Certain modern developments in stochastic extreme value theory, on occasion of 110th birthday of Boris Vladimirovich Gnedenko

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Abstract

We present a short overview of developments of the last decade in asymptotic analysis of extrema of families of random variables. We focus on the methods of investigating the quality of approximations as given by Gnedenko's extreme value theorem, and its generalizations to the case of dependent random variables.

Keywords: Gnedenko, stochastic extreme value theory, modern developments, extreme value theorem.

I. Introduction

Let us consider a sequence $X_1, ..., X_n, ...$ of independent identically distributed random variables with the cumulative distribution function $F(x)$. Let us furthermore assume that there exists sequences of real numbers $a_n > 0$ and b_n such that the limit of the distribution functions of the sequence

$$\frac{\max(X_1, ..., X_n) - b_n}{a_n}$$

as $n \to \infty$ is non-degenerate, so that

$$\lim_{n\to\infty} F^n(a_n x + b_n) = G(x), \tag{1}$$

where the distribution function $G(x)$ takes more than two values. The maximum of random data is one of the key statistics in various applications, and so various possible forms of the function $G(x)$ were established early on in the 20s of the previous century, see [1]. But it was only in 1941 when Gnedenko, in a short note [2], published a rigorous mathematical statement describing all possible types of the distribution function $G(x)$, where the type of a distribution function $G(x)$ is understood to be a class of distributions obtained from $G(x)$ by shifting and scaling its argument. Let us state this result in modern notation.

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Theorem 1. *If* (1) *holds for some non-degenerate* G, *then there exist* $a > 0$ *and* b *such that* $G(ax + b) = G_\gamma(x)$, *where*

$$G_\gamma(x) = \exp\left(-(1 + \gamma x)^{-1/\gamma}\right), \quad 1 + \gamma x > 0, \tag{2}$$

the class of extreme value distributions with γ *real, and for* $\gamma = 0$ *the exponent on the right is interpreted as* $\exp(-e^{-x})$.

When $\gamma > 0$ this is the Frechet class, when $\gamma < 0$ this is the Weibull class, and for $\gamma = 0$ this is the Gumbel class, or, in this case, the standard Gumbel distribution.

The full proof of the theorem was published in 1943, [3] not in the Soviet Union for obvious reasons, but in Annals of Mathematics, in French. The English translation appeared in 1992 in a book Breakthroughs in Statistics published by Springer.

The importance of this work goes far beyond its use in the domain of applied probability theory and statistics. In our view, this is one of the cornerstones of the modern mathematical apparatus of the theory of probability. Every year since the result was discovered 80 years ago, a large number of papers that further develop mathematical methods in this area come out. It would not be an exaggeration to draw strong parallels with the central limit theorem which also came out of the needs of applications but since influenced the development of core mathematical methodologies of the whole of probability theory.

This short overview of the mathematical methods for asymptotic analysis of extrema of families of random variables is dedicated to the latest developments in this area, primarily covering the last decade, since the 100th birthday of Gnedenko that was widely celebrated by the mathematical community. We focus our attention on the areas that can be called a classical extension of the theory. Specifically we look into the quality of approximations given by Gnedenko's theorem, Theorem 1, and generalizations of the limit relation (2) to the case of dependent X_i that form a stochastic sequence or a random field on an integer lattice. There also exist various other generalizations of the original problem statement for limit distributions of maxima. This area of research mostly focuses on distributions of maxima of random processes and random fields in continuous time, extrema of vector sequences, and even functional limit theorems with follow-up analysis of the so-called max-stable stochastic processes. In short, here the focus is on limit distributions of maxima of random variables over various probabilistic structures. Among the latest on these topics the following are worth mentioning: [4] on the distribution of the maximum of a random number of random vectors, and [5], [6], [7] on max-stable processes and fields, as well as vector-valued random processes. All these papers have extensive literature reviews. From a somewhat different angle, considering triangular arrays, rather than sequences, of identically distributed random variables expands not only the class of possible limit distributions of normalized maxima, [8], [9], but also a class of distributions for which the limit distribution of the normalized maxima is non-degenerate, [11], [10]. There also exist results on the limit distribution of the maxima of stochastic sequences under non-linear normalization [12]. Lebedev in [13], [14] considers the problems of limit distributions of maxima of the particle scores in branching processes; the bibliography in these papers should also be perused. The author moves away from the classical conditions of the Gnedenko limit theorem, which is a substantial development of the theory of Lamperti-type maximal branching processes.

