000
MBIII-D	α=124637.2, β=1.7052, γ=121473.2
BIII-D	α=2503.6088, β=1.1982.
LL-D	β=0.2235.
MF-D	β=0.9013, θ=7111.323, λ=0.0021.
F-D	β=1.1984, θ=685.7135

Table 3: *Statistical model selection*

Model	-2LL	AIC	AICC	BIC
TIIPTLD-D	-614.164	624.164	625.9287	632.6084
D-D	651.2760	659.2761	660.4189	666.0316
MBIII-D	638.7774	644.773	645.444	649.844
BIII-D	651.2590	655.2589	655.5832	658.6367
LL-D	825.6310	827.6309	827.7362	829.3198
MF-BIII	701.9472	707.9472	708.6139	713.0138
F-D	651.2778	655.2778	655.6022	658.6556

II. Data set 2: Actuarial science data

The actuarial science data set carried out using the *R* software. The following Tables 4 to 6 explain about summary of statistics, estimated parameters values and statistical model selection for actuarial science data.

We compared statistical models namely type II power Topp-Leone Dagum distribution (TIIPTLDD) with Dagum distribution (D). The statistical model selection based on the minimum value of statistic information theoretic criterion, such as Akaike information criterion (AIC), Bayesian information criterion (BIC), consistent Akaike information criterion (CAIC) and 2log-likelihood. The type II power Topp-Leone Dagum distribution provides better fit and flexibility compare than Dagum distribution based on statistics measures.

Table 4: *Summary of statistics*

n	Mean	Median	Minimum	Maximum	Q_3	Q_3
280	47.79	49.00	22.00	86.00	40.00	55.25

Table 5: *The value of estimated parameters*

Model	Estimated value of the parameters
TIIPTLD-D	α=37.3031, τ=0.6554, σ=26.2491, θ=1.5575, β=25.4281
D-D	σ=1282.2665, β=3.9888, γ=2183.6861

Table 6: *Statistical model selection*

Model	-2LL	AIC	AICC	BIC
TIIPTLD-D	2113.836	2123.836	2124.055	2142.01
D-D	2203.86	2209.860	2209.947	2220.764

VIII. Conclusion

In this article, we introduced new generating probability distribution called type II power Topp-Leone Dagum distribution. Many of reliability measures are investigated including reliability function, hazard rate function, reversed hazard rate function, mean waiting time, mean past life time, mean deviation, second failure rate function and mean residual life function. We have obtained different statistical properties such as moments, moment generating function, characteristic function, cumulant generating function, inverted moments, central moments, conditional moments, probability weighted moments and order statistics. We derived some of income inequality measures like Lorenz curve, Bonferroni index, Zenga index and Generalized entropy for proposed new probability distribution. The parameters of proposed new probability distributions are estimated by method of maximum likelihood. Finally, we fitted the type II power Topp-Leone Dagum distribution for real life time data sets and showed that TIIPTLD-D provide better fit these two data set.

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Decomposable Semi-Regenerative Processes: Review of Theory and Applications to Queueing and Reliability Systems

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Abstract

A review of the Smith's regeneration idea development is proposed. As a generalization of this idea the main definitions and results of decomposable semi-regenerative processes are reminded. Their applications for investigation of various queueing and reliability systems are considered.

Keywords: Decomposable semi-regenerative processes, Stochastic models, time-dependent and steady state probabilities.

1. Introduction, Motivation, and Abbreviations

The main idea of this paper is to give a review of the Smith's *regeneration notion* development. Definition and main results of *decomposable semi-regenerative processes* (DSRPr) will be under our attention. Applications of these processes to investigation of some real-world queueing and reliability systems makes up an essential part of the paper.

As a generalization of classical independence the regeneration idea has been proposed by W. Smith in 1955. The regenerative approach allows not only to calculate the *regenerative process* (RPr) state *time-dependent probabilities* (t.d.p.'s) in terms of its state probabilities at the separate *regeneration period* (RP), but also to prove its stationary regime existence and find the *steady state probabilities* (s.s.p.'s).

However, if the process behavior in the separate regeneration period is enough complex and its distribution can not be analytically represented, the more detailed investigation of the process could be obtained with the help of *semi-regenerative processes* (SRPr's), which joins the regeneration approach with the Markov type dependency. The next step in the generalization of the regeneration idea consists in finding some new regeneration points of time into regeneration period and construction of so called *embedded regenerative process* (ERPr). This idea can be developed for construction of *decomposable semi-regenerative processes* (DSRPr's).

These ideas have been applied for investigation of several stochastic systems such as: priority queueing systems (QS's), one-server QS with recurrent input and service processes, polling systems, complex hierarchical systems, etc. In this paper we remind the results of some of these investigations. Recently the ERPr has been used for investigation of a double-redundant system with general distributions of its component life- and repair times and of a k-out-of-n system. These models also will be in our focus.

Along the paper the following general notations are used:

- $\mathbf{P}\{\cdot\}$, $\mathbf{E}[\cdot]$ — symbols of probability and expectation, symbols $\mathbf{P}_i\{\cdot\}$, $\mathbf{E}_i[\cdot]$ are used for conditional probability and expectation, given initial state of the process is i;
- the vectors are marked with arrows and are understood as column vectors, and transposition of vectors and matrices is indicated by a prime;
- the representatives of any sequence of independent identically distributed random variables A_i $(i = 1, 2, \dots)$ are denoted by appropriate letter without indexes A and their common cumulative distribution functions (c.d.f.'s) are denoted by the same letter with an argument $A(x) = \mathbf{P}\{A \leq x\}$;
- The *moment generating functions* (MGF's) of r.v.'s (the *Laplace-Stiltjes transforms* (LST's) of their c.d.f.'s) are denoted by appropriate small letters with tilde $\tilde{a}(s) = \mathbf{E}\left[e^{-sA}\right] = \int_0^\infty e^{-sx} A(dx)$.

The paper is organized as follows. In the next section the main ideas of the regenerative approach and its developments will be reminded. All other sections are devoted to applications of above methods to investigations of the real world systems which have been considered early and recently that have been proposed without proofs. The paper ends with Conclusion where some further possible investigations are pointed out.

In the paper the following abbreviations are used.

- RPr — regenerative process,
- SRPr — semi-regenerative process,
- ERPr — embedded semi-regenerative process,
- DSRPr — decomposable semi-regenerative process,
- RT — regeneration time,
- SRT — semi-regeneration time,
- ERT — embedded regeneration time,
- RP —regeneration period,
- SRP —semi-regeneration period,
- ESRP — embedded semi-regeneration period,
- RS — regeneration state,
- SRS — semi-regeneration state,
- ESRS — embedded semi-regeneration state,
- SMM — semi-Markov matrix,
- ESMM — embedded semi-Markov matrix,
- RF — regeneration function,
- ERF — embedded regeneration function,
- MRM — Markov renewal matrix,
- EMRM — embedded Markov renewal matrix,
- ERK — embedded renewal kernel,
- QS — queueing system,

- NoC — number of calls,
- PP — poling process,
- IP — idle period,
- BP — busy period,
- MGF — moment generating ff-unction
- LT — Laplace transform,
- LST — Laplace-Stiltjes transform,
- i.i.d. — independent identically distributed,
- r.v. —random variable,
- c.d.f. — cumulative distribution function,
- t.d.p. — time-dependent probability,
- s.s.p. — steady state probability

2. On regenerative approach

Here the main results of regenerative and semi-regenerative processes are reminded. The so-called decomposable semi-regenerative processes are also under our attention. We omit very known strong definitions and represent the main results, which will be used in the paper.

2.1. Regenerative process

As a generalization of classical independence the *regeneration idea* has been proposed by W. Smith in 1955 [1]. Consider a stochastic process $X = \{X(t):\ t \geq 0\}$ with filtration $\mathcal{F}_t^X$. The process X is called the regenerative one if there exist a sequence of points of time, *regeneration times (RT)* S_n, in which the process forgets its past,

$$\mathbf{P}\{X(S_n + t) \in \Gamma\,|\mathcal{F}_{S_n}^X\} = \mathbf{P}\{X(S_n + t) \in \Gamma\} = \mathbf{P}\{X(S_1 + t) \in \Gamma\}.$$

The intervals $[S_{n-1}, S_n)$ and their length $T_n = S_n - S_{n-1}$ is called *regeneration periods* (RP). Note, that the functional elements $W_n = \{X(S_n + t),\, t \leq T_n\}$ are independent. They are called *regeneration cycles* (RC).

The regenerative approach allows to calculate the *regenerative process* (RPr) state probabilities $\pi(t; \Gamma) = \mathbf{P}\{X(t) \in \Gamma\}$ in terms of its state probabilities at the *separate regeneration period* $\pi^{(1)}(t, \Gamma) = \mathbf{P}\{X(S_{n-1} + t) \in \Gamma,\ t < T_n\}$ in the form

$$\pi(t; \Gamma) = \int_0^t H(du)\pi^{(1)}(t-u, \Gamma), \tag{1}$$

where $F(t) = \mathbf{P}\{T_n \leq t\}$ and

$$H(t) = \mathbf{E}\left[\sum_{n\geq 1} 1_{\{S_n \leq t\}}\right] = \sum_{n\geq 1} \mathbf{P}\{S_n \leq t\} = \sum_{n\geq 1} F^{*n}(t).$$

is the *renewal function* (RF) of the process, where symbol "$\star$" means a convolution, The RF satisfies the Winner-Hopf equation

$$H(t) = F(t) + F \star H(t) \equiv F(t) + \int_0^t F(du)H(t-u). \tag{2}$$

This approach allows not only to obtain the representation (1), but also to prove the existence of the stationary probabilities and give its close form representation in terms of the process distribution at separate regeneration periods. Namely it holds

$$\pi(\Gamma) = \lim_{t\to\infty} \pi(t;\Gamma) = \frac{1}{\mathbf{E}[T]} \int\limits_0^\infty \pi^{(1)}(t,\Gamma)dt. \tag{3}$$

2.2. Semi–regenerative process

As a generalization to the Markov's dependency, an idea of *semi-Markov chains* has been proposed by E. Cinlar (1969) [2] and J Jakod (1971) [3]. This idea leaded further to the constructions of *semi-Markov processes* (SMPr) (see, for example [3], [4]). The joining of these notions with the regeneration idea leaded to introduction of *semi-regenerative processes* (SRPr's), which firstly appeared under different titles: as *semi-Markov processes with additional trajectories* in 1966 (Klimov [5]), *regenerative processes with several types of regeneration poins* in 1971 (Rykov & Yastrebenetsky [6]) before it became the name SRPr due to E. Nummeline [7].

The difference of the SRPr from the RPr consists in the assumption that in its semi-regeneration points of time S_n the future of the process does not depend of its past but depends on its present state belonging to some set E of *regeneration states* (RS's),

$$\mathbf{P}\{X(S_n+t)\in\Gamma\,|\mathcal{F}^X_{S_n}\} = \mathbf{P}\{X(S_n+t)\in\Gamma\,|X(S_n)\} = \mathbf{P}\{X(S_1+t)\in\Gamma\,|X(S_1)\}.$$

The main characteristic of the SRP is its *semi-Markov matrix* (SMM) $Q(t) = [Q_{ij}(t)]_{ij\in E}$ with components

$$Q_{ij}(t) = \mathbf{P}\{X(S_n)=j,\ T_n\le t,\ |\ X(S_{n-1})=i\}.$$

For SRPr with denumerable RS's E that starts in RT with an initial distribution $\alpha=\{\alpha_i,\ i\in E\}$ the formula (1) takes the form

$$\pi(t;\Gamma) = \sum_{i\in E}\alpha_i\left[\delta_{ij}\pi_j^{(1)}(t) + \int\limits_0^t H_{ij}(du)\pi_j^{(1)}(t-u,\Gamma)\right]. \tag{4}$$

Here $\pi_j^{(1)}(t,\Gamma) = \mathbf{P}\{X(S_{n+1}+t)\in\Gamma,\ t<T_n\,|\,X(S_n+0)=j\}$ is the process state probability distribution on a separate semi-regeneration period (SRP) of type j, and $H(t)=[H_{ij}(t)]$ is its *Markov renewal matrix* (MRM) with

$$H_{ij}(t) = \mathbf{E}\left[\sum_{n\ge1} 1_{\{S_n\le t,\ X(S_n)=j\}}\,|\,X(0)=i\right] = \left[\sum_{n\ge1} Q^{*n}(t)\right]_{ij}.$$

these functions satisfy the Winner-Hopf equation, in which symbol "$\star$" means the matrix-functional convolution,

$$H(t) = Q(t) + Q\star H(t). \tag{5}$$

The corresponding limit theorem takes the form

$$\pi(\Gamma) = \lim_{t\to\infty}\pi(t,\Gamma) = \frac{1}{m}\int\limits_0^\infty \sum_{j\in E}\bar{\alpha}_j\pi_j^{(1)}(t,\Gamma)dt, \tag{6}$$

where $\bar{\alpha}=\{\bar{\alpha}_i\,(i\in E)\}$ represents the invariant probabilities of the embedded Markov chain $Y=\{Y_n=X(S_n),\ n=1,2,\dots\}$, and $m=\sum_{i\in E}\bar{\alpha}_i\mathbf{E}_i[T]$ is the expected stationary regeneration period.

2.3. Decomposable semi-regenerative process

If the process behavior at the separate regeneration period T_n is too complex and its distribution can not be analytically represented, sometimes it is possible to find some embedded regeneration points of time $S_k^{(1)}$ $(k = 1, 2, \dots)$ into this period, in which the process forgets its past up to the present state $X^{(1)}(S_k^{(1)})$ conditionally to its behavior at the regeneration period T_n. This subset of the process state is called *embedded semi-regeneration states* (ESRS's) and denoted by $E^{(1)}$. This process is called an *embedded regeneration process* (ERPr). Spreading out this procedure for all regeneration periods of the main process leads to construction of *decomposable semi-regenerative process* (DSRPr). This procedure can be extended to several embedding levels. The strong definitions and details can be found in V. Rykov (1975) [8] (see also [9], and [10]).

While analyzing the DSRPr, the role of the ordinary MRM plays the *embedded Markov renewal matrix (EMRM) $H(t) = [H_{ij}(t)]$*, which is given by its components

$$H_{ij}^{(1)}(t) = \mathbf{E}_i \left[\sum_{k\geq 0} 1_{\{[0,t),j\}} \left(S_k^{(1)}, X(S_k^{(1)}) \right) 1_{\{S_k^{(1)} < T\}} \right].$$

This matrix study depends on the type of the embedded regeneration points construction. There are different scenarios of their construction. If they arise as $\min[S_k^{(1)}, T]$, then unlike equations (2), and (5) for EMRM holds the following equation,

$$H^{(1)}(t) = Q^{(1)}(t) + H^{(1)} \star Q^{(1)}(t) - Q(t). \tag{7}$$

Here $Q(t)$ and $Q^{(1)}(t)$ are the SMM of the external and internal embedded periods and the symbol $\star$ denotes as before the matrix-functional convolution.

In most practical situations, both internal and external regeneration points of time coincide with the times of the regeneration states destinations. At that the external regeneration points of time are the moments, when the process exits the subset of the embedded regeneration states. In this case, the transition matrix for the embedded regeneration points of time $Q^{(1)}(t)$ is a sub-matrix of the matrix $Q(t)$ with components from the subset of the embedded states $E^{(1)}$ and therefore it is a degenerative one. In this case the equation for EMRM has the form

$$H^{(1)}(t) = Q^{(1)}(t) + Q^{(1)} \star H^{(1)}(t).$$

Its solution is

$$H^{(1)}(t) = (I - Q^{(1)}(t))^{-1} Q^{(1)}(t) = \sum_{n\geq 1} Q^{(1)*n}(t).$$

It is bounded for all t and approaches the expected number of visits to subset of states $E^{(1)}$ when $t \to \infty$. Naturally when the both scenarios are applicable, the solutions of the last equation coincide with the solution of equation (7).

Similarly to (4) different characteristics of the DSRPr of the first level can be expressed in terms of its corresponding characteristics of the second level. Particularly, for the one-dimensional distributions $\pi_i^{(1)}(t, \Gamma)$ the following representation holds

$$\pi_i^{(1)}(t, \Gamma) = H_i^{(1)} \star \pi^{(2)}(t, \Gamma).$$

Here

$$\pi_i^{(2)}(t, \Gamma) = \mathbf{P}\left\{ X\left(S_{k-1}^{(1)} + t\right) \in \Gamma,\ t < T_k^{(1)} \mid X^{(1)}(S_{k-1}^{(1)}) = i \right\}$$

are the process state probabilities in ERPr and EMRM $H^{(1)}(t)$ satisfies the equation (7). These relations make possible to recover the process distribution by its distribution on a separate minimal periods of embedded regeneration. The limit theorem for SRPr's allows to calculate its stationary distributions, and the system of embedded regeneration periods make possible to calculate them in terms of distributions on smallest regeneration period.

In the next sections several applications of the DSRPr's for investigation of some real word QS's and reliability models are considered.

3. M/GI/1/∞ Queueing System

We apply the DSRPr to investigation of the main processes in $M|G|1|\infty$ QS. Many authors deals with this system. A.Ya. Khinchin [11] found the s.s.p.'s of this system in 1932. D. Kendall [12] in 1953 studied this system with the help of method of embedded Markov chains. G. Klimov [5] uses the method of probabilistic interpretation of generating functions (GF). In next section we propose the results, given by V. Rykov in [10].

3.1. Number of calls as a regenerative process

Consider a $M|GI|1|\infty$ QS with Poisson input $L(t)$ and recurrent service process, where service times are i.i.d. r.v.'s B_n with common c.d.f. $B(t) = \mathbf{P}\{B_n \le t\}$. Denote by $X = \{X(t),\, t \ge 0\}$ the *number of calls* (NoC) process in the system. Evidently, it is DSRPr, and its RP's R consists of *idle period* (IP) Δ and *busy period* (BP) Π, $R = \Delta + \Pi$ (see figure 1), while its RT's are $S_n = \sum_{1\le i\le n} R_i$.

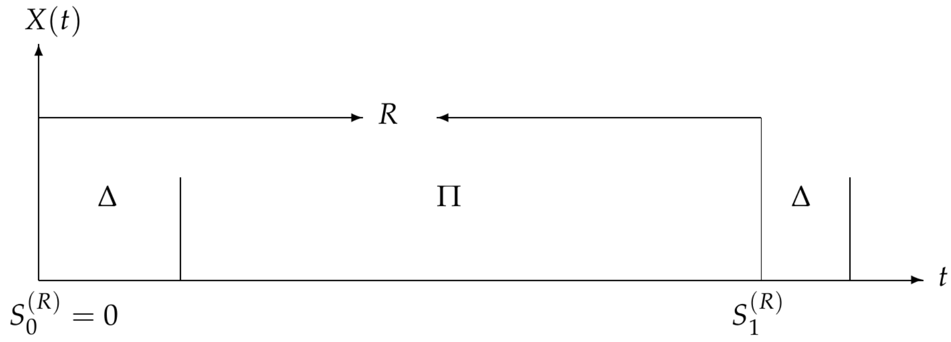

Figure 1: *The structure of a regeneration period*

The RP in turns consists of

- service time B_1 of the first arrived in free system call and
- random number $L(B_1)$ BP's, generated by this call.

Therefore, the BP satisfies to the following stochastic equation

$$\Pi = B_1 + \sum_{0\le i\le L(B_1)} \Pi_i, \tag{8}$$

and thus its MGF satisfied to the Kendall equation

$$\pi(s) = \beta(s + \lambda - \lambda\pi(s)). \tag{9}$$

while the MGF of the RP equals

$$\tau(s) = \mathbf{E}\left[e^{-sR}\right] = \mathbf{E}\left[e^{-s(\Delta+\Pi)}\right] = \frac{\lambda\pi(s)}{\lambda + s}. \tag{10}$$

If the system is idle in initial time, then the process behavior $X(t)$ in any time t can be represented in terms of its behavior at separate RP $X_R(t)$ as follows

$$X(t) = \sum_{n\ge 0} 1_{\{S_n \le t < S_{n+1}\}} X_R(t - S_n).$$

In terms of MGF Laplace transform (LT) of NoC $p(s,z)$ the last expression takes the form

$$\begin{aligned} p(s,z) &\equiv \int_0^\infty e^{-st}\mathbf{E}\left[z^{X(t)}\right] = \int_0^\infty e^{-su} \sum_{n\ge 1} d\mathbf{P}\{S_n \le u\} \int_u^\infty e^{-s(t-u)}\mathbf{E}\left[z^{X_R(t-u)}\right] dt = \\ &= p_R(s,z)\frac{s+\lambda}{s+\lambda-\lambda\pi(s)}. \end{aligned} \tag{11}$$

Here $p_R(s,z) = \int\limits_0^\infty e^{-st}\mathbf{E}\left[z^{X_R(t)}\right]$, and

$$\int\limits_0^\infty e^{-st}\sum_{n\geq 1} d\mathbf{P}\{S_n \leq t\} \equiv \int\limits_0^\infty e^{-st}dH(t) = \tilde{h}(s) = \frac{1}{1-\tau(s)} = \frac{s+\lambda}{s+\lambda-\lambda\pi(s)}.$$

is the Laplace-Stilties Transform (LST) of the RF $H(t)$, generated by RT's S_n.

3.2. NoC on a separate RP

The process $X_R(t)$ behavior on a separate RP in terms of its behavior on separate BP $X_\Pi(t)$ has the form

$$X_R(t) = \begin{cases} 0 & \text{for} \quad t < \Delta, \\ X_\Pi(t-\Delta) & \text{for} \quad \Delta \leq t < \Pi. \end{cases}$$

From here one can find the LT of the MGF of the process $X_R(t)$

$$p_R(s,z) = \int\limits_0^\infty e^{-st}\left[e^{-\lambda t} + \int\limits_0^t \lambda e^{\lambda v}\mathbf{E}z^{X_\Pi(t-v)}dv\right]dt = \frac{1+\lambda p_\Pi(1,s,z)}{s+\lambda}.$$

Jointly with (11) this gives a well known formula for LT MGF of NoC process for $M|GI|1|\infty$ QS in terms of appropriate characteristic at separate BP,

$$p(s,z) = p_R(s,z)\frac{s+\lambda}{s+\lambda-\lambda\pi(s)} = \frac{1+\lambda p_\Pi(1,s,z)}{s+\lambda-\lambda\pi(s)}. \tag{12}$$

For the NoC process on a separate BP, opening with only one call, when $S_0^{(\Pi)} = 0$, $X(S_0^{(\Pi)}) = 1$, it holds

$$X_\Pi(t) = X(S_{n-1}^{(\Pi)}) + L(t - S_{n-1}^{(\Pi)}), \quad \text{for} \quad S_{n-1}^{(\Pi)} \leq t < S_n^{(\Pi)},$$

where $L(t)$ is an input Poisson process and a sequence of embedded RT's $S_n^{(\Pi)}$ is given recursively

$$S_1^{(\Pi)} = B_1, \;\; S_{n+1}^{(\Pi)} = S_n^{(\Pi)} + 1_{\{X(S_n^{(\Pi)})>0\}}B_n.$$

For the LT of NoC MGF on a separate BP $p_\Pi(1,s,z)$ by calculation with the help of conditional expectation formula we get

$$p_\Pi(1,s,z) = \int\limits_0^\infty e^{-st}\mathbf{E}\left[z^{X_\Pi(t)}\right] = [z + h^\Pi(1,s,z)]\frac{1-\beta(s+\lambda-\lambda z)}{s+\lambda-\lambda z}. \tag{13}$$

Here $h^{(\Pi)}(1,s,z)$ denotes LT of ERF, generated by the sequence $S_n^{(\Pi)}$, jointly with the MGF of the process in these times

$$h^{(\Pi)}(1,s,z) = \mathbf{E}\left[\sum_{n\geq 1} e^{-sS_n^\Pi}z^{X(S_n^\Pi)}1_{\{X(S_n^\Pi)>0\}}\right].$$

Due to the expression for NoC at the end of service times

$$X(S_n^\Pi) = \begin{cases} X(S_{n-1}^\Pi) - 1 + L(S_n^\Pi - S_{n-1}^\Pi) & \text{for} \quad X(S_{n-1}^\Pi) \geq 1, \\ 0 & \text{for} \quad X(S_{n-1}^\Pi) = 0. \end{cases}$$

This function satisfies to the equation (details of its derivations see in [10])

$$h^{(\Pi)}(1,s,z) = \beta(s+\lambda-\lambda z) + z^{-1}h^{(\Pi)}(1,s,z)\beta(s+\lambda-\lambda z) - \pi(s).$$

Solution of this equation is

$$h(1,s,z) = z\frac{\beta(s+\lambda-\lambda z)-\pi(s)}{z-\beta(s+\lambda-\lambda z)}, \tag{14}$$

and its substitution in (13) for LT MGF of NoC at separate BP gives

$$p_\Pi(1,s,z) = z\frac{\pi(s)-z}{\beta(s+\lambda-\lambda z)-z} \times \frac{1-\beta(s+\lambda-\lambda z)}{s+\lambda-\lambda z}. \tag{15}$$

Thus, the last expression jointly with (12) allow to find the LT of the non-stationary MGF of the NoC process.

3.3. Stationary regime

For MGF of NoC process in stationary regime from (12) and taking into account that from (9) it follows that $\pi_1 = -\pi'(0) = b_1/(1-\rho)$ we obtain the very known Pollachek-Khinchin formula for stationary queue,

$$P(z) = \lim_{s\to 0} sp(s,z) = (1-\rho)\frac{(1-z)\beta(\lambda-\lambda z)}{\beta(\lambda-\lambda z)-z}.$$

4. Priority queueing systems $M_r/GI_r/1/\infty$

Priority QS arise in many applications. Such systems studied by many authors and by different methods: Klimov (1966) [5], Jaiswell (1968) [13], Gnedenko and all (1973) [14]), Klimov, Mishkoi (1979) [15]. The DSRPr method firstly has been applied for such systems investigation by V. Rykov in [8]. Here we shortly remind these results.

4.1. System description.

Consider a single-server QS $M_r/GI_r/1/\infty$ with r independent Poisson inputs $\vec{L}(t) = (L_1(t),\dots L_r(t))$ intensities λ_k $(k = \overline{1,r}$, with common intensity $\Lambda = \sum_{1\le k\le r}\lambda_k$. Service times are i.i.d. r.v. B_k $(k=\overline{1,r})$ with common for each type of calls c.d.f. $B_k(t) = \mathbf{P}\{B_k \le t\}$. The calls are served with priority discipline in such a manner that the calls of k-th type has a priority before calls of the $(k+1)-st,\ k=\overline{1,r-1}$. There are different types of priorities:

- head-of-the-line;
- preemptive. In this case there are several sub-cases:
  - preemptive resume priority,
  - preemptive repeat priority (with new independent realization of interrupted service times),
  - preemptive repeat priority (with the same realization of first represented service time),
  - preemptive loss priority.

Denote by

- $\vec{x} = (x_1, x_2, \dots, x_r)$ vector, k-th component of which means the number k-th type calls in the system;
- E the set of the system states;
- $\vec{X}(t) = (X_1(t),\dots,X_r(t))$ is the NoC process.

It is evident that under given assumption the process $\vec{X}(t)$ is an DSRPr, and its main RP is the same as for the $M/GI/1/\infty$ QS and consists from an idle Δ and a busy Π periods (see figure 1). Denote by

$$p(s,\vec{z}) = \int_0^\infty e^{-st}\mathbf{E}\left[\vec{z}^{\vec{X}(t)}\right] dt$$

LT of the process $\vec{X}(t)$ MGF. Because the RP structure for the system $M_r/GI_r/1/\infty$ is the same as for the system $M/GI/1/\infty$, LT of NoC process MGF coincide with analogous for the system $M/GI/1/\infty$ (12),

$$p(s,\vec{z}) = p_R(s,\vec{z})\frac{s+\lambda}{s+\lambda-\lambda\pi(s)} = \frac{1+\lambda p_\Pi(1,s,\vec{z})}{s+\lambda-\lambda\pi(s)}. \tag{16}$$

However now $\Pi = \Pi_r$ is a BP, during which all calls are served. This period consists of *k-periods* Π_k, during which all calls of k-th and above priority are served and *k-cycles* during which only one call k-th type and all calls higher priority are served.

4.2. Structure of NoC process in $M_r/GI_r/1/\infty$ QS.

The structure of these periods is shown in the figures 2 and 3

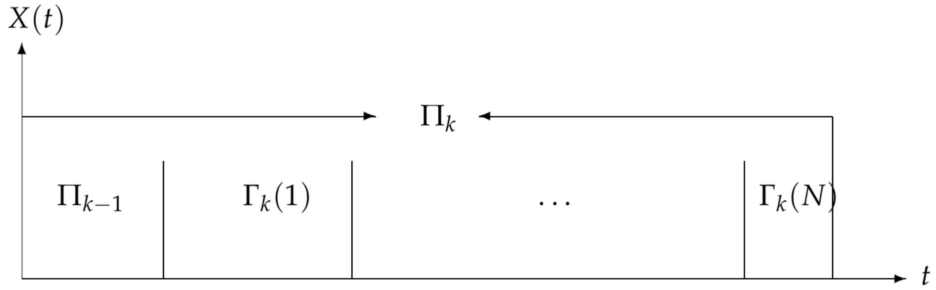

Figure 2: *The structure of k-period*

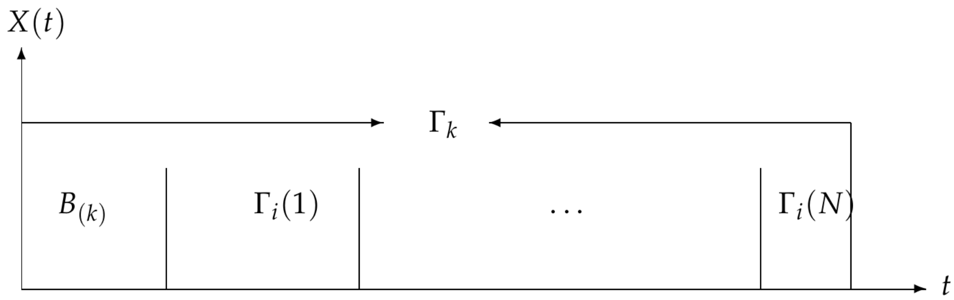

Figure 3: *The structure of k-cycle*

Thus, for calculation of appropriate LT of the MGF $p_\Pi(1,s,\vec{z})$ NoC at separate BP of the system and the MGF $\pi(s)$ of BP introduce the following notations

- $$\sigma_k = \sum_{1\le i\le k}\lambda_i, \quad v_k = \sum_{k\le i\le r}(\lambda_i - \lambda_i z_i), \quad V_k = s + v_k = \sigma_k(1-\pi_k(s+v_k));$$
- the LT MGF of NoC process at separate k-period

$$p_k(\vec{z},s) = \int_0^\infty e^{-st}\mathbf{E}\left[\vec{z}^{\vec{X}(t)} 1_{\Pi_k\le t}\right] dt;$$

- the LT MGF of NoC process at separate k-cycle

$$p^{\gamma_k}(\vec{z},s) = \int_0^\infty e^{-st}\mathbf{E}\left[\vec{z}^{\vec{X}(t)} 1_{\Gamma_k \le t}\right] dt;$$

- the MGF of k-period by $\pi_k(s) = \mathbf{E}\left[e^{-s\Pi_k}\right]$;
- the MGF of k-cycle by $\gamma_k(s) = \mathbf{E}\left[e^{-s\Gamma_k}\right]$.

With these notations using the considered above structure of embedded k-periods and k-cycles with the help of DSRP methods in [8] (see also [10]) the following recursive relation for the MGF LT of NoC of $M_r/GI_r/1/\infty$-priority system QS has been obtained

$$\sigma p_\Pi(\vec{z},s) = \sum_{1\le i\le r} \frac{\lambda_i z_i + \sigma_{i-1}\pi_{i-1}(s+v_i) - \sigma_i \pi_i(s+v_{i+1})}{z_i - \gamma_i(s+v_i)} p^{\gamma_i}(\vec{z},s). \tag{17}$$

Here the LT of NoC process MGF on a separate k-cycle $p^{\gamma_i}(\vec{z},s)$ for different type of priorities satisfies to the recursive relations

$$p^{\gamma_k}(\vec{z},s) = z_k \frac{1-\beta_k(s+v_1)}{s+v_1} + \sum_{1\le i\le k-1} \frac{\beta_k(V_i) - \beta_k(V_{i+1})}{z_i - \gamma_i(s+v_i)} p^{\gamma_i}(\vec{z},s). \tag{18}$$

This provides an algorithm for calculating of all basic characteristics of the system. For details see [8] and [10].

5. Polling Systems

Next important application of DSRP are the polling systems that have a wide sphere of applications. There is vast bibliography on this topic including monographs of Takagi (1986) [16], Borst (1996) [17], Vishnevsky, Semenova (2012) [18] and reviews of Takagi (1997) [19], Vishnevsky, Semenova (2006, 2021) [20], [21]. A general description of the polling model one can find in Fricker & Jabi (1994) [22], where also a stability conditions for the system were presented. Most of polling systems investigations deal with the system stationary regime at point of times when server attends users. Here by following [23] we show the possibility of the DSRP theory to be applied for the poling process (PP) investigation in continuous time.

5.1. The system description

Following to Fricker and Jabi (1994) [22], consider the following model (see Fig 4).

There are r users and a single server. Calls from k-th user (*k-calls*) form Poisson input of intensity λ_k. Therefore, $\vec{L}(t) = (L_1(t), \dots L_k(t))$ is the vector flow with summary intensity $\Lambda = \sum_{1\le k\le r} \lambda_k$. Service times are supposed to be i.i.d. r.v. B_k $(k = \overline{1,r})$ with common for k-calls c.d.f. $B_k(t) = \mathbf{P}\{B_k \le t\}$. Beside for switching from i-th user to the j-th one some random times C_{ij} with c.d.f. $C_{ij}(t) = \mathbf{P}\{C_{ij} \le t\}$ are needed. Note that for up-to-date telecommunication systems the service time has the same order as the switching time, therefore it is important to take into account the switching times. Thus, in order to optimize the system behavior it is useful to introduce some delay for service in such manner that the service of k-calls begins only after their number attains some level, say l_k.

The service consists of several, say $n \ge r$, *stages* that are determined by the *polling table* — function f,

$$f : \{1,2,\dots,n\} \to \{1,2,\dots,r\},$$

where $f(j) = k$ denotes that at the j-th stage the k-th user is served and the full round over all users are accomplished during n stages that composes a *cycle* of service. Note that for given

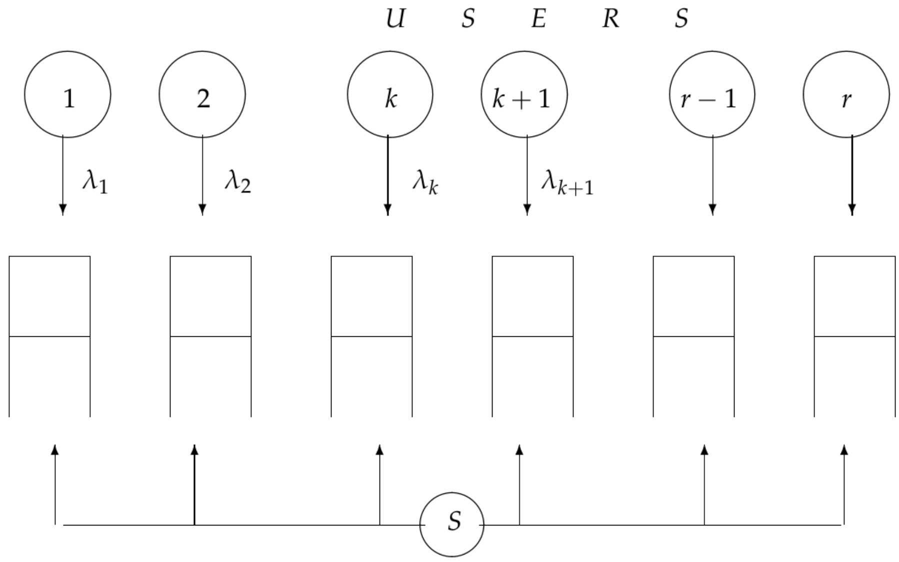

Figure 4: *The Polling model*

polling table the switching time C_j means switching times $C_{f(j),f(j+1)}$ from $f(j)$-th user to $f(j+1)$ one with c.d.f. $C_j(t)$.

There are different service disciplines $\delta(j)$ that could be used at different stages. Some of them are:

- *l-limited service discipline*, for which it is served fix, say l_k number of $f(j) = k$-calls, especially only one call if $l_k = 1$;
- *gated service discipline*, for which all $f(j)$-calls that are present at the very beginning of the stage are served during it;
- *exhaustive service discipline* for which the service of $f(j)$-th user is continued until the queue become empty.

It is supposed that
Assumption 1. All r.v.'s have at least *two finite moments*.
Assumption 2. the stability conditions for the system are fulfilled.

5.2. Structure of the Polling Process

For the Polling System investigation denote by $\vec{x} = (x_1, x_2, \ldots, x_r)$ the states of the polling system, where x_k is a number of k-calls, by E the set of all states, and consider a random process $\vec{X}(t) = \{(X_1(t), \ldots, X_r(t)) : \ t \geq 0\}$ with the states space E to which we will refer as a *polling process* (PP). It is evident that under given assumption the process $\vec{X}(t)$ is a DSRPr, and the structure of its main RP's are the same as for the $M|GI|1|\infty$ QS (see figure 1). However, the structure of its embedded busy periods now are different and in the following figures are represented (see Fig's 5, 6, 7). The BP consists of random number N service cycles G_j, each of which consists of n stages of $A_j \ \ (j = \overline{1, n})$ that also have enough complex structure, which depends on the service discipline at the stage.

5.3. Stochastic relations for Polling Process

Denote by $p(\vec{x}, t) = \mathbf{P}\{\vec{X}(t) = \vec{x}\}$ probability distribution of the PP $\vec{X}(t)$. From our assumptions it follows that the PP is usual multi-dimensional service process for $M_r/GI_r/1/\infty$ QS with r

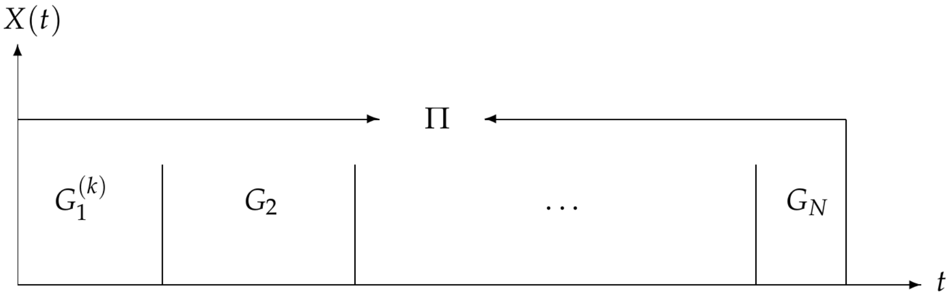

Figure 5: *The structure of a busy period*

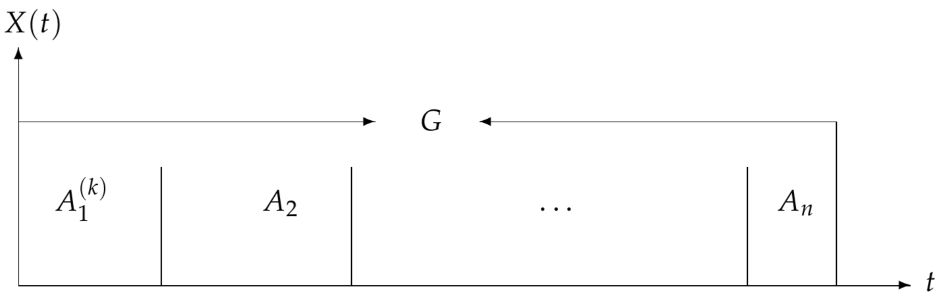

Figure 6: *The structure of a service cycle*

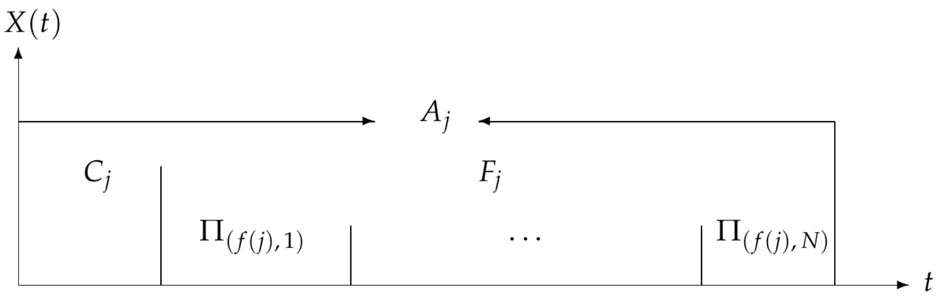

Figure 7: *The structure of a service stage*

types of calls, and therefore LT of the PP MGF accordingly to previous section has the form

$$p(s,\vec{z}) \equiv \int_0^\infty e^{-st}\mathbf{E}\left[\vec{z}^{\vec{X}(t)}\right] = \frac{1+\vec{\Lambda}'\vec{p}_\Pi(s,\vec{z})}{s+\Lambda-\vec{\Lambda}'\vec{\pi}(s)}, \tag{19}$$

where

$$\vec{\Lambda}'\vec{p}_\Pi(s,\vec{z}) = \sum_{1\le k\le r}\lambda_k p_\Pi(\vec{e},s,\vec{z}), \quad \vec{\Lambda}'\vec{\pi}(s) = \sum_{1\le k\le r}\lambda_k\pi_k(s).$$

Here

$$p_\Pi(\vec{e}_k,s,\vec{z}) = \int_0^\infty e^{-st}\mathbf{E}_{\vec{e}_k}\left[\vec{z}^{\vec{X}_\Pi(t)}\right]$$

is the LT MGF PP on a separate BP Π, opening with a single k-call, and $\pi_k(s) = \mathbf{E}_{e_k}\left[e^{-s\Pi}\right]$ is a MGF of a service cycle G, opening with a single k-call.

Therefore to find LT MGF PP $p(s,\vec{z})$ it is necessary to investigate appropriate functions at separate BP $p_\Pi(\vec{e}_k,s,\vec{z})$ and MGF $\pi_k(s)$ of the service cycle Π), opening with a single k-call and a number of calls in its end.

To calculate these functions consider the stochastic relations that associate the process behavior in any point of times and in points of times at separate embedded periods of regeneration, busy, cycles and stages of service which were denoted by

- $\vec{X}_R(t) = \{\vec{X}_R(t),\ t \le R\}$ is the PP at separate RP;
- $\vec{X}_\Pi(t) = \{\vec{X}_\Pi(t),\ t \le \Pi\}$ is the PP at separate BP;
- $\vec{X}_G(t) = \{\vec{X}_G(t),\ t \le G\}$ is the PP at separate Service Cycle; and
- $\vec{X}_j(t) = \{\vec{X}_j(t),\ t \le A_j\}$ is the PP at separate Service Stage.

Using the structure of the embedded RPr's the LT of the MGF of the PP in any time can be represented in terms of appropriate characteristics within the separate service stages. Using the process behavior at different separate stages the LT of MGF of the PP at them also can be calculated in closed form for any poling table and different service disciplines. The details can be found in [23].

6. $GI/GI/1/\infty$ Queueing System

The $GI/GI/1/\infty$ QS is a very interesting model both from theoretical and application point of views. The detailed study of this system one can find in the book of Cohen (1969) [24]. In this section we remind the results about application of DSRP method for investigation of one-server queueing system with recurrent input and generally distributed service time that has been done by Rykov (1983, 1984) [25], [26].

6.1. System description

Consider a $GI/GI/1/\infty$ QS with recurrent arrival and service processes. The system is a regenerative one, its RT's are the arrival times that find the system empty. For simplicity it is supposed that in the initial time $t = 0$ a new call arrive in the empty system Denote by

- A_n — inter-arrival times (i.i.d. r.v.),
- B_n — their service times (i.i.d. r.v.),
- R_k — k-th RP duration,
- Π_k — k-th BP duration,
- ν_k — number of calls, served during k-th BP,
- N_k — number of call, which open k-th RP,
- S_k — arrival times in empty system (RT's),
- $S^{(o)}_{n,k}$ — service completion times at separate k-th RP,
- $S^{(i)}_{n,k}$ — arrival times within separate k-th RP,

The above values are calculated recursively:

$$N_0 = 0, \quad N_k = N_{k-1} + \nu_k, \quad \nu_k = \min\{n :\ n > N_{k-1},\ S^{(o)}_{n,k} < S^{(i)}_{n,k}\},$$

$$S^{(o)}_{n,k} = \sum_{N_k < i \le n} B_i, \quad S^{(i)}_{n,k} = \sum_{N_k < i \le n} A_i.$$

$$\Pi_k = S^{(o)}_{\nu_k,k}, \quad R_k = S^{(i)}_{\nu_k,k}, \quad S_0 = 0, \quad S_k = S_[k-1] + R_k.$$

6.2. Main processes

The main processes of the $GI/GI/1/\infty$ QS are:

- Number of calls (NoC) in the system $X(t)$,
- Virtual waiting time (VWT) $V(t)$,
- Actual waiting time (AWT) W_n.

The first two processes are strongly (in sense of Smith) regenerative, which regeneration times S_k are the times of new calls arrivals in empty system. The discrete time AWT process is also regenerative sequence with respect to numbers N_k of calls arrived into empty system.

The structure of RP's for this system is the same as for the system $M/GI/1/\infty$. However behavior of the main processes at RP's is enough complex, and for their investigation we'll use the construction ERP's. The structure of regeneration and busy periods at the figure 8 are presented.

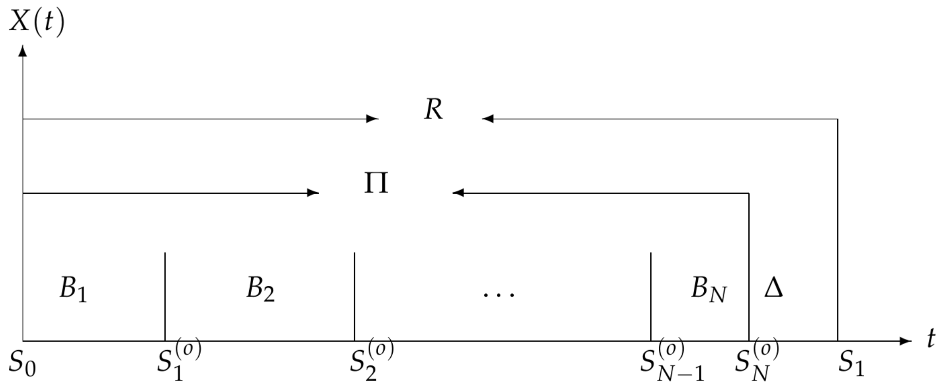

Figure 8: *Structure of the RP for the $GI/GI/1/\infty$-system*

Denote by

- $F(t) = \mathbf{P}\{R_n \leq t\}$ RP c.d.f.;
- $f(s) = \mathbf{E}[e^{-sR_n}] = \int\limits_0^\infty e^{st} F(dt)$ the RP MGF;
- $P(z,t) = \mathbf{E}[z^{X(t)}]$ MGF of the N0C process X;
- $p(z,s) = \int\limits_0^\infty e^{-st} P(z,t)dt$ its LT;
- $P^{(R)}(z,t) = \mathbf{E}\left[z^{X(t)}\,1\{t < R\}\right]$ process $X(t)$ MGF on a separate RP;
- $p(z,s) = \int\limits_0^\infty e^{-st} P^{(R)}(z,t)dt$ its LT;
- $V(x,t) = \mathbf{P}\{V(t) < x\}$ process $V(t)$ distribution;
- $v(x,s) = \int\limits_0^\infty e^{-st} V(x,t)dt$ its MGF;
- $V^{(R)}(x,t) = \mathbf{P}\{V(t) < x,\ t < R\}$ process $V(t)$ distribution on a separate RP;
- $v^{(R)}(x,s) = \int\limits_0^\infty e^{-st} V^{(R)}(x,t)dt$ its MGF within a separate RP;

- $H(t) = \sum\limits_{n\geq 1} \mathbf{P}\{S_n \leq t\} = \sum\limits_{n\geq 1} F^{(*n)}(t)$ the main processes RF.

According to the RPr's theory for LT the NoC and VWT processes it holds

$$\begin{aligned} p(z,s) &= p^{(R)}(z,s)(1+h(s)) = \frac{p^{(R)}(z,s)}{1-f(s)}; \\ v(x,s) &= s^{(R)}(x,s)(1+h(s)) = \frac{v^{(R)}(x,s)}{1-f(s)}; \end{aligned}$$

Analogous expressions take place for AWT process $\{W_n\}$ with respect to discrete RPr $\{N_k\}$, which discrete RF is

$$h_k = \mathbf{E}\left[\sum_{n\geq 0} 1_k(N_n)\right] = \sum_{n\geq 0} \mathbf{P}\{N_n = k\} = \sum_{n\geq 0} g^{*n}(k),$$

where $g(k) = \mathbf{P}\{N_1 = k\}$ is the distribution of the number of calls served at separate RP, and $g^{*n}(k)$ means the discrete n-th convolution of functions g at point k.

Therefore, to investigate the processes in any time point one should investigate them on a separate RP's. Because they are identically distributed it is enough to consider them at the first RP. To do that denote by $N = N_1 = \nu_1$ the number of calls served during the first RP and consider two-dimensional Random Walk $Y_n = (S_n^{(i)}, S_n^{(o)})$. For simplicity here and further the index of the RP number is omitted. It is evident that

$$N = \min\{n: \ S_n^{(o)} < S_n^{(i)}\}, \quad \Pi = S_N^{(o)}, \quad R = S_N^{(i)}.$$

6.3. Busy, idle periods and number of calls, served at BP

Denote by $(R_+^2, \mathcal{R}_+^2)$ the positive quadrant of an Euclid plane (t_1, t_2). Put

$$R_<^2 = \{(t_1, t_2): \ t_1 < t_2\}, \ R_\geq^2 = \{(t_1, t_2): \ t_1 \geq t_2\},$$

and introduce the sequence of measures

$$\begin{aligned} \pi_1(C) &= \mathbf{P}\{(A, B) \in C\} = \int\limits_{R_+^2} 1_C(t_1, t_2) A(dt_1) B(dt_2), \\ \pi_n(C) &= \mathbf{P}\{S_j^{(i)} < S_j^{(o)}: \ j = \overline{1, n-1}, \ (S_n^{(i)}, S_n^{(o)}) \in C\}. \end{aligned}$$

For $C \in R_\geq^2$ the value $\pi_n(C)$ is the probability of the event $\{N = n, \ (R, \Pi) \in C\}$,

$$\pi_n(C) = \mathbf{P}\{N = n, \ (R, \Pi) \in C\}.$$

So the parametric measure

$$\tilde{\pi}(z, C) = \sum_{n\geq 1} \pi_n(C) z^n$$

is the MGF of the number of calls served during RP jointly with RP and BP lengths. Many system characteristics can be expressed in terms of measure $\tilde{\pi}(z, C)$:

- For $C_x^1 = \{(t_1, t_2): \ t_2 - t_1 \geq x\}$ the value $1 - \tilde{\pi}(1, C_x^1)$ is server idle time distribution,

$$1 - \tilde{\pi}(1, C_x^1) = \mathbf{P}\{\Delta = R - \Pi \leq x\};$$

- For $C_x^2 = \{(t_1, t_2) \in R_\geq^2: \ t_1 < x\}$ the value $\tilde{\pi}(1, C_x^2)$ is the RP length distribution,

$$\tilde{\pi}(1, C_x^2) = \mathbf{P}\{R \leq x\};$$

- For $C_x^3 = \{(t_1,\, t_2) \in R^2_{\geq} : t_2 < x\}$ the value $\tilde{\pi}(1, C_x^3)$ is the BP length distribution

$$\tilde{\pi}(1, C_x^3) = \mathbf{P}\{\Pi \leq x\};$$

- For $C_x^0 = \{(t_1,\, t_2) \in R^2_{\geq} : t_2 > x > t_1\}$ one has

$$\tilde{\pi}(1,\, C_x^0) = \mathbf{P}\{\Pi \leq x < R\}.$$

In order to study the parametric measure $\tilde{\pi}(z,\, C)$ introduce a measure $q(c)$ and a kernel $Q(\omega, C)$ for $\omega \in R^2_+,\ C \in \mathcal{R}^2_+$ with $\mathbf{e}_1 = (1,0),\ \mathbf{e}_2 = (0,1)$ as follows:

$$\begin{aligned} q(C) &= \int\limits_{R^2_+} 1_C(t_1,\, t_2) A(dt_1) B(dt_2) = \pi_1(C); \\ Q(\omega, C) &= \int\limits_{R^2_+} 1_C(\omega + t_1\mathbf{e}_1,\, \omega + t_2\mathbf{e}_2) A(dt_1) B(dt_2). \end{aligned}$$

The following Theorem has been proved in Rykov (1983) [25].

Theorem 1. For $\rho = a^{-1}b < 1$ the function $\tilde{\pi}(z,\, C)$ is

(i) a probability measure on R^2_+ and a MGF with respect to z, i.e. $\tilde{\pi}(1,\, R^2_+) = 1$;

(ii) it satisfies to the equation

$$\tilde{\pi}(z,\, C) = z(q(C) + \tilde{\pi} o Q(C)), \tag{20}$$

where an operation $\tilde{\pi} o Q(C))$ means integration under the set $R^2_<$:

$$\tilde{\pi} o Q(C)) = \int\limits_{R^2_<} \tilde{\pi}(z, d\omega) Q(\omega, C).$$

■

It is possible to show that $Q(.,.)$ generates a continuous operator with the norm $\|Q\| = 1$, and so for $z < 1$ the equation (20) has a unique solution that could be represented in the form

$$\tilde{\pi}(z,\, C) = zqo \sum_{k \geq 0} z^k Q^{(ok)} = zqoG(z,\, C),$$

where

$$G(z, \omega, C) = \sum_{k \geq 0} z^k Q^{ok}(\omega,\, C). \tag{21}$$

Unfortunately, the condition $\|Q\| = 1$ does not allow directly to find the solution of this equation for $z = 1$ and does not guarantee its uniqueness. Nevertheless, the following theorem holds:

Theorem 2. (Rykov (1983) [25].) For $\rho = a^{-1}b < 1$ the kernel

$$G(z, \omega, C) = \mathbf{E}_\omega \left[\sum_{n \geq 0} z^n 1_{\{S_k^{(i)} < S_k^{(o)},\ k = \overline{1, n-1},\ (S_n^{(i)}, S_n^{(o)}) \in C\}} \right]$$

exists and is finite for any bounded sets $C \in \mathcal{R}^2_+$ and $|z| \leq 1$. Moreover, the representation (21) holds.

6.4. Investigation of the main processes on a separate BP

For investigation of VWT process on a separate BP consider the arrival epochs S_n^i as ERT's and introduce an ERK

$$U(\omega, C) = \mathbf{E}_\omega \left[\sum_{n<N} 1_C(V(S_n^{(i)}), S_n^{(i)}) \right],$$

that satisfies to the ERE

$$U(\omega, C) = Q^{(1)}(\omega, C) + U * Q^{(1)}(\omega, C) - Q(\omega, C).$$

Here Q and $Q^{(1)}$ are transition probability kernels of corresponding semi-Markov chains:

$$\begin{aligned} Q^{(1)}(\omega, C) &= \mathbf{E}_\omega 1_C(V(A), A) = \int_{R_+^2} 1_C(x+u-v, v) B(du) A(dv); \\ Q(\omega, C) &= \mathbf{E}_\omega 1_C(V(R), R) = \mathbf{P}\{(0, R) \in C\}. \end{aligned}$$

In the special case $\omega = 0$, ERK can be directly calculated

$$\begin{aligned} U(C) &= \mathbf{E}\left[\sum_{n<N} 1_C(V(S_n^i), S_n^i)\right] = \mathbf{E}\left[\sum_{n<N} 1_C(S_n^o - S_n^i), S_n^i)\right] = \\ &= \sum_{n\geq 0} \int_{R_<^2} \pi_n(du, dv) 1_C(v-u, u) = \int_{R_<^2} \tilde{\pi}(1, du, dv) 1_C(v-u, u). \end{aligned} \tag{22}$$

Therefore the DSRP theory leads to the following result.

