Statistical Analysis of Marshall-Olkin inverse Maxwell Distribution: Estimation and Application to Real Data

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Abstract

In this paper, Marshall-Olkin inverse Maxwell distribution is proposed by generalizing the inverse Maxwell distribution under the Marshall-Olkin family of distribution that leads to greater flexibility in modeling various new data types. The basic statistical properties for the proposed distribution including moments, quantile function, median, skewness, kurtosis, and stochastic ordering are derived. Point estimates for the parameters are obtained by using two well known methods maximum likelihood and maximum spacing methods. The confidence intervals are used by using asymptotic properties of maximum likelihood estimators and boot-p methods. We have applied the proposed distribution under different real-life scenarios such as record value problem, system lifetime distributions, stress-strength reliability and random censored problems. For illustration purposes, simulation and real data results are established.

Keywords: Marshall-Olkin Family, Tilt Parameter, Inverse Maxwell Distribution, Stochastic Ordering, Maximum Likelihood Estimate, Maximum Spacing, Record Value, Stress-Strength Reliability, Random Censoring

1. Introduction

1.1. Literature

In lifetime experiments, the problem of finding the appropriate lifetime distributions is a concern for a long time. In literature, we have a lot of distributions that study different types of hazard nature. There are many scenarios in real life where some standard distributions are not applied or less suitable to fit actual data. Another application of generalizing distributions is to gain flexibility and better fit to real-life data. Since the several lifetime distributions had been proposed consisting of increasing, decreasing and bathtub hazard rates. Some well-known distributions consisting of these hazard natures are Exponential, Weibull, Gamma, Normal, etc. but these standard distributions are not always fulfilling our purposes. A situation in which the hazard rate initially increases attains a maximum point and then again starts decreasing generates an upside-down bathtub (UBT) shaped hazard rate. There are some distributions in literature containing UBT shapes as, inverse Gamma distribution, inverse Gaussian distribution, Log-Normal distribution, Log-Logistic distribution, Birnbaum-Saunders distribution, inverse Weibull distribution and inverse Maxwell distribution (InvMWD). For example, the lifetime models that present upside-down bathtub failure rates curves can be observed in the course of a disease whose mortality reaches a peak after some finite period and then declines gradually. Also, it is observed that the risks of dying a patient just after an operation increase due to infection and then decrease with recovery.

Many authors have been discussed the situations where the data shows decreasing, increasing, bathtub and UBT shape hazard rates. Proschan (1963) discussed the air-conditioning systems of planes follow decreasing failure rate. They obtained the reliability characteristics of an aircraft air-conditioning system of the plane in an airline-use environment. In the paper, Kuş (2007) analyzed the earthquake data in the last century in the North Anatolian fault zone and found that the fitting the decreasing. Folks and Chhikara (1978) showed the behavior of the inverse Gaussian distribution as an upside-down bathtub model and reviewed its important statistical properties. Langlands et al. (1979) studied the pattern of mortality for the breast carcinoma data and concluded that it increases initially and after a certain time it started declining. Bennett (1983) studied the failure rate of lung cancer data applied to uni-model Log-Logistic distribution. In article Efron (1988), authors analyzed the data set in the context of head and neck cancer, in which the hazard rate initially increased, attained a maximum and then decreased before it stabilized owing to a therapy. Sharma et al. (2015) discussed the inverted versions of usual distributions are capable of modeling the data with UBT shaped failure rate. In a recent study in article Tomer and Panwar (2020), authors had obtained the estimates of the InvMWD under classical and Bayesian paradigm along with basic statistical properties and applications in different scenarios. In this article, we propose a new lifetime distribution utilizing the well known M-O family proposed by Marshall and Olkin (1997) taking InvMWD as the baseline distribution. The proposed distribution is named as Marshall-Olkin inverse Maxwell distribution (M-O InvMWD). We discuss the statistical properties of the proposed distribution and show the application for lifetime experiment.

1.2. Inverse Maxwell Distribution

Singh and Srivastava (2014) proposed the InvMWD and discusses the basic properties. Recently, Tomer and Panwar (2020) reviewed the InvMWD and established its important statistical properties with its applications in many fields. A real valued random variable Y following InvMWD has probability density function (pdf) as

$$f(y;\theta) = \frac{4}{\sqrt{\pi} y^4 \theta^{\frac{3}{2}}} \exp\left\{-\frac{1}{\theta y^2}\right\}; \quad y > 0, \theta > 0. \tag{1}$$

The survival function of Y is given by

$$S(y;\theta) = \frac{2}{\sqrt{\pi}} \gamma\left(\frac{3}{2}, \frac{1}{\theta y^2}\right) = 1 - \frac{2}{\sqrt{\pi}} \Gamma\left(\frac{3}{2}, \frac{1}{\theta y^2}\right), \tag{2}$$

where, $\gamma(a,z) = \int_0^z u^{a-1} e^{-u}\, du$ and $\Gamma(a,z) = \int_z^\infty u^{a-1} e^{-u}\, du$ are lower and upper incomplete gamma functions, respectively. The hazard function of random variable Y is define as

$$h(y;\theta) = \frac{2\theta^{-\frac{3}{2}}}{y^4 \gamma\left(\frac{3}{2}, \frac{1}{\theta y^2}\right)} \exp\left\{-\frac{1}{\theta y^2}\right\}. \tag{3}$$

The hazard function of InvMWD is a UBT in nature, i.e. it increases sharply in the initial phase of time passes, and then after reaching a peak point it deepens gradually and tends to zero. This means InvMWD represents the lifetime of such individuals who have an increased chance of failing in the early age of life span after survival up to a speci?c age, the rate of failure starts decreasing as age increases.

1.3. Marshall-Olkin Family

Since Marshall and Olkin (1997) proposed a modern methodology to develop a model which is more compatible with real-life experiments than existing lifetime models. In this methodology, a real-valued parameter is added with the existing baseline distribution and constitutes a modern family distribution. This family of distribution is called the Marshall-Olkin (M-O) extended family of distribution. If X is a random variable (rv) having the cdf $F(X,\theta)$ and it is considered the baseline distribution function then the corresponding cdf of the M-O family can be defined as

$$F_{MO}(x;\alpha,\theta) = \frac{F(x;\theta)}{1-(1-\alpha)S(x;\theta)}; \quad -\infty < y < \infty, \alpha > 0, \tag{4}$$

where, α is called a tilt parameter. Here it is to be noticeable that for $\alpha = 1$, we have $F_{MO}(x,\alpha,\theta) = F(x,\theta)$, i.e. the M-O distribution will reduce to the baseline distribution.

In the last decade, many authors studied the different distributions utilizing the M-O family. Ghitany et al. (2012) proposed a two-parameter M-O extended Lindley distribution and derived some basic statistical properties. They obtained the parameter estimate of the proposed distributions and standard error by utilizing the limiting distribution of maximum likelihood (ML) estimate under randomly censored data. A new distribution, namely M-O Frechet distribution was proposed by Krishna et al. (2013) and discussed the basic statistical properties such as moments, quantiles, Renyi entropy and order statistics. They obtained the point estimates by utilizing three different iterative procedures. Finally, the M-O Frechet distribution is applied to the survival time data. Mansour et al. (2017) proposed Kumaraswamy M-O Lindley distribution having four parameters and derived the statistical properties. They also obtained the parameter estimates and two real data sets analyzed for illustration purposes. Benkhelifa (2017) proposed a new three-parameter model called the M-O extended generalized Lindley distribution. They derived various structural properties of the proposed model including expansions for the density function, ordinary moments, moment generating function, quantile function, mean deviations, Bonferroni and Lorenz curves, order statistics and their moments, Renyi entropy and reliability function. The parameter estimates of the given distribution have been obtained and for illustration purposes, the simulation study and two real data sets have been discussed. Pakungwati et al. (2018) studied the M-O extended Inverse Weibull distribution and applied it to wind speed data. The basic properties and ML estimate derived for M-O length-biased exponential distribution discussed by UL Haq et al. (2019). The distribution applied to the tensile strength of 100 carbon fibers data. Maxwell et al. (2019) proposed a new distribution M-O inverse Lomax distribution by adding a new parameter to the existing inverse Lomax distribution which facilitates modeling of various kinds of data sets. Raffiq et al. (2020) derived a new distribution from the M-O family called M-O inverted Nadarajah-Haghighi distribution. The distribution appeared to give flexible shapes of hazard and pdf that existing model. A three-parameter flexible model named M-O extended inverted Kumaraswamy is derived by Usman and UL Haq (2020). They also obtained the point estimates for model parameters. Finally, simulation and real data studies were done for illustration purposes.

In this paper, we used the M-O family and proposed a new distribution named M-O InvMWD, where InvMWD is considered as a baseline distribution. The pdf and cdf of M-O InvMWD have been established with the discussion of the nature of survival and hazard function in Section 2. In Section 3, some important statistical properties of M-O InvMWD have been derived. In Section 4, the point estimation procedure for the parameters has been discussed by using ML and maximum spacing (MS) methods. We also discuss the asymptotic confidence and boot-p method to calculate the exact confidence intervals for the parameters. In Section 5, we show the applicability of the proposed distribution for several statistical problems. The mathematical expression for record data-based problem, series, parallel and k-out-of-n systems, coherent systems, stress-strength reliability and random censoring, the expressions have been derived. Section 6 deals with the simulation study for the proposed models. The flexibility of the proposed distribution is judged based on the likelihood function, AIC and BIC criteria. The real data analysis is done to support the proposed model setup in Section 7.

2. Marshall-Olkin inverse Maxwell Distribution

Now we use the new methodology proposed by Marshall and Olkin (1997) for the construction of a flexible model. So the given section is completely dedicated to establishing the M-O InvMWD.

For this purpose, the InvMWD is considered the baseline distribution for the M-O family. A rv Y is said to follow the M-O InvMWD with cdf $F_{MO}(y;\alpha,\theta),\ y \in \mathcal{R}^+$, if it is defined as follows

$$F_{MO}(y;\alpha,\theta) = \frac{\frac{2}{\sqrt{\pi}}\Gamma\left(\frac{3}{2},\frac{1}{\theta y^2}\right)}{\left[1-\frac{2(1-\alpha)}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{1}{\theta y^2}\right)\right]}; \quad \alpha,\theta > 0. \tag{5}$$

For the incomplete gamma function, we know that $\Gamma(a,0)=\Gamma(a)$ and $\gamma(a,0)=0$ and one can directly write that $\lim_{y\to\infty} F_{MO}(y;\alpha,\theta)=1$. Henceforth, the cdf expressions of $M-O$ InvMWD in (5) represents a proper density.

As it is defined that two parameter points (α_1,θ_1) and (α_2,θ_2) are said to be observationally equivalent if $F(y;\alpha_1,\theta_1) = F(y;\alpha_2,\theta_2)\ \forall\ y \in \mathcal{R}$. Additionally, a parameter point (α^0,θ^0) in $\omega \subset \mathcal{R}^2$ is said to be identifiable if there is no other point (α,θ) in ω which is observationally equivalent.

