Harris Extended Two Parameter Lindley Distribution and Applications in Reliability

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Abstract

This paper introduces a new generalization of the two parameter Lindley distribution distribution namely, Harris extended two parameter Lindley distribution. Various structural properties of the new distribution are derived including moments, quantile function, Renyi entropy, and mean residual life. The model parameters are estimated by maximum likelihood method. The usefulness of the new model is illustrated by means of two real data sets on Wheaton river flood and bladder cancer. Also, we derive a reliability test plan for acceptance or rejection of a lot of products submitted for inspection with lifetimes following this distribution. The operating characteristic functions of the sampling plans are obtained. The producer's risk, minimum sample sizes and associated characteristics are computed and presented in tables. The results are illustrated using two data sets on ordered failure times of products as well as failure times of ball bearings.

Keywords: Acceptance Sampling Plan, Harris extended two parameter Lindley distribution, Lindley distribution, maximum likelihood estimation, hazard rate, extreme order statistics.

1. Introduction

The Lindley distribution was introduced by Lindley (1958) to analyze failure time data, especially in applications to modeling stress-strength reliability. The motivation of the Lindley distribution arises from its ability to model failure time data with increasing, decreasing, unimodal and bathtub shaped hazard rates. The Lindley distribution belongs to the exponential family and it can be written as a mixture of exponential and gamma distributions. The properties and inferential procedure for the Lindley distribution were studied by Ghitany et al. (2008, 2011). It is shown that the Lindley distribution is better than the exponential distribution when hazard rate is unimodal or bathtub shaped. Mazucheli and Achcar (2011) also proposed the Lindley distribution as a possible alternative to exponential and Weibull distributions.

In recent years, there have been many studies to obtain a new distribution based on modifications of the Lindley distribution for modeling data in biology, medicine, finance, and engineering. To name a few extensions, three parameter generalized Lindley suggested by Zakerzadeh and Dolati (2009), New Generalized Lindley Distribution(NGLD) proposed by Abouammoh et al. (2015), two-parameter weighted Lindley distribution developed by Ghitany et al.(2011) , power

Lindley by Ghitany et al. (2013), and Merovci (2013) proposed transmuted Lindley distribution. The Lindley distribution has been generalized by different researchers including Elbatal et al.(2013), Liyanage and Parai (2014), Nadaeajah et al. (2011), Oluyede and Yang (2014), Shanker and Fesshaye (2015), Kemaloglu and Yilmaz (2017) are some among others. Some recent works based on the Lindley distribution are wrapped Lindley distribution by Joshi and Jose (2018), three parameter generalized Lindley by Ekhosuehi and Opone (2018), Lindley Weibull distribution by Cordeiro et al. (2018), modified Lindley distribution by Chesneau et al. (2019a), wrapped modified Lindley distribution by Chesneau et al. (2019b), inverted modified Lindley distribution by Chesneau et al. (2020a), and sum and difference of two Lindley distributions by Chesneau et al. (2020b). The Lindley distribution does not provide enough flexibility for analyzing different types of lifetime data because of having only one parameter. To increase the flexibility for modelling purposes it will be useful to consider further alternatives of this distribution. Therefore, the aim of this study is to introduce a new family of distributions using the Lindley generator. In these directions, one can also study the properties of the Harris Extended generalization of two-parameter Lindley distribution. The main idea of this technique is to get more flexible structures than the base distribution.

The procedure of adding one or two parameters to a family of distributions to obtain more flexibility is a well-known technique in the existing literature. Aly and Benkherouf (2011) introduced a new family of distributions, called the Harris Extended (HE) family by adding two new parameters to a baseline distribution. The new method is based on the probability generating function (pgf) of Harris (1948) distribution. If $\bar{F}(x)$, $f(x)$, and $r_F(x)$ denote the survival function (sf), probability density function (pdf) and hazard rate function (hrf) of a parent distribution, then the sf $\bar{G}(x)$ of HE family of distribution is given by;

$$\bar{G}(x) = \left[\frac{\lambda \bar{F}(x)^k}{1-\bar{\lambda}\bar{F}(x)^k}\right]^{1/k} \quad ; \; x>0, \quad \alpha>0, \quad \bar{\lambda}=1-\lambda, \; k>0. \tag{1}$$

Here, the parameters α and k are additional shape parameters that aims to introduce greater flexibility. The HE density function is;

$$g(x) = \frac{\lambda^{1/k} f(x)}{[1-\bar{\lambda}\bar{F}(x)^k]^{(k+1)/k}} \quad ; \; x>0, \quad \lambda>0, \quad \bar{\lambda}=1-\lambda, \; k>0. \tag{2}$$

The hrf of the HE distribution is given by

$$r(x) = \frac{r_F(x)}{[1-\bar{\lambda}\bar{F}(x)^k]}, \quad x>0$$

where $r_F(x)$ denotes the failure rate function of the baseline distribution. When $k=1$, the above equations reduces to those of Marshall-Olkin family of distributions, introduced by Marshall and Olkin (1997). Hence the HE family of distributions generalizes the well-known Marshall-Olkin class of distributions. Batsidis and Lemonte (2014) considered the HE family of distributions with respect to some lifetime models. Pinho et al. (2015), introduced and studied Harris extended exponential model. More recently, Jose and Paul (2018) derived reliability test plans for percentiles based on Harris generalized linear exponential distribution. Jose et al.(2018) developed reliability test plans for Harris extended Weibull distribution.

In this article, we introduce a new variant of Harris extended model by considering the baseline distribution to be a two parameter lindley distribution suggested by Shankar et al.(2013). The article is organized as follows: The derivation of the Harris extended two-parameter Lindley distribution (HETLD), hazard rate and cumulative hazard rate are presented in Section 2. Linear representation of the density function is presented in Section 3. Statistical properties include moments, quantile function, Renyi entropy, mean residual life and maximum likelihood estimation are explored in Section 4. The new distribution is illustrated on real datasets in Section 5. In section 6, the proposed sampling plans are established for the Harris extended two parameter Lindley distribution. Finally, conclusions are given in Section 7.

2. Harris Extended Two Parameter Lindley Distribution

If $\bar{F}(x)$, $f(x)$, and $r_F(x)$ denote the survival function (sf), probability density function (pdf) and hazard rate function (hrf) of a parent distribution, then the baseline sf of two parameter Lindley distribution (TLD) is given by,

$$\bar{F}(x;\beta,\theta) = \left(\frac{\theta+\beta+\beta\theta x}{\theta+\beta}\right) e^{-\theta x}; \quad x>0, \ \ \beta>0, \ \ \theta>0$$

and the corresponding pdf is given by

$$f(x;\beta,\theta) = \frac{\theta^2}{\theta+\beta}(1+\beta x)e^{-\theta x}; \quad x>0, \ \ \beta>0, \ \ \theta>0$$

For detailed information, see Shanker et al. (2013). Substituting the sf of TLD in (1), the distribution called (Harris extended two-parameter Lindley distribution) HETLD is obtained. It is denoted by HETLD $(\lambda, k, \beta, \theta)$ with sf

$$\bar{G}(x) = \left\{ \frac{\lambda\left(\frac{\theta+\beta+\beta\theta x}{\theta+\beta}\right)^k e^{-k\theta x}}{1-\bar{\lambda}\left(\frac{\theta+\beta+\beta\theta x}{\theta+\beta}\right)^k e^{-k\theta x}} \right\}^{1/k} \tag{3}$$

where, $(\lambda, k, \beta, \theta) > 0$, $\bar{\lambda} = 1-\lambda$
pdf of HETLD is given by;

$$g(x) = \frac{\lambda^{1/k}\frac{\theta^2}{\theta+\beta}(1+\beta x)e^{-\theta x}}{\left\{1-\bar{\lambda}\left(\frac{\theta+\beta+\beta\theta x}{\theta+\beta}\right)^k e^{-k\theta x}\right\}^{1+\frac{1}{k}}} \ ; \ x>0, \quad (\lambda,k,\beta,\theta)>0, \quad \bar{\lambda}=1-\lambda. \tag{4}$$

Figure 1 illustrates some possible shapes of the pdf of HETLD for different values of the parameters λ, k, β, and θ.
The hrf of HETLD is given as

$$r_{HETLD}(x) = \frac{\left(\frac{\theta^2(1+\beta x)}{\theta+\beta+\beta\theta x}\right)}{\left[1-\bar{\lambda}(\frac{\theta+\beta+\beta\theta x}{\theta+\beta})^k e^{-\theta k x}\right]}$$

Plots of hrf of HETLD for different values of parameters β, θ, λ and k are given in Figure 2. It can be seen that, the hrf of HETLD is attractively flexible. Therefore, the new distribution can be used quite effectively for analyzing different types of lifetime data in practice.