Theorem 3. (Rykov (1984), [26]) C.d.f. $V^{(R)}(x,t)$ of VWT process for the $GI/GI/1/\infty$ QS on a separate RP is determined by the expression

$$\begin{aligned} V^{(R)}(x,t) &= (1-A(t))(1-B(x+t)) + \\ &+ \int_{u<t, v<t-u+x} U(dv, du)(1-A(t-u))(1-B(t-u+x-v)), \end{aligned} \tag{23}$$

where ERK satisfies the expression (22).

Analogous argumentation allow to calculate the distribution of an AWT process on a separate RP. A little bit more complex argumentation that include ERT's of the second level is used for calculation of NoC distribution on a separate RP.

The above theorems proofs and detailed investigation of the $GI|GI|1|\infty$ QS with the DSRPr's methods one can find in Rykov (1983, 1984) [25], [26].

In further two sections some recent applications of DSRP will be proposed.

7. Reliability of a double redundant renewable system

Consider a homogeneous cold double redundant repairable system with generally distributed life- and repair times, which, according to modified Kendall's notations [12], will be denoted as $< GI_2/GI/1 >$. The system consists of two identical units which can be in two possible states: operational and failed. The system fails when both units are in a failed state. For the repairable system, different strategies of renovation are possible. In this section we consider a strategy when after the system failure it continues to operate in the previous regime, and after the repair of a failed unit, it returns into the state one, where a new system cycle begins, where one unit starts working while the other one being to repair (see figure 9).

7.1. The problem setup: assumption and notations

Denote by A_i $(i = 1, 2, \dots)$ lifetimes of the system units, by B_i $(i = 1, 2, \dots)$, their repair times, and suppose that all these r.v. are mutually independent and identically distributed for each sequence. Thus, denote by $A(t) = \mathbf{P}\{A_i \le t\}$ and $B(t) = \mathbf{P}\{B_i \le t\}$ the corresponding c.d.f. Suppose that the instantaneous failures and repairs are impossible and their mean times are finite:

$$A(0) = B(0) = 0, \quad a = \int_0^\infty (1 - A(x))dx < \infty, \quad b = \int_0^\infty (1 - B(x))dx < \infty,$$

and in the initial time $t = 0$ both units are in good state.

Denote by $E = \{i = 0, 1, 2\}$ the set of system states, where i means the number of failed units, and introduce a random process $X = \{X(t),\, t \ge 0\}$, where

$$X(t) = \text{number of failed units at time } t.$$

Denote by F the time between system failures, and by by F_1 the time to first system failure (see Figure 9). Their c.d.f. are: $F(t) = \mathbf{P}\{F \le t\}$ and $F_1(t) = \mathbf{P}\{F_1 \le t\}$. We are interesting in calculation of:

- the reliability function $R(t) = \mathbf{P}\{F > t] = 1 - F(t)$;
- the distribution of the time to the first system failure $F_1(t)$;
- the system t.d.p. $\pi_j(t) = \mathbf{P}\{X(t) = j\}$ $(j = 0, 1, 2)$;
- the s.s.p. $\pi_j = \lim\limits_{t\to\infty} \pi_j(t) \equiv \lim\limits_{t\to\infty} \mathbf{P}\{X(t) = j\}$, $(j = 0, 1, 2)$;
- the availability coefficient $K_{\text{av.}} = \pi_0 + \pi_1 = 1 - \pi_2$.

The following notations will be used next:

- a modified LT:
$$\tilde{a}_B(s) = \int_0^\infty e^{-sx} B(x) dA(x), \qquad \tilde{b}_A(s) = \int_0^\infty e^{-sx} A(x) dB(x); \tag{24}$$
- the modified mean values:
$$a_B = -\frac{d}{ds}\tilde{a}_B(s)\big|_{s=0} = \int_0^\infty xB(x)dA(x), \quad b_A = -\frac{d}{ds}\tilde{b}_A(s)\big|_{s=0} = \int_0^\infty xA(x)dB(x); \tag{25}$$
- the probabilities $\mathbf{P}\{B \le A\}$ and $\mathbf{P}\{B \ge A\}$ associated with these transformations through the relations:
$$p \equiv \mathbf{P}\{B \le A\} = \tilde{a}_B(0), \quad q = 1 - p \equiv \mathbf{P}\{B > A\} = \tilde{b}_A(0).$$

Note the property of transformations (24)

$$\tilde{a}_{1-B}(s) = \tilde{a}(s) - \tilde{a}_B(s), \tag{26}$$

7.2. Reliability Function

Process X is a regenerative one. A trajectory of this process is illustrated on Figure 9. Here, F means the time between system failures. The variable G specifies the length of a RP according to a defined renovation policy. The following lemma holds for the LSTs of the time F between failures and the time to the first failure F_1.

Lemma 1. The LST $\tilde{f}(s) = \mathbf{E}\left[e^{-sF}\right]$ of the time F between failures and the LST $\tilde{f}_1(s) = \mathbf{E}\left[e^{-sF_1}\right]$ of the time to the first failure F_1 are of the form:

$$\tilde{f}(s) = \frac{\tilde{a}(s) - \tilde{a}_B(s)}{1 - \tilde{a}_B(s)}, \qquad \tilde{f}_1(s) = \tilde{a}(s)\frac{\tilde{a}(s) - \tilde{a}_B(s)}{1 - \tilde{a}_B(s)}. \tag{27}$$

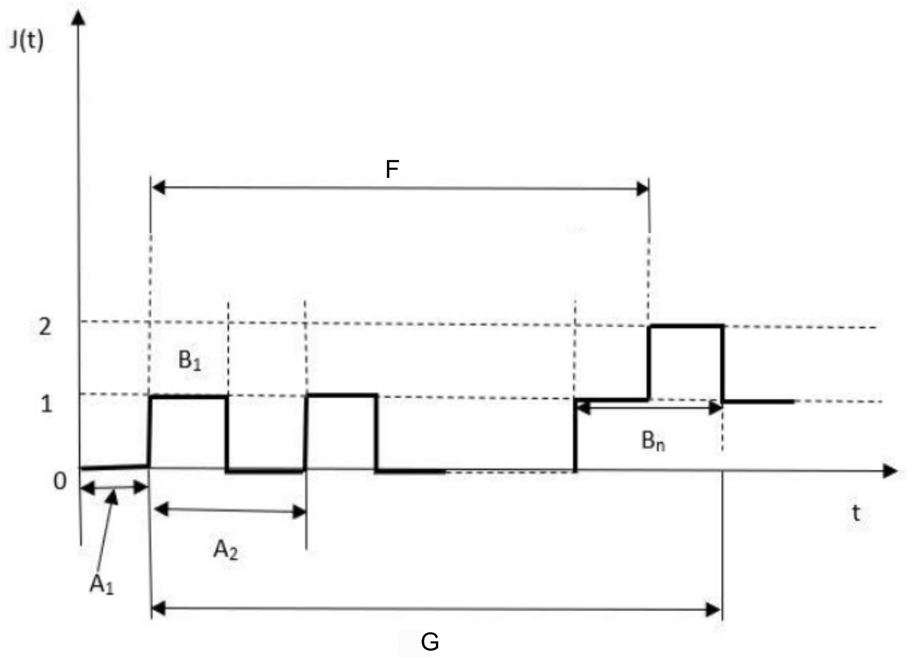

Figure 9: *Trajectory of the process X.*

Proof. From the Figure 9 can be seen that the system lifetime F satisfies the following stochastic equation:

$$F = \begin{cases} A + F & \text{if } B < A, \\ A & \text{if } B > A. \end{cases} \tag{28}$$

Applying the LT to this equation and taking into account notations (24), one can obtain

$$\tilde{f}(s) = \mathbf{E}\left[e^{-sF}\right] = \int_0^\infty e^{-st} dF(t) = \tilde{f}(s)\tilde{a}_B(s) + \tilde{a}_{1-B}(s). \tag{29}$$

From here the first relation in of (27) follows. The second one directly follows from the stochastic relation $F_1 = A + F$. ■

The main result of this subsection is the following theorem:

Theorem 4. The LT $\tilde{R}(s)$ of the system reliability function $R(t) = 1 - F(t)$ is

$$\tilde{R}(s) = \frac{1 - \tilde{a}(s)}{s(1 - \tilde{a}_B(s))}. \tag{30}$$

Proof. Taking into account that the LT of any c.d.f. is connected with its LST by the relation $\tilde{F}(s) = s^{-1}\tilde{f}(s)$, the proof follows directly from (27). ■

The expected system life time between failures $\mathbf{E}[F]$ and mean time to the first failure $\mathbf{E}[F_1]$ are obtained in the form:

$$\mathbf{E}[F] = \frac{a}{q}, \quad \mathbf{E}[F_1] = a + \frac{a}{q}.$$

7.3. Time dependent system state probabilities

For calculation of the system state t.d.p.'s the renewal theory is used. In our case, the process X is a RPr with a delay (see Figure 9) and its RT's are

$$S_0 = 0,\ S_1 = A_1,\ S_2 = S_1 + G_1,\ \ldots,\ S_{k+1} = S_k + G_k,\ \ldots.$$

Here, RP's G_i $(i = 1, 2, \dots)$ are the time intervals between two successive returns of the process X into state 1 after a system failure, when one of the system units begins to operate and the other one begins to repair. Thus, the process X state t.d.p.'s $\pi_j(t) = \mathbf{P}\{X(t) = j\}$ $(j = 0, 1, 2)$ can be represented in terms of its distribution on a separate RP $\pi_j^{(1)}(t) = \mathbf{P}[X(t) = j,\ t < G]$ $(j = 0, 1, 2)$ and the renewal function $H(t)$ as follows:

$$\pi_i(t) = \pi_i^{(1,1)}(t) + \int_0^t dH(u)\pi_j^{(1)}(t-u). \tag{31}$$

Here the distribution of the process X at the first RP is of the form

$$\pi_j^{(1,1)}(t) = \mathbb{P}[X(t) = j,\ t < A_1] = \delta_{j0}(1 - A(t)). \tag{32}$$

In terms of the c.d.f. $G(t) = \mathbf{P}\{G_i \le t\}$ of r.v.'s G_i, the RF $H(t)$ is determined as follows:

$$H(t) = \sum_{k\ge 1} \mathbf{P}\left[\left(\sum_{1\le i\le k} G_i\right) \le t\right] = \sum_{k\ge 1} G^{*k}(t). \tag{33}$$

Consider, first of all, the RP distribution.

Lemma 2. The LST of the RP is of the form:

$$\tilde{g}(s) = \mathbb{E}[e^{-sG}] = \frac{\tilde{b}_A(s)}{1 - \tilde{a}_B(s)}. \tag{34}$$

Proof. The RP is the time between two successive visits of the process to state 1 from the state 2, when two events begin simultaneously: operating of one unit and the repair the other one. Figure 9 shows that r.v. G satisfies the following stochastic equation:

$$G = \begin{cases} A + G & \text{if } A > B, \\ B & \text{if } A \le B. \end{cases} \tag{35}$$

Applying LST to this stochastic equation leads to the equation

$$\begin{aligned} \tilde{g}(s) &= \mathbb{E}\left[e^{-sG}\right] = \int_0^\infty e^{-st} dG(t) = \int_0^\infty dA(x)\left[B(x)e^{-sx}\tilde{g}(s) + \int_{y>x} e^{-sy} dB(y)\right] = \\ &= \tilde{g}(s)\int_0^\infty e^{-sx} B(x) dA(x) + \int_0^\infty e^{-sy} A(y) dB(y) = \tilde{g}(s)\tilde{a}_B(s) + \tilde{b}_A(s), \end{aligned} \tag{36}$$

which implies the expression (34) for the LST of the RP. ∎

By differentiation due to properties of $\tilde{a}_B(0)$ and $\tilde{b}_A(0)$ one can obtain the mean length of the RP as:

$$\mathbf{E}[G] = \frac{a_B + b_A}{q}.$$

Lemma 3. The LST of the system RF is given by

$$\tilde{h}(s) = \frac{\tilde{b}_A(s)}{1 - (\tilde{a}_B(s) + \tilde{b}_A(s))}. \tag{37}$$

Proof. From the renewal theory, it is well known (and follows from (33)) that the LST of the RF $H(t)$ is defined as $\tilde{h}(s) = \tilde{g}(s)(1 - \tilde{g}(s))^{-1}$. Thus, substitution of the expression (34) for $\tilde{g}(s)$ into this one leads to (37). ∎

Theorem 5. The LT's $\tilde{\pi}_j(s)$ of the process X state t.d.p's. $\pi_j(t)$ $(j = 0, 1, 2)$ are of the form:

$$\tilde{\pi}_j(s) = \delta_{j0}\frac{1 - \tilde{a}(s)}{s} + \frac{\tilde{b}_a(s)}{1 - (\tilde{a}_B(s) + \tilde{b}_A(s))}\tilde{\pi}_j^{(1)}(s), \ (j = 0, 1, 2), \tag{38}$$

where $\tilde{\pi}_j^{(1)}(s)$ $(j = 0, 1, 2)$ are the LTs of the t.d.p.'s $\pi_j^{(1)}(t)$ $(j = 0, 1, 2)$ in a separate RP. These probabilities will be calculated in he next subsection.

Proof. Applying LT to equation (31) and taking into account equation (32), one can obtain

$$\tilde{\pi}_j(s) = \delta_{j0}\frac{1 - \tilde{a}(s)}{s} + \tilde{h}(s)\tilde{\pi}_j^{(1)}(s). \tag{39}$$

A substitution into this equality of the expression (37) for $\tilde{h}(s)$ leads to (38). ■

7.4. The state probabilities on a separate RP

Now, we calculate of the process state t.d.p.'s on a separate RP. The probability $\pi_2^{(1)}(t)$ can be calculated easy for the main level RP.

Lemma 4. The LT of the second t.d.p. $\pi_2^{(1)}(t)$ in the main RP is given by

$$\tilde{\pi}_2^{(1)}(s) = \frac{\tilde{a}(s) - (\tilde{a}_B(s) + \tilde{b}_A(s))}{s(1 - \tilde{a}_B(s))}. \tag{40}$$

Proof. Due to representation of the RP by formula (35), and as it is shown in Figure 9, the event $\{X(t) = 2, \ t < G\}$ occurs if and only if either the event $\{A_1 \le t \le B_1\}$ occurs or the events $\{t > u = A_1 > B_1\}$ and $\{X(t-u) = 2, \ t - u < G\}$ occur. Thus, it holds

$$\pi_2^{(1)}(t) = \mathbb{P}[J(t) = 2, \ t < G] = \mathbb{P}[A \le t < B] + \int_0^t dA(u)B(u)\pi_2^{(1)}(t-u).$$

From this equation it follows the LST

$$\tilde{\pi}_2^{(1)}(s) = \int_0^\infty e^{-st}A(t)(1 - B(t))dt + \tilde{a}_B(s)\pi_2^{(1)}(s),$$

and

$$\tilde{\pi}_2^{(1)}(s) = \frac{1}{1 - \tilde{a}_B(s)}\int_0^\infty e^{-st}A(t)(1 - B(t))dt.$$

Calculating the integral in the last expressions by partial integration we get (40). ■

Since the calculation of the probabilities $\pi_j^{(1)}$ $(j = 0, 1)$ is not a trivial task, we intend to apply the theory of DSRP [6, 9, 10]. For this consider the process X as an ERPr at the time interval F, which ERT's are the random number $\nu = \min\{n : \ A_n < B_n\}$ of time epochs

$$S_1^{(1)} = A_1 1_{\{A_1 > B_1\}}, \ S_2^{(1)} = S_1^{(1)} + A_2 1_{\{A_1 > B_1, \ A_2 > B_2\}}, \ldots$$

up to the time, when the event $\{A_n \le B_n\}$ happens for first time. It means that the time interval $G^{(1)}$ between ERT's has a distribution $G^{(1)}(t) = A(t)$ and these epochs lie within the time interval F, which is determined by the equation (28).

According to this theory, the process distribution within the basic RP (RP of the first level) $\pi_i^{(1)}(t)$, similar to the equation (31), can be presented in terms of distributions in ERP's (RP's of the second level) $\pi_j^{(2)}(t)$ and ERF $H^{(1)}(t)$ in the following way:

$$\pi_j^{(1)}(t) = \pi_j^{(2)}(t) + \int_0^\infty dH^{(1)}(u)\pi_j^{(2)}(t-u), \; j = 0, 1, \tag{41}$$

where

$$\pi_j^{(2)}(t) = \mathbf{P}\{X(t) = j, \; t < G^{(2)}\}, \; j = 0, 1$$

is the process t.d.p.'s on a separate RP of the second level, and the ERF $H^{(1)}(t)$ satisfies the equation

$$H^{(1)}(t) = A(t) + \int_0^t dH^{(1)}(u)A(t-u) - F(t), \tag{42}$$

where $F(t)$ is the CDF of the time between system failures determined by its LST (27).

Similar to the basic case, the solution of equations (41) and (42) can be represented in terms of their LTs and LSTs:

$$\tilde{\pi}_j^{(2)}(s) = \int_0^\infty e^{-st}\pi_j^{(2)}(t)dt, \quad \tilde{h}^{(1)}(s) = \int_0^\infty e^{-st}dH^{(1)}(t).$$

The next lemma specifies connections between process distributions in the first and in the second level regeneration cycles in terms of their LTs.

Lemma 5. The LT's of the process t.d.p.'s of the first and in the second level RP's satisfy the relation

$$\tilde{\pi}_j^{(1)}(s) = \tilde{\pi}_j^{(2)}(s)\frac{\tilde{a}_B(s)}{1 - \tilde{a}_B(s)} \; j = 0, 1. \tag{43}$$

Proof. Applying LT to equation (41), we get

$$\tilde{\pi}_j^{(1)}(s) = (1 + \tilde{h}^{(1)}(s))\tilde{\pi}_j^{(2)}(s). \tag{44}$$

Due to (42), the LST $\tilde{h}^{(1)}(s)$ of the embedded renewal function $H^{(1)}(t)$ is of the form

$$\tilde{h}^{(1)}(s) = \tilde{a}(s) + \tilde{h}^{(1)}(s)\tilde{a}(s) - \tilde{f}(s),$$

which leads in turn to

$$\tilde{h}^{(1)}(s) = \frac{\tilde{a}(s) - \tilde{f}(s)}{1 - \tilde{a}(s)}.$$

Substitution of this relation into equation (44) and taking into account the expressions for $\tilde{f}(s)$ from (27) we get the result (40) that completes the proof. ■

We have to calculate now only the $\tilde{\pi}_j^{(2)}(s) \;\; (j = 0, 1)$.

Lemma 6. In notations (24), the LT's of the second level system state probabilities are:

$$\begin{aligned}
\tilde{\pi}_0^{(2)}(s) &= \frac{1}{s}(\tilde{a}(s) - (\tilde{a}_B(s) + \tilde{b}_A(s)));\\
\tilde{\pi}_1^{(2)}(s) &= \frac{1}{s}[1 - (\tilde{a}(s) + \tilde{b}(s)) + (\tilde{a}_B(s) + \tilde{b}_A(s))].
\end{aligned} \tag{45}$$

Proof. For probabilities $\tilde{\pi}_j^{(2)}(t) \;\; (j = 0, 1)$ from Figure 9, it follows that

- The event $\{X(t) = 0, \; t < G^{(1)}$ occurs if and only if $\{B \leq t < A\}$;
- The event $\{X(t) = 1, \; t < G^{(1)}\}$ occurs if and only if $\{t < B \leq A\}$, or if $\{t < A \leq B\}$.

Hence, the respective probabilities are

$$\begin{aligned}\pi_0^{(2)}(t) &= \mathbf{P}\{B < t < A\} = B(t)(1 - A(t)),\\ \pi_1^{(2)}(t) &= \mathbf{P}\{t < B < A\} + \mathbf{P}\{t < A < B\}.\end{aligned} \tag{46}$$

Calculation LT's of these expressions by partial integration in terms of notations (24) leads to (45), that ends the proof. The details of calculation one can find in [28]. ■

By combining all these results in [28], the process t.d.p. have been found. They are too cumbersome and are omitted here. We show here only the system s.s.p.'s. By using a Tauber theorem

$$\pi_j = \lim_{t\to\infty} \pi_j(t) = \lim_{s\to 0} s\tilde{\pi}_j(s) \tag{47}$$

in [28] for the process s.s.p.'s the following results have been obtained.

Theorem 6. The system state stationary probabilities are:

$$\pi_0 = 1 - \frac{b}{a_B + b_A}, \quad \pi_1 = \frac{a+b}{a_B + b_A} - 1, \quad \pi_2 = 1 - \frac{a}{a_B + b_A}. \tag{48}$$

For a Markov model $< M_2|M|1 >$, when $A(t) = 1 - e^{-\alpha t}, \quad B(t) = 1 - e^{-\beta t}$ this result coincides with those calculated by direct approach using Birth and Death process for the Markov case.

8. k-out-of-n system

In this section we apply the theory of DSRP to study of k-out-of-n : F model, which has applications in many real-world phenomena. There are many papers devoted to investigations of this model. A detailed review of previous investigations of the model one can find in [29]. Some special applications of this model to engineering problems in oil and gs industry one can find in [35] and [36].

8.1. Stating the problem. Notations

Consider k-out-of-n : F system, which can be considered as a reparable n-components reliability system in parallel that fails when k of its components fail. It is supposed that the life times of the systems' components are i.i.d. r.v.'s with common exponential distribution of parameter α. Failed components are repaired by a single facility. Repair times are i.i.d. r.v. $B_i \ \ (i = 1, 2, \dots)$ with the common c.d.f. $B(t) = \mathbf{P}\{B_i \le t\}$.

For the system study introduce the following notations:

- $E = \{0, 1, \dots k\}$ is the system set of states, where j means number of failed components and k is the system failure state;
- $\lambda_i = (n - i)\alpha$ intensity of one of components failure, when the system is in the state i;
- define the random process $X = \{X(t), \ t \ge 0\}$ by the relation

$$X(t) = j, \text{ if in time epoch } t \text{ the system is in the state } j \in E;$$

- system (and the process) t.d.s. probabilities $\pi_j(t) = \mathbf{P}\{X(t) = j\}$;
- the process s.s.p. $\pi_j = \lim\limits_{t\to\infty} \mathbf{P}\{X(t) = j\}$;
- the time T to the system failure, $T = \inf\{t : \ X(t) = k\}$;
- reliability function $R(t) = \mathbf{P}\{T > t\}$.

For repairable k-out-of-n : F system there exist at least two possible scenarios of the system repair after its fail:

- **Partial repair regime**, when after the system failure it continues to work in previous regime and after the repair of repaired component it passes to the state $k-1$;
- **Full repair regime**, when after the system failure the repair of whole system begins, after which the system becomes as a new one, and comes to the state 0. After each system failure the full repair duration are i.i.d. r.v. G_i $(i = 1, 2, \dots)$ with the common c.d.f. $G(t) = \mathbf{P}\{G_i \le t\}$.

Suppose that in the very beginning all system components are in good (UP) state, which means that the initial state of the process is zero, $X(0) = 0$. It is also supposed that immediate components failures and repairs are impossible and mean failure and repair times are finite,

$$B(0) = G(0) = 0, \quad \int_0^\infty (1 - B(t))dt < \infty, \quad \int_0^\infty (1 - G(t))dt < \infty. \tag{49}$$

Further in the section the system t.d.p. $\pi_j(t)$, reliability function $R(t)$, and the system s.s.p. π_j are calculated.

8.2. Partial repair regime.

Consider firstly partial repair regime, when after the system failure the repair of previously failed component is continued and after its end the system goes to the state $k-1$. In this case we will consider the process X as a semi-regenerative one (see figure 10). Its regeneration times S_n of the type j are the times of repair end when system occurs in state j, $X(S_n + 0) = j$, SRP's are $T_n = S_n - S_{n-1}$, and the RS's is $E_1 = \{j : (j = \overline{0, k-1})\}$.

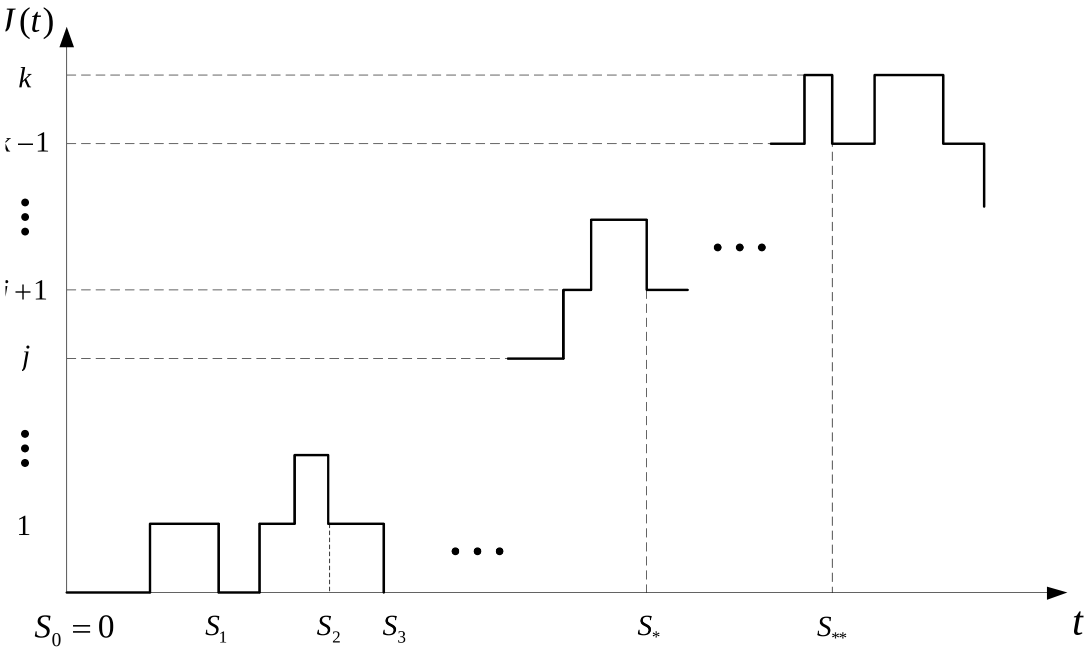

Figure 10: *Trajectory of the process J for system with partial repair*

The SMP X behavior is determined by its SMM $Q(t) = [Q_{ij}(t)]_{ij \in E_1}$ with transition probabilities

$$Q_{ij}(t) = \mathbf{P}\{X(S_n + 0) = j \; T_n \le t \,|X(S_{n-1} + 0) = i\}.$$

For its calculation denote by

$$p_{ij}(t) = \binom{n-i}{j-i}(1-e^{-\alpha t})^{j-i} e^{-(n-j)\alpha t}$$

the probability that during time t the process passes from the state i to the state j. Let

$$P_{ik}(t) = \sum_{j\geq k} p_{ij}(t) = 1 - \sum_{i\leq j\leq k-1} p_{ij}(t)$$

be the probability that starting from state i, the process leaves the subset of states E_1 during time t. Note that the last probability is the lifetime c.d.f. of the non-reparable k-out-of-$n:F$ system starting from the state i. To simplify further calculations, we represent these probabilities by Newton's binomial formula by using for simplicity the substitution $\lambda_i = (n-i)\alpha$ as

$$p_{ij}(t) = \binom{n-i}{j-i} e^{-\lambda_j t} \sum_{m=0}^{j-i} (-1)^m \binom{j-i}{m} e^{-\alpha m t}. \tag{50}$$

and

$$P_{ik}(t) = 1 - \sum_{i\leq j\leq k-1} \binom{n-i}{j-i} e^{-\lambda_j t} \sum_{m=0}^{j-i} (-1)^m \binom{j-i}{m} e^{-\alpha m t}. \tag{51}$$

Using these notations for the SMM in [29] the following lemma has been proved.

Lemma 7. The differentials of the process X SMMs components are:

$$\begin{aligned}
Q_{0j}(dt) &= \int_0^t \lambda_0 e^{-\lambda_0 u}\,du\; p_{1j+1}(t-u)B(dt-u), \;\; j=\overline{0,k-2};\\
Q_{0k-1}(dt) &= \int_0^t \lambda_0 e^{-\lambda_0 u}\,du\; P_{1k}(t-u)B(dt-u);\\
Q_{ij}(dt) &= p_{ij+1}(t)B(dt), \;\; (i=\overline{1,k-2}, \; j=\overline{i-1,k-2});\\
Q_{ik-1}(dt) &= P_{ik}(t)B(dt).
\end{aligned} \tag{52}$$

Their LST $\tilde{q}_{ij}(s) = \int_0^\infty e^{-st} Q_{ij}(dt)$ are:

$$\begin{aligned}
\tilde{q}_{0j}(s) &= \frac{\lambda_0}{s+\lambda_0}\binom{n-1}{j}\sum_{m=0}^{j}(-1)^m\binom{j}{m}\tilde{b}(s+\lambda_{j+1-m}), \;\; j=\overline{0,k-2};\\
\tilde{q}_{0k-1}(s) &= \frac{\lambda_0}{s+\lambda_0}\sum_{j\geq k}\binom{n-1}{j-1}\sum_{m=0}^{j-1}(-1)^m\binom{j-1}{m}\tilde{b}(s+\lambda_{j-m});\\
\tilde{q}_{ij}(s) &= \binom{n-i}{j-i+1}\sum_{m=0}^{j-i+1}(-1)^m\binom{j-i+1}{m}\tilde{b}(s+\lambda_{j+1-m});\\
\tilde{q}_{ik-1}(s) &= \sum_{j\geq k}\binom{n-i}{j-i}\sum_{m=0}^{j-i}(-1)^m\binom{j-i}{m}\tilde{b}(s+\lambda_{j-m}),
\end{aligned} \tag{53}$$

Remind now that the state t.d.p.'s of the process X are determined not only by its SMM but also by the initial distribution $\vec{\alpha}^{(0)} = \{\alpha_i^{(0)}: \; i\in E\}$. Thus, denoting by $\vec{\pi}(t) = \{\pi_j(t): \; j\in E\}$ the vector of the process state probabilities, where

$$\pi_j(t) = \mathbf{P}\{X(t)=j\} \;\; (j\in E),$$

and by $\Pi(t) = [\pi_{ij}(t)]_{ij\in E}$ the probability transition matrix of the process X, where

$$\pi_{ij}(t) = \mathbf{P}\{X(t)=j \,|\, X(0)=i)\} \;\; (i,j\in E)$$

in matrix form it means that $\vec{\pi}(t) = \vec{\alpha}^{(0)}\Pi(t)$.

On the other side, according to the theory of SRPr, the process transition probabilities $\Pi(t)$ in terms of appropriate transition probabilities $\Pi^{(1)}(t) = [\pi_{ij}^{(1)}(t)]_{ij\in E}$ where

$$\pi_{ij}^{(1)}(t) = \mathbf{P}\{X(S_{n-1}+t) = j,\ t \le T_n \mid X(S_{n-1}+0) = i)\}\ \ (i \in E_1,\ j \in E)$$

are the transition probabilities on separate RP's can be represented in the form

$$\Pi(t) = \Pi^{(1)}(t) + H \star \Pi^{(1)}(t). \tag{54}$$

here the MRM $H(t) = \left[H_{ij}(t)\right]_{ij\in E_1}$ with

$$H_{ij}(t) = \mathbf{E}\left[\sum_{n\ge1} 1_{\{S_n\le t,\ J(S_n)=j\}} \mid X(0) = i\right]$$

satisfies to the equations

$$H(t) = Q(t) + Q \star H(t). \tag{55}$$

The above results show that the best way for the system state t.d.p. representation and the equations for MRM solution is its representation in terms of LS and LST. Therefore passing to LT

$$\tilde{\Pi}(s) = \int_0^\infty e^{-st}\Pi(t)dt, \quad \tilde{\Pi}^{(1)}(s) = \int_0^\infty e^{-st}\Pi^{(1)}(t)dt,$$

and LST

$$\tilde{q}(s) = \int_0^\infty e^{-st}Q(dt), \quad \tilde{h}(s) = \int_0^\infty e^{-st}H(dt)$$

from equations (54) and (55) one can obtain the following results:

$$\tilde{\Pi}(s) = \tilde{\Pi}^{(1)}(s) + \tilde{h}(s)\cdot\tilde{\Pi}^{(1)}(s) \tag{56}$$

and

$$\tilde{h}(s) = \tilde{q}(s) + \tilde{q}(s)\cdot\tilde{h}(s). \tag{57}$$

At least the process t.d.p. on separate RP's given in the following lemma (its proof can be found in [29]):

Lemma 8. The process state t.d.p. on a separate SRP are:

$$\begin{aligned}
\pi_{00}^{(1)}(t) &= e^{-\lambda_0 t};\\
\pi_{0j}^{(1)}(t) &= \int_0^t \lambda_0 e^{-\lambda_0 u} p_{1j}(t-u)(1-B(t-u))du\ \ (j=\overline{1,k-1});\\
\pi_{0k}^{(1)}(t) &= \int_0^t \lambda_0 e^{-\lambda_0 u} P_{1k}(t-u)(1-B(t-u))du;\\
\pi_{ij}^{(1)}(t) &= p_{ij}(t)(1-B(t))\ \ (1\le i\le j\le k-1);\\
\pi_{ik}^{(1)}(t) &= P_{ik}(t)(1-B(t))\ \ (i=\overline{1,k-1}).
\end{aligned} \tag{58}$$

Their LT $\tilde{\pi}_{ij}^{(1)}(s)$ are:

$$\begin{aligned}
\tilde{\pi}_{00}^{(1)}(s) &= \frac{1}{s+\lambda_0};\\
\tilde{\pi}_{0j}^{(1)}(s) &= \frac{\lambda_0}{s+\lambda_0}\binom{n-1}{j-1}\sum_{m=0}^{j-1}(-1)^m\binom{j-1}{m}\frac{1-\tilde{b}(s+\lambda_{j-m})}{s+\lambda_{j-m}} \quad (j=\overline{1,k-1});\\
\tilde{\pi}_{0k}^{(1)}(s) &= \frac{\lambda_0}{s+\lambda_0}\sum_{j\geq k}\binom{n-1}{j-1}\sum_{m=0}^{j-1}(-1)^m\binom{j-1}{m}\frac{1-\tilde{b}(s+\lambda_{j-m})}{s+\lambda_{j-m}};\\
\tilde{\pi}_{ij}^{(1)}(s) &= \binom{n-i}{j-i}\sum_{m=0}^{j-i}(-1)^m\binom{j-i}{m}\frac{1-\tilde{b}(s+\lambda_{j-m})}{s+\lambda_{j-m}}, \quad (1\leq i\leq j\leq k-1);\\
\tilde{\pi}_{ik}^{(1)}(s) &= \sum_{j\geq k}\binom{n-i}{j-i}\sum_{m=0}^{j-i}(-1)^m\binom{j-i}{m}\frac{1-\tilde{b}(s+\lambda_{j-m})}{s+\lambda_{j-m}}.
\end{aligned} \tag{59}$$

By joining the above results the following theorem follows

Theorem 7. The LT of the process X state t.d.p. in matrix form given by the equality

$$\tilde{\Pi}(s) = (I-\tilde{q}(s))^{-1}\tilde{\Pi}^{(1)}(s), \tag{60}$$

where components $\tilde{\pi}_{ij}^{(1)}(s)$ of the matrix $\tilde{\Pi}^{(1)}(s)$ are given by formulas (59) in lemma 8, and components $\tilde{q}_{lj}(s)$ of matrix $\tilde{q}(s)$ given by the formula (52) from lemma 7. ■

The s.s.p.'s of the process could be calculated by passing to limits as $t\to\infty$ in the last equality. But it would be preferable to use the limit theorem for transition probabilities of SRPr's.

Theorem 8. The stationary regime of the considered system under partial repair regime exists and its s.s.p.'s equal

$$\pi_j = \frac{1}{m}\sum_{0\leq l\leq j\wedge(k-1)}\alpha_l\tilde{\pi}_{lj}^{(1)}(0) \qquad (j=\overline{0,k}), \tag{61}$$

where $m=\lambda_0^{-1}(\alpha_0+\lambda_0 b)$, $\tilde{\pi}_{ij}^{(1)}(0)$ can be found from the formulas (59) of lemma 8, and $\vec{\alpha}=\{\alpha_l:\ l\in E\}$ is the invariant probability measure of the embedded Markov chain that satisfies the system of equations

$$\vec{\alpha}' = \vec{\alpha}'\tilde{q}(0), \qquad \sum_{l\in E}\alpha_l = 1.$$

■

8.3. Full repair regime.

For study of the system behavior under the full repair regime we will consider the main process X as a regenerative one, whose regeneration points of time $S_n\ (n=1,2,\dots), S_0=0$ are the times when the system fully repaired after its failure. The regeneration periods $\Theta_n = S_n - S_{n-1}$ of the process X consist of two terms: the system life times (times to the system failure after its repair) F_n and the system repair times after its failure $G_n:\ \Theta_n = F_n + G_n$ (see figure 11). Denote by $F(t)=\mathbf{P}\{F_n\leq t\}$ and $\Gamma(t)=\mathbf{P}\{\Theta_n\leq t\}$ the common distribution function of r.v.'s F_n and $\Theta_n\ (n=1,2,\dots)$.

From the RPr's theory in sense of Smith it follows that the process t.d.p. $\pi_j(t)=\mathbf{P}\{X(t)=j\}$ can be represented in terms of corresponding process t.d.p. $\pi_j^{(\Theta)}(t)=\mathbf{P}\{X(t)=j,\ t<\Theta\}$ at separate RP Θ as follows

$$\pi_j(t) = \pi_j^{(\Theta)}(t) + \int_0^t \pi_j^{(\Theta)}(t-u)H(du). \tag{62}$$

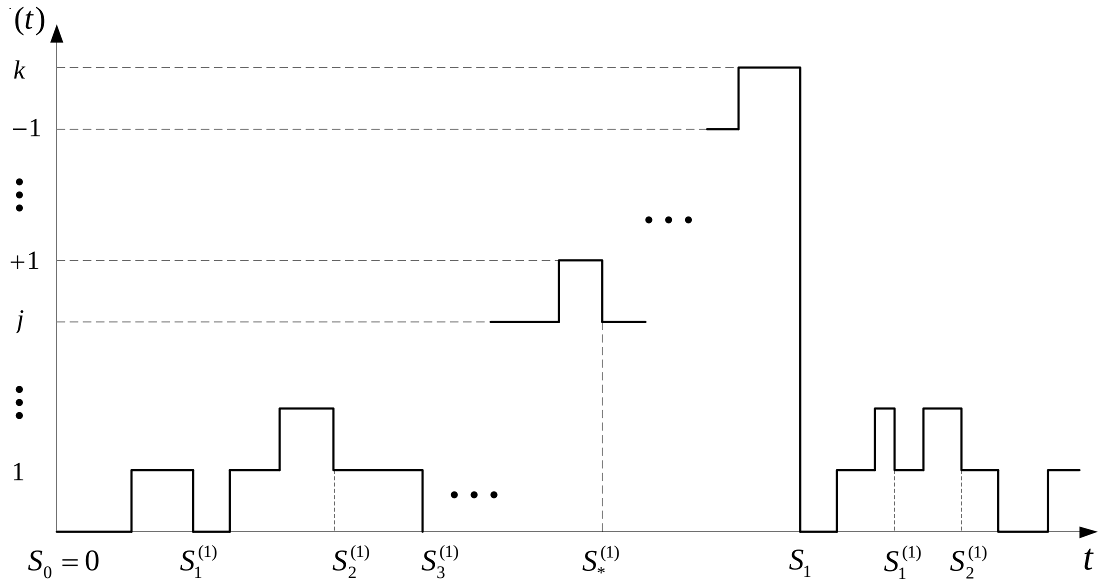

Figure 11: *The main process as a regenerative one.*

Here $H(t)$ is the appropriate RF, which is generated by the distribution $\Gamma(t) = \mathbf{P}\{\Theta_n \le t\}$ of r.v's. Θ_n, and can be calculated as

$$H(t) = \sum_{n\ge1} \mathbf{P}\{S_n \le t\} = \sum_{n\ge1} \Gamma^{(*n)}(t). \tag{63}$$

Its LST equals

$$\tilde{h}(s) = \int_0^\infty e^{-st} H(dt) = \frac{\tilde{\gamma}(s)}{1-\tilde{\gamma}(s)}, \tag{64}$$

where $\tilde{\gamma}(s)$ is the LST of the c.d.f. $\Gamma(t)$.

Thus, for the process analysis we need firstly to calculate the RP Θ_n distribution and the process distribution $\pi_j^{(\Theta)}(t)$ on them. Remind that the r.v. Θ_n is the sum of two independent r.v.'s F_n and G_n, the distributions of the second one is supposed to be known and the distribution of the first one will be done jointly with its LST later in lemma 10 (see Corollary 1 from it).

Let us turn now to calculation of the process t.d.p.'s $\pi_j^{(\Theta)}(t)$ on a separate RP. The process behavior on the separate RP's $\Theta_n = F_n + G_n$ can be divided into two parts: as a process behavior at separate system lifetime F_n and its behavior during the repair time G_n,

$$\{X(S_{n-1}+t) = j,\ t \le \theta_n\} = \begin{cases} X(S_{n-1}+t) = j & \text{for } t \le F_n \ \ (j \ne k), \\ k & \text{for } F_n < t \le \Theta_n. \end{cases} \tag{65}$$

Thus, the process t.d.p.'s on at a separate RP are given in the following lemma.

Lemma 9. The process state t.d.p.'s at the separate RP Θ in terms of according probabilities on separate system life times F are

$$\pi_j^{(\Theta)}(t) = (1-\delta_{jk})\pi_j^{(F)}(t) + \delta_{jk}\mathbf{P}\{F \le t < \Theta\} \tag{66}$$

and have LT

$$\tilde{\pi}_j^{(\Theta)}(s) = (1-\delta_{jk})\tilde{\pi}_j^{(F)}(s) + \delta_{jk}\tilde{f}(s)\frac{1-\tilde{g}(s)}{s}. \tag{67}$$

Proof. The proof follows directly from the relation (65) and details can be found in[29]. ■

The last equality shows that we need in the system lifetime F distribution and the process distribution on it. For this, we turn to the process X analysis on separate system lifetime. Since the process behavior on the system lifetimes F_n is rather complicated, the process X behavior within any of them will be considered as an ESRP $X_n^{(1)} = \{X_n^{(1)}(t) : \ t \geq 0\}$ with

$$X_n^{(1)}(t) = X(S_{n-1} + t), \quad t \leq F_n.$$

Its ESRT $S_l^{(1)}$ of the type j are the times of any repair ends (inside a separate system lifetime F_n) that find the system in state j. They are the same as for the SRP in the subsection 8.2 except the fact that right now the process $X_n^{(1)}$ is considered on a separate system lifetime and therefore never occurs in state $k-1$ after the repair ends. Thus there are only $k-1$ ERS's, $E^{(1)} = \{0, 1, \ldots, k-2\}$.

To study the process behavior on separate system lifetime denote by

- $T_l^{(1)} = S_l^{(1)} - S_{l-1}^{(1)}, \ l = 1, 2, \ldots$ the intervals between ESRT's of the ESRP $X^{(1)}$ (the times between repair ends);
- $Q^{(1)}(t) = [Q_{ij}^{(1)}(t)]_{ij \in E^{(1)}}$ ESMM, which components are the process transition probabilities between ESRT,
$$Q_{ij}^{(1)}(t) = \mathbf{P}\{X^{(1)}(S_l^{(1)} + 0) = j, \ T_l^{(1)} \leq t \,|\, X^{(1)}(S_{l-1}^{(1)} + 0) = i\};$$
- $H^{(1)}(t) = [H_{ij}^{(1)}(t)]_{ij \in E^{(1)}}$ EMRM, which components are the conditional ERF's on a separate lifetime period
$$H_{ij}^{(1)}(t) = \mathbf{E}\left[\sum_{l \geq 1} 1_{\{X^{(1)}(S_l^{(1)}+0) = j, \ S_l^{(1)} \leq t\}} \,|\, X^{(1)}(S_0^{(1)}) = i\right].$$

We start with calculation of the ESMM $Q^{(1)}(t) = [Q_{ij}^{(1)}(t)]_{ij \in E^{(1)}}$ of the ESRP $X^{(1)}$. For this note that its components coincide with those from subsection 8.2 except the fact that now the process $X^{(1)}$ never falls in state $k-1$ after the repair end. Thus they are defined only for $j \leq k-2$ and in terms of notations (50, 51) are represented in differential forms in lemma 7 by formulas (52, 53). Now, since the set of ERS's is a proper subset of the process states, the ESMM is a degenerative matrix in contrast to the matrix of the previous section. Hence, we are ready to represent some useful characteristics of the model. Introduce:

- the vector-function $\vec{F}(dt) = [F_{ik}(dt)]$, components of which are the differentials of the c.d.f.'s of the absorbing state k destination time by the ESRP starting from state i $(i = \overline{0, k-2})$;
- the vector-function $\vec{Q}^{(1)}(t) = [Q_{ik}^{(1)}(t)]$, components of which are differentials f the c.d.f.'s of the absorbing state k destination time by the ESRPr starting from state i along a monotone trajectory.

The components of the last one analogous to lemma 7, satisfy to the expressions

$$\begin{aligned}
Q_{0k}^{(1)}(dt) &= \int_0^t \lambda_0 e^{-\lambda_0 u} \, du \ P_{1k}(t-u) B(dt-u), \\
Q_{ik}^{(1)}(dt) &= P_{ik}(t) B(dt).
\end{aligned} \tag{68}$$

Their LST $\tilde{q}_{ij}^{(1)}(s)$ are

$$\begin{aligned}
\tilde{q}_{0k}^{(1)}(s) &= \frac{\lambda_0}{s+\lambda_0} \sum_{j \geq k-1} \binom{n-1}{j-1} \sum_{m=0}^{j-1} (-1)^m \binom{j-1}{m} \tilde{b}(s + \lambda_{j-m}); \\
\tilde{q}_{ik}^{(1)}(s) &= \sum_{j \geq k-1} \binom{n-i}{j-i} \sum_{m=0}^{j-i} (-1)^m \binom{j-i}{m} \tilde{b}(s + \lambda_{j-m}).
\end{aligned} \tag{69}$$

For the vector $\vec{F}(t)$ the following representation holds.

Lemma 10. The vector $\vec{F}(t)$ satisfies to the equation

$$\vec{F}(dt) = \vec{Q}^{(1)}(dt) + Q^{(1)} \star \vec{F}(dt), \tag{70}$$

whose unique solution in terms of LST is

$$\tilde{\vec{f}}(s) = (I - \tilde{q}^{(1)}(s))^{-1}\tilde{\vec{q}}^{(1)}(s). \tag{71}$$

Proof. The first equation is obtained with the help of the complete probability formula and its solution in terms of LST is evident. ■

From this lemma some useful corollaries follow.

Corollary 1. The first component $F_{0k}(t)$ of vector $\vec{F}(t)$ is c.d.f. of time to the first (and between) failures for system starting from state 0. The first component $\tilde{f}_{0k}(s)$ of vector $\tilde{\vec{f}}(s)$ is the MGF of the respective times. For simplicity we will denote them without indexes as it was done before $\tilde{f}(s) \equiv \tilde{f}_{0k}(s)$. ■

Corollary 2. Since the process regeneration cycle Θ equals to the sum of two independent r.v.'s: time to the system failure and its repair time its MGF is

$$\tau(s) \equiv \mathbf{E}\left[e^{-s\Theta}\right] = \tilde{f}(s)\tilde{g}(s). \tag{72}$$

■

Now the expression (64) leads to the following corollary.

Corollary 3. The LST $\tilde{h}(s)$ of the RF $H(t)$ of the system operating in the full repair regime equals to

$$\tilde{h}(s) = \frac{\tilde{f}(s)\tilde{g}(s)}{1 - \tilde{f}(s)\tilde{g}(s)}. \tag{73}$$

■

Taking into account that

$$R(t) = 1 - F(t) = 1 - \int_0^t f(u)du$$

one can obtain the following corollary

Corollary 4. The LT of the reliability function of the system is

$$\tilde{R}(s) = \frac{1}{s}(1 - \tilde{f}(s)). \tag{74}$$

■

To study the process behavior during a separate system lifetime cycle we consider

- the matrix $\Pi^{(F)}(t) = [\Pi^{(F)}_{ij}(t)]_{ij\in E^{(1)}}$, which components are the transition probabilities of the process on a separate life-cycle,

$$\Pi^{(F)}_{ij}(t) = \mathbf{P}\{X^{(1)}_n(t) = j,\ t < F_n \mid X^{(1)}_n(0) = i\};$$

- the matrix $\Pi^{(1)}(t) = [\Pi^{(1)}_{ij}(t)]_{ij\in E^{(1)}}$, which components are the transition probabilities of the process on a separate ESRP (between successive repair ends),

$$\Pi^{(1)}_{ij}(t) = \mathbf{P}\{X^{(1)}_n(S^{(1)}_{l-1} + t) = j, t < T^{(1)}_l \mid X^{(1)}_n(S^{(1)}_{l-1} + 0) = i\}.$$

In terms of these notations and according to the DSRP theory the following representations take place (symbol $\star$ here means matrix-functional convolution)

$$\Pi^{(F)}(t) = \Pi^{(1)}(t) + H^{(1)} \star \Pi^{(1)}(t). \tag{75}$$

In terms of the LT of matrices $\Pi^{(F)}(t)$, $\Pi^{(1)}(t)$ and LST of matrix $H^{(1)}(t)$ this equation take the form

$$\tilde{\Pi}^{(F)}(s) = \tilde{\Pi}^{(1)}(s) + \tilde{h}^{(1)}(s) \cdot \tilde{\Pi}^{(1)}(s). \tag{76}$$

In our case

$$H^{(1)}(dt) = Q^{(1)}(dt) + Q^{(1)} * H^{(1)}(dt),$$

and therefore

$$\tilde{h}^{(1)}(s) = (I - \tilde{q}^{(1)}(s))^{-1}\tilde{q}^{(1)}(s).$$

From here it follows

$$I + \tilde{h}^{(1)}(s) = I + \sum_{l \geq 1} \tilde{q}^{(1)*l}(s) = (I - \tilde{q}^{(1)}(s))^{-1},$$

and

$$\tilde{\Pi}^{(F)}(s) = \tilde{\Pi}^{(1)}(s) + \tilde{h}^{(1)}(s) \cdot \tilde{\Pi}^{(1)}(s) = (I - \tilde{q}^{(1)}(s))^{-1} \cdot \tilde{\Pi}^{(1)}(s). \tag{77}$$

The system state transition probabilities on a separate embedded repair time $\Pi^{(1)}(t)$ coincide for the embedded set of states $E^{(1)}$ with the corresponding probabilities as in the section 8.2 and represented in the formula (58) in lemma 8. The above results can be summing as follows.

Theorem 9. The LT $\tilde{\pi}_j(s)$ of the process X starting from the state zero state t.d.p.'s equals to

$$\tilde{\pi}_j(s) = \frac{1}{1 - \tilde{f}(s)\tilde{g}(s)} \begin{cases} \tilde{\pi}_j^{(F)}(s) & \text{for } \; j = \overline{1, k-1}, \\ \tilde{f}(s)\frac{1-\tilde{g}(s)}{s} & \text{for } \; j = k, \end{cases} \tag{78}$$

where $\tilde{f}(s)$ is defined by the first component of the vector $\tilde{\tilde{f}}(s)$ from corollary 1, which is represented by formula (71). The LT of time-dependent process state probabilities on a separate process lifetime period $\tilde{\pi}_j^{(F)}(s)$ is the first row of matrix $\tilde{\Pi}^{(F)}(s)$, which can be calculated from (76, 59).

Remark 1. Since any main RP begins with the state 0, for calculation of the process state t.d.p.'s as well as s.s.p.'s we need only in probabilities with the initial zero state.

Remark 2. The representation of the final results in the initial system information in general are too cumbersome and it needs additional study for concrete situations.

From this theorem by using the Smith's key renewal theorem one can obtain the stationary process probabilities.

Theorem 10. The s.s.p. π_j of the process X, starting from any state, equals to

$$\pi_j = \frac{1}{\mathbf{E}[F] + \mathbf{E}[G]} \begin{cases} \tilde{\pi}_j^{(F)}(0) & \text{for } \; j = \overline{1, k-1}, \\ \mathbf{E}[G] & \text{for } \; j = k, \end{cases} \tag{79}$$

where $\mathbf{E}[F] = -\tilde{f}'(0)$ is the expected system lifetime. It can be found from the formula (71), and the values $\tilde{\pi}_j^{(F)}(0)$ are the components of first row of matrix $\tilde{\Pi}^{(F)}(0)$, which can be calculated from (76, 59).

9. Conclusion

A review of the Smith's regeneration idea development is proposed in this paper. It is shown that the DSRPr can be used as a useful method for study of different stochastic models. Several previous and two recent results of application this method are demonstrated. For all considered systems the state t.d.p.'s and the s.s.p.'s are represented in terms of respective probabilities on a separate ERP. Some other applications for complex hierarchical system investigations can be find in [30], [31], [32].

The proposed approach allows to obtain analytical expressions of main quality of service characteristics for various complex stochastic models. Presence of analytical results allows to propose more detailed analysis of such systems. Especially these results can be used for further investigation of output systems characteristics sensitivity to the shape of the input distributions that determine the system behavior. Some of these investigations can be found in series of our papers. Review of these one can see for example in [33] and [34].

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A MAP/PH/1 queue with Setup time, Bernoulli vacation, Reneging, Balking, Bernoulli feedback, Breakdown and repair

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Abstract

A single server classical queueing model with Markovian Arrival Process(MAP), phase-type(PH) distributed service time and rest of the random variables are distributed exponentially is investigated. By making use of matrix analytic method, the resultant QBD process is examined in the stationary state. The practical applicability, objectives and the uniqueness of our model have been provided. The busy period analysis has been done and the distribution function for the waiting time has also been obtained. Some performance measures are enlisted. At last, some graphical and numerical exemplifications are furnished.

Keywords: Markovian Arrival Process, Setup process, Phase type distribution, Feedback, Vacation, Balking of customers, Renege of customers, Breakdown, Repair

I. Introduction

As far as the theory of point processes is concerned, the Markovian Arrival Process(MAP) is one of the most adaptable modelling tools. With an objective to formulate the incoming processes which may not be compulsorily renewal processes, a different thought notably, Versatile Markovian Point Processes(VMPP) has been introduced by Neuts [20]. The new terminologies, specifically Batch MAP and MAP had been coined by Lucantoni *et al.* [16] for the purpose of easy understanding of VMPP. The concept of MAP has been extensively discussed by Chakravarthy [3] in the *"Encyclopaedia of Operations Research and Management Science"*. The parameter matrices (D_0, D_1) characterizes the MAP and these matrices are of dimension m. In particular, the change overs which are related to no arrivals are taken care by D_0, whereas the change overs related to arrivals are taken care by D_1.