Lemma 1. The cdf of M-O InvMWD represents a proper density i.e.

$$\lim_{y\to\infty} F_{MO}(y;\alpha,\theta) = 1.$$

Proof.

$$\begin{aligned}
\lim_{y\to\infty} F_{MO}(y;\theta,\alpha) &= \lim_{y\to\infty}\left[\frac{\frac{2}{\sqrt{\pi}}\Gamma\left(\frac{3}{2},\frac{1}{\theta y^2}\right)}{\left[1-\frac{2(1-\alpha)}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{1}{\theta y^2}\right)\right]}\right] \\
&= \frac{\frac{2}{\sqrt{\pi}}\Gamma\left(\frac{3}{2},\frac{1}{\theta\infty^2}\right)}{\left[1-\frac{2(1-\alpha)}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{1}{\theta\infty^2}\right)\right]} \\
&= \frac{\frac{2}{\sqrt{(\pi)}}\Gamma(\frac{3}{2},0)}{1-\frac{2(1-\alpha)}{\sqrt{\pi}}\gamma(\frac{3}{2},0)} \\
&= \frac{\frac{2}{\sqrt{\pi}}\times\frac{1}{2}\times\Gamma(\frac{1}{2})}{1-0} \\
&= 1,
\end{aligned}$$

From here we can see that MO-InvMWD has a proper pdf.

Lemma 2. The M-O InvMWD is identifiable i.e.

$$F_{MO}(y;\alpha_1,\theta_1) = F_{MO}(y;\alpha_2,\theta_2)\ \forall\ y \in \mathcal{R}^+ \quad \text{iff} \quad (\alpha_1,\theta_1) = (\alpha_2,\theta_2),$$

where (α,θ) in $\omega \subset \mathcal{R}^2$.

Proof. We known that InvMWD is identifiable, see Tomer and Panwar (2020), i.e. $F(y;\theta_1) = F(y;\theta_2),\ y \in \mathcal{R}^+$ iff $\theta_1 = \theta_2$. Let first assume that the two real valued parameter points (α_1,θ_1) and (α_2,θ_2) are such as $(\alpha_1,\theta_1) = (\alpha_2,\theta_2)$. So

$$\begin{aligned}
F_{MO}(y;\alpha_1,\theta_1) &= \frac{\frac{2}{\sqrt{\pi}}\Gamma\left(\frac{3}{2},\frac{1}{\theta_1 y^2}\right)}{\left[1-\frac{2(1-\alpha_1)}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{1}{\theta_1 y^2}\right)\right]} \\
&= \frac{\frac{2}{\sqrt{\pi}}\Gamma\left(\frac{3}{2},\frac{1}{\theta_2 y^2}\right)}{\left[1-\frac{2(1-\alpha_2)}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{1}{\theta_2 y^2}\right)\right]} \qquad \because F(y;\theta_1) = F(y;\theta_2) \\
&= F_{MO}(y;\alpha_2,\theta_2).
\end{aligned}$$

Similarly it can be shown that if $F_{MO}(y;\alpha_1,\theta_1) = F_{MO}(y;\alpha_2,\theta_2)$ then $(\alpha_1,\theta_1) = (\alpha_2,\theta_2)$. Hence the M-O InvMWD is identifiable.

The pdf, $f_{MO}(y;\alpha,\theta)$, of M-O InvMWD can be obtained by using cdf given in (5) such as $f_{MO}(y;\alpha,\theta) = \frac{d}{dy}F_{MO}(y;\alpha,\theta)$. So

$$f_{MO}(y;\alpha,\theta) = \frac{\frac{4\alpha}{\sqrt{\pi}}\frac{1}{y^4\theta^{\frac{3}{2}}}exp\left(-\frac{1}{\theta y^2}\right)}{\left[1-\frac{2(1-\alpha)}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{1}{\theta y^2}\right)\right]^2}; \quad y>0,\alpha,\theta>0. \tag{6}$$

For the arbitrary value of $(\alpha,\theta) = \{(0.5,1.5),(1.0,1.5),(1.8,1.5),(1.8,0.8),(0.8,0.2)\}$, the different patterns of pdf have been drawn in Figure 1. It can be seen from the pdf plot that the M-O InvMWD is a uni-modal and positively skewed distribution.

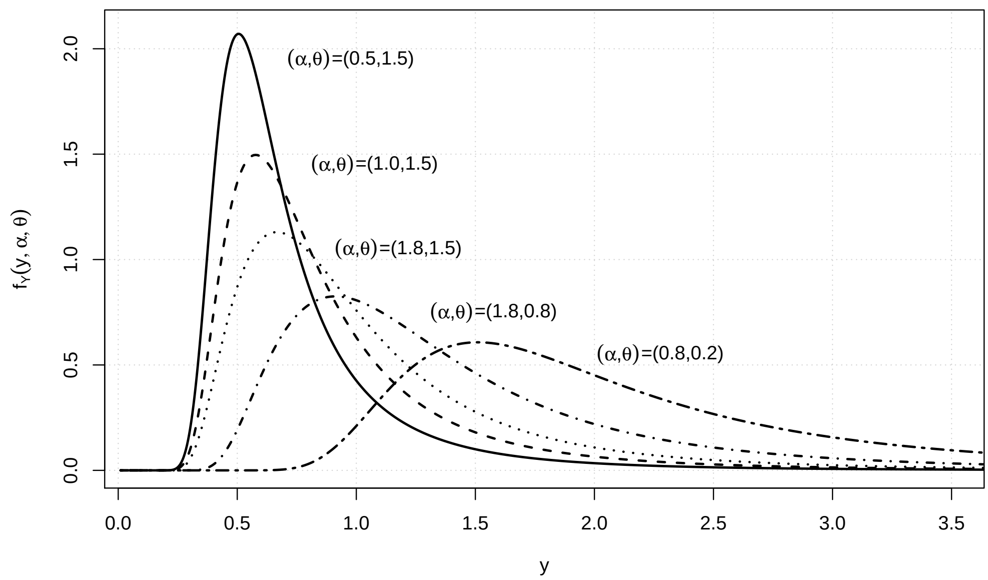


Figure 1: *The probability density function of M-O InvMWD(α,θ)*

The survival function of the M-O InvMWD is given by

$$S_{MO}(y;\alpha,\theta) = 1 - F_{MO}(y;\alpha,\theta) = \frac{\frac{2\alpha}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{1}{\theta y^2}\right)}{\left[1-\frac{2(1-\alpha)}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{1}{\theta y^2}\right)\right]} \tag{7}$$

Similarly, the hazard rate function, say $h_{MO}(y;\theta,\alpha)$, of M-O InvMWD can be defined as

$$h_{MO}(y;\alpha,\theta) = \frac{2}{\theta^{\frac{3}{2}}y^4}\frac{\left[\gamma\left(\frac{3}{2},\frac{1}{\theta y^2}\right)\right]^{-1}}{\left[1-\frac{2(1-\alpha)}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{1}{\theta y^2}\right)\right]}\exp\left\{-\frac{1}{\theta y^2}\right\} \tag{8}$$

For the arbitrary pair values of parameter (α,θ), the hazard rate functions have been sketched in Figure 2. It can be seen that the hazard rate of M-O InvMWD is upside down bathtub shape and quite flexible with respect to parameters of the distribution. The turning point of hazard rate can be obtained easily by solving the following equation

$$\frac{d}{dy}\ln h_{MO}(y;\alpha,\theta) = 0$$

$$\text{or} \quad 2\sqrt{\pi}\left[\exp\left(-\frac{1}{\theta y^2}\right) - (2\theta-1)\theta^{\frac{1}{2}}y^3\gamma\left(\frac{3}{2},\frac{1}{\theta y^2}\right)\right] + 4(1-\alpha)\theta^{\frac{1}{2}}\,(2\theta-1)\,y^3\left\{\gamma\left(\frac{3}{2},\frac{1}{\theta y^2}\right)\right\}^2 = 0.$$

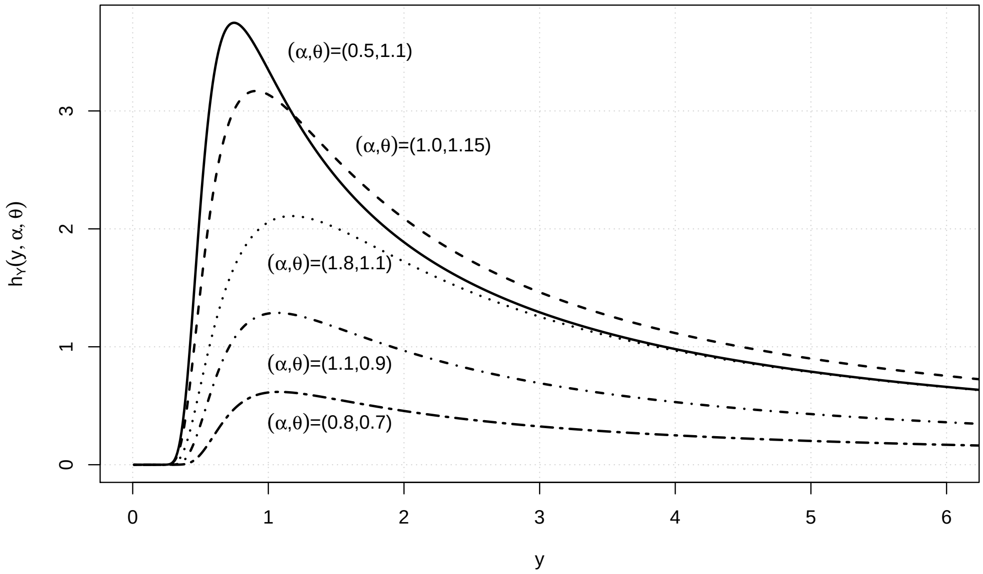


Figure 2: *The hazard rate function of M-O InvMWD(α, θ).*

By using optimization techniques, one can obtain the value of y corresponding to the turning point of the hazard rate function. For the considered choices of $(\theta, \alpha) = \{(0.5, 1.1), (1.0, 1.15), (1.8, 1.1), (1.1, 0.9), (0.8, 0.7)\}$, in Figure 2, the turning points are $0.7473, 0.9085, 1.1730, 1.0642$ and 1.0770, respectively.

3. Statistical Properties

In everyday scenes, lifetime rarely appears in unique applications. Likely researchers are discussing the statistical descriptors including the mean, the range and the variance to understand how these statistics are extracted is one goal for the study of perception. Now we discuss the statistical properties of M-O InvMWD.