The cumulative hazard rate function of HETLD is given by

$$H_{HETLD}(x) = \frac{1}{k}ln\left[1-\bar{\lambda}\left(\frac{\theta+\beta+\beta\theta x}{\theta+\beta}\right)^k e^{-k\theta x}\right] - \frac{1}{k}ln\left[\lambda\left(\frac{\theta+\beta+\beta\theta x}{\theta+\beta}\right)^k e^{-k\theta x}\right]$$

Figure 4 illustrates the behavior of the cumulative hazard rate function of the HETLD distribution at different values of the parameters.

3. Linear representation of the density function

In this section, we obtain a useful expansion for the HETLD. For $|\ z\ |< 1$ and $r>0$, we have

$$(1-z)^{-r} = \sum_{i=0}^{\infty}\binom{r+i-1}{i}z^i \tag{5}$$

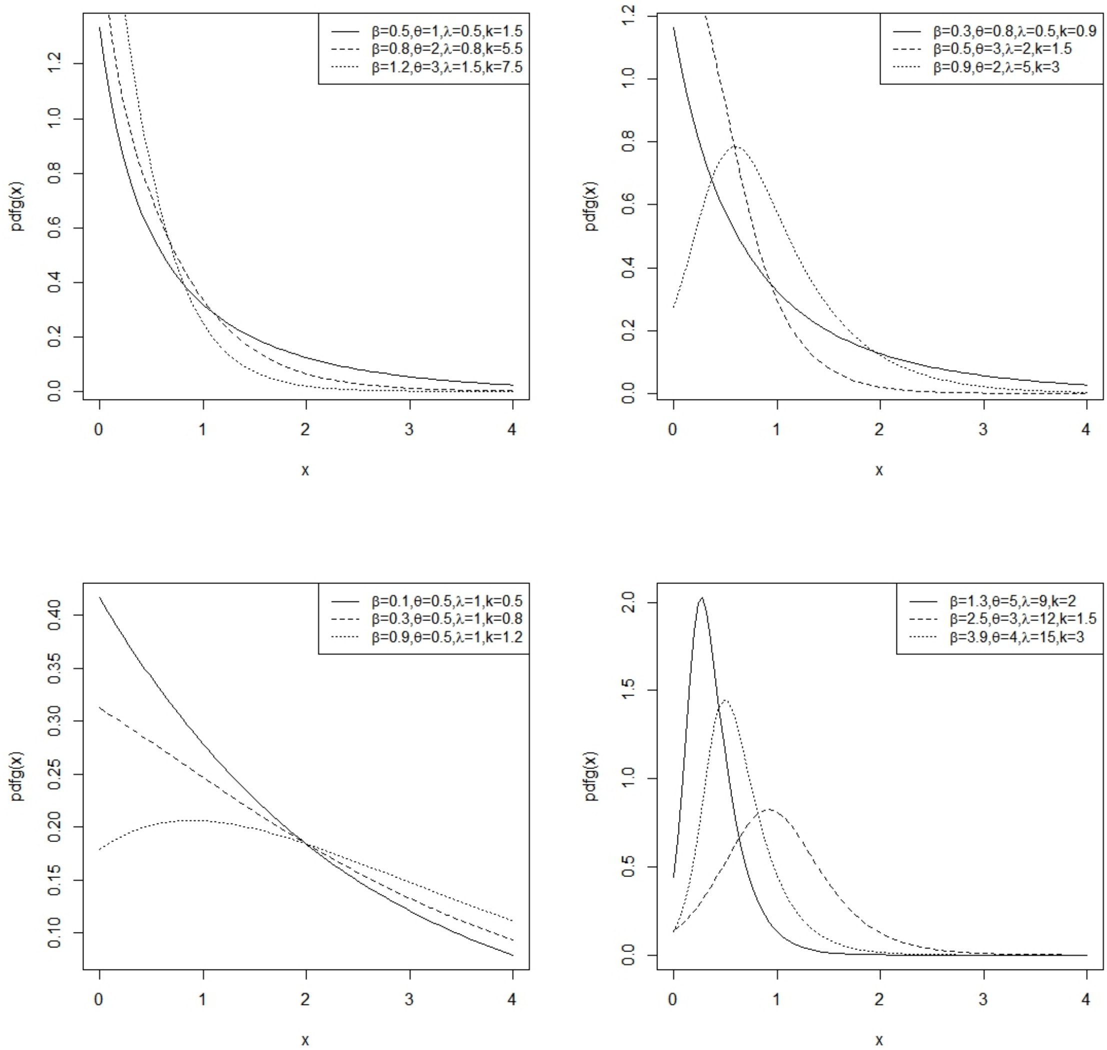

Figure 1: *pdf of HETLD distribution for various values of β, θ, λ and k.*

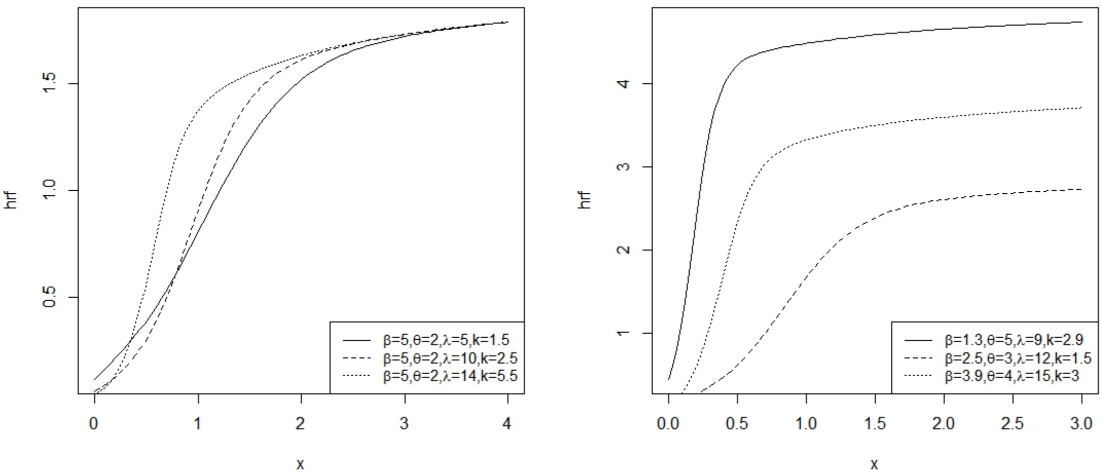

Figure 2: *hrf of HETLD for various values of β, θ, λ and k.*

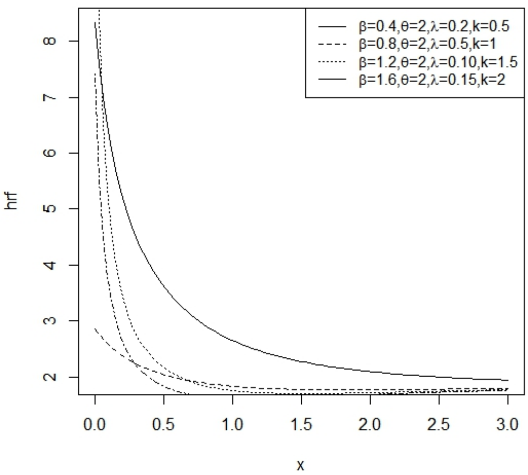

Figure 3: *HETLD for various values of β, θ, λ and k.*

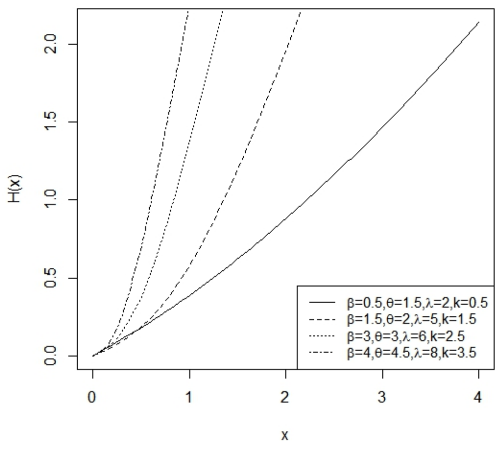

Figure 4: *The cumulative hazard rate function of the HETLD*

Applying Equation (5) in Equation (4), for $\lambda \in (0,1)$, yields

$$g(x) = \sum_{i=0}^{\infty} w_i f(x;\beta,\theta); \quad x > 0 \tag{6}$$

where,

$$w_i = \binom{i+k^{-1}}{i} \frac{\lambda^{1/k}(1-\lambda)^i}{1+ik}$$

otherwise, if $\lambda > 1$, we can obtain

$$g(x) = \sum_{i=0}^{\infty} v_i f(x;\beta,\theta); \quad x > 0 \tag{7}$$

where,

$$v_i = \frac{\frac{1}{\lambda}(-1)^i}{1+ik} \sum_{j=i}^{\infty} \binom{j+k^{-1}}{j} \binom{j}{i} (1-\frac{1}{\lambda})^j$$

and $f(x;\beta,\theta)$ denotes the Two parameter Lindley density function. Thus HETLD as an infinite linear combination of Two parameter Lindley density functions. Equations (6) and (7) have the same form except for the coefficients which are $w_i's$ in Equation (6) and $v_i's$ in Equation (7).

4. Statistical Properties

In this section, statistical properties of the HETLD including the moments, quantile function, Rènyi entropy and limiting distributions of order statistics, stochastic Orders, Lorenz and Bonferroni curves and mean residual lifetime are given. Furthermore, the estimation of the HETLD parameters based on maximum likelihood is discussed.

4.1. Quantiles

Recently, Bensid and Zeghdoudi (2017) showed that the quantile function of the two parameter Lindley distribution is given by

$$F^{-1}(v) = -\frac{\theta+\beta}{\theta\beta} - \frac{1}{\theta} W\left(-\frac{1}{\beta}(1-v)(\theta+\beta)e^{(-\frac{\theta+\beta}{\beta})}\right), \qquad 0 < v < 1,$$

where $W_{-1}(.)$ denotes the negative branch of the Lambert W function (i.e., the solution of the equation $W(z)e^{W(z)} = z$).