It is worth noting that we do not cover other types of convergence, focusing exclusively on convergence in distribution. We mention in passing one of the latest papers here, [15] and literature therein, on the iterated logarithm laws for almost sure convergence of sequences of maxima. Other works of I. Matsak on this topic are also of interest.

II. On the quality of convergence

The question of quantifying the quality of approximations in limit theorems of probability theory has many aspects. Broadly, the main topics of interest include convergence of moments; rates of convergence in limit theorems; rates of convergence for large and growing values of arguments (large deviations); convergence in probability and almost surely; asymptotic expansions and accompanying laws that improve the quality of approximations. An excellent review of relatively latest advances in the areas of moment convergence, rates of convergence, large deviations in the Gnedenko limit theorem, and sequences of normalized maxima can be found in Chapter 5 of a fairly current monograph [16]. Some of the more contemporary works covered in that review are also cited in our bibliography. The area of asymptotic expansions and accompanying measures (laws) is relatively mature, with only a few new developments appearing recently, mostly related to specific distributions important in certain applications, such as the Weibull distribution or the Normal distribution, see e.g. [17], [18].

It is important to point out that establishing asymptotic expansions and their accompanying laws is much easier for limit distributions of maxima of random variables than in the context of the central limit theorem [19], [20]. For maxima of *independent* random variables, deriving asymptotic expansions can basically just follow the approach developed by Gnedenko himself, or its somewhat more contemporary interpretations. A cumulative distribution function $F(x)$ from the maximum domain of attraction of the Gumbel distribution $MDA(\Lambda)$ can be characterized in terms of the von Mises function. As shown in [21], a distribution from $MDA(\Lambda)$ can be described via the von Mises representation. Specifically, under the assumption $F(x) < 1$ for all x, $F \in MDA(\Lambda)$ if and only if there exists $x_0 \geq 0$ such that $F(x)$ can be represented in the form

$$1 - F(x) = c(x) \exp\left\{ -\int_{x_0}^{x} \frac{g(t)}{f(t)} dt \right\}, \quad x \geq x_0, \tag{3}$$

where $f(x)$ is a positive absolutely continuous function on $[x_0, \infty)$, where $f'(x) \to 0$, $g(t) \to 1$ and $c(x) \to c > 0$ for $x \to \infty$. A similar statement can be made for a distribution bounded from the right. Normalizing sequences can be chosen as follows,

$$b_n = F^{\leftarrow}(1 - n^{-1}), \quad a_n = f(b_n).$$

It is obvious then that

$$\begin{aligned} F^n(a_n x + b_n) &= \left(1 - \exp\left(\log c(a_n x + b_n) + \int_{x_0}^{a_n x + b_n} \frac{g(t)}{f(t)} dt\right)\right)^n \\ &= \left(1 - \frac{1}{n} e^{-\gamma_n(x)}\right)^n, \end{aligned}$$

where

$$\gamma_n(x) := -\log n + \int_{b_n}^{a_n x + b_n} \frac{g(t)}{f(t)} dt - \log \frac{c(a_n x + b_n)}{c(b_n)}. \tag{4}$$

Let us denote

$$B_n(x) := e^{-e^{-\gamma_n(x)}} \mathbf{I}_{\{\gamma_n(x) \geq -\log\log n\}}, \tag{5}$$

where $\mathbf{I}$ is the indicator function. Paper [22] uses standard calculus techniques, under the assumption of $F(x) < 1$ for all x, to demonstrate that

$$P(M_n \leq a_n x + b_n) - B_n(x) = O\left(n^{-1} \log^2 n\right) \tag{6}$$

for $n \to \infty$, uniformly in $x \in \mathbb{R}$. This implies that the equality

$$P(M_n \leq a_n x + b_n) - \exp\left(-e^{-x}\right) = \exp\left(-e^{-x}\right) e^{-x} (\gamma_n(x) - x)(1 + o(1)) + O(n^{-1} \log^2 n) \tag{7}$$

holds uniformly on the set $\{x : \gamma_n(x) \geq -\log\log n\}$ as $n \to \infty$. Naturally, the idea of using the Taylor expansion applied to a power of the distribution function appears in various other works on distributions of maxima such as [23] and other references we cite.