The generator matrix of the resultant CTMC is given as $D = D_0 + D_1$. The invariant probability matrix of the MAP which is a particular style of semi-Markov process is as follows:

$$\int_0^t e^{D_0 x} D_1 dx = [I - e^{D_0 t}](-D_0)^{-1} D_1.$$

Suppose π indicates probability vector for the matrix $D = D_0 + D_1$ in the stable state with the condition that $\pi D = 0$ and $\pi e = 1$. Then, $\lambda = \pi D_1 e_m$ provides the average count of arrival for each section of time in the steady state form of the MAP and is named as the fundamental rate. The PH-distributions and QBD process have been intensively examined by Latouche *et al.* [13].

The two researchers who have discussed about different types of vacations namely, single and multiple for a queueing model are Levy and Yechiali [14]. Keilson and Servi [10] have initiated the notion of Bernoulli vacation. A queueing system with multi-server, exponentially distributed vacation times had been examined by Levy and Yechiali [15]. By making use of one of the analysing techniques namely, the partial generating function, the size of the system had been computed by them. Tak*á*cs [24] was the first who introduced the concept of Bernoulli feedback. He has derived distribution for queue size for a stationary process.

Chakravarthy and Agnihothri [4] have analysed a non-Markovian queueing system in which service times are PH-distributed with back up server. The phase type nature of the duration of time in which the server is busy and the sojourn time(system and queue) have been shown by them. A cost model has also been developed by them for the purpose of finding the amicable threshold values. Chang *et al.* [5] have done an analysis of non-Markovian system where the arrivals come in groups with setup times and finite buffer by employing embedded Markov-chain technique. They have also derived the stationary distributions for the length of the waiting line at various instants.

Rajadurai *et al.* [23] have studied a non-Markovian retrial model where arrivals occur in groups with unreliable server, two stages of service and vacation. The probability generating function for the system size at different states have been obtained by them. Jain *et al.* [7] have examined the non-Markovian system where arrivals come in groups with breakdown, feedback and setup. By employing supplementary variable technique, they have established invariant distribution function of the queue length. They have also determined the staying time in the waiting line.

A Markovian queueing system with single server, feedback, vacation and impatient customers has been examined by Marichamy *et al.* [19]. They have employed probability generating function technique to study the invariant probability distribution. A multiserver Markovian retrial queueing system with impatient customers has been examined by Luh *et al.* [17] by using Matrix-Analytic Method. They have provided an analytic solution for their model. They have also employed eigen vector approach for analyzing their system.

Rakesh Kumar *et al.* [11] have done an analysis in transient and steady state for a queueing model with balking, reneging and two heterogeneous servers. They have provided various performance measures and have discussed about some particular cases. Bouchentouf *et al.* [2] have done an economic analysis of a batch arrival multiserver Markovian queueing model with feedback, multiple vacation and impatient customers. They have derived the steady state solution by using probability generating functions. A Markovian queueing system with multiserver and impatient customers along with the provision of additional removable servers has been examined by Jain *et al.* [6]. They have obtained equilibrium queue size distribution by employing recursive approach.

Ke *et al.* [9] have investigated the multiserver Markovian retrial system with balking and vacation. By using the Matrix-Geometric Method, the formulae for evaluating invariant probabilities and rate matrix have been derived by them. They have constructed cost function and have performed optimization tasks by employing various numerical methods. Rakesh Kumar *et al.* [12] have done a transient analysis of a queueing model with correlated inputs and reneging. They have studied the model with the aid of Runge-Kutta method. Bouchentouf *et al.* [1] have analysed a model with single server, feedback, multiple vacation and balking. They have obtained steady state probabilities and have developed the model for cost analysis. A non-Markovian retrial model with feedback, Bernoulli vacation and unreliable server has been investigated by Pavai Madheswari *et al.* [18]. The ergodicity condition for their model has been obtained by them. They have also obtained joint distribution function for various states of the server, system and orbit size. We have utilized the matrix-analytic method for our discussion and it has been introduced by Neuts[21]. The logarithmic reduction algorithm has been utilized to compute the rate matrix and it was described by Latouche *et al.* [13].

Consider a nationalized bank which has more than one serving counter. We may consider any one of those counters. Suppose the server in the counter deals with money transaction in the following ways(phases).

- Demand Draft(DD)
- Challan
- withdrawal/deposit forms

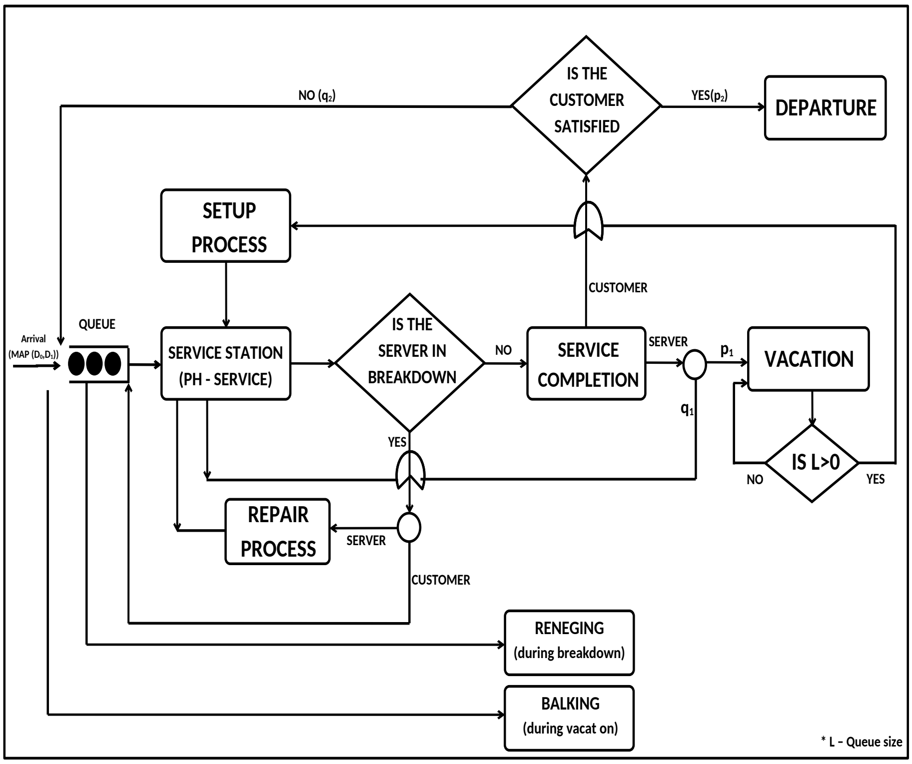

Figure 1: *Schematic representation of our model*

The arriving customer may demand money transaction in any of these ways. At the time of customer's entry, suppose the server is available, then the customer get the service at once. Otherwise, the customer joins the waiting line. Before each transaction, the server will perform some preparatory work(like refreshing the computer, selecting respective computer page for different modes of transaction, etc.). After offering the service, the server can either go for vacation(like attending telephone calls, cross checking the transaction amount, discussing with the adjacent servers, etc.) or may continue to serve the subsequent customers. Similarly, after receiving service, if the customer is not satisfied(like incorrect beneficiary name in the demand draft, deposited/withdrawn extra amount, etc.,), then the customer joins the queue to get the service again. Otherwise, they exit the bank permanently. During the busy period, the server may experience breakdown(like loss of internet connection, internal technical errors, virus attack to the system, etc.). After being repaired, the server will start to provide service to the customer who faced service interruption and is waiting in the anterior end of the queue. In the course of breakdown period, the customer in the queue may depart that particular counter(reneging). Moreover, in the course of vacation period of the server, the incoming customer may balk that particular counter. Our model has been formulated so that it will be on a par with the above circumstance.

The rest of our work is organized as follows: the description of our system is provided in Section II. Section III is devoted to the mathematical formulation of our model. The invariant analysis of our model has been presented in Section IV. The analysis of the active period of our system has been done in Section V. The analysis of the sojourn period of our model has been done in Section VI. Section VII contains a few performance measures of the system. Finally, in Section VIII, some illustrative examples are furnished via., tabular and graphical work.

II. Model Description

A queueing system with single server where the customers reach the system as specified by the MAP whose parameters matrices of dimension n are D_0 and D_1 has been considered. The duration of the service offered by the server is considered to be PH-distributed with notation (α, T) which is of order m, where $T^0 + Te = 0$. At the end of providing service, the server may choose to undergo vacation with p_1 as its probability or commence service to the succeeding customer with q_1 as its probability, where $p_1 + q_1 = 1$. The server always choose to avail vacation provided the system size is zero. The setup process begins at the completion of vacation period with the constraint that there must be a minimum of single customer in the space for the customer's to wait. Or else, the server carry on with his vacation upto a minimal of single customer waiting in the system for service while coming back from vacation. After the completion of setup process, the server commences service to the customer. Similarly, to the end of service completion, suppose a customer is fulfilled, he exits the service station forever with p_2 as its probability. Otherwise, the customer joins the anterior part of the waiting line with q_2 as its probability to acquire the service afresh. During the busy period, the server may get breakdown. As a result, the customer who is obtaining service at that time has to join the anterior end of the waiting line. At the completion of repair process, the server commences service to the customer, if any in the waiting line. Or else, the server undergoes vacation. During the customer's arrival, if the server is in vacation, then the customer may balk the system with b as its probability. Further, in the course of breakdown period of the server, the customer in the waiting line may renege due to their impatience. The vacation times, setup times, breakdown times, repair times and the reneging times are all supposed to follow exponential distribution with parameters η, τ, σ, δ abd r respectively.

III. The Generator Matrix

In this section, the generator matrix of the system under study is constructed. Our work starts with the definition of the desired successive notations.

Notations:

* *N(t)*: Size of the system at epoch t
* I_n: $n \times n$ identity matrix
* **0**: The zero matrix of needed dimension
* e_r: $r \times 1$ vector with all its entities to be 1
* e=e_{3n+mn}
* e_1=e_{2n}
* $e_1(1)$: $2n \times 1$ vector in which initial n entries are 1 and rest of the entries are zero
* $e_2(2)$: $2n \times 1$ vector with $n+1$ to $2n$ entries to be 1 and leftover entries to be zero
* $e(1)$: $(3n+mn) \times 1$ vector with first n entries to be 1 and leftover entries to be zero
* $e(2)$: $(3n+mn) \times 1$ vector with $n+1$ to $2n$ entries to be 1 and leftover entries to be zero
* $e(3)$: $(3n+mn) \times 1$ vector with $2n+1$ to $2n+mn$ entries to be 1 and leftover entries to be zero
* $e(4)$: $(3n+mn) \times 1$ vector with $2n+mn+1$ to $3n+mn$ entries to be 1 and rest of the entries to be zero
* $\otimes$: Symbol for Kronecker multiplication

* $\oplus$: Symbol for Kronecker addition
* *Y(t)* - Server's nature at instant t,
  where

$$Y(t) = \begin{cases} 0, & \text{server undergoes vacation} \\ 1, & \text{server is in setup process} \\ 2, & \text{server is offering service} \\ 3, & \text{server is in breakdown} \end{cases}$$

* $S(t)$: Service phase of the server at epoch t
* M(t): Phase of the MAP at epoch t
* λ: Fundamental rate of arrival and is mentioned by $\lambda = \mathbf{ß} D_1 \mathbf{e}$ in which $\mathbf{ß}$ is the probability vector of the matrix $D = D_0 + D_1$ in the steady state
* γ: The rate at which the server offers service, where $\gamma = [\alpha(-T)^{-1}\boldsymbol{e}]^{-1}$

Clearly, $\{(N(t),Y(t),S(t),M(t)):\ t \geq 0\}$ is a Continuous Time Markov Chain (CTMC) with succeeding state space:

$$\mathbf{\Omega} = U(0) \cup \bigcup_{j\geq 1} U(j)$$

where

$$U(0) = \{(0,0,k) : 1 \leq k \leq n\} \cup \{(0,3,k) : 1 \leq k \leq n\}$$

and

$$U(j) = \{(j,i,k) : i = 0,1; 1 \leq k \leq n\} \cup \{(j,2,l,k) : 1 \leq l \leq m, 1 \leq k \leq n\} \\ \cup \{(j,3,k) : 1 \leq k \leq n\}$$

The generator matrix of our Markov chain is as below:

$$Q = \begin{bmatrix} B_{00} & B_{01} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ B_{10} & C_1 & C_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{0} & C_2 & C_1 & C_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{0} & C_2 & C_1 & C_0 & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & C_2 & C_1 & C_0 & \mathbf{0} & \cdots \\ \cdots & \cdots & \cdots & \ddots & \ddots & \ddots & \cdots & \cdots \end{bmatrix}$$

where

$$B_{00} = \begin{bmatrix} D_0 + bD_1 & \mathbf{0} \\ \delta I_n & D_0 - \delta I_n \end{bmatrix}, B_{01} = \begin{bmatrix} (1-b)D_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & D_1 \end{bmatrix},$$

$$B_{10} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ p_2 T^0 \otimes I_n & \mathbf{0} \\ \mathbf{0} & rI_n \end{bmatrix},\ C_2 = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ p_1 p_2 T^0 \otimes I_n & \mathbf{0} & q_1 p_2 T^0 \alpha \otimes I_n & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & rI_n \end{bmatrix},$$

$$C_0 = \begin{bmatrix} (1-b)D_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & D_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_m \otimes D_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & D_1 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} D_0 - \eta I_n + bD_1 & \eta I_n & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & D_0 - \tau I_n & \alpha \otimes \tau I_n & \mathbf{0} \\ p_1 q_2 T^0 \otimes I_n & \mathbf{0} & (q_1 q_2 T^0 \alpha + T) \oplus (D_0 - \sigma I_n) & e_m \otimes \sigma I_n \\ \mathbf{0} & \mathbf{0} & \alpha \otimes \delta I_n & D_0 - (r+\delta) I_n \end{bmatrix}.$$

IV. System Analysis

I. Stability Condition

Define $C = C_0 + C_1 + C_2$ which results that C is a generator matrix and hence, we could compute it's invariant vector which is indicated by Ψ and it abides

$$\Psi C = \mathbf{0}; \quad \Psi \mathbf{e} = 1$$

where $\Psi = (\psi_0, \psi_1, \psi_2, \psi_3)$.
The vector Ψ, partitioned as $\Psi = (\psi_0, \psi_1, \psi_2, \psi_3)$ is determined by solving the successive equations:

$$\begin{aligned}
&\psi_0[D - \eta I_n] + \psi_2[p_1 T^0 \otimes I_n] = \mathbf{0}, \\
&\psi_0[\eta I_n] + \psi_1[D - \tau I_n] = \mathbf{0}, \\
&\psi_1[\alpha \otimes \tau I_n] + \psi_2[(q_1 T^0 \alpha + T) \oplus (D - \sigma I_n)] + \psi_3[\alpha \otimes \delta I_n] = \mathbf{0}, \\
&\psi_2[e_m \otimes \sigma I_n)] + \psi_3[D - \delta I_n] = \mathbf{0}
\end{aligned}$$

subject to

$$\psi_0 e_n + \psi_1 e_n + \psi_2 e_{mn} + \psi_3 e_n = 1.$$

The necessary and sufficient condition stablility is $\Psi A_0 \mathbf{e} < \Psi A_2 \mathbf{e}$
i.e.,

$$\psi_0[(1-b) D_1 e_n] + \psi_1[D_1 e_n] + \psi_2[e_m \otimes D_1 e_n] + \psi_3[D_1 e_n] < \psi_2[p_2 T^0 \otimes e_n] + \psi_3[r e_n].$$

II. The Invariant Probability Vector

In the steady state, let the probability vector of the generator Q be specified by $\mathbf{x}$ and it is of infinitesimal dimension.
This probability vector is further subdivided in the following fashion: $\mathbf{x} = (\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, ...)$, where the dimension of $\mathbf{x}_0$ and $\mathbf{x}_i$ are $2n$ and $3n + mn$ respectively, for $i \geq 1$.
Since $\mathbf{x}$ is an invariant vector of Q, the subsequent constraints will be abide by it:

$$\mathbf{x} Q = \mathbf{0}, \quad \mathbf{x}\mathbf{e} = 1.$$

Once the stableness is attained, the steady-state probability vector $\mathbf{x}$ may be determined by solving the subsequent equations.

$$\mathbf{x}_{i+1} = \mathbf{x}_1 R^i, \qquad i \geq 1$$

where R is the least non-negative solution of the equation

$$R^2 C_2 + R C_1 + C_0 = 0$$

and the remaining vectors namely, $\mathbf{x}_0$ and $\mathbf{x}_1$ can be determined by solving the subsequent equations:

$$\mathbf{x}_0 B_{00} + \mathbf{x}_1 B_{10} = \mathbf{0},$$

$$\mathbf{x}_0 B_{01} + \mathbf{x}_1 [C_1 + R C_2] = \mathbf{0}$$

with the normalizing condition

$$\mathbf{x}_0 \mathbf{e}_{2n} + \mathbf{x}_1 [I - R]^{-1} \mathbf{e}_{3n+mn} = 1.$$

The rate matrix R may be computed by using “Logarithmic Reduction Algorithm" given by Latouche *et al.* [13].

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V. Busy Period Analysis

The time duration between the advent of the customer to the system no customers and the epoch at which the system size becomes zero for the first time is defined as the active period. Thus, the first passage time from level 1 to 0 and the active period are the same.

Likewise, the first return time to level 0 with minimum one visit to a state in any other level may be defined as the busy cycle. Initially, the idea of the fundamental period is proposed to analyze the active period. The first passage time from the level i to $i-1$,($i \geq 2$) may be defined as the fundamental period for the QBD process. A distinct argumentation has to be carried out for the boundary states viz., $i = 0,$ and 1.

NOTATIONS:

* $G_{jj\prime}(k,x)$ - The probability of the QBD process entering the level $i-1$ by performing precisely k changeovers to the left and also by entering the state $(i, j\prime)$ with the constraint that it started in the state (i, j) at instant $t = 0$.
* $\tilde{G}_{jj\prime}(z,s) = \sum_{k=1}^{\infty} z^k \int_0^{\infty} e^{-sx} dG_{jj\prime}(k,x) : |z| \leq 1,\ Re(s) \geq 0$
* $\tilde{G}(z,s) -$ The matrix $(\tilde{G}_{jj\prime}(z,s))$
* $G = (G_{jj\prime}) = \tilde{G}(1,0)$- The matrix which takes care of the first passage times for the states other than the boundary state.
* $G_{jj\prime}^{(1,0)}(k,x)$ - The probability of the QBD process get into the level 0 by doing exactly k change overs to the left with the condition that it commenced in the level 1 at instant $t = 0$.
* $G_{jj\prime}^{(0,0)}(k,x)$ - The first return time to the level 0.
* $\mathbb{E}_{1j}$, $\mathbb{E}_{2j}$ - The expected first passage time and the expected number of customers who acquired service in the interval of first passage time between the levels i and $i-1$ respectively, with the constraint that the process is in the state (i, j) at the instant $t = 0$.
* $\vec{\mathbb{E}}_1$, $\vec{\mathbb{E}}_2$ - The column vectors with $\mathbb{E}_{1j}$ and $\mathbb{E}_{2j}$ as their entries respectively.
* $\vec{\mathbb{E}}_1^{(1,0)}$, $\vec{\mathbb{E}}_2^{(1,0)}$ - The vectors providing the expected first passage time from level 1 to level 0 and the expected number of service completion in that interval respectively.
* $\vec{\mathbb{E}}_1^{(0,0)}$, $\vec{\mathbb{E}}_2^{(0,0)}$ - The vectors providing the expected first return time to level 0 and the expected number of service completion in that interval respectively.

It is evident that the matrix $\tilde{G}(z,s)$ abides the subsequent equation:

$$\tilde{G}(z,s) = z[sI - A_1]^{-1} A_2 + [sI - A_1]^{-1} A_0 \tilde{G}^2(z,s)$$

If the rate matrix R is obtained, the determination of the matrix G may be done by utilizing the successive result

$$\text{“}G = -[A_1 + RA_2]^{-1} A_2\text{”}.$$

Likewise, the matrix G may be determined by using the logarithmic reduction algorithm(Latouche *et al.* [13]).

The succeeding equations which are fulfilled by $\tilde{G}^{(1,0)}(z,s)$ and $\tilde{G}^{(0,0)}(z,s)$ are for the boundary states viz., 1 and 0 respectively.

$$\tilde{G}^{(1,0)}(z,s) = z[sI - A_1]^{-1} B_{10} + [sI - A_1]^{-1} A_0 \tilde{G}(z,s) \tilde{G}^{(1,0)}(z,s)$$

$$\tilde{G}^{(0,0)}(z,s) = [sI - B_{00}]^{-1} B_{01} \tilde{G}^{(1,0)}(z,s).$$

Since the matrices G, $\tilde{G}^{(1,0)}(1,0)$ and $\tilde{G}^{(0,0)}(1,0)$ are all stochastic, the subsequent moments may be readily computed. At $z = 1$ and $s = 0$,

$$\vec{\mathbb{E}}_1 = -\frac{\partial}{\partial s}\{\tilde{G}(z,s)\} = -[A_0(G+I) + A_1]^{-1}e,$$

$$\vec{\mathbb{E}}_2 = \frac{\partial}{\partial z}\{\tilde{G}(z,s)\} = -[A_0(G+I) + A_1]^{-1}A_2 e,$$

$$\vec{\mathbb{E}}_1^{(1,0)} = -\frac{\partial}{\partial s}\{\tilde{G}^{(1,0)}(z,s)\} = -[A_1 + A_0 G]^{-1}[A_0\vec{\mathbb{E}}_1 + e],$$

$$\vec{\mathbb{E}}_2^{(1,0)} = \frac{\partial}{\partial z}\{\tilde{G}^{(1,0)}(z,s)\} = -[A_1 + A_0 G]^{-1}[B_{10}e + A_0\vec{\mathbb{E}}_2],$$

$$\vec{\mathbb{E}}_1^{(0,0)} = -\frac{\partial}{\partial s}\{\tilde{G}^{(0,0)}(z,s)\} = -B_{00}^{-1}[e + B_{01}\vec{\mathbb{E}}_1^{(1,0)}],$$

$$\vec{\mathbb{E}}_2^{(0,0)} = \frac{\partial}{\partial z}\{\tilde{G}^{(0,0)}(z,s)\} = -B_{00}^{-1}B_{01}\vec{\mathbb{E}}_2^{(1,0)}.$$

VI. Waiting time analysis

With the aid of analysis of first passage time, the distribution function for the waiting time of an arriving customer has been derived in this section.

Let W(t), where $t \geq 0$ be a vector of dimension $1 \times m$ which indicates the waiting time distribution of an arriving tagged customer in the queue. While taking a multi-server model with Bernoulli vacation under study, we could see that $W(0+) = 0$, because each arriving customer has to hold up for the completion of either vacation period or service period. Let $(*) \cup \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \cdots\}$ indicates the state space of an absorbing CTMC. The service for the arriving tagged customer will commence from their arrival into the absorbing state $(*)$. For this absorbing Markov chain, the transition matrix is as follows:

$$\tilde{Q} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ H_0 & F_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ H_1 & F_{10} & F & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{0} & F_2 & F & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & F_2 & F & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & F_2 & F & \cdots \\ \cdots & \cdots & \cdots & \ddots & \ddots & \ddots & \cdots \end{bmatrix}$$

where

$$H_0 = \begin{bmatrix} \eta \\ \delta \end{bmatrix}, F_0 = \begin{bmatrix} -\eta & \mathbf{0} \\ \mathbf{0} & -\delta \end{bmatrix}, H_1 = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ q_1 p_2 T^0 \\ \mathbf{0} \end{bmatrix}, F_{10} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ p_1 p_2 T^0 & \mathbf{0} \\ \mathbf{0} & r \end{bmatrix},$$

$$F = \begin{bmatrix} -\eta & \eta & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\tau & \tau\alpha & \mathbf{0} \\ p_1 q_2 T^0 & \mathbf{0} & q_1 q_2 T^0\alpha + T - \sigma I_m & \sigma e_m \\ \mathbf{0} & \mathbf{0} & \delta\alpha & -(\delta + r) \end{bmatrix}, F_2 = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ p_1 p_2 T^0 & \mathbf{0} & q_1 p_2 T^0\alpha & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & r \end{bmatrix}.$$

With an objective to derive the arriving tagged customer's waiting time distribution $W(t)$, where $(t \geq 0)$, we begin with the process of finding the system size probability vector in the steady state at the arrival instant and it is indicated by $\mathbf{z}(0) = (\mathbf{z}_0(0), \mathbf{z}_1(0), \mathbf{z}_2(0), ...)$. As the arrival process obeys the Markovian property, the system size probability vector in the steady state at the arrival epoch is as follows:

$$\mathbf{z}_0(0) = \mathbf{x}_0[I_2 \otimes \frac{D_1 e_n}{\lambda},]$$

$$\mathbf{z}_i(0) = \mathbf{x}_i[I_{3+m} \otimes \frac{D_1 e_n}{\lambda}], \text{for } i \geq 1$$

where λ indicates the fundamental arrival rate of the MAP.
Define $\mathbf{z}(t) = (\mathbf{z}_*(t), \mathbf{z}_0(t), \mathbf{z}_1(t), \cdots)$,
where
$\mathbf{z}_i(t),\ i \geq 1$ - $1 \times (3+m)$ vector,
$\mathbf{z}_0(t)$ - a 1×2 vector
and their components provide the probability of the CTMC whose generator matrix is $\tilde{Q}$ being in the respective state of level i at epoch t. Since $\mathbf{z}_*(t)$ specifies the probability of the tagged customer being in the absorbing state at epoch t, we get $W(t) = \mathbf{z}_*(t)$, where $t \geq 0$.
The differential equation $\acute{\mathbf{z}}(t) = \mathbf{z}(t)\tilde{Q}$, where $t \geq 0$ reduces to

$$\acute{\mathbf{z}}_*(t) = \mathbf{z}_1(t)H_1$$

$$\acute{\mathbf{z}}_0(t) = \mathbf{z}_0(t)F_0 + \mathbf{z}_1(t)F_{10}$$

$$\acute{\mathbf{z}}_i(t) = \mathbf{z}_i(t)F + \mathbf{z}_{i+1}(t)F_2,\ i \geq 1$$

where $\acute{}$ specifies the derivative with respect to t.

Let us compute the Laplace Stieltjes Transform(LST) for W(t) with the aid of technique indicated by Neuts *et al.* [21]. By commencing the process at the state i with $\mathbf{z}_i(0)$, $i \geq 1$ as initial probability vector, the row vector $\omega(s)$ specifies the LST of the first passage time to level 1. As indicated in [21], we get,

$$\omega(s) = \sum_{i=1}^{\infty} \mathbf{z}_i(0)[(sI - F)^{-1}F_2]^{i-1}. \tag{1}$$

Let the LST of the absorbing time to the state $(*)$ with the constraint that the process commences at level $i = 0, 1, 2$ be specified by $\phi(i, s)$. Just as in [21], we have

$$\phi(0, s) = [sI - F_0]^{-1}H_0, \tag{2}$$

$$\phi(1, s) = [sI - F]^{-1}F_{10}\phi(0, s) + [sI - F]^{-1}H_1. \tag{3}$$

Thus, we may simply observe that the LST for the distribution of waiting time is as below:

$$\bar{W}(s) = \mathbf{z}_0(0)\ \phi(0, s) + \omega(s)\ \phi(1, s). \tag{4}$$

I. Average waiting time

The average waiting time is provided as

$$EW = -\mathbf{z}_0(0)\acute{\phi}(0,0) - \acute{\omega}(0)e_{3+m} - \omega(0)\acute{\phi}(1,0)e_m. \tag{5}$$

The expected time to enter into the absorbing state $(*)$ given that the system is in the level $i = 0$ is provided by the foremost term of the preceding equation. In the same way, if the system is in level $i \geq 1$, then the mean time for accessing the state $(*)$ is provided by the end two terms of the preceding equation.
By differentiating (2) and (3), and setting $s = 0$, we obtain,

$$\acute{\phi}(0,0) = (-1)[-F_0]^{-2}H_0, \tag{6}$$

$$\acute{\phi}(1,0) = (-1)[-F]^{-2}F_{10}\phi(0,0) + [-F]^{-1}F_{10}\acute{\phi}(0,0) - [-F]^{-2}H_1. \tag{7}$$

With the help of (6) along with the vector $\mathbf{z}(0) = (\mathbf{z}_0(0), \mathbf{z}_1(0), \mathbf{z}_2(0), \cdots)$, we may readily compute the first term of (5). From (1), we get

$$\omega(0) = \sum_{i=1}^{\infty} \mathbf{z}_i(0)V^{i-1}, \tag{8}$$

where $V = [-F]^{-1}F_2$. Since the matrix V is stochastic, we get

$$\omega(0)e_{3+m} = 1 - \mathbf{z}_0(0)e_2. \tag{9}$$

With the help of (7) and (9) along with the vector $\mathbf{z}(0) = (\mathbf{z}_0(0), \mathbf{z}_1(0), \mathbf{z}_2(0), \cdots)$, we may readily compute the final term of (5).
By differentiating (1) and making $s = 0$, we get

$$\acute{\omega}(0) = (-1)\sum_{i=1}^{\infty} \mathbf{z}_{1+i}(0) \sum_{j=0}^{i-1} V^j[-F]^{-1}V^{i-j}. \tag{10}$$

As V is stochastic, we get

$$(-1)\acute{\omega}(0)e_{3+m} = (-1)\sum_{i=1}^{\infty} \mathbf{z}_{1+i}(0) \sum_{j=0}^{i-1} V^j[-F]^{-1}e_{3+m}. \tag{11}$$

With the help of the method mentioned in Kao *et al.* [8]. and Neuts *et al.* [22], let us evaluate the value of $(-1)\acute{\omega}(0)e_{3+m}$. We begin with the construction of a matrix V_2 which is such that V_2 is stochastic, generalized inverse of $I - V$ and $I - V + V_2$ is non-singular. The matrix V_2 can be assumed to be $V_2 = e_{1+m_1+m_2+m_1m_2}v_0$ in which v_0 is the stationary probability vector of V. Further, with the help of the property $VV_2 = V_2V = V_2$, we have

$$\sum_{j=0}^{i-1} V^j(I - V + V_2) = I - V^i + iV_2, \quad \text{for } i \geq 1. \tag{12}$$

By using (14) in (13), we obtain,

$$(-1)\acute{\omega}(0)e_{3+m} = \{\mathbf{x}_1[I-R]^{-1}[I_{3+m} \otimes \frac{D_1e_n}{\lambda}] - \omega(0) + \mathbf{x}_1R[I-R]^{-2}[I_{3+m} \otimes \frac{D_1e_n}{\lambda}]V_2\} \times [I - V + V_2]^{-1}[-F]^{-1}e_{3+m}. \tag{13}$$

Hence, all the terms of (5) have been found out and so we may readily obtain the average period of waiting.

VII. Performance Measures

* Probability of server is on vacation:
  $P_{vacation} = \mathbf{x}_0e_1(1) + \mathbf{x}_1(I-R)^{-1}\, e(1)$
* Probability of server is in setup process:
  $P_{setup} = \mathbf{x}_1(I-R)^{-1}e(2)$
* Probability of server is busy:
  $P_{busy} = \mathbf{x}_1(I-R)^{-1}e(3)$
* Probability of server is in breakdown:
  $P_{breakdown} = \mathbf{x}_0e_1(2) + \mathbf{x}_1(I-R)^{-1}e(4)$
* The mean system:
  $E_{system} = \mathbf{x}_1(I-R)^{-2}e$
* The mean system size when the server is undergoing vacation:
  $E_v = \mathbf{x}_1(I-R)^{-2}e(1)$

* Expected system size during setup process:
  $E_s = \mathbf{x}_1(I-R)^{-2}e(2)$
* Average system size when the server is busy:
  $E_b = \mathbf{x}_1(I-R)^{-2}e(3)$
* Expected system size during breakdown:
  $E_{bd} = \mathbf{x}_1(I-R)^{-2}e(4)$

VIII. Numerical Illustrations

The comprehensive aim of this section is to explore the performance of our system through numerical and graphical exemplifications. For the arrival patterns, we took the following distinctive MAP representations so that their mean is 1, and Chakravarthy [3] suggested these values.

Erlang of order 2-(A-Erl):

$$D_0 = \begin{bmatrix} -2 & 2 \\ 0 & -2 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}$$

Exponential-(A-Exp):

$$D_0 = \begin{bmatrix} -1 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 1 \end{bmatrix}$$

Hyperexponential-(A-Hyp-Exp):

$$D_0 = \begin{bmatrix} -1.90 & 0 \\ 0 & -0.19 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 1.710 & 0.190 \\ 0.171 & 0.019 \end{bmatrix}$$

It is evident that they have zero correlation because of the renewal character of these three arrival processes.

MAP - Negative Correlation-(A-MAP-NC):

$$D_0 = \begin{bmatrix} -1.00222 & 1.00222 & 0 \\ 0 & -1.00222 & 0 \\ 0 & 0 & -225.75 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0.01002 & 0 & 0.99220 \\ 223.4925 & 0 & 2.2575 \end{bmatrix}$$

The successive PH - distributions have been taken for service times suggested by Chakravarthy [3] as well.

Erlang of order 2-(S-Erl):

$$\alpha = (1,0), \qquad T = \begin{bmatrix} -2 & 2 \\ 0 & -2 \end{bmatrix}$$

Exponential-(S-Exp):

$$\alpha = (1), \qquad T = \begin{bmatrix} -1 \end{bmatrix}$$

Hyperexponential-(S-Hyp-Exp):

$$\alpha = (0.8, 0.2), \qquad T = \begin{bmatrix} -2.8 & 0 \\ 0 & -0.28 \end{bmatrix}$$

Illustration 1

From the Tables 1-4, we study the impact of the repair rate δ against the probability of server being busy. Fix $\lambda = 1,\ \gamma = 6,\ \sigma = 3,\ \eta = 6,\ \tau = 5,\ r = 1,\ b = 0.6,\ p_2 = 0.5,\ q_2 = 0.5$.
For Bernoulli vacation(Bv): $p_1 = 0.5,\ q_1 = 0.5$.
For 1 - limited vacation(1-Lv): $p_1 = 1,\ q_1 = 0$.
From Tables 1-4, we derive the succeeding observations.

* As the repair rate(δ) maximizes, the probability of server being busy also increases for distinct feasible groupings of service and arrival times.
* While correlating the tabulated values for distinct arrival patterns, the probability of server being busy maximizes more rapidly for A-Hyp-Exp and gradually for A-MAP-NC. In the same way, the probability of server being busy increases gradually for S-Hyp-Exp and rapidly for S-Erl.
* Also, the probability of server being busy maximizes slowly for Bv and quickly for 1-Lv for distinct arrangements of arrival and service patterns.

Illustration 2

From the Tables 5-8, we study the impact of the service rate γ on the expected waiting time(EW). We fix $\lambda = 1,\ \delta = 1,\ \sigma = 3,\ \eta = 6,\ \tau = 5,\ r = 1,\ b = 0.6,\ p_2 = 0.5,\ q_2 = 0.5$.
For Bernoulli vacation(Bv): $p_1 = 0.5;\ q_1 = 0.5$.
For 1 - limited vacation(1-Lv): $p_1 = 1;\ q_1 = 0$.
From Tables 5-8, we get the subsequent interpretation.

* While raising the service rate, EW minimizes for distinct possible combinations of service and arrival patterns.
* While correlating the values of distinct arrival patterns, EW decreases more quickly in the case of A-Hyp-Exp whereas slowly for A-Erl. Similarly, EW decreases gradually for S-Hyp-Exp and more quickly in the case of S-Erl.
* Further, the average waiting time decreases rapidly for 1-Lv and slowly in the case of Bv.

Illustration 3

From the 2D graphs 2-13, we view the effect of the vacation rate η on average system size (E_{system}). Fix $\lambda = 1,\ \delta = 1,\ \sigma = 3,\ \gamma = 6,\ \tau = 5,\ \ r = 1,\ b = 0.6,\ p_2 = 0.5,\ q_2 = 0.5$.
For Bernoulli vacation(Bv): $p_1 = 0.5,\ q_1 = 0.5$.
For 1 - limited vacation(1 - Lv): $p_1 = 1,\ q_1 = 0$.

From Figures 2-13, we could see that while raising the vacation rate (η), the rate of decrement of E_{system} is high in the case of A-Hyp-Exp and low for A-Erl. Also, it is high in the case of S-Erl and low in the case of S-Hyp-Exp. Further, we may view that E_{system} decreases quickly in the case of 1 - Lv and slowly in the case of Bv.

Table 1: *Repair rate vs. Probability of server being busy - A-Exp*

δ	SERVICE					
	S-Exp		S-Erl		S-Hyp-Exp	
	Bv	1-Lv	Bv	1-Lv	Bv	1-Lv
2.0	0.16152	0.17895	0.18126	0.20038	0.10756	0.11959
3.0	0.17270	0.19495	0.19591	0.22133	0.11197	0.12588
4.0	0.17896	0.20409	0.20422	0.23354	0.11437	0.12930
5.0	0.18299	0.21001	0.20961	0.24155	0.11589	0.13147
6.0	0.18581	0.21417	0.21340	0.24721	0.11695	0.13296
7.0	0.18791	0.21725	0.21622	0.25142	0.11773	0.13406

Illustration 4

From the 2D graphs 14-25, we study the impact of the breakdown rate(σ) on the mean period of waiting(EW). Fix $\lambda = 1,\ \delta = 1,\ \eta = 6,\ \gamma = 6,\ \tau = 5,\ r = 1,\ b = 0.6,$ $p_2 = 0.5,\ q_2 = 0.5.$
For Bernoulli vacation(Bv): $p_1 = 0.5,\ q_1 = 0.5.$
For 1 - limited vacation(1 - Lv): $p_1 = 1,\ q_1 = 0.$

From Figures 14-25, we may view that while raising the breakdown rate (σ), the speed of increment of EW is maximum for A-Hyp-Exp and minimum for A-Erl. Also, it is high in the case of S-Erl and low in the case of S-Hyp-Exp. Further, we may view that EW increases quickly in the case of 1 - Lv and slowly in the case of Bv.

Illustration 5:

From the 3D graphs 26-37, we analyse the impact of the setup rate (τ) and the rate of service provided by the server(γ) on the probability of server is availing vacation($P_{vacation}$). Fix $\lambda = 1,\ \delta = 1,$ $\eta = 6,\ \sigma = 3,\ r = 1,\ b = 0.6,\ p_1 = 0.5,\ q_1 = 0.5,\ p_2 = 0.5,\ q_2 = 0.5.$

A quick view of Figures 26-37 reveals the fact that $P_{vacation}$ increases while maximizing both the setup rate and the rate of service offered by the server for various arrangement of arrival and service patterns. Further, it maximizes rapidly for A-MAP-NC and gradually for A-Hyp-Exp. In the same way, the rate of increment is high for S-Erl and low for S-Hyp-Exp.

Illustration 6:

From the 3D graphs 38-49, we observe the consequences of the customer's reneging rate(r) and the server's vacation rate(η) on the Average system size(E_{system}). Fix $\lambda = 1,\ \delta = 1,\ \gamma = 6,$ $\sigma = 3,\ \tau = 5,\ b = 0.6,\ p_1 = 0.5,\ q_1 = 0.5,\ p_2 = 0.5,\ q_2 = 0.5$.

A quick view of Figures 38-49 reveals the point that E_{system} reduces while maximizing both the reneging rate and the vacation rate of the customer and the server respectively for distinct groupings of arrival and service times. Further, it minimizes quickly for A-Hyp-Exp and slowly for A-Erl. In the same way, the rate of decrement of E_{system} is high in the case of S-Erl and low in the case of S-Hyp-Exp.

Table 2: *Repair rate vs. Probability of server being busy - A-Erl*

δ	SERVICE					
	S-Exp		S-Erl		S-Hyp-Exp	
	Bv	1-Lv	Bv	1-Lv	Bv	1-Lv
2.0	0.15626	0.17639	0.17582	0.19840	0.103440	0.11650
3.0	0.16632	0.19163	0.18908	0.21864	0.107370	0.12228
4.0	0.17192	0.20028	0.19653	0.23038	0.109510	0.12541
5.0	0.17552	0.20587	0.20135	0.23805	0.110870	0.12739
6.0	0.17805	0.20978	0.20474	0.24347	0.111830	0.12876
7.0	0.17993	0.21269	0.20727	0.24750	0.112530	0.12977

Table 3: *Repair rate vs. Probability of server being busy - A-Hyp-Exp*

δ	SERVICE					
	S-Exp		S-Erl		S-Hyp-Exp	
	Bv	1-Lv	Bv	1-Lv	Bv	1-Lv
2.0	0.18240	0.18781	0.20235	0.20704	0.12405	0.13085
3.0	0.19805	0.20641	0.22252	0.23028	0.13007	0.13901
4.0	0.20676	0.21714	0.23403	0.24396	0.13318	0.14345
5.0	0.21229	0.22412	0.24145	0.25296	0.13506	0.14625
6.0	0.21610	0.22902	0.24664	0.25934	0.13632	0.14817
7.0	0.21890	0.23265	0.25046	0.26410	0.13723	0.14957

Table 4: *Repair rate vs. Probability of server being busy - A-MAP-NC*

δ	SERVICE					
	S-Exp		S-Erl		S-Hyp-Exp	
	Bv	1-Lv	Bv	1-Lv	Bv	1-Lv
2.0	0.15122	0.17400	0.17037	0.19642	0.099946	0.11412
3.0	0.16029	0.18861	0.18240	0.21606	0.103440	0.11950
4.0	0.16540	0.19691	0.18923	0.22745	0.105400	0.12244
5.0	0.16873	0.20229	0.19369	0.23490	0.106680	0.12432
6.0	0.17110	0.20607	0.19685	0.24017	0.107590	0.12563
7.0	0.17287	0.20887	0.19922	0.24409	0.108280	0.12659

Table 5: *Service rate vs. Average waiting time-A-Exp*

γ	SERVICE					
	S-Exp		S-Erl		S-Hyp-Exp	
	Bv	1-Lv	Bv	1-Lv	Bv	1-Lv
7.0	2.4993	8.3702	2.9533	11.000	1.4773	4.0849
8.0	2.0203	6.1915	2.3009	7.4783	1.2958	3.4923
9.0	1.7030	4.9497	1.8916	5.7077	1.1625	3.0774
10.0	1.4788	4.1511	1.6132	4.6482	1.06040	2.7703
11.0	1.3129	3.5965	1.4130	3.9461	0.97959	2.5337
12.0	1.1857	3.1900	1.2627	3.4485	0.91410	2.3457

Table 6: *Service rate vs. Average waiting time-A-Erl*

γ	SERVICE					
	S-Exp		S-Erl		S-Hyp-Exp	
	Bv	1-Lv	Bv	1-Lv	Bv	1-Lv
7.0	1.87310	6.2099	2.20200	8.1571	1.12500	3.0281
8.0	1.51780	4.5819	1.71990	5.5304	0.98996	2.5852
9.0	1.28300	3.6557	1.41810	4.2119	0.89072	2.2755
10.0	1.11750	3.0610	1.21320	3.4243	0.81472	2.0465
11.0	0.99508	2.6488	1.06610	2.9033	0.75462	1.8703
12.0	0.90133	2.3471	0.95583	2.5346	0.70591	1.7305

Table 7: *Service rate vs. Average waiting time-A-Hyp-Exp*

γ	SERVICE					
	S-Exp		S-Erl		S-Hyp-Exp	
	Bv	1-Lv	Bv	1-Lv	Bv	1-Lv
7.0	7.0763	25.198	8.5479	33.276	3.8226	12.076
8.0	5.5843	18.612	6.5041	22.603	3.2673	10.278
9.0	4.5926	14.841	5.2154	17.213	2.8615	9.0147
10.0	3.8918	12.403	4.3372	13.972	2.5526	8.0765
11.0	3.3742	10.703	3.7059	11.814	2.3102	7.3514
12.0	2.9785	9.4514	3.2334	10.279	2.1151	6.7739

Table 8: *Service rate vs. Average waiting time-A-MAP-NC*

γ	SERVICE					
	S-Exp		S-Erl		S-Hyp-Exp	
	Bv	1-Lv	Bv	1-Lv	Bv	1-Lv
7.0	2.4453	8.3500	2.8876	11.014	1.45970	4.0420
8.0	1.9814	6.1528	2.2527	7.4498	1.28680	3.4530
9.0	1.6761	4.9065	1.8574	5.6667	1.16010	3.0422
10.0	1.4616	4.1086	1.5902	4.6048	1.06330	2.7393
11.0	1.3035	3.5567	1.3989	3.9042	0.98690	2.5066
12.0	1.1826	3.1538	1.2559	3.4097	0.92503	2.3221