3.1. Moments

Many distributions have parameters that control their respective attribute distribution. Central moments are useful because they allow us to quantify properties of distributions in ways that are location-invariant. The moment is the most important characteristic of a distribution function and it can be derived from the functional form of a distribution function. Thus, moments have a great role in defining a distribution theory. If Y follows M-O InvMWD(α, θ), then r^{th} moment about the origin of M-O InvMWD is given by

$$\begin{aligned} E(Y^r) &= \int_{-\infty}^{\infty} y^r f_{MO}(y; \alpha, \theta) dy \\ &= \frac{4\alpha}{\sqrt{\pi}\theta^{\frac{3}{2}}} \int_0^{\infty} \frac{y^{r-4} exp\left(-\frac{1}{\theta y^2}\right)}{\left[1 - \frac{2(1-\alpha)}{\sqrt{\pi}} \gamma\left(\frac{3}{2}, \frac{1}{\theta y^2}\right)\right]^2} dy \\ &= \frac{4\alpha}{\sqrt{\pi}\theta^{\frac{3}{2}}} \mathcal{G}(r; \alpha, \theta); \qquad \text{for } r \leq 2 \end{aligned} \tag{9}$$

where, $\mathcal{G}(y;\alpha,\theta) = \int_0^\infty y^{r-4}\left[1-\frac{2(1-\alpha)}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{1}{\theta y^2}\right)\right]^{-2} exp\left(-\frac{1}{\theta y^2}\right)dy$ and this integral can be computed numerically.

3.2. Quantile Function

The quantile function is one way of prescribing a probability density distribution and based on the inverse distribution function. For statistical applications, researchers are required to know key percentage points of a given distribution. The q^{th} quantile, y_q, of the M-O InvMWD can be derived as follows

$$q = F_{MO}(y_q;\alpha,\theta) = \frac{\frac{2}{\sqrt{\pi}}\Gamma\left(\frac{3}{2},\frac{1}{\theta y_q^2}\right)}{\left[1-\frac{2(1-\alpha)}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{1}{\theta y_q^2}\right)\right]}.$$

Finally, after some calculations, we get

$$y_q(\alpha,\theta) = \left[\theta\Gamma_{3/2}^{-1}\left(\frac{\sqrt{\pi}}{2}\frac{q\alpha}{1-q(1-\alpha)}\right)\right]^{-\frac{1}{2}}. \tag{10}$$

For a particular choice of $q,\ 0 < q < 1$, the corresponding quantile value of the distribution can be obtained. If we put $q = 0.5$ in (10), the median, $\bar{X}_{md}$, of the M-O InvMWD can be obtained such as

$$\bar{X}_{md} = \left[\theta\Gamma_{3/2}^{-1}\left(\frac{\sqrt{\pi}}{2}\frac{0.5\alpha}{1-0.5(1-\alpha)}\right)\right]^{-\frac{1}{2}} \tag{11}$$

The quantile function can also be utilized to generate random numbers from M-O InvMWD. If u is a random number from uniform distribution i.e. $u \sim \mathcal{U}(0,1)$ then one can obtain a random number y from M-O InvMWD as follows

$$y = \left[\theta\Gamma_{3/2}^{-1}\left(\frac{\sqrt{\pi}}{2}\frac{u\alpha}{1-u(1-\alpha)}\right)\right]^{-\frac{1}{2}}. \tag{12}$$

3.3. Measures of Skewness and Kurtosis

Since, the Skewness and Kurtosis are two important characteristics of a distribution function. But in this case, it is noticeable that the higher-order moment of the M-O InvMWD does not exist. For this purpose, to calculate the coefficient of skewness and kurtosis, we are using the approach of quantile functions. By using the expression of quantile function given in (10), we used the formula of Galton's measure of skewness and Moor's measure of kurtosis, Gilchrist (2000) are as follows

$$\mathcal{S}(\alpha,\theta) = \frac{y_{0.75} - 2y_{0.5} + y_{0.25}}{y_{0.75} - y_{0.25}} \quad \text{and} \quad \mathcal{K}(\alpha,\theta) = \frac{y_{0.875} - y_{0.625} - y_{0.325} + y_{0.125}}{y_{0.75} - y_{0.25}} \tag{13}$$

The range of Galton's measure of skewness $S(.)$ is $(-1,1)$ and a perfectly symmetrical distribution at $S(.) = 0$. A large and positive value of $S(.)$ indicates a long tail to the right, i.e. it indicates that the pdf has a positively skewed distribution and vice versa. From Table 1, it is clear that M-O InvMWD is positively skewed. Galton's skewness is varying significantly with different values of α and θ. Also, Moor's measure of kurtosis is not influenced by the (extreme) tails of the distribution.

3.4. Stochastic Ordering

In probability theory, the stochastic orderings are relating to inequalities between expectations of functions concerning the corresponding distribution. Stochastic ordering of positive continuous

Table 1: *The mean($\bar{x}$), variance($var(x)$), median($\bar{x}_{md}$), skewness and kurtosis for different sets of parameters (α,θ).*

α	θ	$\bar{x}$	$var(x)$	$\bar{x}_{md}$	$\mathcal{S}(\alpha,\theta)$	$\mathcal{K}(\alpha,\theta)$
	0.5	1.31	0.87	1.08	0.48	0.81
0.5	1.0	0.92	0.43	0.76	0.51	0.79
	1.5	0.75	0.29	0.62	0.55	0.78
	0.5	1.59	1.45	1.30	0.51	0.82
1.0	1.0	1.12	0.72	0.91	0.59	0.81
	1.5	0.92	0.48	0.75	0.65	0.80
	0.5	1.80	1.94	1.46	0.54	0.83
1.5	1.0	1.27	0.97	1.03	0.64	0.82
	1.5	1.04	0.64	0.84	0.70	0.81

random variables is used to study the comparative behavior. A random variable X is said to be smaller than a random variable Y in the

1. stochastic order $(X \leq_{st} Y)$ if $F_X(x) \geq F_Y(x)$ for all x;
2. hazard rate order $(X \leq_{hr} Y)$ if $h_X(x) \geq h_Y(x)$ for all x;
3. mean residual order $(X \leq_{mrl} Y)$ if $m_X(x) \leq m_Y(x)$ for all x;
4. likelihood ratio order $(X \leq_{lr} Y)$ if $\left(\frac{f_X(x)}{f_Y(x)}\right)$ decreases in x.

The following result is a well known result and are given below:

$$X \leq_{lr} Y \implies X \leq_{hr} Y \implies X \leq_{mrl} Y \implies X \leq_{st} Y$$

That is, we can see that the likelihood ratio ordering implies the rest of all orderings.

Theorem 1. Let X $\sim$ M-O InvMWD (α_1,θ_1) and Y $\sim$ M-O InvMWD (α_2,θ_2). If $\alpha_1 = \alpha_2 = \alpha$ and $\theta_1 \geq \theta_2$, then (Y$\leq_{lr}$X).
Proof. The likelihood ratio is given by

$$\frac{f_X(x)}{f_Y(x)} = \left(\frac{\theta_2}{\theta_1}\right)^{(3/2)} \exp\left(-\frac{\theta_2-\theta_1}{\theta_1\theta_2 x^2}\right)\left[\frac{1-\frac{2(1-\alpha_2)}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{1}{\theta_2 x^2}\right)}{1-\frac{2(1-\alpha_1)}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{1}{\theta_1 x^2}\right)}\right]$$

If $\alpha_1 = \alpha_2 = \alpha$ and $\theta_1 > \theta_2$, then $\frac{d}{dx}\frac{f_X(x)}{f_Y(x)} \leq 0$, which implies that $X \leq_{lr} Y$ and hence $X \leq_{hr} Y$, $X \leq_{mrl} Y$ and $X \leq_{st} Y$.

4. Parameter Estimation

4.1. Maximum Likelihood Estimation

The Maximum likelihood procedure is to determine the values for the unknown parameters of a model. The obtained parameter values are such that they maximize the likelihood function that the process described by the model produced the data that is actually observed. In this section, the unknown parameters of M-O InvMWD are to be estimated by using maximum likelihood estimation techniques. Let $Y_1, Y_2, \ldots, Y_n$ be a random sample of size n from the M-O InvMWD population, and the likelihood function is given by

$$L(\alpha,\theta|y) = \prod_{i=1}^{n} f(y_i;\alpha,\theta)$$
$$= \prod_{i=1}^{n} \frac{\frac{4\alpha}{\sqrt{\pi}} \frac{1}{y_i^4 \theta^{\frac{3}{2}}} exp\left(-\frac{1}{\theta y_i^2}\right)}{\left[1 - \frac{2(1-\alpha)}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{1}{\theta y_i^2}\right)\right]^2}$$

On taking the natural logarithm of $L(\alpha,\theta|y)$ both side, the log-likelihood function is given by

$$l(\alpha,\theta|y) \propto n\ln(\alpha) - \frac{3n}{2}\ln(\theta) - \frac{1}{\theta}\sum_{i=1}^{n}\frac{1}{y_i^2} - 2\sum_{i=1}^{n}\ln\left[1 - \frac{2(1-\alpha)}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{1}{\theta y_i^2}\right)\right]$$

We can use the following normal equations for obtaining the maximum likelihood estimates of θ and α. So, we have

$$-\frac{3n}{2\theta} + \frac{1}{\theta^2}\sum_{i=1}^{n}\frac{1}{y_i^2} + \frac{4(1-\alpha)}{\sqrt{\pi}\theta^{5/2}}\sum_{i=1}^{n}\frac{\frac{1}{y_i^3}\exp\left(-\frac{1}{\theta y_i^2}\right)}{\left[1 - \frac{2(1-\alpha)}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{1}{\theta y_i^2}\right)\right]} = 0 \tag{14}$$

and

$$\frac{n}{\alpha} - \frac{4}{\sqrt{\pi}}\sum_{i=1}^{n}\frac{\gamma\left(\frac{3}{2},\frac{1}{\theta y_i^2}\right)}{\left[1 - \frac{2(1-\alpha)}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{1}{\theta y_i^2}\right)\right]} = 0 \tag{15}$$

The above equations give the roots for unknown parameters of M-O InvMWD. The above nonlinear equations can be solved by using any iterative method. Let $\hat{\theta}$ and $\hat{\alpha}$ be the ML estimate of θ and α, respectively, after solving the above equations.