Thus, the sequence $B_n(x)$ is the natural sequence of accompanying laws, i.e. signed measures, in Gnedenko's limit theorem. It gives an exponential-type rate of convergence to the distribution of the maximum. The same characterisation holds for the two other maximum domains of attraction, Frechét and Weibull. For them, an analogue to the representation (3) is obtained using Karamata representation for regularly varying functions, see for example [16], [24]. Expansions and accompanying laws can be derived along the same lines as our calculations above. We remind the reader that the Frechét maximum domain of attraction consists only of distributions with tails that are regularly varying at infinity, and the Weibull maximum domain of attraction with regularly varying tails at a finite right endpoint.

In [22] the authors consider the Gumbel maximum domain of attraction, where the double exponential gives a logarithmic rate of convergence only, which is often insufficient in applications. Another reason for considering this domain specifically is the fact that it is extremely broad, and various applications require splitting it into reasonable, in some sense, sub-domains. For example, this domain includes distributions whose tails are equivalent, for $x \to \infty$, to the tail of the Weibull distribution $\log(1 - F(x)) \sim -Cx^p$, $C, p > 0$, as well as log-Weibull tail, $\log(1 - F(x)) \sim -C(\log x)^p$, $C > 0, p > 1$. Moreover, the exponents in the asymptotics can be replaced by slowly varying at infinity functions. A wide variety of other distributions with heavier (slower decaying) or lighter (faster decaying) tails belongs to the same domain. The Weibull and log-Weibull classes of distributions are considered in detail in [22] as specific examples.

One of the principal recent approaches to the study of rates of convergence in the limit theorem for the maxima has been an introduction of additional conditions on the distribution tail behavior. Primarily this is the second-order condition suggested by de Haan [26]. Let us state this condition in terms of the function $\gamma_n(x)$.

The second-order condition for functions from $MDA(\Lambda)$ with an infinite right tail. *There exists a sequence $A(n)$ of constant sign, approaching zero as $n \to \infty$ and such that the limit*

$$\lim_{n\to\infty} \frac{e^{-\gamma_n(x)} - e^{-x}}{A(n)} = H(x) \tag{8}$$

exists and is not identically zero or infinite.

This formulation is based on Theorem 2.3.8, [16]. It follows from the second-order condition (see e.g. [16]) that $A(n)$ is a slowly varying at infinity function of non-positive index $\rho \leq 0$. It is also known, see [27], that for the case of convergence to the Gumbel distribution we are considering here, the function H is equal to

$$H(x) = \frac{1}{\rho}\left(\frac{x^\rho - 1}{\rho} - \log x\right), \text{ if } \rho < 0, \tag{9}$$

and

$$H(x) = \frac{1}{2}\log^2 x \text{ if } \rho = 0.$$

Using the aforementioned Theorem 2.3.8, [16], and (4), one can obtain a somewhat different asymptotic expansion,

$$P(M_n \leq a_n x + b_n) = \exp\left\{-e^{-x} - A(n)H(x)(1 + o(1)\right\}$$
$$\times \exp\left(-\frac{1}{n}\sum_{k=0}^{\infty}\frac{1}{(k+2)n^k}\left(\frac{1 - F(a_n x + b_n)}{1 - F(b_n)}\right)^{k+2}\right).$$

We note that if $\rho < -1$, the main contribution to the speed of convergence to the double exponential distribution is given by the second exponent. In the case $\rho = -1$, on the other hand, one needs to know the behavior of the function $A(n) = n^{-1}\ell(n)$ more precisely, i.e. how the slowly varying function $\ell(n)$ behaves. In the case $\rho > -1$, the second term in the first exponent is the main contributing factor to the rate of convergence.

Similar calculations can be carried out for the n-th order condition on the distribution tail introduced in [28]. Let us state a recent estimate by Drees and de Haan (see [29]) for the rate of convergence taking into account the accompanying law.