The quantile function, $Q(u)$, $0 < u < 1$, for the Harris Extended (HE) family of distributions is computed by using the formula of Batsidis and Lemonte (2014) as

$$G^{-1}(u) = F^{-1}\left(1 - (1-u)[\lambda + \bar{\lambda}(1-u)^k]^{-1/k}\right)$$

where $G^{-1}(.)$ and $F^{-1}(.)$ are the inverse functions of G(.) (obtained from Equation (10)) and F(.) (baseline cumulative function), respectively. The quantile function of the HETLD is given by

$$x = G^{-1}(u) = -(\frac{\theta+\beta}{\theta\beta}) - \frac{1}{\theta}W\left[-\frac{1}{\beta}\left(\frac{1-u}{[\lambda+\bar{\lambda}(1-u)^k]^{1/k}}\right)(\theta+\beta)e^{(-\frac{\theta+\beta}{\beta})}\right] \tag{8}$$

Simulation of HETL random variable follows directly from (8). The quantiles of the HETLD can be obtained by setting $u = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$.

4.2. Moments

Here, we shall derive a general expression for the moments of HETLD$(\lambda, k, \beta, \theta)$, in terms of the Probability Weighted Moments (PWMs) of the baseline distribution. From Equation (6) and for $\lambda \in (0,1)$, r^{th} moment of the HETLD can be expressed as

$$\mu'_r = \sum_{j=0}^{\infty} w_j \tau_{r,jk} \tag{9}$$

For $\lambda > 1$, Equation (7) holds replace w_j with v_j
where,

$$\begin{aligned}
\tau_{r,jk} &= \int_0^\infty x^r[\bar{F}x]^{jk} f(x)dx \\
&= \frac{\theta^2}{\beta+\theta}\int_0^\infty x^r\left[\left(1+\frac{\beta\theta x}{\theta+\beta}\right)e^{-\theta x}\right]^{jk}(1+\beta x)e^{-\theta x}dx \\
&= \sum_{i=0}^{\infty}\binom{jk}{i}\left(\frac{\beta\theta}{\theta+\beta}\right)^i \frac{\theta^2}{\beta+\theta}\left(\frac{\Gamma(r+1)}{[\theta(1+jk)]^{r+1}} + \frac{\beta\Gamma(r+2)}{[\theta(1+jk)]^{r+2}}\right)
\end{aligned}$$

Substituting $\tau_{r,jk}$ in Equation (9) μ'_r of HETLD, for $\lambda \in (0,1)$,

$$\mu'_r = E(X^r) = \frac{\lambda^{1/k}\theta^2}{(\beta+\theta)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(1-\lambda)^j\frac{\Gamma(k^{-1}+1+j)}{\Gamma(k^{-1}+1)j!}\binom{jk}{i}\left(\frac{\beta\theta}{\theta+\beta}\right)^i\left\{\frac{\Gamma(r+1)}{[\theta(1+jk)]^{r+1}} + \frac{\beta\Gamma(r+2)}{[\theta(1+jk)]^{r+2}}\right\}$$

For $\lambda > 1$

$$\begin{aligned}
\mu'_r = E(X^r) = \frac{\lambda^{-1}\theta^2}{(\beta+\theta)}\sum_{i=0}^{\infty}\sum_{l=j}^{\infty}\binom{l}{j}(-1)^j(1-\frac{1}{\lambda})^l\frac{\Gamma(k^{-1}+1+l)}{\Gamma(k^{-1}+1)l!}\binom{jk}{i}\left(\frac{\beta\theta}{\theta+\beta}\right)^i \\
\times\left\{\frac{\Gamma(r+1)}{[\theta(1+jk)]^{r+1}} + \frac{\beta\Gamma(r+2)}{[\theta(1+jk)]^{r+2}}\right\}
\end{aligned}$$

and Γ. is the complete gamma function.

