Dynamical Behavior of an SEIS Epidemic Model with Nonlinear Incidence Rate

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Abstract

In this paper, an SEIS epidemic model with nonlinear incidence rate is studied. The basic reproduction number (R_0) is calculated. The local and global stability of the disease free equilibrium E_0 and the endemic equilibrium E^* of the model are discussed and also the global asymptotical stability of the disease free equilibrium and endemic equilibrium are discussed. The stability analysis of the model shows that the system is locally asymptotically stable at disease free equilibrium E_0 and endemic equilibrium E^* under suitable conditions. Moreover, show that the disease free equilibrium and the unique endemic equilibrium of the system is globally asymptotically stable under certain conditions. Finally, numerical simulations are given to support some of the theoretical results.

Keywords: epidemic models, equilibrium, local and global stability.

I. Introduction

Mathematical models describing the population dynamics of infectious diseases have been playing an important role in disease control for a long time. In recent years, various epidemic models have been proposed and explored to prevent and control the spread of the infectious diseases, such as measles, tuberculosis, and flu (see e.g., [2, 12]). Simple mass action bilinear incidence rate βSI was introduced Kermack –Mcendrick [5] in 1927. In many epidemic models, bilinear incidence rate βSI is frequently used [1, 6, 13, 14, 15]. Moreover, nonlinear incidence rates of the form $\beta I^p S^q$ were investigated by Liu, Hethcote and Levin [9], Liu, Levin and Iwasa [10]. In this paper, we consider an $SEIS$ epidemic model with nonlinear incidence rate $\beta I^p S^q$ taking $p = 2$ and $q = 1$ that is βSI^2 have similar repertoires of dynamical behaviors, much wider than of bilinear incidence rate models, and we study the existence and stability of the equilibriums of the $SEIS$ epidemic model.

This manuscript is organized as follows: In Sect. 2, $SEIS$ model is presented. In Sect. 3, basic properties of solutions are discussed. In Sect. 4, we determine all possible equilibria of model. In Sect. 5, we discuss and analyze the local stability of the equilibriums. In Sect. 6, we discuss and analyze the global stability of the equilibriums. We present in Sect. 7, some numerical examples of the dynamics of the model. Finally, in Sect. 8, we discussed the conclusion.

II. Model Formulation

By a standard nonlinear incidence rate βSI^2, we consider an $SEIS$ epidemic model which consists of three compartments corresponding to three population classes, namely, susceptible (S), exposed (but not yet infectious) (E), infectious (I) and the total population (N).
The model is given as follows:

$$\left.\begin{aligned}\frac{dS}{dt} &= A-\beta I^2 S-dS+\gamma I\\ \frac{dE}{dt} &= \beta I^2 S-(d+\varepsilon)E\\ \frac{dI}{dt} &= \varepsilon E-(d+\gamma+\alpha)I\end{aligned}\right\} \tag{2.1}$$

whose state space is the first quadrant $R_3^+=\{(S,E,I): S\geq 0, E\geq 0, I\geq 0\}$ and subject to the initial conditions $S(0)=S_0\geq 0$, $E(0)=E_0\geq 0$, $I(0)=I_0\geq 0$. It is assumed that all the parameters are positive. From the model, the parameters can be summarized in the following list:
A is the recruitment rate of the population, d is the natural death rate of the population, α is the constant rate of infectious hosts suffer an extra disease related death, β is the transmission or contact rate, ε and γ is the transfer rates among the corresponding classes.

III. Basic Properties of the Model

Summing up the four equations of model (2.1) and denoting

$$N(t)=S(t)+E(t)+I(t),$$

Having, $N'(t)=A-dN-\alpha I$. If disease is not present that is $I=0$, then $N'(t)=A-dN$. This shows that population size $N\to\frac{A}{d}$ as $t\to\infty$. Since the spread of the disease in the population will lead to the decrease of N, it follows that $N\in[0,\frac{A}{d}]$. It follows that the solutions of model (2.1) exists in the region defined by

$$\Omega=\left\{(S,E,I)\in R_3^+: S,E,I\geq 0, S+E+I\leq\frac{A}{d}\right\}.$$

This gives the following lemma which shows that the solutions of model (2.1) are bounded, continuous for all positive time and lie in a compact set.