If the condition (8) is satisfied with $\rho < 0$, see (9), then for $b_n = F^{\leftarrow}\left(e^{-1/n}\right)$ and, correspondingly, $a_n = f(b_n)$, and for any $\varepsilon > 0$, the following holds,

$$\sup_x e^{(1-\varepsilon)x} \left| \frac{F^n(a_n x + b_n) - \exp(-e^{-x})}{A(n)} + \frac{1}{\rho} e^{-x+\rho x} e^{-e^{-x}} \right| \to 0$$

for $n \to \infty$.

Note that this can also be derived from the expansion (7).

It is also interesting to use the expansion (7) to study probabilities of large deviations in the Gnedenko limit theorem. For example, Corollary 2.1, [29] and Theorem 5.3.12, [16], under suitable restrictions, follow from the relation (7). [24] uses similar expansions for this purpose.

Scale in $MDA(\Lambda)$. As we already mentioned, the Gumbel maximum domain of attraction is extremely broad, and the idea of splitting it into parts and developing criteria for classifying distributions into these sub-domains is quite reasonable. [22] proposes one such classification of distributions with smooth tails, based on the von Mises representation. The first two "grades" in this scale are the generalized distributions of Weibull and log-Weibull type, defined by functions $f(t) = Ct^{1-p}$, $C, p > 0$, and $f(t) = Ct \log^{1-p} t$, $C > 0$, $p > 1$ in the representation (3), respectively. These distributions play an important role in financial and actuarial mathematics, in reliability theory, and other industrial applications. Yet the information on distribution tails obtained from the approximation provided by the Gnedenko theorem is far from complete. For example, insurance premiums directly depend on the specific type of the tail of the distribution that generates a given insurance event. Recently a number of studies appeared that aim to distinguish tails of Weibull and log-Weibull type distributions, see for example [30], [31], [32] and their bibliographies.

The continuation of the scale that begins with the two aforementioned classes of distributions can proceed as follows. Distribution tails with $f(t) = Ct(\log\log t)^{1-p}$, $p > 1$, are heavier than Weibull and log-Weibull type ones. (Here C denotes some constant that could be different in different contexts.) The number k of iterated logarithms in these expressions for $f(t)$ could be defined to be Gumbel's index for the distribution. Then tails of distributions of Weibull type have Gumbel's index $k = 0$, log-Weibull type distributions have index $k = 1$, and so on. More details can be found in [22].

Shubochkin in his thesis [33] determines convergence rates for approximations of distributions of normalized maxima and their accompanying laws, see (7).

The definition of the scale the we presented above is not the only reasonable option, and alternatives have been proposed. For example, Troshin in his thesis [34] considers an alternative definition of Gumbel's index, defined to be the smallest $k = 0, 1, ...$ such that the integral

$$\int_{x_0}^{\infty} \frac{a(t)dt}{t^2 \log t \log_{(2)} t ... \log_{(k)} t}.$$

converges. (Indices mean numbers of $\log ... \log$ repeating.) The existence of such k for functions from $MDA(\Lambda)$ has been proved.

III. Models with dependence

One of the first follow-up questions that Gnedenko's limit theorem elicits is whether its results could be generalized to sequences of dependent and/or non-identically distributed random variables. First results of this type, mostly concerning distributions of maxima of stationary sequences, appeared back in the 60s and 70s in the works by Berman, Loynes, Cramér, and Leadbetter. The results obtained during this period are comprehensively covered in the monograph [35] by Leadbetter, Lindgren and Rootzén.

A situation when a maximum over some collection of random variables behaves like a maximum of independent random variables has a special name in the extreme value theory, and is called *extremal independence*. For example, if $\{X_i\}_{i\geq 1}$ is a sequence of random variables, with F_i the distribution function of X_i, and $M_n = \max\{X_1, \dots, X_n\}$, then this sequence possesses extremal independence if

$$\sup_{x\in\mathbb{R}} \left| P(M_n \le x) - \prod_{i=1}^{n} F_i(x) \right| \to 0, \quad n \to \infty.$$

We note that if $\{X_i\}_{i\ge 1}$ are identically distributed according to the (common) distribution function F, and this sequence is extremely independent, then the distribution of the normalized maximum converges to one of the three types of limit distributions from Gnedenko's limit theorem, as long as F satisfies the conditions of the theorem.