4.2. Maximum Spacing Estimation

The method of "maximum product of spacings" (MPS) was proposed by Cheng and Amin (1979). The method is based on the maximization of the geometric mean of spacings in the data, which are the differences between the values of the cumulative distribution function at neighboring data points. Here our main aim is to estimate the unknown parameters θ and α of the distribution function. The idea of MPS is to make the observed data as uniform as possible, based on a quantitative measure of uniformity. Let $y_1 < y_2, \ldots, < y_n$ be the complete ordered sample. Also, let us define some quantities

$$D_1 = F(y_1;\alpha,\theta) \tag{16}$$
$$D_{n+1} = 1 - F(y_n;\alpha,\theta) \tag{17}$$

And the general term we can write for the spacings given by,

$$D_i = F(y_{i:n};\alpha,\theta) - F(y_{(i-1):n};\alpha,\theta) = \frac{\frac{2}{\sqrt{\pi}}\Gamma\left(\frac{3}{2},\frac{1}{\theta y_{i:n}^2}\right)}{\left[1 - \frac{2(1-\alpha)}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{1}{\theta y_{i:n}^2}\right)\right]} - \frac{\frac{2}{\sqrt{\pi}}\Gamma\left(\frac{3}{2},\frac{1}{\theta y_{(i-1):n}^2}\right)}{\left[1 - \frac{2(1-\alpha)}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{1}{\theta y_{(i-1):n}^2}\right)\right]} \tag{18}$$

such that $\sum D_i = 1$. The MPS method choose θ which maximizes the product of spacing or other words it maximizes the geometric mean of the spacings, i.e.

$$G = \left(\prod_{i=1}^{n+1} D_i\right)^{\frac{1}{n+1}}$$

Taking the logarithm of above equation, we get

$$S = \frac{1}{n+1}\sum_{i=1}^{n+1} \ln D_i$$

Also we can write S as

$$\begin{aligned}
S =& \frac{1}{n+1}\left\{\ln D_1 + \sum_{i=2}^{n} \ln D_i + \ln D_{n+1}\right\} \\
=& \frac{1}{n+1}\ln\left(\frac{2}{\sqrt{\pi}}\right) + \frac{1}{n+1}\ln\left(\Gamma\left(\frac{3}{2},\frac{1}{\theta y_1^2}\right)\right) - \frac{1}{n+1}\ln\left[1-\frac{2(1-\alpha)}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{1}{\theta y_1^2}\right)\right] \\
&+\frac{1}{n+1}\sum_{i=2}^{n}\ln\left[\frac{\frac{2}{\sqrt{\pi}}\Gamma\left(\frac{3}{2},\frac{1}{\theta y_i^2}\right)}{\left\{1-\frac{2(1-\alpha)}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{1}{\theta y_i^2}\right)\right\}} - \frac{\frac{2}{\sqrt{\pi}}\Gamma\left(\frac{3}{2},\frac{1}{\theta y_{i-1}^2}\right)}{\left\{1-\frac{2(1-\alpha)}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{1}{\theta y_{i-1}^2}\right)\right\}}\right] \\
&+\frac{1}{n+1}\ln\left[1-\frac{\frac{2}{\sqrt{\pi}}\Gamma\left(\frac{3}{2},\frac{1}{\theta y_n^2}\right)}{\left\{1-\frac{2(1-\alpha)}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{1}{\theta y_n^2}\right)\right\}}\right]
\end{aligned}$$

After differentiating the above equation with respect to parameters and equating them to zero, we get the normal equations from which we can obtain the required estimates.

4.3. Variance-Covariance Matrix

We have obtained the variance-covariance matrix to find out the asymptotic confidence intervals. The observed information matrix for the parameter is given by inverting the second derivative matrix with respect to the given parameters. Therefore, we get the observed approximate Fisher's Information matrix which is given by

$$I(\hat{\underline{\zeta}}) = \begin{bmatrix} I_{\alpha\alpha} & I_{\alpha\theta} \\ I_{\theta\alpha} & I_{\theta\theta}\end{bmatrix}\Bigg|_{(\alpha,\theta)=(\hat{\alpha},\hat{\theta})},$$

where, $\underline{\zeta} = (\alpha, \theta)$ is the parameter vector, and

$$I_{\theta\theta} = -\frac{\partial^2 l}{\partial\theta^2} = -\frac{3n}{2\theta^2} + \frac{2}{\theta^3}\sum_{i=1}^{n}\frac{1}{y_i^2} - \frac{4(1-\alpha)}{\sqrt{\pi}}\sum_{i=1}^{n}\psi(y_i;\alpha,\theta),$$

$$I_{\theta\alpha} = -\frac{\partial^2 l}{\partial\theta\partial\alpha} = \frac{4}{\sqrt{\pi}\theta^{\frac{3}{2}}}\sum_{i=1}^{n}\frac{\left[\frac{2(1-\alpha)}{\sqrt{\pi}}\exp\left(-\frac{1}{\theta y_i^2}\right)\gamma\left(\frac{3}{2},\frac{1}{\theta y_i^2}\right) - \frac{1}{y_i^3}\exp\left(-\frac{1}{\theta y_i^2}\right)\left\{1-\frac{2(1-\alpha)}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{1}{\theta y_i^2}\right)\right\}\right]}{\left[1-\frac{2(1-\alpha)}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{1}{\theta y_i^2}\right)\right]^2},$$

$$I_{\alpha\theta} = I_{\theta\alpha},$$

$$I_{\alpha\alpha} = -\frac{\partial^2 l}{\partial\alpha^2} = \frac{n}{\alpha^2} + \frac{8}{\pi}\sum_{i=1}^{n}\frac{\left\{\gamma\left(\frac{3}{2},\frac{1}{\theta y_i^2}\right)\right\}^2}{\left[1-\frac{2(1-\alpha)}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{1}{\theta y_i^2}\right)\right]^2},$$

where, $\psi(y_i;\alpha,\theta)$ is given by

$$\begin{aligned}
\psi(y_i;\alpha,\theta) =& \frac{2(1-\alpha)}{\sqrt{\pi}}\xi(y_i,\theta)\phi(y_i;\alpha,\theta)\left[-\frac{1}{\theta^2 y_i^2} - \frac{5}{2\theta} - \xi(y_i,\theta)\right] \\
&+\left\{\gamma\left(\frac{3}{2},\frac{1}{\theta y_i^2}\right)\right\}^{-1}\xi(y_i,\theta)\phi(y_i;\alpha,\theta)\left(\frac{5}{2\theta}+\frac{1}{\theta y_i^2}\right)
\end{aligned}$$

and $\xi(y_i,\theta) = \theta^{-5/2}y_i^{-3}\exp\left(-\frac{1}{\theta y_i^2}\right)\left\{\gamma\left(\frac{3}{2},\frac{1}{\theta y_i^2}\right)\right\}^{-1}$, $\phi(y_i;\alpha,\theta) = \left[\left\{\gamma\left(\frac{3}{2},\frac{1}{\theta y_i^2}\right)\right\}^{-2} - \frac{2(1-\alpha)}{\sqrt{\pi}}\right]^{-2}$ respectively. The approximate asymptotic variance-covariance matrix for the parameters θ and α

based on MLE can be found by inverting $I(\hat{\underline{\zeta}})$ as

$$I^{-1}(\hat{\underline{\zeta}}) = \begin{bmatrix} Var(\hat{\theta}) & Cov(\hat{\theta}, \hat{\alpha}) \\ Cov(\hat{\alpha}, \hat{\theta}) & Var(\hat{\alpha}) \end{bmatrix}$$

Thus, using above equation, we get the $100(1-\gamma)\%$ confidence limits for $\hat{\theta}$ and $\hat{\alpha}$ given by $\hat{\theta} \pm z_{\frac{\gamma}{2}} SE(\hat{\theta})$ and $\hat{\alpha} \pm z_{\frac{\gamma}{2}} SE(\hat{\alpha})$ respectively, where $z_{(\frac{\gamma}{2})}$ is upper $100(\frac{\gamma}{2})^{th}$ percentile of standard normal variate.

4.4. Boot-p Method

The Bootstrap method is a resampling technique and used to estimate the statistic of the population by using the sampling technique with replacement. In some situations of distribution theory, the ACI does not provide the appropriate confidence interval for the parameters. This technique is used to construct confidence intervals, calculate the standard errors and perform hypothesis testing for several types of sample statistics. We have applied the boot-p method to calculate the confidence intervals for parameters. For more details, one can cite the article Tibshirani and Efron (1993). The necessary steps for applying the parametric bootstrap method are given below:

1. Based on the original sample $\underline{y} = (y_1, y_2, \ldots, y_n)$, obtain the MLE of $\hat{\zeta} = (\hat{\alpha}, \hat{\theta})$.
2. Under the same conditions to generate the sample, say $(x_1, x_2, \ldots, x_m)$, from the underlying distribution M-O InvMWD $(\hat{\zeta})$ with parameter $\hat{\zeta}$.
3. Compute the MLE of $\hat{\zeta}$ based on observed sample $(x_1, x_2, \ldots, x_m)$, say $\hat{\zeta}^*$.
4. Repeat step (2) and (3) B times and obtain $\hat{\zeta}_1^*, \hat{\zeta}_2^*, \ldots, \hat{\zeta}_B^*$.
5. Arrange $\hat{\zeta}_1^*, \hat{\zeta}_2^*, \ldots, \hat{\zeta}_B^*$ in ascending order.
6. A two-sided $100(1-\gamma)\%$ percentile bootstrap confidence interval of ζ, say $\left[\hat{\zeta}_L^*, \hat{\zeta}_U^*\right]$ is given by $\left[\hat{\zeta}_L^*, \hat{\zeta}_U^*\right] = \left[\hat{\zeta}_B^{*(\frac{\gamma}{2})}, \hat{\zeta}_B^{*(1-\frac{\gamma}{2})}\right]$

5. Statistical Application

In this section, we have presented the application of the proposed distribution in various real-life situations. Here we can show the record value estimation procedure for the M-O InvMWD. The reliability function is also derived when the system has arranged in k-out-of-n configuration (a special case of series and parallel system) when all components are *i.i.d.*. The stress-strength reliability is also discussed here. The estimation procedure for the parameter when the data is random censored. We discuss in brief the following the necessary procedures.

5.1. Order Statistics

Let $Y_1, Y_2, \ldots, Y_n$, be a random sample from the M-O InvMWD and $z_1, z_2, \ldots, z_n$ are the ascending order with their magnitude of observed sample. Then pdf of the j^{th} order statistic of M-O InvMWD is given by

$$f_{j:n}(z) = \frac{n!}{(j-1)!(n-j)!} f_{MO}(z) F_{MO}(z)^{(j-1)} \left[1 - F_{MO}(z)\right]^{(n-j)} \tag{19}$$

By putting j=1 and j=n in (19), we can obtain the distributions of minimum and maximum order statistics for M-O InvMWD respectively.

$$\begin{aligned} f_{1:n}(y) &= nf_{MO}(y)\left[1-F_{MO}(y)\right]^{(n-1)} \\ &= n\left(\frac{\frac{4\alpha}{\sqrt{\pi}}\frac{1}{y^4\theta^{\frac{3}{2}}}exp\left(-\frac{1}{\theta y^2}\right)}{\left[1-\frac{2(1-\alpha)}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{1}{\theta y^2}\right)\right]^2}\right)\left[\frac{\frac{2\alpha}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{1}{\theta y^2}\right)}{\left[1-\frac{2(1-\alpha)}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{1}{\theta y^2}\right)\right]}\right]^{(n-1)} \\ &= \frac{2^{n+1}\pi^{-\frac{n}{2}}n\alpha^n\theta^{-\frac{3}{2}}y^{-4}exp\left(-\frac{1}{\theta y^2}\right)\left\{\gamma\left(\frac{3}{2},\frac{1}{\theta y^2}\right)\right\}^{(n-1)}}{\left[1-\frac{2(1-\alpha)}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{1}{\theta y^2}\right)\right]^{(n+1)}} \end{aligned}$$

and

$$\begin{aligned} f_{n:n}(y) &= nf_{MO}(y)F_{MO}(y)^{(n-1)} \\ &= n\left(\frac{\frac{4\alpha}{\sqrt{\pi}}\frac{1}{y^4\theta^{\frac{3}{2}}}exp\left(-\frac{1}{\theta y^2}\right)}{\left[1-\frac{2(1-\alpha)}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{1}{\theta y^2}\right)\right]^2}\right)\left[\frac{\frac{2}{\sqrt{\pi}}\Gamma\left(\frac{3}{2},\frac{1}{\theta y^2}\right)}{\left[1-\frac{2(1-\alpha)}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{1}{\theta y^2}\right)\right]}\right]^{(n-1)} \\ &= \frac{2^{n+1}\pi^{-\frac{n}{2}}n\alpha\theta^{-\frac{3}{2}}y^{-4}exp\left(-\frac{1}{\theta y^2}\right)\left\{\Gamma\left(\frac{3}{2},\frac{1}{\theta y^2}\right)\right\}^{(n-1)}}{\left[1-\frac{2(1-\alpha)}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{1}{\theta y^2}\right)\right]^{(n+1)}} \end{aligned}$$

5.1.1 Record Estimation

In many real-life situations instead of collecting the whole data, we collect data related to record-breaking observations which are known as “records”. An observation is considered an upper (lower) record if it is greater (smaller) than all previous observations. Doostparast and Balakrishnan (2010) used the exponential record data to analyze the optimal sample size and associated optimum cost of the experiment. The record data estimation and prediction for gamma distribution derived by Sultan et al. (2008). Here, we are interested to estimate the parameters under record value setup.