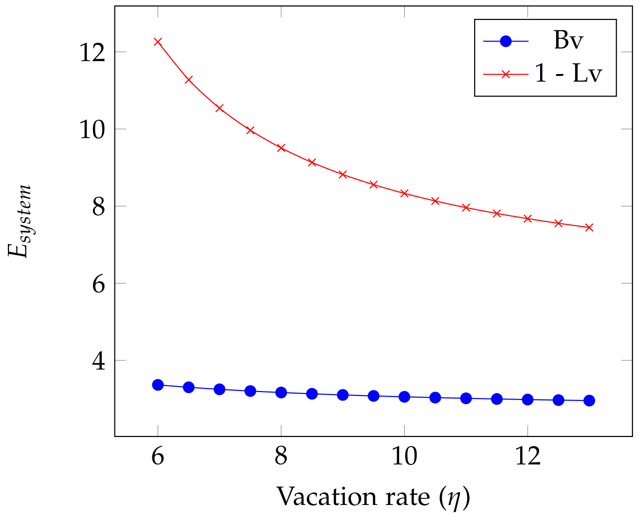

Figure 2: *Vacation rate vs. E_{system} - M/M/1*

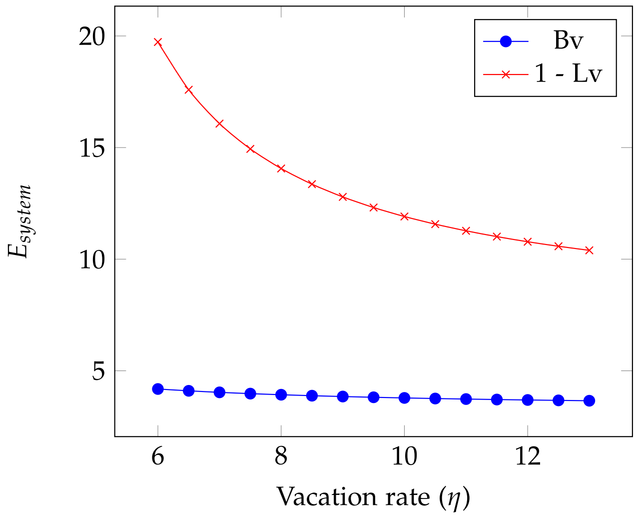

Figure 3: *Vacation rate vs. E_{system} - M/Ek/1*

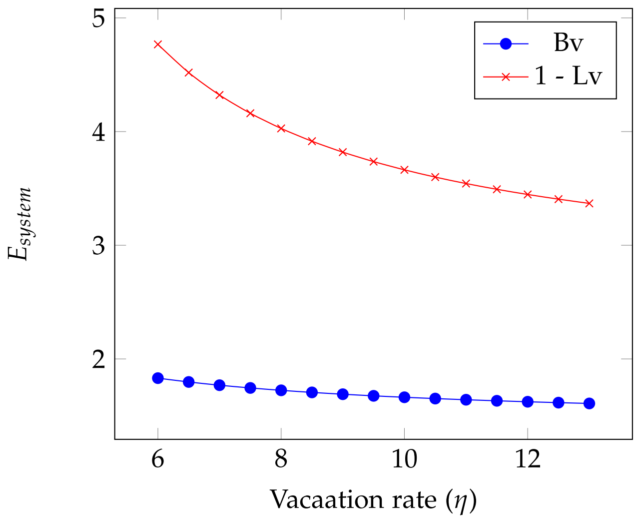

Figure 4: *Vacation rate vs. E_{system} - M/Hk/1*

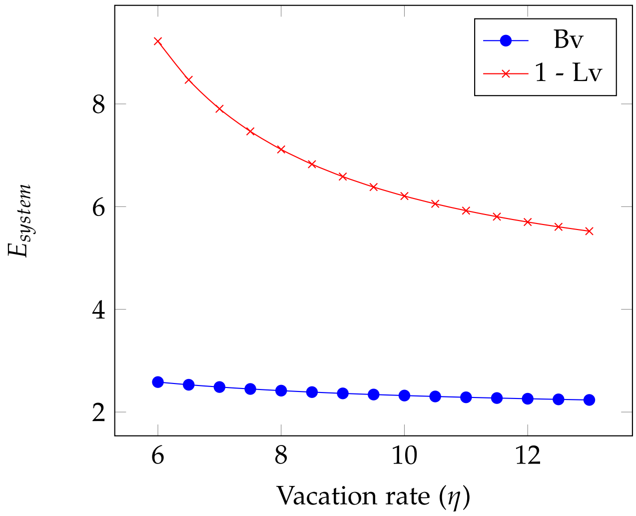

Figure 5: *Vacation rate vs.* E_{system} *- EK/M/1*

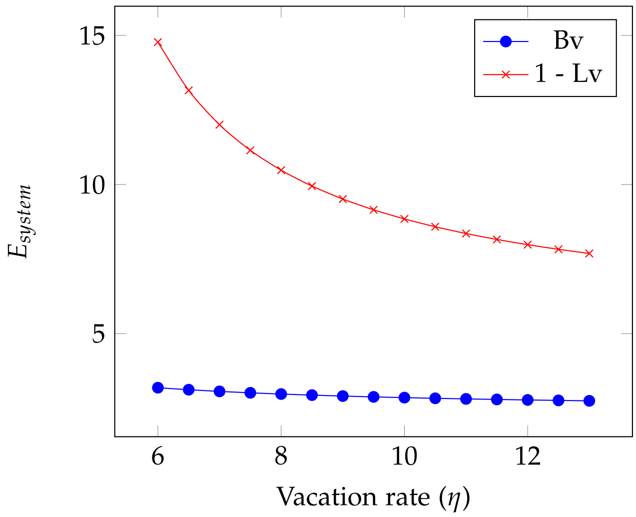

Figure 6: *Vacation rate vs.* E_{system} *- Ek/Ek/1*

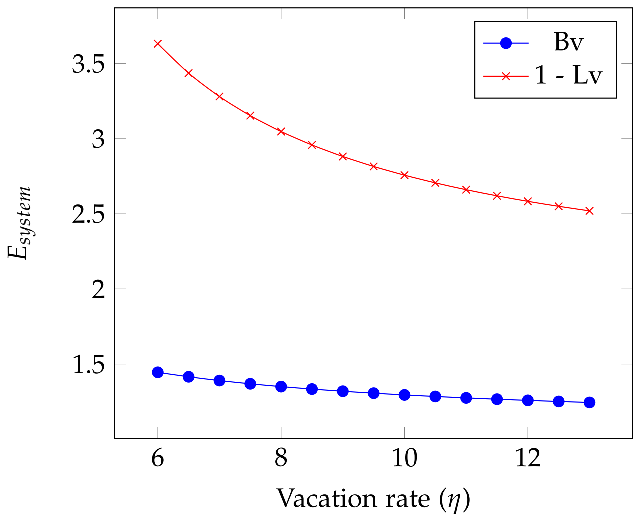

Figure 7: *Vacation rate vs.* E_{system} *- Ek/Hk/1*

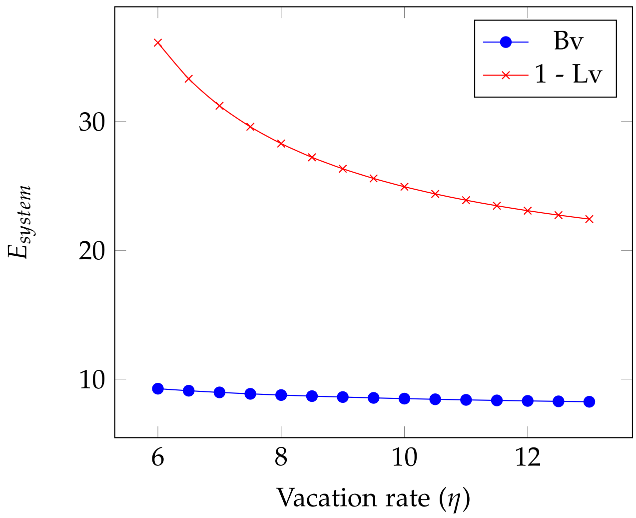

Figure 8: *Vacation rate vs. E_{system} - Hk/M/1*

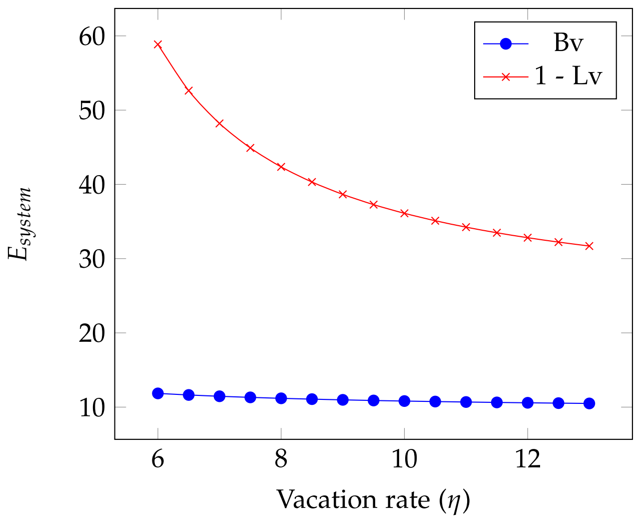

Figure 9: *Vacation rate vs. E_{system} - Hk/Ek/1*

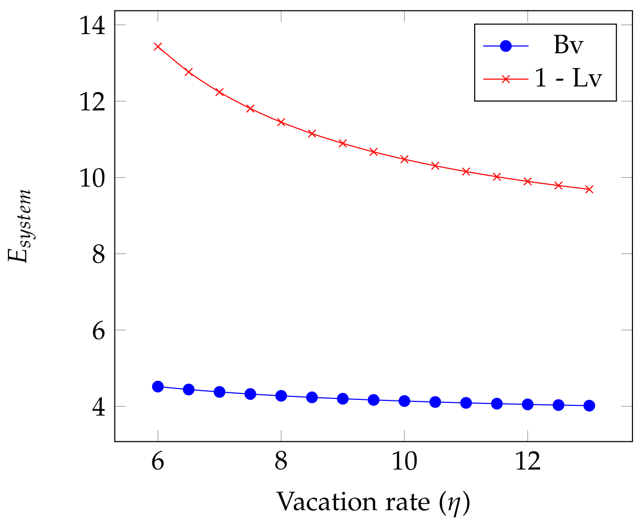

Figure 10: *Vacation rate vs. E_{system} - Hk/Hk/1*

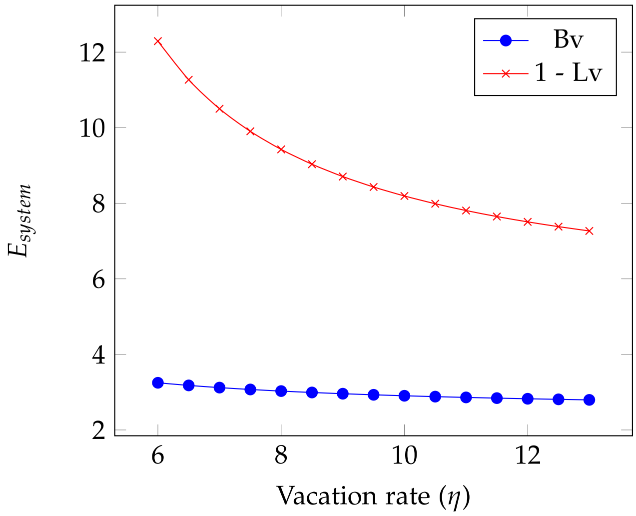

Figure 11: *Vacation rate vs. E_{system} - MAP-NC/M/1*

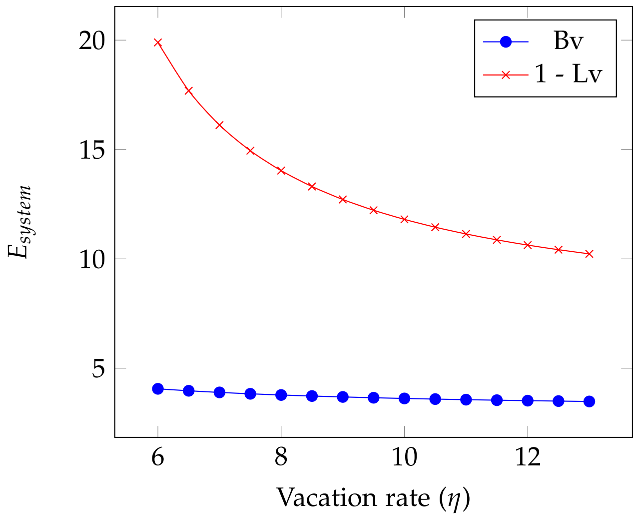

Figure 12: *Vacation rate vs. E_{system} - MAP-NC/Ek/1*

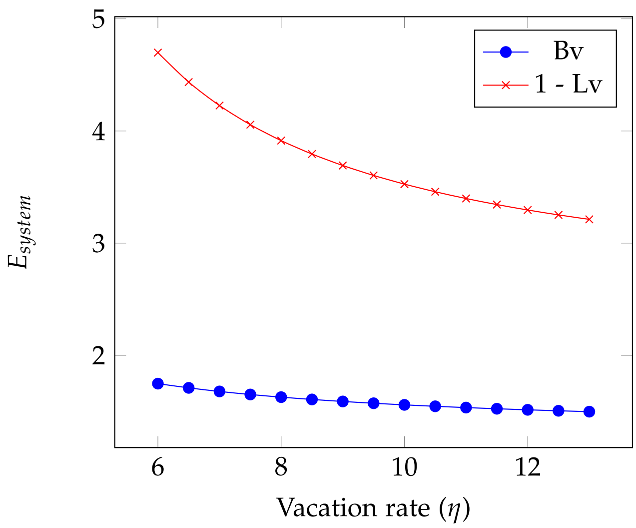

Figure 13: *Vacation rate vs. E_{system} - MAP-NC/Hk/1*

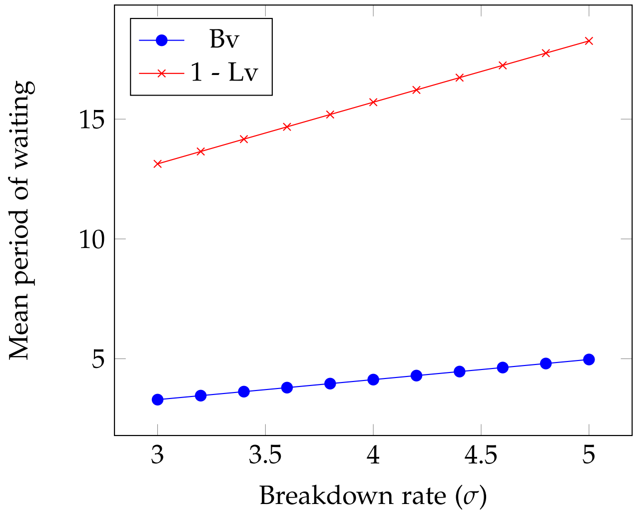

Figure 14: *Breakdown rate vs. Mean period of waiting - M/M/1*

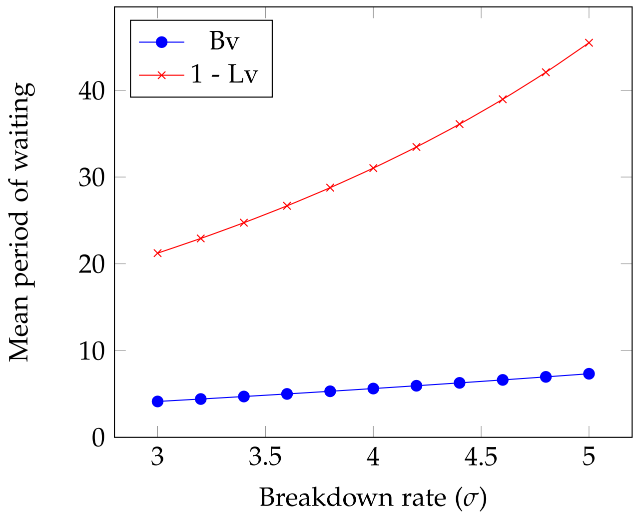

Figure 15: *Breakdown rate vs. Mean period of waiting - M/Ek/1*

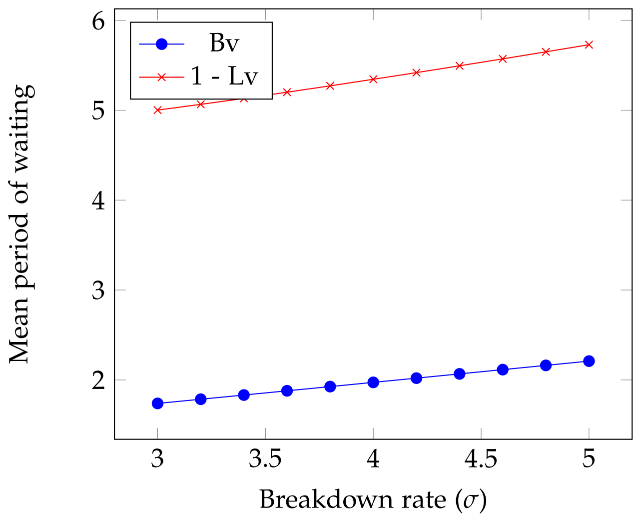

Figure 16: *Breakdown rate vs. Mean period of waiting - M/Hk/1*

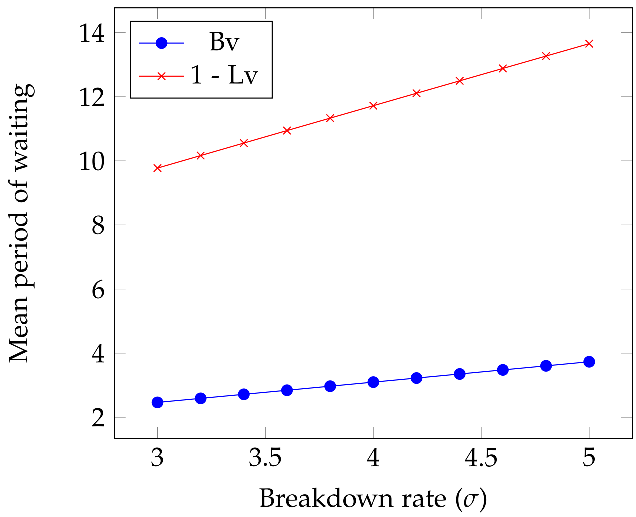

Figure 17: *Breakdown rate vs. Mean period of waiting - Ek/M/1*

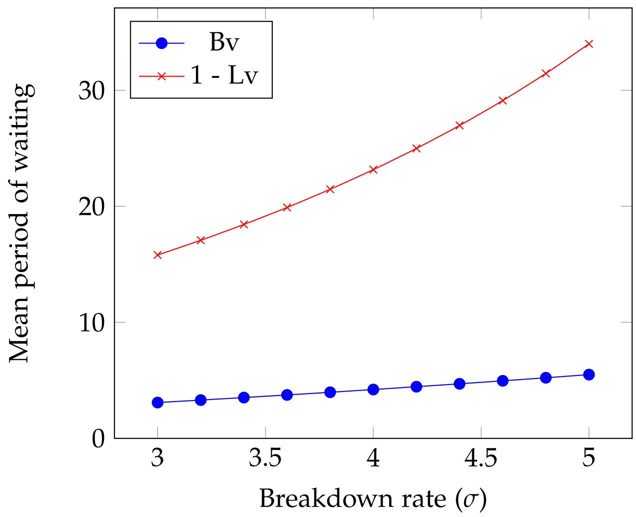

Figure 18: *Breakdown rate vs. Mean period of waiting - Ek/Ek/1*

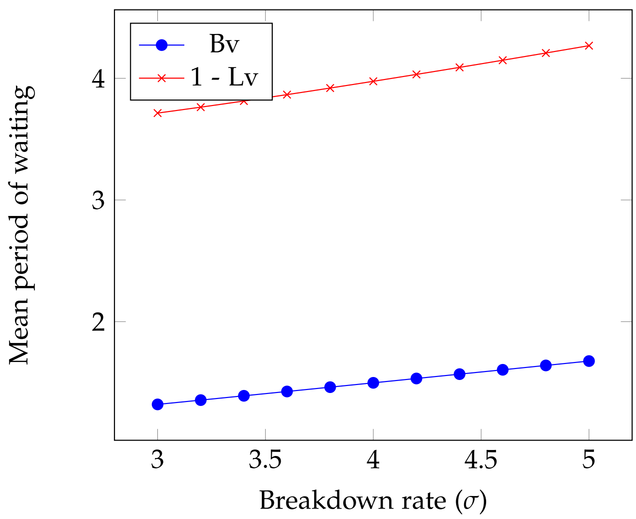

Figure 19: *Breakdown rate vs. Mean period of waiting - Ek/Hk/1*

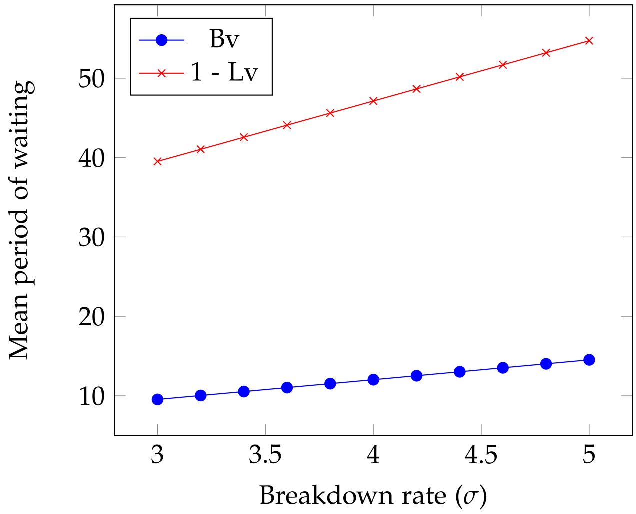

Figure 20: *Breakdown rate vs. Mean period of waiting - Hk/M/1*

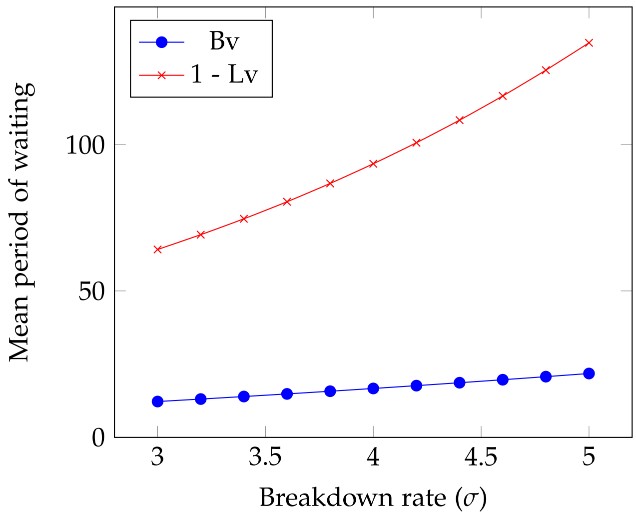

Figure 21: *Breakdown rate vs. Mean period of waiting - Hk/Ek/1*

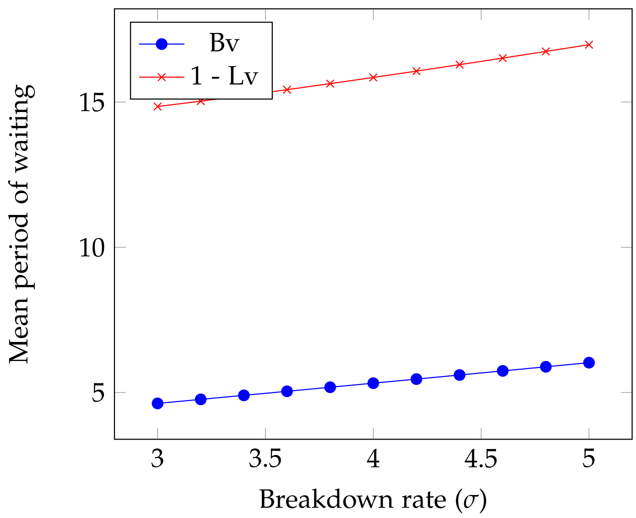

Figure 22: *Breakdown rate vs. Mean period of waiting - Hk/Hk/1*

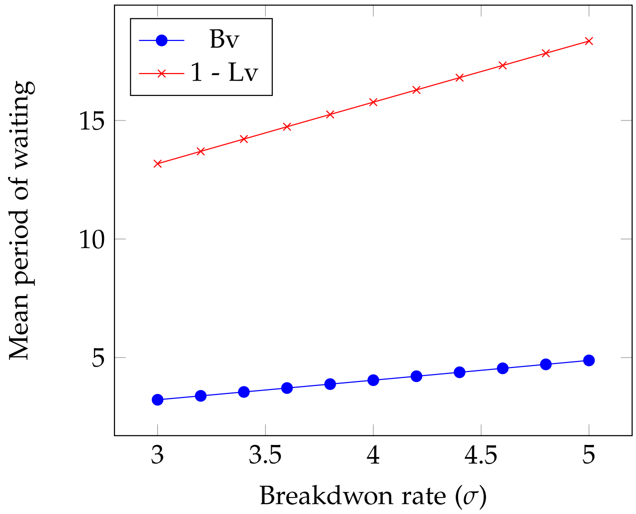

Figure 23: *Breakdown rate vs. Mean period of waiting - MAP-NC/M/1*

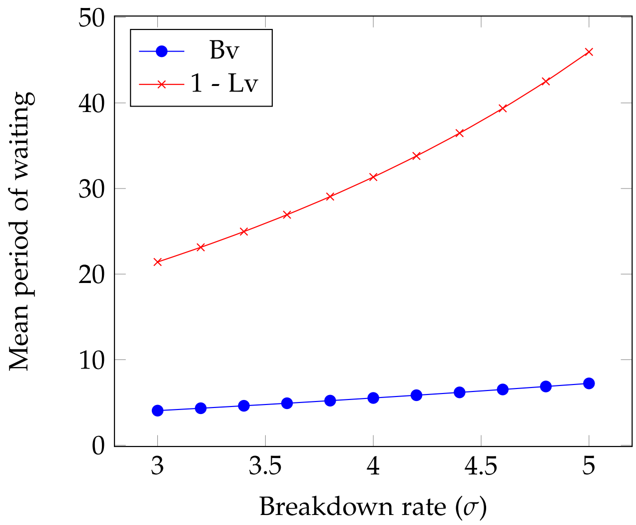

Figure 24: *Breakdown rate vs. Mean period of waiting - MAP-NC/Ek/1*

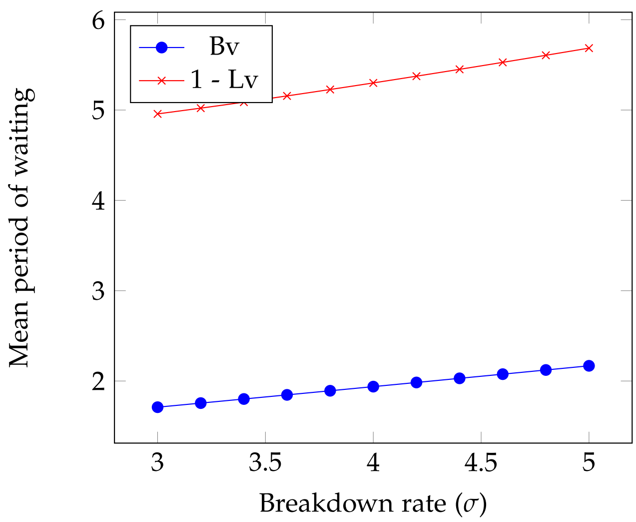

Figure 25: *Breakdown rate vs. Mean period of waiting - MAP-NC/Hk/1*

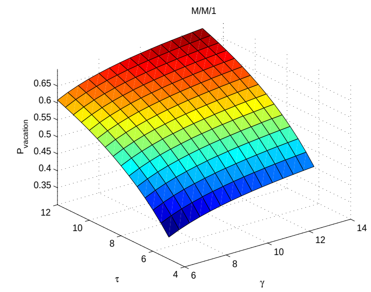

Figure 26: *(Setup and Service rate of the Server) vs.* $P_{vacation}$

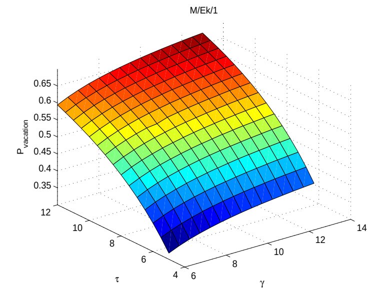

Figure 27: *(Setup and Service rate of the Server) vs.* $P_{vacation}$

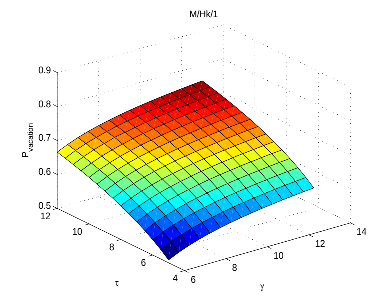

Figure 28: *(Setup and Service rate of the Server) vs.* $P_{vacation}$

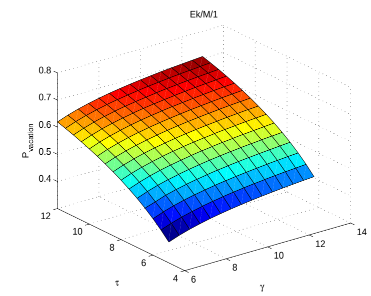

Figure 29: *(Setup and Service rate of the Server) vs.* $P_{vacation}$

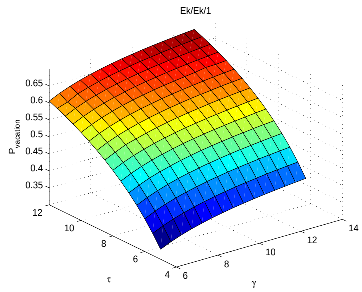

Figure 30: *(Setup and Service rate of the Server) vs.* $P_{vacation}$

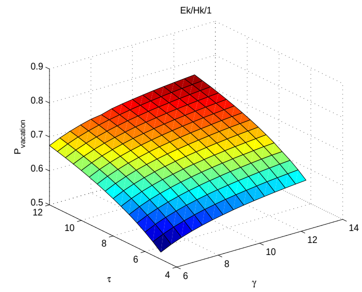

Figure 31: *(Setup and Service rate of the Server) vs.* $P_{vacation}$

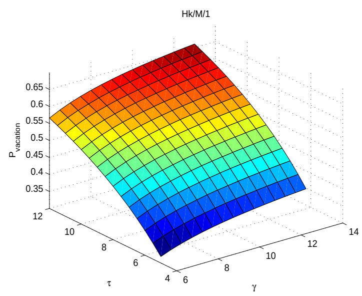

Figure 32: *(Setup and Service rate of the Server) vs.* $P_{vacation}$

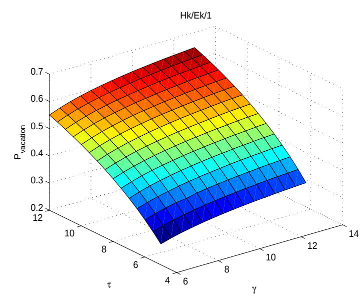

Figure 33: *(Setup and Service rate of the Server) vs.* $P_{vacation}$

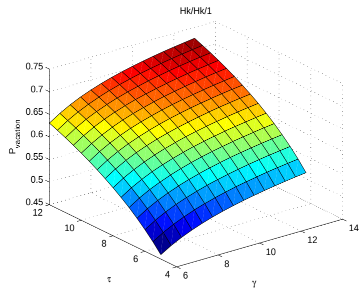

Figure 34: *(Setup and Service rate of the Server) vs. $P_{vacation}$*

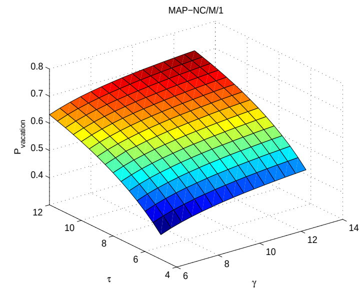

Figure 35: *(Setup and Service rate of the Server) vs. $P_{vacation}$*

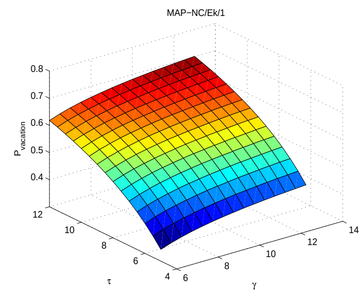

Figure 36: *(Setup and Service rate of the Server) vs. $P_{vacation}$*

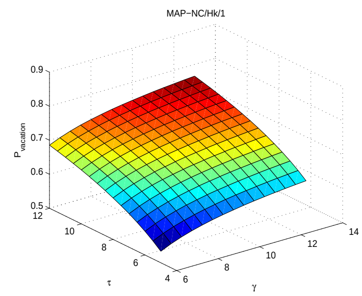

Figure 37: *(Setup and Service rate of the Server) vs. $P_{vacation}$*

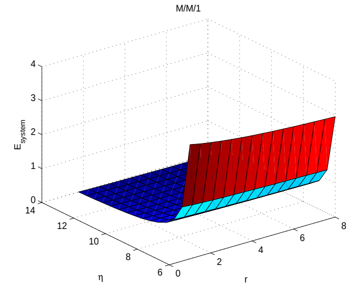

Figure 38: *(Vacation rate and Reneging rate of the server and customer resp.) vs.* E_{system}

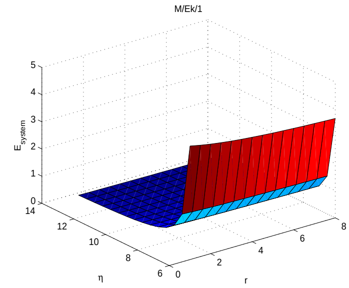

Figure 39: *(Vacation rate and Reneging rate of the server and customer resp.) vs.* E_{system}

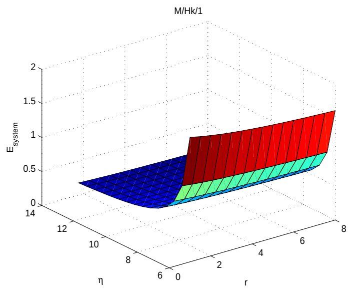

Figure 40: *(Vacation rate and Reneging rate of the server and customer resp.) vs.* E_{system}

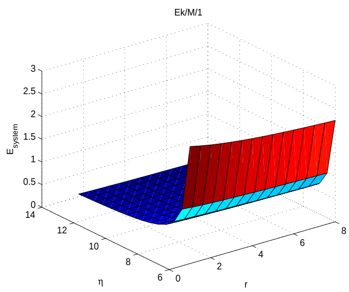

Figure 41: *(Vacation rate and Reneging rate of the server and customer resp.) vs.* E_{system}

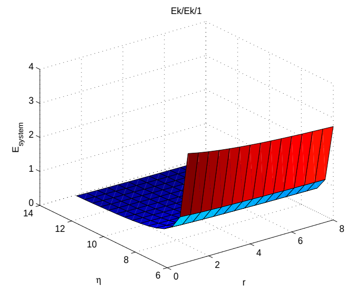

Figure 42: *(Vacation rate and Reneging rate of the server and customer resp.) vs.* E_{system}

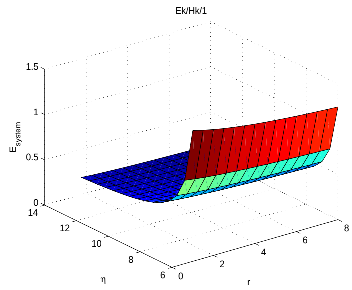

Figure 43: *(Vacation rate and Reneging rate of the server and customer resp.) vs.* E_{system}

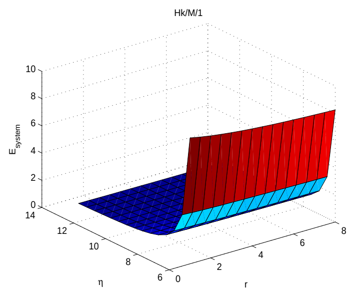

Figure 44: *(Vacation rate and Reneging rate of the server and customer resp.) vs.* E_{system}

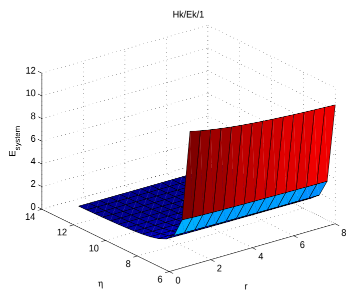

Figure 45: *(Vacation rate and Reneging rate of the server and customer resp.) vs.* E_{system}

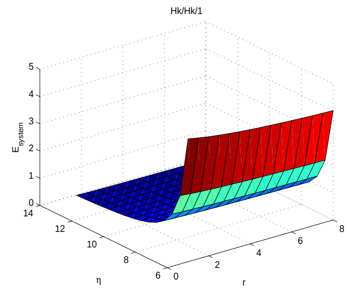

Figure 46: *(Vacation rate and Reneging rate of the server and customer resp.) vs.* E_{system}

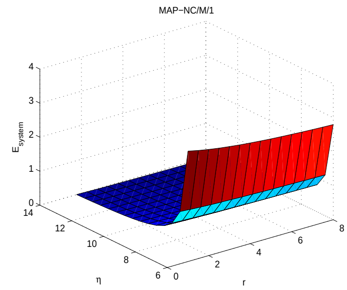

Figure 47: *(Vacation rate and Reneging rate of the server and customer resp.) vs.* E_{system}

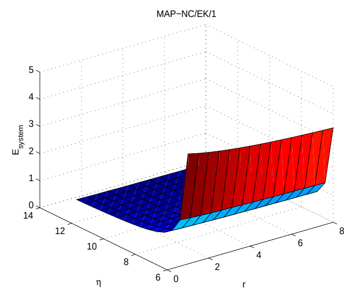

Figure 48: *(Vacation rate and Reneging rate of the server and customer resp.) vs.* E_{system}

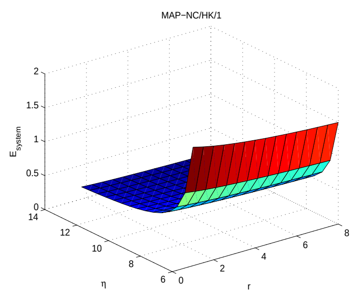

Figure 49: *(Vacation rate and Reneging rate of the server and customer resp.) vs.* E_{system}

IX. Conclusion

Our article deals with a classical queueing model with MAP arrival, single server, PH-service together with vacation, setup time, breakdown, repair, feedback, balking and reneging. The stability condition for our system has been obtained. In addition, the active period of model under study has been explored. The consequences of the vacation rate(η) and the breakdown rate(σ) upon average size of the system and expected waiting time respectively for two different types of vacation, namely 1-limited and Bernoulli vacation have been visualized with the aid of 2D graphs. Further, the impact of both the setup(τ) and service rate(γ) of the server upon probability that the server is undergoing vacation has been pictured with the support of 3D graphs. Also, the consequences of both reneging rate(r) and vacation rate (η) on the average size of the system has been pictured with the support of 3D graphs. In addition, one can perform the cost analysis for our model and can also extend the work by considering BMAP for arrival process.

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A Software Reliability Growth Model Considering Mutual Fault Dependency

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Abstract

Many software reliability growth models (SRGMs) have been introduced since 1970s. Most of the models consider that the faults are independent and debugging method is perfect. In this paper, we present a new SRGM under the assumption that the faults are mutually dependent i.e. repairing a detected fault may introduce new faults or it may simultaneously correct some future faults without any additional effort. The model is validated on two real datasets that are widely used in many studies to demonstrate its applicability. The comparisons with eight established models in terms of Mean Square Error (MSE), Variance, Predictive Ratio Risk (PRR) and R^2 *have been presented.*

Keywords: Software Reliability, SRGM, Software Testing, Debugging, Fault Prediction, Project Management.

I. Introduction

Today we are very much dependent on software systems in many facets of our life. The demand of highly reliable software has rapidly increased. Software development is a time consuming and intensive job that involves many people, process and technology. Thus software systems are error prone. Reliability is an end-user quality feature related to the system-usage. Software reliability can be defined as the probability that no failure occurs up to a specified time interval. Unlike hardware, it is not possible to measure or quantify software reliability directly. With the help of probabilistic and statistical methods, different approaches have been developed for measuring software reliability. However, use of software reliability growth models (SRGMs) is a popular and traditional way to describe the failure patterns and predict the reliability. The SRGMs are represented in abstract forms that include many parameters based on certain assumptions. In the last four decades, a sufficient number of SRGMs have been suggested at regular intervals [1][2]. They are broadly divided into two groups: times between failure models and fault count models [3][4]. The models that recognize MTBF (mean time between failures) as input are referred to as times between failures models and the models that use failure rate are referred to as fault count models. Examples of some times between failure models are The Jelinski-Mornada de-eutrophication model (in short J-M model), Littlewood-Verral model etc. The J-M model, known as one of the earliest models, assumes that the failure rate is constant between failures and reduces in fixed step-size following the repair of each fault [5]. The Littlewood-Verral model which is an updated version of the J-M model considers that the times between failures follow an exponential distribution [6]. Most of the SRGMs fall under the category of fault count models such as follows. Goel-Okumoto model (or G-O model) is a Non Homogeneous Poisson process (NHPP) with an exponentially decaying rate function [7]. Musa Okumoto model represents the cumulative number of failures over time in terms of a logarithmic function [8]. Yamada et. al. proposed the delayed S-Shaped model to describe the increase-decrease failure rate pattern considering the learning process of the testers' skills [9]. Ohba suggested the Inflection S-shaped model with the concept of mutual fault dependency (i.e. some faults are discoverable only after the detection of some specific faults) [10]. Yamada et. al. [11] also suggested a two variant Imperfect Debugging Model by modifying G-O model that incorporates the linear fault introduction rate. H.Pham et. al. suggested PNZ model by considering fault introduction rate is a linear function of testing time [12] and PZ model by considering fault introduction rate is an exponential function of testing time [13]. Recent studies in reliability modelling include different approaches of machine learning techniques or deal with the issue of uncertainties due to random operating environment. Jaiswal and Malhotra [14] tested different ML techniques for software reliability prediction on different datasets collected from industrial projects and compared the results. They concluded that adaptive neuro fuzzy inference system (ANFIS) is the most effective method compared to others in predicting software reliability. Chang et. al. [15] proposed a testing-coverage model considering the uncertainty of operating environment. Pham [16] discussed two NHPP models with and without considering the uncertainty factor based on a log-log distribution function.

Till date, near about 200 software reliability growth models have been suggested [4] and most of them are based on the assumption that the faults are independent. This assumption is not true in real testing environment. The paper presents a model that considers the issue of dependent faults.

II. Proposed Model

A generalized failure intensity function of a software reliability growth model under the assumption that the fault detection rate is proportional to the number of remaining faults is given by [17]:

$$\frac{dm(t)}{dt} = b(t)[a - m(t)] \tag{1}$$

where,

- m(t) : The mean value function (Expected number of faults detected by time t).
- a : Total expected number of faults that exist in the system.
- b(t) : Time dependent fault detection rate per fault.

In practice, it is seen that faults are dependent. Sometimes repairing one fault introduces new faults. Sometimes repairing one fault removes some future faults without any extra effort. Therefore, number of fault detections differs with the number of fault removals. Let us consider that p is the fault removal rate per detected fault. Therefore, number of faults removed at time t is $pm(t)$ and number of remaining faults will be $(a - pm(t))$. From (1), we can write,

$$\frac{dm(t)}{dt} = b(t)[a - pm(t)] \qquad \text{where, } p>0; \tag{2}$$

If p<1 then it means imperfect debugging with high fault introduction rate. If p>1then it indicates the one to many mapping between fault detection and fault removal. P =1 represents perfect debugging with one to one mapping. The solution of eqn. (2) for the mean value function m(t) with the initial condition m(0) = 0, is given by:

$$m(t) = \frac{a}{p}(1 - e^{-p\int_0^t b(t)dt}) \tag{3}$$

We also assume that the fault detection rate per fault will increase with time. Initially the testing team takes time to understand the behavior of the system; hence, fault detection rate is relatively slow. As testing progresses the team gradually becomes familiar with the system leading to higher fault detection rate. In this paper, we consider the following function of b(t):

$$b(t) = b(1+ct) \tag{4}$$

c -is a parameter that reflects the change in fault detection rate with time and b is a constant. Replacing the value of (4) in eqn. (3),

$$m(t) = \frac{a}{p}(1 - e^{-pb(t+ct^2/2)}) \tag{5}$$

This is the mean value function of the proposed model. Now we can derive the failure intensity function from (5),

$$\lambda(t) = \frac{dm(t)}{dt} = ab(ct+1)e^{-pb(t+\frac{ct^2}{2})}) \tag{6}$$

III. Analysis of the Model

We evaluate the performance of the proposed model on two different datasets (DS1 and DS2) and compare the results with the following eight existing models (Table 1).

Table 1. *Software Reliability Models*

Model	m(t)
Goel-Okumoto Model [7]	$a(1-e^{-bt})$
Delayed S-Shaped [9]	$a(1-(1+bt)e^{-bt})$
Inflection S-shaped [10]	$\frac{a(1-e^{-bt})}{1+\beta e^{-bt}}$
Yamada Imperfect Model-1 [11]	$\frac{ab}{\alpha+b}(e^{\alpha t}-e^{-bt})$
Yamada Imperfect Model-2 [11]	$a(1-e^{-bt})\left(1-\frac{\alpha}{b}\right)+\alpha at$
P-N-Z Model [12]	$\frac{a(1-e^{-bt})\left(1-\frac{\alpha}{b}\right)+\alpha at}{1+\beta e^{-bt}}$
Testing Coverage Model [15]	$N\left(1-\left(\frac{\beta}{\beta+(at)^b}\right)^{\alpha}\right)$
Loglog Fault-detection Rate Model [16]	$N(1-e^{-(a^{t^b}-1)})$
Proposed Model	$\frac{a}{p}(1-e^{-pb(t+ct^2/2)})$

A. Comparison Criteria

None of the SRGMs is reliable to get accurate results in all circumstances and thus be selected a priori. It is necessary to compare multiple models and then select the one that match the failure data most accurately. There are many standard criteria known as "Goodness of Fit" criteria available for model comparison and selection [18-20]. In this study, we have used the following four criteria.

- MSE: The mean square error (MSE) is a calculation of how far the estimated values vary from the actual observations, and is defined as [18][19]:

$$\mathrm{MSE} = \frac{\sum_{i=1}^{n}(m_i - m(t_i))^2}{n - k}$$

- Variance: The variance is the standard deviation of the differences between actual and predicted data. It is defined as [18][19]:

$$\mathrm{Variance} = \sqrt{\frac{\sum_{i=1}^{n}(m_i - m(t_i) - \mathrm{Bias})^2}{n - 1}}$$

$$\text{where, Bias} = \frac{\sum_{i=1}^{n}(m(t_i) - m_i)}{n}$$

- PRR: The predictive-ratio risk (PRR) measures the per estimate model deviation from the actual data and is defined as [18][19]:

$$\mathrm{PRR} = \sum_{i=1}^{n}\left(\frac{(m(t_i) - m_i)}{m(t_i)}\right)^2$$

- R^2: It measures how well a model fits the data. It is also known as the "coefficient of determination" and defined as [18][19]:

$$R^2 = 1 - \frac{\sum_{i=1}^{n}(m_i - m(t_i))^2}{\sum_{i=1}^{n}(m_i - \sum_{j=1}^{n}\frac{m_j}{n})^2}$$

The smaller values of MSE, Variance, PRR and AIC criteria indicate fewer numbers of fitting errors and better performance [20] whereas the value of R^2 is expected to be 1 for an ideal model.

B. Dataset Description

The basic approach of the SRGMs is to predict the future faults by analyzing the past failure data. The performance of an SRGM greatly depends on the type of datasets. We consider two datasets from Tandem Technical Report-96.1 [21][22] in our experiment. The Tandem report contains four failure datasets related to the four different releases of Tandem Computer Project. Table 2 presents a failure dataset (DS1) of release 1 having 100 software faults collected over the 20 weeks of testing and the 10000 hours of execution. Table 3 provides the dataset (DS2) of Release 4 having 42 faults collected over the 19 weeks of testing and the 11305 hours CPU execution.

Table 2. *DS1: Tandem computers failure data – Release 1*

Test Week	CPU hrs	Cumulative Faults	Test Week	CPU hrs	Cumulative Faults
1	519	16	11	6539	81
2	968	24	12	7083	86
3	1430	27	13	7487	90
4	1893	33	14	7846	93
5	2490	41	15	8205	96
6	3058	49	16	8564	98
7	3625	54	17	8923	99
8	4422	58	18	9282	100
9	5218	69	19	9641	100
10	5823	75	20	10000	100

Table 3. *DS2: Tandem computers failure data – Release 4*

Test Week	CPU hrs	Cumulative Faults	Test Week	CPU hrs	Cumulative Faults
1	254	1	11	7621	32
2	788	3	12	8783	32
3	1054	8	13	9604	36
4	1393	9	14	10064	38
5	2216	11	15	10560	39
6	2880	16	16	11008	39
7	3593	19	17	11237	41
8	4281	25	18	11243	42
9	5180	27	19	11305	42
10	6003	29	-	-	-

C. Parameter Estimation

The parameters of all the 9-models mentioned in Table 1, have been estimated using the least square estimation (LSE) technique and time weeks. The resultant values of the parameters have been provided in Table 4 for the datasets DS-1 and DS-2 respectively.

Table 4. *Parameter Estimation using LSE*

Model	DS1	DS2
Goel-Okumoto	a = 130.2, b = 0.083	a = 89.63, b = 0.037
Delayed S-Shaped	a = 104, b = 0.265	a = 47.23, b = 0.207
Inflection S-shaped	a =110.829, b =0.172, β =1.205	a =43.36, b =0.279, β = 6.459
Yamada Imperfect Model-1	a =130.2, b =0.083, α =4.25*10^{-4}	a =87.94, b =0.037, α= 0.0001
Yamada Imperfect Model-2	a =130.2, b =0.083, α =1.283*10^{-4}	a =87.69, b =0.038, α=0.0001
P-N-Z Model	a =116.324, b = 0.14, α = 0.001, β = 0.787	a =31.44, b = 0.353, α= 0.023, β = 7.275
Testing Coverage Model	N = 119.205, a = 13.798*10^{-3}, b = 1.111, α = 65.069, β = 7.337	N = 44.398, a = 0.04, b =1.672, α = 25.908, β = 5.356
Loglog Fault-detection Model	N =105.109, a =1.095, b = 0.947	N = 48.72, a = 1.051, b = 1.237
Proposed Model	a = 100.926, b = 0.087, p = 0.937, c = 0.092	a = 41.598, b = 0.027, p = 0.967, c = 0.659

D. Results and Comparison

The criteria values (MSE, Variance, PRR and R^2) of all the models have been provided in Table 5 and 6. For both the datasets, the proposed model provides highest R^2, lowest Variance and PRR and second lowest MSE values. The findings clearly indicate that the proposed model fits better than many existing models studied in the paper. Figure 1 and 2 display two curves representing the deviation of the measured faults according to the proposed model from the actual observed faults for DS-1 and DS-2 respectively.

Table 5. *Model Comparison for DS1*

Model	MSE	Variance	PRR	R^2
Goel-Okumoto	12.915	3.511	0.203	0.986
Delayed S-Shaped	28.065	5.772	1.084	0.969
Inflection S-shaped	10.564	3.177	0.305	0.989
Yamada Imperfect Model-1	13.787	3.514	0.204	0.986
Yamada Imperfect Model-2	13.688	3.504	0.203	0.986
P-N-Z Model	12.662	3.315	0.277	0.988
Testing Coverage Model	14.577	3.445	0.3	0.987
Loglog Fault-detection Rate Model	8.437	2.861	0.238	0.991
Proposed Model	10.688	3.064	0.295	0.990

Table 6. *Model Comparison for DS2*

Model	MSE	Variance	PRR	R^2
Goel-Okumoto	5.1	2.527	6.726	0.976
Delayed S-Shaped	1.095	1.017	0.126	0.995
Inflection S-shaped	1.117	0.999	0.8	0.995
Yamada Imperfect Model-1	5.380	2.222	6.324	0.976
Yamada Imperfect Model-2	5.410	2.519	6.816	0.976
Testing Coverage Model	1.50	1.340	0.111	0.995
Loglog Fault-detection Rate Model	3.75	1.951	5.235	0.983
P-N-Z Model	1.086	0.973	0.424	0.995
Proposed Model	1.150	0.983	0.340	0.995

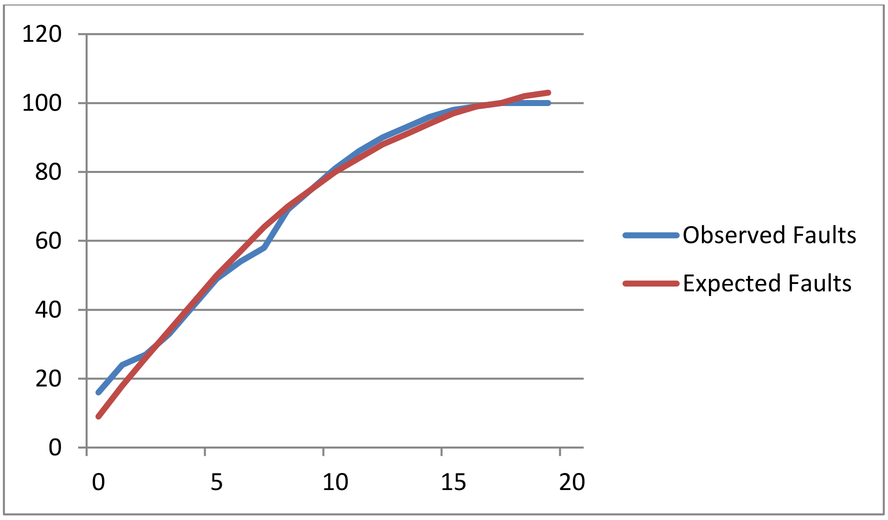

Figure 1: *Expected faults vs. observed faults for DS1*

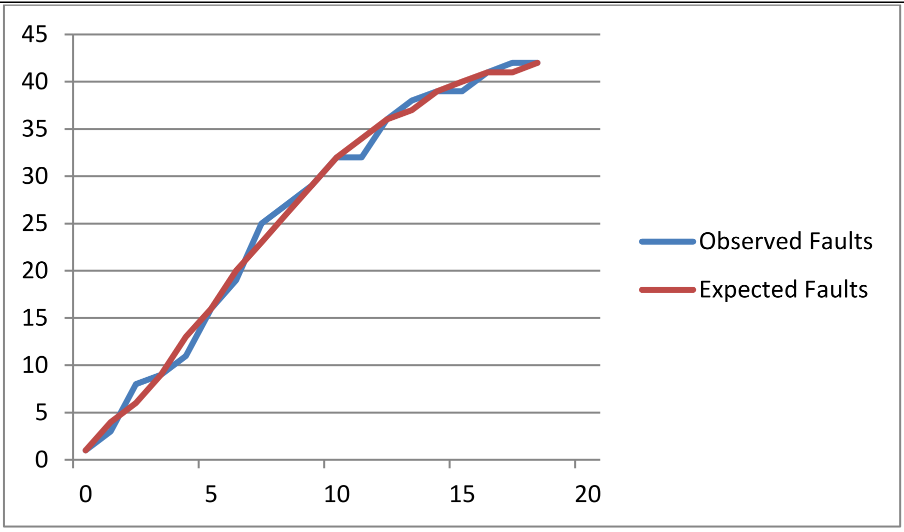

Figure 2: *Expected faults vs. observed faults for DS2*

IV. Conclusion

The paper presents a new software reliability growth model addressing the issue of mapping between fault detection and fault removal processes. The proposed model incorporates a time dependent fault detection rate function. The model has been tested with two actual failure datasets and compared with eight established models using four different criteria. The results are very promising. However, there are some scopes for possible improvements. We only tested the model with two datasets, which is insufficient to claim any superiority about the model performance. Moreover, the datasets are relatively old. Future work will focus on broader validation of the proposed model based on more recent datasets considering different comparison criteria.

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A Compounded Probability Model for Decreasing Hazard and its Inferential Properties

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Abstract

Early failures are generally observed due to latent defects within a product caused by faulty components, faulty assembly, transportation damage and installation damage. Also early life (infant mortality) failures tend to exhibit a decreasing failure rate over time. Such type of problems can be modelled either by a complex distribution having more than one parameter or by finite mixture of some distribution. In this article a single parameter continuous compounded distribution is proposed to model such type of problems. Some important properties of the proposed distribution such as distribution function, survival function, hazard function and cumulative hazard function, entropies, stochastic ordering are derived. The maximum likelihood estimate of the parameter is obtained which is not in closed form, thus iteration procedure is used to obtain the estimate of parameter. The moments of the proposed distribution does not exist. Some real data sets are used to see the performance of proposed distribution with comparison of some other competent distributions of decreasing hazard using Likelihood, AIC, AICc, BIC and KS statistics.

Keywords: Entropy, Hazard function, KS, MLE, Order Statistics, Quantile function.

I. Introduction

Normal, exponential, gamma and weibull distributions are the basic distributions that demonstrated in a number of theoretical results in the distributions theory. Particularly, exponential distribution is an invariable example for a number of theoretical concepts in reliability studies. It is characterized as constant hazard rate. In case of necessity for an increasing/decreasing failure rate model ordinarily the choice falls on weibull distribution. Lindley distribution is an increasing hazard rate distribution and has its own importance as a life testing distribution. The lindley distribution is one parameter distribution that is a mixture of exponential and gamma distributions and was proposed by Lindley [16]. The lindley distribution is used to explain the lifetime phenomenon such as engineering, biology, medicine, ecology and finance. Ghitany et al. [10]. Lindley distribution has generated little attention in excess of the exponential distribution because of its decreasing mean residual life function and increasing hazard rate however exponential distribution has constant mean residual life function and hazard rate.

Adamidis & Loukas [1] introduced a two-parameter exponential-geometric distribution with decreasing hazard rate and Barreto-Souza et al. [5] introduced a decreasing failure rate model, compounding exponential and poisson-lindley distribution (EPL) and the probability density function is given as

$$f_{epl}(x;\beta,\theta) = \frac{\beta\theta^2(1+\theta)^2 e^{-\beta x}}{(1+3\theta+\theta^2)}\frac{(3+\theta-e^{-\beta x})}{(1+\theta-e^{-\beta x})^3}; x>0, \beta>0, \theta>0 \tag{1}$$

Another idea was proposed by Kuş [15] and Tahmasbi & Rezaei [27]. They introduced the exponential Poisson (EP) and exponential logarithmic (EL) distributions and the pdf is given by

$$f_{ep}(x;\beta,\lambda) = \frac{\lambda\beta}{1-e^{-\lambda}} e^{-\lambda-\beta x+\lambda e^{-\beta x}}; x > 0, \beta > 0, \lambda > 0 \tag{2}$$

$$f_{el}(x;\beta,p) = \frac{1}{-\log p} \frac{\beta(1-p)e^{-\beta x}}{1-(1-p)e^{-\beta x}}; x > 0, \beta > 0, p \in (0,1) \tag{3}$$

Chahkandi & Ganjali [8] introduced a class of distributions, which is exponential power series distributions (EPS), where compounding procedure follows the same way that was previously given by Adamidis & Loukas [1]. Weibull [29] a Swedish mathematician describe the weibull distribution that is usefull for increasing as well as decresing hazard and the pdf is defined as

$$f_w(x;\beta,\alpha) = \alpha\beta^\alpha x^{\alpha-1} e^{-\beta x}; x > 0, \beta > 0, \alpha > 0 \tag{4}$$

Natural mixing of exponential populations, giving rise to a decreasing hazard rate distribution, were first introduce by Proschan [23]. Subsequently other distributions with decreasing hazard rates of practical interest were discussed by Cozzolino [7]. The distributions with decreasing failure rate (DFR) are discussed in the works of Lomax [18], Barlow et al. [4], Barlow & Marshall [2, 3], Marshall & Proschan [19], Dahiya & Gurland [9], Saunders & Myhre [25], Nassar [21], Gleser [12], Gurland & Sethuraman [13]. Keeping these ideas in view, in this study, an attempt has been made to develop a new lifetime distribution by compounding exponential and lindley distribution and named as compounded exponential-lindley (CEL) distribution. The distributional properties, estimation of parameters, Fisher information, entropies, stochastic ordering, quantile function, order statistics and simulation study for the proposed distribution have been discussed in detail.

II. Proposed Distribution

Let $X_1, X_2, ..., X_n$ be a random sample from following exponential distribution with scale parameter $\lambda > 0$ and the probability density function (pdf) is in the form

$$f(x|\lambda) = \lambda e^{-\lambda x}; \quad x > 0, \lambda > 0 \tag{5}$$

The parameter $\lambda > 0$ of the above distribution takes continuous value and hazard of the distribution is constant. Now we assume the parameter λ is a random variable follows lindley distribution with pdf given as

$$\phi(\lambda;\theta) = \frac{\theta^2}{(\theta+1)}(1+\lambda)e^{-\theta\lambda}; \quad \theta > 0, \lambda > 0 \tag{6}$$

Now the pdf of the proposed distribution CEL is given by

$$\begin{aligned} g(x;\theta) &= \int_0^\infty f(x|\lambda)\phi(\lambda;\theta)d\lambda = \frac{\theta^2}{(\theta+1)}\int_0^\infty (\lambda+\lambda^2)e^{-\lambda(x+\theta)}d\lambda \\ &= \frac{\theta^2}{(\theta+1)}\frac{(x+\theta+2)}{(x+\theta)^3}; \quad x > 0, \theta > 0 \end{aligned} \tag{7}$$

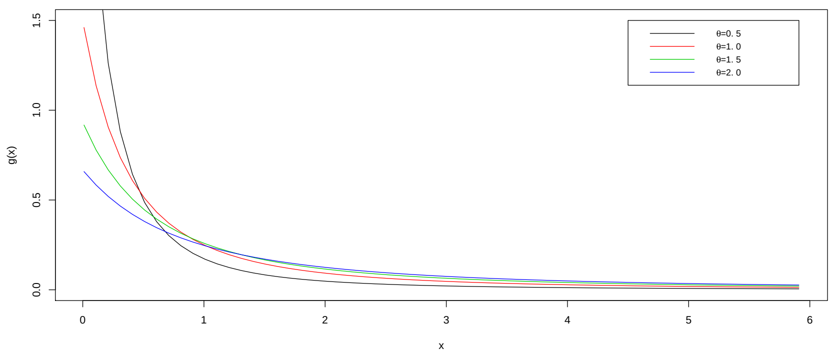

Figure 1: *Probability density function of CEL distribution*

and the cumulative distribution function (cdf) of CEL is obtained as

$$G(x;\theta) = \frac{x\left[x(\theta+1)+\theta(\theta+2)\right]}{(\theta+1)(x+\theta)^2}; x>0, \theta>0 \tag{8}$$

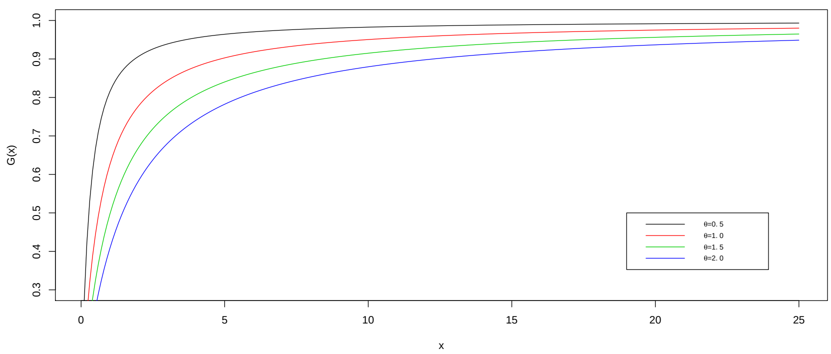

Figure 2: *Cumulative distribution function of CEL distribution*

From the figure 1 and 2, it is clear that the distribution is early failure distribution for smaller value of θ. The survival or reliability function $S(x)$ of CEL having pdf (7), is given as

$$S(x) = \frac{\theta^2(x+\theta+1)}{(\theta+1)(x+\theta)^2} \tag{9}$$

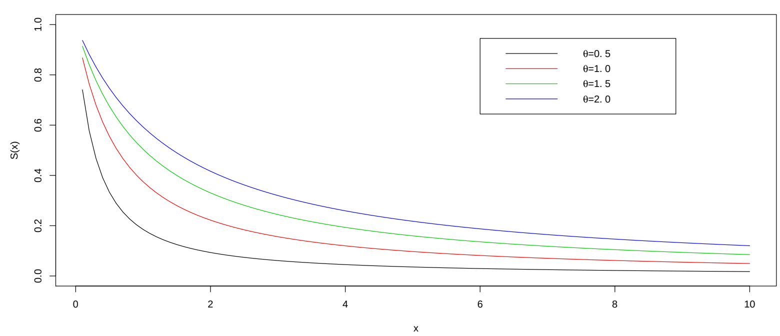

Figure 3: *Survival function of CEL distribution*

The hazard function is defined as

$$h(x) = \frac{g(x)}{1-G(x)} = \frac{g(x)}{S(x)} = \frac{(x+\theta+2)}{(x+\theta)(x+\theta+1)} \tag{10}$$

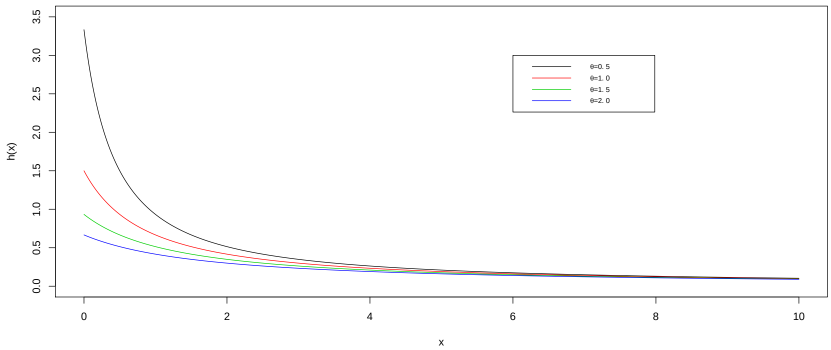

Figure 4: *Hazard rate function of CEL distribution*

According to Glaser [11], $g(t)$ is density function, $g'(t)$ is the first order derivative and $\eta(t) = -\frac{g'(t)}{g(t)}$. If $\eta'(t) > 0 \quad \forall \quad t > 0$, then the distribution has increasing failure rate (IFR) and if $\eta'(t) < 0 \quad \forall \quad t > 0$, then the distribution has decreasing failure rate (DFR). For the proposed CEL distribution

$$\eta(t) = \frac{2t+3\theta+3}{(t+\theta)(t+\theta+2)} \tag{11}$$

Differentiating $\eta(t)$ with respect to t we get

$$\begin{aligned} \eta'(t) = -\frac{2}{(t+\theta)(t+\theta+2)} - \frac{4}{(t+\theta)^2(t+\theta+2)} - \frac{2(\theta-1)}{(t+\theta)(t+\theta+2)^2} \\ - \frac{2(\theta-1)}{[(t+\theta)(t+\theta+2)]^2} \end{aligned} \tag{12}$$

Now from the equation (12) we have $\eta'(t) < 0$ for all $t > 0$, hence distribution has DFR. Also the hazard function of the CEL distribution is

$$h(x) = \frac{(x+\theta+2)}{(x+\theta)(x+\theta+1)} = \frac{2}{(x+\theta)} - \frac{1}{(x+\theta+1)}$$

After differentiating (10) with respect to x we get

$$\begin{aligned} h'(x) &= -\frac{2}{(x+\theta)^2} + \frac{1}{(x+\theta+1)^2} \\ \lim_{x\to 0} h'(x) &= -\frac{2}{\theta^2} + \frac{1}{(\theta+1)^2} < 0 \quad \forall \quad \theta > 0 \end{aligned} \tag{13}$$

Therefore $h'(0) < 0 \quad \forall \quad \theta > 0$, Hence CEL distribution is a distribution of monotonic decreasing hazard with increasing time.
Now Cumulative hazard function $H(t)$ is defined as

$$H(t) = \int_0^t h(x)dx = \log\left[\left(\frac{\theta+1}{t+\theta+1}\right)\left(\frac{t+\theta}{\theta}\right)^2\right] \tag{14}$$

Theorem 1. *The moments of the $CEL(\theta)$ distribution does not exists.*

Proof: Suppose the random variable X comes from $CEL(\theta)$ then the r^{th} moment is given by

$$E(X^r) = \int_0^\infty x^r g(x)dx = \frac{\theta^2}{\theta+1}\int_0^\infty x^r \frac{x+\theta+2}{(x+\theta)^3}dx$$

Now

$$\frac{1}{\theta+1}\int_0^\infty \frac{x^r}{(1+\frac{x}{\theta})^2}dx + \frac{2}{\theta(\theta+1)}\int_0^\infty \frac{x^r}{(1+\frac{x}{\theta})^3}dx$$

Let $\frac{x}{\theta} = z;\quad dx = \theta dz;\quad x \to 0, z \to 0,$ and $\quad x \to \infty, z \to \infty$ above integral become

$$\frac{\theta^{r+1}}{\theta+1}\int_0^\infty \frac{z^r}{(1+z)^2}dz + \frac{2\theta^{r+1}}{\theta(\theta+1)}\int_0^\infty \frac{z^r}{(1+z)^3}dz$$

using Beta integral of second kind i.e $\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx = B(m,n) \quad ; m>0; n>0$, we get

$$\begin{aligned} E(X^r) &= \frac{\theta^{r+1}}{\theta+1}B(r+1,1-r) + \frac{2\theta^{r+1}}{\theta(\theta+1)}B(r+1,2-r) \\ &= \frac{\theta^{r+1}}{\theta+1}\left[B(r+1,1-r) + \frac{2}{\theta}B(r+1,2-r)\right] \end{aligned} \tag{15}$$

Here range is $-1 < r < 1$. But range of r should be $r \geq 1$. Hence $E(X^r)$ does not exists. Therefore mean, variance, SD as well as higher order moments does not edxists for $CEL(\theta)$.