Stationary stochastic sequences provide an important example. Let us recall classical (in extreme value theory) sufficient conditions for extremal independence of a stationary sequence, namely the conditions D and D' from [35]. Let $\{X_i\}_{i\ge 1}$ be a (strictly) stationary sequence with a marginal distribution function F. We say that it satisfies the condition $D(u_n)$ for a sequence u_n if for any integers $1 \le i_1 < \ldots < i_p$ and $j_1 < \ldots < j_q \le n$, for which $j_1 - i_p \ge l$, the following holds,

$$\left| P(\max(X_{i_1}, ..., X_{i_p}, X_{j_1}, ..., X_{j_q}) \le u_n) - P(\max_{k\in[p]} X_{i_k} \le u_n) P(\max_{k\in[q]} X_{j_k} \le u_n) \right| \le \alpha_{l,n}, \quad (10)$$

where $\alpha_{l_n,n} \to 0$ as $n \to \infty$ for some index sequence $l_n = o(n)$. Furthermore, we say that the stationary sequence $\{X_i\}_{i\ge 1}$ satisfies the condition $D'(u_n)$ for a sequence u_n if

$$\limsup_{n\to\infty} n \sum_{j=2}^{[n/k]} P(X_1 > u_n, X_j > u_n) \to 0, \quad k \to \infty \quad (11)$$

holds (here $[\cdot]$ is an integer part of a number). Then, if for independent copies of random variables$\{X_i\}_{i\ge 1}$ the conditions of Gnedenko's limit theorem are satisfied for some sequences a_n and b_n, and the conditions $D(u_n(x))$ and $D'(u_n(x))$ are satisfied for the sequence $u_n = a_n x + b_n$ for any x, then the sequence $\{X_i\}_{i\ge 1}$ is extremely independent.

The conditions D and D' play a foundational role in the extreme value theory for stochastic models with dependence. The conditions have been slightly modified in [36] and [37] to extend the result above to non-stationary sequences. Papers [38] and [39] extended it even further to stationary random fields on integer lattices, and [40] proved an equivalent result for non-stationary random fields in dimension 2. The main technique that was used in all these proofs was the so-called block method, where the domain of the stochastic process is split into non-overlapping intervals in such a way that maxima over the intervals are asymptotically independent. Applications of this method to random fields required very complicated versions of the conditions D and D' which hindered further progress along similar lines of attack. This issue was finally overcome in [41] (see full text in [42]). These papers derived the conditions for extremal independence of random variables that constitute so-called stochastic systems. A stochastic system here is defined as a sequence $(X_1(n), \dots, X_d(n)) \in \mathbb{R}^d$ of random vectors of varying dimensions, where $d = d(n)$ is some sequence of positive integers. We should also mention here [43] whose results can be used to derive asymptotics for the distribution of the maxima of a stochastic system under certain conditions. Stochastic systems generalize many models such as stochastic sequences, stochastic fields and triangular arrays. They are rich enough to even represent complicated objects such as random networks and graphs that are otherwise quite challenging to analyze.

Gaussian stochastic sequences, a special case of stochastic sequences, exhibit extremal independence under rather weak assumptions. Let $\{X_i\}_{i\ge 1}$ be a Gaussian random sequence with mean 0 and covariance function $r(i)$, where $r(0) = 1$. Berman [44] found a simple condition for the convergence of the distribution of the maximum of the sequence $\{X_i\}_{i\ge 1}$, with the same normalization as for the maximum of independent Gaussian variables in Gnedenko's limit theorem, to the Gumbel distribution. The Berman condition simply requires that

$$r(n) \ln n \to 0, \quad n \to \infty.$$

It has been established that the Berman's condition implies the conditions D and D' for stationary Gaussian sequences. Hüsler in [36] and [37] showed that under a certain generalization of the Berman's condition, a non-stationary Gaussian sequence is extremely independent. Pereira [45] obtained this result for Gaussian non-stationary random fields in $\mathbb{R}^2$, while Jakubowski and Soja-Kukieła in [46] extended it to Gaussian stationary fields of arbitrary dimension.