4.3. Rényi entropy

The *Rényi* entropy of a continuous random variabe X distributed according to the HEW is derived by the following formula

$$I_R(\delta) = \frac{1}{1-\delta}log\int_0^\infty g^\delta(x)dx \quad ,\delta > 0$$

$$\int_0^\infty g^\delta(x)dx = (\lambda)^{\delta/k}\left(\frac{\theta^2}{\beta+\theta}\right)^\delta \int_0^\infty \frac{(1+\beta x)^\delta e^{-\delta\theta x}}{[1-\bar{\lambda}(1+\frac{\beta\theta x}{\theta+\beta})^k e^{-\theta k x}]^{\delta+\frac{\delta}{k}}}dx$$

Using the expansions: $(1-z)^{-\delta} = \sum_{j=0}^{\infty}\frac{\Gamma(\rho+j)}{\Gamma(\rho)j!}z^j$ to expand the expression

$$\left[1-\bar{\lambda}(1+\left(\frac{\beta\theta x}{\theta+\beta}\right)^k e^{-\theta kx}\right]^{\delta+\frac{\delta}{k}} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{\Gamma(\delta+\frac{\delta}{k}+j)}{\Gamma(\delta+\frac{\delta}{k})j!}\binom{jk}{i}(\bar{\lambda})^j\left(\frac{\theta\beta}{\theta+\beta}\right)^i x^i e^{-\theta kjx}$$

then

$$\int_0^\infty g^\delta(x)dx = \left[\frac{\lambda^{1/k}\theta^2}{\beta+\theta}\right]^\delta \sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\sum_{n=0}^{\infty}\frac{\Gamma(\delta+\frac{\delta}{k}+j)}{\Gamma(\delta+\frac{\delta}{k})j!}\binom{jk}{i}(\bar{\lambda})^j\left(\frac{\theta\beta}{\theta+\beta}\right)^i\binom{\delta}{n}\beta^n\int_0^\infty x^{n+i}e^{-(j+\delta)\theta x}dx$$

The integral $I = \int_0^\infty x^{n+i}e^{-(j+\delta)\theta x}dx$ can be evaluated as

$$\begin{aligned} I &= \int_0^\infty x^{n+i}e^{-(j+\delta)\theta x}dx \\ &= \frac{\Gamma(n+i+1)}{[(j+\delta)\theta]^{n+i+1}} \end{aligned}$$

Collecting all of the above evaluations, the Renyi entropy of HETLD can be written as

$$I_R(\delta) = \frac{1}{1-\delta}\left\{lnC + ln\left(\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\sum_{n=0}^{\infty}\frac{\Gamma(\delta+\frac{\delta}{k}+j)}{\Gamma(\delta+\frac{\delta}{k})j!}\binom{jk}{i}(\bar{\lambda})^j\left(\frac{\theta\beta}{\theta+\beta}\right)^i\binom{\delta}{n}\beta^n\frac{\Gamma(n+i+1)}{[(j+\delta)\theta]^{n+i+1}}\right)\right\}$$

where, $C = \left[\frac{\lambda^{1/k}\theta^2}{\beta+\theta}\right]^\delta$

4.4. The Mean Residual Lifetime

The additional lifetime given that the component has survived up to time x is called the residual life function of the component, then the expectation of the random variable X that represent the remaining lifetime is called the mean residual lifetime (MRL) and is given by

$$\begin{aligned} m(x) &= E(X-x \mid X\geq x) \\ &= \left\{\frac{1}{\bar{F}(x)}\int_x^\infty tf(t)dt\right\} - x \end{aligned}$$

The MRL function m(x) for HETLD random variable can be derived in the following steps.

$$\int_x^\infty tf(t)dt = \lambda^{1/k}\frac{\theta^2}{\beta+\theta}\int_x^\infty \frac{(t+\beta t^2)e^{-\theta t}}{[1-\bar{\lambda}(1+\frac{\beta\theta t}{\theta+\beta})^k e^{-\theta kt}]^{1+\frac{1}{k}}}dt$$

Using the expansion $(1-z)^{-\delta} = \sum_{j=0}^{\infty}\frac{\Gamma(\delta+j)}{[\Gamma(\delta)j!]}z^j, \ \mid z\mid < 1$ one has

$$[1-\bar{\lambda}(1+\frac{\beta\theta t}{\theta+\beta})^k e^{-\theta kt}]^{1+\frac{1}{k}} = \sum_{j=0}^{\infty}\frac{\Gamma(1+k^{-1}+j)}{[\Gamma(1+k^{-1})j!]}\bar{\lambda}^j(1+\frac{\beta\theta t}{\theta+\beta})^{jk}e^{-\theta kjt}$$

Similarly, using the expansion $(1+b)^n = \sum_{i=0}^{\infty}\binom{n}{i}b^{n-i}$ one can have

$$(1+\frac{\theta\beta t}{\theta+\beta})^{jk} = \sum_{i=0}^{\infty}\binom{jk}{i}(\frac{\theta\beta}{\theta+\beta})^{jk-i}t^{jk-i}$$

$$\int_x^\infty tf(t)dt = \lambda^{1/k}\frac{\theta^2}{\beta+\theta}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{\Gamma(1+\frac{1}{k}+j)}{\Gamma(1+\frac{1}{k})j!}\binom{jk}{i}(\bar{\lambda})^j(\frac{\theta\beta}{\theta+\beta})^{jk-i}$$
$$\times\int_x^\infty (t^{jk-i+1}+\beta t^{jk-i+2})e^{-(1+jk)(\theta t)}dt$$

Using the substitution $u=\theta(j+1)t$

$$\int_x^\infty (t^{jk-i+1}+\beta t^{jk-i+2})e^{-(1+jk)(\theta t)}dt = \frac{\Gamma(jk-i+2,\theta(jk+1)x)}{[\theta(1+jk)]^{jk-i+1}}+\beta\frac{\Gamma(jk-i+3,\theta(jk+1)x)}{[\theta(1+jk)]^{jk-i+2}}$$

Collecting all of the above evaluations and making the necessary simplifications, the MRL can be written as

$$m(x)=\left\{\frac{1-\bar{\lambda}\left(\frac{\theta+\beta+\beta\theta x}{\theta+\beta}\right)^k e^{-k\theta x}}{\left(\frac{\theta+\beta+\beta\theta x}{\theta+\beta}\right)^k e^{-k\theta x}}\right\}^{1/k}\left(\frac{\theta^2}{\theta+\beta}\right)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{\Gamma(1+\frac{1}{k}+j)}{\Gamma(1+\frac{1}{k})j!}\binom{jk}{i}(\bar{\lambda})^j$$
$$\times(\frac{\theta\beta}{\theta+\beta})^{jk-i}\left[\frac{\Gamma(jk-i+2,\theta(jk+1)x)}{[\theta(1+jk)]^{jk-i+1}}+\beta\frac{\Gamma(jk-i+3,\theta(jk+1)x)}{[\theta(1+jk)]^{jk-i+2}}\right]-x$$

4.5. Estimation of Parameters

Let $x=(x_1,....,x_n)$ be a random sample of size n from the HETLD distribution. Then, the log-likelihood function is given by

$$L(\lambda,k,\beta,\theta)=n\left[\frac{1}{k}log\lambda+2log\theta-log(\beta+\theta)\right]+\sum_{i=1}^{n}log(1+\beta x_i)-\theta\sum_{i=1}^{n}x_i$$
$$-(1+\frac{1}{k})\sum_{i=1}^{n}logA_i(k,\lambda,\beta,\theta)$$

where,

$$A_i(k,\lambda,\beta,\theta)=1-\bar{\lambda}\left(\frac{\theta+\beta+\beta\theta x_i}{\theta+\beta}\right)^k e^{-k\theta x_i},\qquad i=1,....n$$

The MLEs $\bar{\lambda},\bar{k},\bar{\beta},\bar{\theta}$ of λ,k,β,θ are then the solutions of the following non-linear equations.