Theorem 2. *The moment generating function of $CEL(\theta)$ does not exists.*

Proof: Let X be the random variable from $NWEL(\theta)$ distribution then the moment generating function (mgf) is given by

$$\begin{aligned} E(e^{tx}) &= \int_0^\infty e^{tx} g(x)dx = \frac{\theta^2}{\theta+1}\int_0^\infty e^{tx}\frac{x+\theta+2}{(x+\theta)^3}dx \\ &= \frac{\theta^2}{\theta+1}\left[\int_0^\infty \frac{e^{tx}}{(x+\theta)^2}dx + \int_0^\infty \frac{2e^{tx}}{(x+\theta)^3}dx\right] \end{aligned} \tag{16}$$

Now

$$\begin{aligned} \int_0^\infty \frac{e^{tx}}{(x+\theta)^2}dx &= \left[\frac{e^{tx}}{-(x+\theta)}\right]_0^\infty + t\int_0^\infty \frac{e^{tx}}{(x+\theta)}dx \\ &= \frac{1}{\theta} + \lim_{\epsilon\to\infty}\left[t\int_0^\epsilon \frac{e^{tx}}{(x+\theta)}dx\right] \end{aligned} \tag{17}$$

Now applying L'Hospital rules we get

$$\lim_{x\to\infty}\frac{e^{tx}}{(x+\theta)} = \lim_{x\to\infty}\frac{te^{tx}}{1} = \infty$$

Hence integrand is divergent, as well as the function is not integrable over R we conclude thta $E(e^{tx})$ does not exists. The characteristic function of CEL distribution is defined as

$$\Phi_x(t) = \int_0^\infty e^{itx}g(x)dx = \frac{1}{\theta+1}\sum_{k=0}^\infty (-1)^k\frac{(k+1)!}{(it)^{k+1}}\left[1+\frac{2}{\theta}(k+2)\right] \tag{18}$$

III. Entropies

An entropy is a measure of randomness occured in any system. Entropy is an important property of probability distributions and it measures the uncertainty in a probability distribution.

I. Rényi Entropye

An entropy is a measure of variation of the uncertainty, Rényi [24] gave an expression of the Entropy function defined by

$$e(\eta) = \frac{1}{1-\eta} \log \left[\int_0^\infty g^\eta(x) dx \right]$$

where $0 < \eta < 1$, Substituting the value of g(x) from (7)

$$\begin{aligned} e(\eta) &= \frac{1}{1-\eta} \log \left[\int_0^\infty \left(\frac{\theta^2}{(\theta+1)} \frac{(x+\theta+2)}{(x+\theta)^3} \right)^\eta dx \right] \\ &= \frac{1}{1-\eta} \log \left[\left(\frac{\theta^2}{\theta+1} \right)^\eta \int_0^\infty \left\{ \frac{1}{(x+\theta)^2} + \frac{2}{(x+\theta)^3} \right\} dx \right] \end{aligned}$$

Now applying Binomial expansion $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$ we get

$$\frac{1}{1-\eta} \log \left[\left(\frac{\theta^2}{\theta+1} \right)^\eta \int_0^\infty \sum_{k=0}^{\eta} \binom{\eta}{k} \left(\frac{1}{x+\theta} \right)^{2k} \left(\frac{2}{(x+\theta)^3} \right)^{\eta-k} dx \right]$$

after simlification we get the Renyi entropy as

$$e(\eta) = \frac{\eta}{1-\eta} \log \left(\frac{\theta^2}{\theta+1} \right) + \frac{1}{1-\eta} \log \left[\sum_{k=0}^{\eta} \binom{\eta}{k} \frac{2^{\eta-k}}{(3\eta-k-1)\theta^{(3\eta-k-1)}} \right] \tag{19}$$

where $0 < \eta < 1, \quad \theta > 0, \quad x > 0$

II. Tsallis Entropy

This is introduced by Tsallis [28] as a basis for generalizing the standard statistical mechanics

$$\begin{aligned} S_\lambda &= \frac{1}{1-\lambda} \left[1 - \int_0^\infty g^\lambda(x) dx \right] \\ &= \frac{1}{1-\lambda} \left[1 - \left(\frac{\theta^2}{(\theta+1)} \right)^\lambda \int_0^\infty \left(\frac{(x+\theta+2)}{(x+\theta)^3} \right)^\lambda dx \right] \end{aligned}$$

Now applying Binomial expansion $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$ and simplifying we get Tsallis Entropy as in (20).

$$e(\eta) = \frac{1}{1-\lambda} \left[1 - \left(\frac{\theta^2}{\theta+1} \right)^\lambda \sum_{k=0}^{\lambda} \binom{\lambda}{k} \frac{2^{\lambda-k}}{(3\lambda-k-1)\theta^{(3\lambda-k-1)}} \right] \tag{20}$$

IV. Quantile Function

The quantile function for CEL distribution is defined in the form $x_q = Q(u) = G^{-1}(u)$ where $Q(u)$ is the quantile function of $G(x)$ in the range $0 < u < 1$. Taking $G(x)$ is the cdf of CEL distribution and inverting it as above will give us the quantile function as follows

$$G(x) = \frac{x\left[x(\theta+1)+\theta(\theta+2)\right]}{(\theta+1)(x+\theta)^2} = u \tag{21}$$

Simplifying equation (21) above gives the following:

$$\left(\frac{x}{x+\theta}\right)^2 + \frac{x\theta(\theta+2)}{(x+\theta)^2} = u$$

Now let $\frac{x}{x+\theta} = z$ we get from above

$$z^2 + \left(\frac{\theta+2}{\theta+1}\right) z(1-z) = u$$
$$z^2 - z(\theta+2) + u(\theta+1) = 0 \tag{22}$$

This is a quadratic equation and after solving we get the solution for x as

$$z = \frac{x}{x+\theta} = \frac{(\theta+2) \pm \sqrt{(\theta+2)^2 - 4u(\theta+1)}}{2}$$
$$Q(u) = \theta\left[\frac{2}{-\theta \pm \sqrt{(\theta+2)^2 - 4u(\theta+1)}} - 1\right] \tag{23}$$

where u is a uniform variate on the unit interval (0,1).
The median of X from the CEL distribution is simply obtained by setting $u = 0.5$ and this substitution of $u = 0.5$ in the above equation (23) gives.

$$Median = \theta\left[\frac{2}{-\theta + \sqrt{(\theta+1)^2 + 1}} - 1\right] \tag{24}$$

Bowley's measure of skewness based on quartiles is defined as:

$$SK = \frac{Q(\frac{3}{4}) - 2Q(\frac{1}{2}) + Q(\frac{1}{4})}{Q(\frac{3}{4}) - Q(\frac{1}{4})} \tag{25}$$

and [20] presented the Moors' kurtosis based on octiles by

$$KT = \frac{Q(\frac{7}{8}) - Q(\frac{5}{8}) - Q(\frac{3}{8}) + Q(\frac{1}{8})}{Q(\frac{6}{8}) - Q(\frac{1}{8})} \tag{26}$$

where Q(.) is calculated by using the quantile function from equation (23).

V. Stochastic Orderings

Stochastic ordering of a continuous random variable is an important tool to judging their comparative behaviour. A random variable X is said to be smaller than a random variable Y.
(i) Stochastic order $X \leq_{st} Y$ if $F_X(x) \geq F_Y(x)$ for all x.
(ii) Hazard rate order $X \leq_{hr} Y$ if $h_X(x) \geq h_Y(x)$ for all x.
(iii) Mean residual life order $X \leq_{mrl} Y$ if $m_X(x) \geq m_Y(x)$ for all x.
(iv) Likelihood ratio order $X \leq_{lr} Y$ if $\frac{f_X(x)}{f_Y(x)}$ decreases in x.

The following results by Shaked & Shanthikumar [26] are well known for introducing stochastic ordering of distributions

$$\begin{aligned} X \leq_{lr} Y \implies X \leq_{hr} Y &\implies X \leq_{mrl} Y \\ i.e \quad & \quad X \leq_{st} Y \end{aligned}$$

with the help of following theorem we claim that CEL distribution is ordered with respect to strongest likelihood ratio ordering

Theorem 3. *Let $X \sim CEL(\theta_1)$ distribution and $Y \sim CEL(\theta_2)$ distribution. If $\theta_1 > \theta_2$ then $X \leq_{lr} Y$ and therefore $X \leq_{hr} Y$, $X \leq_{mrl} Y$ and $X \leq_{st} Y$.*

Proof: We have

$$\frac{f_X(x)}{f_Y(x)} = \frac{\theta_1^2(\theta_2+1)}{\theta_2^2(\theta_1+1)}\left(\frac{x+\theta_1+2}{x+\theta_2+2}\right)\left(\frac{x+\theta_2}{x+\theta_1}\right)^3; \qquad x > 0$$

Now taking log both side we get

$$\log\left[\frac{f_X(x)}{f_Y(x)}\right] = \log\left[\frac{\theta_1^2(\theta_2+1)}{\theta_2^2(\theta_1+1)}\right] + \log\left(\frac{x+\theta_1+2}{x+\theta_2+2}\right) + 3\log\left(\frac{x+\theta_2}{x+\theta_1}\right)$$

By differentiating both side we get

$$\frac{d}{dx}\log\frac{f_X(x)}{f_Y(x)} = \frac{\theta_2-\theta_1}{(2+\theta_1+x)(2+\theta_2+x)} + \frac{3(\theta_2-\theta_1)}{(x+\theta_1)(x+\theta_2)}$$

Thus for $\theta_1 > \theta_2$, $\frac{d}{dx}\log\frac{f_X(x)}{f_Y(x)} < 0$.This means that $X \leq_{lr} Y$ and hence $X \leq_{hr} Y$, $X \leq_{mrl} Y$ and $X \leq_{st} Y$.

VI. Distribution of order statistics

Let $X_1, X_2, ..., X_m$ be a random sample of size m from CEL distribution and let $X_{1;m} \leq X_{2;m} \leq ... \leq X_{m;m}$ represent the corresponding order statistics. The pdf of $X_{m;m}$ i.e r^{th} order statistics is given by

$$\begin{aligned} g_{(r:m)}(x) &= \frac{m!}{(r-1)!(m-r)!}G^{r-1}(x)\left[1-G(x)\right]^{m-r}g(x) \\ &= Z\sum_{l=0}^{m-r}\binom{m-r}{l}(-1)^l G^{r+l-1}(x)g(x) \end{aligned} \tag{27}$$

where $Z = \frac{m!}{(r-1)!(m-r)!}$ and $g(x)$ and $G(x)$ are pdf and cdf of CEL distribution defined in (7) and (8) respectively.
Substituting for $G(x)$ and $g(x)$ in (27) and applying the general binomial expansion, we have

$$\begin{aligned} g_{(r:m)}(x) &= Z\sum_{l=0}^{m-r}\binom{m-r}{l}(-1)^l\left[\frac{x\left[x(\theta+1)+\theta(\theta+2)\right]}{(\theta+1)(x+\theta)^2}\right]^{r+l-1}\frac{\theta^2}{(\theta+1)}\frac{(x+\theta+2)}{(x+\theta)^3} \\ &= Z\sum_{l=0}^{m-r}\sum_{k=0}^{(r+l-1)}\binom{m-r}{l}\binom{r+l-1}{k}C_{l;k}\frac{x^{2r+2l-k-2}(x+\theta+2)}{(x+\theta)^{2r+2l+1}} \end{aligned} \tag{28}$$

where $C_{l;k} = (-1)^l\left(\frac{\theta^2}{(\theta+1)}\right)^{k+1}\left(\frac{\theta+2}{\theta}\right)^k$.
Hence, the pdf of the minimum order statistic $X_{(1)}$ and maximum order statistic $X_{(n)}$ of the CEL distribution are respectively given by, respectively given by

$$g_{(1:m)}(x) = Z\sum_{l=0}^{m-1}\sum_{k=0}^{l}\binom{m-1}{l}\binom{l}{k}C_{l;k}\frac{x^{2l-k}(x+\theta+2)}{(x+\theta)^{2l+3}} \tag{29}$$

VII. Estimation of the Parameter of CEL Distribution

Suppose $X = (X_1, X_2, X_3, ..., X_n)$ be an independently and identically distributed (iid) random variables of size n with pdf (7) from CEL(θ). Then, the likelihood function based on observed sample $X = (x_1, x_2, x_3, ..., x_n)$ is defined as

$$L(\theta; x) = \left(\frac{\theta^2}{\theta+1}\right)^n \prod_{i=0}^{n} \frac{x_i+\theta+2}{(x_i+\theta)^3} \tag{30}$$

The log-likelihood function corresponding to (30) is given by

$$\log L = 2n\log\theta - n\log(\theta+1) + \sum_{i=0}^{n}\{\log(x_i+\theta+2) - 3\log(x_i+\theta)\} \tag{31}$$

Hence, the log-likelihood equation for estimating θ is

$$\frac{2n}{\theta} - \frac{n}{(\theta+1)} + \sum_{i=0}^{n}\left\{\frac{1}{(x_i+\theta+2)} - \frac{3}{(x_i+\theta)}\right\} = 0 \tag{32}$$

Above equation is not solvable analytically for θ. Thus numerical iteration technique is used to get its numerical solution. Fisher Information matrix can be estimated by

$$\begin{aligned} I(\hat{\theta}) &= \left[\frac{-\partial^2}{\partial\theta^2}\log L\right]_{\theta=\hat{\theta}} \\ \frac{\partial^2}{\partial\theta^2}\log L &= -\frac{2n}{\theta^2} + \frac{n}{(\theta+1)^2} + \sum_{i=0}^{n}\left\{\frac{3}{(x_i+\theta)^2} - \frac{1}{(x_i+\theta+2)^2}\right\} \end{aligned} \tag{33}$$

For large samples, we can obtain the confidence intervals based on Fisher information matrix $I^{-1}(\hat{\theta})$ which provides the estimated asymptotic variance for the parameter θ. Thus, a two-sided $100(1-\alpha)\%$ confidence interval of θ and it is defined as $\hat{\theta} \pm Z_\alpha/2\sqrt{var\hat{\theta}}$. Where $Z_\alpha/2$ denotes the upper α-th percentile of the standard normal distribution.

VIII. Simulation study

In this section we evaluate the performance of the MLEs of the model parameter for the CEL distribution. We generate random variables from $CEL(\theta)$ and then obtain m.l.e. of the parameter θ, Now for $\theta = 1.5, 2, 2.5, 3$ we generate the sample size 20, 30, 50, 90, 150, 200. The program is replicated N= 2,500 times to get the maximum likelihood estimate of θ. The simulation results are reported in Table (1).

Table 1: *Simulation results for different values of θ*

	n	Bias	MSE	Var.	Est.
θ=1.5	20	0.07273	0.25756	0.26034	1.57273
	30	0.06864	0.16356	0.16447	1.56864
	50	0.03938	0.09339	0.09211	1.53938
	90	0.01632	0.04662	0.04854	1.51632
	150	0.01270	0.02856	0.02858	1.51270
	200	0.01126	0.02597	0.02152	1.51126
θ=2	20	0.11554	0.53363	0.50319	2.11554
	30	0.07354	0.29175	0.30196	2.07354
	50	0.04340	0.16161	0.16965	2.04340
	90	0.01612	0.08701	0.08982	2.01612
	150	0.01415	0.05453	0.05321	2.01415
	200	0.00896	0.03761	0.03949	2.00896
θ=2.5	20	0.15021	0.82604	0.81882	2.65021
	30	0.11545	0.50885	0.50463	2.61544
	50	0.06234	0.27557	0.27921	2.56234
	90	0.02329	0.14411	0.14678	2.52329
	150	0.02114	0.08766	0.08675	2.52114
	200	-0.00545	0.06452	0.06337	2.49455
θ=2.5	20	0.21267	1.23999	1.25054	3.21267
	30	0.16488	0.82261	0.77061	3.16488
	50	0.09941	0.40727	0.42342	3.09941
	90	0.06733	0.22374	0.22481	3.06733
	150	0.03938	0.12947	0.13061	3.03938
	200	0.03598	0.09335	0.09728	3.03598

It is clearly observed from the Table (1) that the values of bias and mean square error (MSE) of the parameter estimates decreases as the sample size n increases. It indicates the consistency of the estimator.

IX. Goodness of fit

The application of goodness of fit of proposed CEL distribution has been discussed with two real data sets. First data set presents the results of a life-test experiment in which specimens of a type of electrical insulating fluid were subject to a constant voltage stress (34 KV/minutes), this data set is reported by Nelson [22] and other data is represents 30 failure times of the air conditioning system of an airplane has been reported in a paper by Linhart & Zucchini [17] and has also been analyzed by Barreto-Souza & Bakouch [6] and so on. For comparing the suitability of the model, we have considered following criterion's; namely AIC (Akaike Information Criterion), BIC (Bayesian information criterion), AICc (Corrected Akaike information criterion) and KS statistics with associated p-value of the fitted distributions are presented in Table (2) and Table (3).The AIC, BIC, AICc and KS Statistics are computed using the following formulae

$$AIC = -2loglik + 2k, \qquad BIC = -2loglik + k\log n$$
$$AICc = AIC + \frac{2k^2 + 2k}{n-k-1}, \qquad D = \sup_x |F_n(x) - F_0(x)|$$

where k= the number of parameters, n= the sample size, and the $F_n(x)$=empirical distribution function and $F_0(x)$ is the theoretical cumulativedistribution function.

Table 2: *MLE's, - 2ln L, AIC, KS and p-values of the fitted distributions for the 1st dataset.*

Distribution	Estimate	-2LL	AIC	BIC	AICc	KS	p-value
CEL(θ)	7.0385	137.98	139.98	140.92	140.21	0.1131	0.9458
EPL(β,θ)	(0.0334, 0.5521)	136.18	140.18	142.06	140.93	0.1500	0.7312
EL(β,p)	(0.0393, 0.0982)	135.98	139.98	141.87	140.73	0.1382	0.8137
EP(β,λ)	(0.0409, 2.2112)	136.89	140.89	142.78	141.64	0.1611	0.6497
Weibull(β,θ)	(0.0818, 0.7708)	136.77	140.77	142.66	141.52	0.1613	0.6482
Gamma(β,θ)	(0.0480, 0.6897)	137.23	141.23	143.12	141.98	0.1846	0.4802

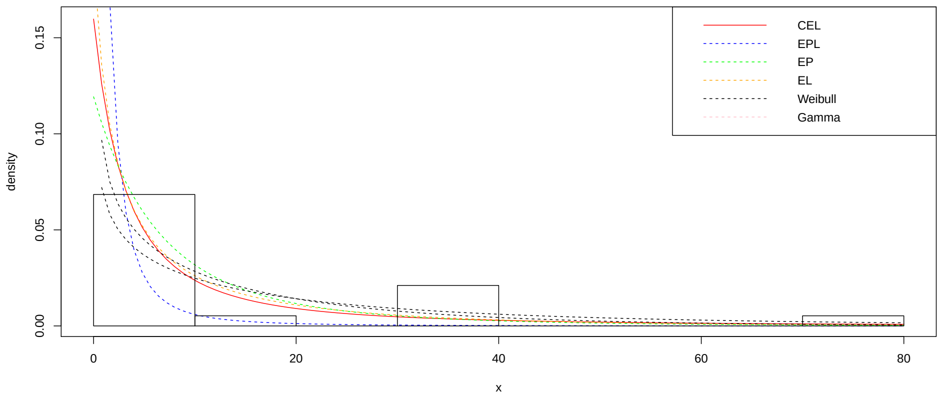

Figure 5: *Fitted pdfs of 1st data set*

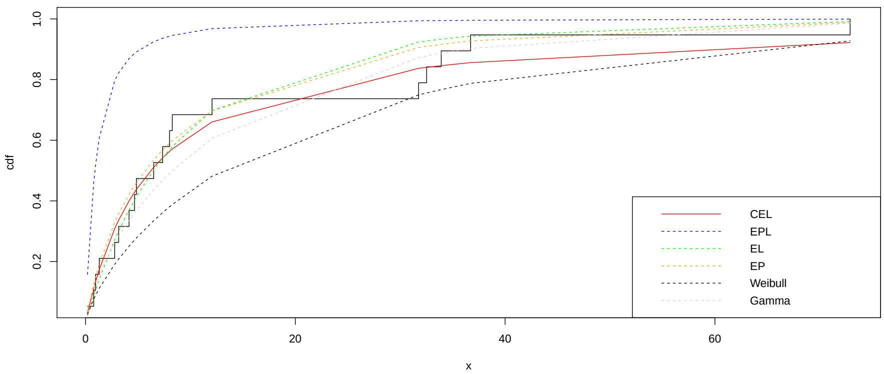

Figure 6: *Fitted cdfs and ecdf of 1st data set*

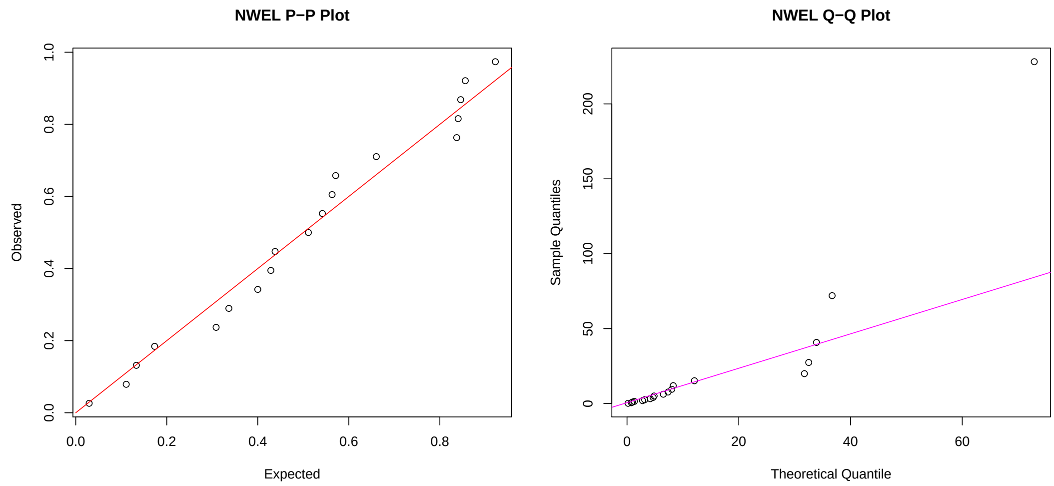

Figure 7: *p-p and q-q plot for the 1st data set.*

Here we notice that all of the considered models fit the data at 5% level of significance but the proposed distribution has minimum KS and maximum p-value among all the fitted models. Therefore, we may say that proposed CEL distribution is the most acceptable model for the present data set among the other considered models. For better visualization of the fitted models the estimated pdfs, cdfs, pp and qq plots are shown in Figure 5, Figure 6, Figure 7 for the first data set.

Table 3: *MLE's, - 2ln L, AIC, KS and p-values of the fitted distributions for the 2nd dataset.*

Distribution	Estimate	-2LL	AIC	BIC	AICc	KS	p-value
CEL(θ)	30.267	307.17	309.17	310.57	309.31	0.1061	0.8695
EPL(β,θ)	(0.0101, 0.9193)	302.87	306.87	309.68	307.32	0.1282	0.7076
EL(β,p)	(0.0111, 0.1932)	302.83	306.83	309.63	307.28	0.1291	0.6986
EP(β,θ)	(0.0105, 1.8243)	303.22	307.22	310.02	307.66	0.1468	0.5375
Weibull(β,θ)	(0.0183, 0.8536)	307.87	310.68	308.32	303.87	0.1534	0.4806
Gamma(β,θ)	(0.0136, 0.8119)	304.33	308.33	311.13	308.78	0.1694	0.3556

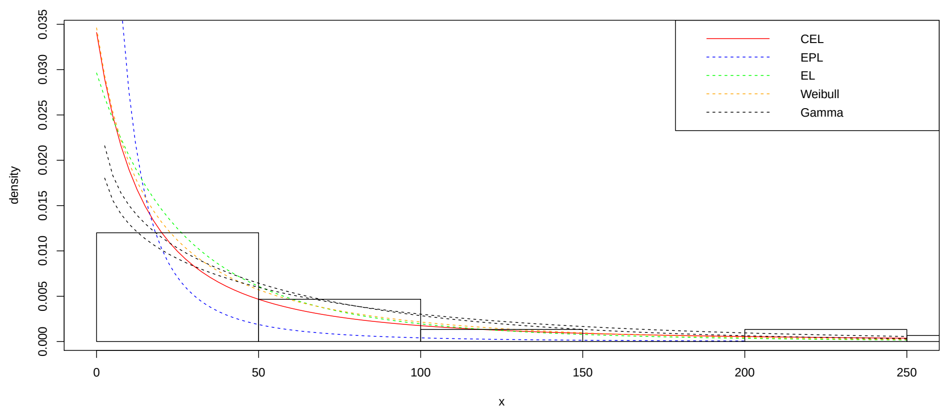

Figure 8: *Fitted pdfs of 2nd data set*

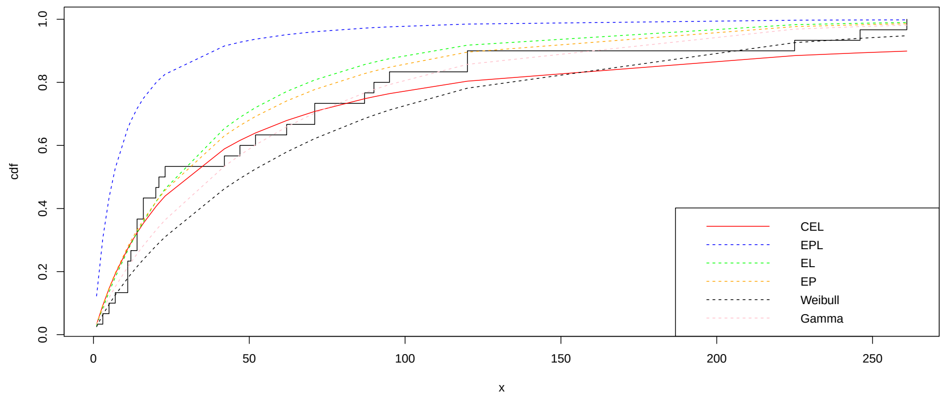

Figure 9: *Fitted cdfs of 2nd data set*

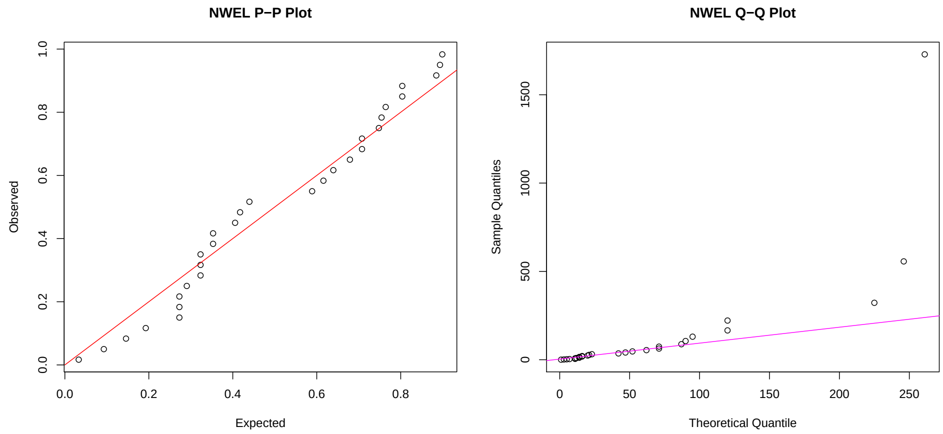

Figure 10: *p-p and q-q plot for the 2nd data set.*

For the second data set also all the considered models fit well. Here also, the value of KS statistics is the minimum for CEL distribution with the maximum p-value. From the above discussion on two real data sets we see that all the considered six decreasing failure models fit to the two data sets. The fitted models the estimated pdfs, cdfs, pp and qq plots are shown in Figure 8, Figure 9, Figure 10 for the second data set.

X. Applications on infant mortality data

Since the CEL distribution is an early failure distribution then this may be suitable for the data of infant deaths. In this study an attemt has been made to apply CEL distribution for the data of infant deaths taken from the fourth round of National Family Health Survey (NFHS-4) for the most poupulous state of India i.e. Uttar Pradesh conducted in 2015-16 (IIPS and ICF, 2017)[14]. The data on infant deaths for four categories have been extacted and CEL distribution with other compitent distributions considered here have been applied. The fitting, estimate of parameters, KS distance and its p-value are provided in table 4-7. The p-value reveals that the CEL distribution is most appropriate among all considered distributions.

Table 4: *Comperison of Goodness of fit for CEL, EPL, EP, EL, Weibull and Gamma on infant mortality data (Infant deaths of mothers aged 20-25)*

Age at Infant Death	Observed frequencies	Expected frequencies of CEL	Expected frequencies of EPL	Expected frequencies of EP	Expected frequencies of EL	Expected frequencies of Weibull	Expected frequencies of Gamma
0-1	104	90.61	83.12	80.71	76.30	72.18	143.00
1-2	17	30.67	33.66	36.74	34.29	36.55	16.75
2-3	2	15.03	17.46	18.91	19.43	21.51	4.39
3-4	10	98.83	10.35	10.69	12.09	13.18	1.28
4-5	5	5.79	6.66	6.49	7.90	8.26	0.39
5-6	7	4.07	4.53	4.16	5.32	5.26	0.12
6-7	7	3.02	3.19	2.78	3.64	3.39	0.04
7-8	2	2.33	2.31	1.92	2.52	2.21	0.01
8-9	3	1.85	1.71	1.36	1.76	1.45	0.00
9-10	4	1.50	1.28	0.98	1.24	0.95	0.00
10-11	2	1.24	0.97	0.71	0.87	0.63	0.00
11-12	3	1.05	0.74	0.53	0.62	0.42	0.00
Total	166	166.00	166.00	166.00	166.00	166.00	166.00
Estimates of parameter	$\theta = 1.4410$		$\theta = 0.6102$ $\beta = 0.2399$	$\lambda = 2.4852$ $\beta = 0.2700$	$p = 0.2378$ $\beta = 0.3399$	$\alpha = 0.8961$ $\beta = 0.5304$	$\alpha = 0.4745$ $\beta = 0.9554$
K-S Distance	0.0807		0.1257	0.1403	0.1668	0.1917	0.2478
p-value	0.2206		0.0094	0.0025	0.0002	0.0000	0.0000

Table 5: *Comperison of Goodness of fit for CEL, EPL, EP, EL, Weibull and Gamma on infant mortality data (Infant deaths of mothers aged 25-30)*

Age at Infant Death	Observed frequencies	Expected frequencies of CEL	Expected frequencies of EPL	Expected frequencies of EP	Expected frequencies of EL	Expected frequencies of Weibull	Expected frequencies of Gamma
0-1	94	86.94	84.05	84.79	76.14	75.60	104.47
1-2	17	22.52	24.63	27.91	28.19	30.36	22.28
2-3	8	9.93	11.02	11.38	14.02	14.74	6.86
3-4	3	5.51	6.01	5.43	7.72	7.55	2.24
4-5	3	3.48	3.67	2.91	4.45	3.99	0.75
5-6	0	2.40	2.41	1.69	2.64	2.15	0.26
6-7	3	1.74	1.66	1.05	1.58	1.18	0.09
7-8	2	1.33	1.18	0.68	0.96	0.66	0.03
8-9	4	1.04	0.86	0.46	0.58	0.37	0.01
9-10	1	0.84	0.64	0.31	0.36	0.21	0.00
10-11	2	0.69	0.48	0.22	0.22	0.12	0.00
11-12	0	0.57	0.37	0.16	0.13	0.07	0.00
Total	137	137.00	137.00	137.00	137.00	137.00	137.00
Estimates of parameter	$\theta = 1.0624$		$\theta = 0.3689$ $\beta = 0.2355$	$\lambda = 3.4829$ $\beta = 0.3033$	$p = 0.2501$ $\beta = 0.4879$	$\alpha = 0.8868$ $\beta = 0.7795$	$\alpha = 0.7081$ $\beta = 0.9833$
K-S Distance	0.0515		0.0726	0.0672	0.1304	0.1342	0.1150
p-value	0.8509		0.4507	0.5508	0.0171	0.0128	0.0492

Table 6: *Comperison of Goodness of fit for CEL, EPL, EP, EL, Weibull and Gamma on infant mortality data (Infant deaths in year 2003)*

Age at Infant Death	Observed frequencies	Expected frequencies of CEL	Expected frequencies of EPL	Expected frequencies of EP	Expected frequencies of EL	Expected frequencies of Weibull	Expected frequencies of Gamma
0-1	76	66.35	64.10	60.46	55.41	55.08	98.89
1-2	9	18.33	19.84	23.23	20.33	21.54	7.25
2-3	3	8.28	9.03	10.54	11.09	11.82	1.43
3-4	2	4.65	4.99	5.42	6.87	7.05	0.32
4-5	1	2.97	3.09	3.06	4.54	4.41	0.07
5-6	2	2.05	2.07	1.86	3.11	2.84	0.02
6-7	3	1.50	1.46	1.19	2.18	1.87	0.00
7-8	1	1.14	1.07	0.80	1.55	1.25	0.00
8-9	3	0.90	0.81	0.55	1.11	0.85	0.00
9-10	2	0.72	0.63	0.39	0.81	0.59	0.00
10-11	3	0.60	0.50	0.28	0.59	0.41	0.00
11-12	3	0.50	0.40	0.20	0.43	0.29	0.00
Total	108	108.00	108.00	108.00	108.00	108.00	108.00
Estimates of parameter	$\theta = 1.1394$		$\theta = 0.2528$ $\beta = 0.1512$	$\lambda = 3.1272$ $\beta = 0.2785$	$p = 0.1183$ $\beta = 0.3040$	$\alpha = 0.7880$ $\beta = 0.6433$	$\alpha = 0.7081$ $\beta = 0.9833$
K-S Distance	0.0894		0.1102	0.1439	0.1907	0.1937	0.2119
p-value	0.3392		0.1361	0.0204	0.0006	0.0005	0.0000

Table 7: *Comperison of Goodness of fit for CEL, EPL, EP, EL, Weibull and Gamma on infant mortality data (Infant death in year 2004)*

Age at Infant Death	Observed frequencies	Expected frequencies of CEL	Expected frequencies of EPL	Expected frequencies of EP	Expected frequencies of EL	Expected frequencies of Weibull	Expected frequencies of Gamma
0-1	54	46.83	42.23	42.22	38.71	36.31	75.22
1-2	15	17.27	18.87	20.22	19.41	20.56	10.87
2-3	3	8.79	11.02	10.77	11.40	12.59	3.25
3-4	2	5.28	6.01	6.28	7.21	7.89	1.08
4-5	1	3.51	3.67	4.11	4.74	5.00	0.37
5-6	2	2.49	2.41	2.82	3.19	3.20	0.13
6-7	4	1.86	1.66	2.00	2.18	2.06	0.05
7-8	2	1.44	1.18	1.45	1.51	1.33	0.02
8-9	2	1.15	0.86	1.07	1.05	0.87	0.01
9-10	2	0.94	0.64	0.08	0.73	0.57	0.00
10-11	2	0.78	0.48	0.60	0.56	0.37	0.00
11-12	2	0.66	0.37	0.46	0.36	0.24	0.00
Total	91	91.00	91.00	91.00	91.00	91.00	91.00
Estimates of parameter	$\theta = 1.6062$		$\theta = 0.7679$ $\beta = 0.2496$	$\lambda = 2.6031$ $\beta = 0.2382$	$p = 0.3311$ $\beta = 0.3477$	$\alpha = 0.9427$ $\beta = 0.4853$	$\alpha = 0.5032$ $\beta = 1.0726$
K-S Distance	0.0788		0.1293	0.1294	0.1680	0.1943	0.2332
p-value	0.6064		0.0876	0.0871	0.0102	0.0017	0.0000

XI. Conclusions

A single parameter lifetime distribution $CEL(\theta)$ has been introduced. The $CEL(\theta)$ distribution is mean free distribution and has decreasing hazard. The moment generating function, r^{th} oreder moments does not exists thus mean, variance, cumulant generating function, mean deviation about mean and median, Bonferroni, Gini index, mean residual life function (MRLF) also does not exists. The beauty of CEL distribution is that, this is a single parameter decreasing hazard distribution and explains the phenomenon better than other two parameter models. Although the moments do not exist, but Figure 1 indicates that, the distribution is highly positively skewed distribution. As the value of θ is increasing the density of the distribution becomes flatten. Hence, we can easily conclude that the proposed CEL distribution may be considered as a suitable model for the case of decreasing failure rate scenario with a hope to get better model in various disciplines such as medical, engineering, and social sciences.

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A Two Non-Identical Unit Parallel System Subject to Two Types of Failure and Correlated Life Times

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Abstract

The paper deals with the reliability and cost-benefit analysis of a two non-identical unit system with two types of failure. The units are named as unit-1 and unit-2 and they are arranged in a parallel configuration. Unit-1 can fail due to hardware or due to human error failure whereas unit-2 fails due to normal cause. A single repairman is considered with the system for all types of failure in the units and unit-1 gets priority in repair over the unit-2. The repair time distributions of unit-1 are taken as general with different c.d.fs and the repair time distribution of unit-2 is taken as exponential. Failure time distribution of unit-1 due to human error is taken exponential. Whereas the random variable denoting the failure time of unit-1 due to hardware failure and random variable denoting the failure time of unit-2 are assumed to be correlated random variables having their joint distribution as bivariate exponential (B.V.E.).

Keywords: Transition probabilities, mean sojourn time, bi-variate exponential distribution, regenerative point, reliability, MTSF, availability, expected busy period of repairman, net expected profit.

I. Introduction

Reliability is an important concept in the planning design and operation stages of various complex systems. Gupta et al. (2014) analysed a two non-identical unit parallel system with two independent repairmen-skilled and ordinary. A failed unit is first attended by skilled repairman to perform first phase repair and then it goes for second phase repair by ordinary repairman. Both types of repair discipline are FCFS. Chaudhary et al. (2015) analysed a two non-identical unit parallel system model assuming that an administrative delay occurs in getting the repairman available with the system whenever needed. Upon failure of a unit, the other operating unit shares the load of failed unit. Chopra and Ram (2017) analysed a two non-identical unit parallel system with two types of failures: common cause failure and partial failure. The repairman is not always available with the system to repair a failed unit. Whenever a unit fails, the repairman is called to come at the system and he takes some significant time to reach at the system. This time is known as waiting time of repairman during which the failed unit waits for repair. Chandra et al. (2020) performed the reliability and cost benefit analysis of the two identical and non-identical unit parallel system models by using Semi – Markov Process in regenerative point. A study of comparison is made between the reliability characteristics for both the system models under study. In these papers, the authors did not consider the concept of human error failure. In all the above system models the authors have considered single cause of failure in a unit i.e. normal cause.

Mahmoud and Moshref (2010) analysed a two-unit cold standby system by considering two cause of failure in a unit namely-Due to hardware and Due to human error. It has also been assumed that an operating unit goes for preventive maintenance (PM) to increase the system effectiveness. All the distributions of random variables involved in the system model are taken to follow arbitrary distributions. Kumar and Malik (2011) carried out the profit analysis of a computer system model with software and hardware failure subject to maximum operation time (MOT) and maximum repair time (MRT). An operating unit goes for preventive maintenance (PM) after completing MOT, if it is not failed during this time. Further if a failed unit is not repaired during MRT, it is replaced by new one. The priority to software replacement is given over hardware repair. Singh et al. (2016) analysed a two-unit warm standby system with two types of repairman. The first type of repairman, usually called regular repairman who is always remains available with the system to attend a failed unit. If he might not be able to do some complex repairs within some tolerable (patience) time, an expert repairman is called from the outside to complete the repair of the failed unit and he takes some significant time to reach at the system. Further an operating unit may fail either due to hardware or due to human error. In all the above system models the common assumptions considered is that the failure and repair times of the units are taken to uncorrelated random variables.

Gupta and co-workers [2008,2018] analysed two unit parallel and standby system models under different sets of assumptions by taking the failure and repair times as correlated random variables having their joint distribution as bivariate exponential. They have considered only single type of failure in an operating unit. Some authors including [1999, 2013] analysed two-unit parallel system models by taking the joint distribution of life times of the units working in parallel as bivariate exponential. They have also considered the single type of failure in an operating unit. **The objective of the present paper is to study a two non-identical unit parallel system subject to two causes of failure in an operating unit-Due to hardware and Due to human error**. Human failure is defined as a failure to perform a prescribed task which could result in damage to the equipment and property. There exist a number of causes for human error; e.g., lack of good job environments, poor training or skill of the operating personnel and so on. On the other hand, hardware failure occurs due to flaws in design, poor quality control and poor maintenance.

The life time of the units working in parallel form are taken to be correlated random variables having their joint distribution as bivariate exponential with different parameters as the form of joint p.d.f. given below.

$$f(x_1,x_2)=\alpha_1\alpha_2(1-r)e^{-\alpha_1x_1-\alpha_2x_2}\,I_0\left(2\sqrt{\alpha_1\alpha_2 r x_1 x_2}\right);\quad x_1,x_2,\alpha_1,\alpha_2>0;\quad 0\le r<1$$

where, $$I_0(z)=\sum_{k=0}^{\infty}\frac{(z/2)^{2k}}{(k!)^2}$$

is the modified Bessel function of type-I and order zero.

By using regenerative point technique, the following measures of system effectiveness are obtained-

i. Transient-state and steady-state transition probabilities.
ii. Mean sojourn time in various regenerative states.
iii. Reliability and mean time to system failure (MTSF).
iv. Point-wise and steady-state availabilities of the system as well as expected up time of the system during time interval (0, t).
v. The expected busy period of repairman in time interval (0, t).
vi. Net expected profit earned by the system in time interval (0, t) and in steady-state.

II. System Description and Assumptions

1. The system comprises of two non-identical units-unit-1 and unit-2. Initially, both the units are operative in parallel configuration.
2. Each unit has two modes-Normal (N) and Total failure (F).
3. Unit-1 can fail either due to hardware or human error. Whereas unit-2 can fail only due to its normal cause.
4. The system failure occurs when both the units are totally failed.
5. A single repairman is always available to repair the failed unit-1 either due to hardware or human error and the failed unit-2. The unit-1 gets priority in repair over the unit-2.
6. Failure time of unit-1 due to human error is taken exponential distribution whereas the failure time of unit-1 due to hardware and failure time of unit-2 due to normal cause are assumed to be correlated random variables having their joint distribution as bivariate exponential (B.V.E.) with density function as follows-

$$f(x_1, x_2) = \alpha_1\alpha_2(1-r)e^{-\alpha_1 x_1 - \alpha_2 x_2} I_0\left(2\sqrt{\alpha_1\alpha_2 r x_1 x_2}\right); \quad x_1, x_2, \alpha_1, \alpha_2 > 0; \quad 0 \le r < 1$$

where, $I_0(z) = \sum_{k=0}^{\infty} \frac{(z/2)^{2k}}{(k!)^2}$

7. The repair time distribution of unit-1 failed either due to hardware or due to human error are taken as general with different c.d.fs whereas the repair time distribution of unit-2 failed due to normal cause is taken as exponential.
8. A repaired unit always works as good as new.

III. Notations and States of the System

We define the following symbols for generating the various states of the system-

N_{o1}^1, N_{o2}^2	:	Unit-1 and Unit-2 in normal (N) mode and operative.
F_{r1}^1	:	Unit-1 is in failure (F) mode and repair which is failed due to hardware failure.
F_{r2}^1	:	Unit-1 is in failure (F) mode and repair which is failed due to human error.
F_r^2, F_{wr}^2	:	Unit-2 is in failure (F) mode and under repair/waits for repair.

Considering the above symbols in view of assumptions stated in section-2, the possible states of the system are shown in the transition diagram represented by **Figure. 1**. It is to be noted that the epochs of transitions into the state S_4 from S_1, S_5 from S_2 are non-regenerative, whereas all the other entrance epochs into the states of the systems are regenerative.

The other notations used are defined as follows:

E : Set of regenerative states.

$X_i\left(i=1,2\right)$: Random variables representing the failure time of uni1-1 in N-mode and unit-2 respectively for i=1,2.

$f\left(x_1,x_2\right)$: Joint p.d.f. of $\left(x_1,x_2\right)$.

$$f(x_1,x_2)=\alpha_1\alpha_2\left(1-r\right)e^{-\alpha_1x_1-\alpha_2x_2}\,I_0\left(2\sqrt{\alpha_1\alpha_2rx_1x_2}\right)$$

$;\ x_1,x_2,\alpha_1,\alpha_2>0\ ;\ 0\le r<1$

$$\text{where, } I_0\left(z\right)=\sum_{k=0}^{\infty}\frac{(z/2)^{2k}}{(k!)^2}$$

$g_i\left(x\right)$: Marginal p.d.f. of $X_i=x$

$$=\alpha_i\left(1-r_i\right)e^{-\alpha_i(1-r)x}$$

$k_1\left(x_1|X_2=x_2\right)$: Conditional p.d.f. of $X_1|X_2=x$.

$$=\alpha_1e^{-\left(\alpha_1x_1+\alpha_2rx\right)}I_0\left(2\sqrt{\alpha_1\alpha_2rxx_1}\right)$$

$k_2\left(x_2|X_1=x\right)$: Conditional p.d.f. of $X_2|X_1=x$.

$$=\alpha_2e^{-\left(\alpha_2x_2+\alpha_1rx\right)}I_0\left(2\sqrt{\alpha_1\alpha_2rxx_2}\right)$$

$K_i\left(\cdot|x\right)$: Conditional c.d.f. of $X_i|X_j=x,\ i\ne j\ ;\ i,j=1,2$.

λ : Constant failure rate of unit-1 due to Human error.

β : Constant repair rate of unit-2 due to normal cause.

$G_1\left(\cdot\right),G_2\left(\cdot\right)$: c.d.f. of repair time of unit-1 failed due to hardware failure and unit-1 failed due to human error.

$q_{ij}(\cdot),q_{ij}^{(k)}\left(\cdot\right)$: p.d.f. of transition time from state S_i to S_j and S_i to S_j via S_k.

$p_{ij},p_{ij}^{(k)}$: Steady-state transition probabilities from state S_i to S_j and S_i to S_j via S_k.

$p_{ij|x},p_{ij|x}^{(k)}$: Steady-state transition probabilities from state S_i to S_j and S_i to S_j via S_k when it is known that the unit has worked for time x before its failure.

* : †Symbol for Laplace Transform i.e. $q_{ij}^*\left(s\right)=\int e^{-st}q_{ij}\left(t\right)dt$

~ : Symbol for Laplace Stieltjes Transform i.e. $\tilde{Q}_{ij}(s) = \int e^{-st} dQ_{ij}(t)$

© : Symbol for ordinary convolution i.e.

$$A(t) © B(t) = \int_0^t A(u)B(t-u)du$$

†The limits of integration are 0 to ∞ whenever they are not mentioned.

TRANSITION DIAGRAM

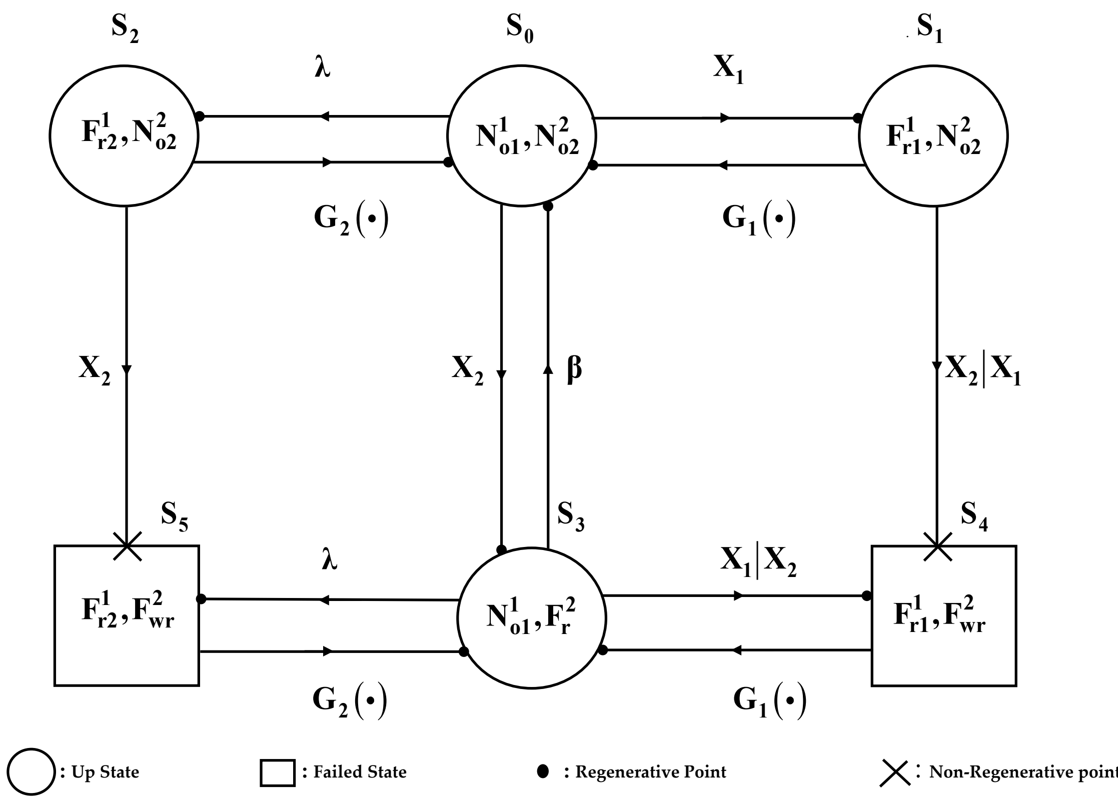

Figure. 1

IV. Transition Probabilities and Sojourn Times

Let $X(t)$ be the state of the system at epoch t, then $\{X(t); t \geq 0\}$ constitutes a continuous parametric Markov-Chain with state space $E = \{S_0 \text{ to } S_5\}$ .The various measures of system effectiveness are obtained in terms of steady-state transition probabilities and mean sojourn times in various states. First we obtain the direct conditional and unconditional transition probabilities in terms of

$$\alpha_1' = \frac{\alpha_1}{\alpha_1 + \lambda + \beta}, \qquad \alpha_2' = \frac{\alpha_2}{\alpha_2 + \theta_1}$$

as follows-

$$p_{01} = \int \alpha_1 (1-r) e^{-\{\lambda+\alpha_1(1-r)+\alpha_2(1-r)\}t} dt = \frac{\alpha_1(1-r)}{\lambda+\alpha_1(1-r)+\alpha_2(1-r)}$$

Similarly,

$$p_{02} = \frac{\lambda}{\lambda+\alpha_1(1-r)+\alpha_2(1-r)}, \qquad p_{03} = \frac{\alpha_2(1-r)}{\lambda+\alpha_1(1-r)+\alpha_2(1-r)}$$

$$p_{20} = \tilde{G}_2\left[\alpha_2(1-r)\right], \qquad p_{23}^{(5)} = 1-\tilde{G}_2\left[\alpha_2(1-r)\right]$$

$$p_{43} = \int dG_1(t) = 1, \qquad p_{53} = \int dG_2(t) = 1$$

$$p_{10|X} = \int dG_1(u)\bar{K}_2(u|x)$$

Similarly,

$$p_{13|X}^{(4)} = \int \bar{G}_1(u) dK_2(u|x)$$

$$p_{30|X} = \int \beta e^{-(\lambda+\beta)u}\left(\int_u^\infty \alpha_1 e^{-(\alpha_1 y+\alpha_2 rx)}\sum_{j=0}^{\infty}\frac{(\alpha_1\alpha_2 rxy)^j}{(j!)^2}dy\right)du = \frac{\beta}{\lambda+\beta}\left[1-\alpha_1' e^{-\alpha_2 rx(1-\alpha_1')}\right]$$

$$p_{34|X} = \alpha_1' e^{-\alpha_2 rx(1-\alpha_1')}, \qquad p_{35|X} = \frac{\lambda}{\lambda+\beta}\left[1-\alpha_1' e^{-\alpha_2 rx(1-\alpha_1')}\right]$$

The unconditional transition probabilities with correlation coefficient from some of the above conditional transition probabilities can be obtained as follows:

$$p_{10} = \int p_{10|X} g_1(x) dx = \int p_{10|x}\{\alpha_1(1-r)e^{-\alpha_1(1-r)x}\}dx$$

Similarly,

$$p_{13}^{(4)} = \int p_{13|x}^{(4)}\{\alpha_1(1-r)e^{-\alpha_1(1-r)x}\}dx, \qquad p_{30} = \frac{\beta}{\lambda+\beta}\left\{1-\frac{\alpha_1'(1-r)}{(1-r\alpha_1')}\right\}$$

$$p_{34} = \frac{\alpha_1'(1-r)}{(1-r\alpha_1')}, \qquad p_{35} = \frac{\lambda}{\lambda+\beta}\left\{1-\frac{\alpha_1'(1-r)}{(1-r\alpha_1')}\right\}$$

It can be easily verified that,

$$p_{01}+p_{02}+p_{03}=1, \qquad p_{10}+p_{13}^{(4)}=1, \qquad p_{20}+p_{23}^{(5)}=1$$
$$p_{30}+p_{34}+p_{35}=1, \qquad p_{43}=p_{53}=1 \tag{1-5}$$

V. Mean Sojourn Times

The mean sojourn time ψ_i in state S_i is defined as the expected time taken by the system in state S_i before transiting into any other state. If random variable U_i denotes the sojourn time in state S_i then,

$$\psi_i = \int P\left[U_i > t\right]dt$$

Therefore, its values for various regenerative states are as follows-

$$\psi_0 = \int e^{-\{\lambda+\alpha_1(1-r)+\alpha_2(1-r)\}t} dt = \frac{1}{\lambda+\alpha_1(1-r)+\alpha_2(1-r)} \tag{6}$$

$$\psi_{1|X} = \int \bar{G}_1(t)\bar{K}_2(t|x)dt = \int \bar{G}_1(t)\left(\int_t^\infty \alpha_2 e^{-(\alpha_2 u+\alpha_1 rx)}\sum_{j=0}^{\infty}\frac{(\alpha_1\alpha_2 rxu)^j}{(j!)^2}du\right)dt$$

So that,

$$\psi_1 = \int \psi_{1|x} g_1(x)dx = \int \psi_{1|x}\alpha_1(1-r)e^{-\alpha_1(1-r)x}dx \tag{7}$$

$$\psi_2 = \int \bar{G}_2(t)e^{-\alpha_2(1-r)t}dt \tag{8}$$

$$\psi_{3|X} = \frac{1}{\beta+\lambda}\left\{1-\alpha_1' e^{-\alpha_2 rx(1-\alpha_1')}\right\}$$

So that,

$$\psi_3 = \frac{1}{\beta+\lambda}\left[1-\frac{\alpha_1'(1-r)}{(1-r\alpha_1')}\right] \tag{9}$$

$$\psi_4 = \int \bar{G}_1(t)dt \tag{10}$$

$$\psi_5 = \int \bar{G}_2(t)dt \tag{11}$$

VI. Analysis of Characteristics

a) Reliability and MTSF

Let $R_i(t)$ be the probability that the system operates during (0, t) given that at t=0 system starts from $S_i \in E$. To obtain it we assume the failed states S_4 and S_5 as absorbing. By simple probabilistic arguments, the value of $R_0(t)$ in terms of its Laplace Transform (L.T.) is given by

$$R_0^*(s) = \frac{Z_0^* + q_{01}^* Z_1^* + q_{02}^* Z_2^* + q_{03}^* Z_3^*}{1-q_{01}^* q_{10}^* - q_{02}^* q_{20}^* - q_{03}^* q_{30}^*} \tag{12}$$

We have omitted the argument's from $q_{ij}^*(s)$ and $Z_i^*(s)$ for brevity. $Z_i^*(s); i = 0,1,2,3$ are the L. T. of

$$Z_0(t) = e^{-\{\lambda+\alpha_1(1-r)+\alpha_2(1-r)\}t}, \qquad Z_1(t) = \bar{G}_1(t)\int \bar{K}_2(t \mid x)g_1(x)dx$$

$$Z_2(t) = e^{-\alpha_2(1-r)t}\bar{G}_2(t), \qquad Z_3(t) = e^{-(\lambda+\beta)t}\int \bar{K}_1(t \mid x)g_2(x)dx$$

Taking the Inverse Laplace Transform of (12), one can get the reliability of the system when system initially starts from state S_0.