It is interesting that the Berman's condition is close to being necessary (as well as sufficient) for extremal independence of stationary Gaussian sequences. Specifically, Mittal and Ylvisaker in [47] showed that if $r(n)\ln n \to \gamma > 0$ as $n \to \infty$, then the limit distribution of the normalized maximum of a Gaussian stationary sequence is completely different, and is a convolution of the Gumbel distribution and the Gaussian one. In this case the sequence does not even possess a phantom distribution function (to be defined shortly).

The extremal independence property is far from being always satisfied, and processes that appear in applications often exhibit a high degree of dependence. It turns out that in many cases, the behavior of the maximum of a stationary sequence can be described in terms of the so-called extremal index. According to the definition from [35], a stationary sequence $\{X_i\}_{i\geq 1}$ has the extremal index $\theta \in [0,1]$, if for any $\tau > 0$ there exists a sequence $u_n(\tau)$ such that

$$n(1 - F(u_n(\tau))) \to \tau \quad \text{and } P(M_n \leq u_n(\tau)) \to e^{-\theta\tau}.$$

It follows, in particular, that

$$\left|P(M_n \leq u_n(\tau)) - F^{\theta n}(u_n(\tau))\right| \to 0, \quad n \to \infty,$$

so that the maximum of n terms of the stationary sequence behaves like the maximum of θn independent copies of X_1. It should now be obvious that the situation we considered just before corresponds to the case $\theta = 1$. The notion of the extremal index is of paramount importance in applications of extreme value theory, because one can often reduce a sequence of real-world observations to a stationary sequence, or simply consider it to be such.

Sufficient conditions for the existence of the extremal index were found by Chernick [48], and they look like this. Let us assume that for $\tau > 0$ a sequence $u_n(\tau)$ is defined such that $n(1 - F(u_n(\tau))) \to \tau$ as $n \to \infty$, and the condition $D(u_n(\tau))$ is satisfied for any such $\tau > 0$. Then, if for some τ the sequence $P(M_n \leq u_n(\tau))$ converges, then the extremal index exists for $\{X_i\}_{i\geq 1}$. It is interesting to note that the criterion for the existence of the extremal index has only been found relatively recently, [49], see also Proposition 11.4, [23]. Various other properties of the extremal index, as well as methods for its estimation, are covered in Chapter 10 of [50].

The extremal index provides a remarkably convenient mechanism for describing extremal dependence in stationary sequences. A single index, however, is not sufficient for describing extremal dependence of stationary random fields on integer lattices. Various attempts to extend the idea of the extremal index to random fields and use it to analyze extremal dependence have been undertaken in, for example, [51] and [52], with more complex models considered in [53]. However, [54] showed that the extremal index of a stationary random field on $\mathbb{Z}^d$ can materially depend on the direction of growth of a multi-index $\mathbf{n} = (n_1, \ldots, n_d)$.

A natural generalization of the notion of the extremal index is provided by the notion of a phantom distribution function, as discussed in [55]. We say that the distribution function G is the phantom distribution function for the stationary sequence$\{X_i\}_{i\geq 1}$ with the marginal distribution function F if

$$\sup_{x\in\mathbb{R}} |P(M_n \leq x) - G^n(x)| \to 0, \quad n \to \infty.$$

It is not hard to see that if G could be chosen to be of the form F^θ, then the extremal index of the sequence $\{X_i\}_{i\geq 1}$ is θ. The existence of a phantom distribution function for a stationary sequence is quite a common property. For example, it is shown in [49] that any α-mixing stationary sequence with a continuous marginal distribution function has a phantom distribution function. The same paper suggests a simple condition for the existence of the phantom distribution function: it exists if and only if for some sequence ν_n and $\gamma \in (0,1)$ the convergence$P(M_n \leq \nu_n) \to \gamma$ holds as $n \to \infty$, and for all $T > 0$ the condition $B_T(\{v_n\})$:

$$\sup_{p,q \in \mathbb{N}: p+q \le T \cdot n} \left| P(M_{p+q} \le v_n) - P(M_p \le v_n) P(M_q \le v_n) \right| \to 0, \quad n \to \infty$$

is satisfied. Clearly the condition $B_T(\{v_n\})$ resembles Leadbetter's condition D. Theory of phantom distribution functions for models other than stationary sequences is still in its infancy. In this regard it is worth mentioning [54] where, for the first time ever, the question of existence of phantom distribution functions for stationary random fields on integer lattices is considered.

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