Let $Y_1, Y_2, \ldots$ be a sequence of *i.i.d.* random variables having cdf and pdf given by (5) and (6). An observation Y_j is considered as an upper record value if it exceeds that of all previous observations. In other words, we say that if Y_j is an upper record value if $Y_j > Y_i$ for all $i < j$. Let $r = (r_1, r_2, \ldots, r_m)$ be the first observed m upper record values from the parent distribution with pdf given in (6). The joint pdf of given data can be constructed by method given by Arnold et al. (2011) as below

$$f(r;\theta,\alpha) = \prod_{i=1}^{m-1} h_{MO}(r_i;\alpha,\theta) f_{MO}(r_m;\alpha,\theta); \quad -\infty < r_1 < r_2 < \ldots < r_m < \infty, \tag{20}$$

where, $h_{MO}(r_i;\alpha,\theta) = \frac{f_{MO}(r_i;\alpha,\theta)}{1-F_{MO}(r_i;\alpha,\theta)}$. Thus, the likelihood function under upper record value is given by

$$\begin{aligned} L(\alpha,\theta|r) &= \bar{F}_{MO}(r_m;\alpha,\theta)\prod_{i=1}^{m}\frac{f_{MO}(r_i;\alpha,\theta)}{\bar{F}_{MO}(r_i;\alpha,\theta)} \\ &= \frac{\frac{2\alpha}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{1}{\theta r_m^2}\right)}{\left[1-\frac{2(1-\alpha)}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{1}{\theta r_m^2}\right)\right]}\prod_{i=1}^{m}\frac{2\exp\left(-\frac{1}{\theta r_i^2}\right)\left\{\gamma\left(\frac{3}{2},\frac{1}{\theta r_i^2}\right)\right\}^{-1}}{\theta^{\frac{3}{2}}r_i^4\left[1-\frac{2(1-\alpha)}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{1}{\theta r_i^2}\right)\right]} \end{aligned}$$

The log-likelihood function is given as

$$l(\alpha,\theta|r) \propto \ln(\alpha) + \ln\left\{\gamma\left(\frac{3}{2},\frac{1}{\theta r_m^2}\right)\right\} - \ln\left[1-\frac{2(1-\alpha)}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{1}{\theta r_m^2}\right)\right] - \frac{1}{\theta}\sum_{i=1}^{m}\frac{1}{r_i^2}$$
$$-\frac{3m}{2}\ln(\theta) - \sum_{i=1}^{m}\ln\left\{\gamma\left(\frac{3}{2},\frac{1}{\theta r_i^2}\right)\right\} - \sum_{i=1}^{m}\ln\left[1-\frac{2(1-\alpha)}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{1}{\theta r_i^2}\right)\right]$$

Differentiating above log-likelihood function with respect to θ and α, we get

$$\hat{\theta}^{\frac{3}{2}} = \frac{\frac{r_m^{-3}\exp\left(-\frac{1}{\theta r_m^2}\right)}{\gamma\left(\frac{3}{2},\frac{1}{\theta r_m^2}\right)} + \frac{\frac{2(1-\alpha)}{\sqrt{\pi}r_m^3}\exp\left(-\frac{1}{\theta r_m^2}\right)}{\left[1-\frac{2(1-\alpha)}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{1}{\theta r_m^2}\right)\right]} - \sum_{i=1}^{m}\frac{r_i^{-3}\exp\left(-\frac{1}{\theta r_i^2}\right)}{\gamma\left(\frac{3}{2},\frac{1}{\theta r_i^2}\right)} + \sum_{i=1}^{m}\frac{\frac{2(1-\alpha)}{\sqrt{\pi}r_i^3}\exp\left(-\frac{1}{\theta r_i^2}\right)}{\left[1-\frac{2(1-\alpha)}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{1}{\theta r_i^2}\right)\right]}}{\left[\frac{1}{\theta}\sum_{i=1}^{m}\frac{1}{r^2}-\frac{3m}{2}\right]} \tag{21}$$

$$\hat{\alpha}^{-1} = \frac{\frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{1}{\theta r_m^2}\right)}{\left[1-\frac{2(1-\alpha)}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{1}{\theta r_m^2}\right)\right]} + \sum_{i=1}^{m}\frac{\frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{1}{\theta r_i^2}\right)}{\left[1-\frac{2(1-\alpha)}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{1}{\theta r_i^2}\right)\right]} \tag{22}$$

The equations (21) and (22) are not obtained in closed form, so required estimates can be obtained by using any iterative technique.

5.2. System Lifetime Distribution

5.2.1 k out of n System

The maximum likelihood estimate for reliability of k-out-of-n systems which are composed of n *iid* components having M-O InvMWD lifetimes are discussed. The system is operational if and only if at least k of out the n components are operational at the given time. In other words, as soon as $(n-k+1)$ components fail, the system fails. For example, we have three generators at an electric power plant station in which a minimum of two must be operational at all times in order to deliver the required power. Such a system is called the 2-out-of-3 system. The system reliability estimation procedure for the failure of uncensored cases (where there are n units put on test which is terminated when all the units have failed) is discussed. In particular, the system having configuration 1-out-of-n is known as parallel while n-out-of-n is known as series systems. By using this configuration, we assume that the failure time distributions of the components are independent. Then we can define the probability of exactly k out of n components functioning as:

$$P(X=k) = \binom{n}{k}\{R_{MO}(t)\}^k\{1-R_{MO}(t)\}^{(n-k)}; \quad for \quad k=0,1,2,\ldots,n,$$

where $R_{MO}(t)$ is defined as the reliability function of each component having M-O InvMWD. Also, the system failure time density is given as

$$f_s(t) = \frac{n!}{(n-k)!(k-1)!}\{R_{MO}(t)\}^k\{1-R_{MO}(t)\}^{(n-k)}f_{MO}(t); \quad for \quad k=0,1,2,...,n.$$

Thus, the reliability of k-out-of-n system, simply, is

$$R_s(t) = \sum_{i=k}^{n}\binom{n}{i}\{R_{MO}(t)\}^i\{1-R_{MO}(t)\}^{(n-i)} \tag{23}$$

5.2.2 Series and Parallel System

The series and parallel systems can be viewed as special cases of k-out-of-n systems. Suppose we have n such systems that each have k-components in series or parallel attachment. Let Y_j, $j \in \{1,2,...,k\}$, denote the sequence of failure times of all components in a system. We assume that the sequence is composed of independent but not identical from M-O InvMWD. In this case, at a failed component, there are two observed quantities are recorded say (T, δ), where $T = min(y_1, y_2, \ldots y_n)$ for the series system and for parallel system $T = max(y_1, y_2, \ldots y_n)$ with $\delta = j$ if $T = Y_j$ for $j = 1, 2, \ldots, k$. The δ quantity can be viewed as an indicator function of the component that caused the system failure. Consider a sample of size n be independent and identically distributed systems (either all series or all parallel systems). The observations are represented by $(T, \delta) = \{(T_i, \delta_i) : i = 1, ..., n\}$. Then, the reliability of the j^{th} component is given by $R_j(t) = P(X_j > t), j = 1, 2, ..., k$. Let us define the random variables $Y_j's$ for component's reliability with M-O InvMWD distributions parameterized by $\zeta_j = (\alpha_j, \theta_j)$, that is,

$$P(Y_j > y|\zeta_j) = R(y|\zeta_j) = \frac{\frac{2\alpha_j}{\sqrt{\pi}}\gamma\left(\frac{3}{2}, \frac{1}{\theta_j y^2}\right)}{\left[1 - \frac{2(1-\alpha_j)}{\sqrt{\pi}}\gamma\left(\frac{3}{2}, \frac{1}{\theta_j y^2}\right)\right]} \tag{24}$$

When all components of a system are connected in series then

$$R(t) = \prod_{j=1}^{k} \frac{\frac{2\alpha_j}{\sqrt{\pi}}\gamma\left(\frac{3}{2}, \frac{1}{\theta_j y^2}\right)}{\left[1 - \frac{2(1-\alpha_j)}{\sqrt{\pi}}\gamma\left(\frac{3}{2}, \frac{1}{\theta_j y^2}\right)\right]}$$

and system reliability when component put in parallel setup

$$R(t) = 1 - \prod_{j=1}^{k} \frac{\frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2}, \frac{1}{\theta_j y^2}\right)}{\left[1 - \frac{2(1-\alpha_j)}{\sqrt{\pi}}\gamma\left(\frac{3}{2}, \frac{1}{\theta_j y^2}\right)\right]}$$

By using the component reliability given in (24), we can obtain the values for the reliability of series and parallel systems.