The MTSF is given by,

$$E(T_0) = \int R_0(t)dt = \lim_{s\to 0} R_0^*(s) = \frac{\psi_0 + p_{01}\psi_1 + p_{02}\psi_2 + p_{03}\psi_3}{1-p_{01}p_{10} - p_{02}p_{20} - p_{03}p_{30}} \tag{13}$$

b) Availability Analysis

Let $A_i(t)$ be the probability that the system is up at epoch t, when initially it starts operation from state $S_i \in E$. Using the regenerative point technique and the tools of Laplace transform, one can obtain the value of $A_0(t)$ in terms of its Laplace transforms i.e. $A_0^*(s)$ given as follows-

$$A_0^*(s) = \frac{N_1(s)}{D_1(s)} \tag{14}$$

where,

$$N_1(s)=\left[1-q_{34}^*q_{43}^*-q_{35}^*q_{53}^*\right]\left(Z_0^*+q_{01}^*Z_1^*+q_{02}^*Z_2^*\right)+\left\lfloor q_{01}^*q_{13}^{(4)*}+q_{02}^*q_{23}^{(5)*}+q_{03}^*\right\rfloor Z_3^*$$

and

$$D_1(s)=\left[1-q_{34}^*q_{43}^*-q_{35}^*q_{53}^*\right]\left(1-q_{01}^*q_{10}^*-q_{02}^*q_{20}^*\right)-q_{30}^*\left\lfloor q_{01}^*q_{13}^{(4)*}+q_{02}^*q_{23}^{(5)*}+q_{03}^*\right\rfloor \tag{15}$$

where, $Z_i(t)$, i=0,1,2,3 are same as given in section VI(a).

The steady-state availability of the system is given by

$$A_0=\lim_{t\to\infty}A_0(t)=\lim_{s\to 0}sA_0^*(s) \tag{16}$$

We observe that

$$D_1(0)=0$$

Therefore, by using L. Hospital's rule the steady state availability is given by

$$A_0=\lim_{s\to 0}\frac{N_1(s)}{D_1'(s)}=\frac{N_1}{D_1'} \tag{17}$$

where,

$$N_1=p_{30}\left(\psi_0+p_{01}\psi_1+p_{02}\psi_2\right)+\left(1-p_{01}p_{10}-p_{02}p_{20}\right)\psi_3$$

and

$$D_1'=p_{30}\left\lfloor \psi_0+p_{01}\left(\psi_1+p_{14}\psi_4\right)+p_{02}\left(\psi_2+p_{25}\psi_5\right)\right\rfloor+\left(1-p_{01}p_{10}-p_{02}p_{20}\right)\left[\psi_3+p_{34}\psi_4+p_{35}\psi_5\right] \tag{18}$$

The expected up time of the system in interval (0, t) is given by

$$\mu_{up}(t)=\int_0^t A_0(u)\,du$$

So that, $$\mu^*_{up}(s)=\frac{A_0^*(s)}{s} \tag{19}$$

c) Busy Period Analysis

Let $B_i^1(t)$, $B_i^2(t)$ and $B_i^3(t)$ be the respective probabilities that the repairman is busy in the repair of unit-1 failed due to hardware, unit-1 failed due to human error and unit-2 failed due to normal cause at epoch t, when initially the system starts operation from state $S_i\in E$. Using the regenerative point technique and the tools of L. T., one can obtain the values of above three probabilities in terms of their L. T. i.e. $B_i^{1*}(s)$, $B_i^{2*}(s)$ and $B_i^{3*}(s)$ as follows-

$$B_i^{1*}(s)=\frac{N_2(s)}{D_1(s)},\qquad B_i^{2*}(s)=\frac{N_3(s)}{D_1(s)}\qquad\text{and}\qquad B_i^{3*}(s)=\frac{N_4(s)}{D_1(s)} \tag{20-22}$$

where,

$$N_2(s)=q_{01}^*\left[1-q_{34}^*q_{43}^*-q_{35}^*q_{53}^*\right]\left(Z_1^*+q_{14}^*Z_4^*\right)+q_{34}^*\left\lfloor q_{01}^*q_{13}^{(4)*}+q_{02}^*q_{23}^{(5)*}+q_{03}^*\right\rfloor Z_4^*$$

$$N_3(s)=q_{02}^*\left[1-q_{34}^*q_{43}^*-q_{35}^*q_{53}^*\right]\left(Z_2^*+q_{25}^*Z_5^*\right)+q_{35}^*\left(q_{01}^*q_{13}^{(4)*}+q_{02}^*q_{23}^{(5)*}+q_{03}^*\right)Z_5^*$$

$$N_4(s)=\left\lfloor q_{01}^*q_{13}^{(4)*}+q_{02}^*q_{23}^{(5)*}+q_{03}^*\right\rfloor Z_3^*$$

and $D_1(s)$ is same as defined by the expression (15) of section VI(b).

Also Z_4^* and Z_5^* are the L. T. of

$$Z_4(t)=\bar{G}_1(t), \quad Z_5(t)=\bar{G}_2(t)$$

The steady state results for the above three probabilities are given by-

$$B_o^1=\lim_{s\to 0} sB_0^*(s)=N_2/D_1', \qquad B_o^2=N_3/D_1' \qquad \text{and} \qquad B_0^3=N_4/D_1' \tag{23-25}$$

$$N_2=p_{30}p_{01}(\psi_1+p_{14}\psi_4)+\lfloor p_{34}(1-p_{01}p_{10}-p_{02}p_{20})\rfloor\psi_4$$

$$N_3=p_{30}p_{02}(\psi_2+p_{25}\psi_5)+\lfloor p_{35}(1-p_{01}p_{10}-p_{02}p_{20})\rfloor\psi_5$$

$$N_4=[1-p_{01}p_{10}-p_{02}p_{20}]\psi_3$$

and D_1' is same as given in the expression (18) of section VI(b).

The expected busy period in repair of unit-1 failed due to hardware, unit-1 failed due to human error and unit-2 failed due to normal cause during time interval (0, t) are respectively given by-

$$\mu_b^1(t)=\int_0^t B_0^1(u)du, \qquad \mu_b^2(t)=\int_0^t B_0^2(u)du \qquad \text{and} \qquad \mu_b^3(t)=\int_0^t B_0^3(u)du$$

So that,

$$\mu_b^{1*}(s)=B_0^{1*}(s)/s \qquad \mu_b^{2*}(s)=B_0^{2*}(s)/s \qquad \text{and} \qquad \mu_b^{3*}(s)=B_0^{3*}(s)/s \tag{26-28}$$

d) Profit Function Analysis

The net expected total cost incurred in time interval (0, t) is given by

$P(t)=$ Expected total revenue in (0, t) - Expected cost of repair in (0, t)

$$=K_0\mu_{up}(t)-K_1\mu_b^1(t)-K_2\mu_b^2(t)-K_2\mu_b^3(t) \tag{29}$$

Where, K_0 is the revenue per- unit up time by the system during its operation. K_1, K_2 and K_3 are the amounts paid to the repairman per-unit of time when he is busy in repair of unit-1 failed due to hardware, unit-1 failed due to human error and unit-2 failed due to normal cause respectively.

The expected total profit incurred in unit interval of time is $P=K_0A_0-K_1B_0^1-K_2B_0^2-K_2B_0^3$

VII. Particular Case

When the repair time of unit-1 failed due to hardware and human error also follow exponential with p.d.fs as follows-

$$g_1(t)=\theta_1 e^{-\theta_1 t}, \qquad g_2(t)=\theta_2 e^{-\theta_2 t}$$

The Laplace Transform of above density function are as given below-

$$g_1^*(s)=\tilde{G}_1(s)=\frac{\theta_1}{s+\theta_1}, \qquad g_2^*(s)=\tilde{G}_2(s)=\frac{\theta_2}{s+\theta_2}$$

Here, $\tilde{G}_1(s)$ and $\tilde{G}_2(s)$ are the Laplace-Stieltjes Transforms of the c.d.fs $G_1(t)$ and $G_2(t)$ corresponding to the p.d.fs $g_1(t)$ and $g_2(t)$.

In view of above, the changed values of transition probabilities and mean sojourn times.

$$p_{10}=1-\frac{\alpha_2'(1-r)}{(1-r\alpha_2')}, \qquad p_{13}^{(4)}=\frac{\alpha_2'(1-r)}{(1-r\alpha_2')}, \qquad p_{20}=\frac{\theta_2}{\alpha_2(1-r)+\theta_2}$$

$$p_{23}^{(5)}=\frac{\alpha_2(1-r)}{\alpha_2(1-r)+\theta_2}, \qquad \psi_1=\frac{1}{\alpha_2(1-r)+\theta_1}, \qquad \psi_2=\frac{1}{\alpha_2(1-r)+\theta_2}$$

VIII. Graphical Study of Behaviour and Conclusions

For a more clear view of the behaviour of system characteristics with respect to the various parameters involved, we plot curves for **MTSF** and **profit function** in **Fig. 2** and **Fig. 3** w.r.t. $\boldsymbol{\alpha_1}$ for three different values of correlation coefficient $\mathbf{r}$ =**0.25, 0.35** and **0.45** and two different values of repair parameter $\boldsymbol{\theta_1}$=**0.7** and **0.9** while the other parameters are kept fixed as $\boldsymbol{\lambda = 0.09}$, $\boldsymbol{\alpha_2 = 0.045}$, $\boldsymbol{\beta = 0.8}$, $\boldsymbol{\theta_2 = 0.7}$ .From the curves of **Fig. 2,** we observe that **MTSF** increases uniformly as the values of $\mathbf{r}$ and $\boldsymbol{\theta_1}$ increase and it decreases with the increase in $\boldsymbol{\alpha_1}$. Further, to achieve **MTSF** at least **94** units we conclude from smooth curves that the value of $\boldsymbol{\alpha_1}$ must be less than **0.118, 0.190 and 0.332** respectively for $\mathbf{r = 0.25}$ **, 0.35 , 0.45** when $\boldsymbol{\theta_1 = 0.9}$. Whereas from dotted curves we conclude that the value of $\boldsymbol{\alpha_1}$ must be less than **0.100, 0.171, 0.294** for $\mathbf{r = 0.25}$ **, 0.35** and **0.45** when $\boldsymbol{\theta_1 = 0.7}$.

Similarly, **Fig. 3** reveals the variations in **profit (P)** w.r.t. α_1 for varying values of $\mathbf{r}$ and $\boldsymbol{\theta_1}$, when the values of other parameters are kept fixed as $\boldsymbol{\lambda = 0.09}$, $\boldsymbol{\alpha_2 = 0.045}$, $\boldsymbol{\beta = 0.8}$, $\boldsymbol{\theta_2 = 0.7}$, $\mathbf{K_0 = 160}$, $\mathbf{K_1 = 400}$, $\mathbf{K_2 = 250}$ and $\mathbf{K_3 = 350}$. Here also the same trends in respect of $\boldsymbol{\alpha_1}$, $\mathbf{r}$ and $\boldsymbol{\theta_1}$ are observed as in case of **MTSF**. Moreover, we conclude from the smooth curves that the system is profitable only if $\boldsymbol{\alpha_1}$ is less than **0.581, 0.700** and **0.850** respectively for $\mathbf{r = 0.25}$ **, 0.35 , 0.45** when $\boldsymbol{\theta_1 = 0.9}$. From dotted curves, we conclude that the system is profitable only if $\boldsymbol{\alpha_1}$ is less than **0.520, 0.612** and **0.759** respectively for $\mathbf{r = 0.25}$ **, 0.35** and **0.45** when $\boldsymbol{\theta_1 = 0.7}$.

Behaviour of MTSF w.r.t. $\boldsymbol{\alpha_1}$ for different values of $\mathbf{r}$ and $\boldsymbol{\theta_1}$

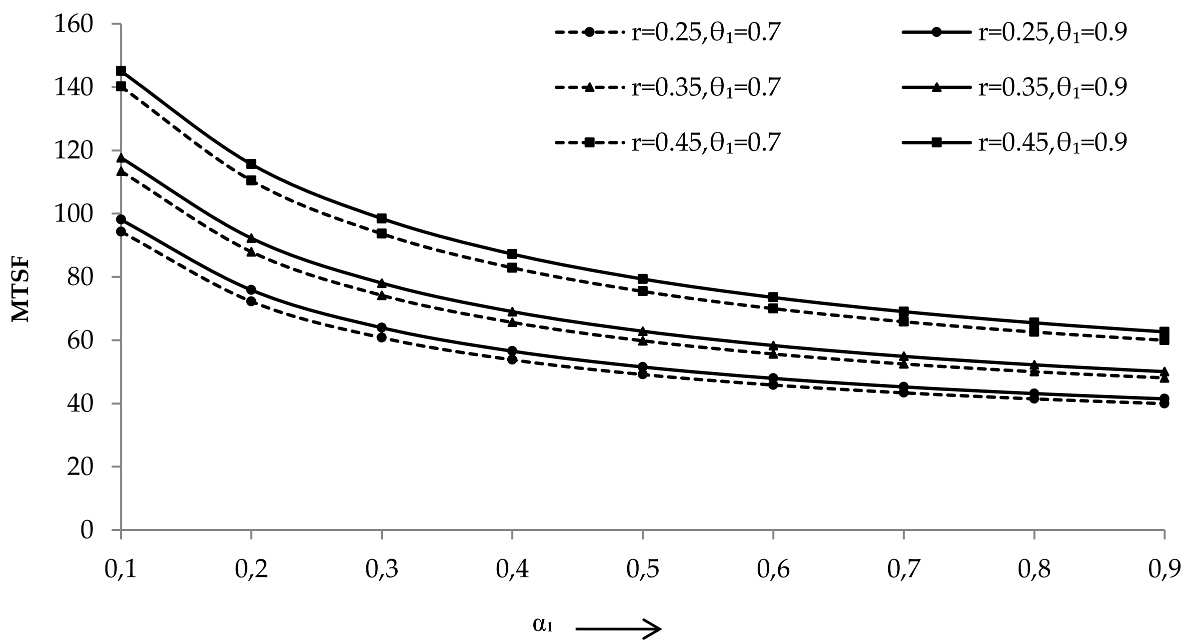

Figure.2

Behaviour of PROFIT (P) w.r.t. $\boldsymbol{\alpha_1}$ for different values of $\mathbf{r}$ and $\boldsymbol{\theta_1}$

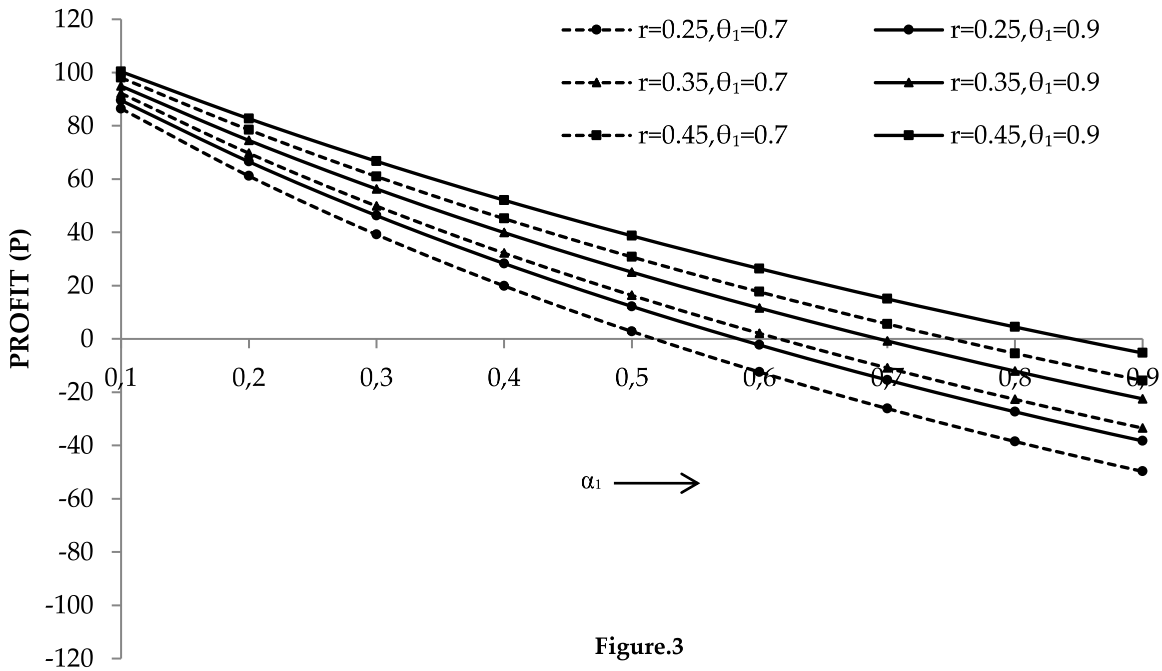

Figure.3

IX. Acknowledgment

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Optimal System for Five Units Serial Systems under Partial and Complete Failure

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Abstract

The present paper studies and compared some reliability characteristics of series-parallel systems containing five units each under partial and complete failure. Four different system configurations are considered in this paper. It is assumed that both the failure and repair rates of each system configuration follow exponential distribution. The steady-state availability, busy period of repairman due to partial and complete failure, profit function, mean time to failure (MTTF) have been derived, examined and compared. The system configurations are compared analytically in terms of their availability and mean time to failure (MTTF). Cost-benefit measure has been evaluated for all the system configurations. The computed results are presented in tables and figures. From the analysis, system configuration II is observed to be the optimal configuration.

Keywords: Reliability, availability, standby, partial, complete failure, configuration, cost benefit.

I. Introduction

In reliability analysis, the performance evaluation of repairable systems is a matter of great importance. Maintaining the reliability of the system is indispensable. System performance can be measure through some reliability characteristics such as availability, mean time to failure, profit and benefit-cost analysis. The system availability of some engineering systems depends on the system structure, preventive maintenance, redundancy and also on the component availability. To affirm system failure and high system performance of complex systems, it is necessary to have a system component of higher availability. Generally, increasing redundant units or using units with high availability can also enhance system performance. Performance of systems increases significantly through redundancy optimization, using components with high availability, system's structural design and maintenance through repair and preventive maintenance.

To achieve high system reliability and availability, the system must be maintained at the highest order. To achieve this end, numerous researchers have designed different types of mathematical models to study and compare their reliability, availability and mean time to failure. For instance, Singh and Abdul Kareem [8] discussed the cost assessment of complex repairable systems consisting two subsystems in series configuration using Gumbel Hougaard family copula. Berk et al [2] have discussed the reliability assessment of safety-critical sensor information. Sanusi *et al* [12] have recently studied the performance evaluation of an industrial configured as series-parallel system. Wang *et al* [16] have presented the reliability analysis of two-dissimilar unit warm standby repairable system with priority in use. Singh and Ayagi [15] discussed the study of availability of standby complex system under waiting repair and human failure using Gumbel-Hougaard family copula distribution.

Harish Garg [4] discussed the study of the multi objective non-linear programming problem for reliability (GSA) and the results have been compared with the results computed by practice swarm optimization (PSO) methodology. Malik and Tewari [10] analyzed the performance of a system and maintenance priorities decision for the water flow system of a coal-based thermal power plant. Kumar et al [1] have recently studied the reliability analysis of a redundant system with 'FCFS' repair policy subject to weather conditions. Niwas and Garg [11] presented an approach for analyzing the reliability and profit of an industrial system based on the cost-free warranty repair policy. More recently, Sanusi and Yusuf [13] have presented the study of cost analysis of 2-out-of-4 system connected to two-unit parallel supporting device for operation. Mortazavi et al. [9] have evaluated the MTBF and other reliability parameters for a 2-out-of-3: G redundant repairable systems with common cause failures considering fuzzy rates for failures and repair via a case study of a centrifugal water pumping system. Saini et al [14] have investigated microprocessor systems using RAMD approach. Yang et al [19] discussed the reliability assessment of system with inconsistent priors and multi-level data. Gahlot et al. [5] investigated the performance assessment of serial system with different types of failure and repair policy. Zhang [23] dealt with the reliability analysis of computer networks based on intelligent cloud computing methods. Zhao et al [24] have discussed the reliability analysis of aero-engine compressor rotor system considering cruise characteristics. Ibrahim et al [6] have studied the reliability assessment of complex system consisting two subsystems connected in series configuration using Gumbel-Hougaard family copula distribution. Kakkar et al [7] have examined availability analysis of two parallel unit system under the provision of maintenance. Yusuf et al [22] have analyzed some reliability characteristics of a linear consecutive 2-out-of-4system connected to 2-out-of-4 supporting device for operation.

Some research works in the field of reliability and performance analysis of systems with standby components/units have shown that optimality among the systems under considerations is not unique and it depends on the value of some parameters. Some studies such as Wang and Learn [17], Wang et al. [18], EL-Sherbeny [3] and Yen and Wang [20] did take into the effect of cost benefit such as cost/availability and cost/mean time to failure on system reliability. Wang and Learn [17], Wang et al. [18], EL-Sherbeny [3] and Yen and Wang [20] have studied cost benefit analysis of various standby systems in which the optimality among configurations in the study depends only on particular parameter using cost/MTTF and depends on the other parameter using cost/availability. The present paper is motivated by the work of Wang and Learn [17], Wang et al. [18], EL-Sherbeny [3] and Yen and Wang [20] to study reliability of four 30MW power plant systems consisting of five units each arranged in series parallel and to determine the unique optimal system among the systems under study.

The contributions of this paper are as follow:

(i) To develop the explicit expressions for availability, busy period of repairman due partial and complete failure, mean time to failure and profit function.
(ii) To perform analytical comparison between the systems in order to rank them in terms of their availability and mean time to failure.
(iii) To study and compare the four systems in terms of their profit and cost benefit.
(iv) To determine the optimal system among the systems with cold standby.

The structure of this paper is organized as follows. Section 2 gives the description of the systems considered and their reliability block diagrams. Section 3 deals with the formulation of the models. Comparison between the systems analytically in terms of their availability and mean time to failure and numerically in terms of their profit and cost benefit are presented in Section 4. Conclusions are given in Sections 5.

II. Description of the System Configurations

The present study considers $30MW$ power plant arranged in the following four series parallel configurations as shown in Figures 1-4 below: System configuration I is a series parallel system which has three $10MW$ primary units and two $10MW$ cold standby components. System configuration II is a series parallel system having two $15MW$ primary components and three $15MW$ cold standby components. System configuration III is a series parallel system with three $10MW$ primary units and two $10MW$ cold standby units. System configuration IV is a series parallel system which consists of three subsystems with two subsystems arranged in parallel and serial to the other subsystem with two $15MW$ primary components and three $10MW$ cold standby components. It is assumed that all switchover time are instantaneous and switching is perfect. It is also assumed that the switch from standby to operation is perfect. Each of the primary units fails independently of the state of the others and has an exponential failure time with parameter β_0 and is replace with cold standby unit if available while the failed unit is immediately sent for repairs and the time to repair is exponential with parameter α_0 . All failures are assumed to be repairable. System failure occurs when all units in the same subsystem have failed. A failure is partial if the system has not failed completely otherwise the failure is complete (system failure).

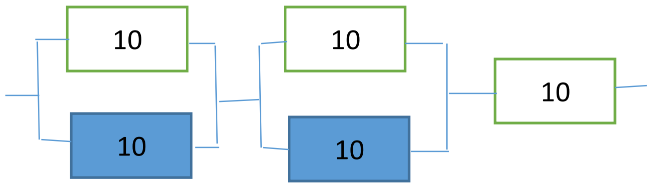

Figure 1: *Reliability block diagram of System Configuration I*

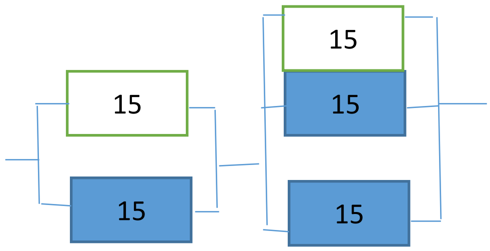

Figure 2: *Reliability block diagram of System Configuration II*

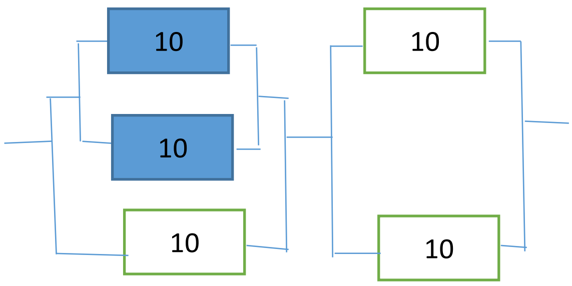

Figure 3: *Reliability block diagram of System Configuration III*

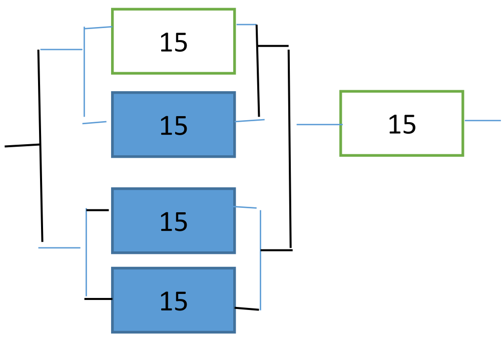

Figure 4: *Reliability block diagram of Configuration IV*

III. Reliability Models Formulation

Formulation of System Configuration I

Mean time to failure (MTTF) Analysis of System Configuration I

Let the probability that the system is in state i at time t be $h_i(t)$ and $H(t)=\left[h_1(t),h_2(t),\ldots,h_{11}(t)\right]$ be the probability row vector time t with initial conditions

$$h_k(0)=\begin{cases}1, & k=0\\ 0, & k=1,2,3,\ldots,11\end{cases}$$

The differential-difference equations derived from system configuration I are given by:

$$\left.\begin{aligned}
\frac{d}{dt}h_0(t)&=-3\beta_0h_0(t)+\alpha_0h_1(t)+\alpha_0h_2(t)+\alpha_0h_{10}(t)\\
\frac{d}{dt}h_1(t)&=-(3\beta_0+\alpha_0)h_1(t)+\beta_0h_0(t)+\alpha_0h_3(t)+\alpha_0h_5(t)+\alpha_0h_9(t)\\
\frac{d}{dt}h_2(t)&=-(3\beta_0+\alpha_0)h_2(t)+\beta_0h_0(t)+\alpha_0h_3(t)+\alpha_0h_4(t)+\alpha_0h_{11}(t)\\
\frac{d}{dt}h_3(t)&=-(3\beta_0+2\alpha_0)h_3(t)+\beta_0h_1(t)+\beta_0h_2(t)+\alpha_0h_6(t)+\alpha_0h_7(t)+\alpha_0h_8(t)\\
\frac{d}{dt}h_i(t)&=-\alpha_0h_i(t)+\beta_0h_j(t)
\end{aligned}\right\} \tag{1}$$

$i=4,5,6,\ldots,11$ and $j=0,1,2,3$

Equation (1) can be written in matrix form as:

$$H'(t)=M_1H(t) \tag{2}$$

Where

$$M_1=\begin{pmatrix}
-3\beta_0 & \alpha_0 & \alpha_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_0 & 0\\
\beta_0 & -\delta_0 & 0 & \alpha_0 & 0 & \alpha_0 & 0 & 0 & 0 & \alpha_0 & 0 & 0\\
\beta_0 & 0 & -\delta_0 & \alpha_0 & \alpha_0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_0\\
0 & \beta_0 & \beta_0 & -\delta_1 & 0 & 0 & \alpha_0 & \alpha_0 & \alpha_0 & 0 & 0 & 0\\
0 & 0 & \beta_0 & 0 & -\alpha_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & \beta_0 & 0 & 0 & 0 & -\alpha_0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & \beta_0 & 0 & 0 & -\alpha_0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & \beta_0 & 0 & 0 & 0 & -\alpha_0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & \beta_0 & 0 & 0 & 0 & 0 & -\alpha_0 & 0 & 0 & 0\\
0 & \beta_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\alpha_0 & 0 & 0\\
\beta_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\alpha_0 & 0\\
0 & 0 & \beta_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\alpha_0
\end{pmatrix}$$

To compute the *MTTF* of system configuration I, the procedure requires deleting rows and columns of absorbing states of matrix M_1 and take the transpose to produce a new matrix, Q_1 as adopted in Wang and Kuo [17], Wang et al [16] and Wang et al [18]. The time expected to reach the absorbing state is calculated from:

$$MTTF_1 = H(0)\left(-Q_1^{-1}\right)(1,1,1,1)^T \tag{3}$$

Thus, the *MTTF* expression for system configuration I is:

$$MTTF_1 = \frac{2\alpha_0^2 + 11\alpha_0\beta_0 + 17\beta_0^2}{\beta_0\left(2\alpha_0^2 + 15\alpha_0\beta_0 + 27\beta_0^2\right)} \tag{4}$$

Where $H(0) = [1,0,0,0]$ and $Q_1 = \begin{pmatrix} -3\beta_0 & \beta_0 & \beta_0 & 0 \\ \alpha_0 & -\delta_0 & 0 & \beta_0 \\ \alpha_0 & 0 & -\delta_0 & \beta_0 \\ 0 & \alpha_0 & \alpha_0 & -\delta_1 \end{pmatrix}$

Availability and Busy period of System Configuration I

To compute the availability of system configuration I, the differential difference equation given in (2) are expressed in the form:

$$\begin{pmatrix} h_0' \\ h_1' \\ h_2' \\ h_3' \\ h_4' \\ h_5' \\ h_6' \\ h_7' \\ h_8' \\ h_9' \\ h_{10}' \\ h_{11}' \end{pmatrix} = \begin{pmatrix} -3\beta_0 & \alpha_0 & \alpha_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_0 & 0 \\ \beta_0 & -\delta_0 & 0 & \alpha_0 & 0 & \alpha_0 & 0 & 0 & 0 & \alpha_0 & 0 & 0 \\ \beta_0 & 0 & -\delta_0 & \alpha_0 & \alpha_0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_0 \\ 0 & \beta_0 & \beta_0 & -\delta_1 & 0 & 0 & \alpha_0 & \alpha_0 & \alpha_0 & 0 & 0 & 0 \\ 0 & 0 & \beta_0 & 0 & -\alpha_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \beta_0 & 0 & 0 & 0 & -\alpha_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta_0 & 0 & 0 & -\alpha_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta_0 & 0 & 0 & 0 & -\alpha_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta_0 & 0 & 0 & 0 & 0 & -\alpha_0 & 0 & 0 & 0 \\ 0 & \beta_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\alpha_0 & 0 & 0 \\ \beta_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\alpha_0 & 0 \\ 0 & 0 & \beta_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\alpha_0 \end{pmatrix} \begin{pmatrix} h_0 \\ h_1 \\ h_2 \\ h_3 \\ h_4 \\ h_5 \\ h_6 \\ h_7 \\ h_8 \\ h_9 \\ h_{10} \\ h_{11} \end{pmatrix}$$

The steady state availability (i.e. the sum of the probabilities of all the operational states), busy period due to partial failure and complete failure are respectively given by:

$$A_{V1}(\infty) = h_0(\infty) + h_1(\infty) + h_2(\infty) + h_3(\infty) \tag{5}$$

$$B_{h1} = h_1(\infty) + h_2(\infty) + h_3(\infty) \tag{6}$$

$$B_{h2} = h_4(\infty) + h_5(\infty) + h_6(\infty) + h_7(\infty) + h_8(\infty) + h_9(\infty) + h_{10}(\infty) + h_{11}(\infty) \tag{7}$$

All the derivatives of state probabilities are set equal to zero in the steady state, therefore equation (2) becomes:

$$M_1 H(t)^T = 0 \tag{8}$$

Which is

$$\begin{pmatrix} -3\beta_0 & \alpha_0 & \alpha_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_0 & 0 \\ \beta_0 & -\delta_0 & 0 & \alpha_0 & 0 & \alpha_0 & 0 & 0 & 0 & \alpha_0 & 0 & 0 \\ \beta_0 & 0 & -\delta_0 & \alpha_0 & \alpha_0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_0 \\ 0 & \beta_0 & \beta_0 & -\delta_1 & 0 & 0 & \alpha_0 & \alpha_0 & \alpha_0 & 0 & 0 & 0 \\ 0 & 0 & \beta_0 & 0 & -\alpha_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \beta_0 & 0 & 0 & 0 & -\alpha_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta_0 & 0 & 0 & -\alpha_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta_0 & 0 & 0 & 0 & -\alpha_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta_0 & 0 & 0 & 0 & 0 & -\alpha_0 & 0 & 0 & 0 \\ 0 & \beta_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\alpha_0 & 0 & 0 \\ \beta_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\alpha_0 & 0 \\ 0 & 0 & \beta_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\alpha_0 \end{pmatrix} \begin{pmatrix} h_0 \\ h_1 \\ h_2 \\ h_3 \\ h_4 \\ h_5 \\ h_6 \\ h_7 \\ h_8 \\ h_9 \\ h_{10} \\ h_{11} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Using the following normalizing condition:

$$\sum_{n=0}^{11} h_n(\infty) = 1 \tag{9}$$

To compute the state probabilities $h_i(t)$ $i = 0,1,2,...,11$, (9) is substituted in the last of (8) to give:

$$\begin{pmatrix} -3\beta_0 & \alpha_0 & \alpha_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_0 & 0 \\ \beta_0 & -\delta_0 & 0 & \alpha_0 & 0 & \alpha_0 & 0 & 0 & 0 & \alpha_0 & 0 & 0 \\ \beta_0 & 0 & -\delta_0 & \alpha_0 & \alpha_0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_0 \\ 0 & \beta_0 & \beta_0 & -\delta_1 & 0 & 0 & \alpha_0 & \alpha_0 & \alpha_0 & 0 & 0 & 0 \\ 0 & 0 & \beta_0 & 0 & -\alpha_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \beta_0 & 0 & 0 & 0 & -\alpha_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta_0 & 0 & 0 & -\alpha_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta_0 & 0 & 0 & 0 & -\alpha_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta_0 & 0 & 0 & 0 & 0 & -\alpha_0 & 0 & 0 & 0 \\ 0 & \beta_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\alpha_0 & 0 & 0 \\ \beta_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\alpha_0 & 0 \\ 0 & 0 & \beta_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\alpha_0 \end{pmatrix} \begin{pmatrix} h_0 \\ h_1 \\ h_2 \\ h_3 \\ h_4 \\ h_5 \\ h_6 \\ h_7 \\ h_8 \\ h_9 \\ h_{10} \\ h_{11} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

(10)

Solving (10) using MATLAB package to obtain $h_i(t)$, the expressions for the steady-state availability, busy period due to partial failure and complete failure given in (5) to (7) are respectively given by:

$$A_{V1}(\infty) = \frac{\alpha_0^3 + 2\alpha_0^2\beta_0 + \alpha_0\beta_0^2}{\alpha_0^3 + 3\alpha_0^2\beta_0 + 5\alpha_0\beta_0^2 + 3\beta_0^3} \tag{11}$$

$$B_{h1}(\infty) = \frac{2\alpha_0^2\beta_0 + \alpha_0\beta_0^2}{\alpha_0^3 + 3\alpha_0^2\beta_0 + 5\alpha_0\beta_0^2 + 3\beta_0^3} \tag{12}$$

$$B_{h2}(\infty) = \frac{3\beta_0^3 + \alpha_0^2\beta_0 + 4\alpha_0\beta_0^2}{\alpha_0^3 + 3\alpha_0^2\beta_0 + 5\alpha_0\beta_0^2 + 3\beta_0^3} \tag{13}$$

Profit Analysis of System Configuration I

The units are exposed to corrective maintenance due to partial and complete failure, while the repairman is busy performing maintenance action to the failed units. Let K_0, K_1 and K_2 be the revenue generated when the system is in working state and no income when in failed state, cost of each repair due to partial and complete failure respectively. The expected total profit of system configuration I per unit time incurred to the system in the steady-state is given by:
Profit =total revenue generated – cost incurred by the repair man due to partial failure – cost incurred due to complete failure.

$$PF_1 = K_0 A_{V1} - \left(K_1 B_{h1} + K_2 B_{h2}\right) \tag{14}$$

Formulation of System Configuration II

Mean time to failure (MTTF) Analysis of System Configuration II

Let $H(t) = \left[h_0(t), h_1(t), h_2(t), \ldots, h_{10}(t)\right]$ be the probability row vector at time t. The initial condition is given by: $h_j(0) = \begin{cases} 1, & j = 0 \\ 0, & j = 1,2,3,\ldots,10 \end{cases}$

The corresponding set of differential-difference equations for system configuration II are expressed as:

$$H'(t) = M_2 H(t) \tag{15}$$

Where:

$$M_2 = \begin{pmatrix}
-2\beta_0 & \alpha_0 & \alpha_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\beta_0 & -(2\beta_0+\alpha_0) & 0 & 0 & \alpha_0 & 0 & 0 & \alpha_0 & 0 & 0 & 0 \\
\beta_0 & 0 & -(2\beta_0+\alpha_0) & \alpha_0 & \alpha_0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \beta_0 & -(2\beta_0+\alpha_0) & 0 & \alpha_0 & \alpha_0 & 0 & 0 & 0 & 0 \\
0 & \beta_0 & \beta_0 & 0 & -(2\beta_0+2\alpha_0) & \alpha_0 & 0 & 0 & \alpha_0 & 0 & 0 \\
0 & 0 & 0 & \beta_0 & \beta_0 & -(2\beta_0+2\alpha_0) & 0 & 0 & 0 & \alpha_0 & \alpha_0 \\
0 & 0 & 0 & \beta_0 & 0 & 0 & -\alpha_0 & 0 & 0 & 0 & 0 \\
0 & \beta_0 & 0 & 0 & 0 & 0 & 0 & -\alpha_0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\alpha_0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \beta_0 & 0 & 0 & 0 & -\alpha_0 & 0 \\
0 & 0 & 0 & 0 & 0 & \beta_0 & 0 & 0 & 0 & 0 & -\alpha_0
\end{pmatrix}$$

Using similar procedure presented in subsection 3.1.1, the expression for the mean time to failure $(MTTF)$ of system configuration II is obtained through:

$$MTTF_2 = H(0)\left(-Q_2^{-1}\right)(1,1,1,1,1,1)^T \tag{16}$$

Thus,

$$MTTF_2 = \frac{4\alpha_0^5 + 29\alpha_0^4\beta_0 + 97\alpha_0^3\beta_0^2 + 119\alpha_0^2\beta_0^3 + 211\alpha_0\beta_0^4 + 100\beta_0^5}{\beta_0^2\left(4\alpha_0^4 + 25\alpha_0^3\beta_0 + 76\alpha_0^2\beta_0^2 + 112\alpha_0\beta_0^3 + 64\beta_0^4\right)} \tag{17}$$

Where $H(0) = [1,0,0,0,0,0]$ and

$$Q_2=\begin{pmatrix} -2\beta_0 & \beta_0 & \beta_0 & 0 & 0 & 0 \\ \alpha_0 & -(2\beta_0+\alpha_0) & 0 & 0 & \beta_0 & 0 \\ \alpha_0 & 0 & -(2\beta_0+\alpha_0) & \beta_0 & \beta_0 & 0 \\ 0 & 0 & \alpha_0 & -(2\beta_0+\alpha_0) & 0 & \beta_0 \\ 0 & \alpha_0 & \alpha_0 & 0 & -(2\beta_0+2\alpha_0) & \beta_0 \\ 0 & 0 & 0 & \alpha_0 & \alpha_0 & -(2\beta_0+2\alpha_0) \end{pmatrix}$$

Availability and Busy period Analysis of System Configuration II

To compute the availability of system configuration II, the differential-difference equations given in (14) are expressed in the form:

$$\begin{pmatrix} h_0' \\ h_1' \\ h_2' \\ h_3' \\ h_4' \\ h_5' \\ h_6' \\ h_7' \\ h_8' \\ h_9' \\ h_{10}' \end{pmatrix}=\begin{pmatrix} -2\beta_0 & \alpha_0 & \alpha_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \beta_0 & -(2\beta_0+\alpha_0) & 0 & 0 & \alpha_0 & 0 & 0 & \alpha_0 & 0 & 0 & 0 \\ \beta_0 & 0 & -(2\beta_0+\alpha_0) & \alpha_0 & \alpha_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta_0 & -(2\beta_0+\alpha_0) & 0 & \alpha_0 & \alpha_0 & 0 & 0 & 0 & 0 \\ 0 & \beta_0 & \beta_0 & 0 & -(2\beta_0+2\alpha_0) & \alpha_0 & 0 & 0 & \alpha_0 & 0 & 0 \\ 0 & 0 & 0 & \beta_0 & \beta_0 & -(2\beta_0+2\alpha_0) & 0 & 0 & 0 & \alpha_0 & \alpha_0 \\ 0 & 0 & 0 & \beta_0 & 0 & 0 & -\alpha_0 & 0 & 0 & 0 & 0 \\ 0 & \beta_0 & 0 & 0 & 0 & 0 & 0 & -\alpha_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\alpha_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta_0 & 0 & 0 & 0 & -\alpha_0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta_0 & 0 & 0 & 0 & 0 & -\alpha_0 \end{pmatrix}\begin{pmatrix} h_0 \\ h_1 \\ h_2 \\ h_3 \\ h_4 \\ h_5 \\ h_6 \\ h_7 \\ h_8 \\ h_9 \\ h_{10} \end{pmatrix}$$

The steady state availability (the proportion of the time the system is functioning or equivalently the sum of the probabilities of operational state), busy period due to partial failure and complete failure are given by:

$$A_{V2}(\infty)=h_0(\infty)+h_1(\infty)+h_2(\infty)+h_3(\infty)+h_4(\infty)+h_5(\infty) \tag{18}$$

$$B_{h3}=h_1(\infty)+h_2(\infty)+h_3(\infty)+h_4(\infty)+h_5(\infty) \tag{19}$$

$$B_{h4}(\infty)=h_6(\infty)+h_7(\infty)+h_8(\infty)+h_9(\infty)+h_{10}(\infty) \tag{20}$$

In the steady state, the derivatives of states probabilities become zero and therefore (15) becomes:

$$M_2H(t)^T=0 \tag{21}$$

In matrix form, we have:

$$\begin{pmatrix} -2\beta_0 & \alpha_0 & \alpha_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \beta_0 & -(2\beta_0+\alpha_0) & 0 & 0 & \alpha_0 & 0 & 0 & \alpha_0 & 0 & 0 & 0 \\ \beta_0 & 0 & -(2\beta_0+\alpha_0) & \alpha_0 & \alpha_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta_0 & -(2\beta_0+\alpha_0) & 0 & \alpha_0 & \alpha_0 & 0 & 0 & 0 & 0 \\ 0 & \beta_0 & \beta_0 & 0 & -(2\beta_0+2\alpha_0) & \alpha_0 & 0 & 0 & \alpha_0 & 0 & 0 \\ 0 & 0 & 0 & \beta_0 & \beta_0 & -(2\beta_0+2\alpha_0) & 0 & 0 & 0 & \alpha_0 & \alpha_0 \\ 0 & 0 & 0 & \beta_0 & 0 & 0 & -\alpha_0 & 0 & 0 & 0 & 0 \\ 0 & \beta_0 & 0 & 0 & 0 & 0 & 0 & -\alpha_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\alpha_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta_0 & 0 & 0 & 0 & -\alpha_0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta_0 & 0 & 0 & 0 & 0 & -\alpha_0 \end{pmatrix}\begin{pmatrix} h_0 \\ h_1 \\ h_2 \\ h_3 \\ h_4 \\ h_5 \\ h_6 \\ h_7 \\ h_8 \\ h_9 \\ h_{10} \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Using the following normalizing condition

$$\sum_{n=0}^{10} h_n(\infty) = 1 \tag{22}$$

To obtain the state probabilities $h_i(t)$ $i = 0,1,2,...,10$, we substitute (22) in (21) to get

$$\begin{pmatrix} -2\beta_0 & \alpha_0 & \alpha_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \beta_0 & -(2\beta_0+\alpha_0) & 0 & 0 & \alpha_0 & 0 & 0 & \alpha_0 & 0 & 0 & 0 \\ \beta_0 & 0 & -(2\beta_0+\alpha_0) & \alpha_0 & \alpha_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta_0 & -(2\beta_0+\alpha_0) & 0 & \alpha_0 & \alpha_0 & 0 & 0 & 0 & 0 \\ 0 & \beta_0 & \beta_0 & 0 & -(2\beta_0+2\alpha_0) & \alpha_0 & 0 & 0 & \alpha_0 & 0 & 0 \\ 0 & 0 & 0 & \beta_0 & \beta_0 & -(2\beta_0+2\alpha_0) & 0 & 0 & 0 & \alpha_0 & \alpha_0 \\ 0 & 0 & 0 & \beta_0 & 0 & 0 & -\alpha_0 & 0 & 0 & 0 & 0 \\ 0 & \beta_0 & 0 & 0 & 0 & 0 & 0 & -\alpha_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\alpha_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta_0 & 0 & 0 & 0 & -\alpha_0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta_0 & 0 & 0 & 0 & 0 & -\alpha_0 \end{pmatrix} \begin{pmatrix} h_0 \\ h_1 \\ h_2 \\ h_3 \\ h_4 \\ h_5 \\ h_6 \\ h_7 \\ h_8 \\ h_9 \\ h_{10} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{23}$$

Solving (23) using MATLAB package to obtain $h_i(t)$, the explicit expressions for the steady-state availability, busy period due to partial failure and complete failure are given by:

$$A_{V2}(\infty) = \frac{\alpha_0^4 + 2\alpha_0^3\beta_0 + 2\alpha_0^2\beta_0^2 + \alpha_0\beta_0^3}{\alpha_0^4 + 2\alpha_0^3\beta_0 + 3\alpha_0^2\beta_0^2 + 3\alpha_0\beta_0^3 + 2\beta_0^4} \tag{24}$$

$$B_{h3}(\infty) = \frac{2\alpha_0^3\beta_0 + 2\alpha_0^2\beta_0^2 + \alpha_0\beta_0^3}{\alpha_0^4 + 2\alpha_0^3\beta_0 + 3\alpha_0^2\beta_0^2 + 3\alpha_0\beta_0^3 + 2\beta_0^4} \tag{25}$$

$$B_{h4}(\infty) = \frac{2\beta_0^4 + \alpha_0^2\beta_0^2 + \alpha_0\beta_0^3}{\alpha_0^4 + 2\alpha_0^3\beta_0 + 3\alpha_0^2\beta_0^2 + 3\alpha_0\beta_0^3 + 2\beta_0^4} \tag{26}$$

Profit Analysis of System Configuration II

Using similar procedure presented in subsection 3.1.3, the explicit expression for profit function of system configuration II is given by:

$$PF_2 = K_0 A_{V2} - (K_1 B_{h3} + K_2 B_{h4}) \tag{27}$$

Formulation of System Configuration III

Mean time to failure of Analysis System Configuration III

Let $H(t) = [h_0(t), h_1(t), ..., h_8(t)]$ be the probability row vector at time t with initial conditions

$$h_n(0) = \begin{cases} 1, & n = 0 \\ 0, & n = 1,2,3,...,8 \end{cases}$$

The corresponding set of differential-difference equations for system configuration III are expressed as:

$$H'(t) = M_3 H(t) \tag{28}$$

Where

$$M_3 = \begin{pmatrix} -3\beta_0 & \alpha_0 & \alpha_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \beta_0 & -(3\beta_0+\alpha_0) & 0 & \alpha_0 & \alpha_0 & 0 & 0 & 0 & 0 \\ 2\beta_0 & 0 & -(3\beta_0+\alpha_0) & 0 & 0 & \alpha_0 & \alpha_0 & 0 & 0 \\ 0 & \beta_0 & 0 & -(3\beta_0+\alpha_0) & 0 & 0 & 0 & \alpha_0 & \alpha_0 \\ 0 & 2\beta_0 & 0 & 0 & -\alpha_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta_0 & 0 & 0 & -\alpha_0 & 0 & 0 & 0 \\ 0 & 0 & 2\beta_0 & 0 & 0 & 0 & -\alpha_0 & 0 & 0 \\ 0 & 0 & 0 & 2\beta_0 & 0 & 0 & 0 & -\alpha_0 & 0 \\ 0 & 0 & 0 & \beta_0 & 0 & 0 & 0 & 0 & -\alpha_0 \end{pmatrix}$$

Using similar procedure presented in subsection 3.1.1, the explicit expression for the mean time to failure $(MTTF)$ of System Configuration III is obtained through:

$$MTTF_3 = H(0)\left(-Q_3^{-1}\right)(1,1,1,1)^T \tag{29}$$

Thus,

$$MTTF_3 = \frac{\alpha_0^3 + 11\alpha_0^2\beta_0 + 41\alpha_0\beta_0^2 + 57\beta_0^3}{\beta_0^2\left(8\alpha_0^2 + 45\alpha_0\beta_0 + 81\beta_0^2\right)} \tag{30}$$

Where $H(0) = [1,0,0,0]$ and $Q_3 = \begin{pmatrix} -3\beta_0 & \beta_0 & 2\beta_0 & 0 \\ \alpha_0 & -(3\beta_0+\alpha_0) & 0 & \beta_0 \\ \alpha_0 & 0 & -(3\beta_0+\alpha_0) & 0 \\ 0 & \alpha_0 & 0 & -(3\beta_0+\alpha_0) \end{pmatrix}$

Availability and Busy period Analysis of System Configuration III

To compute the availability of system configuration III, the differential difference equations given in (28) are expressed in the form:

$$\begin{pmatrix} h_0' \\ h_1' \\ h_2' \\ h_3' \\ h_4' \\ h_5' \\ h_6' \\ h_7' \\ h_8' \end{pmatrix} = \begin{pmatrix} -3\beta_0 & \alpha_0 & \alpha_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \beta_0 & -y_0 & 0 & \alpha_0 & \alpha_0 & 0 & 0 & 0 & 0 \\ 2\beta_0 & 0 & -y_0 & 0 & 0 & \alpha_0 & \alpha_0 & 0 & 0 \\ 0 & \beta_0 & 0 & -y_0 & 0 & 0 & 0 & \alpha_0 & \alpha_0 \\ 0 & 2\beta_0 & 0 & 0 & -\alpha_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta_0 & 0 & 0 & -\alpha_0 & 0 & 0 & 0 \\ 0 & 0 & 2\beta_0 & 0 & 0 & 0 & -\alpha_0 & 0 & 0 \\ 0 & 0 & 0 & 2\beta_0 & 0 & 0 & 0 & -\alpha_0 & 0 \\ 0 & 0 & 0 & \beta_0 & 0 & 0 & 0 & 0 & -\alpha_0 \end{pmatrix} \begin{pmatrix} h_0 \\ h_1 \\ h_2 \\ h_3 \\ h_4 \\ h_5 \\ h_6 \\ h_7 \\ h_8 \end{pmatrix}$$

The steady state availability (the proportion of the time the system is functioning), busy period due to partial failure and complete failure are given by:

$$A_{V3}(\infty) = h_0(\infty) + h_1(\infty) + h_2(\infty) + h_3(\infty) \tag{31}$$

$$B_{h5} = h_1(\infty) + h_2(\infty) + h_3(\infty) \tag{32}$$

$$B_{h6}(\infty) = h_4(\infty) + h_5(\infty) + h_6(\infty) + h_7(\infty) + h_8(\infty) \tag{33}$$