IV. Existence of Equilibria

In this section, we obtain the existence of the disease-free equilibrium E_0 and the endemic equilibrium E^* of model (2.1).
Set the right sides of model (2.1) equal zero, that is,

$$\begin{cases}\frac{dS}{dt} = A-\beta I^2 S-dS+\gamma I=0\\ \frac{dE}{dt} = \beta I^2 S-(d+\varepsilon)E=0\\ \frac{dI}{dt} = \varepsilon E-(d+\gamma+\alpha)I=0\end{cases} \tag{4.1}$$

The model (2.1) always has the disease-free equilibrium point $E_0(A/d,0,0)$. Solving (4.1) we also get a unique positive, endemic equilibrium point $E^*(S^*,E^*,I^*)$of the model (2.1), where $S_*=\frac{(d+\varepsilon)(d+\gamma+\alpha)}{\beta\varepsilon I}$, $E_*=\frac{(d+\gamma+\alpha)}{\varepsilon}I$ and I^* is given as a root of the quadratic equation

$$\Omega_1 I^2 + \Omega_2 I + \Omega_3 = 0,$$

where, $\Omega_1 = \beta(d+\varepsilon)(d+\alpha+\gamma) - \beta\varepsilon\gamma$, $\Omega_2 = -A\beta\varepsilon$ and $\Omega_3 = d(d+\varepsilon)(d+\alpha+\gamma)$.

Now, $I^* = \dfrac{A\beta\varepsilon + \sqrt{\Delta}}{2\beta(d+\varepsilon)(d+\alpha+\gamma) - 2\beta\varepsilon\gamma}$,

where, $\Delta^2 = 4\beta\varepsilon\gamma d(d+\varepsilon)(d+\alpha+\gamma) + 4\beta d(d+\varepsilon)^2(d+\alpha+\gamma)^2[R_0 - 1]$.

Obviously $\Delta_1 > 0$ when $R_0 > 1$.

According to a direct computation, define the basic reproduction number as follows:

$$R_0 = \frac{A^2\beta^2\varepsilon}{4d(d+\varepsilon)^2(d+\gamma+\alpha)^2}.$$

It means the average new infections caused by a single infected individual in a whole susceptible population.

V. Local Stability Analysis

In this section, we study the local stability of the disease-free equilibrium E_0 and the endemic equilibrium E^* of model (2.1).

Theorem 5.1 If $R_0 < 1$, the disease-free equilibrium E_0 of model (2.1) is locally asymptotically stable. If $R_0 > 1$, the disease-free equilibrium E_0 is unstable.

Proof. The Jacobian matrix of model (2.1) at the disease-free equilibrium E_0 is

$$J(E_0) = \begin{pmatrix} -d & 0 & \gamma \\ 0 & -(d+\varepsilon) & 0 \\ 0 & \varepsilon & -(d+\gamma+\alpha) \end{pmatrix}$$

The characteristic equation of $J(E_0)$ is $(d+\lambda)(d+\varepsilon+\lambda)(d+\gamma+\alpha+\lambda) = 0$.

This equation has the following roots: $\lambda_1 = -d$, $\lambda_2 = -(d+\varepsilon)$ and $\lambda_3 = -(d+\gamma+\alpha)$ are always negative. Hence E_0 is locally asymptotically stable for $R_0 < 1$, while it is unstable for $R_0 > 1$.

Theorem 5.2 If $R_0 > 1$, the endemic equilibrium E^* of model (2.1) is locally asymptotically stable.

Proof. The Jacobian matrix of system (2.1) at E^* is

$$J(E^*) = \begin{pmatrix} -(m+d) & 0 & -n+\gamma \\ m & -(d+\varepsilon) & n \\ 0 & \varepsilon & -(d+\gamma+\alpha) \end{pmatrix}$$

where $m = \beta I^{*2}$ and $n = 2\beta S^* I^*$

The characteristic equation of $J(E^*)$ is

$$\lambda^3 + \lambda^2\{(d+\gamma+\alpha)(m+2d+\varepsilon)\} + \lambda\{(d+\gamma+\alpha)(m+2d+\varepsilon) + (m+d)(\varepsilon+d) - \varepsilon n\} + (d+\gamma+\alpha)(m+d)(\varepsilon+d) - \varepsilon(nd+m\gamma) = 0 \quad (5.1)$$

from numerical computation, we realized the real part of equation (5.1) cannot be positive. This indicates that, the steady state(s) will also be stable.

VI. Global Stability Analysis

In this section, we study the global stability of the disease-free equilibrium E_0 and the endemic equilibrium E^* of model (2.1).

Theorem 6.1 If $R_0 < 1$, the disease-free equilibrium E_0 of model (2.1) is globally asymptotically stable.