5.3. Stress-Strength Reliability

In reliability theory, the stress-strength reliability is denoted by quantity $R = P(W > V)$, where W and V denotes the strength and stress of the system. In this regard, when the stress is greater than strength, the system will fail. The applicability of probability R is that it can be used to compare the two random variables encountered in various applied fields so the estimation of R is a great concern for a long time. Kundu and Gupta (2006) discussed the point and interval estimation procedure for stress-strength Weibull model under classical and Bayesian approaches. Chaudhary et al. (2017) analyzed the stress-strength reliability estimates when stress and strength both follow Maxwell lifetime. So for the proposed model, we have calculated expressions for stress-strength reliability estimates. Consider $W \sim$ M-O InvMWD(α,θ) and $V \sim$ M-O InvMWD(β,θ) and W and V are independently distributed. The expression for R comes out to be as:

$$\begin{aligned}
R = P(W > V) &= \int_0^\infty f_{MO}(w; \alpha, \theta) F_{MO}(w; \beta, \theta) dw \\
&= \int_0^\infty \left(\frac{\frac{4\alpha}{\sqrt{\pi}} \frac{1}{w^4 \theta^{\frac{3}{2}}} exp\left(-\frac{1}{\theta w^2}\right)}{\left[1 - \frac{2(1-\alpha)}{\sqrt{\pi}}\gamma\left(\frac{3}{2}, \frac{1}{\theta w^2}\right)\right]^2} \right) \times \left(\frac{\frac{2}{\sqrt{\pi}}\Gamma\left(\frac{3}{2}, \frac{1}{\theta w^2}\right)}{\left[1 - \frac{2(1-\beta)}{\sqrt{\pi}}\gamma\left(\frac{3}{2}, \frac{1}{\theta w^2}\right)\right]} \right) dw \\
&= \frac{8\alpha}{\pi \theta^{\frac{3}{2}}} \int_0^\infty \frac{\Gamma\left(\frac{3}{2}, \frac{1}{\theta w^2}\right) \exp\left(-\frac{1}{\theta w^2}\right)}{\left[1 - \frac{2(1-\alpha)}{\sqrt{\pi}}\gamma\left(\frac{3}{2}, \frac{1}{\theta w^2}\right)\right]^2 \left[1 - \frac{2(1-\beta)}{\sqrt{\pi}}\gamma\left(\frac{3}{2}, \frac{1}{\theta w^2}\right)\right]} dw \\
&= \frac{8\alpha}{\pi \theta^{\frac{3}{2}}} I(r; \alpha, \beta, \theta).
\end{aligned} \tag{25}$$

The quantity $I(r;\alpha,\beta,\theta)=\int_0^\infty \frac{\Gamma\left(\frac{3}{2},\frac{1}{\theta r^2}\right)\exp\left(-\frac{1}{\theta r^2}\right)}{\left[1-\frac{2(1-\alpha)}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{1}{\theta r^2}\right)\right]^2\left[1-\frac{2(1-\beta)}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{1}{\theta r^2}\right)\right]}dr$ can be calculated with help of statistical software for given values of parameters and thus numerical value of R can be also be obtained.

5.4. Random Censored Data

In real-life experiments, we are unable to observe the complete failure of the sample due to the insufficient necessary resource. Since, conducting life experiments is time taking and more expensive which demands a large amount of money, labor, and time. For reducing the cost and time of the experiments, various kinds of censoring schemes are developed in the statistical literature. In this case, when the researchers are interested in analyzing the partial part of the sample, say censored data. A special type of censoring scheme known as random censoring occurs in literature when the item is lost or removed randomly from the experiment before its failure under the study. A sample is randomly censored when the experimental unit and censoring time points are random and independent of each other outcomes. In real-life situations, especially in clinical trials, the patients do not complete the course of treatment and they leave due to several factors before the termination point of the experiment. Nandi and Dewan (2010) discussed the parameter estimation procedure for bivariate Weibull distribution under random censoring. In the article authors Kumar and Garg (2014), the authors obtained ML and Bayes estimates of parameters for generalized Inverse Rayleigh distribution under random censoring. Krishna et al. (2015) discussed the classical and Bayesian estimation procedure for the Maxwell distribution random censored sample. Kumar and Kumar (2019) obtained the ML and Bayes estimates of Inverse Weibull distribution parameters under random censoring. Here our interest lies in dealing with M-O InvMWD distribution under the random censoring setup.

Suppose n items are put on test with their lifetimes as X_1, X_2,..., X_n which are *iid* random variables with cdf and pdf given in (5) and (6). Also, let T_1, T_2,..., T_n be the random censoring times. Let pdf and cdf of $T_i's$ be $f_T(t)$ and $F_T(t)$, respectively. Further, let $X_i's$ and $T_i's$ be mutually independent. Note that, between $X_i's$ and $T_i's$, only one will actually be observed. Let the actual observation time be $Y_i=min(X_i,T_i); i=1,2,\ldots,n$. Also, define the indicator variable δ_i as

$$\delta_i=\begin{cases}1, & \text{if } X_i\le T_i\\ 0, & \text{if } X_i>T_i\end{cases}$$

Note that δ_i is a random variable with Bernoulli probability mass function given by

$$P[\delta_i=j]=p^j(1-p)^{(1-j)};\quad j=0,1\quad \&\quad p=P[X_i\le T_i]$$

Since $X_i's$ and $T_i's$ are independent, so will be $Y_i's$ and $\delta_i's$. Now, it is simple to show that the joint probability density function of Y and Δ is

$$f_{Y,\Delta}(y,\delta)=\{f_X(y)\,(1-F_T(y))\}^{\delta}\,\{f_T(y)\,(1-F_X(y))\}^{1-\delta};y,\lambda,\theta\ge 0,\delta=0,1.$$

Taking the distribution function of X & T as M-O InvMWD(α,θ_1) and M-O InvMWD(α,θ_2), respectively, and putting the expressions for their pdf & cdf in above equation, we get

$$\begin{aligned}f_{Y,\Delta}(y,\delta;\alpha,\theta_1,\theta_2)=&\frac{8\alpha^2}{\pi y^4}\theta_1^{-\frac{3\delta}{2}}\theta_2^{-\frac{3(1-\delta)}{2}}\exp\left\{-\left(\frac{\delta}{\theta_1 y^2}+\frac{(1-\delta)}{\theta_2 y^2}\right)\right\}\left\{\gamma\left(\frac{3}{2},\frac{1}{\theta_2 y^2}\right)\right\}^{\delta}\\&\left\{\gamma\left(\frac{3}{2},\frac{1}{\theta_1 y^2}\right)\right\}^{(1-\delta)}\left[1-\frac{2(1-\alpha)}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{1}{\theta_1 y^2}\right)\right]^{-(\delta+1)}\\&\left[1-\frac{2(1-\alpha)}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{1}{\theta_2 y^2}\right)\right]^{-(\delta-2)}\end{aligned}$$

Let $(y,\delta) = \{(y_1,\delta_1),(y_2,\delta_2),\ldots,(y_n,\delta_n)\}$ be a randomly censored sample from the above model. Now, the likelihood function can be given by

$$L(\alpha,\theta_1,\theta_2|y,\delta) \propto \alpha^{2n}\theta_1^{-\frac{3m}{2}}\theta_2^{-\frac{3(n-m)}{2}}\prod_{i=1}^{n}\frac{1}{y^4}\exp\left\{-\left(\sum_{i=1}^{n}\frac{\delta_i}{\theta_1 y_i^2}+\sum_{i=1}^{n}\frac{(1-\delta_i)}{\theta_2 y_i^2}\right)\right\}$$
$$\prod_{i=1}^{n}\left\{\gamma\left(\frac{3}{2},\frac{1}{\theta_2 y_i^2}\right)\right\}^{\delta_i}\prod_{i=1}^{n}\left\{\gamma\left(\frac{3}{2},\frac{1}{\theta_1 y_i^2}\right)\right\}^{(1-\delta_i)}$$
$$\prod_{i=1}^{n}\left[1-\frac{2(1-\alpha)}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{1}{\theta_1 y_i^2}\right)\right]^{-(\delta_i+1)}\prod_{i=1}^{n}\left[1-\frac{2(1-\alpha)}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{1}{\theta_2 y_i^2}\right)\right]^{-(\delta_i-2)}$$

Taking natural logarithm above expression, we get the log-likelihood function as

$$l(\alpha,\theta_1,\theta_2) \propto 2n\ln(\alpha)-\frac{3m}{2}\ln(\theta_1)-\frac{3(n-m)}{2}\ln(\theta_2)-\frac{1}{\theta_1}\sum_{i=1}^{n}\frac{\delta_i}{y_i^2}-\frac{1}{\theta_2}\sum_{i=1}^{n}\frac{(1-\delta_i)}{y_i^2}$$
$$+\sum_{i=1}^{n}\delta_i\ln\left\{\gamma\left(\frac{3}{2},\frac{1}{\theta_2 y_i^2}\right)\right\}+\sum_{i=1}^{n}(1-\delta_i)\ln\left\{\gamma\left(\frac{3}{2},\frac{1}{\theta_1 y_i^2}\right)\right\}$$
$$-\sum_{i=1}^{n}(\delta_i+1)\ln\left[1-\frac{2(1-\alpha)}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{1}{\theta_1 y_i^2}\right)\right]-\sum_{i=1}^{n}(\delta_i-2)\ln\left[1-\frac{2(1-\alpha)}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{1}{\theta_2 y_i^2}\right)\right]$$

Here, quantity $m=\sum_{i=1}^{n}\delta_i$ denotes the number of observed failures. On differentiating the log-likelihood equation with respect to θ_1, θ_2 and α, we get the normal equations for the parameters

$$\hat{\theta}_1 = \frac{\sum_{i=1}^{n}\frac{(1-\delta_i)y_i^{-3}\exp\left(-\frac{1}{\theta_1 y_i^2}\right)}{\gamma\left(\frac{3}{2},\frac{1}{\theta_1 y_i^2}\right)}-\frac{2(1-\alpha)}{\sqrt{\pi}}\sum_{i=1}^{n}\frac{(1+\delta_i)y_i^{-3}\exp\left(-\frac{1}{\theta_1 y_i^2}\right)}{\left[1-\frac{2(1-\alpha)}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{1}{\theta_1 y_i^2}\right)\right]}}{\left[-\frac{3m}{2}+\frac{1}{\theta_1}\sum_{i=1}^{n}\frac{\delta_i}{y_i^2}\right]} \tag{26}$$

$$\hat{\theta}_2 = \frac{\sum_{i=1}^{n}\frac{\delta_i y_i^{-3}\exp\left(-\frac{1}{\theta_2 y_i^2}\right)}{\gamma\left(\frac{3}{2},\frac{1}{\theta_2 y_i^2}\right)}-\frac{2(1-\alpha)}{\sqrt{\pi}}\sum_{i=1}^{n}\frac{(\delta_i-2)y_i^{-3}\exp\left(-\frac{1}{\theta_2 y_i^2}\right)}{\left[1-\frac{2(1-\alpha)}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{1}{\theta_2 y_i^2}\right)\right]}}{\left[-\frac{3(n-m)}{2}+\frac{1}{\theta_2}\sum_{i=1}^{n}\frac{(1-\delta_i)}{y_i^2}\right]} \tag{27}$$

$$\hat{\alpha}^{-1} = \frac{1}{n\sqrt{\pi}}\left[\sum_{i=1}^{n}\frac{(1+\delta_i)\gamma\left(\frac{3}{2},\frac{1}{\theta_1 y_i^2}\right)}{\left[1-\frac{2(1-\alpha)}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{1}{\theta_1 y_i^2}\right)\right]}+\sum_{i=1}^{n}\frac{(\delta_i-2)\gamma\left(\frac{3}{2},\frac{1}{\theta_2 y_i^2}\right)}{\left[1-\frac{2(1-\alpha)}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{1}{\theta_2 y_i^2}\right)\right]}\right] \tag{28}$$

These equations are obtained after some simplifications, but we are not able to get closed form, so for finding roots of above equations we have to apply some iterative methods.