$$\frac{\partial LogL}{\partial k}=-nk^{-2}log\lambda+\frac{1}{k^2}\sum_{i=1}^{n}logA_i(k,\lambda,\beta,\theta)+(1+\frac{1}{k})\bar{\lambda}\sum_{i=1}^{n}\frac{A_{i,k}(k,\lambda,\beta,\theta)}{A_i(k,\lambda,\beta,\theta)}=0$$

$$\frac{\partial LogL}{\partial \lambda}=\frac{n}{k\lambda}-(1+\frac{1}{k})\sum_{i=1}^{n}\frac{A_{i,\lambda}(k,\lambda,\beta,\theta)}{A_i(k,\lambda,\beta,\theta)}=0$$

$$\frac{\partial LogL}{\partial \beta}=\frac{n}{\beta+\theta}+\sum_{i=1}^{n}\frac{x_i}{1+\beta x_i}+\bar{\lambda}(1+k)\sum_{i=1}^{n}\frac{A_{i,\beta}(k,\lambda,\beta,\theta)}{A_i(k,\lambda,\beta,\theta)}=0$$

$$\frac{\partial LogL}{\partial \theta}=\frac{2n}{\theta}-\frac{n}{\beta+\theta}-\sum_{i=1}^{n}x_i+\frac{\bar{\lambda}(1+k)}{(\beta+\theta)^2}\sum_{i=1}^{n}\frac{A_{i,\theta}(k,\lambda,\beta,\theta)}{A_i(k,\lambda,\beta,\theta)}=0$$

where,

$$A_{i,k}(k,\lambda,\beta,\theta)=\frac{\partial A_i(k,\lambda,\beta,\theta)}{\partial k}=\left(\frac{(\theta+\beta+\beta\theta x_i)e^{-\theta x_i}}{\theta+\beta}\right)^k log\left(\frac{(\theta+\beta+\beta\theta x_i)e^{-\theta x_i}}{\theta+\beta}\right)$$

Table 1: *Wheaton River Data.*

1.7	2.2	14.4	1.1	0.4	20.6	5.3	0.7
1.9	13.0	12.0	9.3	1.4	18.7	8.5	25.5
11.6	14.1	22.1	1.1	2.5	14.4	1.7	37.6
0.6	2.2	39.0	0.3	15.0	11.0	7.3	22.9
1.7	0.1	1.1	0.6	9.0	1.7	7.0	20.1
0.4	2.8	14.1	9.9	10.4	10.7	30.0	3.6
5.6	30.8	13.3	4.2	25.5	3.4	11.9	21.5
27.6	36.4	2.7	64.0	1.5	2.5	27.4	1.0
27.1	20.2	16.8	5.3	9.7	27.5	2.5	27.

Table 2: *Bladder Cancer Patients Data.*

0.08	2.09	3.48	4.87	6.94	8.66	13.11	23.63
0.20	2.23	3.52	4.98	6.97	9.02	13.29	0.40
2.26	3.57	5.06	7.09	9.22	13.80	25.74	0.50
2.46	3.64	5.09	7.26	9.47	14.24	25.82	0.51
2.54	3.70	5.17	7.28	9.74	14.76	26.31	0.81
2.62	3.82	5.32	7.32	10.06	14.77	32.15	2.64
3.88	5.32	7.39	10.34	14.83	34.26	0.90	2.69
4.18	5.34	7.59	10.66	15.96	36.66	1.05	2.69
4.23	5.41	7.62	10.75	16.62	43.01	1.19	2.75
4.26	5.41	7.63	17.12	46.12	1.26	2.83	4.33
5.49	7.66	11.25	17.14	79.05	1.35	2.87	5.62
7.87	11.64	17.36	1.40	3.02	4.34	5.71	7.93
11.79	18.10	1.46	4.40	5.85	8.26	11.98	19.13
1.76	3.25	4.50	6.25	8.37	12.02	2.02	3.31
4.51	6.54	8.53	12.03	20.28	2.02	3.36	6.76
12.07	21.73	2.07	3.36	6.93	8.65	12.63	22.69

$$A_{i,\lambda}(k,\lambda,\beta,\theta) = \frac{\partial A_i(k,\lambda,\beta,\theta)}{\partial \lambda} = \left(\frac{(\theta+\beta+\beta\theta x_i)e^{-\theta x_i}}{\theta+\beta}\right)^k$$

$$A_{i,\beta}(k,\lambda,\beta,\theta) = \frac{\partial A_i(k,\lambda,\beta,\theta)}{\partial \beta} = \left(\frac{(\theta+\beta+\beta\theta x_i)e^{-\theta x_i}}{\theta+\beta}\right)^{k-1} \frac{\theta^2 x_i e^{-\theta x_i}}{(\theta+\beta)^2}$$

$$A_{i,\theta}(k,\lambda,\beta,\theta) = \frac{\partial A_i(k,\lambda,\beta,\theta)}{\partial \theta} = \left(\frac{(\theta+\beta+\beta\theta x_i)e^{-\theta x_i}}{\theta+\beta}\right)^{k-1} x_i e^{-\theta x_i}[-\theta^2 - 2\beta\theta - \beta\theta^2(\beta+\theta)]$$

Here, it is not possible to find the exact solution of the estimators for $\bar{\lambda}$, $\bar{k}$, $\bar{\beta}$, and $\bar{\theta}$, so the MLEs are obtained numerically by using the appropriate optimization methods.

5. Application

In this section, we illustrate the applicability of HETLD by considering 2 real data sets. The first data set correspond to the exceedances of flood peaks (in m3=s) of the Wheaton River near Carcross in Yukon Territory, Canada. The data consist of 72 exceedances for the years 1958-1984, rounded to one decimal place. They were analysed by Choulakian and Stephens (2001) and are listed in Table 1. The second data set given by Lee and Wang (2003) which represent remission times (in months) of a random sample of 128 bladder cancer patients. Its application in survival analysis has been identified and Table2 lists the remission times of the bladder cancer.

The HETL distribution was compared to four other distributions, namely,

- Harris Extended Weibull (HEW) distribution (Batsidis and Lemonte, 2014), with cumulative distribution function (cdf) $F(x) = 1 - \left(\frac{\lambda e^{-k(\eta x)^{\beta}}}{1-\bar{\lambda}e^{-k(\eta x)^{\beta}}}\right)^{1/k}$
- exponentiated Weibull (EW) distribution (Mudholkar and Srivastava, 1993), with $F(x) = \{1 - e^{-(\eta x)^{\beta}}\}^{\alpha}$
- TLD distribution, which is the HETLD with $k = 1, \lambda = 1$
- Weibull distribution with $F(x) = 1 - e^{-(\eta x)^{\beta}}$

The parameters of the above distributions are estimated by the maximum-likelihood method and, then, the values of Akaike information criterion (AIC)and Bayesian information criterion (BIC) are calculated. A summary of computations are given in Table 3 and Table 4. Since the values of the AIC and BIC are smaller for the HETLD compared with those values of the other models,the new distribution seems to be a very competitive model to these data sets. For both data sets, the fitted densities and the empirical cdf plots of the HETLD model are shown in Figures 5 and 6, respectively. The figures indicate a satisfactory fit for the HETLD model too. Thus We conclude that HETLD is the best fit for bladder cancer patients data and wheaton river data.