Proof. We prove the global stability of the model (2.1) at the equilibrium E_0 when $R_0 < 1$. Taking the Lyapunov function

$$V(t) = E(t)$$

Calculating the derivative of $V(t)$ along the positive solution of model (2.1), it follows that

$$\dot{V}(t) = \beta SI^2 - (d+\varepsilon)E = \beta SI^2 - \frac{(d+\varepsilon)(d+\alpha+\gamma)}{\varepsilon} I$$

Since the incidence function

$$\beta SI^2 \le \frac{\beta AI^2}{d} \text{ for } 0 \le S \le \frac{A}{d}.$$

$$\dot{V}(t) \le \left[\frac{\beta AI}{d} - \frac{(d+\varepsilon)(d+\alpha+\gamma)}{\varepsilon} \right] I \le 0$$

Furthermore, $\dot{V} = 0$ only if $I = 0$, so the largest invariant set contained in $\{(S, E, I) \in \Omega : \dot{V} = 0\}$ is the plane $I = 0$. By Lassalle's invariance principle [7], this implies that all solution in Ω approach the plane $I = 0$ as $t \to \infty$. On the other hand, solutions of (2.1) contained in such plane satisfy $\frac{dS}{dt} = A - dS$, $\frac{dE}{dt} = -(d+\varepsilon)E$, which implies that $S \to \frac{A}{d}$ and $E \to 0$ as $t \to \infty$, that is, all of these solutions approach E_0 is globally asymptotically stable in Ω.

Next, we analysis the global stability of an endemic equilibrium E^* by using geometric approach method described by Li and Muldowney in [8]. We first briefly explain the geometric approach method.

Theorem 6.2 (Li & Muldowney [8]). Suppose that the system $x' = f(x)$, with $f : D \subset \tilde{}^n \to \tilde{}^n$, satisfies the following:

(H1) D is a simply connected open set,

(H2) there is a compact absorbing set $K \subset D$,

(H3) x^* is the only equilibrium in D.

Then the equilibrium x^* is globally stable in D if there exists a *Lozinskiĭ* measure η such that

$$\lim_{t\to\infty} \sup \sup_{x_0 \in K} \frac{1}{t} \int_0^t \eta(B(x(s, x_0)))ds < 0,$$

$$B = P_f P^{-1} + PJ^{[2]} P^{-1}$$

and $Q \to Q(x)$ is an $\binom{n}{2} \times \binom{n}{2}$ matrix valued function.

In our case, model (2.1) can be written as $x' = f(x)$ with $f : D \subset \tilde{}^n \to \tilde{}^n$ and D being the interior of the feasible region Ω. The existence of a compact absorbing set $K \subset D$ is equivalent to proving that (2.1) is uniformly persistent (see [8]) and the proof for this in the case when $R_0 > 1$ is similar to that of proposition 4.2 of [8]. Hence, **(H1)** and **(H2)** hold for system (2.1), and by assuming the uniqueness of the endemic equilibrium in D, we can prove its global stability with the aid of *Theorem 6.2*.

Theorem 6.3 If $R_0 > 1$ then the endemic equilibrium E^* of the system (2.1) is globally asymptotically stable in the feasible region Ω..

Proof. Let J be the Jacobian matrix of the system (2.1), i.e.

$$J = \begin{pmatrix} -\beta I^2 - d & 0 & -2\beta IS + \gamma \\ \beta I^2 & -(d+\varepsilon) & 2\beta IS \\ 0 & \varepsilon & -(d+\gamma+\alpha) \end{pmatrix}$$

Then the second additive compound matrix of J is given by

$$J^{[2]} = \begin{pmatrix} -\beta I^2 - 2d - \varepsilon & 2\beta IS & 2\beta IS - \gamma \\ \varepsilon & -\beta I^2 - 2d - \alpha - \gamma & 0 \\ 0 & \beta I^2 & -2d - \alpha - \varepsilon - \gamma \end{pmatrix}$$

Choose the function $P = P(S, E, I) = \text{diag}(1, \frac{E}{I}, \frac{E}{I})$; then $P^{-1} = \text{diag}(1, \frac{I}{E}, \frac{I}{E})$ and $P_f = \text{diag}(0, \frac{E'I - I'E}{I^2}, \frac{E'I - I'E}{I^2})$. Also, $P_f P^{-1} = \text{diag}(0, \frac{E'}{E} - \frac{I'}{I}, \frac{E'}{E} - \frac{I'}{I})$

$$PJ^{[2]}P^{-1}=\begin{pmatrix} -\beta I^2-2d-\varepsilon & \dfrac{2\beta I^2 S}{E} & \dfrac{2\beta I^2 S}{E}-\dfrac{\gamma I}{E} \\ \dfrac{\varepsilon E}{I} & -\beta I^2-2d-\alpha-\gamma & 0 \\ 0 & \beta I^2 & -2d-\alpha-\varepsilon-\gamma \end{pmatrix}$$