6. Simulation Study

In this section, we give illustrations based on the simulation study. We generate the random sample for the assumption the failure observation follows M-O InvMWD with corresponding parameters α and θ. A simulation study has been performed by setting different initial values of parameters α and θ. In this regard, we generate the random sample by using the inverse cdf method. The necessary steps for generating the random sample from M-O InvMWD are given below:

1. Set the initial values of n, α and θ.
2. Generate a standard uniform number, $u \sim U(0,1)$.

3. By using the quantile formula in (12), we obtain the value of the random variable y.
4. Repeat (2) to (3) n times to get a sample $(y_1, y_2, \ldots, y_n)$ of size n from M-O InvMWD(α,θ).

6.1. Simulation Analysis (For section - 4.1 to 4.4)

The simulated sample is generated by using the necessary steps described above and using the initial values of parameters $\alpha = 0.50, 1.20$ and $\theta = 1.50, 0.85$. The simulation study is done for different sample sizes $n = 40$, 50 and 60 for 5000 iterations. Here, we discuss two methods for obtaining the point and interval estimates for the unknown parameters. For this, two estimation procedures i.e. (i) maximum likelihood method and (ii) maximum product spacing and for interval estimation (i) asymptotic confidence interval and (ii) boot-p methods are used. We present the estimated values of unknown parameters along with their mean square error (MSE) and absolute bias (AB) in Tables 2 and 3. The MLE's are consistent as we see the value of MSE and AB decrease with increase in sample size. The interval estimates of parameters under asymptotic normality assumption and boot-p method are given in Tables 4 and 5.

Table 2: *Average values of MSE and Absolute Bias under varying sample sizes* 40, 50 *and* 60 *for simulated data under ML and MPS estimates and parameter values* $\alpha = 0.50$ *and* $\theta = 1.50$.

		ML		MPS	
n		$\hat{\theta}$	$\hat{\alpha}$	$\hat{\theta}^*$	$\hat{\alpha}^*$
40	Estimate	1.4999	0.5401	1.3293	0.4197
	MSE	0.1334	0.1215	0.1224	0.0748
	AB	0.2861	0.2423	0.2903	0.2220
50	Estimate	1.4845	0.5327	1.3832	0.4513
	MSE	0.1041	0.1012	0.1032	0.0730
	AB	0.2463	0.2244	0.2586	0.2044
60	Estimate	1.4944	0.5187	1.3841	0.4361
	MSE	0.0906	0.0699	0.0832	0.0527
	AB	0.2340	0.1944	0.2339	0.1773

Table 3: *Average values of MSE and Absolute Bias under varying sample sizes* 40, 50 *and* 60 *for simulated data under ML and MPS estimates and parameter values* $\alpha = 1.20$ *and* $\theta = 0.85$.

		ML		MPS	
n		$\hat{\theta}$	$\hat{\alpha}$	$\hat{\theta}^*$	$\hat{\alpha}^*$
40	Estimate	0.8527	1.3164	0.7207	0.9671
	MSE	0.0861	1.0600	0.0623	0.5011
	AB	0.2132	0.6542	0.2091	0.5708
50	Estimate	0.8520	1.2998	0.7407	0.9892
	MSE	0.0699	0.8172	0.0522	0.4070
	AB	0.1941	0.5824	0.1921	0.5166
60	Estimate	0.8518	1.2632	0.7660	1.0381
	MSE	0.0513	0.5179	0.0481	0.3849
	AB	0.1788	0.5164	0.1794	0.4975

It can be observed from the simulation Tables 2, 3, 4 and 5 that

1. The average values of MSE and AB decreases as the sample size increases for both parameters.

Table 4: *Interval estimates under ML and Boot-p methods along with their average lengths, shape and coverage probabilities (CP) for varying sample sizes* 40, 50 *and* 60 *and parameter values* $\alpha = 0.50$ *and* $\theta = 1.50$.

		ACIs		Boot-p	
n		$\hat{\theta}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\alpha}$
	Length	1.5123	1.6333	1.5149	1.5356
40	Shape	1.0000	1.0000	1.5180	2.6335
	CP	0.94	0.95	0.92	0.93
	Length	1.3146	1.3633	1.3044	1.2420
50	Shape	1.0000	1.0000	1.4427	2.3569
	CP	0.93	0.94	0.95	0.96
60	Length	1.1910	1.1622	1.1783	1.0975
	Shape	1.0000	1.0000	1.4033	2.2023
	CP	0.93	0.95	0.92	0.93

Table 5: *Interval estimates under ML and Boot-p methods along with their average lengths, shape and coverage probabilities (CP) for varying sample sizes* 40, 50 *and* 60 *and parameter values* $\alpha = 1.20$ *and* $\theta = 0.85$.

		ACIs		Boot-p	
n		$\hat{\theta}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\alpha}$
	Length	1.2360	5.1358	1.1441	4.1830
40	Shape	1.0000	1.0000	1.7499	3.0104
	CP	0.93	0.95	0.90	0.93
	Length	1.0681	4.1173	1.0567	3.8256
50	Shape	1.0000	1.0000	1.6795	2.7408
	CP	0.92	0.94	0.92	0.93
60	Length	0.9392	3.3435	0.9315	3.2209
	Shape	1.0000	1.0000	1.6105	2.5208
	CP	0.92	0.95	0.93	0.95

2. ML and MPS methods perform almost equally well.
3. The average values of lengths decreases as sample size increases.
4. The value of shape under the boot-p method indicates that the proposed distribution is positively skewed.
5. The ACIs and boot-p confidence intervals give almost equal performances.
6. High coverage probability values indicate the proportion of the true value to lie in interval estimate is good.

7. Real Data Study

In this section, two real data sets have been analyzed to show the applicability of the proposed model in real life situations. The first data is based on measurements of glycosaminoglycans (GAG) concentration in urine and second data is the relief time of patients receiving an analgesic.

Data 1: GAGurine Data

This GAGurine [glycosaminoglycans (GAG) concentration in urine] data discussed by Ripley (2002) and explain the concentration of GAG(in units of milligrams per millimole creatinine) in the urine of children aged up to 17 years. Analysis of such data may be helpful for pediatricians to diagnose whether the GAG concentration is normal for a child or not. Here we are considering the results for 40 children of age between 12 to 17 years given as: 5.8, 5.4, 5.7, 3.1, 6.4, 7.0, 5.7, 3.9, 9.4, 4.4, 5.0, 15.9, 3.7, 9.1, 4.7, 3.6, 3.7, 4.1, 7.9, 3.3, 6.6, 1.9, 3.0, 5.7, 3.2, 3.8, 5.3, 3.2, 4.2, 6.0, 9.7, 3.4, 3.2, 2.5, 2.0, 4.0, 4.3, 2.8, 2.2, 4.7.

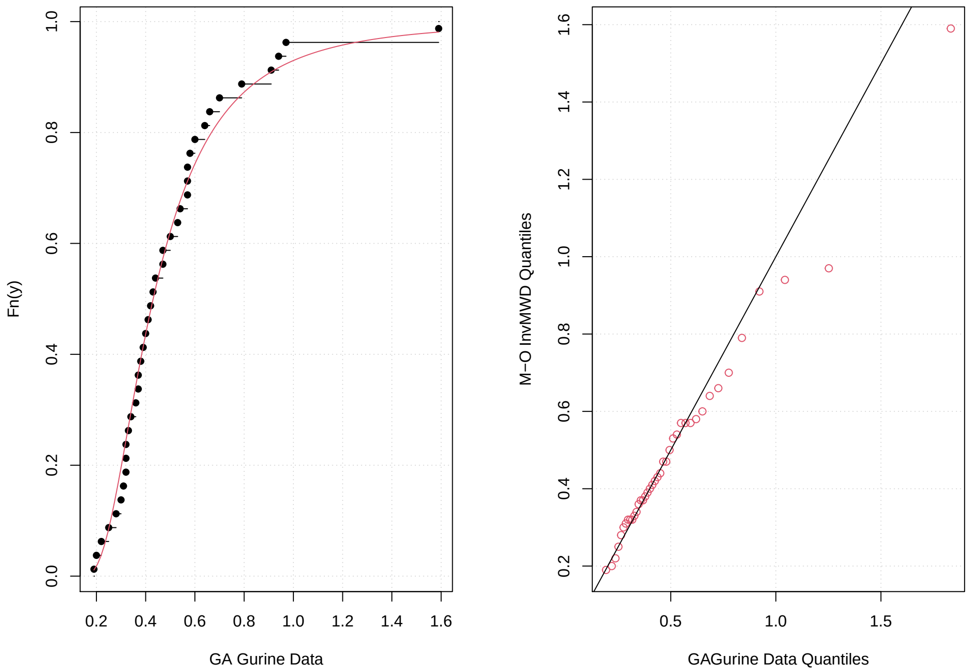
Table 6: *Descriptive Statistics for GAG Data.*

Median	Mean	Variance	Skewness	Kurtosis	Min	Max
4.25	4.99	6.90	2.07	8.74	1.90	15.90

In order, we used Anderson Darling (AD) test to show the fitting of this data to M-O InvMWD. The observed Anderson darling test statistic comes out to be 0.2458 along with high p value. For the estimated parameter, the compatibility of M-O InvMWD to GAGurine data is shown graphically by empirical cdf (ECDF) plot and Q-Q plots in Figure 3 which shows well data fitting. Here for the goodness of fit we have applied three criteria: Negative Log-Likelihood, Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC). The distribution for which AIC and BIC values are smallest is known as the best model. The AIC and BIC are the criterion based on the likelihood and explain the information lost while fitting the model to the given data. We can see from the Table 7 that our model performs better than all other distributions.

Table 7: *Model comparison based on Negative log-likelihood (LL), AIC and BIC for GA Gurine data.*

Sr. No.	Model	Negative LL	AIC	BIC
1.	**M-O InvMWD**	**84.15**	**172.30**	**172.63**
2.	Log-Normal	84.38	172.77	176.15
3 .	Inverse Weibull	84.74	173.49	176.87
4.	Inverse Rayleigh	86.35	174.72	176.41
5.	Maxwell	90.21	182.44	184.13
6.	Inverse Lindley	103.25	208.51	210.20
7.	Inverse Exponential	103.76	209.52	211.21



(a) *ECDF plot for GA Gurine data.*

(b) *QQ plot for GA Gurine data.*

Figure 3: *ECDF and QQ plots for GA Gurine data.*

Data 2: Relief time Data

This data set represents the relief times (in minutes) of 20 patients receiving an analgesic reported by article Gross and Clark (1975). This data-set is studied and fitted with different models by authors Fayomi (2019). The sample values are given as follows: 1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1, 1.8, 1.5, 1.2, 1.4, 3.0, 1.7, 2.3, 1.6, 2.0.