Table 3: *MLE, maximized log-likelihood, AIC, and BIC for the Wheaton River Data.*

Distribution	MLEs of parameters	-Log L	AIC	BIC
HETLD	$(\hat{\alpha}, \hat{k}, \hat{\beta}, \hat{\theta})$=(0.0417, 7.4455,3.1565,0.1119)	247.8641	503.7282	512.8349
HEW	$(\hat{\alpha}, \hat{k}, \hat{\eta}, \hat{\beta})$ =(5.5322,0.0900,0.5750,0.7844)	250.8359	509.6719	518.7785
EW	$(\hat{\alpha}, \hat{\eta}, \hat{\beta})$=(0.5180,0.0501,1.3879)	251.0251	508.0502	514.8802
TLD	$(\hat{\beta}, \hat{\theta})$=$(7.359766e-05, 8.200532e-02)$	252.128	508.2559	512.8093
Weibull	$(\hat{\eta}, \hat{\beta})$=(0.0859,0.9010)	251.4986	506.9973	511.5506

Table 4: *MLE, maximized log-likelihood, AIC, and BIC for the Bladder Cancer Patients Data.*

Distribution	MLEs of parameters	-Log L	AIC	BIC
HETLD	$(\hat{\alpha}, \hat{k}, \hat{\beta}, \hat{\theta})$=(0.0783,0.8450,1.3110,0.0615)	409.2563	826.514	837.8932
HEW	$(\hat{\alpha}, \hat{k}, \hat{\eta}, \hat{\beta})$ =(8.8429,4.3278,0.1997,0.7319)	409.7029	827.4058	838.814
EW	$(\hat{\alpha}, \hat{\eta}, \hat{\beta})$=(2.7989,0.2993,0.6540)	410.6801	827.3602	835.9163
TLD	$(\hat{\beta}, \hat{\theta})$=(0.000014,0.1068)	414.3419	832.6838	838.3879
Weibull	$(\hat{\eta}, \hat{\beta})$=(0.1046,1.0475)	414.0869	832.1738	837.8778

6. Reliability Test Plan

As a sequel, in this section, we discuss the application of the HETLD in acceptance sampling. It is assumed that the probability distribution of life times follow the HETLD. Acceptance sampling is an inspection procedure used to determine whether to accept or reject a specific quantity of material. The decision law to accept or reject a lot according to the results of a random sample from the population is called acceptance sampling plan . The procedure is to take a random sample of size (n) and inspect each item. If the number of defectives does not exceed a specified acceptance number c, the consumer accepts the entire set of products. Any defectives found in the sample are either repaired or returned to the producer. If the number of defectives in the sample is greater than c, the consumer subjects the entire products to 100 percent inspection or

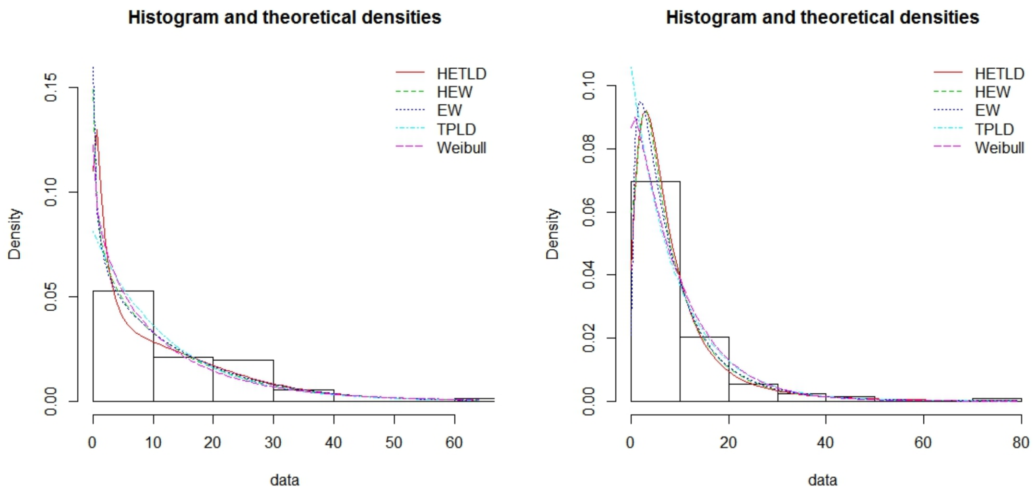

Figure 5: *Fitted densities plots for the first and the second data sets.*

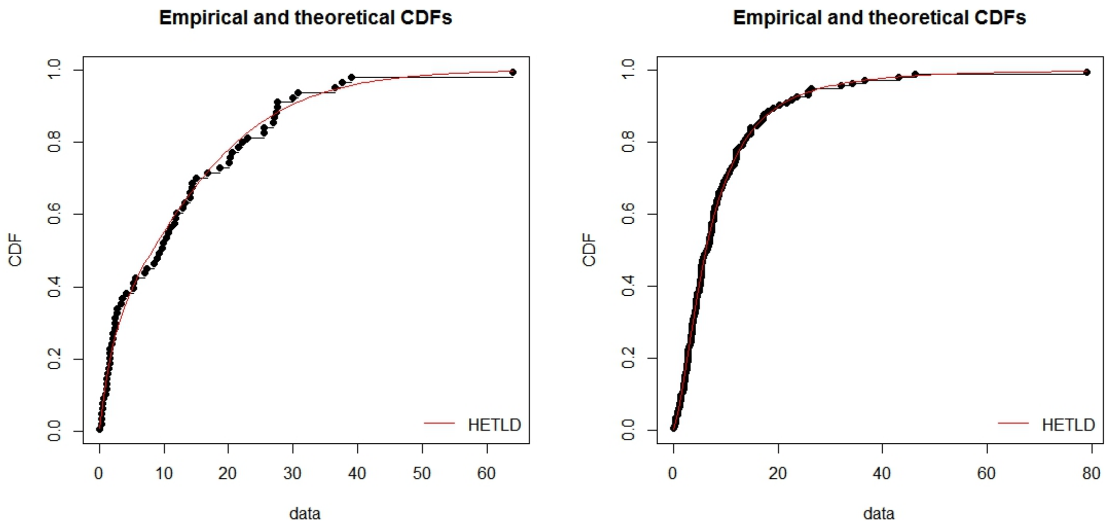

Figure 6: *Empirical cdf plots of HETLD distribution for the first and the second data sets.*

rejects the entire product and returns to the producer. There are two kinds of errors which may arise when a lot is either accepted or rejected. If a good lot is rejected it is called producer's risk and if a bad lot is accepted it is known as consumer's risk. An acceptance sampling plan should be designed in such a way that both risks are minimal. Then the procedure is termed as 'acceptance sampling based on life tests 'or 'reliability test plans'.

Gupta and Groll (1961), Good and Kao (1961), Kantam and Rosaiah (1998), Kantam et al. (2001), Rosaiah and Kantam (2005), Rosaiah et al. (2006), Lio et al. (2010), Krishna et al. (2013) etc. have discussed acceptance sampling plans for various distributions. Jose and Sivadas (2015) developed a reliability test plan for acceptance or rejection of a lot of products submitted for inspection with lifetimes governed by Negative Binomial Marshall-Olkin Rayleigh distribution. Recently, Jose and Joseph (2018) introduced reliability test plan for Gumbel - Uniform life time model. Jose and Paul (2018) derived the reliability test plans for percentiles based on Harris generalized linear exponential distribution. Jose et al. (2018) developed the reliability test plan for Harris extended Weibull distribution.

6.1. The Sampling Plans

In statistical quality control acceptance sampling plan is concerned with the inspection of a sample of products taken from a lot and the decision whether to accept or not to accept the lot based on the quality of product.

Here we discuss the reliability test plan for accepting or rejecting a lot where the life time of the product follows HETLD. In a life testing experiment the procedure is to terminate the test by a pre-determined time t and note the number of failures. If the number of failures at the end of time t does not exceed a given number c, called acceptance number then we accept the lot with a given probability of at least p^*. But if the number of failures exceeds c before time t then the test is terminated and the lot is rejected. For such truncated life test and the associated decision rule we are interested in obtaining the smallest sample size to arrive at a decision. Assume that the lifetime of a product follows HETLD. If a scale parameter $\eta > 0$ is introduced then cumulative distribution function (cdf) of HETLD is given by,

$$\bar{G}(t) = \left\{ \frac{\lambda\left(\theta+\beta+\beta\theta(\frac{t}{\eta})\right)^k e^{-k\theta(\frac{t}{\eta})}}{(\theta+\beta)^k - \bar{\lambda}\left(\theta+\beta+\beta\theta(\frac{t}{\eta})\right)^k e^{-k\theta(\frac{t}{\eta})}} \right\}^{1/k} \tag{10}$$

where,$\bar{\lambda} = 1-\lambda$ and $(\lambda, k, \beta, \theta) > 0$are shape parameters and $\eta > 0$ is the scale parameter. The average life time depends only on η if λ, β, θ and k are known. Let η_0 be the required minimum average life time. Then

$$G(t, \lambda, k, \beta, \theta, \eta) \leq G(t, \lambda, k, \beta, \theta, \eta_0) \Leftrightarrow \eta \geq \eta_0.$$

A sampling plan is specified by the following quantities:

1) the number of units n on test,
2) the acceptance number c,
3) the maximum test duration t, and
4) the minimum average lifetime represented by η_0.