In the steady state, the derivatives of states probabilities become zero and therefore (28) becomes:

$$M_3 H(t)^T = 0 \tag{34}$$

In matrix form, we have:

$$\begin{pmatrix} -3\beta_0 & \alpha_0 & \alpha_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \beta_0 & -y_0 & 0 & \alpha_0 & \alpha_0 & 0 & 0 & 0 & 0 \\ 2\beta_0 & 0 & -y_0 & 0 & 0 & \alpha_0 & \alpha_0 & 0 & 0 \\ 0 & \beta_0 & 0 & -y_0 & 0 & 0 & 0 & \alpha_0 & \alpha_0 \\ 0 & 2\beta_0 & 0 & 0 & -\alpha_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta_0 & 0 & 0 & -\alpha_0 & 0 & 0 & 0 \\ 0 & 0 & 2\beta_0 & 0 & 0 & 0 & -\alpha_0 & 0 & 0 \\ 0 & 0 & 0 & 2\beta_0 & 0 & 0 & 0 & -\alpha_0 & 0 \\ 0 & 0 & 0 & \beta_0 & 0 & 0 & 0 & 0 & -\alpha_0 \end{pmatrix} \begin{pmatrix} h_0 \\ h_1 \\ h_2 \\ h_3 \\ h_4 \\ h_5 \\ h_6 \\ h_7 \\ h_8 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Using the following normalizing condition

$$\sum_{n=0}^{8} h_n(\infty) = 1 \tag{35}$$

To obtain the state probabilities $h_i(t)$ $i = 0,1,2,...,8$, (35) is substituted in (34) to give:

$$\begin{pmatrix} -3\beta_0 & \alpha_0 & \alpha_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \beta_0 & -y_0 & 0 & \alpha_0 & \alpha_0 & 0 & 0 & 0 & 0 \\ 2\beta_0 & 0 & -y_0 & 0 & 0 & \alpha_0 & \alpha_0 & 0 & 0 \\ 0 & \beta_0 & 0 & -y_0 & 0 & 0 & 0 & \alpha_0 & \alpha_0 \\ 0 & 2\beta_0 & 0 & 0 & -\alpha_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta_0 & 0 & 0 & -\alpha_0 & 0 & 0 & 0 \\ 0 & 0 & 2\beta_0 & 0 & 0 & 0 & -\alpha_0 & 0 & 0 \\ 0 & 0 & 0 & 2\beta_0 & 0 & 0 & 0 & -\alpha_0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \\ p_7 \\ p_8 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{36}$$

Solving (36) using MATLAB package to obtain $h_i(t)$,the explicit expressions for the steady-state availability, busy period due to partial failure and complete failure are given by:

$$A_{V3}(\infty) = \frac{\alpha_0^3 + 3\alpha_0^2\beta_0 + \alpha_0\beta_0^2}{\alpha_0^3 + 3\alpha_0^2\beta_0 + 9\alpha_0\beta_0^2 + 3\beta_0^3} \tag{37}$$

$$B_{h5}(\infty) = \frac{3\alpha_0^2\beta_0 + \alpha_0\beta_0^2}{\alpha_0^3 + 3\alpha_0^2\beta_0 + 9\alpha_0\beta_0^2 + 3\beta_0^3} \tag{38}$$

$$B_{h6}(\infty) = \frac{3\beta_0^3 + 8\alpha_0\beta_0^2}{\alpha_0^3 + 3\alpha_0^2\beta_0 + 9\alpha_0\beta_0^2 + 3\beta_0^3} \tag{39}$$

Profit analysis of System Configuration III

Using similar procedure presented in subsection 3.1.3, the explicit expression for profit function of configuration III is given by:

$$PF_3 = K_0 A_{V3} - \left(K_1 B_{h5} + K_2 B_{h6}\right) \tag{40}$$

Formulation of System Configuration IV

Mean time to failure of Analysis System Configuration IV

Define $H(t) = \left[h_0(t), h_1(t), h_2(t), ..., h_8(t)\right]$ to be the probability row vector at time t with initial conditions:

$$h_n(0)=\begin{cases}1, & n=0\\ 0, & n=1,2,3,\ldots,8\end{cases}.$$

The corresponding set of differential-difference equations for system configuration IV is expressed as:

$$H'(t)=M_4H(t) \tag{41}$$

Where:

$$M_4=\begin{pmatrix}-2\beta_0 & \alpha_0 & \alpha_0 & 0 & 0 & 0 & 0 & 0 & 0\\ \beta_0 & -(2\beta_0+\alpha_0) & 0 & \alpha_0 & \alpha_0 & 0 & 0 & 0 & 0\\ \beta_0 & 0 & -\alpha_0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & \beta_0 & 0 & -(2\beta_0+\alpha_0) & 0 & \alpha_0 & \alpha_0 & 0 & 0\\ 0 & \beta_0 & 0 & 0 & -\alpha_0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & \beta_0 & 0 & -(2\beta_0+\alpha_0) & 0 & \alpha_0 & \alpha_0\\ 0 & 0 & 0 & \beta_0 & 0 & 0 & -\alpha_0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & \beta_0 & 0 & -\alpha_0 & 0\\ 0 & 0 & 0 & 0 & 0 & \beta_0 & 0 & 0 & -\alpha_0\end{pmatrix}$$

Using similar procedure presented in subsection 3.1.1, the explicit expression for the mean time to failure $(MTTF)$ of System Configuration IV is obtained through:

$$MTTF_4=H(0)\left(-Q_4^{-1}\right)(1,1,1,1)^T \tag{42}$$

Thus,

$$MTTF_4=\frac{\alpha_0^3+5\alpha_0^2\beta_0+12\alpha_0\beta_0^2+15\beta_0^3}{\alpha_0^3+5\alpha_0^2\beta_0+12\alpha_0\beta_0^2+16\beta_0^3} \tag{43}$$

Where $H(0)=[1,0,0,0]$ and $Q_4=\begin{pmatrix}-2\beta_0 & \beta_0 & 0 & 0\\ \alpha_0 & -(2\beta_0+\alpha_0) & \beta_0 & 0\\ 0 & \alpha_0 & -(2\beta_0+\alpha_0) & \beta_0\\ 0 & 0 & \alpha_0 & -(2\beta_0+\alpha_0)\end{pmatrix}$

Availability and Busy period Analysis of System Configuration IV

To compute the availability of system configuration IV, the differential difference equations given in (41) are expressed in the form:

$$\begin{pmatrix}h_0'\\ h_1'\\ h_2'\\ h_3'\\ h_4'\\ h_5'\\ h_6'\\ h_7'\\ h_8'\end{pmatrix}=\begin{pmatrix}-2\beta_0 & \alpha_0 & \alpha_0 & 0 & 0 & 0 & 0 & 0 & 0\\ \beta_0 & -(2\beta_0+\alpha_0) & 0 & \alpha_0 & \alpha_0 & 0 & 0 & 0 & 0\\ \beta_0 & 0 & -\alpha_0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & \beta_0 & 0 & -(2\beta_0+\alpha_0) & 0 & \alpha_0 & \alpha_0 & 0 & 0\\ 0 & \beta_0 & 0 & 0 & -\alpha_0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & \beta_0 & 0 & -(2\beta_0+\alpha_0) & 0 & \alpha_0 & \alpha_0\\ 0 & 0 & 0 & \beta_0 & 0 & 0 & -\alpha_0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & \beta_0 & 0 & -\alpha_0 & 0\\ 0 & 0 & 0 & 0 & 0 & \beta_0 & 0 & 0 & -\alpha_0\end{pmatrix}\begin{pmatrix}h_0\\ h_1\\ h_2\\ h_3\\ h_4\\ h_5\\ h_6\\ h_7\\ h_8\end{pmatrix}$$

The steady state availability (the proportion of the time the system is functioning or equivalently the sum of the probabilities of operational states), busy period due to partial failure and complete failure are given by:

$$A_{V4}(\infty) = h_0(\infty) + h_1(\infty) + h_3(\infty) + h_5(\infty) \tag{44}$$

$$B_{h7} = h_1(\infty) + h_3(\infty) + h_5(\infty) \tag{45}$$

$$B_{h8}(\infty) = h_2(\infty) + h_4(\infty) + h_6(\infty) + h_7(\infty) + h_8(\infty) \tag{46}$$

In the steady state, the derivatives of states probabilities become zero and therefore (41) becomes:

$$M_4 H(t)^T = 0 \tag{47}$$

These can be expressed in matrix form as:

$$\begin{pmatrix} -2\beta_0 & \alpha_0 & \alpha_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \beta_0 & -(2\beta_0+\alpha_0) & 0 & \alpha_0 & \alpha_0 & 0 & 0 & 0 & 0 \\ \beta_0 & 0 & -\alpha_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \beta_0 & 0 & -(2\beta_0+\alpha_0) & 0 & \alpha_0 & \alpha_0 & 0 & 0 \\ 0 & \beta_0 & 0 & 0 & -\alpha_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta_0 & 0 & -(2\beta_0+\alpha_0) & 0 & \alpha_0 & \alpha_0 \\ 0 & 0 & 0 & \beta_0 & 0 & 0 & -\alpha_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta_0 & 0 & -\alpha_0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta_0 & 0 & 0 & -\alpha_0 \end{pmatrix} \begin{pmatrix} h_0 \\ h_1 \\ h_2 \\ h_3 \\ h_4 \\ h_5 \\ h_6 \\ h_7 \\ h_8 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Using the following normalizing condition

$$\sum_{n=0}^{8} h_n(\infty) = 1 \tag{48}$$

To obtained the state probabilities $h_i(t)\ i = 0,1,2,\ldots,8$ using (48) in the last row of (47) we get:

$$\begin{pmatrix} -2\beta_0 & \alpha_0 & \alpha_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \beta_0 & -(2\beta_0+\alpha_0) & 0 & \alpha_0 & \alpha_0 & 0 & 0 & 0 & 0 \\ \beta_0 & 0 & -\alpha_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \beta_0 & 0 & -(2\beta_0+\alpha_0) & 0 & \alpha_0 & \alpha_0 & 0 & 0 \\ 0 & \beta_0 & 0 & 0 & -\alpha_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta_0 & 0 & -(2\beta_0+\alpha_0) & 0 & \alpha_0 & \alpha_0 \\ 0 & 0 & 0 & \beta_0 & 0 & 0 & -\alpha_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta_0 & 0 & -\alpha_0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -\alpha_0 \end{pmatrix} \begin{pmatrix} h_0 \\ h_1 \\ h_2 \\ h_3 \\ h_4 \\ h_5 \\ h_6 \\ h_7 \\ h_8 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{49}$$

Explicit expressions for the steady-state availability, busy period due to partial failure and complete failure are given by:

$$A_{V4} = \frac{\alpha_0^4 + \alpha_0^3\beta_0 + \alpha_0^2\beta_0^2 + \alpha_0\beta_0^3}{\alpha_0^4 + 2\alpha_0^3\beta_0 + 2\alpha_0^2\beta_0^2 + 2\alpha_0\beta_0^3 + 2\beta_0^4} \tag{50}$$

$$B_{h7} = \frac{\alpha_0^3\beta_0 + 2\alpha_0^2\beta_0^2}{\alpha_0^4 + 2\alpha_0^3\beta_0 + 2\alpha_0^2\beta_0^2 + 2\alpha_0\beta_0^3 + 2\beta_0^4} \tag{51}$$

$$B_{h8} = \frac{\alpha_0^3\beta_0 + \alpha_0^2\beta_0^2 + \alpha_0\beta_0^3 + 2\beta_0^4}{\alpha_0^4 + 2\alpha_0^3\beta_0 + 2\alpha_0^2\beta_0^2 + 2\alpha_0\beta_0^3 + 2\beta_0^4} \tag{52}$$

Profit analysis of System Configuration IV

Using similar procedure presented in subsection 3.1.3, the explicit expression for profit function of system configuration IV is given by:

$$PF_4 = K_0 A_{V4} - \left(K_1 B_{h7} + K_2 B_{h8}\right) \tag{53}$$

IV. Discussion

I. Comparison between the System Configurations

Analytical Comparisons

Here, the configurations are compared analytically in terms of their availability and mean time to failure to determine the optimal configuration by taking the difference between the configurations $\alpha_0, \beta_0 > 0$ using MAPLE software package.

$$A_{V2} - A_{V1} = \frac{\alpha_0\beta_0\left(\alpha_0^4 + 4\alpha_0^3\beta_0 + 5\alpha_0^2\beta_0^2 + 3\alpha_0\beta_0^3 + \beta_0^4\right)}{\left(\alpha_0^3 + 3\alpha_0^2\beta_0 + 5\alpha_0\beta_0^2 + 3\beta_0^3\right)\left(\alpha_0^4 + 2\alpha_0^3\beta_0 + 3\alpha_0^2\beta_0^2 + 3\alpha_0\beta_0^3 + 2\beta_0^4\right)} > 0 \tag{54}$$

$$\Rightarrow A_{V2} > A_{V1} \qquad \forall\ \alpha_0, \beta_0 > 0$$

$$A_{V2} - A_{V3} = \frac{\alpha_0\beta_0^2\left(7\alpha_0^4 + 14\alpha_0^3\beta_0 + 13\alpha_0^2\beta_0^2 + 6\alpha_0\beta_0^3 + \beta_0^4\right)}{\left(\alpha_0^4 + 2\alpha_0^3\beta_0 + 3\alpha_0^2\beta_0^2 + 3\alpha_0\beta_0^3 + 2\beta_0^4\right)\left(\alpha_0^3 + 3\alpha_0^2\beta_0 + 9\alpha_0\beta_0^2 + 3\beta_0^3\right)} > 0 \tag{55}$$

$$\Rightarrow A_{V2} > A_{V3} \qquad \forall\ \alpha_0, \beta_0 > 0$$

$$A_{V2} - A_{V4} = \frac{\alpha_0^2\beta_0\left(\alpha_0^5 + 2\alpha_0^4\beta_0 + 2\alpha_0^3\beta_0^2 + 2\alpha_0^2\beta_0^3 + 2\alpha_0\beta_0^4 + \beta_0^5\right)}{\left(\alpha_0^4 + 2\alpha_0^3\beta_0 + 3\alpha_0^2\beta_0^2 + 3\alpha_0\beta_0^3 + 2\beta_0^4\right)\left(\alpha_0^4 + 2\alpha_0^3\beta_0 + 2\alpha_0^2\beta_0^2 + 2\alpha_0\beta_0^3 + 2\beta_0^4\right)} > 0 \tag{56}$$

$$\Rightarrow A_{V2} > A_{V4} \qquad \forall\ \alpha_0, \beta_0 > 0$$

$$A_{V3} - A_{V4} = \frac{\alpha_0\beta_0\left(\alpha_0^5 - 4\alpha_0^4\beta_0 - 6\alpha_0^3\beta_0^2 - 5\alpha_0^2\beta_0^3 - 4\alpha_0\beta_0^4 - \beta_0^5\right)}{\left(\alpha_0^3 + 3\alpha_0^2\beta_0 + 9\alpha_0\beta_0^2 + 3\beta_0^3\right)\left(\alpha_0^4 + 2\alpha_0^3\beta_0 + 2\alpha_0^2\beta_0^2 + 2\alpha_0\beta_0^3 + 2\beta_0^4\right)} > 0 \tag{57}$$

$\Rightarrow A_{V3}(\infty) > A_{V4}(\infty)$ if and only if $\alpha_0^5 > \left(4\alpha_0^4\beta_0 + 6\alpha_0^3\beta_0^2 + 5\alpha_0^2\beta_0^3 + 4\alpha_0\beta_0^4 + \beta_0^5\right)$ for some $\alpha_0 > \beta_0$

$$A_{V3} - A_{V1} = \frac{\alpha_0\beta_0^2\left(\alpha_0^2 - 2\alpha_0\beta_0 - \beta_0^2\right)}{\left(\alpha_0^3 + 3\alpha_0^2\beta_0 + 9\alpha_0\beta_0^2 + 3\beta_0^3\right)\left(\alpha_0^3 + 3\alpha_0^2\beta_0 + 5\alpha_0\beta_0^2 + 3\beta_0^3\right)} > 0 \tag{58}$$

$\Rightarrow A_{V3}(\infty) > A_{V1}(\infty)$ if and only if $\alpha_0^2 > \left(2\alpha_0\beta_0 + \beta_0^2\right)$ for some $\alpha_0 > \beta_0$

$$A_{V4} - A_{V1} = \frac{\alpha_0\beta_0^2\left(2\alpha_0^3 + 2\alpha_0^2\beta_0 + \alpha_0\beta_0^2 + \beta_0^3\right)}{\left(\alpha_0^4 + 2\alpha_0^3\beta_0 + 2\alpha_0^2\beta_0^2 + 2\alpha_0\beta_0^3 + 2\beta_0^4\right)\left(\alpha_0^2 + 2\alpha_0\beta_0 + 3\beta_0^2\right)} > 0 \tag{59}$$

$\Rightarrow A_{V4} > A_{V1} \qquad \forall\ \alpha_0, \beta_0 > 0$

Using availability models of all the system configurations, it is clear from (54) – (59) that

$$A_{V2} > A_{V3} > A_{V4} > A_{V1}$$

Thus, the optimal system configuration is configuration II

$$MTTF_2 - MTTF_1 = \frac{8\alpha_0^7 + 110\alpha_0^6\beta_0 + 643\alpha_0^5\beta_0^2 + 2125\alpha_0^4\beta_0^3 + 4421\alpha_0^3\beta_0^4 + 5870\alpha_0^2\beta_0^5 + 4589\alpha_0\beta_0^6 + 1617\beta_0^7}{\beta_0^2\left(2\alpha_0^2 + 15\alpha_0\beta_0 + 27\beta_0^2\right)\left(4\alpha_0^4 + 25\alpha_0^3\beta_0 + 76\alpha_0^2\beta_0^2 + 112\alpha_0\beta_0^3 + 64\beta_0^4\right)} > 0 \tag{60}$$

$\Rightarrow MTTF_2 > MTTF_1 \qquad \forall\ \alpha_0, \beta_0 > 0$

$$MTTF_2 - MTTF_3 = \frac{28\alpha_0^7 + 343\alpha_0^6\beta_0 + 1890\alpha_0^5\beta_0^2 + 6041\alpha_0^4\beta_0^3 + 12303\alpha_0^3\beta_0^4 + 16138\alpha_0^2\beta_0^5 + 12585\alpha_0\beta_0^6 + 4452\beta_0^7}{\beta_0^2\left(4\alpha_0^4 + 25\alpha_0^3\beta_0 + 76\alpha_0^2\beta_0^2 + 112\alpha_0\beta_0^3 + 64\beta_0^4\right)\left(8\alpha_0^2 + 45\alpha_0\beta_0 + 81\beta_0^2\right)} > 0 \quad (61)$$

$$\Rightarrow MTTF_2 > MTTF_3 \qquad \forall\ \alpha_0, \beta_0 > 0$$

$$MTTF_2 - MTTF_4 = \frac{4\alpha_0^8 + 45\alpha_0^7\beta_0 + 245\alpha_0^6\beta_0^2 + 839\alpha_0^5\beta_0^3 + 1942\alpha_0^4\beta_0^4 + 3088\alpha_0^3\beta_0^5 + 3284\alpha_0^2\beta_0^6 + 2128\alpha_0\beta_0^7 + 640\beta_0^8}{\beta_0^2\left(\alpha_0^3 + 5\alpha_0^2\beta_0 + 12\alpha_0\beta_0^2 + 16\beta_0^3\right)\left(4\alpha_0^4 + 25\alpha_0^3\beta_0 + 76\alpha_0^2\beta_0^2 + 112\alpha_0\beta_0^3 + 64\beta_0^4\right)} > 0 \quad (62)$$

$$\Rightarrow MTTF_2 > MTTF_4 \qquad \forall\ \alpha_0, \beta_0 > 0$$

$$MTTF_3 - MTTF_1 = \frac{2\alpha_0^5 + 21\alpha_0^4\beta_0 + 96\alpha_0^3\beta_0^2 + 233\alpha_0^2\beta_0^2 + 306\alpha_0\beta_0^4 + 162\beta_0^5}{\beta_0^2\left(2\alpha_0^2 + 15\alpha_0\beta_0 + 27\beta_0^2\right)\left(8\alpha_0^2 + 45\alpha_0\beta_0 + 81\beta_0^2\right)} > 0 \quad (63)$$

$$\Rightarrow MTTF_3 > MTTF_1 \qquad \forall\ \alpha_0, \beta_0 > 0$$

$$MTTF_3 - MTTF_4 = \frac{\alpha_0^6 + 8\alpha_0^5\beta_0 + 23\alpha_0^4\beta_0^2 + 8\alpha_0^3\beta_0^3 - 112\alpha_0^2\beta_0^4 - 307\alpha_0\beta_0^5 - 305\beta_0^6}{\beta_0^2\left(\alpha_0^3 + 5\alpha_0^2\beta_0 + 12\alpha_0\beta_0^2 + 16\beta_0^3\right)\left(8\alpha_0^2 + 45\alpha_0\beta_0 + 81\beta_0^2\right)} > 0 \quad (64)$$

$\Rightarrow MTTF_3 > MTTF_4$ if and only if

$\left(\alpha_0^6 + 8\alpha_0^5\beta_0 + 23\alpha_0^4\beta_0^2 + 8\alpha_0^3\beta_0^3\right) > \left(112\alpha_0^2\beta_0^4 + 307\alpha_0\beta_0^5 + 305\beta_0^6\right)$ for some $\alpha_0, \beta_0 > 0$

$$MTTF_4 - MTTF_1 = \frac{4\alpha_0^4 + 30\alpha_0^3\beta_0 + 96\alpha_0^2\beta_0^2 + 8\alpha_0^3\beta_0^3 + 169\alpha_0\beta_0^3 + 133\beta_0^4}{\left(\alpha_0^3 + 5\alpha_0^2\beta_0 + 12\alpha_0\beta_0^2 + 16\beta_0^3\right)\left(2\alpha_0^2 + 15\alpha_0\beta_0 + 27\beta_0^2\right)} > 0 \quad (65)$$

$$\Rightarrow MTTF_4 > MTTF_1 \qquad \forall\ \alpha_0, \beta_0 > 0$$

Similarly, using mean time to failure $(MTTF)$ models of all the system configurations, it is clear from (60) – (65) that

$$MTTF_2 > MTTF_3 > MTTF_4 > MTTF_1$$

Thus, the optimal system configuration is configuration II

II. Numerical examples

The purpose of this section is to rank the system configurations in terms of their availability and mean time to failure using MATLAB software package. The results are summarized in tables below

Table 1: *Ranking between the systems configurations in terms of their availability and mean time to failure.*

Case	Parameter Range	Results	Constant Value
1	$0 < \beta_0 < 0.3$	$A_2(\infty) > A_4(\infty) > A_1(\infty) > A_3(\infty)$	$\alpha_0 = 0.6$ $C_0 = 100000$ $C_1 = 500$ $C_2 = 1000$
	$0.3 < \beta_0 < 0.6$	$A_2(\infty) > A_4(\infty) > A_1(\infty) > A_3(\infty)$	
	$0.6 < \beta_0 < 0.9$	$A_2(\infty) > A_4(\infty) > A_1(\infty) > A_3(\infty)$	
	$0.9 < \beta_0 < 1.2$	$A_2(\infty) > A_4(\infty) > A_1(\infty) > A_3(\infty)$	
2	$0 < \beta_0 < 0.3$	$MTTF_2 > MTTF_3 > MTTF_4 > MTTF_1$	
	$0.3 < \beta_0 < 0.6$	$MTTF_2 > MTTF_4 > MTTF_3 > MTTF_1$	
	$0.6 < \beta_0 < 0.9$	$MTTF_2 > MTTF_4 > MTTF_3 > MTTF_1$	
	$0.9 < \beta_0 < 1.2$	$MTTF_2 > MTTF_4 > MTTF_3 > MTTF_1$	
3	$0 < \alpha_0 < 0.3$	$A_2(\infty) > A_3(\infty) > A_4(\infty) > A_1(\infty)$	
	$0.3 < \alpha_0 < 0.6$	$A_2(\infty) > A_3(\infty) > A_4(\infty) > A_1(\infty)$	

	$0.6 < \alpha_0 < 0.9$	$A_2(\infty) > A_3(\infty) > A_4(\infty) > A_1(\infty)$	$\beta_0 = 0.01$
	$0.9 < \alpha_0 < 1.2$	$A_2(\infty) > A_3(\infty) > A_4(\infty) > A_1(\infty)$	$C_0 = 100000$
4	$0 < \alpha_0 < 0.3$	$MTTF_2 > MTTF_3 > MTTF_4 = MTTF_1$	$C_1 = 500$
	$0.3 < \alpha_0 < 0.6$	$MTTF_2 > MTTF_3 > MTTF_4 = MTTF_1$	$C_2 = 1000$
	$0.6 < \alpha_0 < 0.9$	$MTTF_2 > MTTF_3 > MTTF_4 = MTTF_1$	
	$0.9 < \alpha_0 < 1.2$	$MTTF_2 > MTTF_3 > MTTF_4 = MTTF_1$	

Profit Comparison

In this section, numerically comparison with respect to the profit functions for all configurations are discussed. For consistency, we fix the following set of parameters values throughout the simulations: $\alpha_0 = 0.6$, $\beta_0 = 0.01$, $K_0 = 100,000$, $K_1 = 500$ and $K_2 = 1,000$

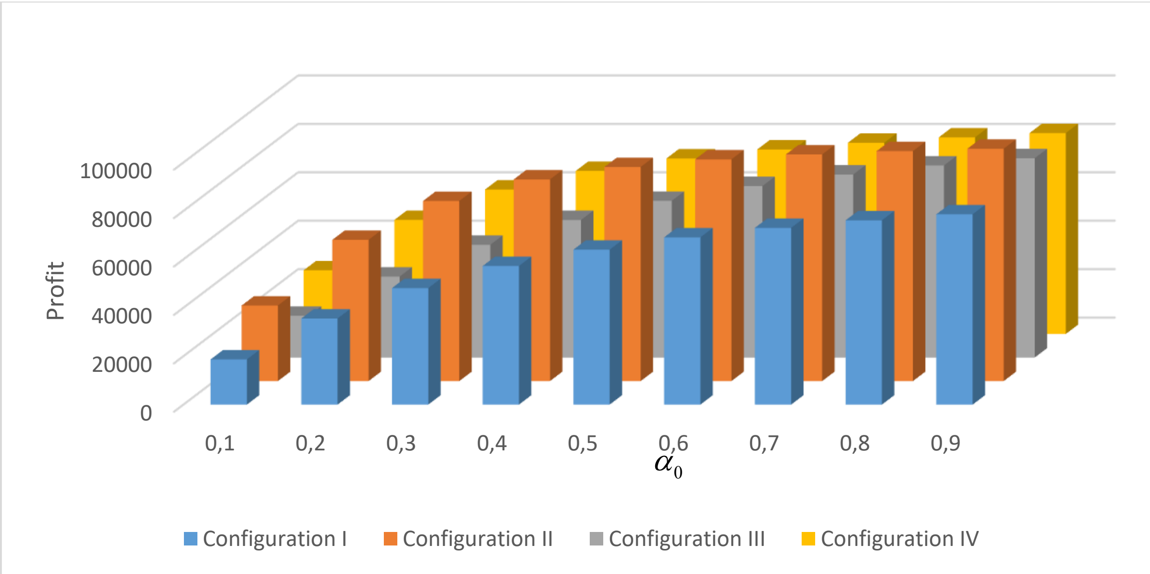

Figure 5: *Profit Comparison for all configuration using* α_0

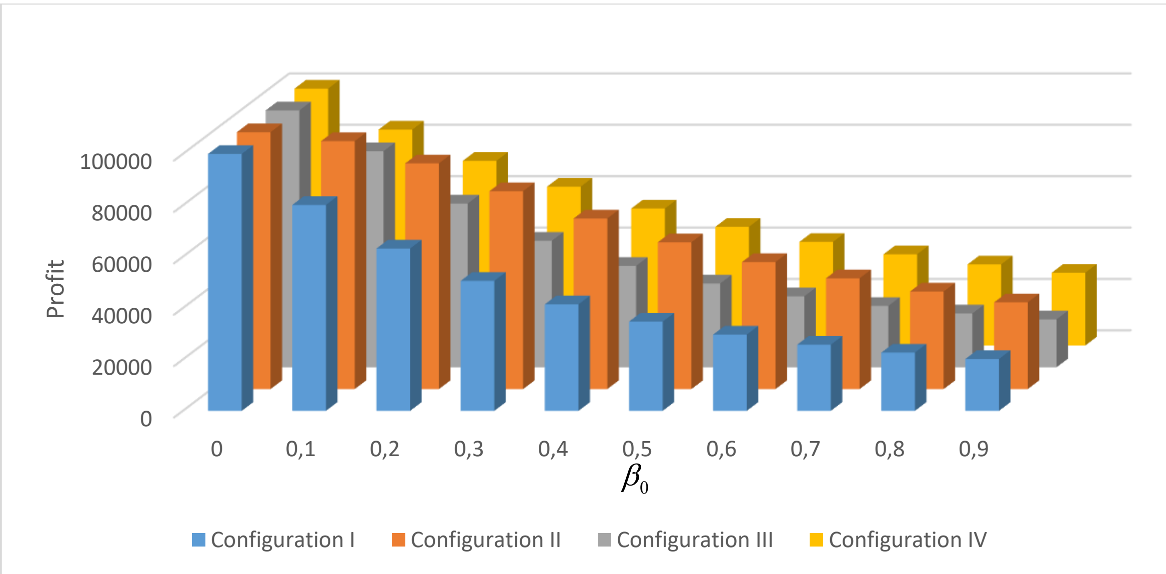

Figure 6: *Profit Comparison for all configuration using* β_0

Figures 5 and 6 depict the trends of profit for all the system configurations against the repair and failure rates α_0 and β_0 respectively. In both figures, it is seen that as repair rate α_0 increases, the profit increases, while with increase in failure rate β_0, the profit decreases. This means that preventive and major maintenance is significant in maximizing the system profit. It is also evident from these Figures that Configuration II has the highest profit as compared to the other three configurations.

Cost Benefit Comparison

In this section, the system configurations are compared based on their cost benefit, where the benefit is either availability or mean time to failure. Numerical values of Wang el al. (2006) parameter values are used to compare the configurations.

$C_1 = 48,000,000$, $C_2 = 39,000,000$, $C_3 = 42,000,000$ and $C_4 = 39,000,000$

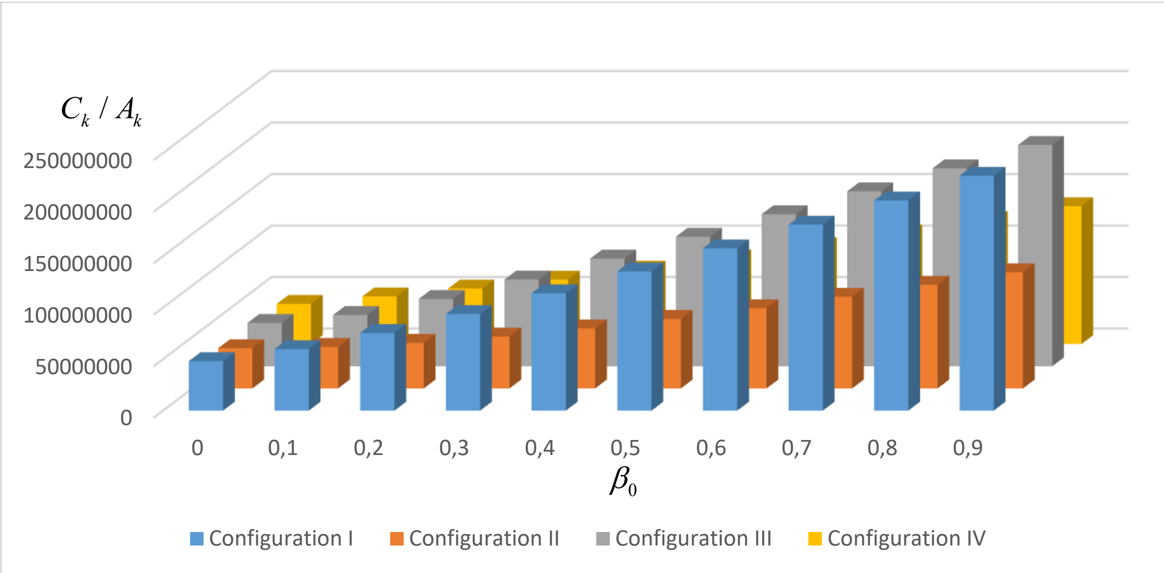

Figure 7: C_k / A_k *Comparison for all configuration using* β_0

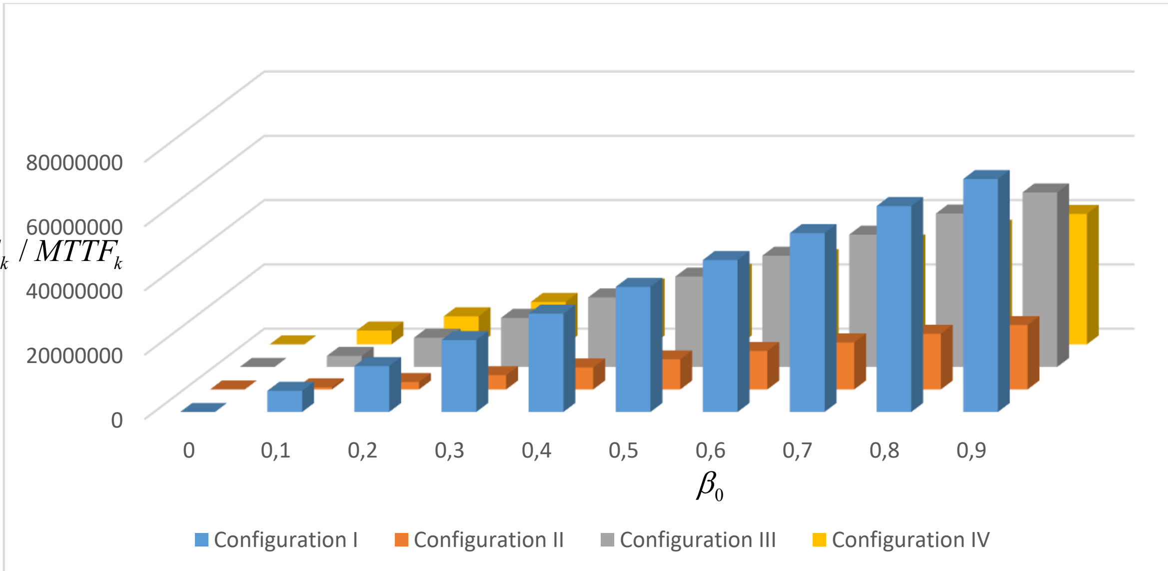

Figure 8: $C_k / MTTF_k$ *Comparison for all configuration using* β_0

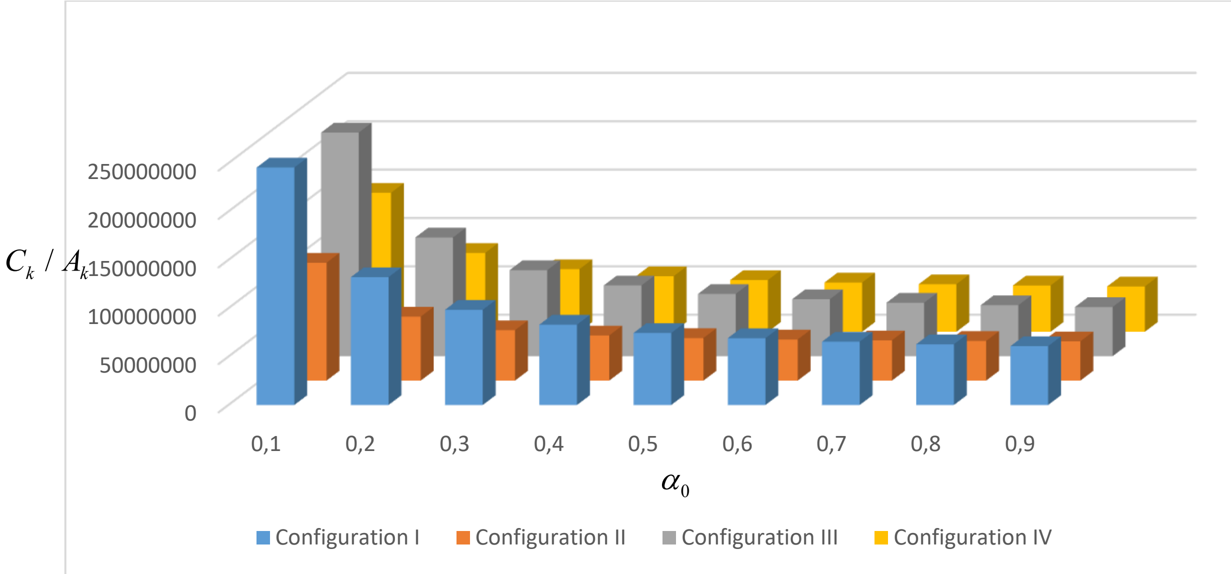

Figure 9: C_k / A_k ***Comparison for all configuration using*** α_0

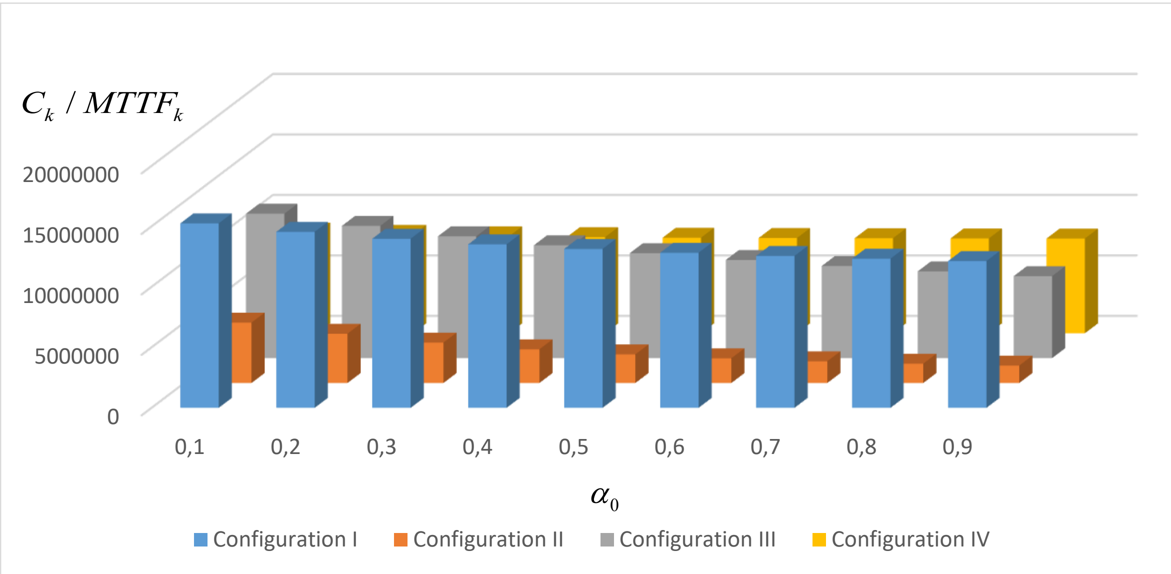

Figure 10: $C_k / MTTF_k$ *Comparison for all configuration using* α_0

Figures 7-10 present the trends of cost benefit C_k / A_k and $C_k / MTTF_k$ for all configurations against the repair and failure rates α_0 and β_0 respectively**.** It is observed from Figures 7 and 8 that C_k / A_k and $C_k / MTTF_k$ increases as β_0 increases for any system configuration. It is also observed from these figures that the optimal system configuration is configuration II. On the other hand, Figures 9 and 10 display the effects of C_k / A_k and $C_k / MTTF_k$ for all the system configurations against the repair rate α_0. These figures revealed that C_k / A_k and $C_k / MTTF_k$ decreases as α_0 increases for any system configuration. Also, from these figures, it is clear that the optimal configuration is configuration II.

V. CONCLUSION

In this paper, we have constructed four different standby serial systems each consisting of five units. The expressions for the system characteristics such as system availability, busy period of repairman due to partial and complete failure as well as profit functions for all the configurations have been obtained and validated by performing numerical experiments. Analysis of the effect of various system parameters on profit function and availability was performed. These are the main contributions of this study. On the basis of the numerical results obtained in Figures 5 – 10 and Tables 1-4 for a particular case, it is evident that the optimal system configuration is configuration II. This is supported from analytical comparison presented in terms of the availability and mean time to failure models obtained in which configuration II is the optimal configuration for all $\alpha_0, \beta_0 > 0$ contrary to some studies where the optimality among the system configuration is not uniform as it depends on some system parameters.

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Product Of n Independent Maxwell Random Variables

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Abstract

We derive the exact probability density functions (pdf) of a product of n *independent Maxwell distributed random variables. The distribution functions are derived by using an inverse Mellin transform technique from statistics, and are given in terms of a special function of mathematical physics, the Meijer G-function.*

Keywords: Product Distribution, Maxwell Distribution, Mellin transform technique, Meijer G-function, probability density function.

1 Introduction

Engineering, Physics, Economics, Order statistics, Classification, Ranking, Selection, Number theory, Genetics, Biology, Medicine, Hydrology, Psychology, these all applied problems depend on the distribution of product of random variables[1][2].

As an example of use of the product of random variables in physics, Sornette [27] mentions:

"…To mimic system size limitation, Takayasu, Sato, and Takayasu introduced a threshold x_c …and found a stretched exponential truncating the power-law pdf beyond x_c. Frisch and Sornette recently developed a theory of extreme deviations generalizing the central limit theorem which, when applied to multiplication of random variables, predicts the generic presence of stretched exponential pdfs. The problem thus boils down to determining the tail of the pdf for a product of random variables …"

Several authors have studied the product distributions for independent random variables come from the same family or different families, see [21] for t and Rayleigh families, [4] for Pareto and Kumaraswamy families, [6] for the t and Bessel families, and [22] for the independent generalized gamma-ratio family. In this paper, we find analytically the probability distributions of the product $\prod_{i=1}^{n} X_i$, when X_i is a Maxwell random variable with probability density function (p.d.f)

$$f_{X_i}(x_i) = \sqrt{\frac{2}{\pi}} \frac{x_i^2}{b_i^3} e^{\frac{-x_i^2}{2(b_i)^2}}, \quad x_i \geq 0. \tag{1}$$

The functions are derived by using an inverse Mellin transform technique from statistics and given in terms of the Meijer G-function.

2 Basic Definitions

2.1 Mellin integral transform

The Mellin integral transform of $f(x)$ is defined only for $x \geq 0$, as:

$$M\{f(x)/s\} = E[x^{s-1}] = \int_0^\infty x^{s-1} f(x)dx \tag{2}$$

The inverse transform is:

$$f(x) = \frac{1}{2j\pi}\int_{c-j\infty}^{c+j\infty} x^{-s} M\{f(x)/s\}ds \tag{3}$$

The path of integration is any line parallel to the imaginary axis and lying within the strip of analyticity of $M\{f(x)/s\}$.

The Mellin integral transform of the density function $f(x)$ of the product $X = X_1.X_2 \ldots X_n$ of n independent random variables X_i with the density function $f_{X_i}(x_i)$ is defined as:

$$M\{f_X(x)/s\} = \prod_{i=1}^n M\{f_{X_i}(x_i)/s\} \tag{4}$$

Using the inverse transform formula we obtain the density function of the product distribution as:

$$f_X(x) = \frac{1}{2j\pi}\int_{c-j\infty}^{c+j\infty} x^{-s} \prod_{i=1}^n M\{f_{X_i}(x_i)/s\}ds \tag{5}$$

2.2 Meijer G-function

The Meijer G-function is defined by the contour integral:

$$G_{pq}^{mn}\left(z\middle|\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}\right) = \frac{1}{2j\pi}\int_{c-j\infty}^{c+j\infty} z^{-s}\frac{\prod_{i=1}^m \Gamma(s+b_i)\prod_{i=1}^n \Gamma(1-a_i-s)}{\prod_{i=n+1}^p \Gamma(s+a_i)\prod_{i=m+1}^q \Gamma(1-b_i-s)}ds \tag{6}$$

where z, $\{a_i\}_i, and \{b_i\}_i$ are in general, complex-valued. The contour is chosen so that it separates the poles of the gamma products in the numerator. The Meijer G-function has been implemented in some commercial mathematics software packages.

3 Product of n Independent Maxwell Random Variables

Theorem 1: Suppose X_i, $i = 1,..,n$ are independent random variables distributed according to (1). Then for $x > 0$ the probability density function *p.d.f.* of $X = \prod_{i=1}^n X_i$ can be expressed as:

$$f_X(x) = 2(\sqrt{\frac{2}{\pi}})^n \frac{1}{\prod_{i=1}^n b_i} G_{0n}^{n0}\left(x^2 2^{-n}\prod_{i=1}^n b_i^{-2}\middle|_{1,\ldots,1}\right) \tag{7}$$

ProofConsider a product of n independent random variables

$$X = \prod_{i=1}^n X_i \tag{8}$$

where X_i is a Maxwell distributed random variable with probability density function according to (1), The Mellin integral transform of $f_{X_i}(x_i)$ is:

$$\begin{aligned} M\{f_{X_i}(x_i)/s\} &= \int_0^\infty x_i^{s-1} f_{X_i}(x_i)dx_i \\ &= \frac{1}{b_i^3}\sqrt{\frac{2}{\pi}}\int_0^\infty x_i^{s+1} e^{\frac{-x_i^2}{2(b_i)^2}}dx_i \\ &= \sqrt{\frac{2}{\pi}}2^{s/2}\frac{1}{b_i}(b_i^{-2})^{-\frac{s}{2}}\Gamma(1+s/2) \end{aligned} \tag{9}$$

Where we have used the definition of the gamma function

$$\Gamma(t) = \int_0^\infty x^{t-1}e^{-x} \tag{10}$$

The Mellin integral transform of $f_X(x)$

$$\begin{aligned} M\{f_X(x)/s\} &= \prod_{i=1}^n M\{f_{X_i}(x_i)/s\} \\ &= \prod_{i=1}^n [\sqrt{\frac{2}{\pi}}2^{s/2}\frac{1}{b_i}(b_i^{-2})^{-\frac{s}{2}}\Gamma(1+s/2)] \end{aligned}$$

We can find the pdf of X as the inverse Mellin transform

$$
\begin{aligned}
f_X(x) &= \frac{1}{2j\pi}\int_{c-j\infty}^{c+j\infty} x^{-s}[\prod_{i=1}^{n}[\sqrt{\frac{2}{\pi}}2^{s/2}\frac{1}{b_i}(b_i^{-2})^{-\frac{s}{2}}\Gamma(1+s/2)]]ds \\
&= \frac{1}{2j\pi}\int_{c-j\infty}^{c+j\infty}(x^2)^{-\frac{s}{2}}(\sqrt{\frac{2}{\pi}})^n(2^{-n})^{-\frac{s}{2}}\frac{1}{\prod_{i=1}^{n}b_i}(\prod_{i=1}^{n}(b_i)^{-2})^{-\frac{s}{2}}\prod_{i=1}^{n}\Gamma(1+\frac{s}{2})2\frac{ds}{2} \qquad (11)\\
&= 2(\sqrt{\frac{2}{\pi}})^n\frac{1}{\prod_{i=1}^{n}b_i}\frac{1}{2j\pi}\int_{c-j\infty}^{c+j\infty}(x^2 2^{-n}\prod_{i=1}^{n}b_i^{-2})^{-\frac{s}{2}}\prod_{i=1}^{n}\Gamma(1+\frac{s}{2})\frac{ds}{2}
\end{aligned}
$$

Finally using the definition of the Meijer G-function we get

$$
f_X(x) = 2(\sqrt{\frac{2}{\pi}})^n\frac{1}{\prod_{i=1}^{n}b_i}G_{0n}^{n0}\left(x^2 2^{-n}\prod_{i=1}^{n}b_i^{-2}\Big|_{1,\ldots,1}\right) \qquad (12)
$$

Corollary 1: Suppose $X_i,\ i=1,..,n$ are independent random variables distributed according to (1). Then for $t>0$ the cumulative distribution function *c.d.f.* of $X=\prod_{i=1}^{n}X_i$ can be expressed as:

$$
F_X(t) = 2(\sqrt{\frac{2}{n}})^n\frac{1}{\prod_{i=1}^{n}b_i}\frac{t}{2}G_{1n+1}^{n1}\left(t^2 2^{-n}\prod_{i=1}^{n}b_i^{-2}\Big|_{1,\ldots,1,-\frac{1}{2}}^{\frac{1}{2}}\right) \qquad (13)
$$

Proof The cumulative distribution function $F_X(t)=\int_0^t f_X(x)dx$ is obtained by integrating (7) with respect to x inside the contour integral by using:

$$
\int_0^t x^{-s}dx = \frac{t^{1-s}}{1-s} = t^{1-s}\frac{1}{2}(\frac{2}{1-s}) = t^{1-s}\frac{1}{2}(\frac{1}{2}-\frac{s}{2})^{-1} \qquad (14)
$$

And

$$
\frac{1}{2}-\frac{s}{2} = \frac{(\frac{1}{2}-\frac{s}{2})\Gamma(\frac{1}{2}-\frac{s}{2})}{\Gamma(\frac{1}{2}-\frac{s}{2})}
$$

Then we get

$$
\int_0^t x^{-s}dx = t^{1-s}\frac{1}{2}\frac{\Gamma(\frac{1}{2}-\frac{s}{2})}{\Gamma(\frac{3}{2}-\frac{s}{2})}
$$

Let $\beta_n = 2(\sqrt{\frac{2}{n}})^n\frac{1}{\prod_{i=1}^{n}b_i}$

$$
F_X(t) = t\beta_n\frac{1}{2j\pi}\int_{c-j\infty}^{c+j\infty}\frac{1}{2}(t^2)^{-\frac{s}{2}}(2^{-n}\prod_{i=1}^{n}b_i^{-2})^{-s/2}\frac{\Gamma(\frac{1}{2}-\frac{s}{2})\prod_{i=1}^{n}(\Gamma(1+\frac{s}{2}))}{\Gamma(\frac{3}{2}-\frac{s}{2})} \qquad (15)
$$

Finally using the definition of the Meijer G-function we obtain

$$
F_X(t) = 2(\sqrt{\frac{2}{n}})^n\frac{1}{\prod_{i=1}^{n}b_i}\frac{t}{2}G_{1n+1}^{n1}\left(t^2 2^{-n}\prod_{i=1}^{n}b_i^{-2}\Big|_{1,\ldots,1,-\frac{1}{2}}^{\frac{1}{2}}\right)
$$

Corollary 2: Suppose $X_i,\ i=1,..,n$ are independent random variables distributed according to (1). Then for $r>0, \alpha>0$ the moment of order r of $X=\prod_{i=1}^{n}X_i$ can be expressed as:

$$
E[X^r] = 2(\sqrt{\frac{2}{\pi}})^n\frac{\alpha^{r+1}}{\prod_{i=1}^{n}b_i}\frac{1}{2}G_{1n+1}^{n1}\left(\alpha^2 2^{-n}\prod_{i=1}^{n}b_i^{-2}\Big|_{1,\ldots,1,-\frac{r}{2}-\frac{1}{2}}^{\frac{1}{2}-\frac{r}{2}}\right) \qquad (16)
$$

Proof

$$
\begin{aligned}
E[X^r] &= \int_{-\infty}^{+\infty}x^r f_X(x)dx \\
&= \int_{\alpha}^{+\infty}x^r f_X(x)dx \\
&= \beta_n\frac{1}{2j\pi}\int_{c-j\infty}^{c+j\infty}\int_{\alpha}^{\infty}x^{r-s}(2^{-n}\prod_{i=1}^{n}b_i^{-2})^{-s/2}\prod_{i=1}^{n}\Gamma(1+\frac{s}{2})dx\frac{ds}{2}
\end{aligned}
$$

We have

$$
\int_{\alpha}^{\infty}x^{-s+r}dx = \frac{\alpha^{1+r-s}}{s-r-1} \qquad (17)
$$

Then

$$
E[X^r] = \beta_n\frac{1}{2j\pi}\int_{c-j\infty}^{c+j\infty}\frac{\alpha^{1+r}}{s-r-1}(2^{-n}\prod_{i=1}^{n}b_i^{-2}\alpha^2)^{-s/2}\prod_{i=1}^{n}\Gamma(1+\frac{s}{2})\frac{ds}{2} \qquad (18)
$$

Also we have

$$\frac{1}{s-r-1} = -\frac{1}{2}\frac{\Gamma(\frac{-s}{2}+\frac{r}{2}+\frac{1}{2})}{\Gamma(\frac{-s}{2}+\frac{r}{2}+\frac{3}{2})} \tag{19}$$

Finally using (19) and the definition of the Meijer G-function we obtain

$$\begin{aligned} E[X^r] &= \beta_n \frac{1}{2j\pi}\alpha^{r+1}(-\tfrac{1}{2})\int_{c-j\infty}^{c+j\infty} (2^{-n}\prod_{i=1}^n b_i^{-2}\alpha^2)^{-\frac{s}{2}}\frac{\Gamma(-\frac{s}{2}+\frac{r}{2}+\frac{1}{2})}{\Gamma(-\frac{s}{2}+\frac{r}{2}+\frac{3}{2})}\prod_{i=1}^n \Gamma(1+\tfrac{s}{2}) \\ &= 2(\sqrt{\tfrac{2}{\pi}})^n \frac{\alpha^{r+1}}{\prod_{i=1}^n b_i}\frac{1}{2} G_{1n+1}^{n1}\left(\alpha^2 2^{-n}\prod_{i=1}^n b_i^{-2}\Big|{}^{\frac{1}{2}-\frac{r}{2}}_{1,\dots,1,-\frac{r}{2}-\frac{1}{2}}\right) \end{aligned} \tag{20}$$

Corollary 3: Suppose $X_i,\ i=1,..,n$ are independent random variables distributed according to (1). Then for $\alpha>0$ the expected value of $X=\prod_{i=1}^n X_i$ can be expressed as: For $r=1$

$$E[X] = 2(\sqrt{\tfrac{2}{\pi}})^n \frac{\alpha^2}{\prod_{i=1}^n b_i}\frac{1}{2} G_{1n+1}^{n1}\left(\alpha^2 2^{-n}\prod_{i=1}^n b_i^{-2}\Big|{}^{0}_{1,\dots,1,-1}\right) \tag{21}$$

Corollary 4: Suppose $X_i,\ i=1,..,n$ are independent random variables distributed according to (1). Then for $\alpha>0$ the expected value of $X=\prod_{i=1}^n X_i$ can be expressed as:

$$\begin{aligned} \sigma^2 &= 2(\sqrt{\tfrac{2}{\pi}})^n \frac{\alpha^3}{\prod_{i=1}^n b_i}\frac{1}{2} G_{1n+1}^{n1}\left(\alpha^2 2^{-n}\prod_{i=1}^n b_i^{-2}\Big|{}^{-\frac{1}{2}}_{1,\dots,1,-\frac{3}{2}}\right) \\ &\quad -[2(\sqrt{\tfrac{2}{\pi}})^n \frac{\alpha^2}{\prod_{i=1}^n b_i}\frac{1}{2} G_{1n+1}^{n1}\left(\alpha^2 2^{-n}\prod_{i=1}^n b_i^{-2}\Big|{}^{0}_{1,\dots,1,-1}\right)]^2 \end{aligned} \tag{22}$$

Proof. By definition the variance of X/Y is:

$$\sigma^2 = E[Z^2]-E[Z]^2 \tag{23}$$

Corollary 5: Suppose $X_i,\ i=1,..,n$ are independent random variables distributed according to (1). Then for $x>0$ the survival function of $X=\prod_{i=1}^n X_i$ can be expressed as:

$$S_X(x) = \begin{pmatrix} 1 & \text{if} x\le 0 \\ 1-2(\sqrt{\tfrac{2}{n}})^n \frac{1}{\prod_{i=1}^n b_i}\frac{x}{2} G_{1n+1}^{n1}\left(x^2 2^{-n}\prod_{i=1}^n b_i^{-2}\Big|{}^{\frac{1}{2}}_{1,\dots,1,-\frac{1}{2}}\right) & \text{if} x>0 \end{pmatrix} \tag{24}$$

Proof By definition of the survival function

$$S_X(x) = 1-F_X(x) \tag{25}$$

Corollary 6: Suppose $X_i,\ i=1,..,n$ are independent random variables distributed according to (1). Then for $x>0$ the hazard function of $X=\prod_{i=1}^n X_i$ can be expressed as:

$$h_X(x) = \begin{pmatrix} 0 & \text{if} x\le 0 \\ \dfrac{2(\sqrt{\frac{2}{\pi}})^n \frac{1}{\prod_{i=1}^n b_i} G_{0n}^{n0}\left(x^2 2^{-n}\prod_{i=1}^n b_i^{-2}\Big|_{1,\dots,1}\right)}{1-2(\sqrt{\frac{2}{n}})^n \frac{1}{\prod_{i=1}^n b_i}\frac{x}{2} G_{1n+1}^{n1}\left(x^2 2^{-n}\prod_{i=1}^n b_i^{-2}\Big|{}^{\frac{1}{2}}_{1,\dots,1,-\frac{1}{2}}\right)} & \text{if} x>0 \end{pmatrix} \tag{26}$$

4 Examples and special cases

4.1 Product of two independent Maxwell random variables

1. **Probability density function:** Suppose $X_i,\ i=1,2$ are independent Maxwell random variables with scale parameters $b_1=1, b_2=2$ respectively, the probability density function of X is

$$f_X(x) = \begin{pmatrix} 0 & if x\le 0 \\ \frac{2}{\pi} G_{02}^{20}\left(\frac{x^2}{16}\Big|_{1,1}\right) & if x>0 \end{pmatrix} \tag{27}$$

2. **Cumulative distribution function:** Suppose $X_i,\ i=1,2$ are independent Maxwell random variables with scale parameters $b_1=1, b_2=2$ respectively, the cumulative distribution function of X is

For $t>0$

$$F_X(t)=\begin{cases}0 & if t\leq 0\\ \frac{t}{\pi}G_{13}^{21}\left(\frac{t^2}{16}\middle|\begin{matrix}\frac{1}{2}\\ 1,1,-\frac{1}{2}\end{matrix}\right) & if t>0\end{cases} \tag{28}$$

3. **Moment of order "r":** Suppose $X_i,\ i=1,2$ are independent Maxwell random variables with scale parameters $b_1=1, b_2=2$ respectively, the moment of order r of X is

For $\alpha>0$

$$E[X^r]=-\frac{\alpha^{r+1}}{\pi}G_{13}^{21}\left(\frac{\alpha^2}{16}\middle|\begin{matrix}\frac{1}{2}-\frac{r}{2}\\ 1,1,-\frac{r}{2}-\frac{1}{2}\end{matrix}\right) \tag{29}$$

4. **Expected value:** Suppose $X_i,\ i=1,2$ are independent Maxwell random variables with scale parameters $b_1=1, b_2=2$ respectively, the Expected value of X is

$$E[X]=-\frac{\alpha^2}{\pi}G_{13}^{21}\left(\frac{\alpha^2}{16}\middle|\begin{matrix}0\\ 1,1,-1\end{matrix}\right) \tag{30}$$

5. **Variance:** Suppose $X_i,\ i=1,2$ are independent Maxwell random variables with scale parameters $b_1=1, b_2=2$ respectively, the Variance of X is

$$\begin{aligned}\sigma^2 &= -\frac{\alpha^3}{\pi}G_{13}^{21}\left(\frac{\alpha^2}{16}\middle|\begin{matrix}-\frac{1}{2}\\ 1,1,-\frac{3}{2}\end{matrix}\right)\\ &-[-\frac{\alpha^2}{\pi}G_{13}^{21}\left(\frac{\alpha^2}{16}\middle|\begin{matrix}0\\ 1,1,-1\end{matrix}\right)]^2\end{aligned} \tag{31}$$

6. **Survival function:** Suppose $X_i,\ i=1,2$ are independent Maxwell random variables with scale parameters $b_1=1, b_2=2$ respectively, the Survival function of X is

$$S_X(t)=\begin{cases}1 & \text{if} t\leq 0\\ 1-\frac{t}{\pi}G_{13}^{21}\left(\frac{t^2}{16}\middle|\begin{matrix}\frac{1}{2}\\ 1,1,-\frac{1}{2}\end{matrix}\right) & \text{if} t>0\end{cases} \tag{32}$$

7. **Hazard function:** Suppose $X_i,\ i=1,2$ are independent Maxwell random variables with scale parameters $b_1=1, b_2=2$ respectively, the Hazard function of X is

For $t>0$

$$h_X(x)=\begin{cases}0 & if x\leq 0\\ \dfrac{\frac{2}{\pi}G_{02}^{20}\left(\frac{x^2}{16}\middle|_{1,1}\right)}{1-\frac{x}{\pi}G_{13}^{21}\left(\frac{x^2}{16}\middle|\begin{matrix}\frac{1}{2}\\ 1,1,-\frac{1}{2}\end{matrix}\right)} & if x>0\end{cases} \tag{33}$$

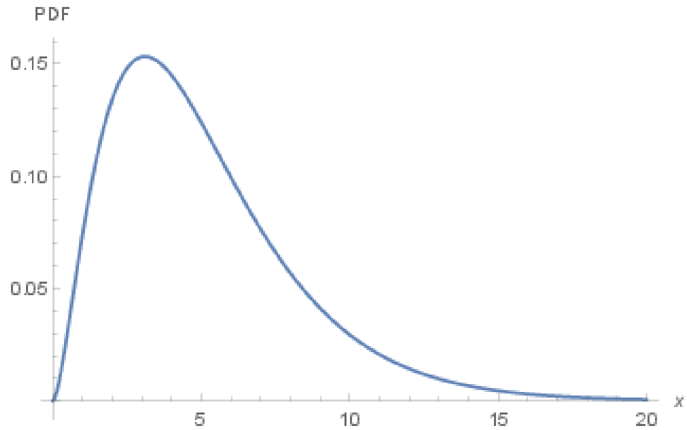

Figure 1: Plot of the probability density function for two independent Maxwell random variables for $b_1 = 1, b_2 = 2$.