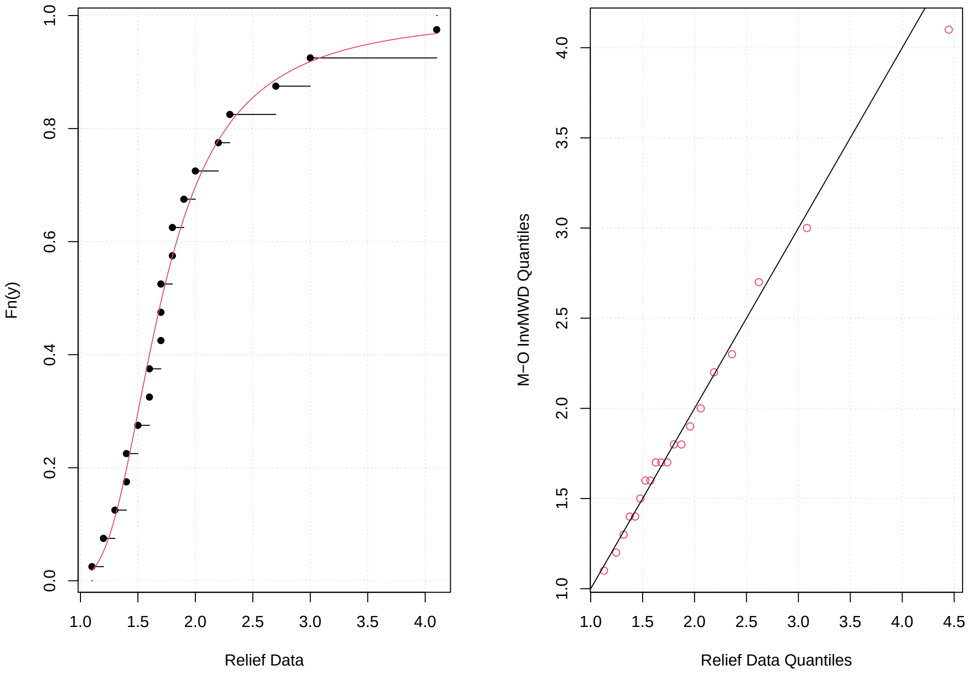
Table 8: *Descriptive Statistics for relief time Data.*

Median	Mean	Variance	Skewness	Kurtosis	Min	Max
1.70	1.90	0.50	1.72	5.92	1.10	4.10

The observed Anderson Darling test statistics comes out to be 0.1438 along with a high p value. For the estimated parameter the compatibility of M-O InvMWD to relief time data is shown graphically by QQ plot and ECDF plots in Figure 4 which shows well data fitting. We can see from the Table 9, that our model performs better than all other considered models.

Table 9: *Model comparison based on Negative log-likelihood (LL), AIC and BIC for relief data.*

Sr. No.	Model	Negative LL	AIC	BIC
1.	**M-O InvMWD**	**15.52**	**35.04**	**37.03**
2.	Log-Normal	16.77	37.54	39.53
3.	Maxwell	20.18	42.36	43.35
5.	Inverse Rayleigh	21.18	44.36	45.36
6.	Inverse Lindley	31.76	65.51	66.51
7.	Inverse Maxwell	32.35	66.69	67.68
8.	Inverse Exponential	32.67	67.34	68.33



(a) *ECDF plot for relief time data.*

(b) *QQ plot for relief time data.*

Figure 4: *ECDF and QQ plots for patient relief data.*

Table 10: *Point and interval estimates for GA Gurine data under ML and Boot-p methods.*

	$\hat{\theta}$	$\hat{\alpha}$
ML	0.0523	1.2698
MPS	0.0422	1.2098
ACIs	(0.0378,0.0668)	(0.5725,1.9672)
Boot-p	(0.0241,0.0885)	(0.2623,3.4904)

8. Results and Findings

In the simulation study, we see that MSE and AB decrease as the sample size increases. This indicates that the parameters are consistent. Also, the estimated confidence interval obtained by

Table 11: *Point and interval estimates for relief data under ML and Boot-p methods.*

	$\hat{\theta}$	$\hat{\alpha}$
ML	0.1135	0.1232
MPS	0.0955	0.1032
ACIs	(0.0901,0.1369)	(0.0366,0.2098)
Boot-p	(0.0644,0.1739)	(0.0132,0.4447)

asymptotic confidence and boot-p methods contains the true parameters. Two real data sets were analyzed in support of the proposed model and provide a good result. We observed satisfactory results for both simulation and real data.

9. Concluding Remarks

In this paper, a new life-time distribution named M-O InvMWD is proposed by generalizing InvMWD. The nature of hazard of distribution is uni-modal which can be applicable in real life situations in which rate of failure is higher in initial phases and with the passes of time it reduces attaining a maximum point. Basic statistical characteristics for the related distribution are derived and parameters are estimated by using the maximum likelihood estimation and maximum product spacing methods. For calculation of asymptotic confidence intervals, the observed information matrix is derived. Also, the boot-p method is also discussed for obtaining interval estimates. The simulation and real data study are shown for the applicability of the proposed model. We have presented the applications of the proposed distribution under different real life situations. One can extend the work in desired directions based on their choice and availability of real data situations.

Disclosure statement

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References

[1] Arnold, B. C., Balakrishnan, N., and Nagaraja, H. N. (2011). *Records*, volume 768. John Wiley & Sons.

[2] Benkhelifa, L. (2017). The Marshall-Olkin extended generalized Lindley distribution: Properties and applications. *Communications in Statistics-Simulation and Computation*, 46(10):8306–8330.

[3] Bennett, S. (1983). Log-logistic regression models for survival data. *Journal of the Royal Statistical Society: Series C (Applied Statistics)*, 32(2):165–171.

[4] Chaudhary, S., Kumar, J., and Tomer, S. K. (2017). Estimation of $P[Y < X]$ for Maxwell distribution. *Journal of Statistics and Management Systems*, 20(3):467–481.

[5] Cheng, R. and Amin, N. (1979). Maximum product-of-spacings estimation with applications to the Lognormal distribution. *Math report*, 79.

[6] Doostparast, M. and Balakrishnan, N. (2010). Optimal sample size for record data and associated cost analysis for Exponential distribution. *Journal of Statistical Computation and Simulation*, 80(12):1389–1401.

[7] Efron, B. (1988). Bootstrap confidence intervals: Good or bad? *Psychological Bulletin*, 104(2):293.

[8] Fayomi, A. (2019). The Odd Frechet Inverse Weibull distribution with application. *Journal of Nonlinear Sciences and Applications*, 12:165–172.

[9] Folks, J. L. and Chhikara, R. S. (1978). The inverse Gaussian distribution and its statistical application- A review. *Journal of the Royal Statistical Society: Series B (Methodological)*, 40(3):263–275.

[10] Ghitany, M., Al-Mutairi, D., Al-Awadhi, F., and Al-Burais, M. (2012). Marshall-Olkin extended Lindley distribution and its application. *International Journal of Applied Mathematics*, 25(5):709–721.

[11] Gilchrist, W. G. (2000). *Statistical modeling with quantile functions*. Chapman and Hall/CRC.

[12] Gross, A. J. and Clark, V. A. (1975). *Survival distributions: Reliability applications in the biomedical sciences*. University microfilm international.

[13] Krishna, E., Jose, K., Alice, T., and Ristić, M. M. (2013). The marshall-olkin Fréchet distribution. *Communications in Statistics-Theory and Methods*, 42(22):4091–4107.

[14] Krishna, H., Vivekanand, and Kumar, K. (2015). Estimation in Maxwell distribution with randomly censored data. *Journal of Statistical Computation and Simulation*, 85(17):3560–3578.

[15] Kumar, K. and Garg, R. (2014). Estimation of the parameters of randomly censored generalized Inverted Rayleigh distribution. *International Journal of Agricultural and Statistical Sciences*, 10:147–155.

[16] Kumar, K. and Kumar, I. (2019). Estimation in Inverse Weibull distribution based on randomly censored data. *Statistica*, 79(1):47–74.

[17] Kundu, D. and Gupta, R. D. (2006). Estimation of $P[Y < X]$ for Weibull distributions. *IEEE Trans. Reliability*, 55(2):270–280.

[18] Kuş, C. (2007). A new lifetime distribution. *Computational Statistics & Data Analysis*, 51(9):4497–4509.

[19] Langlands, A. O., Pocock, S. J., Kerr, G. R., and Gore, S. M. (1979). Long-term survival of patients with breast cancer: A study of the curability of the disease. *Br med J*, 2(6200):1247–1251.

[20] Mansour, M. M., Nofal, Z. M., Noori, A. N., and El Gebaly, Y. M. (2017). Kumaraswamy Marshall-Olkin Lindley distribution: Properties and applications. *International Journal of Business and Statistical Analysis*, 4(01):21–28.

[21] Marshall, A. W. and Olkin, I. (1997). A new method for adding a parameter to a family of distributions with application to the Exponential and Weibull families. *Biometrika*, 84(3):641–652.

[22] Maxwell, O., Chukwu, A. U., Oyamakin, O. S., and Khaleel, M. A. (2019). The Marshall-Olkin Inverse Lomax distribution (MO-ILD) with application on cancer stem cell. *Journal of Advances in Mathematics and Computer Science*, pages 1–12.

[23] Nandi, S. and Dewan, I. (2010). An EM algorithm for estimating the parameters of bivariate Weibull distribution under random censoring. *Computational Statistics & Data Analysis*, 54(6):1559–1569.

[24] Pakungwati, R., Widyaningsih, Y., and Lestari, D. (2018). Marshall-Olkin extended inverse Weibull distribution and its application. In *Journal of Physics: Conference Series*, volume 1108, page 012114. IOP Publishing.

[25] Proschan, F. (1963). Theoretical explanation of observed decreasing failure rate. *Technometrics*, 5(3):375–383.

[26] Raffiq, G., Dar, I. S., Haq, M. A. U., Ramos, E., et al. (2020). The Marshall-Olkin Inverted Nadarajah-Haghighi distribution: Estimation and applications. *Annals of Data Science*, pages 1–16.

[27] Ripley, B. D. (2002). *Modern applied statistics with S*. Springer.

[28] Sharma, V. K., Singh, S. K., Singh, U., and Agiwal, V. (2015). The Inverse Lindley distribution: A stress-strength reliability model with application to head and neck cancer data. *Journal of Industrial and Production Engineering*, 32(3):162–173.

[29] Singh, K. L. and Srivastava, R. (2014). Inverse Maxwell distribution as a survival model, genesis and parameter estimation. *Research Journal of Mathematical and Statistical Sciences*, 2320:6047.

[30] Sultan, K., Al-Dayian, G., and Mohammad, H. (2008). Estimation and prediction from gamma distribution based on record values. *Computational Statistics & Data Analysis*, 52(3):1430–1440.

[31] Tibshirani, R. J. and Efron, B. (1993). An introduction to the bootstrap. *Monographs on Statistics and Applied Probability*, 57:1–436.

[32] Tomer, S. K. and Panwar, M. (2020). A review on Inverse Maxwell distribution with its statistical properties and applications. *Journal of Statistical Theory and Practice*, 14:1–25.

[33] UL Haq, M. A., Usman, R. M., Hashmi, S., and Al-Omeri, A. I. (2019). The Marshall-Olkin length-biased exponential distribution and its applications. *Journal of King Saud University-Science*, 31(2):246–251.

[34] Usman, R. M. and UL Haq, M. A. (2020). The Marshall-Olkin extended inverted Kumaraswamy distribution: Theory and applications. *Journal of King Saud University-Science*, 32(1):356–365.