The matrix $B=P_f P^{-1}+PJ^{[2]}P^{-1}$ can be written in matrix form

$$B=\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

where

$$B_{11}=-\beta I^2-2d-\varepsilon,\quad B_{12}=\left(\frac{2\beta I^2 S}{E},\frac{2\beta I^2 S}{E}-\frac{\gamma I}{E}\right),\quad B_{21}=\left(\frac{\varepsilon E}{I},0\right)^T,$$

$$B_{22}=\begin{pmatrix} -\beta I^2-2d-\alpha-\gamma+\dfrac{E'}{E}-\dfrac{I'}{I} & 0 \\ \beta I^2 & -2d-\alpha-\varepsilon-\gamma+\dfrac{E'}{E}-\dfrac{I'}{I} \end{pmatrix}$$

Let (u,v,w) be a vector in R^3; its norm $\|.\|$ is defined as $\|(u,v,w)\|=\max\{|u|,|v|+|w|\}$

Let $\mu(B)$ be the Lozinskiĭ measure with respect to this norm.

$$\mu(B)\le \sup\{g_1,g_2\}$$

where

$g_1=\mu_1(B_{11})+|B_{12}|$, $g_2=\mu_1(B_{22})+|B_{21}|$, $|B_{12}|$, $|B_{21}|$ are matrix norm with respect to l_1 vector norm and μ_1 denotes the Lozinskiĭ measure with respect to l_1 vector norm, then $\mu_1(B_{11})=-\beta I^2-2d-\varepsilon$, $|B_{12}|=\max\left(\frac{2\beta I^2 S}{E},\frac{2\beta I^2 S}{E}-\frac{\gamma I}{E}\right)=\frac{2\beta I^2 S}{E}$, $|B_{21}|=\frac{\varepsilon E}{I}$.

Therefore,

$$\left.\begin{aligned} g_1&=\mu_1(B_{11})+|B_{12}|=-\beta I^2-2d-\varepsilon+\frac{2\beta I^2 S}{E} \\ g_2&=\mu_2(B_{22})+|B_{21}|=-2d-\alpha-\gamma+\frac{E'}{E}-\frac{I'}{I}+\frac{\varepsilon E}{I} \end{aligned}\right\}.$$

where

$$\mu_2(B_{22})=\max\{-2d-\alpha-\gamma+\frac{E'}{E}-\frac{I'}{I},-2d-\alpha-\varepsilon-\gamma+\frac{E'}{E}-\frac{I'}{I}\}=-2d-\alpha-\gamma+\frac{E'}{E}-\frac{I'}{I}.$$

From system (2.1) we have $\frac{E'}{E}=\frac{\beta SI^2}{E}-(\varepsilon+d)$ and $\frac{I'}{I}=\frac{\varepsilon E}{I}-(d+\gamma+\alpha)$

Then

$$\begin{aligned} g_1&=-\beta I^2-2d-\varepsilon+\frac{2\beta I^2 S}{E} \\ &\le \frac{\beta I^2 S}{E}-(\varepsilon+2d)\le \frac{E'}{E}-d \\ g_2&=-2d-\alpha-\gamma+\frac{E'}{E}-\frac{I'}{I}+\frac{\varepsilon E}{I}=\frac{E'}{E}-d \end{aligned}$$

Furthermore, obtain

$$\begin{aligned} \mu(B)&\le \sup\{g_1,g_2\} \\ &\le \left\{\frac{E'}{E}-d,\frac{E'}{E}-d\right\}\le \frac{E'}{E}-d \end{aligned}$$

By integrating both sides at the same time,

$$\frac{1}{t}\int_0^t \mu(B)\mathrm{ds}\le \frac{1}{t}\mathrm{In}\frac{E(t)}{E(0)}-d$$

$$q = \lim_{t\to\infty} \sup\sup \frac{1}{t}\int_0^t \mu(B)\mathrm{ds} < -d < 0.$$

Hence, E^* is globally asymptotically stable in Ω.

VII. Numerical Simulations

To see the dynamical behavior of system (2.1) some numerical simulations are given. For this, consider the Hypothetical set of parameter values as the following.