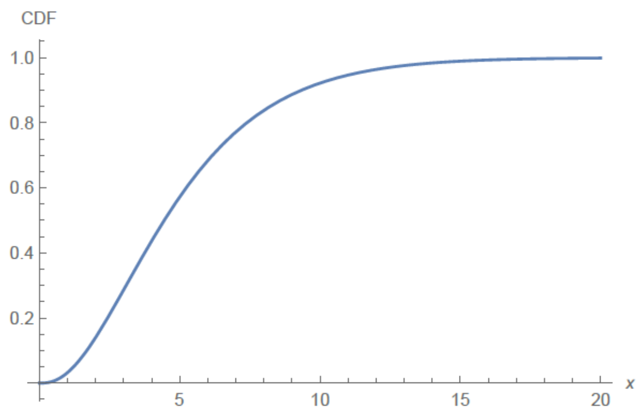

Figure 2: Plot of the cumulative distribution function for two independent Maxwell random variables for $b_1 = 1, b_2 = 2$.

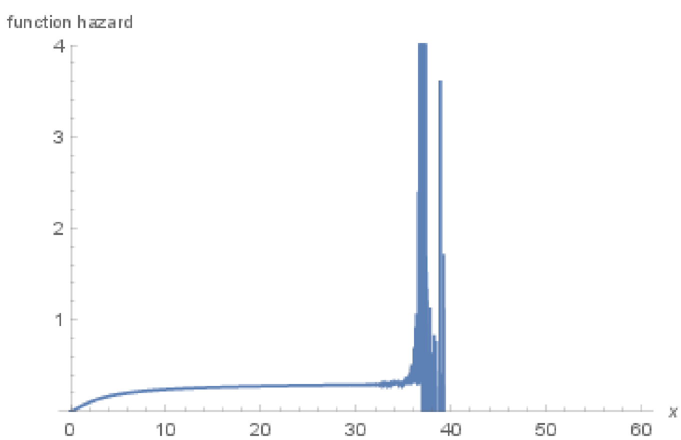

Figure 3: Plot of the hazard function for two independent Maxwell random variables for $b_1 = 1, b_2 = 2$.

5 Applications

The air molecules surrounding us are not all traveling at the same speed, even if the air is all at a single temperature. Some of the air molecules will be moving extremely fast, some will be moving with moderate speeds, and some of the air molecules will hardly be moving at all. Because of this, we can't ask questions like "What is the speed of an air molecule in a gas?" since a molecule in a gas could have any one of a huge number of possible speeds.

So instead of asking about any one particular gas molecule, we ask questions like, "What is the distribution of speeds in a gas at a certain temperature?" In the mid to late 1800s, James Clerk Maxwell and Ludwig Boltzmann figured out the answer to this question. Their result is referred to as the Maxwell-Boltzmann distribution, because it shows how the speeds of molecules are distributed for an ideal gas. The Maxwell-Boltzmann distribution is often represented with the following graph.

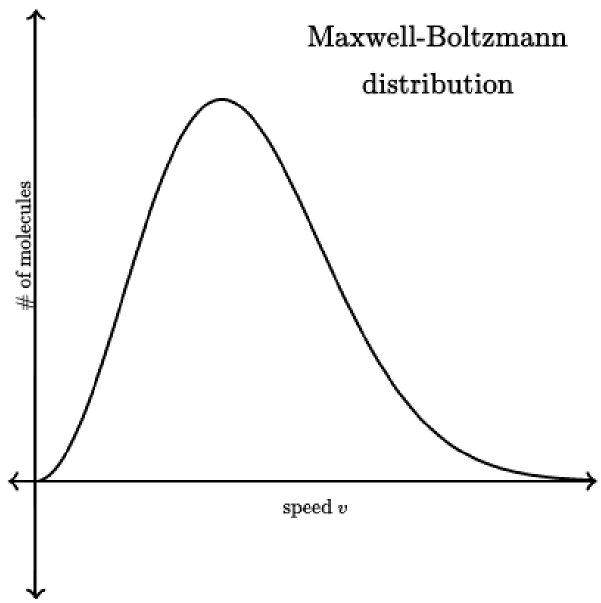

Figure 4: Maxwell-Boltzmann distribution

The y-axis of the Maxwell-Boltzmann graph can be thought of as giving the number of molecules per unit speed. So, if the graph is higher in a given region, it means that there are more gas molecules moving with those speeds.

Let take the following example: we are interested to find the distribution $X = X_1X_2X_3X_4X_5X_6X_7X_8X_9X_{10}$, where X_i are independent Maxwell random variables with scale parameters b_i,$b_1 = 1, b_2 = 2, b_3 = 3, b_4 = 4, b_5 = 5, b_6 = 6, b_7 = 7, b_8 = 8, b_9 = 9, b_{10} = 10$.

So the speeds of molecules are distributed for an ideal gas with respect to the probability density function of X

$$f_X(x) = 2(\sqrt{\frac{2}{\pi}})^{10} \frac{1}{(12345678910)} G_{010}^{100}\left(x^2 2^{-10}(\frac{1}{13168189440000})|_{1,1,1,1,1,1,1,1,1,1}\right) \quad (34)$$

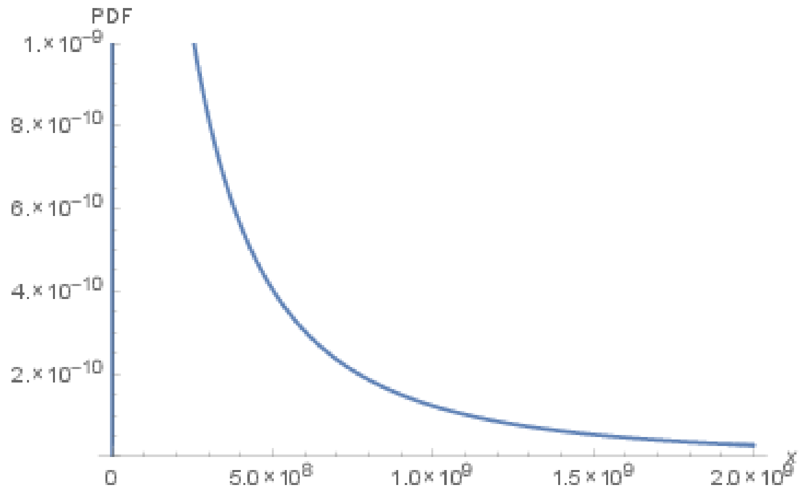

Figure 5: Plot of n=10 independent Maxwell rondom variables for $b_1 = 1, b_2 = 2, b_3 = 3, b_4 = 4, b_5 = 5, b_6 = 6, b_7 = 7, b_8 = 8, b_9 = 9, b_{10} = 10$.

6 Monte Carlo simulation:

Monte Carlo simulations are used to model the probability of different outcomes in a process that cannot easily be predicted due to the intervention of random variables. It is a technique used to understand the impact of risk and uncertainty in prediction and forecasting models.

A Monte Carlo simulation can be used to tackle a range of problems in virtually every field such as finance, engineering, supply chain, and science. It is also referred to as a multiple probability simulation.

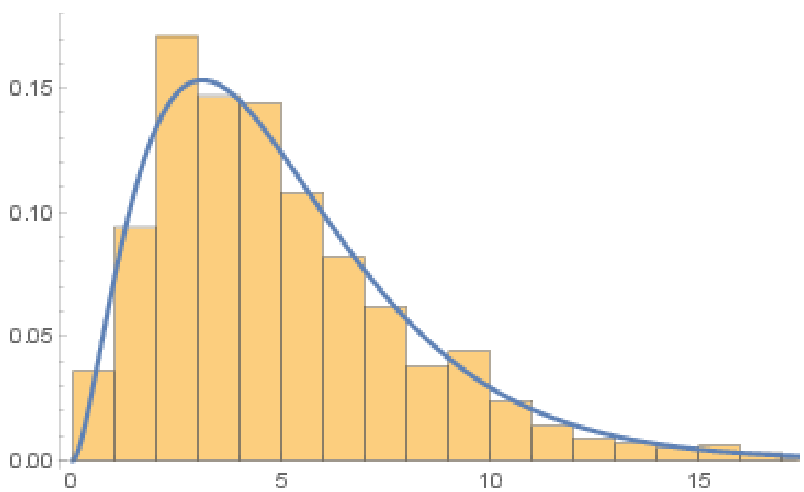

Figure 6: Monte Carlo simulation for the product of two independent maxwell random variables for scale parameters $b_1 = 1, b_2 = 2$.

7 Conclusion

This paper has derived the analytical expressions of the PDF, CDF, the moment of order r, the survival function, and the hazard function, for the distribution of $X = \prod_{i=1}^{n} X_i$ when X_i are Maxwell random variables distributed independently of each other, we have illustrated our results for $n = 2$ as a special case, then we have discussed an application of the distribution of product $X = \prod_{i=1}^{n} X_i$, finally, we have confirmed our result using Monte Carlo simulation.

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Quadratic Fractional Bi-level Fuzzy Probabilistic Programming Problem When b_i Follows Exponential Distribution

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Abstract

Some of the actual life decisions are made in decentralized manner under uncertainty. This paper formulates a quadratic fractional bi-level (QFBL) programming problem with probabilistic constraints in both first (leader) and second level (follower) having two parameter exponential random variables with known probability distributions and fuzziness is considered as triangular and trapezoidal fuzzy number. These fuzzy numbers of the membership functions related with the proportional probability density function has been used to introduce a defuzzification approach for finding the crisp values of fuzzy numbers. In the proposed model the problem is first converted into an equivalent deterministic quadratic fractional fuzzy bi level programming model by applying chance constrained programming technique. Secondly, in the suggested model, each objective function of the bi-level quadratic fractional programming problem has its own non-linear membership function. The fuzzy goal programming (FGP) approach is used to find a compromise solution for the BLQFP problem. Finally, to demonstrate the applicability and performance of the proposed approach an illustrative numerical example is given.

Keywords: Bi-level programming, Quadratic programming, Fractional programming, Two parameter exponential distribution, Fuzzy chance constrained programming, Fuzzy goal programming.

I. Introduction

In actual life decision-making situations, decision-makers are frequently confronted with different kinds of vagueness, the most important of which are randomness and fuzziness. There are two common approaches to dealing with such uncertainties: probability theoretical approach and fuzzy set theoretical approach. Chance constrained programming is a well-defined approach described by Charnes and Cooper [1] for dealing with problems involving probabilistic data (CCP). Gardening, capability planning, banking, forestry, army, manufacture control and arrangement, sports, broadcastings, transport, and eco-friendly management planning are just a few of the fields where it is commonly used.

Quadratic programming (QP) is a technique of solving certain mathematical optimization problems involving quadratic function. Specially a quadratic programming minimized or maximized a multivariate quadratic function subject to linear constraint on the variables.

Quadratic programming is a type of nonlinear programming of the several. A wide range of applications of quadratic programming is portfolio selection, electrical energy growth, agriculture, and harvest selection. Probabilistic quadratic programming is applicable for financial and risk management, and various important related literatures are found in this direction, which are mentioned below.

Mc Carl *et al*. [2] presented some of the approaches during which QP are often used. Interval parameters are used to represent the cost coefficients, constraint coefficients, and right-hand sides in an interval quadratic programming problem proposed by Liu and Wang [6]. A fuzzy quadratic programming problem was introduced in [7] where fuzzy data represents the cost coefficients, constraint coefficients, and right-hand side values. Kausar and Adhami [18] have given a fuzzy goal programming approach for solving chance constrained bi-Level multi-objective quadratic fractional programming problems. Ammar [9] proposed a multi-objective quadratic programming problem based on the fuzzy random coefficient matrix's goals and constraints where the decision vector interpreted as a fuzzy vector. The problem of fuzzy quadratic programming was introduced by Liu [10] in which convex fuzzy numbers represent the cost coefficients, constraint coefficients, and constraints parameters on the right side. He [11] also specified a stream water quality control solution process. Qin and Huang [12] suggested an inexact chance constrained quadratic programming model. Nasseri [13] outlined a fuzzy quadratic programming problem with trapezoidal and/or triangular fuzzy numbers for the cost coefficients, constraint coefficients, and right-hand parameter values. Guo and Huang [14] developed an incorrect fuzzy-stochastic quadratic programming approach to efficiently distribute waste to a municipal solid waste management scheme while accounting for the nonlinear objective function and various parameter uncertainties in the constraints. Bi-level multi-objective stochastic linear fractional programming with general form of distribution has been developed by Kausar and Adhami [17]

In this paper, we are constructing a different approach for solving quadratic fractional bi level programming problems with probabilistic constraints having two parameter exponential distributed fuzzy random variables with known probability distributions. The probabilistic problem is changed into an equivalent deterministic model. Under the quadratic fractional programming outline, both uncertainty and fuzziness are considered. Poularikas [16] proposes a defuzzification technique for determining the crisp values of fuzzy numbers using the Mellin transformation.

I. Probabilistic Fuzzy Quadratic Fractional Bi-Level Programming Problem

In some circumstances, quadratic fractional programming with a quadratic fractional objective function and few linear constraints including fuzziness and randomness is called a probabilistic fuzzy quadratic fractional programming problem. When a number of the input parameters of QFP are described by stochastic and fuzzy parameters, the problem is treated as a probabilistic fuzzy quadratic fractional programming problem. A general probabilistic fuzzy quadratic fractional bi-level programming problem is presented as follows

$$\underset{X_1}{\text{Max}}\, \tilde{f}_1(x_1, x_2) = \frac{f_{11}}{f_{12}} = \frac{\tilde{c}_{i1}x_j + \frac{1}{2}x^T\tilde{d}_{i1}x_j \ + \alpha_{i1}}{\tilde{c}_{i2}x_j + \frac{1}{2}x^T\tilde{d}_{i2}x_j \ + \beta_{i2}} \tag{1.1}$$

where for given X_1, X_2

$$\underset{X_2}{\text{Max}}\, \tilde{f}_2(x_1, x_2) = \frac{f_{21}}{f_{22}} = \frac{\tilde{c}_{i1}x_j + \frac{1}{2}x^T\tilde{d}_{i1}x_j \ + \alpha_{i1}}{\tilde{c}_{i2}x_j + \frac{1}{2}x^T\tilde{d}_{i2}x_j \ + \beta_{i2}} \tag{1.2}$$

$$\mathrm{X} \in \mathrm{G} = \left\{X \in R^n \middle| P(\textstyle\sum_{j=1}^{n} \tilde{a}_{ij}x_j \le b_i) \ge 1 - \lambda_i\,, \qquad i = 1,2, \ldots\ldots m\right\} \tag{1.3}$$

$$x_j \ge 0, \quad j = 1,2, \ldots, n \tag{1.4}$$

where $0 < \lambda_i < 1$ and $\tilde{b}_i$ (i=1,2,…,m) represent two parameter exponential distribution fuzzy random variables; $\tilde{c}_{i1}, \tilde{c}_{i2}, \tilde{d}_{i1}, \tilde{d}_{i2}$ (i=1,2,…,m) and $\tilde{a}_{ij}$ are considered as fuzzy number and $\lambda_i \epsilon[0,1]$.

The decision vector $X_1 = (x_{11}, x_{12}, \dots, x_{1n_1})$ is controlled by the leader and decision vector $X_2 = (x_{21}, x_{22}, \dots, x_{2n_2})$ is controlled by the follower; $X_1 \cup X_2 = X = (x_1, x_2, \dots, x_n) \in R^n$ with $n_1 + n_2 = n$.

II. Some Preliminaries

In the model formulation the triangular and trapezoidal membership functions are discussed in this section. We can also introduce the Mellin transform to find the expected value of a random variable's function by using proportional probability density function related with membership functions of fuzzy numbers.

Definition 2.1 (triangular fuzzy number): A triangular fuzzy number is one that is represented by the triplet $\tilde{A} = (a_1, a_2, a_3)$ and has a piecewise linear membership function $\mu_{\tilde{A}}(x)$ is given by

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{x - a_1}{a_2 - a_1}, & a_1 \le x \le a_2 \\ \dfrac{a_3 - x}{a_3 - a_2}, & a_2 \le x \le a_3 \\ 0, & otherwise \end{cases} \tag{2.1}$$

Definition 2.2 (trapezoidal fuzzy number): A fuzzy number represented by the quadruplet $\tilde{A} = (a_1, a_2, a_3, a_4)$ and has a piecewise linear membership function $\mu_{\tilde{A}}(x)$ is given by

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{x - a_1}{a_2 - a_1}, & a_1 \le x \le a_2 \\ 1, & a_2 \le x \le a_3 \\ \frac{a_4 - x}{a_4 - a_3}, & a_3 \le x \le a_4 \\ 0, & otherwise \end{cases} \tag{2.2}$$

I. Defuzzification with Probability Density Function and Membership Function

Assume that $F(\mathbb{R})$ represent the sum of all fuzzy numbers. In$(\mathbb{R})$ the triangular and trapezoidal fuzzy numbers are (a_1, a_2, a_3) and (a_1, a_2, a_3, a_4) respectively. Now the method associated with a probability density function for the membership function of $\tilde{A}$ is defined as follows ([15], [4]).

Proportional probability distribution: describe a probability density function $f_1 = c\mu_{\tilde{A}}(x)$ associated with $\tilde{A}$, where the constant c is obtained by using the property of probability density function, where $\int_{-\infty}^{\infty} f_1(x)dx = 1$ and $\int_{-\infty}^{\infty} c\mu_{\tilde{A}}(x)dx = 1$.

II. Mellin Transform

The Mellin transform ([15], [4]) is used to find this expected value since any probability density function with finite support is associated with expected value.

Definition 2.3: The Mellin transform $M_X(t)$ of a probability density function $f(x)$, where x denote the positive, is given as

$$M_X(t) = \int_0^{\infty} x^{t-1} f(x)\, dx \tag{2.3}$$

Now in terms of expected values we find the Mellin transform. Remember that the expected value of any function $g(X)$ of the random variable X whose probability density function is $f(x)$, is given as

$$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f(x) dx \tag{2.4}$$

Thus, it follows that $M_X(t) = \mathbb{E}[X^{t-1}] = \int_0^{\infty} x^{t-1} f(x)\, dx$.

Hence, $\mathbb{E}[X^t] = M_X(t+1)$. Thus the expected value of random variable X is $\mathbb{E}[X] = M_X(2)$.

For example, if the triangular and trapezoidal fuzzy numbers are $\tilde{A}_1 = (a_1, a_2, a_3)$ and $\tilde{A}_2 = (a_1, a_2, a_3, a_4)$ respectively, then their crisp values are determined by finding expected values

using the probability density function that corresponding to the membership functions of the given fuzzy number.
Thus, the probability density function corresponding to triangular fuzzy number $\tilde{A}_1 = (a_1, a_2, a_3)$ is given as

$$f\tilde{A}_1(x) = c_1 \mu \tilde{A}_1(x) \tag{2.5}$$

where $\mu\tilde{A}_1(x)$ is defined as

$$\mu_{\tilde{A}_1}(x) = \begin{cases} \dfrac{x - a_1}{a_2 - a_1}, & a_1 \le x \le a_2 \\ \dfrac{a_3 - x}{a_3 - a_2}, & a_2 \le x \le a_3 \\ 0, & otherwise \end{cases} \tag{2.6}$$

Now c_1 is calculated as

$$\int_{-\infty}^{\infty} f_{\tilde{A}_1}(x)dx = 1 \tag{2.7}$$

that is

$$\int_{-\infty}^{\infty} c_1 \mu_{\tilde{A}_1}(x)dx = 1 \tag{2.8}$$

that is

$$c_1 \int_{a_1}^{a_2} \frac{x - a_1}{a_2 - a_1} dx + c_1 \int_{a_2}^{a_3} \frac{a_3 - x}{a_3 - a_2} dx = 1 \tag{2.9}$$

On integration, we get

$$c_1 = \frac{2}{a_3 - a_1} \tag{2.10}$$

The proportional probability function corresponding to triangular fuzzy number $\widetilde{A}$ is given by

$$f_{X_{\tilde{A}_1}}(x) = \begin{cases} \dfrac{2(x - a_1)}{(a_2 - a_1)(a_3 - a_1)}, & a_1 \le x \le a_2 \\ \dfrac{2(a_3 - x)}{(a_3 - a_2)(a_3 - a_1)}, & a_2 \le x \le a_3 \\ 0, & otherwise \end{cases} \tag{2.11}$$

Graphically it is presented in Figure 1

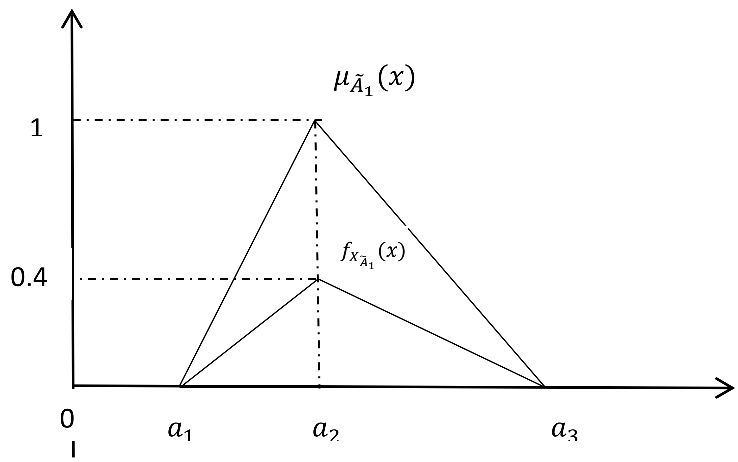

Figure 5.1: Proportional probability density function of triangular fuzzy number

By using the Mellin transform, we obtain

$$M_X(t) = \int_0^{\infty} x^{t-1} f_{X_{\tilde{A}_1}}(x)\, dx = \int_{a_1}^{a_2} x^{t-1} \frac{2(x-a_1)}{(a_2-a_1)(a_3-a_1)} dx + \int_{a_2}^{a_3} x^{t-1} \frac{2(x-a_1)}{(a_2-a_1)(a_3-a_1)} dx \tag{2.12}$$

On integration, we obtain

$$M_{X_{\tilde{A}_1}}(t) = \frac{2}{(a_3 - a_1)t(t+1)} \left[\frac{a_3(a_3^t - a_2^t)}{(a_3 - a_2)} - \frac{a_1(a_2^t - a_1^t)}{(a_2 - a_1)} \right] \tag{2.13}$$

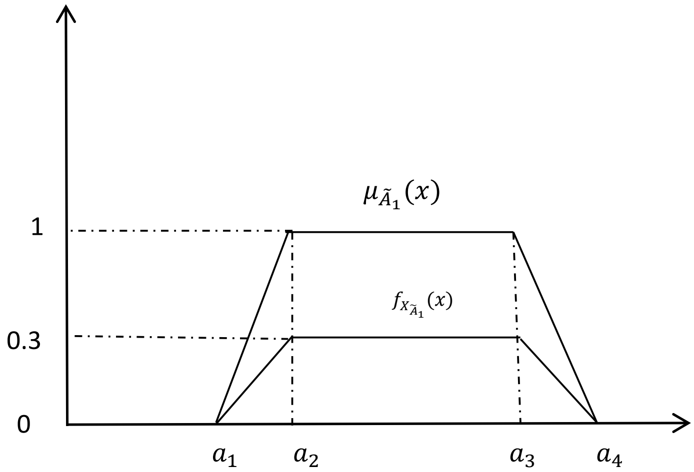

Figure 5.2: Proportional probability density function of trapezoidal fuzzy number

Thus the random variable $X_{\tilde{A}_1}$ has mean $(\mu_{X_{\tilde{A}_1}})$ and variance $\left(\sigma^2_{X_{\tilde{A}_1}}\right)$ can be obtained as

$$\mu_{X_{\tilde{A}_1}} = \mathbb{E}\left[X_{\tilde{A}_1}\right] = M_{X_{\tilde{A}_1}}(2) = \frac{a_1 + a_2 + a_3}{3} \tag{2.14}$$

$$\sigma^2_{X_{\tilde{A}_1}} = M_{X_{\tilde{A}_1}}(3) - \left[M_{X_{\tilde{A}_1}}\right]^2 = \frac{a_1^2 + a_2^2 + a_3^2 - a_1a_2 - a_2a_3 - a_3a_1}{18} \tag{2.15}$$

Further, the probability density function corresponding to trapezoidal fuzzy number $\tilde{A}_2 = (a_1, a_2, a_3, a_4)$ is given as $f_{\tilde{A}_2}(x) = c_2\mu_{\tilde{A}_2}(x)$, where $\mu_{\tilde{A}_2}(x)$ is defined as

$$\mu_{\tilde{A}_2}(x) = \begin{cases} \dfrac{x - a_1}{a_2 - a_1}, & a_1 \le x \le a_2 \\ 1, & a_2 \le x \le a_3 \\ \dfrac{a_4 - x}{a_4 - a_3}, & a_3 \le x \le a_4 \\ 0, & otherwise \end{cases} \tag{2.16}$$

Now c_2 is calculated as

$$\int_{-\infty}^{\infty} f_{\tilde{A}_2}(x)dx = 1 \tag{2.17}$$

That is,

$$\int_{-\infty}^{\infty} c_2\mu_{\tilde{A}_1}(x)dx = 1 \tag{2.18}$$

That is,

$$c_2\int_{a_1}^{a_2} \frac{x - a_1}{a_2 - a_1}dx + c_2\int_{a_2}^{a_3} dx + c_2\int_{a_3}^{a_4} \frac{a_4 - x}{a_4 - a_3}dx = 1 \tag{2.19}$$

On integration, we get

$$c_2 = \frac{2}{a_4 + a_3 - a_1 - a_2} \tag{2.20}$$

The proportional probability density function corresponding to triangular fuzzy number $\tilde{A}_1$ is given by

$$f_{X_{\tilde{A}_2}}(x) = \begin{cases} \dfrac{2(x - a_1)}{(a_2 - a_1)(a_4 + a_3 - a_1 - a_2)}, & a_1 \le x \le a_2, \\ \dfrac{2}{(a_4 + a_3 - a_1 - a_2)}, & a_2 \le x \le a_3 \\ \dfrac{2(x - a_1)}{(a_2 - a_1)(a_4 + a_3 - a_1 - a_2)} & a_3 \le x \le a_4 \\ 0, & otherwise \end{cases} \tag{2.21}$$

Graphically it is shown in figure 5.2

Using the Mellin transform, we get

$$M_{\tilde{A}_2}(t) = \int_0^\infty x^{t-1} f_{X_{\tilde{A}_2}}(x)\,dx = \int_{a_1}^{a_2} x^{t-1} \frac{2(x-a_1)}{(a_2-a_1)(a_4+a_3-a_1-a_2)}dx + \int_{a_2}^{a_3} x^{t-1}\frac{2}{(a_4+a_3-a_1-a_2)}dx + \int_{a_3}^{a_4}\frac{2(a_4-x)}{(a_4-a_3)(a_4+a_3-a_1-a_2)}dx \tag{2.22}$$

On integration, we obtain

$$M_{\tilde{A}_2}(t) = \frac{2}{(a_4+a_3-a_1-a_2)t(t+1)}\left[\frac{(a_4^{t+1}-a_3^{t+1})}{(a_4-a_3)} - \frac{a_2^{t+1}-a_1^{t+1}}{(a_2-a_1)}\right] \tag{2.23}$$

Thus, the random variable $X_{\tilde{A}_2}$has mean ($\mu_{X_{\tilde{A}_2}}$) and variance ($\sigma^2_{X_{\tilde{A}_2}}$) can be obtained as

$$\mu_{X_{\tilde{A}_2}} = \mathbb{E}\left[X_{\tilde{A}_2}\right] = M_{X_{\tilde{A}_2}}(2) = \frac{1}{3}\left[(a_1+a_2+a_3+a_4) + \frac{(a_1a_2-a_3a_4)}{(a_4+a_3-a_2-a_1)}\right] \tag{2.24}$$

$$\sigma^2_{X_{\tilde{A}_2}} = M_{X_{\tilde{A}_1}}(3) - \left[M_{X_{\tilde{A}_2}}(2)\right]^2 = \frac{1}{6}\left[(a_1^2+a_2^2+a_3^2+a_4^2) + \frac{(a_1+a_2)(a_3^2+a_4^2)-(a_3+a_4)(a_1^2+a_2^2)}{(a_4+a_3-a_2-a_1)}\right] + (\mu_{X_{\tilde{A}_2}})^2 \tag{2.25}$$

III. Probabilistic Fuzzy Quadratic Programming Problem and Its Crisp Model

Let $\tilde{c}_j = (c_j^1, c_j^2, c_j^3), j = 1,2,\dots,n, \tilde{a}_{ij} = (a_{ij}^1, a_{ij}^2, a_{ij}^3), i = 1,2,\dots,m,\ j = 1,2,\dots,n$ and $\tilde{q}_{ij} = (q_{ij}^1, q_{ij}^2, q_{ij}^3)$, $i = 1,2,\dots,n, j = 1,2,\dots,n$ denote the triangular fuzzy numbers. By using the method of defuzzification the crisp values of these fuzzy numbers can be obtained with probability density function of given membership function given as follows

$$\hat{c}_j = \frac{c_j^1+c_j^2+c_j^3}{3},\quad j = 1,2,\dots,n, \qquad \hat{a}_{ij} = \frac{a_{ij}^1+a_{ij}^2+a_{ij}^3}{3},\quad i = 1,2,\dots,m, j = 1,2,\dots,n, \tag{2.26}$$

$$\hat{q}_{ij} = \frac{q_{ij}^1+q_{ij}^2+q_{ij}^3}{3},\quad i = 1,2,\dots,m,\qquad j = 1,2,\dots,n,$$

where the crisp value of the given fuzzy number $\tilde{c}_j$ is represented by $\hat{c}_j$ and so on.
Similarly if all the coefficients have trapezoidal fuzzy numbers such as, $\tilde{c}_j = c_j^1 + c_j^2 + c_j^3 + c_j^4, j = 1,2,\dots,n,\ \tilde{a}_{ij} = (a_{ij}^1 + a_{ij}^2 + a_{ij}^3 + a_{ij}^4), i = 1,2,\dots,m, j = 1,2,\dots,n$ and $\tilde{q}_{ij} = (q_{ij}^1, q_{ij}^2, q_{ij}^3, q_{ij}^4), i = 1,2,\dots,n, j = 1,2,\dots,n$, then the crisp values are given as

$$\hat{c}_j = \frac{1}{3}\left[(c_j^1+c_j^2+c_j^3+c_j^4) + \frac{(c_j^1c_j^2 - c_j^3c_j^4)}{(c_j^3+c_j^4-c_j^1-c_j^2)}\right],\ j = 1,2,\dots,n$$

$$\hat{a}_{ij} = \frac{1}{3}\left[(a_{ij}^1+a_{ij}^2+a_{ij}^3+a_{ij}^4) + \frac{(a_{ij}^1a_{ij}^2 - a_{ij}^3a_{ij}^4)}{(a_{ij}^3+a_{ij}^4-a_{ij}^1-a_{ij}^2)}\right],\ i = 1,2,\dots,m, j = 1,2,\dots,n$$

$$\hat{q}_{ij} = \frac{1}{3}\left[(q_{ij}^1+q_{ij}^2+q_{ij}^3+q_{ij}^4) + \frac{(q_{ij}^1q_{ij}^2 - q_{ij}^3q_{ij}^4)}{(q_{ij}^3+q_{ij}^4-q_{ij}^1-q_{ij}^2)}\right],\ i = 1,2,\dots,n, j = 1,2,\dots,n \tag{2.27}$$

Thus the probabilistic quadratic fractional bi-level programming can be stated as follows

$$\underset{X_1}{\text{Max}}\,\tilde{f}_1(x_1,x_2) = \frac{f_{11}}{f_{12}} = \frac{\tilde{c}_{i1}x_j + \frac{1}{2}x^T\tilde{d}_{i1}x_j + \alpha_{i1}}{\tilde{c}_{i2}x_j + \frac{1}{2}x^T\tilde{d}_{i2}x_j + \beta_{i2}} \tag{2.28}$$

where for given X_1, X_2

$$\underset{X_2}{\text{Max}}\,\tilde{f}_2(x_1,x_2) = \frac{f_{21}}{f_{22}} = \frac{\tilde{c}_{i1}x_j + \frac{1}{2}x^T\tilde{d}_{i1}x_j + \alpha_{i1}}{\tilde{c}_{i2}x_j + \frac{1}{2}x^T\tilde{d}_{i2}x_j + \beta_{i2}} \tag{2.29}$$

$$\mathrm{X} \in \mathrm{G} = \left\{X \in R^n \middle| P(\textstyle\sum_{j=1}^n \tilde{a}_{ij}x_j \le b_i) \ge 1-\lambda_i,\quad i = 1,2,\dots\dots m\right\} \tag{2.30}$$

$$x_j \ge 0,\quad j = 1,2,\dots,n \tag{2.31}$$

where $0 < \lambda_i < 1, i = 1,2,\dots,m$

IV. Deterministic Model of the Probabilistic Quadratic Fractional Programming Problem

A technique for converting the quadratic fractional bi-level fuzzy probabilistic programming into its equivalent quadratic fractional bi-level fuzzy programming is discussed. We assume that $b_i(i = 1,2,\dots,m)$ in the model (2.28)-(2.31) are independent random variables following two-parameter exponential distribution [3] with parameters θ_i, σ_i where mean and variance of random variable b_i are given by:

$$E(b_i) = \theta_i + \sigma_i \quad i = 1,2,\dots,m \tag{2.32}$$

$$V(b_i) = \sigma_i^2 \quad i = 1,2,\dots,m \tag{2.33}$$

The probability density function of the i^{th} two-parameter exponential variable b_i is given by

$$f(b_i) = \frac{1}{\sigma_i} exp\left(\frac{-(b_i - \theta_i)}{\sigma_i}\right), \qquad i = 1,2,\dots,m \tag{2.34}$$

where $b_i \geq \theta_i, \sigma_i > 0$

To solve the problem (2.28)-(2.31), the deterministic form of the problem is established. Then from the chance-constraint (2.30), we have

$$\Pr\left(\sum_{j=1}^{n} a_{ij}x_j \leq b_i\right) \geq (1 - \gamma_i)$$

$$\Pr\left(b_i \geq \sum_{j=1}^{n} a_{ij}x_j\right) \geq (1 - \gamma_i)$$

$$\int_{\sum_{j=1}^{n} a_{ij}x_j}^{\infty} f(b_i)db_i \geq (1 - \gamma_i)$$

$$\int_{\sum_{j=1}^{n} a_{ij}x_j}^{\infty} \frac{1}{\sigma_i} exp\left(\frac{-(b_i - \theta_i)}{\sigma_i}\right) db_i \geq (1 - \gamma_i) \tag{2.35}$$

After integrating the above equation, following result is obtained

$$\sum_{j=1}^{n} a_{ij}x_j \leq \theta_i - \sigma_i \ln(1 - \gamma_i) \tag{2.36}$$

Hence the quadratic fractional bi-level fuzzy probabilistic programming into its equivalent deterministic quadratic fractional bi-level fuzzy programming by using the derived methodology is given as follows

$$\underset{X_1}{\text{Max}}\, \tilde{f}_1(x_1, x_2) = \frac{\tilde{c}_{i1}x_j + \frac{1}{2}x^T\tilde{d}_{i1}x_j \ + \alpha_{i1}}{\tilde{c}_{i2}x_j + \frac{1}{2}x^T\tilde{d}_{i2}x_j \ + \beta_{i2}}$$

where for given X_1, X_2

$$\underset{X_2}{\text{Max}}\, \tilde{f}_2(x_1, x_2) = \frac{\tilde{c}_{i1}x_j + \frac{1}{2}x^T\tilde{d}_{i1}x_j \ + \alpha_{i1}}{\tilde{c}_{i2}x_j + \frac{1}{2}x^T\tilde{d}_{i2}x_j \ + \beta_{i2}}$$

Subject to

$$\sum_{j=1}^{n} \tilde{a}_{ij}x_j \leq \theta_i - \sigma_i \ln(1 - \gamma_i) \tag{2.37}$$

$$0 < \gamma_i < 1, \quad i = 1,2,\dots.,m$$

$$x_j > 0, \qquad j = 1,2,\dots.,m$$

III. Transformation of bi-level quadratic fractional programming problem to nonlinear programming problem

The objective function of each decision maker is transformed from fractional form to nonlinear form with the procedure described below in order to find the best solution for each level without considering other levels. Consider the most basic model of the quadratic fractional programming problem, which is defined as follows.

$$\operatorname*{Max}_{X_i} \tilde{f}_i = \frac{f_{i1}}{f_{i2}} = \frac{\tilde{c}_{i1}x_j + \frac{1}{2}x^T\tilde{d}_{i1}x_j \ + \alpha_{i1}}{\tilde{c}_{i2}x_j + \frac{1}{2}x^T\tilde{d}_{i2}x_j \ + \beta_{i2}}$$

Subject to

$$\sum_{j=1}^{n} a_{ij}x_j \le \theta_i - \sigma_i \ln(1-\gamma_i)$$
$$0 < \gamma_i < 1, \quad i = 1,2, \ldots, m$$
$$x_j > 0, \quad j = 1,2, \ldots, n$$

To convert the above type of quadratic fractional programming problem into the nonlinear programming problem, take $\frac{1}{f_{i2}} = \frac{`1}{\tilde{c}_{i2}x_j + \frac{1}{2}x^T\tilde{d}_{i2}x_j + \beta_{i2}} = y_i \Rightarrow f_{i2}y_i = 1$

Thus, problem becomes as the following:

$$\operatorname*{Max}_{X_i} \tilde{f}_i = f_{i1}y_i = \left(\tilde{c}_{i1}x_j + \frac{1}{2}x^T\tilde{d}_{i1}x_j \ + \alpha_{i1}\right)y_i$$

Subject to

$$\left(\tilde{c}_{i2}x_j + \frac{1}{2}x^T\tilde{d}_{i2}x_j \ + \beta_{i2}\right)y_i = 1$$
$$\sum_{j=1}^{n} a_{ij}x_j \le \theta_i - \sigma_i \ln(1-\gamma_i)$$
$$0 < \gamma_i < 1, \quad i = 1,2, \ldots, m$$
$$x_j > 0, \quad j = 1,2, \ldots, n$$

IV. Formulation of fuzzy goal programming approach for bi-level quadratic fractional programming problem

The decision makers of a bi-level quadratic fractional programming problem (BLQFPP) are fundamentally cooperative and make sequential decisions. By optimizing each decision maker individually for the given set of constraints for each objective functions f_i, $i = 1,2, \ldots m$ of the problem the fuzzy goal are taken as the maximum and minimum value of each decision maker. . Let $f_i^{U_i}, f_i^{L_i}$ is the maximum and minimum value of each objective function which are obtained by optimizing them as individually i.e. $f_i^{U_i} = \max_{x \in X} f_i$ with the solution set of decision variables as $(x_1^{U_i}, x_2^{U_i}, \ldots, x_n^{U_i})$
$f_i^{L_i} = \min_{x \in X} f_i$ with the solution set of decision variables as $(x_1^{L_i}, x_2^{L_i}, \ldots, x_n^{L_i})$
For $i = 1,2, \ldots, m$ now, for the given model, our objective was to maximize the objective function of each decision maker such that maximum value of objective function for first decision maker is $f_1^{U_1}$ at the point $(x_1^{U_1}, x_2^{U_1}, \ldots, x_n^{U_1})$, similarly maximum value of objective function for second decision maker is $f_2^{U_2}$ at the point $(x_1^{U_2}, x_2^{U_2}, \ldots, x_n^{U_2})$. It can be assumed reasonably that the value of $f_i \ge f_i^{U_i}$ are acceptable and all values less than $f_i^{L_i} = \min_{x \in X} f_i$ are absolutely unacceptable. Then the membership function $\mu_i f_i$ for the i^{th} fuzzy goal can be formulated as

$$\mu_i f_i = \begin{cases} 1, & if \;\; f_i \geq f_i^{U_i} \\ \dfrac{f_i - f_i^{L_i}}{f_i^{U_i} - f_i^{L_i}} & if \; f_i^{L_i} \leq f_i \leq f_i^{U_i} \;\; i = 1,2, \dots m \\ 0 & if \;\; f_i \leq f_i^{L_i} \end{cases}$$

Each decision maker seeks to maximize his or her own objective function when making a decision. When each decision maker's optimum solution is computed separately, it is considered as the best solution, and the associated value of the objective function is regarded as the aspiration level of the corresponding fuzzy goal. In fuzzy programming approach, the highest degree of membership is one Mohamed [5]. The flexible membership goal having the aspired level unity can be represented as follows:

$$\mu_{f_i}(f_i) + d_i^- - d_i^+ = 1, \quad i = 1,2, \dots, m$$

Or equivalently as

$$\frac{f_i - f_i^{L_i}}{f_i^{U_i} - f_i^{L_i}} + d_i^- - d_i^+ = 1, \quad i = 1,2, \dots, m,$$

where $d_i^-, d_i^+ \geq 0$, with $d_i^- \times d_i^+ = 0$, represent the under and over deviations, respectively, from the aspired levels Pramanik and Roy [8]

In the formulation of fuzzy goal programming, the under and over deviational variables in the achievement function for minimizing them depends up on the type of objective functions to be optimized. To reach the aspiration level in the proposed fuzzy goal programming approach, the sum of under deviational variables must be minimized. The proposed fuzzy goal programming model for BLQFP problem follows as:

$$MinZ = \sum_{i=1}^{m} w_i d_i^-$$

$$\frac{f_i - f_i^{L_i}}{f_i^{U_i} - f_i^{L_i}} + d_i^- - d_i^+ = 1$$

$$\left(\tilde{c}_{i2} x_j + \frac{1}{2} x^T \tilde{d}_{i2} x_j \; + \beta_{i2}\right) y_i = 1$$

$$\sum_{j=1}^{n} a_{ij} x_j \leq \theta_i - \sigma_i \ln\,(1 - \gamma_i)$$

$$0 < \gamma_i < 1, \quad i = 1,2, \dots, m$$

$$x_j = x_j^* \qquad j = 1,2, \dots, n$$

and $d_i^-, d_i^+ = 0, d_i^-, d_i^+ \geq 0, \forall i = 1,2, \dots, k$

where Z is the fuzzy goal's achievement function which is made up of the weighted under-deviational variables. The numerical weights w_i represent the relative importance of achieving the aspired levels of the respective fuzzy goals. The values of w_i are determined as Mohamed [5]:

$$w_i = \frac{1}{f_i^{U_i} - f_i^{L_i}} \qquad i = 1,2, \dots, m$$

V. Numerical Examples

To demonstrate the proposed FGP approach, consider the following BLQFP problem with probabilistic nature in the constraints.

[FLDM]

$$MaxF_1 = \frac{\tilde{6}x_1 + \tilde{3}x_2 + \widetilde{-1}x_1^2 + \widetilde{-1}x_2^2 + 6}{\tilde{1}x_1^2 + \tilde{1}x_2^2 + 4}$$

where x_2 solves

[SLDM]

$$MaxF_2 = \frac{\tilde{1}x_1 + \tilde{5}x_2 + \widetilde{-1}x_2^2 + 8}{\tilde{1}x_1^2 + \tilde{1}x_2 + 6}$$

Subject to

$$P(\tilde{1}x_1 + \tilde{1}x_2 \leq b_1) \geq 0.99$$
$$P(\tilde{3}x_1 + \tilde{2}x_2 \leq b_2) \geq 0.95$$
$$P(\tilde{2}x_1 + \tilde{1}x_2 \geq b_3) \geq 0.90$$
$$x_1, x_2 \geq 0 \quad (5.1)$$

Here, we assume that $b_i\ (i = 1,2,3)$ are random variables following two parameter exponential distributions with following parameters:

$E(b_1) = 161, \quad E(b_2) = 144, \quad E(b_3) = 106$
$Var(b_1) = 25, \quad Var(b_2) = 36, \quad Var(b_3) = 64$

Using (2.32) and (2.33), the parameters are calculated as follows:

$\theta_1 = 156, \quad \sigma_1 = 5, \quad \theta_2 = 138, \quad \sigma_2 = 6, \quad \theta_3 = 98\ , \sigma_3 = 8$

The coefficients of the objectives are taken as triangular fuzzy number with the values $-\tilde{1} = (-1.5, -1, -0.5), \tilde{6} = (4,6,9), -\tilde{1} = (-1.1, -1, -0.15), \tilde{5} = (4.5,5,5.8),\ \tilde{1} = (0.95,1,1.05), \tilde{3} = (2,3,5), -\tilde{1}\ = (-1.04, -1,0.03), -\tilde{2}(-1.5, -2, -2.5), -\tilde{2} = (-1.8, -2, -2.2), \tilde{5} = (4.2,5,5.8), \tilde{1} = (0.8,1,1.2), \tilde{3} = (2,3,4)$

The coefficients of the probabilistic constraints are taken as trapezoidal fuzzy number with the values

$$\tilde{1} = (0.2,0.8,1,1.2), \tilde{1} = (0.4,1,1.6,2), \tilde{3} = (1,2,3,4), \tilde{2} = (0.5,1.5,2.5,3), \tilde{2} = (1,1.8,2.2,3), \tilde{1} = (0.5,1,1.5,2).$$

On the basis of the method of defuzzification with probability density function and CCP technique the above model (5.1) can be expressed as

[FLDM]

$$MaxF_1 = \frac{6.33x_1 + 3.33x_2 - x_1^2 - 0.75x_2^2 + 8}{0.8x_1^2 + 0.93x_2^2 + 4}$$

where x_2 solves

[SLDM]

$$MaxF_2 = \frac{x_1^2 + 5.1x_2 - 0.61x_1 + 10}{1.06x_1^2 + 1.2x_2 + 6}$$

Subject to

$$0.78x_1 + 1.24x_2 \leq 156.05$$
$$2.67x_1 + 1.86x_2 \leq 138.308$$
$$2x_1 + 1.25x_2 \leq 98.41$$
$$x_1, x_2 \geq 0$$

The bi-level multi-objective quadratic fractional programming problem is transformed into the bi-level quadratic programming model as follows.

[FLDM]

$$MaxF_1 = (6.33x_1 + 3.33x_2 - x_1^2 - 0.75x_2^2 + 8)y_1$$

where x_2 solves

[SLDM]

$$MaxF_2 = (x_1^2 + 5.1x_2 - 0.61x_1 + 10)y_2$$

Subject to

$$(0.8x_1^2 + 0.93x_2^2 + 4)y_1 = 1$$
$$(1.06x_1^2 + 1.2x_2 + 6)y_2 = 1$$
$$0.78x_1 + 1.24x_2 \leq 156.05$$

$2.67x_1 + 1.86x_2 \leq 138.3082x_1 + 1.25x_2 \leq 98.41$
$x_1, x_2 \geq 0$

We first obtain the value of $F_1^{max} = 2.945$, $F_2^{max} = 4.088, F_1^{min} = 2, F_2^{min} = 0.924$
and $f_1^U = 2.945$, $f_2^U = 4.088$, $f_1^L = 2, f_2^L = 0.924$
To solve FGP models to get $x_1 = x_1^*$. Thus the first level FGP model follows as:

$MinZ = 1.058d_1^-$
Subject to
$(6.33x_1 + 3.33x_2 - x_1^2 - 0.75x_2^2 + 8)y_1 + 0.9448d_1^- - 0.9448d_1^+ = 2.945$
$(0.8x_1^2 + 0.93x_2^2 + 4)y_1 = 1$
$(1.06x_1^2 + 1.2x_2 + 6)y_2 = 1$
$0.78x_1 + 1.24x_2 \leq 156.05$
$2.67x_1 + 1.86x_2 \leq 138.3082x_1 + 1.25x_2 \leq 98.41$
$x_1, x_2 \geq 0$
$d_1^+, d_1^- \geq 0$

Using Lingo software, the compromise solution of first level decision maker problem is obtained as; (x_1, x_2) = (0.954, 0.478). Then assuming that the FLDM set $x_1^* = 0.954$
$MinZ = 1.058d_1^- + 0.316d_2^-$
$(6.33x_1 + 3.33x_2 - x_1^2 - 0.75x_2^2 + 8)y_1 + 0.945d_1^- - 0.945d_1^+ = 2.945$
$(x_1^2 + 5.1x_2 - 0.61x_1 + 10)y_2 + 3.164d_2^- - 3.164d_2^+ = 4.0877$
$(0.8x_1^2 + 0.93x_2^2 + 4)y_1 = 1$
$(1.06x_1^2 + 1.2x_2 + 6)y_2 = 1$
$0.78x_1 + 1.24x_2 \leq 156.05$
$2.67x_1 + 1.86x_2 \leq 138.3082x_1 + 1.25x_2 \leq 98.41$
$x_1^* = 0.943$
$d_1^+, d_1^-, d_2^+, d_2^- \geq 0$

Using Lingo software, the compromise solution of the BLQFP problem is obtained as (x_1, x_2) = (0.954, 0.798) with the corresponding objective function.

VI. Discussion

In this paper, the technique for solving a Quadratic Fractional Bi-level Fuzzy Probabilistic Programming (QFBLFP) programming is outlined by considering the fact that the random variable follows two parameter exponential distributions and using a combination of probabilistic and fuzzy concepts. Firstly the probabilistic nature of the problem is transformed into an equivalent deterministic problem and then a fuzzy goal programming technique is used to solve the bi-level deterministic model.

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