The consumer's risk , i.e. the probability of accepting a bad lot should not exceed the value $1-p^*$, where p^* is a lower bound for the probability that a lot of true value η below η_0 is rejected by the sampling plan. For fixed p^* the sampling plan is characterized by $(n, c, t/\eta_0)$. By sufficiently large lots we can apply binomial distribution to find acceptance probability. The problem is to determine the smallest positive integer n for given value of c and t/η_0 such that

$$L(p_0) = \sum_{i=0}^{c} \binom{n}{i} p_0^i (1-p_0)^{n-i} \leq 1-p*, \tag{11}$$

where $p_0 = G(t, \lambda, k, \beta, \theta, \eta_0)$. The function $L(p)$ is called operating characteristic function of the sampling plan and it gives, i.e. the acceptance probability of the lot as a function of the failure probability $p(\eta) = G(t, \lambda, k, \beta, \beta, \eta)$. The average life time of the product is increasing with η and therefore the failure probability $p(\eta)$ decreases implying that the operating characteristic function is increasing in η. The minimum values of n satisfying (11) are obtained for $\lambda = 2$, $k = 2$, $\beta = 2$, $\theta = 2$ and $p*$=0.75, 0.90, 0.95, 0.99 and t/η_0=0.40, 0.56, 0.72, 0.88, 1, 1.5, 2 and 2.5. The results are displayed in Table 5. If $p_0 = G(t, \lambda, k, \beta, \theta, \eta_0)$ is very small and n is large, the binomial probability may be approximated by Poisson probability with parameter $\delta = np_0$ so that Equation (11) becomes

$$L_1(p_0) = \sum_{i=0}^{c} \frac{\delta^i}{i!} e^{-\delta} \leq 1 - p* . \tag{12}$$

The minimum values of n satisfying Equation (12) are obtained for the same combination of values of δ, β and t/η_0 for various values of p^* are presented in Table 6. The operating characteristic function of the sampling plan $(n, c, t/\eta_0)$gives the probability L(p) of accepting the lot with

$$L(p) = \sum_{i=0}^{c} \binom{n}{i} p_0^i (1 - p_0)^{n-i} \tag{13}$$

where $p = G(t, \eta)$ is considered as a function of η. For given p^* ,t/η_0 the choice of c and n are made on the basis of operating characteristics. Considering the fact that

$$p = G(\frac{t}{\eta_0} / \frac{\eta_0}{\eta}) \tag{14}$$

values of operating characteristic for a few sampling plans are computed and presented in Table 7.

6.2. Illustration

Assume that the life distribution is HETLD with $\lambda = 2$, $k = 2$, $\beta = 2$, $\theta = 2$. Suppose that the experimenter is interested in establishing that the true unknown average life is at least 1000 hours. Let the consumer's risk is set to be $1 - p^* = .25$. It is desired to stop the experiment at t=560 hrs. Then, for an acceptance number $c = 2$, the required n from Table 5 is 10. So the sampling plan is $(n = 10, c = 2, t/\eta_0 = .56)$. ie; if during 560 hours, not more than 2 failures out of 10 are observed then the experimenter can assert with confidence limit 0.75 that the average life is at least 1000 hours. If we use Poisson approximation to binomial the corresponding value is n=11 for the sampling plan $(n = 11, c = 2, t/\eta_0 = 0.56)$ with the consumer risk 0.25 under the HETLD, the operating characteristic values from Table 7 are,

$\frac{\eta}{\eta_0}$	2	4	6	8	10	12
L(p)	0.7790	0.9638	0.9886	0.9950	0.9974	0.9985

Application 1

Consider the following ordered failure times of the product, gathered from a software development project (Wood, 1996). The data set regarding the software reliability, presented by Wood (1996), was analyzed in the context of acceptance sampling by Rosaiah and Kantam (2005), Rao et al. (2008), Rao et al. (2009a, 2009b), Lio et al. (2010) and Rao and Kantam (2010). This data can be regarded as an ordered sample of size 12 with observations x_i, i=1,2,...12. The observations in the sample are

519, 968, 1430, 1893, 2490, 3058, 3625, 4422, 5218, 5823, 6539, 7083.

Let the specified average life be 1000 hours and the testing time is 560 hours, this leads to the ratio of $t/\eta_0 = 0.56$ with corresponding n and c as 12 and 1 from Table 7.1 for p^* =0.95. Therefore, the sampling plan for above sample data is $(n = 12, c = 1, t/\eta_0 = 0.56)$. We accept the lot only if the number of failures before 560 hours is less than or equal to 1. However, the confidence level is assured by the sampling plan only if the given lifetimes follow the HETLD. We have

Table 5: *Minimum sample size for the specified ratio t/η_0, confidence level p^*, acceptance number c, $\lambda = 2$, $k = 2$, $\beta = 2$, $\theta = 2$ using binomial approximation*

p^*	c	t/η_0							
		0.4	0.56	0.72	0.88	1	1.5	2	2.5
0.75	0	5	4	3	2	2	1	1	1
	1	11	7	5	4	4	3	2	2
	2	15	10	8	6	6	4	3	3
	3	20	14	10	8	7	5	5	4
	4	25	17	13	10	9	7	6	5
	5	29	20	15	12	11	8	7	6
	6	34	23	17	14	13	9	8	7
	7	38	26	20	16	14	10	9	8
	8	43	29	22	18	16	12	10	10
	9	47	32	24	20	18	13	11	11
	10	52	35	27	22	19	14	12	12
0.90	0	9	6	4	3	3	2	1	1
	1	15	10	7	6	5	3	3	2
	2	20	14	10	8	7	5	4	3
	3	26	17	13	10	9	6	5	5
	4	31	21	15	12	11	8	6	6
	5	36	24	18	14	13	9	7	7
	6	41	27	21	17	15	10	8	8
	7	46	31	23	19	17	12	9	9
	8	50	34	26	21	18	13	11	10
	9	55	37	28	23	20	14	12	11
	10	60	41	30	25	22	16	13	12
0.95	0	11	7	5	4	4	2	2	1
	1	18	12	9	7	6	4	3	3
	2	24	16	12	9	8	5	4	4
	3	29	20	14	11	10	7	5	5
	4	35	23	17	14	12	8	7	6
	5	40	27	20	16	14	10	8	7
	6	45	30	23	18	16	11	9	8
	7	50	34	25	20	18	12	10	9
	8	56	37	28	22	20	14	11	10
	9	61	41	30	24	21	15	13	12
	10	66	44	33	26	23	16	14	13
0.99	0	17	11	8	6	5	3	2	2
	1	24	16	12	9	8	5	4	3
	2	31	21	15	12	10	7	5	4
	3	37	25	18	14	12	8	6	6
	4	43	29	21	17	14	10	8	7
	5	49	33	24	19	16	11	9	8
	6	55	36	27	21	19	13	10	9
	7	61	40	30	24	21	14	11	10
	8	66	44	33	26	22	15	13	11
	9	71	48	35	28	24	17	14	12
	10	77	51	38	30	26	18	15	13