Case I. $A = 10$, $d = 0.2$, $\beta = 0.04$, $\alpha = 1.25, \varepsilon = 1.2$, $\gamma = 0.4$ then the basic reproduction number $R_0 = 0.035777641 < 1$, $S(t)$ approaches to its steady state value while $E(t)$ and $I(t)$ approach zero as time goes to infinity, the disease dies out **(Fig. 1)**.

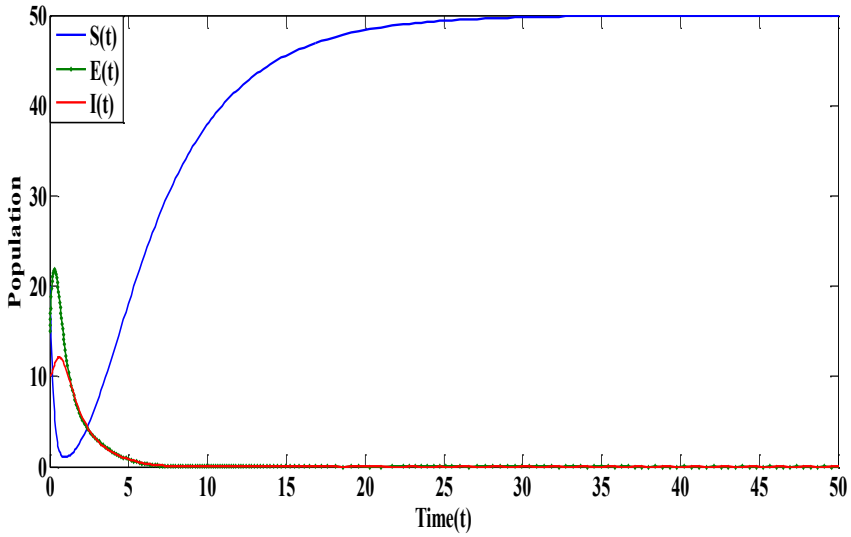

Figure 1: *The figure represents that the disease dies out*

Case II. $A = 10$, $d = 0.2$, $\beta = 1$, $\alpha = 1.25, \varepsilon = 1.2$, $\gamma = 0.4$ then the basic reproduction number $R_0 = 22.36102622 > 1$, all the three component $S(t)$, $E(t)$ and $I(t)$ approach to their steady state values as time goes to infinity, the disease becomes endemic **(Fig. 2)**.

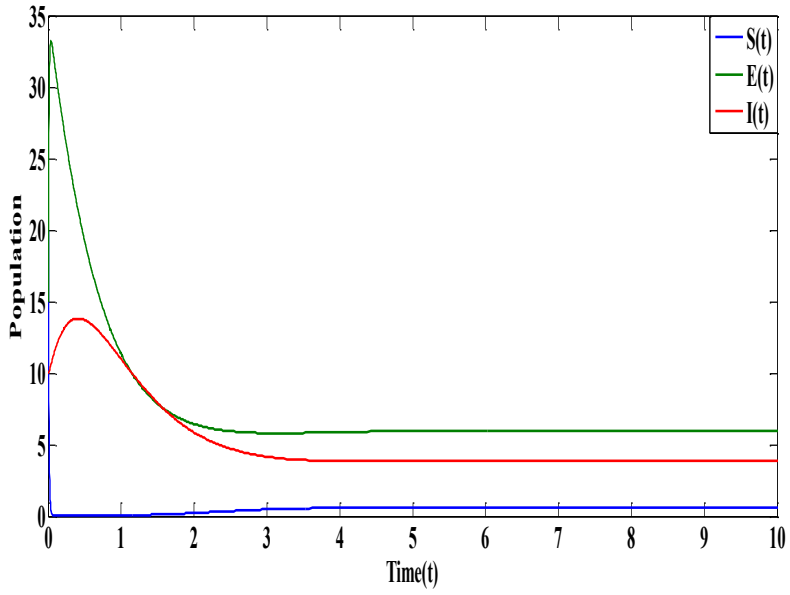

Figure 2: *The figure represents that the disease endemic*

VIII. Conclusion

This paper presents a mathematical study on the dynamics of an $SEIS$ epidemic model that incorporates constant recruitment, exponential natural death as well as the disease related rate, so that the population size may vary in time. The incidence rate is of the non-linear incidences frequently used in the literature. Also, we see that if the basic reproduction number R_0 is less one the disease free equilibrium E_0 is locally and globally asymptotically stable in feasible region Ω and disease always dies out (see **Fig. 1**). If the basic reproduction number R_0 is greater than one the unique endemic equilibrium E^* is locally and globally under certain condition if the interior of Ω. In this case, the disease cannot control easily (see **Fig. 2**).

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