Table 6: *Minimum sample size for the specified ratio t/η_0, confidence level p^*, acceptance number c, $\lambda = 2$, $k = 2$, $\beta = 2$, $\theta = 2$ using Poisson approximation*

p^*	c	t/η_0							
		0.4	0.56	0.72	0.88	1	1.5	2	2.5
0.75	0	6	4	3	3	3	2	2	2
	1	11	8	6	5	5	4	3	3
	2	16	11	9	7	7	5	5	4
	3	21	15	11	10	9	6	6	6
	4	26	18	14	12	10	8	7	7
	5	31	21	16	14	12	9	9	8
	6	35	24	19	16	14	11	10	9
	7	40	28	21	18	16	12	11	10
	8	44	31	24	20	18	13	12	11
	9	49	34	26	22	19	15	13	13
	10	53	37	28	24	21	16	15	14
0.90	0	10	7	5	5	4	3	3	3
	1	16	11	9	7	7	5	5	4
	2	22	15	12	10	9	7	6	6
	3	28	19	15	12	11	9	8	7
	4	33	23	18	15	13	10	9	9
	5	38	26	20	17	15	12	11	10
	6	43	30	23	19	17	13	12	11
	7	48	33	26	21	19	15	13	12
	8	53	37	28	24	21	16	15	14
	9	58	40	31	26	23	18	16	15
	10	63	44	34	28	25	19	17	16
0.95	0	13	9	7	6	5	4	4	4
	1	20	14	11	9	8	6	6	5
	2	26	18	14	12	10	8	7	7
	3	32	22	17	14	13	10	9	8
	4	38	26	20	17	15	12	10	10
	5	43	30	23	19	17	13	12	11
	6	48	34	26	21	19	15	13	13
	7	54	37	29	24	21	16	15	14
	8	59	41	31	26	23	18	16	15
	9	64	45	34	28	25	20	18	16
	10	69	48	37	31	27	21	19	18
0.99	0	19	13	10	9	8	6	5	5
	1	27	19	15	12	11	9	8	7
	2	35	24	19	15	14	11	10	9
	3	41	29	22	18	16	13	11	11
	4	47	33	25	21	19	15	13	12
	5	54	37	29	24	21	16	15	14
	6	59	41	32	26	24	18	16	15
	7	65	45	35	29	26	20	18	17
	8	71	49	38	31	28	22	19	18
	9	78	53	41	34	30	23	21	20
	10	82	57	44	36	32	25	22	21

Table 7: *Values of the operating characteristic function of the sampling plan*$(n, c, \frac{t}{\eta_0})$

				$\frac{\eta}{\eta_0}$					
p^*	n	c	$\frac{t}{\eta_0}$	2	4	6	8	10	12
0.75	15	2	0.4	0.7613	0.9568	0.9858	0.9937	0.9967	0.9980
	10	2	0.56	0.7790	0.9638	0.9886	0.9950	0.9974	0.9985
	8	2	0.72	0.7528	0.9607	0.9879	0.9948	0.9973	0.9984
	6	2	0.88	0.7894	0.9700	0.9912	0.9963	0.9981	0.9989
	6	2	1	0.7127	0.9562	0.9870	0.9946	0.9972	0.9984
	4	2	1.5	0.7087	0.9598	0.9890	0.9956	0.9978	0.9988
	3	2	2	0.7493	0.9682	0.9918	0.9969	0.9985	0.9991
	3	2	2.5	0.5895	0.9344	0.9827	0.99346	0.9969	0.9983
0.90	20	2	0.4	0.5992	0.9111	0.9687	0.9857	0.9923	0.9954
	14	2	0.56	0.5850	0.9126	0.9702	0.9866	0.9929	0.9958
	10	2	0.72	0.6188	0.9278	0.9765	0.9897	0.9946	0.9968
	8	2	0.88	0.6159	0.9309	0.9782	0.9906	0.9951	0.9972
	7	2	1	0.6088	0.9315	0.9788	0.9910	0.9954	0.9973
	5	2	1.5	0.5269	0.9167	0.9754	0.9899	0.9949	0.9971
	4	2	2	0.4715	0.9030	0.9723	0.9890	0.9946	0.9970
	3	2	2.5	0.5895	0.9344	0.9827	0.9934	0.9969	0.9983
0.95	24	2	0.4	0.4764	0.8649	0.9499	0.9765	0.9872	0.9922
	16	2	0.56	0.4927	0.8799	0.9574	0.9805	0.9895	0.9937
	12	2	0.72	0.4908	0.8861	0.9610	0.9825	0.9907	0.9945
	9	2	0.88	0.5314	0.9058	0.9693	0.9866	0.9930	0.9959
	8	2	1	0.5094	0.9020	0.9685	0.9863	0.9929	0.9959
	5	2	1.5	0.5269	0.9167	0.9754	0.9899	0.9949	0.9971
	4	2	2	0.4715	0.9030	0.9723	0.9890	0.9946	0.9970
	4	2	2.5	0.2733	0.8172	0.9442	0.9774	0.9890	0.9938
0.99	31	2	0.4	0.3000	0.7699	0.9069	0.9543	0.9744	0.9843
	21	2	0.56	0.3013	0.7837	0.9158	0.9596	0.9777	0.9864
	15	2	0.72	0.3290	0.8114	0.9302	0.9675	0.9824	0.9894
	12	2	0.88	0.3178	0.8140	0.9330	0.9693	0.9836	0.9902
	10	2	1	0.3394	0.8315	0.9413	0.9736	0.9861	0.9918
	7	2	1.5	0.2508	0.7973	0.9315	0.9701	0.9845	0.9910
	5	2	2	0.2661	0.8140	0.9411	0.9754	0.9877	0.9930
	4	2	2.5	0.2733	0.8172	0.9442	0.9774	0.9890	0.9938

correlated the sample quantiles and the corresponding population quantiles to confirm that the given sample is generated by lifetimes following the HETLD and found a satisfactory agreement. Thus, the adoption of the decision rule of the sampling plan seems to be justified.
In the above sample there is only one failure at 519 hours before termination t=560 hours. Hence we accept the product. Here, we may note that termination time t is smaller than that of the sampling plan suggested by Krishna et al. (2013) and Rosaiah and Kantam (2005). Hence, the cost and the experimental time can be saved considerably by using present sampling plans.
Application 2
The second data set is obtained from tests on endurance of deep groove ball bearings (Lawless, 1982). The data are the number of million revolutions before failure for each of the 11 ball bearings in life test and they are,
51.84, 51.96, 54.12, 55.56, 67.80, 68.44, 68.64, 68.88, 84.12, 93.12.
Let the specified average life be 93 hours and the testing time is 53 hours, this leads to the ratio of $t/\eta_0 = 0.56$ with corresponding n and c as 10, 2 from Table 7.1 for p^* =0.75. Therefore, the sampling plan for above sample data is $(n = 10, c = 2, t/\eta_0 = 0.56)$. We accept the lot only if the number of failures before 53 hours is less than or equal to 2. Thus in the above sample of 10 failures there are 2 failures at 51.84, 51.96 before 53 hours, therefore we accept the product.

7. Conclusion

In this article, we consider the HETLD from the Harris extended family and discussed its properties. We derived expansions for the moments, hazard rate function, reversed hazard rate function, cumulative hazard rate function, mean residual lifetime distribution, quantiles and Renyi entropy. The estimation of parameters is approached by the method of maximum likelihood. The applicability and usefulness of the proposed model are presented by using the real data sets. Also a reliability test plan is developed when the lifetimes of the items follow the HETLD. The results are illustrated using two data sets on ordered failure times of products as well as failure times of ball bearings.

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